Truth in a Logic of Formal Inconsistency: How classical can it get?

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Abstract

Weakening classical logic is one of the most popular ways of dealing with semantic paradoxes. Their advocates often claim that such weakening does not affect nonsemantic reasoning. Recently, however, Halbach and Horsten (2006) have shown that this is actually not the case for Kripke's fixed-point theory based on the Strong Kleene evaluation scheme. Feferman's axiomatization KF in classical logic is much stronger than its paracomplete counterpart PKF, not only in terms of semantic but also arithmetical content. This paper compares the proof-theoretic strength of an axiomatization of Kripke's construction based on the paraconsistent evaluation scheme of LP formulated in classical logic with that of an axiomatization directly formulated in LP extended with a consistency operator. The ultimate goal is to find out whether paraconsistent solutions to the paradoxes that employ consistency operators fare better in this respect than paracomplete ones.

Keywords— Kripke fixed points, LFIs, sequent-calculus truth theories, proof-theoretic strength

The semantic paradoxes reveal a conflict between classical reasoning and intuitively correct truth principles such as transparency, that is, the equivalence between each sentence and its truth predication. For they allow us to derive every sentence of the language of our naïve truth theories, trivialising them.

More often than not logicians have advocated the weakening of classical logic to block the paradoxes and avoid triviality. Some hold independent reasons to believe classical logic is incorrect, while others, such as Kripke [21], Field [16], and Beall [7], somehow regret the conflict and seek to keep the trimming of classical reasoning to a minimum. This paper is addressed to the latter. To justify their drastic move, logicians of this view often claim that the restrictions imposed on classical inferences need not affect non-semantic reasoning, but just that involving the truth predicate. In other words, classical logic would still apply unrestrictedly to mathematic, scientific, and every other kind of non-semantic discourse.

However, we can see this is not always the case. In [19], Halbach & Horsten compare two different axiomatisations of Kripke's family of fixed-point models with the strong Kleene evaluation scheme: KF, formulated in classical logic, and PKF, formulated in basic De Morgan logic. They show that (the internal logic of) KF is stronger than PKF, not only with respect to their truth-theoretic content, but also with respect to their truth-free consequences. As Halbach & Nicolai [20, p. 2] argue, "[. . .] the deductive weakness of PKF arises from the mutilation of classical logic, which invalidates certain patterns of mathematical reasoning that cannot be regained in PKF in any way." To show that neither the calculus nor the truth principles of the non-classical theory are to blame for its weakness, they provide two results. First, the completeness of the calculus in which PKF is formulated with respect to basic De Morgan logic. Second, that both systems prove the same sentences to be true and determine the same models if induction—arguably not a truth-theoretic but an arithmetical principle—is restricted to formulae not containing the truth predicate. Moreover, Halbach & Nicolai show that an analogous argument can be made for two extensions of PKF, a paracomplete and a paraconsistent one. They result from the addition of truth-theoretic principles so as to block the possibility of truth-value gluts and truth-value gaps, respectively.

A novel and growing field of inquiry in logic is given by a family of paraconsistent logics, that is, systems in which contradictions do not necessarily entail triviality, called 'Logics of Formal Inconsistency' (LFIs).[1] They enrich the language of paraconsistent logics to express consistency or classicality operators. When applied to a sentence, these operators indicate that the sentence together with its negation does actually entail everything. According to Carnielli, Coniglio, & Marcos [12, p. 1], the fundamental feature of the LFIs "is the ability to recover all consistent reasoning *right on demand*, while still allowing for some inconsistency to linger, otherwise" (my italics).

The obvious question is whether an axiomatisation of Kripke's fixed-point models with gluts and no gaps in a suitable LFI could overcome the weaknesses the other theories suffer from, saving mathematical reasoning in full. I show that this is not the case and extract a disjunctive conclusion: either consistency operators are not capable of recovering classical reasoning *in every case* after all, contrary to what its promoters maintain, or the weakening of classical logic in LP is so severe that the consistency of arithmetical principles is simply impossible to recover.

In section 1 I give a semantic presentation of the Logic of Paradox, LP, and introduce Kripke's fixed-point models based on this logic. In section 2 I give a sequent calculus for LP, and semantic and proof-theoretic presentations of the LFI LP°, that results from extending LP with a consistency operator. Soundness and completeness proofs for LP and LP° can be found in the appendix. I then provide an axiomatisation in LP° of the fixed-point models introduced in section 1, and dub it CKF. Finally, an axiomatisation, KFG, of the same family of models in classical logic is given. In 3 I show that (the internal logic of) KFG is stronger than CKF, not only with respect to its truth-theoretic consequences, but also with respect to its non-semantic content. I also provide a result that shows that the weakness of CKF cannot be blamed on its truth-theoretic axioms. In 4 I infer, as Halbach & Nicolai [20] do, that the deductive weakness of the non-classical axiomatisation is the result of crippling classical logic. I further conclude that the consistency operators featured in LFIs do not do the job they were originally intended to do.

1 Complete paraconsistent fixed-point models

LP was originally introduced by Asenjo [1] in 1966 for the study of paradoxes, and adopted years later by Priest [25] for the same purpose. In models of LP, sentences can be true, false, or both true and false. The paradoxes are treated as sentences of the latter kind.

Kripke's original fixed-point construction as conceived in [21] provides a family of

[1] The LFIs were first introduced as such by Carnielli & Marcos [13] and further developed by Carnielli, Coniglio, & Marcos [12].

non-classical models for a language containing its own truth predicate based on Strong Kleene logic. In such models sentences can be true, false, or neither true nor false. The paradoxes fall under the latter category. The models are said to be incomplete, for they admit truth-value gaps. Making slight modifications to Kripke's construction, one can obtain models based on LP for a language containing its own truth predicate, that is, complete models that do not admit truth-value gaps but do admit truth-value gluts instead.

In this section I first introduce the semantics of LP for arbitrary first-order languages. Then, I present the languages $\mathscr{L}$ and $\mathscr{L}_{\mathrm{T}}$, with which we work in the remainder of the paper. $\mathscr{L}_{\mathrm{T}}$ contains a monadic predicate symbol T intended to serve as a truth predicate. Finally, I briefly sketch a version of Kripke's fixed-point construction to provide models for $\mathscr{L}_{\mathrm{T}}$ based on LP in which T is a truth predicate, in a sense to be specified.

1.1 The Logic of Paradox

Let $\mathcal{L}$ be a first-order language. $\mathcal{L}$ contains $\neg, \vee, \exists$, and $=$ as primitive logical symbols. Other symbols such as $\wedge, \rightarrow, \leftrightarrow, \forall$ are defined as usual.
Definition 1. An LP-*model* $\mathcal{M}$ of $\mathcal{L}$ consists of a non-empty set $|\mathcal{M}|$, the domain, and an interpretation function $.^{\mathcal{M}}$ such that

- if c is an individual constant, $c^{\mathcal{M}} \in |\mathcal{M}|$,
- if f is an n-ary function symbol, $f^{\mathcal{M}} \in \{f \mid f : |\mathcal{M}|^n \to |\mathcal{M}|\}$,
- if R is an n-ary relation symbol, $R^{\mathcal{M}} = \langle R^+, R^- \rangle$ and $R^+ \cup R^- = |\mathcal{M}|^n$,
- $=^+$ is the set $\{\langle a, a \rangle \mid a \in |\mathcal{M}|\}$.

R^+ and R^- are to be understood as the extension and the anti-extension of R in $\mathcal{M}$. Note that nothing precludes the extension and anti-extension of a predicate to overlap, but together they must exhaust the whole domain.
Definition 2. A *variable assignment* σ on an LP-model $\mathcal{M}$ of $\mathcal{L}$ is a function that assigns members of $|\mathcal{M}|$ to each variable of the language. We extend σ recursively to every term of the language as follows:

- if c is an individual constant, $\sigma(c) = c^{\mathcal{M}}$,
- if f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, $\sigma(f(t_1, \ldots, t_n)) = f^{\mathcal{M}}(\sigma(t_1), \ldots, \sigma(t_n))$.

Definition 3. Let $\mathcal{M}$ be an LP-model of $\mathcal{L}$, and let σ be an assignment on $\mathcal{M}$. A *valuation* $v_\sigma^{\mathcal{M}}$ in $\mathcal{M}$ is a function that assigns values from the set $\{0, \frac{1}{2}, 1\}$ to each formula of $\mathcal{L}$ as follows:

- $v_\sigma^{\mathcal{M}}(Rt_1 \ldots t_n) = \begin{cases} 1, & \text{if } \langle \sigma(t_1), \ldots, \sigma(t_n) \rangle \in R^+ \setminus R^- \\ 0, & \text{if } \langle \sigma(t_1), \ldots, \sigma(t_n) \rangle \in R^- \setminus R^+ \\ \frac{1}{2}, & \text{otherwise,} \end{cases}$
- $v_\sigma^{\mathcal{M}}(\neg\varphi) = 1 - v_\sigma^{\mathcal{M}}(\varphi)$,
- $v_\sigma^{\mathcal{M}}(\varphi \vee \psi) = max\{v_\sigma^{\mathcal{M}}(\varphi), v_\sigma^{\mathcal{M}}(\psi)\}$,
- $v_\sigma^{\mathcal{M}}(\exists v\varphi) = max\{v_{\sigma'}^{\mathcal{M}}(\varphi) \mid \sigma'(u) = \sigma(u) \text{ for all } u \neq v\}$.

If $\Gamma \subseteq \mathcal{L}$, we write $v_\sigma^{\mathcal{M}}(\Gamma) = i$, $i = 0, \frac{1}{2}, 1$, to indicate that $v_\sigma^{\mathcal{M}}$ maps all members of Γ to i. If φ contains no free variables, the assignment becomes irrelevant, that is, $v_\sigma^{\mathcal{M}}(\varphi) = v_{\sigma'}^{\mathcal{M}}(\varphi)$ for every σ, σ'. In these cases, we sometimes omit the subscript and simply write $v^{\mathcal{M}}(\varphi) = i$, $i = 0, \frac{1}{2}, 1$. We then say $v^{\mathcal{M}}$ is *the* valuation function in $\mathcal{M}$. In a valuation, 1 stands for truth, 0 for falsity, and $\frac{1}{2}$ for both truth and falsity.

Definition 4. Let $\mathcal{M}$ be an LP-model of $\mathcal{L}$. A sentence $\varphi \in \mathcal{L}$ is *true in* $\mathcal{M}$ ($\mathcal{M} \vDash_{\mathsf{LP}} \varphi$) iff $v^{\mathcal{M}}(\varphi) \geqslant \frac{1}{2}$.

If $\neg\varphi$ is true in an LP-model, we say φ is false in that model. Note that in an LP-model $\mathcal{M}$ there are no truth-value gaps. For every sentence φ either $\mathcal{M} \vDash_{\mathsf{LP}} \varphi$ or $\mathcal{M} \vDash_{\mathsf{LP}} \neg\varphi$. Since the extension and anti-extension of every predicate (including the identity predicate) are allowed to overlap, in some models both a sentence and its negation will turn out to be true, that is, there will be sentences that are both true and false in some models. This means that in LP-models truth-value gluts are allowed. In particular, all identity statements of the form $t = t$ will come out true in every model, but in some models also false.

Definition 5. Let $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$. φ is a *semantic consequence* of Γ in LP ($\Gamma \vDash_{\mathsf{LP}} \varphi$) iff, for every LP-model $\mathcal{M}$ and assignment σ on $\mathcal{M}$, if $v_{\sigma}^{\mathcal{M}}(\Gamma) \geqslant \frac{1}{2}$, then $v_{\sigma}^{\mathcal{M}}(\varphi) \geqslant \frac{1}{2}$. If $\Delta \subseteq \mathcal{L}$, we say Δ is a semantic consequence of Γ in LP ($\Gamma \vDash_{\mathsf{LP}} \Delta$) iff $\Gamma \vDash_{\mathsf{LP}} \varphi$ for some $\varphi \in \Delta$.

Since $\frac{1}{2}$ represents a kind of truth in LP-models, it must also be preserved from premises to conclusion in valid arguments.

LP accommodates arguments with multiple conclusions. If Δ is finite, we can say that the argument from Γ to Δ is semantically valid just in case the disjunction of the formulae in Δ is a semantic consequence of Γ.

The following definition will become handy later in this section.

Definition 6. The *complexity* $c(\varphi)$ of a formula $\varphi \in \mathcal{L}$ is defined inductively as follows:[2]

- $c(Rt_1 \dots t_n) = 0$,
- $c(\neg\varphi) = c(\varphi) + 1$,
- $c(\varphi \vee \psi) = c(\varphi) + c(\psi) + 1$,
- $c(\exists v\varphi) = c(\varphi) + 1$.

1.2 The language of truth

$\mathscr{L}$ is the language of first-order arithmetic. It contains an individual constant 0, and function symbols S, $+$, and $\times$ for successor, addition, and multiplication, respectively. It also contains finitely many function symbols for primitive recursive (p.r.) functions, to be specified. The only atomic formulae of $\mathscr{L}$ are of the form $s = t$, where s and t are terms. Let $\mathbb{N}$ be the standard model of $\mathscr{L}$ and let ω be its domain. For every $n \in \omega$, the term given by 0 preceded by n occurrences of S is the *numeral* of n. We denote it $\bar{n}$.

Let $\mathscr{L}_{\mathrm{T}} := \mathscr{L} + \mathrm{T}$ be obtained by adding a primitive monadic predicate symbol T to $\mathscr{L}$, and let $\mathscr{L}_{\mathrm{T}}^{\circ} := \mathscr{L}_{\mathrm{T}} + \circ$ be obtained by adding a primitive monadic operator $\circ$ to $\mathscr{L}_{\mathrm{T}}$. We assume a standard (i.e. effective and monotonic) Gödel coding for expressions of $\mathscr{L}_{\mathrm{T}}^{\circ}$. If ϵ is an expression of the language, we write $\#\epsilon$ for its code, and $\ulcorner\epsilon\urcorner$ for the numeral of its code. $\mathscr{L}_{\mathrm{T}}$ contains formulae $\mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(x)$, $\mathrm{Cterm}_{\mathscr{L}_{\mathrm{T}}}(x)$, and $\mathrm{Var}_{\mathscr{L}_{\mathrm{T}}}(x)$ strongly representing the sets of sentences, closed terms, and variables of $\mathscr{L}_{\mathrm{T}}$, respectively. $\mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}$ also features in the metalanguage as a name for the corresponding set. We often write $\varphi(\mathbf{t})$ and $\varphi(\mathbf{v})$ as short for $\mathrm{Cterm}_{\mathscr{L}_{\mathrm{T}}}(v) \wedge \varphi(v)$ and $\mathrm{Var}_{\mathscr{L}_{\mathrm{T}}}(v) \wedge \varphi(v)$, correspondingly. Let $\exists\mathbf{t}\varphi$ abbreviate $\exists x(\mathrm{Cterm}_{\mathscr{L}_{\mathrm{T}}}(x) \wedge \varphi)$. $\mathscr{L}_{\mathrm{T}}$ also contains a formula strongly representing the recursive function that maps the code of

[2]Since $=$ is a binary relation symbol, for any two terms s, t, $c(s = t) = 0$.

a term of $\mathscr{L}$ to its value. Although it is not possible to have a function symbol for it in the language on pain of triviality, we write $\text{val}(x) = y$, for perspicuity.[3]

$\mathscr{L}_{\text{T}}$ contains a function symbol $\underset{\cdot}{=}$ representing the p.r. function that maps the codes of two terms s and t to the code of $s = t$. $\underset{\cdot}{\text{T}}$ does a similar job for T. $\underset{\cdot}{\neg}$ represents in $\mathscr{L}_{\text{T}}$ the p.r. function that assigns the code of $\neg\varphi$ to the code of each formula φ, and similarly for $\underset{\cdot}{\vee}$. $\underset{\cdot}{\exists}$ represents the function that takes the code of a formula φ and a variable v and returns the code of $\exists v\varphi$. Let $\text{num}(x)$ represent the function that maps each number to the code of its numeral. Finally, s represents the function that takes the codes of a formula φ, term t, and variable v, and returns the code of $\varphi[t/v]$, that is, the result of substituting all free occurrences of v in φ with t, if v is free for t in φ. Let $\ulcorner\varphi(\dot{x})\urcorner$ abbreviate $\text{s}(\ulcorner\varphi(x)\urcorner, \text{num}(x), \ulcorner x\urcorner)$, so that x is free in $\ulcorner\varphi(\dot{x})\urcorner$, and let $\ulcorner\varphi(\underset{\cdot}{\mathbf{t}})\urcorner$ abbreviate $\text{s}(\ulcorner\varphi(x)\urcorner, \mathbf{t}, \ulcorner x\urcorner)$, so we can quantify over $\mathbf{t}$ in $\ulcorner\varphi(\underset{\cdot}{\mathbf{t}})\urcorner$ as well.

Due to Tarski's theorem, we know there is no classical expansion of $\mathbb{N}$ to $\mathscr{L}_{\text{T}}$ in which $\text{T}\ulcorner\varphi\urcorner$ and φ receive the same truth value for every sentence φ of $\mathscr{L}_{\text{T}}$. The traditional way to see this is via the liar paradox. Let $\langle\mathbb{N}, S\rangle$ be a classical expansion of $\mathbb{N}$ to $\mathscr{L}_{\text{T}}$, where $S \subseteq \omega$ is the extension of T in the model. By the diagonal lemma, provable in Robinson arithmetic, there is a sentence $\lambda \in \mathscr{L}_{\text{T}}$ such that $\lambda \leftrightarrow \neg\text{T}\ulcorner\lambda\urcorner$ is true in $\langle\mathbb{N}, S\rangle$. Thus, we cannot have a transparent truth predicate in $\langle\mathbb{N}, S\rangle$.

1.3 Fixed-point models

Unlike classical logic, LP does contain expansions of $\mathbb{N}$ to $\mathscr{L}_{\text{T}}$ where each sentence and its truth predication have the same truth value, including λ. Following Kripke's method, we can construct such models, as shown in what follows.

Let $\langle\mathbb{N}, S^+, S^-\rangle$ be an LP-model that expands $\mathbb{N}$ to $\mathscr{L}_{\text{T}}$. S^+ is the extension of T and S^- its anti-extension. Note that, as in $\mathbb{N}$, $=$ is interpreted classically, so all identity statements will get a classical truth value in $\langle\mathbb{N}, S^+, S^-\rangle$. Truth-value gluts are confined to the fragment of the language containing the truth predicate.

Let $\overline{\text{Sent}_{\mathscr{L}_{\text{T}}}}$ be $\omega \setminus \text{Sent}_{\mathscr{L}_{\text{T}}}$, and let $\Phi : \wp(\omega)^2 \to \wp(\omega)^2$ be such that

$$\begin{aligned}\Phi(S^+, S^-) = \langle &\{\#\varphi \in \text{Sent}_{\mathscr{L}_{\text{T}}} \mid \langle\mathbb{N}, S^+, S^-\rangle \vDash_{\mathsf{LP}} \varphi\}, \\ &\{\#\varphi \in \text{Sent}_{\mathscr{L}_{\text{T}}} \mid \langle\mathbb{N}, S^+, S^-\rangle \vDash_{\mathsf{LP}} \neg\varphi\} \cup \overline{\text{Sent}_{\mathscr{L}_{\text{T}}}}\rangle\end{aligned}$$

Φ maps the extension of T in an LP-model of $\mathscr{L}_{\text{T}}$ to the set of codes of true sentences in the model, and the anti-extension to the set of codes of false sentences plus the numbers that don't codify a sentence. Note that the resulting model $\langle\mathbb{N}, \Phi(S^+, S^-)\rangle$ is an LP-model as well. If $\Phi(S^+, S^-) = \langle S^+, S^-\rangle$, we say $\langle\mathbb{N}, S^+, S^-\rangle$ is a *fixed-point model* of $\mathscr{L}_{\text{T}}$.

Proposition 7. *If $\langle\mathbb{N}, S^+, S^-\rangle$ is a fixed-point model and φ a sentence of $\mathscr{L}_{\text{T}}$, then $v^{\langle\mathbb{N}, S^+, S^-\rangle}(\varphi) = v^{\langle\mathbb{N}, S^+, S^-\rangle}(\text{T}\ulcorner\varphi\urcorner)$.*

Proof. Let us write v for $v^{\langle\mathbb{N}, S^+, S^-\rangle}$, for perspicuity. If $v(\varphi) = 1$, then $\langle\mathbb{N}, S^+, S^-\rangle \vDash_{\mathsf{LP}} \varphi$ but $\langle\mathbb{N}, S^+, S^-\rangle \nvDash_{\mathsf{LP}} \neg\varphi$. Thus, $\varphi \in S^+ \setminus S^-$, so $v(\text{T}\ulcorner\varphi\urcorner) = 1$. Similarly, if $v(\varphi) = 0$, then $\langle\mathbb{N}, S^+, S^-\rangle \nvDash_{\mathsf{LP}} \varphi$ and $\langle\mathbb{N}, S^+, S^-\rangle \vDash_{\mathsf{LP}} \neg\varphi$. Thus, $\varphi \in S^- \setminus S^+$, so $v(\text{T}\ulcorner\varphi\urcorner) = 0$. Finally, if $v(\varphi) = \frac{1}{2}$, then $\langle\mathbb{N}, S^+, S^-\rangle \vDash_{\mathsf{LP}} \varphi$ and $\langle\mathbb{N}, S^+, S^-\rangle \vDash_{\mathsf{LP}} \neg\varphi$. Thus, $\varphi \in S^+ \cap S^-$, so $v(\text{T}\ulcorner\varphi\urcorner) = \frac{1}{2}$. □

[3] Since $\mathscr{L}$ has a function symbol s for the substitution function, as stated in the following paragraph, we can obtain a version of the strong diagonal lemma. If $\text{val}(x)$ were a term of the language, strong diagonalisation would entail the existence of a term t such that $t = \ulcorner\text{S}(\text{val}(t))\urcorner$, which would imply that $\text{val}(t) = \text{S}(\text{val}(t))$.

Therefore, if there were a fixed-point model, we would have an interpretation of $\mathscr{L}_\mathrm{T}$ in which T is a transparent truth predicate. Next we construct such model. First, we define a transfinite sequence of ordered pairs of subsets of ω as follows:

$$\langle \mathrm{S}^+_\alpha, \mathrm{S}^-_\alpha \rangle = \begin{cases} \langle \omega, \omega \rangle, & \text{if } \alpha = 0 \\ \Phi(\mathrm{S}^+_{\alpha-1}, \mathrm{S}^-_{\alpha-1}), & \text{if } \alpha \text{ is a successor ordinal} \\ \langle \bigcap_{\beta<\alpha} \mathrm{S}^+_\beta, \bigcap_{\beta<\alpha} \mathrm{S}^-_\beta \rangle, & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

Note that $\langle \mathbb{N}, \mathrm{S}^+_0, \mathrm{S}^-_0 \rangle$ is an LP-model of $\mathscr{L}_\mathrm{T}$. For every ordinal α, let v_α be the valuation function in $\langle \mathbb{N}, \mathrm{S}^+_\alpha, \mathrm{S}^-_\alpha \rangle$.

Lemma 8. *(i) Let $\alpha < \beta$. If $\mathrm{v}_\alpha(\varphi) = 1$, then $\mathrm{v}_\beta(\varphi) = 1$, and if $\mathrm{v}_\alpha(\varphi) = 0$, then $\mathrm{v}_\beta(\varphi) = 0$.*

(ii) If $\alpha < \beta$, then $\mathrm{S}^+_\beta \subseteq \mathrm{S}^+_\alpha$ and $\mathrm{S}^-_\beta \subseteq \mathrm{S}^-_\alpha$, that is, the sequence is monotonically decreasing.

Proof. (i) By induction on the complexity of φ. If φ is an atomic sentence, then it is either an identity statement or of the form $\mathrm{T}\bar{n}$. In the former case the result follows trivially from the fact that identity statements receive the same value in every model of the sequence. We prove the latter case by a transfinite induction on α. If $\alpha = 0$, $\mathrm{v}_\alpha(\mathrm{T}\bar{n}) = \frac{1}{2}$. Assume the result holds for all $\gamma < \alpha$, and let $\mathrm{v}_\alpha(\mathrm{T}\bar{n}) = 1$. Then, $n \in \mathrm{S}^+_\alpha \setminus \mathrm{S}^-_\alpha$, so $n = \#\psi \in \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}$. If α is a successor ordinal, $\langle \mathbb{N}, \mathrm{S}^+_{\alpha-1}, \mathrm{S}^-_{\alpha-1} \rangle \nvDash_{\mathsf{LP}} \neg\psi$, that is, $\mathrm{v}_{\alpha-1}(\psi) = 1$. By inductive hypothesis, $\mathrm{v}_\beta(\psi) = 1$ for all $\beta > \alpha - 1$. Therefore, $\mathrm{v}_\beta(\mathrm{T}\ulcorner\psi\urcorner) = 1$ for all $\beta > \alpha$. If α is a limit ordinal, there exists $\gamma < \alpha$ such that $\#\psi \in \mathrm{S}^+_\gamma \setminus \mathrm{S}^-_\gamma$. Thus, $\mathrm{v}_{\gamma-1}(\psi) = 1$. By inductive hypothesis, $\mathrm{v}_\beta(\psi) = 1$ for all $\beta > \gamma - 1$. Therefore, $\mathrm{v}_\beta(\mathrm{T}\ulcorner\psi\urcorner) = 1$ for all $\beta > \alpha$. The proof for $\mathrm{v}_\alpha(\mathrm{T}\bar{n}) = 0$ is symmetrical, except for $n \in \overline{\mathrm{Sent}_{\mathscr{L}_\mathrm{T}}}$. Note that in this case $\mathrm{v}_\beta(\mathrm{T}\bar{n}) = 0$ for every $\beta > 0$.

Assume the result holds for every sentence of complexity less than $c(\varphi)$, and let $\varphi := \neg\psi$. If $\mathrm{v}_\alpha(\varphi) = 1$, $\mathrm{v}_\alpha(\psi) = 0$. By inductive hypothesis, for every $\beta > \alpha$ we have that $\mathrm{v}_\beta(\psi) = 0$. Thus, for every $\beta > \alpha$ we also have that $\mathrm{v}_\alpha(\neg\psi) = 1$. If $\mathrm{v}_\alpha(\varphi) = 0$, the proof is symmetrical. The cases in which φ is a disjunction or an existential claim can be dealt with in a similar way.

(ii) Let $\alpha < \beta$. If $\beta = 0$, the proof is trivial. Thus, let $\beta > 0$, and assume $n \notin \mathrm{S}^+_\alpha$. If $n \in \overline{\mathrm{Sent}_{\mathscr{L}_\mathrm{T}}}$, then $n \notin \mathrm{S}^+_\beta$ either. If $n = \#\varphi \in \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}$, then $\mathrm{v}_{\alpha-1}(\varphi) = 0$. By 1, $\mathrm{v}_{\beta-1}(\varphi) = 0$. Therefore, $n \notin \mathrm{S}^+_\beta$. The proof of $\mathrm{S}^-_\beta \subseteq \mathrm{S}^-_\alpha$ is symmetrical.

□

Thus, once a sentence in the sequence of models $\langle \mathbb{N}, \mathrm{S}^+_\alpha, \mathrm{S}^-_\alpha \rangle$ acquires a classical truth value at some ordinal, it receives that value in every later stage in the sequence. Also, once a sentence leaves the extension or anti-extension of the truth predicate in the sequence at an ordinal, it never re-enters the extension or anti-extension, respectively, at a later stage. Therefore, we have the following result.

Proposition 9. *For some ordinal α, $\Phi(\mathrm{S}^+_\alpha, \mathrm{S}^-_\alpha) = \langle \mathrm{S}^+_\alpha, \mathrm{S}^-_\alpha \rangle$, that is, the sequence reaches a fixed point.*

Proof. By point 2 of Lemma 8 and cardinality considerations. □

Let ξ be the smallest ordinal such that $\Phi(\mathrm{S}^+_\xi, \mathrm{S}^-_\xi) = \langle \mathrm{S}^+_\xi, \mathrm{S}^-_\xi \rangle$. Therefore, $\langle \mathbb{N}, \mathrm{S}^+_\xi, \mathrm{S}^-_\xi \rangle$ is a fixed-point model of LP. This shows that LP is able to accommodate a transparent truth predicate in the presence of paradoxical expressions.

Paradoxical expressions such as λ cannot receive a classical truth value in any model in which every sentence and its truth ascription have the same truth value.

Therefore, in fixed-point LP-models, all paradoxical expressions receive truth value $\frac{1}{2}$. In less technical terms, they are all both true and false.

2 Two axiomatisations

In this section a calculus for LP is given. Then, the logic LP° is introduced both semantically and proof-theoretically. I show that LP° is an LFI, as it extends LP with a consistency operator $\circ$, intended as a device for recovering classical reasoning on demand. Then, I present two axiomatisations of the family of fixed-point LP-models introduced in 1.3: a paraconsistent one, CKF, and a classical one, KFG. As we will see in the next section, it turns out that (the internal logic of) KFG is stronger than CKF, not because CKF's underlying logic is incomplete, and not with respect to the truth-theoretic content of each theory, but with respect to their arithmetical consequences. This implies that abandoning classical logic for a paraconsistent one can affect the non-semantic content of a theory even in the presence of consistency operators, contrary to what is normally believed.

2.1 Axiomatising the Logic of Paradox

The obvious way of axiomatising the fixed-point LP-models introduced in the previous section is over a sound and complete calculus with respect to the class of all LP-models. In this section I provide a Gentzen-style multiple-conclusion sequent calculus with such characteristics. However, the theory will be formulated in LP°. We enrich the language with a consistency operator to enforce classicality when needed, that is, when no truth-theoretic content is involved.

There are several ways to give a sequent calculus for LP.[4] My presentation is close to Avron's [2, 3], but contains rules for the quantifiers and the identity predicate. Let $\mathcal{L}$ be a first-order language as in section 1.1. Recall a literal is a formula that is either atomic or the negation of an atomic formula. Let $\Gamma, \Delta \subseteq \mathcal{L}$ be finite sets of formulae, φ, ψ be formulae, t be a term, and u, v variables of $\mathcal{L}$. A sequent of $\mathcal{L}$ has the form $\Gamma \Rightarrow \Delta$. For perspicuity, we write Γ, φ and Γ, Δ for $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \Delta$, respectively. The calculus for LP consists of the following axiom and rules:

$$\Gamma, \varphi \Rightarrow \varphi, \Delta \tag{I}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}\ (\text{Cut})$$

$$\frac{\Gamma, t = t \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\text{Ref}) \qquad\qquad \frac{\Gamma, s = t, \varphi[s/v], \varphi[t/v] \Rightarrow \Delta}{\Gamma, s = t, \varphi[s/v] \Rightarrow \Delta}\ (\text{Repl})$$

$$\frac{\Gamma, \varphi \Rightarrow \neg\varphi, \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}\ (\neg\text{R})$$

[4] See, for instance, Beall [6] for a formulation of a multi-conclusion 2-sided sequent calculus for the propositional fragment of LP inspired in tableaux, and Ripley [28] for a 3-sided sequent calculus for first-order languages with identity.

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}\ (\neg\neg\text{L}) \qquad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \neg\neg\varphi, \Delta}\ (\neg\neg\text{R})$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta \qquad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ (\vee\text{L}) \qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ (\vee\text{R})$$

$$\frac{\Gamma, \neg\varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}\ (\neg\vee\text{L}) \qquad \frac{\Gamma \Rightarrow \neg\varphi, \Delta \qquad \Gamma \Rightarrow \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \vee \psi), \Delta}\ (\neg\vee\text{R})$$

$$\frac{\Gamma, \varphi[u/v] \Rightarrow \Delta}{\Gamma, \exists v\varphi \Rightarrow \Delta}\ (\exists\text{L}) \qquad \frac{\Gamma \Rightarrow \varphi[t/v], \Delta}{\Gamma \Rightarrow \exists v\varphi, \Delta}\ (\exists\text{R})$$

$$\frac{\Gamma, \neg\varphi[t/v] \Rightarrow \Delta}{\Gamma, \neg\exists v\varphi \Rightarrow \Delta}\ (\neg\exists\text{L}) \qquad \frac{\Gamma \Rightarrow \neg\varphi[u/v], \Delta}{\Gamma \Rightarrow \neg\exists v\varphi, \Delta}\ (\neg\exists\text{R})$$

The axiom and (Repl) are restricted to cases in which φ is a literal. In ($\neg$R) φ must be an atomic formula. In ($\exists$L) and ($\neg\exists$R), u cannot be free in the conclusion. Restrictions on (I), (Repl), and ($\neg$R) are just there to facilitate the proofs of cut-elimination, soundness, and completeness given in the appendix. Unrestricted versions of these rules can be easily shown to be admissible in the system.

Given the availability of (Cut) and that we allow for contexts in (I), the structural rule of weakening, that is,

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ (\text{W})$$

is also admissible in LP.

(Ref) and (Repl) are inter-derivable with the sequents $\Rightarrow t = t$ and $s = t, \varphi[s/v] \Rightarrow \varphi[t/v]$, respectively. Although the latter are more commonly used, I choose the rules over the sequents to facilitate proofs of metatheoretic results, following Takeuti [30].[5]

The unrestricted version of ($\neg$R), the sequents $\Gamma \Rightarrow \varphi, \neg\varphi, \Delta$, and the following inference

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}$$

are all inter-derivable. They express that the negation in LP is exhaustive, that is, for every sentence φ we have that either φ or $\neg\varphi$; there are no truth-value gaps. Indeed, together with ($\vee$R), they entail all sequents of the form $\Rightarrow \varphi \vee \neg\varphi$, that is, the Law of Excluded Middle.

On the other hand, since we lack a rule that allows us to introduce the negation symbol to the left, we cannot derive any sequent of the form $\varphi, \neg\varphi \Rightarrow$.[6] As a consequence, the rule of Explosion, given by the sequents $\varphi, \neg\varphi \Rightarrow \psi$, and according to which a contradiction entails everything, is not valid in LP.

Note as well that not all sequents of the form $\varphi, \varphi \rightarrow \psi \Rightarrow \psi$—i.e. $\varphi, \neg\varphi \vee \psi \Rightarrow \psi$—are derivable in LP, for they are not all sound with respect to the semantics introduced in section 1.1. If φ is both true and false in a model and assignment

[5]See also Troelstra & Schwichtenberg [31] and Negri & von Plato [22].

[6]This follows from the soundness theorem, Proposition 37.

and ψ is just false, $\varphi \to \psi$ ($\neg\varphi \vee \psi$) turns out both true and false in that model and assignment. In other words, neither Modus Ponens nor Disjunctive Syllogism are valid inferences in LP.

Let the formula with the connectives in the conclusion of a rule be the *principal* formula of that rule. (Ref) and (Repl) have no principal formulae. The calculus for LP is not canonical in the following sense: more than one connective occurs in the principal formulae of some rules, connectives occur sometimes also in the premisses, and formulae in the premisses are not always subformulae of those in the conclusion, that is, the subformula property doesn't hold. Unfortunately, this is not a defect of our formulation but an intrinsic feature of LP's consequence relation (cf. Avron [3]).

If a sequent $\Gamma \Rightarrow \Delta$ is derivable in LP from a (possibly empty) set of sequents $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$, let us write $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n \vdash_{\mathsf{LP}} \Gamma \Rightarrow \Delta$, and similarly for LP°, the system that will be introduced in the next section.
Definition 10. Let $\Gamma \subseteq \mathcal{L}$ and $\varphi \in \mathcal{L}$. φ is a *proof-theoretic consequence* of Γ in LP ($\Gamma \vdash_{\mathsf{LP}} \varphi$) iff $\vdash_{\mathsf{LP}} \Gamma \Rightarrow \varphi$.

2.2 A Logic of Formal Inconsistency

We now expand the language and the calculus for LP with a consistency operator. Let $\mathcal{L}^\circ := \mathcal{L} + \circ$ extend $\mathcal{L}$ with a monadic primitive operator $\circ$. If φ is a formula of $\mathcal{L}^\circ$, so is $\circ\varphi$. An LP°-model is just an LP-model, and assignments are defined as before.
Definition 11. Let $\mathcal{M}$ be an LP°-model of $\mathcal{L}^\circ$, and let σ be an assignment on $\mathcal{M}$. A *valuation* $v_\sigma^\mathcal{M}$ in $\mathcal{M}$ is a function that assigns values from the set $\{0, \frac{1}{2}, 1\}$ to each formula in $\mathcal{L}^\circ$ according to the clauses in Definition 1 plus the following:

- $v_\sigma^\mathcal{M}(\circ\varphi) = \begin{cases} 1, & \text{if } v_\sigma^\mathcal{M}(\varphi) \neq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$

Definition 12. Let $\mathcal{M}$ be an LP°-model of $\mathcal{L}^\circ$. A sentence $\varphi \in \mathcal{L}^\circ$ is *true in* $\mathcal{M}$ ($\mathcal{M} \vDash_{\mathsf{LP}^\circ} \varphi$) iff $v_\sigma^\mathcal{M}(\varphi) \geqslant \frac{1}{2}$ for every assignment σ on $\mathcal{M}$.

As before, a sentence φ can be both true and false in a model. In that case, every assignment on the model will map φ to $\frac{1}{2}$. As a consequence, $\circ\varphi$ will receive the value 0. If, on the contrary, φ has a classical truth value in a model, $\circ\varphi$ receives the value 1. Thus, $\circ$ is a *consistency* operator: it applies precisely to those formulae that are not both true and false. Moreover, no formula of the form $\circ\varphi$ can be both true and false. Consistency is a classical matter.
Definition 13. Let $\Gamma \subseteq \mathcal{L}^\circ$ and $\varphi \in \mathcal{L}^\circ$. φ is a *semantic consequence* of Γ in LP° ($\Gamma \vDash_{\mathsf{LP}^\circ} \varphi$) iff, for every LP°-model $\mathcal{M}$ and assignment σ on $\mathcal{M}$, if $v_\sigma^\mathcal{M}(\Gamma) \geqslant \frac{1}{2}$, then $v_\sigma^\mathcal{M}(\varphi) \geqslant \frac{1}{2}$.

The calculus for LP° extends LP's with the following rules:

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma, \circ\varphi \Rightarrow \Delta}\ (\circ\mathrm{L}) \qquad \frac{\Gamma, \varphi, \neg\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \circ\varphi, \Delta}\ (\circ\mathrm{R})$$

$$\frac{\Gamma, \varphi, \neg\varphi \Rightarrow \Delta}{\Gamma, \neg\circ\varphi \Rightarrow \Delta}\ (\neg\circ\mathrm{L}) \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Gamma \Rightarrow \neg\varphi, \Delta}{\Gamma \Rightarrow \neg\circ\varphi, \Delta}\ (\neg\circ\mathrm{R})$$

Definition 14. Let $\Gamma \subseteq \mathcal{L}^\circ$ and $\varphi \in \mathcal{L}^\circ$. φ is a *proof-theoretic consequence* of Γ in LP° ($\Gamma \vdash_{\mathsf{LP}^\circ} \varphi$) iff $\vdash_{\mathsf{LP}^\circ} \Gamma \Rightarrow \varphi$.

In the appendix I prove cut-elimination, soundness, and completeness for this calculus with respect to the semantics just given. The rule (W) is also admissible in LP°.

LP° is equivalent to the well-known system $\mathsf{LFI1}^*$ of Carnielli, Marcos, & de Amo [14], later axiomatised by Omori & Waragai [23] in a Hilbert-style calculus. LP° and $\mathsf{LFI1}^*$ are also equivalent to the system CLuNs with a primitive symbol $\perp$ for the 'Falsehood', introduced semantically and axiomatised in a Hilbert-style calculus in Batens & De Clercq [5]. The logical operators of each of these three systems are definable in the other two in a straightforward manner.

$\mathsf{LFI1}^*$ and CLuNs are often seen as expansions of classical logic rather than weakenings, as all classical operators can be easily defined in them. A fortiori, the same can be said about our LP°. However, LP° must be understood here as a subclassical logic, since the *official* negation can only behave as our primitive one $\neg$ and the *official* conditional as the material conditional of LP, to guarantee the existence of fixed-point models.

As a consequence, just like in LP, neither Explosion nor Modus Ponens (or, what is the same, Disjunctive Syllogism) can be said to be valid in LP°. Nonetheless, it's easy to show that all sequents of the form

$$\circ\varphi, \varphi, \varphi \to \psi \Rightarrow \psi$$

that is,

$$\circ\varphi, \varphi, \neg\varphi \vee \psi \Rightarrow \psi$$

are derivable in the system. If consistency of the antecedent is ensured, then Modus Ponens is a valid inference. Thus, so is Disjunctive Syllogism.

An LFI can be seen as a paraconsistent logic in which Explosion is allowed only locally. More formally, we have the following two definitions:[7]

Definition 15. A logic L formulated in $\mathcal{L}^\circ$ is an *LFI* iff

1. $\varphi, \neg\varphi \nvdash_L \psi$ for some formulae $\varphi, \psi \in \mathcal{L}^\circ$;
2. there are $\varphi, \psi \in \mathcal{L}^\circ$ such that
   (a) $\circ\varphi, \varphi \nvdash_L \psi$, and
   (b) $\circ\varphi, \neg\varphi \nvdash_L \psi$; and
3. $\circ\varphi, \varphi, \neg\varphi \vdash_L \psi$ for all $\varphi, \psi \in \mathcal{L}^\circ$.

Definition 16. A logic L formulated in $\mathcal{L}^\circ$ is a *strong LFI* iff

1. there are formulae $\varphi, \psi \in \mathcal{L}^\circ$ such that
   (a) $\varphi, \neg\varphi \nvdash_L \psi$,
   (b) $\circ\varphi, \varphi \nvdash_L \psi$, and
   (c) $\circ\varphi, \neg\varphi \nvdash_L \psi$; and
2. $\circ\varphi, \varphi, \neg\varphi \vdash_L \psi$ for all $\varphi, \psi \in \mathcal{L}^\circ$.

Every strong LFI is an LFI, but the other direction of the implication is not true. This is because in Definition 16, φ and ψ have to be the same witnesses of clauses 2.(a), 2.(b), and 2.(c), whereas in Definition 15 only clauses 2.(a) and 2.(b) have to satisfy this condition; the witnesses of clause 1 can be different formulae.

Proposition 17. LP° *is a strong LFI.*

Proof. Clauses 1.(a)-(c) of Definition 16 follow by the soundness of the system, if we take φ and ψ to be different atomic formulae of $\mathcal{L}^\circ$. For clause (2), consider the following derivation in LP°:

$$\frac{\varphi, \neg\varphi \Rightarrow \varphi, \psi \qquad \varphi, \neg\varphi \Rightarrow \neg\varphi, \psi}{\circ\varphi, \varphi, \neg\varphi \Rightarrow \psi}\ (\circ\mathrm{L})$$

[7] See, e.g. Carnielli, Coniglio & Marcos [12] and Carnielli & Coniglio [11, chap. 2].

□

2.3 Kripke's fixed-point models in an LFI

In the present section I introduce the system CKF, standing for 'Complete Kripke-Feferman'. CKF is a variant of the system PKF of Halbach & Horsten [19], an axiomatisation of Kripke's fixed-point models with the Strong Kleene evaluation scheme formulated in basic De Morgan logic. PKF is in turn a variant of the axiomatisation of these models in classical logic introduced by Feferman [15], dubbed 'KF' after Kripke and Feferman.[8]

We want $\mathsf{CKF} \subseteq \mathscr{L}_{\mathrm{T}}^{\circ}$ to axiomatise the class of fixed-point LP-models of $\mathscr{L}_{\mathrm{T}}$ in LP°. This is plausible, since every LP-model is an LP°-model. Note that, while it will still be true that $v^{\mathcal{M}}(\varphi) = v^{\mathcal{M}}(\mathrm{T}\ulcorner\varphi\urcorner)$ for every valuation $v^{\mathcal{M}}$ in a fixed-point model $\mathcal{M}$ and every sentence $\varphi \in \mathscr{L}_{\mathrm{T}}$, some sentences containing the consistency operator $\circ$ will not satisfy this equivalence. For only codes of sentences of $\mathscr{L}_{\mathrm{T}}$ are allowed in the extension of the truth predicate in such models. Unfortunately, it is not possible to redefine the operator Φ in order to achieve fixed points, that is, a transparent truth predicate for the whole language $\mathscr{L}_{\mathrm{T}}^{\circ}$. The diagonal lemma applied to the formula $\circ\mathrm{T}x \wedge \neg\mathrm{T}x$ delivers a sentence λ', such that, in every model $\langle\mathbb{N}, S^{+}, S^{-}\rangle$ of $\mathscr{L}_{\mathrm{T}}$, $v^{\langle\mathbb{N},S^{+},S^{-}\rangle}(\lambda') = v^{\langle\mathbb{N},S^{+},S^{-}\rangle}(\circ\mathrm{T}\ulcorner\lambda'\urcorner \wedge \neg\mathrm{T}\ulcorner\lambda'\urcorner)$. Thus, λ' 'says' of itself that it's just false, excluding the possibility of it being both true and false. This implies that $v^{\langle\mathbb{N},S^{+},S^{-}\rangle}(\lambda') \neq v^{\langle\mathbb{N},S^{+},S^{-}\rangle}(\mathrm{T}\ulcorner\lambda'\urcorner)$.[9] Sentences like λ' are known as 'revenge paradoxes'. They are paradoxical expressions that emerge from the very machinery we introduce to deal with the ordinary paradoxes. To avoid them, we settle for applying the truth predicate to expressions of $\mathscr{L}_{\mathrm{T}}$ only.

To axiomatise the class of fixed-point models $\langle\mathbb{N}, S^{+}, S^{-}\rangle$ in LP° we need to add axioms for both the arithmetical and the truth-theoretic vocabulary. Let CKF extend LP° with an initial sequent $\Gamma \Rightarrow \varphi, \Delta$ for each basic axiom φ of Peano arithmetic, including definitions for each function symbol other than S, $+$, and $\times$, and the rule of induction

$$\frac{\Gamma, \varphi(x) \Rightarrow \varphi(\mathrm{S}x), \Delta}{\Gamma, \varphi(0) \Rightarrow \varphi(t), \Delta}\ (\mathrm{IND})$$

for each formula $\varphi(v) \in \mathscr{L}_{\mathrm{T}}^{\circ}$, if x is not free in Γ, Δ, and $\varphi(0)$ and t is arbitrary. We need to add the contexts Γ, Δ to guarantee the admissibility of (W) in the theory.

Since we want to keep classical reasoning for the T-free fragment of the language, we also add initial sequents

$$\Gamma \Rightarrow \circ\varphi, \Delta \qquad (\mathrm{CON})$$

for each $\varphi \in \mathscr{L}$. This means that we can apply, for instance, Modus Ponens to sentences in $\mathscr{L}$ when needed. In particular, we can reason classically from the basic axioms of Peano arithmetic of conditional form. However, the same cannot be said about the induction schema. Some instances of this principle do contain T. Given that the material conditional does not necessarily detach in LP° for formulae containing T, we need to formulate induction as a rule instead. In any case, it is clear that we can derive $\Rightarrow \varphi$ for every theorem $\varphi \in \mathscr{L}$ of Peano arithmetic.

Beall [9, 8] puts forward a so-called 'shrieking' rule in order to equip (a Hilbert-style calculus presentation of) LP with a device for recovering classical reasoning when

[8] 'KF' is not the original name Feferman gave to the system. See Halbach [18, chap. 15].

[9] This means that CKF contains what Barrio, Pailos, & Szmuc [4] call a 'strong procedure for self-reference'. Note that the identity of truth values in every extension of $\mathbb{N}$ already follows from standard (weak) versions of diagonalisation. The strong diagonal lemma is not required.

needed.[10] His proposal consists, roughly, in closing a theory formulated over LP under the rule[11]

$$\varphi, \neg\varphi \vdash \bot$$

for every φ from which we wish to reason classically. Unfortunately, Halbach & Nicolai [20] show that this strategy cannot succeed in bringing us back classical reasoning for all arithmetical principles. Our hope is that enriching the language with a consistency operator, which is not definable in LP, has better results.

CKF also contains the following initial sequents governing the truth predicate:

CKF1 (i) $\Gamma, \mathrm{val}(\mathbf{s}) = \mathrm{val}(\mathbf{t}), \neg \mathrm{T}(\mathbf{s} \underset{\cdot}{=} \mathbf{t}) \Rightarrow \Delta$
(ii) $\Gamma, \mathrm{val}(\mathbf{s}) \neq \mathrm{val}(\mathbf{t}), \mathrm{T}(\mathbf{s} \underset{\cdot}{=} \mathbf{t}) \Rightarrow \Delta$

CKF2 (i) $\Gamma, \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(x), \mathrm{T}\underset{\cdot}{\neg} x \Rightarrow \neg \mathrm{T}x, \Delta$
(ii) $\Gamma, \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(x), \neg \mathrm{T}x \Rightarrow \mathrm{T}\underset{\cdot}{\neg} x, \Delta$

CKF3 (i) $\Gamma, \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(x \underset{\cdot}{\vee} y), \mathrm{T}(x \underset{\cdot}{\vee} y) \Rightarrow \mathrm{T}x \vee \mathrm{T}y, \Delta$
(ii) $\Gamma, \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(x \underset{\cdot}{\vee} y), \mathrm{T}x \vee \mathrm{T}y \Rightarrow \mathrm{T}(x \underset{\cdot}{\vee} y), \Delta$

CKF4 (i) $\Gamma, \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(x \underset{\cdot}{\vee} y), \mathrm{T}\underset{\cdot}{\neg}(x \underset{\cdot}{\vee} y) \Rightarrow \neg(\mathrm{T}x \vee \mathrm{T}y), \Delta$
(ii) $\Gamma, \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(x \underset{\cdot}{\vee} y), \neg(\mathrm{T}x \vee \mathrm{T}y) \Rightarrow \mathrm{T}\underset{\cdot}{\neg}(x \underset{\cdot}{\vee} y), \Delta$

CKF5 (i) $\Gamma, \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(\underset{\cdot}{\exists} \mathbf{v} x), \mathrm{T}(\underset{\cdot}{\exists} \mathbf{v} x) \Rightarrow \exists \mathbf{t} \mathrm{T} s(x, \mathbf{t}, \mathbf{v}), \Delta$
(ii) $\Gamma, \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(\underset{\cdot}{\exists} \mathbf{v} x), \exists \mathbf{t} \mathrm{T} s(x, \mathbf{t}, \mathbf{v}) \Rightarrow \mathrm{T}(\underset{\cdot}{\exists} \mathbf{v} x), \Delta$

CKF6 (i) $\Gamma, \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(\underset{\cdot}{\exists} \mathbf{v} x), \mathrm{T}(\underset{\cdot}{\neg} \underset{\cdot}{\exists} \mathbf{v} x) \Rightarrow \neg \exists \mathbf{t} \mathrm{T} s(x, \mathbf{t}, \mathbf{v}), \Delta$
(ii) $\Gamma, \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(\underset{\cdot}{\exists} \mathbf{v} x), \neg \exists \mathbf{t} \mathrm{T} s(x, \mathbf{t}, \mathbf{v}) \Rightarrow \mathrm{T}(\underset{\cdot}{\neg} \underset{\cdot}{\exists} \mathbf{v} x), \Delta$

CKF7 (i) $\Gamma, \mathrm{T}\mathrm{val}(\mathbf{t}) \Rightarrow \mathrm{T}\underset{\cdot}{\mathrm{T}}\mathbf{t}, \Delta$
(ii) $\Gamma, \mathrm{T}\underset{\cdot}{\mathrm{T}}\mathbf{t} \Rightarrow \mathrm{T}\mathrm{val}(\mathbf{t}), \Delta$

CKF8 (i) $\Gamma, \neg \mathrm{T}\mathrm{val}(\mathbf{t}) \Rightarrow \mathrm{T}\underset{\cdot}{\neg}\underset{\cdot}{\mathrm{T}}\mathbf{t}, \Delta$
(ii) $\Gamma, \mathrm{T}\underset{\cdot}{\neg}\underset{\cdot}{\mathrm{T}}\mathbf{t} \Rightarrow \neg \mathrm{T}\mathrm{val}(\mathbf{t}), \Delta$

CKF9 $\Gamma, \mathrm{T}x \Rightarrow \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(x), \Delta$

CKF1 guarantees that true identity statements belong just to the extension of T, and false ones just to the anti-extension. CKF2-CKF6 ensure that the extension and anti-extension of the truth predicate are closed under logical consequence. CKF7 and CKF8 allow us to iterate the truth predicate. Finally, CKF9 ensures that only sentences are in the extension of the truth predicate. Despite having initial sequents CKF2.(i) and (ii) indicating that truth commutes with negation, we need special axioms for negated disjunctions, negated existential claims, and negated truth ascriptions to ensure that formulae on each side of the sequent arrow have the same truth value—e.g. $\mathrm{T}(x \underset{\cdot}{\vee} y)$ and $\mathrm{T}x \vee \mathrm{T}y$. The reason is the sequent arrow in LP° does not contra-pose: $\varphi \Rightarrow \psi$ does not necessarily imply $\neg\psi \Rightarrow \neg\varphi$.

We say φ is a theorem of CKF ($\mathsf{CKF} \vdash_{\mathsf{LP}^\circ} \varphi$) if the sequent $\Rightarrow \varphi$ is derivable in CKF, and similarly for the other theories that will be introduced later. The system $\mathsf{CKF}\restriction$ is obtained from CKF by restricting (IND) to formulae of $\mathscr{L}$ only.

Proposition 18. *The following sequents are derivable in* $\mathsf{CKF}\restriction$ *for every formula* $\varphi(v_1, \dots, v_n) \in \mathscr{L}_\mathrm{T}$*:*

$$\Gamma, \varphi(\mathrm{val}(\mathbf{t_1}), \dots, \mathrm{val}(\mathbf{t_n})) \Rightarrow \mathrm{T}\ulcorner \varphi(\underset{\cdot}{\mathbf{t_1}}, \dots, \underset{\cdot}{\mathbf{t_n}}) \urcorner, \Delta \qquad \text{(T-In)}$$

$$\Gamma, \mathrm{T}\ulcorner \varphi(\underset{\cdot}{\mathbf{t_1}}, \dots, \underset{\cdot}{\mathbf{t_n}}) \urcorner \Rightarrow \varphi(\mathrm{val}(\mathbf{t_1}), \dots, \mathrm{val}(\mathbf{t_n})), \Delta \qquad \text{(T-Out)}$$

$$\Gamma, \neg \mathrm{T}\ulcorner \varphi(\underset{\cdot}{\mathbf{t_1}}, \dots, \underset{\cdot}{\mathbf{t_n}}) \urcorner \Rightarrow \neg \varphi(\mathrm{val}(\mathbf{t_1}), \dots, \mathrm{val}(\mathbf{t_n})), \Delta \qquad \text{(CT-In)}$$

$$\Gamma, \neg \varphi(\mathrm{val}(\mathbf{t_1}), \dots, \mathrm{val}(\mathbf{t_n})) \Rightarrow \neg \mathrm{T}\ulcorner \varphi(\underset{\cdot}{\mathbf{t_1}}, \dots, \underset{\cdot}{\mathbf{t_n}}) \urcorner, \Delta \qquad \text{(CT-Out)}$$

[10] See also Beall [6].

[11] $\bot$ is any sentence that is never true.

Proof. By induction on the complexity of φ. Recall that if t is a term, $\mathrm{val}(\ulcorner t \urcorner) = t$ is a theorem of Peano arithmetic. Also, $\mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(\ulcorner \varphi(\dot{\mathbf{t}}_1, \ldots, \dot{\mathbf{t}}_\mathbf{n}) \urcorner)$ is a theorem of Peano arithmetic, since $v_1, \ldots, v_n$ are the only free variables in $\varphi(v_1, \ldots, v_n)$ and $\mathbf{t_1}, \ldots, \mathbf{t_n}$ are codes of closed terms. As initial cases we have to consider literals. Then, T-In and T-Out can be derived easily from CKF1, CKF2, CKF7, and CKF8. Cases in which φ is of the form $\neg\neg\psi$, $\psi \vee \chi$, $\neg(\varphi \vee \chi)$ follow directly from the inductive hypothesis and the initial sequents CKF2-CKF4. For $\exists v \psi$ and $\neg\exists v \psi$, the sequents CKF5, CKF6, and CKF9 are needed. As an example, I consider the case for T-Out in which φ is of the form $\neg(\psi \vee \chi)$. The first sequents of the following two derivations are obtained by inductive hypothesis:

$$\frac{\Gamma, \mathrm{T}\ulcorner \psi(\dot{\mathbf{t}}_1, \ldots, \dot{\mathbf{t}}_\mathbf{n}) \urcorner \Rightarrow \psi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})), \chi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})), \Delta}{\Gamma, \mathrm{T}\ulcorner \psi(\dot{\mathbf{t}}_1, \ldots, \dot{\mathbf{t}}_\mathbf{n}) \urcorner \Rightarrow \psi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})) \vee \chi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})), \Delta} (\vee\mathrm{R})$$

$$\frac{\Gamma, \mathrm{T}\ulcorner \chi(\dot{\mathbf{t}}_1, \ldots, \dot{\mathbf{t}}_\mathbf{n}) \urcorner \Rightarrow \psi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})), \chi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})), \Delta}{\Gamma, \mathrm{T}\ulcorner \chi(\dot{\mathbf{t}}_1, \ldots, \dot{\mathbf{t}}_\mathbf{n}) \urcorner \Rightarrow \psi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})) \vee \chi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})), \Delta} (\vee\mathrm{R})$$

By an application of $(\vee\mathrm{L})$ to the last sequents of these derivations, we obtain

$$\Gamma, \mathrm{T}\ulcorner \psi(\dot{\mathbf{t}}_1, \ldots, \dot{\mathbf{t}}_\mathbf{n}) \urcorner \vee \mathrm{T}\ulcorner \chi(\dot{\mathbf{t}}_1, \ldots, \dot{\mathbf{t}}_\mathbf{n}) \urcorner \Rightarrow \varphi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})), \Delta$$

Finally, by CKF3 and an application of (Cut), we get

$$\Gamma, \mathrm{T}\ulcorner \varphi(\dot{\mathbf{t}}_1, \ldots, \dot{\mathbf{t}}_\mathbf{n}) \urcorner \Rightarrow \varphi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})), \Delta$$

□

Call a sequent $\Gamma \Rightarrow \Delta$ *sound* with respect to a class of pairs of models and corresponding assignments iff for every model $\mathcal{M}$ and assignment σ in the class, if $v_\sigma^\mathcal{M}(\Gamma) \geqslant \frac{1}{2}$, then $v_\sigma^\mathcal{M}(\varphi) \geqslant \frac{1}{2}$ for some $\varphi \in \Delta$. Both CKF and CKF$\restriction$ are sound with respect to the class of fixed-point models introduced in section 1.3.

Lemma 19 (Soundness of CKF)**.** *If $\Gamma \Rightarrow \Delta$ is derivable in* CKF *and $\langle \mathbb{N}, S^+, S^- \rangle$ is a fixed-point model, $\Gamma \Rightarrow \Delta$ is sound in $\langle \mathbb{N}, S^+, S^- \rangle$ (relative to every assignment).*

Proof. Let $\langle \mathbb{N}, S^+, S^- \rangle$ be a fixed-point LP-model. We have to show that all the initial sequents $\Gamma \Rightarrow \Delta$ of CKF are sound in $\langle \mathbb{N}, S^+, S^- \rangle$ relative to every assignment, and that (IND) preserves soundness in this model, relative to every assignment. This is clearly the case for every sequent that corresponds to a basic axioms of Peano arithmetic, since the axioms are all true in $\mathbb{N}$. The rule of induction must preserve soundness as well, for the domain of the model is ω. Given that this model assigns classical truth values to all sentences in $\mathscr{L}$, the initial sequents $\Gamma \Rightarrow \circ\varphi, \Delta$ are also sound.

Let σ be an assignment on $\langle \mathbb{N}, S^+, S^- \rangle$, and let us write v_σ for $v_\sigma^{\langle \mathbb{N}, S^+, S^- \rangle}$. The soundness of CKF1 follows directly from Proposition 7 and the fact that $\langle \mathbb{N}, S^+, S^- \rangle$ extends $\mathbb{N}$. For CKF2.(i), assume $v_\sigma(\mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(x)) \geqslant \frac{1}{2}$ and $v_\sigma(\mathrm{T}\dot{\neg} x) \geqslant \frac{1}{2}$. Let $\sigma(x) = \#\varphi$. If $v_\sigma(\mathrm{T}\dot{\neg} x) = 1$, then $v_\sigma(\neg\varphi) = 1$, so $\#\varphi \in S^-$. Thus, $v_\sigma(\neg \mathrm{T}x) \geqslant \frac{1}{2}$. If $v_\sigma(\mathrm{T}\dot{\neg} x) = \frac{1}{2}$, then $v_\sigma(\neg\varphi) = \frac{1}{2}$, so $\#\varphi \in S^-$ again. Therefore, $v_\sigma(\neg \mathrm{T}x) \geqslant \frac{1}{2}$. The proof for CKF2.(ii) is symmetrical. The proofs of soundness of the other initial sequents are similar. □

If *Th* is a theory over a logic L and $\mathcal{M}$ is a model for that logic, we write $\mathcal{M} \vDash_L$ *Th* to indicate that all sequents derivable in *Th* are sound in $\mathcal{M}$ relative to every assignment on the model.

Lemma 20. *If* $\langle \mathbb{N}, S^+, S^- \rangle \vDash_{\mathsf{LP}^\circ} \mathsf{CKF}\!\upharpoonright$*, then* $\langle \mathbb{N}, S^+, S^- \rangle$ *is a fixed-point model.*

Proof. Assume every sequent derivable in CKF is sound in $\langle \mathbb{N}, S^+, S^- \rangle$ relative to every assignment. We need to show that $\Phi(S^+, S^-) = \langle S^+, S^- \rangle$, that is,

$$S^+ = \{\#\varphi \in \mathrm{Sent}_{\mathscr{L}_\mathrm{T}} \mid \langle \mathbb{N}, S^+, S^- \rangle \vDash_{\mathsf{LP}} \varphi\}$$
$$S^- = \{\#\varphi \in \mathrm{Sent}_{\mathscr{L}_\mathrm{T}} \mid \langle \mathbb{N}, S^+, S^- \rangle \vDash_{\mathsf{LP}} \neg\varphi\} \cup \overline{\mathrm{Sent}_{\mathscr{L}_\mathrm{T}}}$$

We know that $n \in S^+$ iff $n = \#\varphi \in \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}$ for some φ, by CKF9, and $\mathrm{T}\ulcorner\varphi\urcorner$ is true in the model. By Proposition 18, this is the case iff φ is also true in the model, which holds iff $n \in \{\#\varphi \in \mathrm{Sent}_{\mathscr{L}_\mathrm{T}} \mid \langle \mathbb{N}, S^+, S^- \rangle \vDash_{\mathsf{LP}} \varphi\}$.

On the other hand, $n \in S^-$ iff either $n = \#\varphi \in \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}$ for some $\varphi \in \mathscr{L}_\mathrm{T}$ and $\neg\mathrm{T}\ulcorner\varphi\urcorner$ is true in the model, or $n \in \overline{\mathrm{Sent}_{\mathscr{L}_\mathrm{T}}}$. By CT-In in Proposition 18, this is equivalent to having that either $n = \#\varphi \in \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}$ for some $\varphi \in \mathscr{L}_\mathrm{T}$ and $\neg\varphi$ is true in the model, or $n \in \overline{\mathrm{Sent}_{\mathscr{L}_\mathrm{T}}}$, which means that $n \in \{\#\varphi \in \mathrm{Sent}_{\mathscr{L}_\mathrm{T}} \mid \langle \mathbb{N}, S^+, S^- \rangle \vDash_{\mathsf{LP}} \neg\varphi\} \cup \overline{\mathrm{Sent}_{\mathscr{L}_\mathrm{T}}}$. □

Proposition 21. $\langle \mathbb{N}, S^+, S^- \rangle \vDash_{\mathsf{LP}^\circ}$ CKF *(*$\mathsf{CKF}\!\upharpoonright$*) iff* $\langle \mathbb{N}, S^+, S^- \rangle$ *is a fixed-point model.*

Proof. Directly from Lemmata 19 and 20. □

2.4 Kripke-Feferman with gluts

Since the fixed-point models introduced in section 1.3 are not classical but paraconsistent, there is little point in trying to capture them in classical logic, at least in the sense of Proposition 21. This would require a transparent truth predicate over arithmetic, which is not possible. If we want to remain within the limits of classical logic, what we can do instead is aim at an *external* axiomatisation of these models, as suggested by Reinhardt [27].

Given a theory *Th* formulated in $\mathscr{L}_\mathrm{T}$, its *internal logic ITh* is the set of sentences $\varphi \in \mathscr{L}_\mathrm{T}$ such that $\mathrm{T}\ulcorner\varphi\urcorner$ is a theorem of *Th*, whereas its *external logic* is the set of theorems of *Th*. Thus, the internal logic of a truth theory is the set of sentences the theory proves to be true, and the external logic is simply the set of sentences that can be derived in the system. In an external axiomatisation of the class of fixed-points models, only the internal logic will be sound with respect to this class, whilst the classicality of the external logic will reflect the classicality of the metalanguage in which these models are specified.

Feferman's famous KF is an external axiomatisation of Kripke's paracomplete and paraconsistent fixed-point models with the strong Kleene evaluation scheme. In this section I introduce KFG, for 'Kripke-Feferman with Gluts'. KFG is an extension of KF with initial sequents that preclude the possibility of truth-value gaps, ensuring complete extensions of the truth predicate. I show that the internal logic of KFG, IKFG, captures Kripke's paraconsistent fixed-point models in such a way that KFG can indeed be seen as an external axiomatisation of this class of models.

An alternative way of understanding classical theories such as KF and KFG is as axiomatisations, not of Kripke's fixed-point models, but of a closely related family: the one that results from *closing off* the non-classical models. The closing-off of an LP-model $\langle \mathbb{N}, S^+, S^- \rangle$ is the classical model $\langle \mathbb{N}, S^+ \rangle$. Since, for instance, λ belongs to S^+ and S^-, both λ and $\neg\lambda$ are in S^+, so some sentences are declared to be both true and false in the classical model as well. In this section I show that KFG can be said to capture the class of closing-offs of fixed-point LP-models.

A sequent calculus for first-order classical logic with identity, CL, is obtained by extending the calculus for LP with

$$\frac{\Gamma, \neg\varphi \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta}\ (\neg\text{L})$$

for every atomic sentence φ of the language. As before, we can show that an unrestricted version of this rule is admissible in the system. Moreover, the unrestricted version of $(\neg\text{L})$, the sequents $\Gamma, \varphi, \neg\varphi \Rightarrow \Delta$, and the inference

$$\frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta}$$

are inter-derivable, so the rule of explosion becomes valid. As a consequence, so does Disjunctive Syllogism (i.e. Modus Ponens).

Let KFG $\subseteq \mathscr{L}_{\mathrm{T}}$ extend CL with all initial sequents $\Gamma \Rightarrow \varphi, \Delta$ where φ is a basic axiom of Peano arithmetic, the rule (IND) for all formulae in $\mathscr{L}_{\mathrm{T}}$, and the following:

KFG1 (i) $\Gamma, \mathrm{val}(\mathbf{s}) = \mathrm{val}(\mathbf{t}) \Rightarrow \mathrm{T}(\mathbf{s} \dot{=} \mathbf{t}), \Delta$
(ii) $\Gamma, \mathrm{T}(\mathbf{s} \dot{=} \mathbf{t}) \Rightarrow \mathrm{val}(\mathbf{s}) = \mathrm{val}(\mathbf{t}), \Delta$

KFG2 (i) $\Gamma, \mathrm{val}(\mathbf{s}) \neq \mathrm{val}(\mathbf{t}) \Rightarrow \mathrm{T}(\dot{\neg}\mathbf{s} \dot{=} \mathbf{t}), \Delta$
(ii) $\Gamma, \mathrm{T}(\dot{\neg}\mathbf{s} \dot{=} \mathbf{t}) \Rightarrow \mathrm{val}(\mathbf{s}) \neq \mathrm{val}(\mathbf{t}), \Delta$

KFG3 (i) $\Gamma, \mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(x), \mathrm{T}\dot{\neg}\dot{\neg}x \Rightarrow \mathrm{T}x, \Delta$
(ii) $\Gamma, \mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(x), \mathrm{T}x \Rightarrow \mathrm{T}\dot{\neg}\dot{\neg}x, \Delta$

KFG4 (i) $\Gamma, \mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(x \dot{\vee} y), \mathrm{T}(x \dot{\vee} y) \Rightarrow \mathrm{T}x \vee \mathrm{T}y, \Delta$
(ii) $\Gamma, \mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(x \dot{\vee} y), \mathrm{T}x \vee \mathrm{T}y \Rightarrow \mathrm{T}(x \dot{\vee} y), \Delta$

KFG5 (i) $\Gamma, \mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(x \dot{\vee} y), \mathrm{T}\dot{\neg}(x \dot{\vee} y) \Rightarrow \mathrm{T}\dot{\neg}x \wedge \mathrm{T}\dot{\neg}y, \Delta$
(ii) $\Gamma, \mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(x \dot{\vee} y), \mathrm{T}\dot{\neg}x, \mathrm{T}\dot{\neg}y \Rightarrow \mathrm{T}\dot{\neg}(x \dot{\vee} y), \Delta$

KFG6 (i) $\Gamma, \mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(\dot{\exists}\mathbf{v}x), \mathrm{T}(\dot{\exists}\mathbf{v}x) \Rightarrow \exists\mathbf{t}\mathrm{T}s(x, \mathbf{t}, \mathbf{v}), \Delta$
(ii) $\Gamma, \mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(\dot{\exists}\mathbf{v}x), \exists\mathbf{t}\mathrm{T}s(x, \mathbf{t}, \mathbf{v}) \Rightarrow \mathrm{T}(\dot{\exists}\mathbf{v}x), \Delta$

KFG7 (i) $\Gamma, \mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(\dot{\exists}\mathbf{v}x), \mathrm{T}(\dot{\neg}\dot{\exists}\mathbf{v}x) \Rightarrow \forall\mathbf{t}\mathrm{T}s(\dot{\neg}x, \mathbf{t}, \mathbf{v}), \Delta$
(ii) $\Gamma, \mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(\dot{\exists}\mathbf{v}x), \forall\mathbf{t}\mathrm{T}s(\dot{\neg}x, \mathbf{t}, \mathbf{v}) \Rightarrow \mathrm{T}(\dot{\neg}\dot{\exists}\mathbf{v}x), \Delta$

KFG8 (i) $\Gamma, \mathrm{T}\mathrm{val}(\mathbf{t}) \Rightarrow \mathrm{T}\dot{\mathrm{T}}\mathbf{t}, \Delta$
(ii) $\Gamma, \mathrm{T}\dot{\mathrm{T}}\mathbf{t} \Rightarrow \mathrm{T}\mathrm{val}(\mathbf{t}), \Delta$

KFG9 (i) $\Gamma, \mathrm{T}\dot{\neg}\mathrm{val}(\mathbf{t}) \vee \neg\mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(\mathrm{val}(\mathbf{t})) \Rightarrow \mathrm{T}\dot{\neg}\dot{\mathrm{T}}\mathbf{t}, \Delta$
(ii) $\Gamma, \mathrm{T}\dot{\neg}\dot{\mathrm{T}}\mathbf{t} \Rightarrow \mathrm{T}\dot{\neg}\mathrm{val}(\mathbf{t}) \vee \neg\mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(\mathrm{val}(\mathbf{t})), \Delta$

KFG10 $\Gamma, \mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(x), \neg\mathrm{T}x \Rightarrow \mathrm{T}\dot{\neg}x, \Delta$

KFG11 $\Gamma, \mathrm{T}x \Rightarrow \mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(x), \Delta$

KF is defined as KFG minus KFG10. The system KFG$\restriction$ is obtained by restricting in KFG (IND) to formulae of $\mathscr{L}$ only. Although now the sequent arrow does contrapose, we don't have that T commutes with $\neg$ anymore, on pain of triviality. Thus, negations have to be considered separately once more, in this case to guarantee that the extension of the truth predicate is closed under disjunction, existential quantifiers, and iterations of T. KFG10 ensures that there are no gaps in the extension of the truth predicate, that is, if a sentence is not in the extension, then its negation must be.

The following results establish that KFG$\restriction$, and a fortiori KFG, are able to prove a great deal of transparency. They will be needed in later proofs.

Proposition 22. *The following sequents are derivable in* $\mathsf{KFG}\!\restriction$ *for every formula* $\varphi(v_1,\ldots,v_n)\in\mathscr{L}$*:*

$$\Gamma,\varphi(\mathrm{val}(\mathbf{t_1}),\ldots,\mathrm{val}(\mathbf{t_n}))\Rightarrow \mathrm{T}\ulcorner\varphi(\dot{\mathbf{t}}_{\mathbf{1}},\ldots,\dot{\mathbf{t}}_{\mathbf{n}})\urcorner,\Delta \tag{T-In$\restriction$}$$

$$\Gamma,\mathrm{T}\ulcorner\varphi(\dot{\mathbf{t}}_{\mathbf{1}},\ldots,\dot{\mathbf{t}}_{\mathbf{n}})\urcorner\Rightarrow\varphi(\mathrm{val}(\mathbf{t_1}),\ldots,\mathrm{val}(\mathbf{t_n})),\Delta \tag{T-Out$\restriction$}$$

Proof. The result follows easily from the sequents KFG1-KFG7 by induction on the complexity of φ. □

Proposition 23. *The following sequents are derivable in* $\mathsf{KFG}\!\restriction$ *for every formula* $\varphi(v_1,\ldots,v_n)\in\mathscr{L}_{\mathrm{T}}$*:*

$$\Gamma,\varphi(\mathrm{val}(\mathbf{t_1}),\ldots,\mathrm{val}(\mathbf{t_n}))\Rightarrow \mathrm{T}\ulcorner\varphi(\dot{\mathbf{t}}_{\mathbf{1}},\ldots,\dot{\mathbf{t}}_{\mathbf{n}})\urcorner,\Delta \tag{T-In}$$

Proof. By induction on the complexity of φ. For the base case we need to consider literals instead of just atomic formulae. The interesting case is for formulae of the form $\neg\mathrm{T}t$. The following is a derivation in $\mathsf{KFG}\!\restriction$ (we omit logical steps):

$$\dfrac{\dfrac{\Gamma,\neg\mathrm{T}t\Rightarrow\neg\mathrm{T}t,\Delta}{\Gamma,\neg\mathrm{T}t\Rightarrow\mathrm{T}\dot{\neg}t,\Delta}\,(\mathsf{KFG}10)}{\Gamma,\neg\mathrm{T}t\Rightarrow\mathrm{T}\ulcorner\neg\mathrm{T}\dot{t}\urcorner,\Delta}\,(\mathsf{KFG}9.(\mathrm{i}))$$

For the inductive step cases in which φ a disjunction, an existential claim, or a negation of these two, or a double negation must be considered. They all follow easily from the truth axioms of $\mathsf{KFG}\!\restriction$ and the inductive hypothesis. □

Following the same reasoning as Halbach & Nicolai [20], the subsequent proposition suggests that KFG and $\mathsf{KFG}\!\restriction$ embody somehow the same conception of truth as CKF and $\mathsf{CKF}\!\restriction$, for the capture the closing-offs of the same family of models.

Proposition 24 ($\mathbb{N}$-categoricity of KFG and $\mathsf{KFG}\!\restriction$)**.** $\langle\mathbb{N},S\rangle\vDash_{\mathsf{CL}}\mathsf{KFG}$ *(*$\mathsf{KFG}\!\restriction$*) iff* $\langle\mathbb{N},S\rangle$ *is the closing-off of a fixed-point LP-model.*

Proof. See Lemma 4 and section 5 of Halbach & Nicolai [20]. □

Fischer et al. [17] ponder several ways in which an axiomatic truth system could be said to capture a certain semantic construction. Despite there being certain drawbacks, they lean towards the criterion of $\mathbb{N}$-categoricity. Given an axiomatic theory *Th* formulated over a logic L in an extension $\mathcal{L}$ of $\mathscr{L}_{\mathrm{T}}$, and a class of models M for that language expanding $\mathbb{N}$, *Th* is $\mathbb{N}$-categorical with respect to this class of models iff, for every model $\mathcal{M}$ that expands $\mathbb{N}$ to the whole language $\mathcal{L}$, $\mathcal{M}\vDash_L Th$ iff $\mathcal{M}\in M$. In other words, we say that an axiomatic theory of truth is $\mathbb{N}$-categorical if, once the interpretation of the arithmetical vocabulary is fixed, the axioms of the system pin down the interpretation of the semantic vocabulary exactly in the desired way. Of course, categoricity *simpliciter* is an impossible goal. $\mathbb{N}$-categoricity, on the other hand, is a perfectly reasonable requirement. CKF and $\mathsf{CKF}\!\restriction$, for instance, are both trivially $\mathbb{N}$-categorical with respect to the class of fixed-point models introduced in 1.3, as Proposition 21 shows. In this sense, we say CKF and $\mathsf{CKF}\!\restriction$ axiomatise or capture the class of paraconsistent fixed-point models. More interestingly, Proposition 24 establishes the $\mathbb{N}$-categoricity of KFG and $\mathsf{KFG}\!\restriction$ with respect to the class of closing-offs of fixed-point LP-models. Therefore, KFG and $\mathsf{KFG}\!\restriction$ can be said to capture this class of interpretations.

Nonetheless, neither KFG nor $\mathsf{KFG}\!\restriction$ are sound with respect to the class of paraconsistent fixed-point models of the language. The internal logics of the classical systems, by contrast, can be shown to be not only sound, but also $\mathbb{N}$-categorical with respect to Kripke's paraconsistent fixed-point models. Therefore, it is far more reasonable to compare, e.g. the internal logic of KFG with CKF, rather than KFG itself.

Recall IKFG is the set $\{\varphi \in \mathscr{L}_\mathrm{T} \mid \mathsf{KFG} \vdash_{\mathsf{CL}} \mathrm{T}\ulcorner\varphi\urcorner\}$, and let $\mathsf{IKFG}\upharpoonright := \{\varphi \in \mathscr{L}_\mathrm{T} \mid \mathsf{KFG}\upharpoonright \vdash_{\mathsf{CL}} \mathrm{T}\ulcorner\varphi\urcorner\}$. To establish our $\mathbb{N}$-categoricity result, we first need the following lemma.

Lemma 25. *For every formula $\varphi(v_1, \ldots, v_n) \in \mathscr{L}_\mathrm{T}$, the following belong to $\mathsf{IKFG}\upharpoonright$:*

$$\mathrm{T}\ulcorner\varphi(\dot{\mathbf{t}}_\mathbf{1}, \ldots, \dot{\mathbf{t}}_\mathbf{n})\urcorner \leftrightarrow \varphi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})) \quad \text{(T-schema)}$$

$$\neg\mathrm{T}\ulcorner\varphi(\dot{\mathbf{t}}_\mathbf{1}, \ldots, \dot{\mathbf{t}}_\mathbf{n})\urcorner \leftrightarrow \neg\varphi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})) \quad \text{(CT-schema)}$$

Proof. The result follows trivially from the definition of $\mathsf{IKFG}\upharpoonright$ and the sequents KFG8 and KFG9. □

Proposition 26 ($\mathbb{N}$-categoricity of IKFG and $\mathsf{IKFG}\upharpoonright$)**.** *$\langle \mathbb{N}, S^+, S^- \rangle \vDash_{\mathsf{LP}}$ IKFG ($\mathsf{IKFG}\upharpoonright$) iff $\langle \mathbb{N}, S^+, S^- \rangle$ a fixed-point LP-model.*

Proof. Let $\varphi \in$ IKFG and $\langle \mathbb{N}, S^+, S^- \rangle$ be a fixed-point model of $\mathscr{L}_\mathrm{T}$. By Proposition 24, $\langle \mathbb{N}, S^+ \rangle \vDash_{\mathsf{CL}}$ KFG. Since KFG $\vdash_{\mathsf{CL}} \mathrm{T}\ulcorner\varphi\urcorner$, we have that $\#\varphi \in S^+$. Therefore, $\langle \mathbb{N}, S^+, S^- \rangle \vDash_{\mathsf{LP}} \varphi$.

For the converse, assume $\langle \mathbb{N}, S^+, S^- \rangle \vDash_{\mathsf{LP}} \mathsf{IKFG}\upharpoonright$, and let v be the valuation function in $\langle \mathbb{N}, S^+, S^- \rangle$. Then, we have that $\#\varphi \in S^+$ iff $v(\mathrm{T}\ulcorner\varphi\urcorner) \geqslant \frac{1}{2}$, which is the case, by the (T-schema) in Lemma 25, iff $v(\varphi) \geqslant \frac{1}{2}$, which means that $\langle \mathbb{N}, S^+, S^- \rangle \vDash_{\mathsf{LP}} \varphi$. Also, $\#\varphi \in S^-$ iff $v(\mathrm{T}\ulcorner\varphi\urcorner) \leqslant \frac{1}{2}$, which is the case, by the (CT-schema) in Lemma 25, iff $v(\varphi) \leqslant \frac{1}{2}$, which means that $\langle \mathbb{N}, S^+, S^- \rangle \vDash_{\mathsf{LP}} \neg\varphi$. Therefore, $\langle \mathbb{N}, S^+, S^- \rangle$ is a fixed-point model. □

3 The classical vs. the non-classical

In this section I compare CKF and $\mathsf{CKF}\upharpoonright$ with the internal logics IKFG and $\mathsf{IKFG}\upharpoonright$ of their respective classical counterparts, KFG and $\mathsf{KFG}\upharpoonright$. I provide two main results. First, I show that CKF is proof-theoretically much weaker than IKFG. Second, I prove that $\mathsf{CKF}\upharpoonright$ and $\mathsf{IKFG}\upharpoonright$ prove the same theorems in $\mathscr{L}_\mathrm{T}$.

3.1 A low upper bound for the non-classical

Let BTG $\subseteq \mathscr{L}_\mathrm{T}$ be the result of replacing in KFG the rule of induction (IND) with the weaker rule of internal induction,

$$\frac{\Gamma, \mathrm{T}\ulcorner\varphi(\dot{x})\urcorner \Rightarrow \mathrm{T}\ulcorner\varphi(\mathrm{S}\dot{x})\urcorner, \Delta}{\Gamma, \mathrm{T}\ulcorner\varphi(0)\urcorner \Rightarrow \mathrm{T}\ulcorner\varphi(\dot{\mathbf{t}})\urcorner, \Delta} \text{(IIND)}$$

for every formula $\varphi(v) \in \mathscr{L}_\mathrm{T}$, if x is not free in Γ, Δ, and $\varphi(0)$ and $\mathbf{t}$ is arbitrary. BTG is an extension of the system BT studied by Cantini [10], which is defined as BTG minus KFG10. Cantini shows that BT is a subsystem of KF. Thus, BTG is a subsystem of KFG. Extending a result by Halbach & Horsten [19], in this section I show that BTG can internalise the proof of the soundness of CKF. Together with Cantini's proof-theoretic analysis for BT and some extensions, this gives an upper bound for what CKF can prove. As it turns out, this upper bound is far lower than IKFG's.

Since the language of CKF, $\mathscr{L}_\mathrm{T}^\circ$, contains symbols that are not in $\mathscr{L}_\mathrm{T}$, we need to provide a translation of $\mathscr{L}_\mathrm{T}^\circ$ to $\mathscr{L}_\mathrm{T}$ before we can prove the soundness of CKF in BTG. Let $\vec{v}$ abbreviate $v_1, \ldots, v_n$, and let $\tau : \mathscr{L}_\mathrm{T}^\circ \to \mathscr{L}_\mathrm{T}$ be defined by the following clauses:

- $\tau(\varphi) := \varphi$, if φ is an atomic formula
- $\tau(\neg\varphi) := \neg\tau(\varphi)$
- $\tau(\varphi \vee \psi) := \tau(\varphi) \vee \tau(\psi)$
- $\tau(\exists v\varphi) := \exists v\tau(\varphi)$
- $\tau(\circ\varphi(\vec{v})) := \neg\mathrm{T}\ulcorner\varphi(\vec{\underset{\cdot}{v}})\urcorner \vee \neg\mathrm{T}\ulcorner\neg\varphi(\vec{\underset{\cdot}{v}})\urcorner$, if φ is an atomic formula
- $\tau(\circ\neg\varphi) := \tau(\circ\varphi)$
- $\tau(\circ(\varphi \vee \psi)) := \tau((\circ\varphi \vee (\psi \wedge \circ\psi)) \wedge (\circ\psi \vee (\varphi \wedge \circ\varphi)))$
- $\tau(\circ\exists v\varphi) := \tau(\exists v(\circ\varphi \wedge \varphi) \vee \forall v(\circ\varphi \wedge \neg\varphi))$
- $\tau(\circ\circ\varphi) := (0 = 0)$

We can say in $\mathscr{L}_{\mathrm{T}}$ that an atomic sentence of $\mathscr{L}_{\mathrm{T}}$ is consistent by indicating that either the sentence or its negation are not in the extension of the truth predicate. The consistency of other formulae will depend on the consistency of their atomic components. We need to consider all cases for consistency statements because our truth predicate is not defined for sentences containing $\circ$, so the clause for the translation of $\circ\varphi$ where φ is atomic cannot be directly extended to all formulae of $\mathscr{L}_{\mathrm{T}}^{\circ}$.

To prove our first result, we need the following observation.

Proposition 27. *The following sequents are derivable in* BTG *for every formula* $\varphi(v_1, \ldots, v_n) \in \mathscr{L}$*:*

$$\Gamma, \varphi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})) \Rightarrow \mathrm{T}\ulcorner\varphi(\underset{\cdot}{\mathbf{t_1}}, \ldots, \underset{\cdot}{\mathbf{t_n}})\urcorner, \Delta \qquad \text{(T-In}\restriction)$$

$$\Gamma, \mathrm{T}\ulcorner\varphi(\underset{\cdot}{\mathbf{t_1}}, \ldots, \underset{\cdot}{\mathbf{t_n}})\urcorner \Rightarrow \varphi(\mathrm{val}(\mathbf{t_1}), \ldots, \mathrm{val}(\mathbf{t_n})), \Delta \qquad \text{(T-Out}\restriction)$$

Proof. The proof is the same as that of Proposition 22. □

If $\Gamma \subseteq \mathscr{L}_{\mathrm{T}}^{\circ}$ is finite, let $\tau(\Gamma) := \{\tau(\varphi) : \varphi \in \Gamma\}$, $\bigwedge\Gamma$ be the conjunction of members of Γ, and $\bigvee\Gamma$ their disjunction.

Proposition 28. *If* $\Gamma \Rightarrow \Delta$ *is derivable in* CKF*, then* $\mathrm{T}\ulcorner\bigwedge\tau(\Gamma)\urcorner \Rightarrow \mathrm{T}\ulcorner\bigvee\tau(\Delta)\urcorner$ *is derivable in* BTG*.*

Proof. It is enough to show that all initial sequents in CKF are provably sound in BTG and that all rules of CKF provably preserve soundness in BTG. Halbach & Horsten [19] show that BT can prove the soundness of CKF's logical axiom and rules except for $(\neg\mathrm{R})$, all of CKF's initial arithmetical sequents, as well as (IND) for formulae of $\mathscr{L}_{\mathrm{T}}$, CKF2, CKF3, CKF5, CKF7, and CKF9. It remains to be seen that $(\neg\mathrm{R})$ and the instances of (IND) for formulae containing $\circ$ provably preserves soundness, and that (CON), CKF1, CKF4, CKF6, and CKF8 are provably sound sequents in BTG.

For $(\neg\mathrm{R})$, let $\varphi(\vec{v}) \in \mathscr{L}_{\mathrm{T}}^{\circ}$ be an atomic formula. Thus, $\tau(\varphi(\vec{v})) = \varphi(\vec{v}) \in \mathscr{L}_{\mathrm{T}}$. Therefore, we have the following derivation in BTG:

$$\cfrac{\cfrac{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma, \varphi(\vec{\underset{\cdot}{v}}))\urcorner \Rightarrow \mathrm{T}\ulcorner\bigvee\tau(\neg\varphi(\vec{\underset{\cdot}{v}}), \Delta)\urcorner}{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma)\urcorner, \mathrm{T}\ulcorner\varphi(\vec{\underset{\cdot}{v}})\urcorner \Rightarrow \mathrm{T}\underset{\cdot}{\neg}\ulcorner\varphi(\vec{\underset{\cdot}{v}})\urcorner, \mathrm{T}\ulcorner\bigvee\tau(\Delta)\urcorner}\ (\mathsf{KFG}4{,}5)}{\cfrac{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma)\urcorner \Rightarrow \neg\mathrm{T}\ulcorner\varphi(\vec{\underset{\cdot}{v}})\urcorner, \mathrm{T}\underset{\cdot}{\neg}\ulcorner\varphi(\vec{\underset{\cdot}{v}})\urcorner, \mathrm{T}\ulcorner\bigvee\tau(\Delta)\urcorner}{\cfrac{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma)\urcorner \Rightarrow \mathrm{T}\underset{\cdot}{\neg}\ulcorner\varphi(\vec{\underset{\cdot}{v}})\urcorner, \mathrm{T}\underset{\cdot}{\neg}\ulcorner\varphi(\vec{\underset{\cdot}{v}})\urcorner, \mathrm{T}\ulcorner\bigvee\tau(\Delta)\urcorner}{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma)\urcorner \Rightarrow \mathrm{T}\ulcorner\bigvee\tau(\neg\varphi(\vec{\underset{\cdot}{v}}), \Delta)\urcorner}\ (\mathsf{KFG}4)}\ (\mathsf{KFG}10)}$$

For the instances of (IND) containing $\circ$, note that the translation $\tau(\varphi)$ of any $\varphi \in \mathscr{L}_{\mathrm{T}}^{\circ}$ is a formula of $\mathscr{L}_{\mathrm{T}}$. We prove the soundness of (CON) by induction on the complexity of φ. Note that $\varphi \in \mathscr{L}$, so $\tau(\varphi) = \varphi$. If φ is atomic, we have the following derivation in BTG:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma)\urcorner \Rightarrow \neg\varphi(\vec{v}), \varphi(\vec{v}), \mathrm{T}\ulcorner\bigvee\tau(\Delta)\urcorner}{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma)\urcorner \Rightarrow \mathrm{T}\ulcorner\neg\varphi(\dot{\vec{v}})\urcorner, \mathrm{T}\ulcorner\varphi(\dot{\vec{v}})\urcorner, \mathrm{T}\ulcorner\bigvee\tau(\Delta)\urcorner}\,(\text{T-In}\upharpoonright)}{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma)\urcorner \Rightarrow \mathrm{T}\ulcorner\neg\varphi(\dot{\vec{v}})\urcorner, \mathrm{T}\ulcorner\neg\neg\varphi(\dot{\vec{v}})\urcorner, \mathrm{T}\ulcorner\bigvee\tau(\Delta)\urcorner}\,(\mathsf{KFG}3)}{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma)\urcorner \Rightarrow \mathrm{T}\dot{\neg}\dot{\mathrm{T}}\ulcorner\varphi(\dot{\vec{v}})\urcorner, \mathrm{T}\dot{\neg}\dot{\mathrm{T}}\ulcorner\neg\varphi(\dot{\vec{v}})\urcorner, \mathrm{T}\ulcorner\bigvee\tau(\Delta)\urcorner}\,(\mathsf{KFG}9)}{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma)\urcorner \Rightarrow \mathrm{T}\ulcorner\bigvee\tau(\neg\mathrm{T}\ulcorner\varphi(\dot{\vec{v}})\urcorner \vee \neg\mathrm{T}\ulcorner\neg\varphi(\dot{\vec{v}})\urcorner, \Delta)\urcorner}\,(\mathsf{KFG}4)$$

Assume (CON) is provably sound in BTG for every formulae of complexity less than φ's. If $\varphi := \neg\psi$, $\tau(\circ\varphi) = \tau(\circ\psi)$, so the result follows trivially from the inductive hypothesis. The cases in which φ is a disjunction or an existential claim follow similarly from the definition of τ and the inductive hypothesis.

Finally, we prove the soundness of CKF1.(i) in BTG. The case for CKF1.(ii) is analogous. The soundness of the remaining initial sequents of CKF follows from the axioms of BTG in a similar fashion.

$$\dfrac{\dfrac{\dfrac{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma)\urcorner, \mathrm{val}(\mathbf{s}) = \mathrm{val}(\mathbf{t}), \mathrm{val}(\mathbf{s}) \neq \mathrm{val}(\mathbf{t}) \Rightarrow \mathrm{T}\ulcorner\bigvee\tau(\Delta)\urcorner}{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma)\urcorner, \mathrm{T}\ulcorner\mathrm{val}(\mathbf{s}) = \mathrm{val}(\mathbf{t})\urcorner, \mathrm{T}\dot{\neg}(\mathbf{s}\dot{=}\mathbf{t}) \Rightarrow T\ulcorner\bigvee\tau(\Delta)\urcorner}\,(\text{T-Out}\upharpoonright)}{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma)\urcorner, \mathrm{val}(\mathbf{s}) = \mathrm{val}(\mathbf{t}), \mathrm{T}\ulcorner\neg\mathrm{T}(\mathbf{s}\dot{=}\mathbf{t})\urcorner \Rightarrow \mathrm{T}\ulcorner\bigvee\tau(\Delta)\urcorner}\,(\mathsf{KFG}9)}{\mathrm{T}\ulcorner\bigwedge\tau(\Gamma, \mathrm{val}(\mathbf{s}) = \mathrm{val}(\mathbf{t}), \neg\mathrm{T}(\mathbf{s}\dot{=}\mathbf{t}))\urcorner \Rightarrow \mathrm{T}\ulcorner\bigvee\tau(\Delta)\urcorner}\,(\mathsf{KFG}5)$$

□

Corollary 29. *For every $\varphi \in \mathscr{L}$, if* CKF $\vdash_{\mathsf{LP}^\circ} \varphi$*, then* BTG $\vdash_{\mathsf{CL}} \varphi$.

Proof. If $\Rightarrow \varphi$ is derivable in CKF, then $\Rightarrow \mathrm{T}\ulcorner\varphi\urcorner$ is derivable in BTG by Proposition 28. By Proposition 27, we have that BTG $\vdash_{\mathsf{CL}} \varphi$, since φ is arithmetical. □

As Cantini points out, the addition of principles blocking the occurrence of gaps or gluts to BT, such as KFG10, does not increase the strength of the resulting system. Thus, the proof-theoretic power of BTG coincides with that of BT. This, together with Cantini's proof-theoretic analysis of BT, entails the following result.[12]

Proposition 30 (Cantini)**.** BTG *is proof-theoretically equivalent to ramified truth* $\mathsf{RT}_{<\omega^\omega}$ *up to* ω^ω.

The theory of ramified truth $\mathsf{RT}_{<\alpha}$ up to ordinal α consists of α-many iterations of the axiomatisation of Tarski's truth definition over Peano arithmetic (cf. Halbach [18, chap. 9]). Proposition 30 sets an upper bound on the proof-theoretic power of CKF.

Corollary 31. CKF *cannot prove more than ω^ω iterations of the truth predicate.*

On the other hand, it is well known that KF, and a fortiori KFG, can prove iterations of the truth predicate up to the ordinal ϵ_0, the limit of all the ordinals $\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$ (cf. Halbach [18, chap. 15.3]). As a consequence, KFG is much stronger than its non-classical counterpart, CKF. As Halbach & Nicolai [20] point out, this difference can also be spelled out in terms of instances of arithmetical transfinite induction. While KFG can prove instances of transfinite induction for $\mathscr{L}_\mathrm{T}$ up to any ordinal smaller than ϵ_0, CKF only contains transfinite induction for $\mathscr{L}_\mathrm{T}$ up to any ordinal smaller than ω^ω. Moreover, this translates into differences in the purely arithmetical vocabulary. In particular, KFG proves arithmetical instances of transfinite induction up to $\varphi_{\epsilon_0}(0)$, whereas CKF proves arithmetical transfinite induction only up to ordinals smaller than $\varphi_\omega(0)$ in the Veblen hierarchy.[13]

[12] See Cantini [10, §3, 4 & 9].

[13] For an introduction to the Veblen functions and the Veblen hierarchy, see Pohlers [24].

Note that, since BTG is a subsystem of KFG, Proposition 28 implies that CKF is contained in IKFG.

Corollary 32. *For every $\varphi \in \mathscr{L}_{\mathrm{T}}$, if $\mathsf{CKF} \vdash_{\mathsf{LP}^\circ} \varphi$, $\varphi \in \mathsf{IKFG}$.*

Proof. If $\Rightarrow \varphi$ is derivable in CKF, then $\Rightarrow \mathrm{T}\ulcorner\varphi\urcorner$ is derivable in BTG by Proposition 28. Since $\mathsf{BTG} \subseteq \mathsf{KFG}$, we have that $\mathsf{KFG} \vdash_{\mathsf{CL}} \mathrm{T}\ulcorner\varphi\urcorner$, so $\varphi \in \mathsf{IKFG}$. □

Furthermore, note that KFG and IKFG have the same arithmetical theorems, by Proposition 22. Therefore, the discrepancy in arithmetical content also holds between CKF and IKFG. The latter can prove, for instance, the consistency of CKF, as well as iterated reflection principles for this system. IKFG is proof-theoretically much stronger than CKF, the latter is a proper subsystem of the former.

3.2 Setting induction aside

In this section I show that when the rule of induction is restricted to formulae not containing the truth predicate, the differences in arithmetical content between the classical and the non-classical axiomatisations of Kripke's construction fade away. In other words, I prove that $\mathsf{CKF}{\restriction}$ and $\mathsf{IKFG}{\restriction}$ are equivalent over $\mathscr{L}_{\mathrm{T}}$. The proofs are a variant of Halbach & Nicolai's [20, §4.2].

Proposition 33. *If $\varphi \in \mathscr{L}_{\mathrm{T}}$ and $\mathsf{CKF}{\restriction} \vdash_{\mathsf{LP}^\circ} \varphi$, then $\varphi \in \mathsf{IKFG}{\restriction}$.*

Proof. It is enough to prove in $\mathsf{KFG}{\restriction}$ the soundness of all initial sequents and rules of $\mathsf{CKF}{\restriction}$. For then, if $\varphi \in \mathscr{L}_{\mathrm{T}}$ and $\Rightarrow \varphi$ is derivable in $\mathsf{CKF}{\restriction}$, we have that $\mathsf{KFG}{\restriction} \vdash_{\mathsf{CL}} \mathrm{T}\ulcorner\varphi\urcorner$, so $\varphi \in \mathsf{IKFG}{\restriction}$. In the proof of Proposition 28 we have seen that all logical rules and all truth-theoretic axioms of $\mathsf{CKF}{\restriction}$, as well as (CON), can be proved to be sound in BTG without turning to (IIND). By Proposition 22, we can also prove the soundness of every basic arithmetical sequent in $\mathsf{KFG}{\restriction}$, as well as that of (IND) restricted to arithmetical formulae. □

Let $\langle \mathcal{M}, S^+, S^- \rangle$ be an LP-model of $\mathscr{L}_{\mathrm{T}}$ such that $\mathcal{M}$ interprets the non-logical vocabulary of $\mathscr{L}_{\mathrm{T}}$ plus the identity predicate, and S^+, S^- are the extension and anti-extension of T, respectively. Then, the classical model $\langle \mathcal{M}, S^+ \rangle$ is the closing-off of $\langle \mathcal{M}, S^+, S^- \rangle$. To show that the converse of Proposition 33 also holds we first need to prove the following result. Let σ be an assignment on a model $\mathcal{M}$ for a logic L. We write $\mathcal{M}, \sigma \vDash_L \varphi$ to indicate that φ is true in $\mathcal{M}$ under σ.

Lemma 34. *If $\langle \mathcal{M}, S^+, S^- \rangle, \sigma \vDash_{\mathsf{LP}^\circ} \mathsf{CKF}{\restriction}$, then $\langle \mathcal{M}, S^+ \rangle, \sigma \vDash_{\mathsf{CL}} \mathsf{KFG}{\restriction}$.*

Proof. It is enough to show that if $\langle \mathcal{M}, S^+, S^- \rangle, \sigma \vDash_{\mathsf{LP}^\circ} \mathsf{CKF}{\restriction}$, then all initial sequents and rules of $\mathsf{KFG}{\restriction}$ are sound in $\langle \mathcal{M}, S^+ \rangle$ under σ. We only consider assignments explicitly when needed. Note first that

$$\langle \mathcal{M}, S^+, S^- \rangle, \sigma \vDash_{\mathsf{LP}^\circ} \mathrm{T}\ulcorner\varphi\urcorner \text{ iff } \langle \mathcal{M}, S^+ \rangle, \sigma \vDash_{\mathsf{CL}} \mathrm{T}\ulcorner\varphi\urcorner$$

$\mathsf{KFG}{\restriction}$'s logical axiom and rules are sound in $\langle \mathcal{M}, S^+ \rangle$, for this is a classical model. Arithmetical initial sequents and (IND) restricted to formulae of $\mathscr{L}$ are also sound in $\langle \mathcal{M}, S^+ \rangle$, for they obtain in $\mathsf{CKF}{\restriction}$ and $\mathcal{M}$ assigns arithmetical formulae a classical truth value, due to (CON). Regarding the remaining initial sequents in $\mathsf{KFG}{\restriction}$, we just consider KFG9.(i) and KFG10. The other cases are similar.

For KFG9.(i), let $\langle \mathcal{M}, S^+ \rangle \vDash_{\mathsf{CL}} \mathrm{T}\dot{\neg}\mathrm{val}(\mathbf{t}) \vee \neg\mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(\mathrm{val}(\mathbf{t}))$. If $\langle \mathcal{M}, S^+ \rangle \vDash_{\mathsf{CL}} \neg\mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(\mathrm{val}(\mathbf{t}))$, by (CON) we know that $\langle \mathcal{M}, S^+, S^- \rangle \nvDash_{\mathsf{LP}^\circ} \mathrm{Sent}_{\mathscr{L}_{\mathrm{T}}}(\mathrm{val}(\mathbf{t}))$. By CKF9, $\langle \mathcal{M}, S^+, S^- \rangle \nvDash_{\mathsf{LP}^\circ} \mathrm{Tval}(\mathbf{t})$, which means that $\langle \mathcal{M}, S^+, S^- \rangle \vDash_{\mathsf{LP}^\circ} \neg\mathrm{Tval}(\mathbf{t})$. By (T-In) in Proposition 18, $\langle \mathcal{M}, S^+, S^- \rangle \vDash_{\mathsf{LP}^\circ} \mathrm{T}\dot{\neg}\dot{\mathrm{T}}\mathbf{t}$, so $\langle \mathcal{M}, S^+ \rangle \vDash_{\mathsf{CL}} \mathrm{T}\dot{\neg}\dot{\mathrm{T}}\mathbf{t}$. If

$\langle \mathcal{M}, S^+ \rangle \vDash_{\mathsf{CL}} \mathrm{T}\dot{\neg}\mathrm{val}(\mathbf{t}) \wedge \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(\mathrm{val}(\mathbf{t}))$, by $\mathsf{CKF}8$ we have that $\langle \mathcal{M}, S^+, S^- \rangle \vDash_{\mathsf{LP}^\circ}$ $\mathrm{T}\dot{\neg}\dot{\mathrm{T}}\mathbf{t}$, so $\langle \mathcal{M}, S^+ \rangle \vDash_{\mathsf{CL}} \mathrm{T}\dot{\neg}\dot{\mathrm{T}}\mathbf{t}$ as well.

For $\mathsf{KFG}10$ assume $\langle \mathcal{M}, S^+ \rangle, \sigma \vDash_{\mathsf{CL}} \mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(x) \wedge \neg \mathrm{T}x$. Thus, we have that $\langle \mathcal{M}, S^+, S^- \rangle, \sigma \vDash_{\mathsf{LP}^\circ}$ $\mathrm{Sent}_{\mathscr{L}_\mathrm{T}}(x)$ and $\sigma(x) \notin S^+$. Therefore, $\sigma(x) \in S^-$. By $\mathsf{CKF}2$, $\langle \mathcal{M}, S^+, S^- \rangle, \sigma \vDash_{\mathsf{LP}^\circ}$ $\mathrm{T}\dot{\neg}x$, which implies that $\langle \mathcal{M}, S^+ \rangle, \sigma \vDash_{\mathsf{CL}} \mathrm{T}\dot{\neg}x$. □

Proposition 35. *If* $\varphi \in \mathsf{IKFG}\!\restriction$*, then* $\mathsf{CKF}\!\restriction \vdash_{\mathsf{LP}^\circ} \varphi$.

Proof. Assume $\mathsf{CKF}\!\restriction \nvdash_{\mathsf{LP}^\circ} \varphi$. By the strong completeness of LP° (cf. Proposition 49 in the appendix), there is a model $\langle \mathcal{M}, S^+, S^- \rangle$ of $\mathsf{CKF}\!\restriction$ such that $\langle \mathcal{M}, S^+, S^- \rangle \nvDash_{\mathsf{LP}^\circ} \varphi$. Thus, $\langle \mathcal{M}, S^+ \rangle$ is a model of $\mathsf{KFG}\!\restriction$ by Proposition 33, and $\#\varphi \notin S^+$. Therefore, $\langle \mathcal{M}, S^+ \rangle \nvDash_{\mathsf{CL}} \mathrm{T}\ulcorner\varphi\urcorner$, so $\varphi \notin \mathsf{IKFG}\!\restriction$. □

We are now ready to prove the last result of the section, that $\mathsf{IKFG}\!\restriction$ and $\mathsf{CKF}\!\restriction$ are equivalent with respect to their consequences in $\mathscr{L}_\mathrm{T}$.

Corollary 36. *If* $\varphi \in \mathscr{L}_\mathrm{T}$*, then* $\mathsf{CKF}\!\restriction \vdash_{\mathsf{LP}^\circ} \varphi$ *iff* $\varphi \in \mathsf{IKFG}\!\restriction$.

4 Conclusion

Halbach & Nicolai [20] consider the class of Kripke's fixed-point models of $\mathscr{L}_\mathrm{T}$ with both gaps and gluts, including models in which gaps and gluts occur simultaneously. They provide an axiomatisation of this class in a sound and complete calculus for basic De Morgan logic, and compare the resulting system, PKF, with the internal logic of KF, IKF. This comparison is fair, as both theories have a transparent truth predicate and are sound and $\mathbb{N}$-categorical with respect to the models under consideration. They show that while $\mathsf{PKF}\!\restriction$, the theory that results from restricting the rule of induction (IND) to arithmetical formulae in PKF, is equivalent to the internal logic of $\mathsf{KF}\!\restriction$, IKF is much stronger than PKF, both with respect to its truth-theoretic and its arithmetical content. Moreover, they provide analogous results comparing classical and non-classical axiomatisations of classes of Kripke's fixed-point models where gaps or gluts are excluded. In each case, arithmetical content is lost when we move from the classical to the non-classical setting.

Halbach & Nicolai infer that the arithmetical weakness of PKF and the other non-classical systems cannot be blamed on any other factor than the mutilation of classical logic. It cannot be blamed on the incompleteness of the calculus, since the calculi they work with are shown to be complete. Nor can it be blamed on the truth-theoretic content of the non-classical systems, for otherwise the differences in strength would also show when comparing $\mathsf{PKF}\!\restriction$ and $\mathsf{IKF}\!\restriction$, and the other corresponding restricted theories.

One could object that the instances of induction that contain the truth predicate are actually truth principles of some sort. However, as Halbach & Nicolai point out, induction is not to be considered a truth-theoretic but a mathematical principle. Our intuitive understanding of natural numbers comes with the disposition to accept any instance of this principle, including those in which the truth predicate occurs. There is nothing special about the truth predicate that contributes to the adoption of those instances of induction. Another way of seeing this is to consider instances of schematic principles of a different, e.g. logical nature. As we wouldn't think of the instances of the Law of Excluded Middle in which the truth predicate occurs as truth-theoretic principles, we shouldn't think of the corresponding instances of induction that way.

Consequently, Halbach & Nicolai [20, p. 2] conclude that, contrary to what many truth-theorists have claimed, abandoning classical logic can have an impact on non-semantic reasoning. In their own words, "[...] the incisions to classical logic, when applied to sentences with the truth predicate, severely impede schematic reasoning with the truth predicate. More specifically, classical patterns of mathematical reasoning are no longer licensed in PKF." These are instances of the rule of induction.

In this paper I have considered what initially seemed to be a more promising scenario for those who wish to circumscribe the restrictions to classical logic to truth-theoretic reasoning and other areas prone to paradoxes. While Halbach & Nicolai axiomatise the family of Kripke's fixed-point models with gluts but no gaps in basic De Morgan logic, I have done so in an extension of LP, LP°. In basic De Morgan logic without gaps, models are the same as LP's. However, for a sequent $\Gamma \Rightarrow \Delta$ to be sound it is not only required that in all models in which all members of Γ are true (i.e. receive truth-value 1 or $\frac{1}{2}$) at least one member of Δ is true as well, but also that falsity is preserved from the conclusions to the premises. That is, if all members of Δ are false (i.e. receive truth-value 0 or $\frac{1}{2}$), then at least one element in Γ must also be false. In LP the latter condition is dropped, as indicated in Definition 5. Thus, more sequents turn out to be valid in this logic; LP is already stronger than basic De Morgan logic.

Moreover, LP° extends LP with the consistency operator $\circ$, deemed a recovery operator, designed specifically for restoring classical reasoning whenever needed within paraconsistent logics such as LP.[14] According to Carnielli, Coniglio, & Marcos [12, p. 19], "This feature will permit consistent reasoning to be recovered from inside an inconsistent environment". I've provided a sound and complete calculus for LP° (see Propositions 44 and 38 in the appendix). In the axiomatisation of Kripke's paraconsistent fixed-point models in LP° sequents indicating the consistency of the arithmetical fragment of $\mathscr{L}_{\mathrm{T}}$ in terms of $\circ$ were added as axioms.

Nonetheless, we were still able to prove analogous results as those obtained by Halbach & Horsten [19] and Halbach & Nicolai [20]. We have seen that, on the one hand, IKFG can prove more arithmetical sentences than CKF. On the other hand, $\mathsf{IKFG}\restriction$ and $\mathsf{CKF}\restriction$ are equivalent with respect to their consequences in $\mathscr{L}_{\mathrm{T}}$. Therefore, we must draw similar conclusions. The weakness of CKF with respect to IKFG is not due to the incompleteness of the calculus, nor to CKF's truth-theoretic content. It is a result of the mutilation of classical logic, that shrinks the consequences we can draw from the instances of the arithmetical principle of induction in which the truth predicate occurs. These consequences cannot be retrieved in LP°, despite the availability of consistency operators. Given that this weakness of CKF also concerns expressions with purely mathematical content, it is clear that weakening the logic this way does have an impact on mathematical reasoning.

The impossibility of retrieving classical reasoning for arithmetical principles by means of consistency operators could have two different reasons, neither of which flatters those who embrace LP° as the base logic for their truth theories and, at the same time, wish to reason classically from arithmetical principles. One reason could be the way LP deals with paradoxical expressions, that is, it treats them as dialetheias (i.e. as both true and false). Given how connectives and quantifiers work in LP, this extends to many other sentences in which the truth predicate occurs, including several instances of induction. It could be that this *irreversibly* precludes us from reasoning classically from these instances, no matter what expressive resources, principles, or rules we extend the theory with (cf. Priest [26, §7.3]). In other words, it could be that the incisions made to classical logic are so severe that classical reasoning for all arithmetical principles is irretrievable.

If, instead, classical reasoning were retrievable for all instances of induction con-

[14]See Carnielli & Marcos [13], Carnielli, Coniglio, & Marcos [12], and Carnielli & Coniglio [11].

taining the truth predicate in some way or other, the blame for LP°'s failure would be on the consistency operator's performance. This would mean that consistency operators, within the framework of an LFI, are not *always* able to recover classical reasoning whenever desired or, as its promoters maintain, *right on demand*.

Appendix

In what follows I offer proofs of the soundness and completeness of the calculi for LP and LP° given in sections 2.1 and 2.2 with respect to the corresponding notions of semantic consequence introduced in Definitions 5 and 13, respectively.

Proposition 37 (Soundness of LP)**.** *If* $\Gamma \Rightarrow \Delta$ *is derivable in* LP*, then* $\Gamma \vDash_{\mathsf{LP}} \Delta$.

Proof. Note that every sequent obtained by (I) contains a literal φ both in Γ and Δ, so is trivially sound. One can easily check that all the rules in LP preserve soundness. I show it for (Cut), (Repl), and ($\neg$R).

For (Cut), assume there is a model $\mathcal{M}$ and an assignment σ on $\mathcal{M}$ such that $v_\sigma^{\mathcal{M}}(\Gamma) \geqslant \frac{1}{2}$ and $v_\sigma^{\mathcal{M}}(\Delta) = 0$. Either $v_\sigma^{\mathcal{M}}(\varphi) = 0$ or $v_\sigma^{\mathcal{M}}(\varphi) \geqslant \frac{1}{2}$. If $v_\sigma^{\mathcal{M}}(\varphi) = 0$, then $\Gamma \nvDash_{\mathsf{LP}} \varphi, \Delta$. If $v_\sigma^{\mathcal{M}}(\varphi) \geqslant \frac{1}{2}$, then $\Gamma, \varphi \nvDash_{\mathsf{LP}} \Delta$.

For (Repl), note that for every literal φ, model $\mathcal{M}$ and assignment σ on $\mathcal{M}$ in which $v_\sigma^{\mathcal{M}}(\{s = t, \varphi[s/v]\}) \geqslant \frac{1}{2}$, it is also the case that $v_\sigma^{\mathcal{M}}(\{s = t, \varphi[s/v], \varphi[t,v]\}) \geqslant \frac{1}{2}$. For if $v_\sigma^{\mathcal{M}}(s = t) \geqslant \frac{1}{2}$, $\langle \sigma(s), \sigma(t) \rangle \in =^+$, so $\sigma(s) = \sigma(t)$.

For ($\neg$R), assume $\Gamma, \varphi \vDash_{\mathsf{LP}} \Delta \cup \{\neg\varphi\}$, and let $\mathcal{M}$ and σ on $\mathcal{M}$ be such that $v_\sigma^{\mathcal{M}}(\Gamma) \geqslant \frac{1}{2}$. If $v_\sigma^{\mathcal{M}}(\varphi) \geqslant \frac{1}{2}$, by assumption there must be a $\psi \in \Delta \cup \{\neg\varphi\}$ such that $v_\sigma^{\mathcal{M}}(\psi) \geqslant \frac{1}{2}$. If $v_\sigma^{\mathcal{M}}(\varphi) = 0$, then $v_\sigma^{\mathcal{M}}(\neg\varphi) = 1$. □

Proposition 38 (Soundness of LP°)**.** *If* $\Gamma \Rightarrow \Delta$ *is derivable in* LP°*, then* $\Gamma \vDash_{\mathsf{LP}^\circ} \Delta$.

Proof. We extend the proof of Proposition 37, showing that rules of LP° that govern the consistency operator also preserve soundness.

For ($\circ$L), assume $\Gamma, \circ\varphi \nvDash_{\mathsf{LP}^\circ} \Delta$. Then, there is a model $\mathcal{M}$ and an assignment σ on $\mathcal{M}$ such that $v_\sigma^{\mathcal{M}}(\Gamma) \geqslant \frac{1}{2}$, $v_\sigma^{\mathcal{M}}(\circ\varphi) \geqslant \frac{1}{2}$, and $v_\sigma^{\mathcal{M}}(\Delta) = 0$. Thus, $v_\sigma^{\mathcal{M}}(\circ\varphi) = 1$, so either $v_\sigma^{\mathcal{M}}(\varphi) = 1$ or $v_\sigma^{\mathcal{M}}(\varphi) = 0$. In the former case, $v_\sigma^{\mathcal{M}}(\neg\varphi) = 0$, so $\Gamma \nvDash_{\mathsf{LP}^\circ} \neg\varphi, \Delta$. If $v_\sigma^{\mathcal{M}}(\varphi) = 0$, $\Gamma \nvDash_{\mathsf{LP}^\circ} \varphi, \Delta$. In each case, one of the premises of the rule is unsound.

For ($\circ$R), assume there is a model $\mathcal{M}$ and an assignment σ on $\mathcal{M}$ such that $v_\sigma^{\mathcal{M}}(\Gamma) \geqslant \frac{1}{2}$, $v_\sigma^{\mathcal{M}}(\circ\varphi) = 0$, and $v_\sigma^{\mathcal{M}}(\Delta) = 0$. Thus, $v_\sigma^{\mathcal{M}}(\varphi) = v_\sigma^{\mathcal{M}}(\neg\varphi) = \frac{1}{2}$. As a consequence, $\Gamma, \varphi, \neg\varphi \nvDash_{\mathsf{LP}^\circ} \Delta$.

The proofs for the remaining cases are analogous. □

Let's now turn to completeness. These results are established via a cut-elimination theorem for LP°. Let $\mathcal{L}$ be any first-order language with $\neg, \vee, \exists$, and $=$ as its only primitive logical symbols, and $\mathcal{L}^\circ$ extend $\mathcal{L}$ with $\circ$, as before.

Definition 39. The *complexity* of a formula of $\mathcal{L}^\circ$ is defined inductively just as in Definition 6, with the addition of the following clause:

$c(\circ\varphi) = c(\varphi) + 2$.

Note that the consistency operator $\circ$ adds more complexity to a formula than the other logical operators in the language. This will become important in the proof of Proposition 44.

Definition 40. The *height* of a derivation is the number of rules that have been applied in it. The height of the application of a rule in a derivation is the number of rules that have been applied before in the derivation increased by 1.

We write $\vdash^n_{\mathsf{LP}^\circ} \Gamma \Rightarrow \Delta$ to indicate there is a derivation in LP° of $\Gamma \Rightarrow \Delta$ of at most hight n. If t is free for v in all members $\varphi_1, \ldots, \varphi_n$ of Γ, $\Gamma[t/v]$ is the set $\{\varphi_1[t/v], \ldots, \varphi_n[t/v]\}$. It is easily shown by a standard argument that if $\vdash^n_{\mathsf{LP}^\circ} \Gamma \Rightarrow \Delta$, then $\vdash^n_{\mathsf{LP}^\circ} \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ for every finite $\Gamma', \Delta' \subseteq \mathcal{L}^\circ$. Also, if t is free for v in Γ, Δ, then $\vdash^n_{\mathsf{LP}^\circ} \Gamma[t/v] \Rightarrow \Delta[t/v]$.[15] If $\mathcal{D}$ is a derivation of $\Gamma \Rightarrow \Delta$, let $\mathcal{D}[t/v]$ be a derivation of $\Gamma[t/v] \Rightarrow \Delta[t/v]$ and $\Gamma'|\mathcal{D}|\Delta'$ a derivation of $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$, both of the same height as $\mathcal{D}$.

Lemma 41. *If there is a derivation $\mathcal{D}$ in LP° of a sequent $\Gamma \Rightarrow \Delta$ ending with a cut of height n, then there is a derivation $\mathcal{D}'$ of the same sequent without cuts or in which all cuts are of height less than n.*

Proof. Assume $\mathcal{D}$ is a derivation of $\Gamma \Rightarrow \Delta$ ending with an application of cut of height n on φ. The proof is by cases.

1. *One of the premisses is an axiom.* If $\Gamma, \varphi \Rightarrow \Delta$ is an axiom, then either there is a literal common to Γ and Δ or $\varphi \in \Delta$. In the former case, $\Gamma \Rightarrow \Delta$ is an axiom as well, so let $\mathcal{D}'$ be $\Gamma \Rightarrow \Delta$. If $\varphi \in \Delta$, then the premiss on the right *is* $\Gamma \Rightarrow \Delta$, so we can obtain $\mathcal{D}'$ by erasing the premiss on the left and the last cut in $\mathcal{D}$. The case where $\Gamma \Rightarrow \varphi, \Delta$ is an axiom is symmetrical.
2. *No premiss is an axiom.* There are three possibilities:
   (a) *φ is not principal in the derivation of* $\Gamma, \varphi \Rightarrow \Delta$. No matter which rule has been applied to obtain this sequent, we can reduce the height of the cuts by 1. We show it for (Ref), ($\neg$R), ($\vee$L), and ($\circ$R). The other cases are similar. Assume $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, t = t, \varphi \Rightarrow \Delta}}{\Gamma, \varphi \Rightarrow \Delta}\,(\text{Ref}) \qquad \dfrac{\mathcal{D}_2}{\Gamma \Rightarrow \varphi, \Delta}}{\Gamma \Rightarrow \Delta}\,(\text{Cut})$$

Then, let $\mathcal{D}'$ be:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, t = t, \varphi \Rightarrow \Delta} \qquad \dfrac{t = t|\mathcal{D}_2}{\Gamma, t = t \Rightarrow \varphi, \Delta}}{\Gamma, t = t \Rightarrow \Delta}\,(\text{Cut})}{\Gamma \Rightarrow \Delta}\,(\text{Ref})$$

Assume $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, \varphi, \psi \Rightarrow \neg\psi, \Delta'}}{\Gamma, \varphi \Rightarrow \neg\psi, \Delta'}\,(\neg\text{R}) \qquad \dfrac{\mathcal{D}_2}{\Gamma \Rightarrow \varphi, \neg\psi, \Delta'}}{\Gamma \Rightarrow \neg\psi, \Delta'}\,(\text{Cut})$$

where $\Delta = \Delta' \cup \{\neg\psi\}$. Then, let $\mathcal{D}'$ be:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, \varphi, \psi \Rightarrow \neg\psi, \Delta'} \qquad \dfrac{\psi|\mathcal{D}_2}{\Gamma, \psi \Rightarrow \varphi, \neg\psi, \Delta'}}{\Gamma, \psi \Rightarrow \neg\psi, \Delta'}\,(\text{Cut})}{\Gamma \Rightarrow \neg\psi, \Delta'}\,(\neg\text{R})$$

Assume now $\mathcal{D}$ is of the form:

[15] See, for instance, Negri & von Plato [22].

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma',\varphi,\psi\Rightarrow\Delta}\qquad\dfrac{\mathcal{D}_2}{\Gamma',\varphi,\chi\Rightarrow\Delta}}{\Gamma',\varphi,\psi\vee\chi\Rightarrow\Delta}\,(\vee\text{L})\qquad\dfrac{\mathcal{D}_3}{\Gamma',\psi\vee\chi\Rightarrow\varphi,\Delta}}{\Gamma',\psi\vee\chi\Rightarrow\Delta}\,(\text{Cut})$$

where $\Gamma=\Gamma'\cup\{\psi\vee\chi\}$. Let $\mathcal{D}'$ be the result of replacing in $\mathcal{D}$ the branch ending with the left premiss of (Cut) with:

$$\dfrac{\dfrac{\psi\vee\chi|\mathcal{D}_1}{\Gamma',\varphi,\psi,\psi\vee\chi\Rightarrow\Delta}\qquad\dfrac{\psi|\mathcal{D}_3}{\Gamma',\psi,\psi\vee\chi\Rightarrow\varphi,\Delta}}{\Gamma',\psi,\psi\vee\chi\Rightarrow\Delta}\,(\text{Cut})$$

and the branch ending with the right premiss with:

$$\dfrac{\dfrac{\psi\vee\chi|\mathcal{D}_2}{\Gamma',\varphi,\chi,\psi\vee\chi\Rightarrow\Delta}\qquad\dfrac{\chi|\mathcal{D}_3}{\Gamma',\chi,\psi\vee\chi\Rightarrow\varphi,\Delta}}{\Gamma',\chi,\psi\vee\chi\Rightarrow\Delta}\,(\text{Cut})$$

and then applying $(\vee\text{L})$ to obtain $\Gamma',\psi\vee\chi\Rightarrow\Delta$.

Assume now $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma,\psi,\neg\psi,\varphi\Rightarrow\Delta'}}{\Gamma,\varphi\Rightarrow\circ\psi,\Delta'}\,(\circ\text{R})\qquad\dfrac{\mathcal{D}_2}{\Gamma\Rightarrow\varphi,\circ\psi,\Delta'}}{\Gamma\Rightarrow\circ\psi,\Delta'}\,(\text{Cut})$$

where $\Delta=\Delta'\cup\{\circ\psi\}$. Then, let $\mathcal{D}'$ be:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1|\circ\psi}{\Gamma,\psi,\neg\psi,\varphi\Rightarrow\circ\psi,\Delta'}\qquad\dfrac{\psi,\neg\psi|\mathcal{D}_2}{\Gamma,\psi,\neg\psi\Rightarrow\varphi,\circ\psi,\Delta'}}{\Gamma,\psi,\neg\psi\Rightarrow\circ\psi,\Delta'}\,(\text{Cut})}{\Gamma\Rightarrow\circ\psi,\Delta'}\,(\circ\text{R})$$

(b) *φ is not principal in the derivation of* $\Gamma\Rightarrow\varphi,\Delta$. Cases are analogous to those in 2.(a).

(c) *φ is principal in the derivation of both* $\Gamma,\varphi\Rightarrow\Delta$ *and* $\Gamma\Rightarrow\varphi,\Delta$. We prove this sub-case by induction on the complexity of φ. Since φ is principal, it cannot be atomic or the negation of an atomic formula (in the latter case, there is no rule that introduces the negation of an atomic formula to the left).

Assume the height of the last cut of the derivation of a sequent can be reduced for formulae of complexity smaller than n, and let $c(\varphi)=n$. If $\varphi:=\psi\vee\chi$, $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma,\psi\Rightarrow\Delta}\qquad\dfrac{\mathcal{D}_2}{\Gamma,\chi\Rightarrow\Delta}}{\Gamma,\psi\vee\chi\Rightarrow\Delta}\,(\vee\text{L})\qquad\dfrac{\dfrac{\mathcal{D}_3}{\Gamma\Rightarrow\psi,\chi,\Delta}}{\Gamma\Rightarrow\psi\vee\chi,\Delta}\,(\vee\text{R})}{\Gamma\Rightarrow\Delta}\,(\text{Cut})$$

Replace it with:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1|\chi}{\Gamma,\psi\Rightarrow\chi,\Delta}\qquad\dfrac{\mathcal{D}_3}{\Gamma\Rightarrow\psi,\chi,\Delta}}{\Gamma\Rightarrow\chi,\Delta}\,(\text{Cut})\qquad\dfrac{\mathcal{D}_2}{\Gamma,\chi\Rightarrow\Delta}}{\Gamma\Rightarrow\Delta}\,(\text{Cut})$$

If $\varphi:=\exists v\psi$, $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma,\psi[u/v]\Rightarrow\Delta}}{\Gamma,\exists v\psi\Rightarrow\Delta}\,(\exists\text{L})\qquad\dfrac{\dfrac{\mathcal{D}_2}{\Gamma\Rightarrow\psi[t/v],\Delta}}{\Gamma\Rightarrow\exists v\psi,\Delta}\,(\exists\text{R})}{\Gamma\Rightarrow\Delta}\,(\text{Cut})$$

Replace it with:

$$\dfrac{\dfrac{\mathcal{D}_1[t/u]}{\Gamma, \psi[t/v] \Rightarrow \Delta} \qquad \dfrac{\mathcal{D}_2}{\Gamma \Rightarrow \psi[t/v], \Delta}}{\Gamma \Rightarrow \Delta}\ (\mathrm{Cut})$$

$\mathcal{D}_1[t/u]$ is a derivation of $\Gamma, \psi[t/v] \Rightarrow \Delta$, since u is not free in Γ, Δ or ψ.
If $\varphi := \circ\psi$, $\mathcal{D}$ is of the form

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma \Rightarrow \psi, \Delta} \qquad \dfrac{\mathcal{D}_2}{\Gamma \Rightarrow \neg\psi, \Delta}}{\Gamma, \circ\psi \Rightarrow \Delta}\ (\circ\mathrm{L}) \qquad \dfrac{\dfrac{\mathcal{D}_3}{\Gamma, \psi, \neg\psi \Rightarrow \Delta}}{\Gamma \Rightarrow \circ\psi, \Delta}\ (\circ\mathrm{R})}{\Gamma \Rightarrow \Delta}\ (\mathrm{Cut})$$

Replace it with:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1|\neg\psi}{\Gamma \Rightarrow \psi, \neg\psi, \Delta} \qquad \dfrac{\mathcal{D}_3}{\Gamma, \psi, \neg\psi \Rightarrow \Delta}}{\Gamma, \neg\psi \Rightarrow \Delta}\ (\mathrm{Cut}) \qquad \dfrac{\mathcal{D}_2}{\Gamma \Rightarrow \neg\psi, \Delta}}{\Gamma \Rightarrow \Delta}\ (\mathrm{Cut})$$

Now let φ be a negation. If $\varphi := \neg\neg\psi$, $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, \psi \Rightarrow \Delta}}{\Gamma, \neg\neg\psi \Rightarrow \Delta}\ (\neg\neg\mathrm{L}) \qquad \dfrac{\dfrac{\mathcal{D}_2}{\Gamma \Rightarrow \psi, \Delta}}{\Gamma \Rightarrow \neg\neg\psi, \Delta}\ (\neg\neg\mathrm{R})}{\Gamma \Rightarrow \Delta}\ (\mathrm{Cut})$$

Replace it with:

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, \psi \Rightarrow \Delta} \qquad \dfrac{\mathcal{D}_2}{\Gamma \Rightarrow \psi, \Delta}}{\Gamma \Rightarrow \Delta}\ (\mathrm{Cut})$$

If $\varphi := \neg\circ\psi$, $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, \psi, \neg\psi \Rightarrow \Delta}}{\Gamma, \neg\circ\psi \Rightarrow \Delta}\ (\neg\circ\mathrm{L}) \qquad \dfrac{\dfrac{\mathcal{D}_2}{\Gamma \Rightarrow \psi, \Delta} \qquad \dfrac{\mathcal{D}_3}{\Gamma \Rightarrow \neg\psi, \Delta}}{\Gamma \Rightarrow \neg\circ\psi\Delta}\ (\neg\circ\mathrm{R})}{\Gamma \Rightarrow \Delta}\ (\mathrm{Cut})$$

Replace it with:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, \psi, \neg\psi \Rightarrow \Delta} \qquad \dfrac{\neg\psi|\mathcal{D}_2}{\Gamma, \neg\psi \Rightarrow \psi, \Delta}}{\Gamma, \neg\psi \Rightarrow \Delta}\ (\mathrm{Cut}) \qquad \dfrac{\mathcal{D}_3}{\Gamma \Rightarrow \neg\psi, \Delta}}{\Gamma \Rightarrow \Delta}\ (\mathrm{Cut})$$

Other cases are similar. We omit their proof.

□

Proposition 42 (Cut elimination for LP°)**.** *If a sequent* $\Gamma \Rightarrow \Delta$ *is derivable in* LP° *then there is a derivation of this sequent without cuts.*

Proof. By induction on the height of the last application of (Cut) and Lemma 41. □

We can use the eliminability of (Cut) to provide a completeness proof for LP°. Following Schütte [29], we start with a sequent and construct a *reduction tree* for this sequent, applying the rules of the calculus for LP° in all possible ways. Since proofs need not contain cuts according to Proposition 42, this rule will not be considered in the construction. If all branches of the tree reach the form of an axiom, the tree is a proof of the given sequent. Otherwise, the construction does not terminate. Turning to König's Lemma, we define a refuting model for the sequent.

Definition 43. The *reduction tree* of a sequent $\Gamma \Rightarrow \Delta$ in LP° is constructed in steps as follows, starting with $\Gamma \Rightarrow \Delta$ as the root:

Step 1. Write on top of each topmost sequent of the form $\Gamma' \Rightarrow \Delta'$ the sequent

$$\Gamma', t_1 = t_1, \ldots, t_n = t_n \Rightarrow \Delta'$$

where t_i, $1 \leqslant i \leqslant n$, occurs in Γ' or in Δ'. This restriction is necessary, for if the language contains infinitely many terms and we added all instances of $t = t$ to the antecedent, we would get an infinite set, which cannot be part of a sequent.

Step 2. Write on top of each topmost sequent of the form

$$\Gamma', s = t, \varphi[s/v] \Rightarrow \Delta'$$

where φ is a literal, the sequent

$$\Gamma', s = t, \varphi[s/v], \varphi[t/v] \Rightarrow \Delta'.$$

Step 3. For each topmost sequent of the form

$$\Gamma' \Rightarrow \neg\varphi_1, \ldots, \neg\varphi_n, \Delta'$$

where $\varphi_1, \ldots, \varphi_n$ are atomic formulae, write on top

$$\Gamma', \varphi_1, \ldots, \varphi_n \Rightarrow \neg\varphi_1, \ldots, \neg\varphi_n, \Delta'.$$

Step 4. For each topmost sequent of the form

$$\Gamma', \neg\neg\varphi_1, \ldots, \neg\neg\varphi_n \Rightarrow \Delta'$$

write on top the sequent

$$\Gamma', \varphi_1, \ldots, \varphi_n \Rightarrow \Delta'.$$

Step 5. For each topmost sequent of the form

$$\Gamma' \Rightarrow \neg\neg\varphi_1, \ldots, \neg\neg\varphi_n, \Delta'$$

write on top the sequent

$$\Gamma' \Rightarrow \varphi_1, \ldots, \varphi_n, \Delta'.$$

Step 6. Write on top of each topmost sequent of the form

$$\Gamma' \Rightarrow \varphi_1 \vee \psi_1, \ldots, \varphi_n \vee \psi_n, \Delta'$$

the sequent

$$\Gamma' \Rightarrow \varphi_1, \psi_1, \ldots, \varphi_n, \psi_n, \Delta'.$$

Step 7. For each topmost sequent of the form

$$\Gamma', \varphi_1 \vee \psi_1, \ldots, \varphi_n \vee \psi_n \Rightarrow \Delta'$$

write the 2^n sequents

$$\Gamma', \chi_1, \ldots, \chi_n \Rightarrow \Delta'$$

on top of it, where χ_i is either φ_i or ψ_i.

Step 8. Write on top of each topmost sequent of the form

$$\Gamma', \neg(\varphi_1 \vee \psi_1), \ldots, \neg(\varphi_n \vee \psi_n) \Rightarrow \Delta'$$

the sequent

$$\Gamma', \neg\varphi_1, \neg\psi_1, \ldots, \neg\varphi_n, \neg\psi_n \Rightarrow \Delta'.$$

Step 9. For each topmost sequent of the form

$$\Gamma' \Rightarrow \neg(\varphi_1 \vee \psi_1), \ldots, \neg(\varphi_n \vee \psi_n), \Delta'$$

write the 2^n sequents

$$\Gamma' \Rightarrow \neg\chi_1, \ldots, \neg\chi_n, \Delta'$$

on top of it, where χ_i is either φ_i or ψ_i.

Step 10. Consider each topmost sequent of the form

$$\Gamma', \exists v_1\varphi_1, \ldots, \exists v_n\varphi_n \Rightarrow \Delta'$$

and let $u_1, \ldots, u_n$ be variables not used yet in the reduction tree. Then write on top the sequent

$$\Gamma', \varphi_1[u_1/v_1], \ldots, \varphi_n[u_n/v_n] \Rightarrow \Delta'.$$

Step 11. Consider each topmost sequent of the form

$$\Gamma' \Rightarrow \exists v_1\varphi_1, \ldots, \exists v_n\varphi_n, \Delta'$$

and let t_i, $1 \leqslant i \leqslant n$, be the first term not used yet for a reduction of $\exists v_i\varphi_i$. Then write on top the sequent

$$\Gamma' \Rightarrow \varphi_1[t_1/v_1], \ldots, \varphi_n[t_n/v_n], \exists v_1\varphi_1, \ldots, \exists v_n\varphi_n, \Delta'.$$

We have to keep the existential formulae in the succedent for a possible further application of the reduction step, for we must consider all possible terms that could help us derive these formulae. The same applies to negated existential claims in the following reduction step.

Step 12. Consider each topmost sequent of the form

$$\Gamma', \neg\exists v_1\varphi_1, \ldots, \neg\exists v_n\varphi_n \Rightarrow \Delta'$$

and let t_i, $1 \leqslant i \leqslant n$, be the first term not used yet for a reduction of $\neg\exists v_i\varphi_i$. Then write on top the sequent

$$\Gamma', \neg\varphi_1[t_1/v_1], \ldots, \neg\varphi_n[t_n/v_n], \neg\exists v_1\varphi_1, \ldots, \neg\exists v_n\varphi_n \Rightarrow \Delta'.$$

Step 13. Consider each topmost sequent of the form

$$\Gamma' \Rightarrow \neg\exists v_1\varphi_1, \ldots, \neg\exists v_n\varphi_n, \Delta'$$

and let $u_1, \ldots, u_n$ be variables not used yet in the reduction tree. Then write on top the sequent

$$\Gamma' \Rightarrow \neg\varphi_1[u_1/v_1], \ldots, \neg\varphi_n[u_n/v_n], \Delta'.$$

Step 14. For each topmost sequent of the form

$$\Gamma', \circ\varphi_1, \ldots, \circ\varphi_n \Rightarrow \Delta'$$

write the 2^n sequents

$$\Gamma' \Rightarrow \psi_1, \ldots, \psi_n, \Delta'$$

on top of it, where ψ_i is either φ_i or $\neg\varphi_i$.

Step 15. Write on top of each topmost sequent of the form

$$\Gamma' \Rightarrow \circ\varphi_1, \ldots, \circ\varphi_n, \Delta'$$

the sequent

$$\Gamma', \varphi_1, \neg\varphi_1, \ldots, \varphi_n, \neg\varphi_n \Rightarrow \Delta'.$$

Step 16. Write on top of each topmost sequent of the form

$$\Gamma', \neg\circ\varphi_1, \ldots, \neg\circ\varphi_n \Rightarrow \Delta'$$

the sequent

$$\Gamma', \varphi_1, \neg\varphi_1, \ldots, \varphi_n, \neg\varphi_n \Rightarrow \Delta'.$$

Step 17. For each topmost sequent of the form

$$\Gamma' \Rightarrow \neg\circ\varphi_1, \ldots, \neg\circ\varphi_n, \Delta'$$

write the 2^n sequents

$$\Gamma' \Rightarrow \psi_1, \ldots, \psi_n, \Delta'$$

on top of it, where ψ_i is either φ_i or $\neg\varphi_i$.

Steps 1-17 are repeated in order until every topmost sequent is an axiom, if that happens. Then, the process terminates. When no step is applicable to a topmost sequent that is not an axiom, the same sequent is repeated on top.

Each step in the construction of the reduction tree for a sequent corresponds to a logical rule of LP° (except for repetition). As a consequence, if the process terminates, we reach a proof of the starting sequent.

Proposition 44 (Completeness of LP°)**.** *Let $\Gamma, \Delta \subseteq \mathcal{L}^\circ$ be finite sets. Either $\vdash_{\mathsf{LP}^\circ} \Gamma \Rightarrow \Delta$ or $\Gamma \nvDash_{\mathsf{LP}^\circ} \Delta$.*

Proof. Assume $\Gamma \Rightarrow \Delta$ is not derivable, and let $\mathcal{D}$ be its reduction tree. Thus, $\mathcal{D}$ is not a proof of this sequent. If $\mathcal{D}$ were finite, then its topmost sequents would be axioms, and read top-down every inference would correspond to a rule in LP°. Thus, $\mathcal{D}$ would be a proof of $\Gamma \Rightarrow \Delta$, contrary to our assumption. Therefore, $\mathcal{D}$ must be infinite. By König's lemma, since $\mathcal{D}$ is a finitely branching tree, it must have an infinite branch. Let this branch consist of the sequents $\Gamma_0 \Rightarrow \Delta_0, \ldots, \Gamma_n \Rightarrow \Delta_n, \ldots$, where $\Gamma_0 \Rightarrow \Delta_0$ is $\Gamma \Rightarrow \Delta$. Let $\mathbf{\Gamma} = \cup_{i\in\omega}\Gamma_i$ and $\mathbf{\Delta} = \cup_{i\in\omega}\Delta_i$. I provide an LP°-model $\mathcal{M}$ and an assignment σ on $\mathcal{M}$ such that $v_\sigma^{\mathcal{M}}(\mathbf{\Gamma}) \geqslant \frac{1}{2}$ and $v_\sigma^{\mathcal{M}}(\mathbf{\Delta}) = 0$, showing that $\Gamma \nvDash_{\mathsf{LP}^\circ} \Delta$.

Let $\sim$ be a relation between terms of $\mathcal{L}^\circ$ such that $s \sim t$ iff $s = t \in \mathbf{\Gamma}$ or s and t are the same term. Note that $\sim$ is an equivalence relation. Reflexivity is trivially satisfied. For symmetry, let $s \sim t$. If s and t are the same term, the result follows trivially. If $s = t \in \mathbf{\Gamma}$, by step 1 of Definition 43, we have a sequent on the branch of the form

$$\Gamma', s = t, s = s \Rightarrow \Delta'.$$

Taking φ to be $v = s$, so that $s = s$ is $\varphi[s/v]$ and $\varphi[t/v]$ is $t = s$, by step 2 of Definition 43, the set $\{s = t, s = s, t = s\}$ must be included in the antecedent of a sequent higher up on the branch. As a consequence, $t = s \in \mathbf{\Gamma}$, so $t \sim s$. Finally, for transitivity assume that $s \sim t$ and $t \sim r$. If s and t or t and r are the same terms, the result is trivial. Let $s = t$ and $t = r$ be in $\mathbf{\Gamma}$. Then, there is a sequent of the form

$$\Gamma', t = r, s = t \Rightarrow \Delta'.$$

on the branch. Taking φ to be $s = v$, so that $s = t$ is $\varphi[t/v]$ and $\varphi[r/v]$ is $s = r$, by step 2 of Definition 43, the set $\{t = r, s = t, s = r\}$ must be included in the antecedent of a sequent on the branch. As a consequence, $s = r \in \mathbf{\Gamma}$, so $s \sim r$.

Let $[t]$ be the equivalence class to which the term t belongs, and let members of $|\mathcal{M}|$ consist of these equivalence classes. If R is an n-ary relation symbol, $R^{\mathcal{M}}$ is such that

- $R^+ = \{\langle [t_1], \dots, [t_n] \rangle \mid Rt_1 \dots t_n \notin \mathbf{\Delta}\}$;
- $R^- = \{\langle [t_1], \dots, [t_n] \rangle \mid \neg Rt_1 \dots t_n \in \mathbf{\Gamma} \text{ or } Rt_1 \dots t_n \in \mathbf{\Delta}\}$,

unless R is $=$, in whose case R^+, i.e. $=^+$, is just the set of ordered pairs of the form $\langle [t], [t] \rangle$. Note that R^+ and R^- are exhaustive, that is, $R^+ \cup R^- = |\mathcal{M}|$. If c is an individual constant and f an n-ary function symbol, let

- $c^{\mathcal{M}} = [c]$;
- $f^{\mathcal{M}}([t_1], \dots, [t_n]) = [f(t_1, \dots, t_n)]$.

Since $[t]$ is an equivalence class, existence and uniqueness for $f^{\mathcal{M}}$ are guaranteed. Finally, let σ be the assignment in $\mathcal{M}$ in which $\sigma(v) = [v]$, for every variable v.

I show that $v_\sigma^{\mathcal{M}}(\varphi) \geqslant \frac{1}{2}$ for every $\varphi \in \mathbf{\Gamma}$ and $v_\sigma^{\mathcal{M}}(\varphi) = 0$ for every $\varphi \in \mathbf{\Delta}$ by induction on the complexity of φ. Let $\varphi := (s = t)$. If $s = t \in \mathbf{\Gamma}$, then $[s] = [t]$. Thus, $\langle [s], [t] \rangle \in =^+$, which means that $v_\sigma^{\mathcal{M}}(s = t) \geqslant \frac{1}{2}$. If $s = t \in \mathbf{\Delta}$, it cannot be that $s = t \in \mathbf{\Gamma}$, because the branch would have reached an axiom and be finite. Thus, it cannot be that s and t are the same term either, because, by step 1 in the construction of the reduction tree, $s = t$ would be in $\mathbf{\Gamma}$. So, $[s] \neq [t]$, which means that $\langle [s], [t] \rangle \notin =^+$. Therefore, $v_\sigma^{\mathcal{M}}(s = t) = 0$.

Let $\varphi := Rt_1 \dots t_n$, with R distinct from $=$. If $Rt_1 \dots t_n \in \mathbf{\Gamma}$, $Rt_1 \dots t_n \notin \mathbf{\Delta}$, or the branch would have reached an axiom. Thus, $\langle [t_1], \dots, [t_n] \rangle \in R^+$, so $v_\sigma^{\mathcal{M}}(Rt_1 \dots t_n) \geqslant \frac{1}{2}$. If $Rt_1 \dots t_n \in \mathbf{\Delta}$, $\langle [t_1], \dots, [t_n] \rangle \notin R^+$, so $v_\sigma^{\mathcal{M}}(Rt_1 \dots t_n) = 0$.

Let $\varphi := \neg Rt_1 \dots t_n$. If $\neg Rt_1 \dots t_n \in \mathbf{\Gamma}$, by construction $\langle [t_1], \dots, [t_n] \rangle \in R^-$. Then, $v_\sigma^{\mathcal{M}}(Rt_1 \dots t_n) \leqslant \frac{1}{2}$, which implies that $v_\sigma^{\mathcal{M}}(\neg Rt_1 \dots t_n) \geqslant \frac{1}{2}$. If $\neg Rt_1 \dots t_n \in \mathbf{\Delta}$, then $\neg Rt_1 \dots t_n \notin \mathbf{\Gamma}$. But we also know that $Rt_1 \dots t_n \in \mathbf{\Gamma}$, by step 3 of Definition 43. Thus, $Rt_1 \dots t_n \notin \mathbf{\Delta}$. Therefore, we have that $\langle [t_1], \dots, [t_n] \rangle \notin R^-$, i.e. $v_\sigma^{\mathcal{M}}(Rt_1 \dots t_n) = 1$, which entails that $v_\sigma^{\mathcal{M}}(\neg Rt_1 \dots t_n) = 0$.

Cases in which φ is of the form $\neg\neg\psi$, $\psi \vee \chi$, $\neg(\varphi \vee \chi)$, $\exists v \psi$, $\neg\exists v \psi$, $\circ\psi$, and $\neg\circ\psi$ follow directly from the inductive hypothesis. I consider the cases in which φ is of the form $\neg(\psi \vee \chi)$, $\exists v \psi$, and $\circ\psi$ by way of example.

Let $\varphi := \neg(\psi \vee \chi)$. If $\neg(\psi \vee \chi) \in \mathbf{\Gamma}$, by step 8 of Definition 43, both $\neg\psi \in \mathbf{\Gamma}$ and $\neg\chi \in \mathbf{\Gamma}$. By inductive hypothesis, both $v_\sigma^{\mathcal{M}}(\neg\psi) \geqslant \frac{1}{2}$ and $v_\sigma^{\mathcal{M}}(\neg\chi) \geqslant \frac{1}{2}$. Thus, $v_\sigma^{\mathcal{M}}(\neg(\psi \vee \chi)) \geqslant \frac{1}{2}$. If $\neg(\psi \vee \chi) \in \mathbf{\Delta}$, by step 9 of Definition 43, either $\neg\psi$ or $\neg\chi$ is in $\mathbf{\Delta}$. By inductive hypothesis, either $v_\sigma^{\mathcal{M}}(\neg\psi) = 0$ or $v_\sigma^{\mathcal{M}}(\neg\chi) = 0$. Thus, either $v_\sigma^{\mathcal{M}}(\psi) = 1$ or $v_\sigma^{\mathcal{M}}(\chi) = 1$, which means that $v_\sigma^{\mathcal{M}}(\psi \vee \chi) = 1$. Therefore, $v_\sigma^{\mathcal{M}}(\neg(\psi \vee \chi)) = 0$.

Let $\varphi := \exists v \psi$. If $\exists v \psi \in \mathbf{\Gamma}$, by step 10 of Definition 43, $\psi[u/v] \in \mathbf{\Gamma}$ for some variable u. By inductive hypothesis, $v_\sigma^{\mathcal{M}}(\psi[u/v]) \geqslant \frac{1}{2}$. Thus, $v_\sigma^{\mathcal{M}}(\exists v \psi) \geqslant \frac{1}{2}$ as well. If $\exists v \psi \in \mathbf{\Delta}$, by step 11 of Definition 43, we have that $\psi[t/v] \in \mathbf{\Delta}$ for every term t. By inductive hypothesis, $v_\sigma^{\mathcal{M}}(\psi[t/v]) = 0$ for every term t. Since all members of $|\mathcal{M}|$ are denoted by some term, $v_\sigma^{\mathcal{M}}(\exists v \psi) = 0$.

Let $\varphi := \circ\psi$. If $\circ\psi \in \mathbf{\Gamma}$, by step 14 of Definition 43, either $\psi \in \mathbf{\Delta}$ or $\neg\psi \in \mathbf{\Delta}$. By inductive hypothesis, either $v_\sigma^{\mathcal{M}}(\psi) = 0$ or $v_\sigma^{\mathcal{M}}(\neg\psi) = 0$, that is, either $v_\sigma^{\mathcal{M}}(\psi) = 0$ or $v_\sigma^{\mathcal{M}}(\psi) = 1$. Thus, $v_\sigma^{\mathcal{M}}(\circ\psi) = 1$. If $\circ\psi \in \mathbf{\Delta}$, by step 15 of Definition 43, we have that $\psi, \neg\psi \in \mathbf{\Gamma}$. By inductive hypothesis, $v_\sigma^{\mathcal{M}}(\psi) \geqslant \frac{1}{2}$ and $v_\sigma^{\mathcal{M}}(\neg\psi) \geqslant \frac{1}{2}$, which implies that $v_\sigma^{\mathcal{M}}(\varphi) = \frac{1}{2}$. Thus, $v_\sigma^{\mathcal{M}}(\circ\psi) = 0$. □

The completeness of LP is a corollary of Proposition 44 and the following conservativity result.

Lemma 45. *Let $\Gamma, \Delta \subseteq \mathcal{L}$ be finite sets. If $\vdash_{\mathsf{LP}^\circ} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathsf{LP}} \Gamma \Rightarrow \Delta$.*

Proof. Assume $\vdash_{\mathsf{LP}^\circ} \Gamma \Rightarrow \Delta$. By the cut-elimination result for LP° (Proposition 42), there is a derivation of $\Gamma \Rightarrow \Delta$ in LP° without cuts. Note that in this derivation no rules for $\circ$ have been applied. Otherwise, $\circ$ would occur either in Γ or in Δ, for no rule of LP° besides (Cut) allows for the elimination of a formula containing this operator. Therefore, only LP-rules were employed in the derivation of $\Gamma \Rightarrow \Delta$. □

Proposition 46 (Completeness of LP)**.** *Let $\Gamma, \Delta \subseteq \mathcal{L}$ be finite sets. Either $\vdash_{\mathsf{LP}} \Gamma \Rightarrow \Delta$ or $\Gamma \nvDash_{\mathsf{LP}} \Delta$.*

Proof. Assume $\Gamma \Rightarrow \Delta$ is not derivable in LP. By the previous lemma, it is also not derivable in LP°. By Proposition 44, $\Gamma \Rightarrow \Delta$ is not sound with respect to the class of LP°-models. But LP°-models are just LP-models (recall that $\circ$ doesn't occur in $\Gamma \Rightarrow \Delta$). Therefore, $\Gamma \nvDash_{\mathsf{LP}} \Delta$. □

Finally, I provide a strong completeness result for LP°. Let $\varphi \mathrel{\circ\!\!\rightarrow} \psi$ abbreviate $\neg(\varphi \vee \neg\circ\varphi) \vee \psi \vee \neg\circ\psi$ in $\mathcal{L}^\circ$. Thus, in every LP°-model $\mathcal{M}$ relative to any assignment σ we have that

$$v_\sigma^{\mathcal{M}}(\varphi \mathrel{\circ\!\!\rightarrow} \psi) = \begin{cases} 1, & \text{if } v_\sigma^{\mathcal{M}}(\varphi) = 0 \text{ or } v_\sigma^{\mathcal{M}}(\psi) \geqslant \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

In other words, $\varphi \mathrel{\circ\!\!\rightarrow} \psi$ is true in a model $\mathcal{M}$ relative to an assignment σ iff $\varphi \Rightarrow \psi$ is sound with respect to $\mathcal{M}$ and σ. Therefore, we can prove the following lemma for the strong completeness of LP°.

Lemma 47. *If the sequent $\Gamma \Rightarrow \Delta$ is sound with respect to every LP°-model and assignment the sequents $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$ are all sound with respect to, then $\bigwedge \Gamma_1 \mathrel{\circ\!\!\rightarrow} \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \mathrel{\circ\!\!\rightarrow} \bigvee \Delta_n, \Gamma \vDash_{\mathsf{LP}^\circ} \Delta$.*

Proof. Assume for contradiction that there is an LP°-model $\mathcal{M}$ and an assignment σ on $\mathcal{M}$ such that $v_\sigma^{\mathcal{M}}(\bigwedge \Gamma_i \mathrel{\circ\!\!\rightarrow} \bigvee \Delta_i) \geqslant \frac{1}{2}$, $1 \leqslant i \leqslant n$, $v_\sigma^{\mathcal{M}}(\Gamma) \geqslant \frac{1}{2}$, and $v_\sigma^{\mathcal{M}}(\Delta) = 0$. Thus, for $1 \leqslant i \leqslant n$, $v_\sigma^{\mathcal{M}}(\bigwedge \Gamma_i \mathrel{\circ\!\!\rightarrow} \bigvee \Delta_i) = 1$, which means that the sequents $\Gamma_1 \Rightarrow \Delta_1, \ldots, \Gamma_n \Rightarrow \Delta_n$ are all sound with respect to $\mathcal{M}$ and σ. Therefore, $\Gamma \Rightarrow \Delta$ is so too. But then we should have that $v_\sigma^{\mathcal{M}}(\Delta) \geqslant \frac{1}{2}$, which contradicts our initial assumption. □

We need one last lemma to establish our completeness result.

Lemma 48. *For every finite $\Gamma, \Delta \subseteq \mathcal{L}^\circ$, the sequent $\Rightarrow \bigwedge \Gamma \mathrel{\circ\!\!\rightarrow} \bigvee \Delta$ is derivable in LP° from $\Gamma \Rightarrow \Delta$.*

Proof. Let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$. Recall that $\bigwedge \Gamma$ is (a permutation of) $\neg(\neg\gamma_1 \vee \cdots \vee \neg\gamma_m)$. Thus, the following is a derivation of $\bigwedge \Gamma \mathrel{\circ\!\!\rightarrow} \bigvee \Delta$ from $\Gamma \Rightarrow \Delta$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\gamma_1, \ldots, \gamma_m \Rightarrow \Delta}{\neg\neg\gamma_1, \ldots, \neg\neg\gamma_m \Rightarrow \Delta}\,(\neg\neg\mathrm{L})}{\bigwedge \Gamma \Rightarrow \Delta}\,(\neg\vee\mathrm{L})}{\bigwedge \Gamma \Rightarrow \bigvee \Delta}\,(\vee\mathrm{R})}{\Rightarrow \neg \bigwedge \Gamma, \bigvee \Delta}\,(\neg\mathrm{R}) \qquad \dfrac{\dfrac{\dfrac{\bigwedge \Gamma \Rightarrow \bigvee \Delta}{\bigwedge \Gamma, \neg \bigwedge \Gamma \Rightarrow \bigvee \Delta}\,(\mathrm{W})}{\Rightarrow \circ \bigwedge \Gamma, \bigvee \Delta}\,(\circ\mathrm{R})}{\Rightarrow \neg\neg\circ \bigwedge \Gamma, \bigvee \Delta}\,(\neg\neg\mathrm{R})}{\dfrac{\Rightarrow \neg(\bigwedge \Gamma \vee \neg\circ \bigwedge \Gamma), \bigvee \Delta}{\dfrac{\Rightarrow \neg(\bigwedge \Gamma \vee \neg\circ \bigwedge \Gamma), \bigvee \Delta, \neg\circ \bigvee \Delta}{\Rightarrow \neg(\bigwedge \Gamma \vee \neg\circ \bigwedge \Gamma) \vee \bigvee \Delta \vee \neg\circ \bigvee \Delta}\,(\vee\mathrm{R})}\,(\mathrm{W})}\,(\neg\vee\mathrm{R})$$

□

Proposition 49 (Strong completeness of LP°)**.** *Every sequent $\Gamma \Rightarrow \Delta$ is either derivable in LP° from the sequents $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n$ or not sound with respect to the class of LP°-models and assignments $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n$ are all sound with respect to.*

Proof. Assume $\Gamma \Rightarrow \Delta$ is sound with respect to the class of LP°-models and assignments $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n$ are all sound with respect to. By Lemma 47, $\bigwedge \Gamma_1 \circ\!\!\rightarrow \bigvee \Delta_1, \dots, \bigwedge \Gamma_n \circ\!\!\rightarrow \bigvee \Delta_n, \Gamma \Rightarrow \Delta$ is sound in LP°. By the completeness of LP° (Proposition 44), this sequent is derivable in LP°. By Lemma 48, for each $1 \leqslant i \leqslant n$, $\Rightarrow \bigwedge \Gamma_i \circ\!\!\rightarrow \bigvee \Delta_i$ is derivable from $\Gamma_i \Rightarrow \Delta_i$. Thus, by n successive applications of (Cut), we can derive $\Gamma \Rightarrow \Delta$ from $\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n$. □

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