A cancellation theorem for generalized Swan modules

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Abstract The module cancellation problem asks whether, given modules X, X' and Y over a ring Λ, the existence of an isomorphism $X \oplus Y \cong X' \oplus Y$ implies that $X \cong X'$. When Λ is the integral group ring of a metacyclic group $G(p,q)$, results of Klingler show that the answer to this question is generally negative. By contrast, in this case we show that cancellation holds when $Y = \Lambda$ and X is a generalized Swan module.

Introduction

Let Λ be the integral group ring $\Lambda = \mathbb{Z}[G]$ of a finite group G. For Λ-modules X, X', Y we consider the following cancellation question:

$$(*) \qquad \text{If } X \oplus Y \cong X' \oplus Y \text{ is it true that } X \cong X'?$$

In this paper we focus on this question when G is a metacyclic group $G(p,q)$ defined as the semidirect product

$$G(p,q) = C_p \rtimes C_q$$

where p is an odd prime, q is a positive integral divisor of $p-1$ and C_q acts via the canonical imbedding $C_q \hookrightarrow \mathrm{Aut}(C_p)$. We first analyze the group ring Λ; the projection $G(p,q) \twoheadrightarrow C_q$ induces a surjective ring homomorphism $\eta : \Lambda \twoheadrightarrow \mathbb{Z}[C_q]$. The two-sided ideal $\mathrm{Ker}(\eta)$ has the following non-obvious description; take A to be the fixed ring $A = \mathbb{Z}[\zeta_p]^{C_q}$ under the Galois action of C_q on the ring of cyclotomic integers $\mathbb{Z}[\zeta_p]$; A is a Dedekind domain in which p ramifies completely. We take $\pi \in A$ to be the unique prime over p. Then $\mathrm{Ker}(\eta)$ can be identified with $\mathcal{T}_q$, the subring of *quasi-triangular* matrices in the ring $M_q(A)$ of $q \times q$ matrices over A; thus,

$$\mathcal{T}_q = \bigl\{X = (x_{rs})_{1 \le r,s \le n} \in M_q(A) \bigm| x_{rs} \in (\pi) \text{ if } r > s\bigr\}.$$

A *generalized Swan module* X is one which occurs in an extension of the form

$$(\mathcal{X}) \qquad 0 \to \mathcal{T}_q \to X \to \mathbb{Z}[C_q] \to 0.$$

In particular, given the above description of $\mathrm{Ker}(\eta)$, we see that Λ itself is a generalized Swan module. We shall prove the following:

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THEOREM A
Let Z, Z' be Λ-modules such that $Z \oplus \Lambda \cong Z' \oplus \Lambda$; if Z is a generalized Swan module, then $Z \cong Z'$.

We note that $\mathcal{T}_q$ decomposes as a direct sum $\mathcal{T}_q = R(1) \oplus \cdots \oplus R(q)$ where $R(i)$ consists of elements in the ith-row of $\mathcal{T}_q$. The modules $R(i)$ are isomorphically distinct and $\mathrm{Ext}^1(\mathbb{Z}[C_q], R(i)) \cong \mathbb{F}_p$, the field with p elements. The extension $\mathcal{X}$ is classified up to congruence by a sequence $\mathbf{c} = (c_i)_{1 \le i \le q}$ where $c_i \in \mathbb{F}_p = \mathrm{Ext}^1(\mathbb{Z}[C_q], R(i))$. We write

$$\delta(\mathcal{X}) = \{i \mid c_i = 0\}.$$

The set $\delta(\mathcal{X})$ is called the *degeneracy* of $\mathcal{X}$; we will show that $\delta(\mathcal{X})$ is an invariant of the isomorphism class of the module X not merely of the congruence class of the extension $\mathcal{X}$. Consequently, we may write $\delta(X) = \delta(\mathcal{X})$. We say that the generalized Swan module X is *degenerate* when $\delta(X) \neq \emptyset$ and *nondegenerate* when $\delta(X) = \emptyset$. Nondegenerate modules are necessarily projective and for these the conclusion of Theorem A already follows from the theorem of Swan–Jacobinski (cf. [4, 12]). However, the (more numerous) degenerate modules are not projective and lie outside the scope of the Swan–Jacobinski theorem. In these cases, Theorem A is a consequence of the following, which can be viewed as a rigidity property.

THEOREM B
Let X, X' be degenerate generalized Swan modules; then $X \cong X' \Longleftrightarrow \delta(X) = \delta(X')$.

In formulating our approach we make use of the *derived module category*; that is, the quotient of the category of Λ-modules, by setting projective $= 0$. The salient features are reviewed briefly in Section 1. A fuller account can be found in Chapter 5 of [7].

There is already a considerable literature on the general question of cancellation; see, for example, [13]. In the case of the metacyclic groups considered here, the results of Klingler [9] show that the question (*) has a generally negative answer. Thus, the cancellation statement of Theorem A is atypical and, to that extent, unexpected.

1. The derived module category

In what follows, Λ will denote the integral group ring $\Lambda = \mathbb{Z}[G]$ of a finite group, as yet unspecified. As a ring, Λ is both left and right Noetherian. The category of right Λ-modules is denoted by $\mathcal{M}\mathrm{od}_\Lambda$. If $f : M \to N$ is a morphism in $\mathcal{M}\mathrm{od}_\Lambda$, we write $f \approx 0$, when f can be written as a composite of Λ-homomorphisms $f = \xi \circ \eta$ via a projective module P; thus,

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\eta"'] & & N \\ & P \arrow[ur, "\xi"'] & \end{tikzcd}$$

We define $\langle M, N\rangle = \{f \in \mathrm{Hom}_\Lambda(M, N) : f \approx 0\}$; $\langle M, N\rangle$ is an additive subgroup of $\mathrm{Hom}_\Lambda(M, N)$. We extend $\approx$ to a binary relation on $\mathrm{Hom}_\Lambda(M, N)$ by

$$f \approx g \iff f - g \approx 0.$$

So extended, $\approx$ is an equivalence relation compatible with composition; that is, given Λ-homomorphisms $f, f' : M_0 \to M_1$, $g, g' : M_1 \to M_2$ then

$$f \approx f' \quad \text{and} \quad g \approx g' \implies g \circ f \approx g' \circ f'. \tag{1.1}$$

We denote by $\mathcal{D}\mathrm{er} = \mathcal{D}\mathrm{er}(\Lambda)$ the *derived module category* (cf. [6, 7]); that is, the quotient category of $\mathcal{M}\mathrm{od}_\Lambda$ in which the set of morphisms $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(M, N)$ is given by

$$\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(M, N) = \mathrm{Hom}_\Lambda(M, N)/\langle M, N\rangle.$$

Since $\langle M, N\rangle$ is a subgroup of $\mathrm{Hom}_\Lambda(M, N)$, it follows that

(1.2) $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(M, N)$ has the natural structure of an abelian group.

It is important to distinguish, both notationally and conceptually, between isomorphism in $\mathcal{M}\mathrm{od}_\Lambda$, which we write as $\cdots \cong_\Lambda \cdots$ and isomorphism in $\mathcal{D}\mathrm{er}(\Lambda)$, which we write as $\cdots \cong_{\mathcal{D}\mathrm{er}} \cdots$. For finitely generated Λ-modules the relationship between the two notions is as follows (see [7, p. 120]):

$$D \cong_{\mathcal{D}\mathrm{er}} D' \iff D \oplus P \cong_\Lambda D' \oplus P' \tag{1.3}$$

for some finitely generated projective Λ-modules. P, P'.

There is a related notion, *stable equivalence*, written $D \sim D'$, and defined by

$$D \sim D' \iff D \oplus \Lambda^m \cong_\Lambda D' \oplus \Lambda^n \tag{1.4}$$

for some positive integers m, n.

Clearly we have

$$D \sim D' \implies D \cong_{\mathcal{D}\mathrm{er}} D'. \tag{1.5}$$

The converse to (1.5) is, however, false.

Given a finitely generated Λ-module M, we consider exact sequences in $\mathcal{M}\mathrm{od}_\Lambda$; thus,

$$0 \to D \xrightarrow{i} P \xrightarrow{p} M \to 0 \tag{$\mathcal{E}$}$$

where P is finitely generated projective. Clearly such sequences always exist; we may even take P to be free. Moreover, as Λ is Noetherian then D is also finitely generated. Given another such exact sequence,

$$0 \to D' \xrightarrow{i'} P' \xrightarrow{p'} M \to 0,$$

then Schanuel's Lemma shows that $D \oplus P' \cong_\Lambda D' \oplus P$ so that $D \cong_{\mathcal{D}\mathrm{er}} D'$. We denote by $D_1(M)$ the isomorphism class in $\mathcal{D}\mathrm{er}$ of any module D which occurs in an exact sequence of the above form $(\mathcal{E})$. We may think of $D_1(M)$ as a *first derivative* of M. The correspondence $M \mapsto D_1(M)$ is functorial in the following way. Given any such exact sequence $(\mathcal{E})$ and a Λ-homomorphism $f : M \to M$ then the universal property of projective modules allows us to construct a commutative diagram of Λ-homomorphisms:

$$\begin{array}{ccccccccc} 0 \to & D & \stackrel{i}{\to} & P & \stackrel{p}{\to} & M & \to 0 \\ & \downarrow f_- & & \downarrow f_0 & & \downarrow f & \\ 0 \to & D & \stackrel{i}{\to} & P & \stackrel{p}{\to} & M & \to 0. \end{array}$$

While the Λ-homomorphism f_- is not uniquely determined, nevertheless its class in $\mathcal{D}\mathrm{er}$ *is uniquely determined*. In particular, given another such commutative diagram,

$$\begin{array}{ccccccccc} 0 \to & D & \stackrel{i}{\to} & P & \stackrel{p}{\to} & M & \to 0 \\ & \downarrow f'_- & & \downarrow f'_0 & & \downarrow f' & \\ 0 \to & D & \stackrel{i}{\to} & P & \stackrel{p}{\to} & M & \to 0 \end{array}$$

then we have

$$f \approx f' \implies f_- \approx f'_-. \tag{1.6}$$

Further discussion will be simplified by confining attention to Λ-*lattices*, that is, to Λ-modules which are finitely generated and torsion free as additive groups. For the remainder of this section, all Λ-modules considered will be subject to this restriction. When M is a Λ-lattice then $\mathrm{Ext}^1(M, \Lambda) = 0$, in consequence of which (cf. [7, p. 133]) (1.6) can be improved to

$$f \approx f' \iff f_- \approx f'_-. \tag{1.7}$$

Given $f \in \mathrm{End}_\Lambda(M)$, we denote by $\rho(f) = [f_-]$ the class of f_- in $\mathcal{D}\mathrm{er}$. By (1.7), the correspondence $[f] \mapsto \rho(f) = [f_-]$ determines a ring isomorphism

$$\rho : \mathrm{End}_{\mathcal{D}\mathrm{er}}(M) \xrightarrow{\simeq} \mathrm{End}_{\mathcal{D}\mathrm{er}}\big(D_1(M)\big). \tag{1.8}$$

The extension theory of Λ-lattices can be formulated in terms of the derived module category. Given the exact sequence $\mathcal{E}$ above and a Λ-homomorphism $\alpha : D \to N$, we construct the pushout diagram

$$\begin{array}{c} \mathcal{E} \\ \downarrow \natural \\ \alpha_*(\mathcal{E}) \end{array} = \left(\begin{array}{ccccccc} 0 \to & D & \stackrel{i}{\to} & P & \stackrel{p}{\to} & M & \to 0 \\ & \downarrow \alpha & & \downarrow \natural & & \downarrow \mathrm{Id} & \\ 0 \to & N & \stackrel{i}{\to} & \varinjlim(\alpha, i) & \stackrel{\pi}{\to} & M & \to 0 \end{array} \right).$$

Then $\alpha_*(\mathcal{E}) = (0 \to N \xrightarrow{i} \varinjlim(\alpha, i) \xrightarrow{\pi} M \to 0)$ defines an extension class in $\mathrm{Ext}^1(M, N)$. When P is projective, the correspondence $\alpha \mapsto [\alpha_*(\mathcal{E})]$ defines a mapping $\delta : \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(D, N) \to \mathrm{Ext}^1(M, N)$. With this notation we have

(1.9) $$\delta : \mathrm{Hom}_{\mathcal{D}\mathrm{er}}\big(D_1(M), N\big) \xrightarrow{\simeq} \mathrm{Ext}^1(M, N) \quad \text{is an isomorphism.}$$

The isomorphism of (1.9) is a *corepresentation formula*; thereby the covariant functor $\mathrm{Ext}^1(M, -)$ is represented by the Hom functor $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(D_1(M), -)$. Given the exact sequence $(\mathcal{E})$, then for any Λ-module N we have exact sequences for $k \geq 1$,

$$\mathrm{Ext}^k(P, N) \xrightarrow{i^*} \mathrm{Ext}^k\big(D_1(M), N\big) \xrightarrow{\delta} \mathrm{Ext}^{k+1}(M, N) \xrightarrow{p^*} \mathrm{Ext}^{k+1}(P, N).$$

As P is projective, then $\mathrm{Ext}^k(P, N) \cong \mathrm{Ext}^{k+1}(P, N) = 0$ and we obtain the usual *dimension shifting* isomorphisms

(1.10) $$\mathrm{Ext}^{k+1}(M, N) \cong \mathrm{Ext}^k\big(D_1(M), N\big).$$

We may regard the corepresentation formula (1.9) as the degenerate case of (1.10) corresponding to the case $k = 0$.

We say that M has *periodic cohomology* when, for some positive integer d, there is an exact sequence

$$0 \to M \to P_{d-1} \to \cdots \to P_0 \to M \to 0,$$

where each P_i is projective. As M is a lattice it can be assumed, in addition, that each P_i is finitely generated. The integer d is then said to be a *cohomological period* for M. If M has periodic cohomology, it has a minimal cohomological period denoted by $\mu(M)$ and any cohomological period of M is an integral multiple of $\mu(M)$.

Finally we recall the tensor product construction for Λ-modules; thus, if M, N are right Λ-modules by $M \otimes N$, we mean the abelian group $M \otimes_{\mathbb{Z}} N$ endowed with the diagonal right action of Λ, $(m \otimes n) \cdot \lambda = m\lambda \otimes n\lambda$. The following is well known (cf. [2, p. 11]).

(1.11) If P is finitely generated projective, then so also is $M \otimes P$.

Suppose, given an exact sequence $0 \to \mathbb{Z} \to P_{\mu(\mathbb{Z})-1} \to \cdots \to P_0 \to \mathbb{Z} \to 0$ where each P_i is finitely generated projective. Applying $M \otimes -$ gives an exact sequence

$$0 \to M \otimes \mathbb{Z} \to M \otimes P_{\mu(\mathbb{Z})-1} \to \cdots \to M \otimes P_0 \to M \otimes \mathbb{Z} \to 0.$$

By (1.11), each $M \otimes P_i$ is finitely generated projective; as $M \otimes \mathbb{Z} \cong M$, then:

(1.12) If $\mathbb{Z}$ has periodic cohomology, then $\mu(\mathbb{Z})$ is a cohomological period of every Λ-lattice M.

2. Modules over the metacyclic group $G(p, q)$

For each integer $n \geq 2$, we denote by C_n the cyclic group $C_n = \langle x \mid x^n = 1 \rangle$. For the remainder of this paper, we fix an odd prime p, an integral divisor q of $p - 1$ and write $d = (p-1)/q$. Recalling that $\mathrm{Aut}(C_p) \cong C_{p-1}$, then there exists an element

$\theta \in \mathrm{Aut}(C_p)$ such that $\mathrm{ord}(\theta) = q$. Taking y to be a generator of C_q and making a once and for all choice of θ with order q, we construct the semi-direct product $G(p,q) = C_p \rtimes_h C_q$ where $h : C_q \to \mathrm{Aut}(C_p)$ is the homomorphism $h(y) = \theta$. There is then a unique integer a in the range $1 \le a \le p-1$ such that $\theta(x) = x^a$, and $G(p,q)$ then has the presentation

$$G(p,q) = \langle x, y \mid x^p = y^q = 1; yxy^{-1} = x^a \rangle.$$

A theorem of Zassenhaus–Artin–Tate (cf. [3, Chapter 12]) shows that, over a finite group G, the trivial module $\mathbb{Z}$ has periodic cohomology if and only if for each prime π, every subgroup of order π^2 is cyclic. By this criterion, $\mathbb{Z}$ has periodic cohomology when $G = G(p,q)$; indeed, it can be shown (cf. [8]) that

(2.1) $$\mu(\mathbb{Z}) = 2q \quad \text{when } G = G(p,q).$$

We denote by Λ the integral group ring $\Lambda = \mathbb{Z}[G(p,q)]$ and by $i : \mathbb{Z}[C_p] \hookrightarrow \Lambda$ and $j : \mathbb{Z}[C_q] \hookrightarrow \Lambda$, the respective inclusions. Depending on context, $\mathbb{Z}$ may denote the trivial module over any of the group rings Λ, $\mathbb{Z}[C_p]$ or $\mathbb{Z}[C_q]$. We denote by I_C the augmentation ideal of $\mathbb{Z}[C_p]$; I_C is defined by the exact sequence of $\mathbb{Z}[C_p]$-modules

(2.2) $$0 \to I_C \xhookrightarrow{\iota} \mathbb{Z}[C_p] \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

On dualizing, we get an exact sequence $0 \to \mathbb{Z} \xrightarrow{\epsilon^*} \mathbb{Z}[C_p] \xrightarrow{\iota^*} I_C^* \to 0$ where $\epsilon^*(1) = \Sigma_x = 1 + x + x^2 + \cdots + x^{p-1}$. It is a standard and easily verified fact that

(2.3) I_C^* and I_C are isomorphic as $\mathbb{Z}[C_p]$-modules.

As I_C^* and I_C are not actually identical, we find it convenient to distinguish between them. We identify the dual I_C^* with the quotient $\mathbb{Z}[C_p]/(\Sigma_x)$. As (Σ_x) is a two-sided ideal in $\mathbb{Z}[C_p]$, then I_C^* is naturally a ring; indeed, putting $\zeta = \exp(2\pi i/p)$, then

(2.4) There is a ring isomorphism $I_C^* \cong \mathbb{Z}[\zeta]$.

As is well known, $\mathbb{Z}[C_p]$ has a canonical fiber product decomposition

(2.5) $$\begin{array}{ccc} \mathbb{Z}[C_p] & \to & I_C^* \\ \epsilon \downarrow & & \downarrow \\ \mathbb{Z} & \to & \mathbb{Z}_p \end{array}$$

where $\epsilon : \mathbb{Z}[C_p] \to \mathbb{Z}$ is the augmentation map and $\mathbb{F}_p$ is the field with p elements. To proceed, we briefly recall the cyclic algebra construction. Let S denote a commutative ring and $\theta : S \to S$ a ring automorphism of finite order dividing q; in particular, θ satisfies the identity $\theta^q = \mathrm{Id}$. The *cyclic ring* $\mathcal{C}_q(S,\theta)$ is then the (two-sided) free S-module

$$\mathcal{C}_q(S,\theta) = S\mathbf{1} \dot{+} S\mathbf{y} \dot{+} \cdots \dot{+} S\mathbf{y}^{q-1}$$

of rank q with basis $\{\mathbf{1}, \mathbf{y}, \dots \mathbf{y}^{q-1}\}$ and with multiplication defined by

$$\mathbf{y}^q = \mathbf{1}; \qquad \mathbf{y}\xi = \theta(\xi)\mathbf{y} \quad (\xi \in S).$$

So defined, $\mathcal{C}_q(S,\theta)$ is an extension ring of S. In the fiber product (2.5), θ induces a ring automorphism of order q on $\mathbb{Z}[C_p]$. As θ fixes Σ_x, then θ induces a ring automorphism on the quotient $I_C^* = \mathbb{Z}[C_p]/(\Sigma_x)$. Likewise θ stabilizes the augmentation ideal I_C and induces the identity automorphism both on the quotient $\mathbb{Z} = \mathbb{Z}[C_p]/I_C$ and $\mathbb{F}_p$. As the homomorphisms in (2.5) are equivariant with respect to these ring automorphisms, we may apply the cyclic algebra construction $\mathcal{C}_q(-,\theta)$ to (2.5). Identifying $\mathcal{C}_q(\mathbb{Z}[C_p]) = \mathbb{Z}[G(p,q)]$, $\mathcal{C}_q(\mathbb{Z}) = \mathbb{Z}[C_q]$, $\mathcal{C}_q(\mathbb{F}_p) = \mathbb{F}_p[C_q]$, we obtain a fiber product

$$\begin{array}{ccc} \mathbb{Z}[G(p,q)] & \to & \mathcal{C}_q(I_C^*,\theta) \\ \downarrow & & \downarrow \\ \mathbb{Z}[C_q] & \to & \mathbb{Z}_p[C_q]. \end{array} \tag{2.6}$$

To proceed to a more tractable description of $\mathcal{C}_q(I_C^*,\theta)$, we first make the identification $\mathcal{C}_q(I_C^*,\theta) \otimes \mathbb{Q} \cong \mathcal{C}_q(\mathbb{Q}(\zeta),\theta)$ where, as above, ζ is a primitive pth root of unity. Then θ acts on $\mathbb{Z}[\zeta]$ via the isomorphism $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong C_{p-1}$. Let $A = \mathbb{Z}[\zeta]^\theta$ denote the subring fixed by θ. We note (see [1, Lemma 3]) that $p = (\zeta-1)^{p-1}u$ for some unit $u \in \mathbb{Z}[\zeta]^*$. Putting $\pi = (\zeta-1)^q$, then

(2.7) p ramifies completely in A, and π is the unique prime in A over p.

We denote by $\mathcal{T}_q(A,\pi)$, the subring of *quasi-triangular* matrices in the ring $M_q(A)$ of $q \times q$ matrices over A defined as follows:

$$\mathcal{T}_q(A,\pi) = \left\{X = (x_{rs})_{1\le r,s\le n} \in M_q(A) \;\middle|\; x_{rs} \in (\pi) \text{ if } r > s\right\}.$$

Likewise, we define

$$\mathcal{T}_q(A/\pi) = \left\{X = (x_{rs})_{1\le r,s\le n} \in M_q(A/\pi) \;\middle|\; x_{rs} = 0 \text{ if } r > s\right\}.$$

Taking the quotient by π defines a surjective ring homomorphism

$$\nu : \mathcal{T}_q(A,\pi) \twoheadrightarrow \mathcal{T}_q(A/\pi). \tag{2.8}$$

In turn, the correspondence $X \mapsto (x_{11},\dots,x_{qq})$ gives a surjective ring homomorphism

$$\varphi : \mathcal{T}_q(A/\pi) \twoheadrightarrow \underbrace{A/\pi \times \cdots \times A/\pi}_{q}. \tag{2.9}$$

The following structural theorem is fundamental in what follows:

THEOREM 2.1
There exists a ring isomorphism $\widehat{\lambda}_* : \mathcal{C}_q(I_C^*,\theta) \to \mathcal{T}_q(A,\pi)$.

This can be regarded as an explicit form of Rosen's Theorem (see [11]; see also [10, p. 373]; a proof in the above form may be found in [8]). Theorem 2.1 allows us to re-interpret (2.6) as a fiber square of the form

$$\begin{array}{ccc} \mathbb{Z}[G(p,q)] & \to & \mathcal{T}_q(A,\pi) \\ \downarrow & & \downarrow \\ \mathbb{Z}[C_q] & \to & \mathbb{Z}_p[C_q] \end{array} \tag{2.10}$$

If $i_*(-)$ denotes extension of scalars from $\mathbb{Z}[C_p]$-modules to Λ-modules, then

(2.11) $i_*(I_C)$ and $i_*(I_C^*)$ are isomorphic as Λ-modules.

We note that $\mathcal{C}_q(I_C^*,\theta)$ is simply another description of the induced module $i_*(I_C^*)$. As $\mathcal{T}_q(A,\pi) \cong \mathcal{C}_q(I_C^*,\theta)$, it follows from (2.11) that

$$i_*(I_C) \cong i_*(I_C^*) \cong \mathcal{T}_q(A,\pi). \tag{2.12}$$

Applying i_* to the exact sequence (2.2), we obtain an exact sequence

$$0 \to i_*(I_C) \xhookrightarrow{\iota} i_*\big(\mathbb{Z}[C_p]\big) \xrightarrow{\epsilon} i_*(\mathbb{Z}) \to 0.$$

However, $i_*(I_C) \cong \mathcal{T}_q(A,\pi)$, $i_*(\mathbb{Z}[C_p]) \cong \Lambda$ and $i_*(\mathbb{Z}) \cong \mathbb{Z}[C_q]$, so giving an exact sequence

$$0 \to \mathcal{T}_q(A,\pi) \xhookrightarrow{\iota} \Lambda \xrightarrow{\epsilon} \mathbb{Z}[C_q] \to 0. \tag{2.13}$$

Moreover, from this construction it follows easily that

$$\mathrm{Hom}_\Lambda\big(\mathcal{T}_q(A,\pi), \mathbb{Z}[C_q]\big) = 0. \tag{2.14}$$

Applying $-\otimes\mathbb{Q}$ to (2.14), the semisimplicity of $\mathcal{T}_q(A,\pi)\otimes\mathbb{Q}$ implies that

$$\begin{aligned} &\mathrm{Hom}_\Lambda\big(K, \mathbb{Z}[C_q]\big) = 0 \\ &\quad \text{if } K \text{ is a } \Lambda\text{-submodule of } \mathcal{T}_q(A,\pi)\oplus\cdots\oplus\mathcal{T}_q(A,\pi). \end{aligned} \tag{2.15}$$

We decompose $\mathcal{T}_q(A,\pi)$ as direct sum of right Λ-modules; thus,

$$\mathcal{T}_q(A,\pi) \cong R(1)\oplus R(2)\oplus\cdots\oplus R(q) \tag{2.16}$$

where $R(i)$ is the ith row of $\mathcal{T}_q(A,\pi)$. We note that

$$R(i) \cong_\Lambda R(j) \iff i = j, \tag{2.17}$$

$$\mathrm{Hom}_\Lambda\big(R(i), \mathbb{Z}[C_q]\big) = 0 \quad \text{for all } i \in \{1,\dots,q\}. \tag{2.18}$$

Of the above, (2.17) is proved in Section 4 of [8], while (2.18) follows directly from (2.14).

3. Preliminary cancellation

Let $\mathbb{K}$ be a finite extension field of $\mathbb{Q}$ and let A denote the ring of algebraic integers in $\mathbb{K}$. Let $\mathfrak{B}$ be a finite dimensional semisimple $\mathbb{K}$-algebra. By Wedderburn's Theorem,

$\mathfrak{B} \otimes_{\mathbb{Q}} \mathbb{R}$ decomposes as a direct product of matrix rings

$$\mathfrak{B} \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{i=1}^{m} M_{d_i}(D_i),$$

where each D_i is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $\Omega \subset \mathfrak{B}$ be an A-order; that is, Ω is an A-subalgebra of $\mathfrak{B}$ such that $\Omega \otimes_A \mathbb{K} \cong \mathfrak{B}$. We say that Ω satisfies the *Eichler condition* when, in the above Wedderburn decomposition, $D_i \cong \mathbb{H} \Longrightarrow d_i \geq 2$. We have the following much simplified version of Jacobinski's Cancellation Theorem [5]:

(3.1) Let L, M be Ω-lattices such that $L \oplus M \cong_\Omega M \oplus M$; if Ω satisfies the Eichler condition, then $L \cong_\Omega M$.

An account of the more general version can be found on page 324 in [4].

We apply (3.1) to two of the modules considered in Section 2. In the first case we take $\Omega = \mathcal{T}_q(A,\pi)$ and $\mathfrak{B} = M_q(\mathbb{K})$ where $\mathbb{K}$ is the field of fractions of A. Then for some integers a, b, we have $\mathbb{K} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{(a)} \times \mathbb{C}^{(b)}$ and hence $\mathfrak{B} \otimes_{\mathbb{Q}} \mathbb{R} \cong M_q(\mathbb{R})^{(a)} \times M_q(\mathbb{C})^{(b)}$. In particular, Ω satisfies Eichler's condition. Applying (3.1) gives the following:

(3.2) Let L be a $\mathcal{T}_q(A,\pi)$-lattice such that $L \oplus \mathcal{T}_q(A,\pi) \cong_{\mathcal{T}_q(A,\pi)} \mathcal{T}_q(A,\pi) \oplus \mathcal{T}_q(A,\pi)$; then $L \cong_{\mathcal{T}_q(A,\pi)} \mathcal{T}_q(A,\pi)$.

We extend this to certain Λ-lattices where $\Lambda = \mathbb{Z}[G(p,q)]$. We have a surjective ring homomorphism $\mu : \mathbb{Z}[G(p,q)] \twoheadrightarrow \mathcal{T}_q(A,\pi)$ and induction and co-induction functors

$$\mu_* : \mathcal{M}\mathrm{od}_\Lambda \to \mathcal{M}\mathrm{od}_{\mathcal{T}_q(A,\pi)}; \qquad \mu^* : \mathcal{M}\mathrm{od}_{\mathcal{T}_q(A,\pi)} \to \mathcal{M}\mathrm{od}_\Lambda.$$

By regarding $\mathcal{T}_q(A,\pi)$ as a module over Λ, we are abusing notation; the correct symbol for the intended Λ-module is $\mu^*(\mathcal{T}_q(A,\pi))$. To avoid this confusion in the discussion that follows, we write $\mathcal{T} = \mu^*(\mathcal{T}_q(A,\pi))$. Moreover, it is straightforward to check that

$$\mu_*(\mathcal{T}) = \mathcal{T}_q(A,\pi). \tag{3.3}$$

As Λ satisfies the Eichler condition, it follows directly from (3.1) that

(3.4) If $\mathcal{K}$ is a Λ-lattice such that $\mathcal{K} \oplus \mathcal{T} \cong_\Lambda \mathcal{T} \oplus \mathcal{T}$, then $\mathcal{K} \cong_\Lambda \mathcal{T}$.

Next we take $\Omega = \mathbb{Z}[C_q]$, $\mathbb{K} = \mathbb{Q}$ and $\mathfrak{B} \cong \prod_{d|q} \mathbb{Q}[x]/(c_d(x))$ where $c_d(x)$ is the dth cyclotomic polynomial. Then $\mathfrak{B} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{(a)} \times \mathbb{C}^{(b)}$ for some integers a, b so that again Ω satisfies the Eichler condition. Applying (3.1) gives

(3.5) Let $\mathcal{L}$ be a lattice over $\mathbb{Z}[C_q]$ such that $\mathcal{L} \oplus \mathbb{Z}[C_q] \cong_{\mathbb{Z}[C_q]} \mathbb{Z}[C_q] \oplus \mathbb{Z}[C_q]$; then $\mathcal{L} \cong_{\mathbb{Z}[C_q]} \mathbb{Z}[C_q]$.

We may modify this statement slightly in the context of Λ-lattices. We also have a surjective ring homomorphism $\eta : \mathbb{Z}[G(p,q)] \twoheadrightarrow \mathbb{Z}[C_q]$ and functors

$$\eta_* : \mathcal{M}\mathrm{od}_\Lambda \to \mathcal{M}\mathrm{od}_{\mathbb{Z}[C_q]}; \qquad \eta^* : \mathcal{M}\mathrm{od}_{\mathbb{Z}[C_q]} \to \mathcal{M}\mathrm{od}_\Lambda.$$

In regarding $\mathbb{Z}[C_q]$ as a module over Λ, we should really write $\eta^*(\mathbb{Z}[C_q])$. To avoid this confusion in the discussion that follows, we write $\mathcal{Q} = \eta^*(\mathbb{Z}[C_q])$. With this modification, as Λ satisfies the Eichler condition, we have

(3.6) Let $\mathcal{Q}'$ be a Λ-lattice such that $\mathcal{Q}' \oplus \mathcal{Q} \cong_\Lambda \mathcal{Q} \oplus \mathcal{Q}$; then $\mathcal{Q}' \cong_\Lambda \mathcal{Q}$.

4. Cohomology calculations

For the remainder of this paper, we fix an odd prime p and a positive integral divisor q of $p-1$. As in Section 2, we put $G = G(p,q)$ and write $\Lambda = \mathbb{Z}[G(p,q)]$. In addition, we put $\Gamma = \mathbb{Z}[C_p]$. We proceed to calculate the cohomology of the Λ-modules introduced in Section 1. In doing so, we will employ restriction and extension of scalars to and from the subring $\Gamma \subset \Lambda$. To this end, we shall use boldface symbols **Hom**, **End** and $\mathbf{Ext}^a$, when describing homomorphisms, endomorphisms and extensions of Λ-modules; and standard Roman font, Hom, End and Ext^k, when referring to the corresponding notions over Γ. The calculations that follow are essentially a summary of those of [8], to which paper we refer the reader for fuller details.

$$\mathrm{Ext}^k(\mathbb{Z}, I_C) \cong \begin{cases} \mathbb{F}_p & k=1 \\ 0 & k=2, \end{cases}$$

$$\begin{aligned} \mathbf{Ext}^1\big(\mathbb{Z}[C_q], i_*(I_C),\big) &\cong \mathrm{Ext}^1\big(i^*\big(\mathbb{Z}[C_q]\big), I_C\big) \\ &\cong \bigoplus_{i=1}^{q} \mathrm{Ext}^1(\mathbb{Z}, I_C) \\ &\cong \underbrace{\mathbb{F}_p \times \cdots \times \mathbb{F}_p}_{q}. \end{aligned} \tag{4.1}$$

As $i_*(I_C) \cong \mathcal{T}_q$, then

$$\mathbf{Ext}^1\big(\mathbb{Z}[C_q], \mathcal{T}_q\big) \cong \underbrace{\mathbb{F}_p \times \cdots \times \mathbb{F}_p}_{q}. \tag{4.2}$$

As there is an exact sequence $0 \to \mathcal{T}_q \to \Lambda \to \mathbb{Z}[C_q] \to 0$, it follows by the corepresentation formula that

$$\mathbf{Ext}^1\big(\mathbb{Z}[C_q], \mathcal{T}_q\big) \cong \mathbf{End}_{\mathcal{D}\mathrm{er}}(\mathcal{T}_q). \tag{4.3}$$

It follows that

$$\mathbf{End}_{\mathcal{D}\mathrm{er}}(\mathcal{T}_q) \cong \underbrace{\mathbb{F}_p \times \cdots \times \mathbb{F}_p}_{q}. \tag{4.4}$$

From the decomposition $\mathcal{T}_q \cong \bigoplus_{i=1}^q R(i)$, it follows from (4.4) that

$$\mathbf{End}_{\mathcal{D}\mathrm{er}}\Big(\bigoplus_{i=1}^{q} R(i)\Big) \cong \underbrace{\mathbb{F}_p \times \cdots \times \mathbb{F}_p}_{q}.$$

Consequently,

$$\bigoplus_{i,j=1}^{q} \mathbf{Hom}_{\mathcal{D}\mathrm{er}}\big(R(i), R(j)\big) \cong \underbrace{\mathbb{F}_p \times \cdots \times \mathbb{F}_p}_{q}.$$

As $R(i)$ is not projective over Λ, then $\mathbf{Hom}_{\mathcal{D}\mathrm{er}}(R(i), R(i)) \neq 0$. Hence, we have

$$\mathbf{Hom}_{\mathcal{D}\mathrm{er}}\big(R(i), R(j)\big) \cong \begin{cases} \mathbb{F}_p & i = j \\ 0 & i \neq j. \end{cases}$$

Note that

$$\mathbf{Ext}^1\big(\mathbb{Z}[C_q], R(k)\big) \cong \mathbb{F}_p \quad \text{for all } k \ (1 \le k \le q). \tag{4.5}$$

$$\mathbf{Ext}^2\big(R(i), R(j)\big) \cong \begin{cases} \mathbb{F}_p & j = i+1 \\ 0 & j \neq i+1, \end{cases} \tag{4.6}$$

$$\mathbf{Ext}^{2j}\big(R(i), R(q)\big) \cong \begin{cases} \mathbb{F}_p & j \equiv i \bmod q \\ 0 & j \neq i+1, \end{cases} \tag{4.7}$$

$$\mathbf{Ext}^{2j+1}\big(R(i), R(q)\big) = 0 \quad \text{for all } i, j. \tag{4.8}$$

The above formulae exemplify the $2q$-fold cohomological periodicity of Λ-lattices. If i is a positive integer, then for any Λ-lattice X we put

$$\mathcal{G}^i(X) = \mathbf{Ext}^{2i}\big(X, R(q)\big); \qquad \mathcal{G}^*(X) = \bigoplus_{i=1}^{q} \mathbf{Ext}^{2i}\big(X, R(q)\big);$$

$$\mathcal{H}^i(X) = \mathbf{Ext}^{2i+1}\big(X, R(q)\big); \qquad \mathcal{H}^*(X) = \bigoplus_{i=1}^{q} \mathbf{Ext}^{2i+1}\big(X, R(q)\big).$$

By the dimension shifting argument of (1.10), we see that:

PROPOSITION 4.1
Let $0 \to K \to P \to Q \to 0$ be an exact sequence of Λ-lattices; if P is projective then $\mathcal{H}^i(Q) \cong \mathcal{G}^i(K)$.

For future reference we note that:

PROPOSITION 4.2
Let $\alpha, \beta \subset \{1, \ldots, q\}$; then

$$\mathcal{G}^*\big(R(\alpha)\big) \cong \mathcal{G}^*\big(R(\beta)\big) \quad \Longleftrightarrow \quad \alpha = \beta.$$

Proof
It suffices to show $(\Longrightarrow)$. As $\mathbb{Z}$ has cohomological period $2q$ then by (1.12), it suffices to compare the values $\mathcal{G}^i(R(\alpha))$, $\mathcal{G}^i(R(\beta))$ in the range $1 \le i \le q$. It follows from

(4.6) and (4.7) that

$$\mathcal{G}^i\big(R(j)\big) \cong \begin{cases} \mathbb{F}_p & j = i \\ 0 & j \neq i, \end{cases} \qquad \mathcal{G}^i\big(R(\alpha)\big) \cong \begin{cases} \mathbb{F}_p & i \in \alpha \\ 0 & i \notin \alpha. \end{cases}$$

Thus, if $\mathcal{G}^*(R(\alpha)) \cong \mathcal{G}^*(R(\beta))$, then $\alpha = \beta$. □

We note also, immediately from (4.8), that

(4.9) $$\mathcal{H}^i\big(R(\alpha)\big) = 0 \quad \text{for all } i.$$

5. Invariance of degeneracy

If $\alpha \subset \{1, \dots, q\}$, we put $R(\alpha) = \bigoplus_{i \in \alpha} R(i)$. An extension of the form

($\mathcal{Z}$) $$0 \to R(\alpha) \xrightarrow{i} Z \xrightarrow{p} \mathbb{Z}[C_q] \to 0$$

is said to have *kernel type* α. In fact, the kernel type of the extension $\mathcal{Z}$ depends only on the isomorphism class of the module Z. To see this, suppose that the module Z' occurs in an exact sequence

($\mathcal{Z}'$) $$0 \to R(\beta) \xrightarrow{j} Z' \xrightarrow{q} \mathbb{Z}[C_q] \to 0$$

and that there exists an isomorphism $h : Z \to Z'$. Then the homomorphism

$$q \circ h \circ i : R(\alpha) \to \mathbb{Z}[C_q]$$

is zero by (2.18). Consequently, h induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & R(\alpha) & \xrightarrow{i} & Z & \xrightarrow{p} & \mathbb{Z}[C_q] & \to 0 \\ & \downarrow h_- & & \downarrow h & & \downarrow h_+ & \\ 0 \to & R(\beta) & \xrightarrow{j} & Z' & \xrightarrow{q} & \mathbb{Z}[C_q] & \to 0. \end{array}$$

Moreover one sees easily that the induced homomorphism $h_+ : \mathbb{Z}[C_q] \to \mathbb{Z}[C_q]$ is surjective. As the underlying additive group of $\mathbb{Z}[C_q]$ is free abelian of finite rank, it follows that h_+ is an isomorphism. Extending the above diagram one place to the left by zeroes, it follows from the Five Lemma that $h_- : R(\alpha) \to R(\beta)$ is also an isomorphism. Consequently, $\mathcal{G}^*(R(\alpha)) \cong \mathcal{G}^*(R(\beta))$, so that by Proposition 4.2 it follows that $\beta = \alpha$; that is,

(5.1) In $\mathcal{Z}$ above the kernel type α is an isomorphism invariant of the module Z.

Now consider extensions of the form

($\mathcal{X}$) $$0 \to \mathcal{T}_q \xrightarrow{i} X \xrightarrow{p} \mathbb{Z}[C_q] \to 0;$$

that is, where $= \{1, \ldots, q\}$ so that the module X is a generalized Swan module. Then $\mathcal{X}$ is classified up to congruence by a cohomology class

$$\mathbf{c} \in \mathrm{Ext}^1\big(\mathbb{Z}[C_q], \mathcal{T}_q\big) \cong \bigoplus_{i=1}^{q} \mathrm{Ext}^1\big(\mathbb{Z}[C_q], R(i)\big)$$

described as an $|\alpha|$-tuple $\mathbf{c} = (c_i)_{1 \le i \le q}$ where $c_i \in \mathrm{Ext}^1(\mathbb{Z}[C_q], R(i)) \cong \mathbf{F}_p$. We shall then say that $\mathcal{X}$ is *nondegenerate* when each $c_i \neq 0$.

PROPOSITION 5.1

Let $\mathcal{X} = (0 \to \mathcal{T}_q \xrightarrow{i} X \xrightarrow{p} \mathbb{Z}[C_q] \to 0)$ *be an extension defining a generalized Swan module* X*; then*

$$\mathcal{X} \text{ is nondegenerate} \iff X \text{ is projective}.$$

Proof

($\Longrightarrow$) $\mathcal{X}$ is classified by $\mathbf{c} = (c_i)_{1 \le i \le q} \in \mathrm{Ext}^1(\mathbb{Z}[C_q], \mathcal{T}_q) \cong \underbrace{\mathbb{F}_p \times \cdots \times \mathbb{F}_p}_{q}$.

As we have seen in (4.3), $\mathrm{Ext}^1(\mathbb{Z}[C_q], \mathcal{T}_q) \cong \mathrm{End}_{\mathcal{D}\mathrm{er}}(\mathcal{T}_q)$. As each $c_i \neq 0$, then $\mathbf{c} \in \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(\mathcal{T}_q)$ and we may construct X by means of the pushout construction

$$\begin{array}{ccccccccc} 0 \to & \mathcal{T}_q & \to & \Lambda & \to & \mathbb{Z}[C_q] & \to 0 \\ & \downarrow \mathbf{c} & & \downarrow \natural & & \downarrow \mathrm{Id} & \\ 0 \to & \mathcal{T}_q & \to & X & \to & \mathbb{Z}[C_q] & \to 0. \end{array}$$

As $\mathbf{c} \in \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(\mathcal{T}_q)$, then X is projective by Swan's criterion (see [7, p. 115]).

($\Longleftarrow$) Conversely, suppose that some $c_j = 0$. Let X' be the module described by the extension $0 \to \bigoplus_{i \neq j} R(i) \to X' \to \mathbb{Z}[C_q] \to 0$ with cohomology class $\mathbf{c}' = (c_i)_{i \neq j}$. Then $X \cong R(j) \oplus X'$. As $R(j)$ is not projective, then neither is X. In the contrapositive, if X is projective then $\mathcal{X}$ is nondegenerate. □

The more general extension $\mathcal{Z}$ is classified up to congruence by a cohomology class

$$\mathbf{c} \in \mathrm{Ext}^1\big(\mathbb{Z}[C_q], R(\alpha)\big) \cong \bigoplus_{i \in \alpha} \mathrm{Ext}^1\big(\mathbb{Z}[C_q], R(i)\big)$$

described as an $|\alpha|$-tuple $\mathbf{c} = (c_i)_{i \in \alpha}$ where $c_i \in \mathrm{Ext}^1(\mathbb{Z}[C_q], R(i)) \cong \mathbf{F}_p$. We say that $\mathcal{Z}$ is *nondegenerate relative to* α when $c_i \neq 0$ for each $i \in \alpha$. If some $c_i = 0$, we say that $\mathcal{Z}$ is *degenerate relative to* α. If $\alpha \subset \{1, \ldots, q\}$, write $\overline{\alpha} = \{1, \ldots, q\} - \alpha$. From Proposition 5.1 we derive:

PROPOSITION 5.2

Let $\mathcal{Z} = (0 \to R(\alpha) \xrightarrow{i} Z \xrightarrow{p} \mathbb{Z}[C_q] \to 0)$ *be an extension of kernel type* α*, nondegenerate with respect to* α*. Then* $R(\overline{\alpha})$ *represents* $D_1(Z)$*.*

Proof
Suppose that $\mathcal{Z}$ is classified by $\mathbf{c} = (c_i)_{i\in\alpha}$ and consider the cohomology class $\gamma = (\gamma_i)_{i\in\overline{\alpha}}$, $\gamma_i \in \mathrm{Ext}^1(\mathbb{Z}[C_q], R(i))$ defined by $\gamma_i = 1$ for $i \in \overline{\alpha}$. Consider the extension $\mathcal{P} = (0 \to R(\overline{\alpha}) \oplus R(\alpha) \to P \to \mathbb{Z}[C_q] \to 0)$ defined by $(\gamma, \mathbf{c})$. We note that $R(\overline{\alpha}) \oplus R(\alpha) \cong \mathcal{T}_q$. As each $\gamma_i \neq 0$ and each $c_j \neq 0$, then $\mathcal{P}$ is nondegenerate so that P is projective by Proposition 5.1. Putting $\widetilde{Z} = P/R(\overline{\alpha})$ gives an extension

$$0 \to R(\overline{\alpha}) \to P \to \widetilde{Z} \to 0,$$

where $\widetilde{Z}$ occurs in the extension $(0 \to R(\alpha) \to \widetilde{Z} \to \mathbb{Z}[C_q] \to 0)$ classified by $\mathbf{c}$. Hence, $\widetilde{Z} \cong Z$ so that Z occurs in an extension $(0 \to R(\overline{\alpha}) \to P \to Z \to 0)$ where P is projective. Consequently, $R(\overline{\alpha})$ represents $D_1(Z)$ as claimed. □

Let $\mathcal{E} = (0 \to \mathcal{T}_q \to E \to \mathbb{Z}[C_q] \to 0)$ be an extension defining a generalized Swan module E and classified by $\mathbf{c} = (c_i)_{1\le i\le q}$ where $c_i \in \mathrm{Ext}^1(\mathbb{Z}[C_q], R(i)) \cong \mathbb{F}_p$. The *degeneracy* $\delta(\mathcal{E})$ of $\mathcal{E}$ is defined by $\delta(\mathcal{E}) = \{i \mid c_i = 0\}$, and the *support* of $\mathcal{E}$ is defined by $\mathrm{supp}(\mathcal{E}) = \{i \mid c_i \neq 0\}$. Evidently $\delta(\mathcal{E})$ and $\mathrm{supp}(\mathcal{E})$ are complementary subsets of $\{1, \dots, q\}$, $\mathrm{supp}(\mathcal{E}) = \overline{\delta(\mathcal{E})}$. Given such an extension $\mathcal{E}$, we may decompose the cohomology class as $\mathbf{c} = (\mathbf{c}^-, \mathbf{c}^+)$ where $\mathbf{c}^- = (c_i)_{i\in\delta(\mathcal{E})}$ is identically zero and where $\mathbf{c}^+ = (c_i)_{i\in\mathrm{supp}(\mathcal{E})}$ determines an extension

$$\mathcal{X} = \left(0 \to R(\overline{\alpha}) \xrightarrow{i} X \xrightarrow{p} \mathbb{Z}[C_q] \to 0\right)$$

of kernel type $\mathrm{supp}(\mathcal{E})$ which is nondegenerate with respect to $\mathrm{supp}(\mathcal{E})$. As $\mathbf{c}^-$ is identically zero, then $E \cong R(\alpha) \oplus X$; that is,

PROPOSITION 5.3
Let $\mathcal{E} = (0 \to \mathcal{T}_q \to E \to \mathbb{Z}[C_q] \to 0)$ be an extension defining a generalized Swan module E; then $E \cong R(\alpha) \oplus X$ where X occurs in an extension

$$\mathcal{X} = \left(0 \to R\big(\mathrm{supp}(\mathcal{E})\big) \xrightarrow{i} X \xrightarrow{p} \mathbb{Z}[C_q] \to 0\right)$$

of kernel type $\mathrm{supp}(\mathcal{E})$ *which is nondegenerate with respect to* $\mathrm{supp}(\mathcal{E})$.

PROPOSITION 5.4
Let $\mathcal{E} = (0 \to \mathcal{T}_q \to E \to \mathbb{Z}[C_q] \to 0)$ be an extension defining a generalized Swan module E; then $\mathcal{H}^*(E) \cong \mathcal{G}^*(R(\delta(\mathcal{E}))$.

Proof
Decompose $E \cong R(\delta(\mathcal{E})) \oplus X$ as in Proposition 5.3. Then

$$\mathcal{H}^*(E) \cong \mathcal{H}^*\big(R\big(\delta(\mathcal{E})\big)\big) \oplus \mathcal{H}^*(X).$$

It follows from (4.8) that $\mathcal{H}^*(R(\delta(\mathcal{E})) = 0$ so that $\mathcal{H}^*(E) \cong \mathcal{H}^*(X)$. Thus, it suffices to show that $\mathcal{H}^*(X) \cong \mathcal{G}^*(R(\delta(\mathcal{E})))$. However, $R(\overline{\mathrm{supp}(\mathcal{E})})$ represents $D_1(X)$ by Proposition 5.2 and $\overline{\mathrm{supp}(\mathcal{E})} = \delta(\mathcal{E})$. Thus, $R(\delta(\mathcal{E}))$ represents $D_1(X)$, and hence $\mathcal{H}^*(X) \cong \mathcal{G}^*(R(\delta(\mathcal{E})))$ by Proposition 4.1. □

Clearly $\delta(\mathcal{E})$ is an invariant of the congruence class of the extension $\mathcal{E}$. In fact, it is also an invariant of the isomorphism class of the module E in the derived module category. Formally we have the following:

PROPOSITION 5.5

Let $E(1)$, $E(2)$ be generalized Swan modules; then

$$E(1) \cong_{\mathcal{D}\mathrm{er}} E(2) \quad \Longrightarrow \quad \delta\big(\mathcal{E}(1)\big) = \delta\big(\mathcal{E}(2)\big).$$

Proof

If $E(1) \cong_{\mathcal{D}\mathrm{er}} E(2)$, then for some projective modules $P(1)$, $P(2)$ we have

$$E(1) \oplus P(1) \cong E(2) \oplus P(2)$$

so that $\mathcal{H}^*(E(1) \oplus \mathcal{H}^*(P(1)) \cong \mathcal{H}^*(E(2) \oplus \mathcal{H}^*(P(2))$. As $P(1)$, $P(2)$ are projective, then $\mathcal{H}^*(P(1)) \cong \mathcal{H}^*(P(2)) = 0$, and so $\mathcal{H}^*(E(1)) \cong \mathcal{H}^*(E(2))$. By Proposition 4.1 it follows that $\mathcal{G}^*(\delta(\mathcal{E}(1))) \cong \mathcal{G}^*(\delta(\mathcal{E}(2)))$ so that, by Proposition 4.2, $\delta(\mathcal{E}(1)) = \delta(\mathcal{E}(2))$. □

From (1.5) we obtain the following special case of Proposition 5.5:

COROLLARY 5.1

For $k = 1, 2$, let $\mathcal{E}(k) = (0 \to \mathcal{T}_q \to E(k) \to \mathbb{Z}[C_q] \to 0)$ be extensions defining generalized Swan modules $E(1)$, $E(2)$; then

$$E(1) \sim E(2) \quad \Longrightarrow \quad \delta\big(\mathcal{E}(1)\big) = \delta\big(\mathcal{E}(2)\big).$$

6. Proof of Theorem B

In what follows, $\mathbb{F}_p$ will denote the field with p elements where p is an odd prime, and a will denote an integer in the range $1 \le a \le p-1$ chosen so that the residue class $[a] \in \mathbb{F}_p^*$ generates the multiplicative group $\mathbb{F}_p^*$. For each integer k in the range $1 \le k \le q$, we define elements $\upsilon_1^{(k)}, \ldots, \upsilon_q^{(k)}$ in $\underbrace{\mathbb{F}_p^* \times \cdots \times \mathbb{F}_p^*}_{q}$ as follows:

$$(\upsilon_j^{(k)})_r = \begin{cases} [a] & r = j \\ [a]^{-1} & r = k \\ 1 & r \notin \{j, k\}, \end{cases} \qquad (j < k);$$

$$(\upsilon_j^{(k)})_r = \begin{cases} [a]^{-1} & r = k \\ [a] & r = j \\ 1 & r \notin \{j, k\}, \end{cases} \qquad (k < j).$$

Moreover, we define $(\upsilon_k^{(k)})_r = 1$ for all r and denote by $U(k = \langle \upsilon_1^{(k)}, \ldots, \upsilon_q^{(k)} \rangle$ the subgroup of $\underbrace{\mathbb{F}_p^* \times \cdots \times \mathbb{F}_p^*}_{q}$ generated by $\upsilon_1^{(k)}, \ldots, \upsilon_q^{(k)}$. For $\lambda \in \mathbb{F}_p^{(q)}$ we define the *degeneracy* $\delta(\lambda)$ of $\lambda = (\lambda_1, \ldots, \lambda_q)$ by $\delta(\lambda) = \{i : 1 \le i \le q : \lambda_i = 0\}$.

For $\alpha \subset \{1, \ldots, q\}$, we define $\mathcal{H}(\alpha) \subset \mathbb{F}_p^{(q)}$ by $\mathcal{H}(\alpha) = \{\lambda \in \mathbb{F}_p^{(q)} : \delta(\lambda) = \alpha\}$. The proofs of the following two statements are straightforward:

(6.1) $\quad \mathcal{H}(\alpha)$ is stable under the action of $\underbrace{\mathbb{F}_p^* \times \cdots \times \mathbb{F}_p^*}_{q}$.

(6.2) $\quad$ If $k \in \alpha$, then $U(k)$ acts transitively on $\mathcal{H}(\alpha)$.

Recall the surjective ring homomorphisms $\nu : \mathcal{T}_q(A, \pi) \to \mathcal{T}_q(A/\pi)$ and $\varphi : \mathcal{T}_q(A/\pi) \to \underbrace{A/\pi \times \cdots \times A/\pi}_{q}$ defined in (2.8) and (2.9), respectively. Taking the composition gives a surjective ring homomorphism $\natural : \mathcal{T}_q(A, \pi) \to \underbrace{A/\pi \times \cdots \times A/\pi}_{q}$. In the present case, the inclusion $\mathbb{Z} \hookrightarrow A$ has the property that $\mathbb{Z} \cap (\pi) = (p)$ and $A/\pi = \mathbb{Z}/p = \mathbb{F}_p$. Thus, there is a commutative diagram of ring homomorphisms

$$\begin{tikzcd} \mathcal{T}_q(\mathbb{Z}, p) \arrow[rr, hook, "j"] \arrow[dr, "\natural_1"'] & & \mathcal{T}_q(A, \pi) \arrow[dl, "\natural_2"] \\ & \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{q} & \end{tikzcd}$$

in which $\natural_1$ and $\natural_2$ are surjective. Maintaining the previous choice of a in the range $1 \le a \le p$, then as a and p are coprime, appealing to Bezout's Theorem we can find integers b, d such that

(6.3) $$ad + pb = 1.$$

For k in the range $1 \le k \le q$, we define elements $\widehat{\upsilon}_1^{(k)}, \ldots, \widehat{\upsilon}_q^{(k)}$ in $\mathcal{T}_q(\mathbb{Z}, p)$ as follows:

$$\text{if } j < k \text{ then} \quad (\widehat{\upsilon}_j^{(k)})_{rs} = \begin{cases} a & r = s = j \\ -p & r = k, s = j \\ b & r = j, s = k \\ d & r = s = k \\ 1 & r = s, r \notin \{j, k\} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{if } k < j \text{ then} \quad (\widehat{\upsilon}_j^{(k)})_{rs} = \begin{cases} d & r = s = k \\ b & r = k, s = j \\ -p & r = j, s = k \\ a & r = s = j \\ 1 & r = s, r \notin \{j, k\} \\ 0 & \text{otherwise} \end{cases}$$

Moreover, we define $\upsilon_k^{(k)} = \mathrm{Id}_q$. Using row and column operations and appealing to (6.3), one sees easily that $\det(\widehat{\upsilon}_k^{(k)}) = 1$ for all j. Hence, we have

$$\widehat{\upsilon}_j^{(k)} \in \mathcal{T}_q(\mathbb{Z}, p)^*.$$

Under the homomorphism $\natural_1 : \mathcal{T}_q(\mathbb{Z}, p) \to \underbrace{\mathbb{F}_p \times \cdots \times \mathbb{F}_p}_{q}$, we see that

$$\natural_1(\widehat{\upsilon}_j^{(k)}) = \upsilon_j^{(k)} \tag{6.4}$$

Define $\widehat{U}(k) = \langle \widehat{\upsilon}_1^{(k)}, \dots, \widehat{\upsilon}_q^{(k)} \rangle \subset \mathcal{T}_q(\mathbb{Z}, p)^*$. As $\mathcal{T}_q(\mathbb{Z}, p)$ is a subring of $\mathcal{T}_q(A, \pi)$, we regard $\widehat{U}(k)$ as a subgroup of $\mathcal{T}_q(A, \pi)^*$. In consequence of (6.4), we see that

$$\natural_2\big(\widehat{U}(k)\big) = U(k). \tag{6.5}$$

For any subset $\alpha \subset \{1, \dots, q\}$, we define $\widetilde{\mathcal{H}}(\alpha) \subset \mathrm{Ext}^1(\mathbb{Z}[C_q], \mathcal{T}_q(A, \pi))$ by

$$\widetilde{\mathcal{H}}(\alpha) = \big\{\mathbf{c} = (c_i)_{1 \le i \le q} : \delta(\mathbf{c}) = \alpha\big\}.$$

It follows from (6.2) and (6.5) that

$$\widehat{U}(k) \text{ acts transitively on } \widetilde{\mathcal{H}}(\alpha) \text{ if } k \in \alpha. \tag{6.6}$$

To proceed we fix an extension

$$0 \to \bigoplus_{k=1}^{q} R(k) \to X \to \mathbb{Z}[C_q] \to 0. \tag{$\mathcal{X}$}$$

The corepresentation formula (1.9) then shows that

$$\mathrm{Ext}^1\big(\mathbb{Z}[C_q], \mathcal{T}_q(A, \pi)\big) \cong \mathrm{End}_{\mathcal{D}\mathrm{er}}\big(\mathcal{T}_q(A, \pi)\big) \cong \mathrm{End}_{\mathcal{D}\mathrm{er}}\Big(\bigoplus_{k=1}^{q} R(k)\Big). \tag{6.7}$$

Thus, the extension $\mathcal{X}$ is classified by a matrix

$$\mathbf{c} = (c_{ij})_{1 \le i, j \le q}; \qquad c_{ij} \in \mathrm{Hom}_{\mathcal{D}\mathrm{er}}\big(R(j), R(i)\big).$$

For each i, j choose a Λ-homomorphism $\gamma_{ij} \in \mathrm{Hom}_\Lambda(R(j), R(i))$ which represents c_{ij} after passing to the derived module category. Then $\gamma = (\gamma_{ij})_{1 \le i, j \le q}$ is a Λ-endomorphism of $\bigoplus_{k=1}^q R(k)$ which represents $\mathbf{c}$ in the derived module category. Let $\iota : \bigoplus_{k=1}^q R(k) \to \Lambda$ denote the inclusion; then X can be described as $X \cong X(\gamma)$ via the pushout diagram

$$\begin{array}{ccccccccc} 0 \to & \bigoplus_{k=1}^{q} R(k) & \xrightarrow{\iota} & \Lambda & \to & \mathbb{Z}[C_q] & \to 0 \\ & \downarrow \gamma & & \downarrow \natural & & \downarrow \mathrm{Id} & \\ 0 \to & \bigoplus_{k=1}^{q} R(k) & \to & X(\gamma) & \to & \mathbb{Z}[C_q] & \to 0 \end{array}$$

where $X(\gamma) = \varinjlim(\iota, \gamma)$. However, by (4.5), $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(R(j), R(i)) = 0$ if $j \neq i$ so that $c_{ij} = 0$ for $i \neq j$ and the classifying matrix $\mathbf{c}$ is diagonal

$$\mathbf{c} = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_q \end{pmatrix},$$

where $c_k = c_{kk} \in \mathrm{End}_{\mathcal{D}\mathrm{er}}(R(i)) \cong \mathbb{F}_p$. Making the identification

$$\bigoplus_{i=1}^{q} \mathrm{Ext}^1\big(\mathbb{Z}[C_q], R(i)\big) \overset{\simeq}{\longleftrightarrow} \underbrace{\mathbb{F}_p \times \cdots \times \mathbb{F}_p}_{q}.$$

Then, following Section 5, we associate with $\mathbf{c}$ its degeneracy $\delta(\mathbf{c}) = \{i \mid c_i = 0\}$. Now suppose given another such extension,

$$(\mathcal{X}') \qquad 0 \to \bigoplus_{i=1}^{q} R(i) \to X' \to \mathbb{Z}[C_q] \to 0,$$

parametrized by $\mathbf{c}' = (c_i')_{1 \le i \le q}$. It follows from (6.6) that

(6.8) If $k \in \delta(\mathbf{c}) = \delta(\mathbf{c}')$, then there exists $\widehat{\alpha} \in \widehat{U}(k)$ such that $\widehat{\alpha} \cdot \mathbf{c} = \mathbf{c}'$.

We come to the following, which is Theorem B of the Introduction.

THEOREM 6.1

Let X, X' be degenerate generalized Swan modules; then

$$X \cong X' \iff \delta(X) = \delta(X').$$

Proof

($\Longrightarrow$). Suppose that $X \cong X'$; then $X \cong_{\mathcal{D}\mathrm{er}} X'$. It follows from Proposition 5.5 that $\delta(X) = \delta(X')$.

($\Longleftarrow$) We may suppose that X, X' are described by extensions

$$\mathcal{X} = \big(0 \to \mathcal{T}_q \to X \to \mathbb{Z}[C_q] \to 0\big); \qquad \mathcal{X}' = \big(0 \to \mathcal{T}_q \to X' \to \mathbb{Z}[C_q] \to 0\big)$$

classified by $\mathbf{c}$, $\mathbf{c}'$ respectively. By the above discussion, X can be described as a pushout $X \cong X(\gamma)$

$$\begin{array}{ccccccccc} 0 \to & \mathcal{T}_q & \to & \Lambda & \to \mathbb{Z}[C_q] \to 0 \\ & \gamma \downarrow & & \widehat{\gamma} \downarrow & \mathrm{Id} \downarrow \\ 0 \to & \mathcal{T}_q & \to & X(\gamma) & \to \mathbb{Z}[C_q] \to 0 \end{array}$$

where γ is a Λ-endomorphism of $\mathcal{T}_q \cong \bigoplus_{k=1}^{q} R(k)$ which represents $\mathbf{c}$ on passage to the derived module category. Suppose that $\delta(X) = \delta(X') = \alpha$ where $\alpha \neq \{1, \ldots, q\}$. Choose $i \in \{1, \ldots, q\}$ such that $c_i = c_i' = 0$. Then by (6.8), there exists $\alpha \in \mathrm{Aut}_\Lambda(\mathcal{T}_q)$ such that $\alpha \cdot \mathbf{c} = \mathbf{c}'$.

Now $\alpha \circ \gamma : \mathcal{T}_q \to \mathcal{T}_q$ is a Λ-endomorphism which represents $\mathbf{c}' = \alpha \circ \mathbf{c}$ in the derived module category so that, on forming the pushout extension

$$\begin{array}{ccccccccc} 0 \to & \mathcal{T}_q & \to & \Lambda & \to \mathbb{Z}[C_q] \to 0 \\ & \alpha \circ \gamma \downarrow & & \widehat{\alpha \circ \gamma} \downarrow & \mathrm{Id} \downarrow \\ 0 \to & \mathcal{T}_q & \to & X(\alpha \circ \gamma) & \to \mathbb{Z}[C_q] \to 0 \end{array}$$

we see that $X(\alpha \circ \gamma) \cong X'$. There is now a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \to & \mathcal{T}_q & \to & X(\gamma) & \to \mathbb{Z}[C_q] \to 0 \\ & \alpha \downarrow & & \widehat{\alpha} \downarrow & \mathrm{Id} \downarrow \\ 0 \to & \mathcal{T}_q & \to & X(\alpha \circ \gamma) & \to \mathbb{Z}[C_q] \to 0 \end{array}$$

in which α and Id are isomorphisms. By the Five Lemma, $\widehat{\alpha} : X(\gamma) \to X(\alpha \circ \gamma)$ is also an isomorphism. The conclusion follows as $X \cong X(\gamma)$ and $X(\alpha \circ \gamma) \cong X'$. □

7. Proof of Theorem A

We first consider the notion of separating an extension as a direct sum. Given an extension $\mathcal{E} = (0 \to K \xrightarrow{i} X \xrightarrow{p} C \to 0)$ where X is the internal direct sum $X = X_1 \dotplus X_2$ of submodules X_1, X_2, we put $K_i = X_i \cap \mathrm{Ker}(p)$ so that we have extensions $\mathcal{E}_r = (0 \to K_r \xrightarrow{i_r} X_r \xrightarrow{p_r} p(X_r) \to 0)$. The direct sum extension $\mathcal{E}_1 \oplus \mathcal{E}_2$ is then defined by

$$\mathcal{E}_1 \oplus \mathcal{E}_2 = (0 \to K_1 \oplus K_2 \xrightarrow{\begin{pmatrix} i_1 & 0 \\ 0 & i_2 \end{pmatrix}} X_1 \oplus X_2 \xrightarrow{\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}} p(X_1) \oplus p(X_2) \to 0)$$

and there is a mapping of extensions $\mu : \mathcal{E}_1 \oplus \mathcal{E}_2 \to \mathcal{E}$

$$\begin{array}{c} \mathcal{E}_1 \oplus \mathcal{E}_2 \\ \\ \downarrow \mu \\ \\ \mathcal{E} \end{array} \quad = \left(\begin{array}{ccccccc} 0 \to K_1 \oplus K_2 & \stackrel{\left(\begin{smallmatrix} i_1 & 0 \\ 0 & i_2 \end{smallmatrix}\right)}{\longrightarrow} & X_1 \oplus X_2 & \stackrel{\left(\begin{smallmatrix} p_1 & 0 \\ 0 & p_2 \end{smallmatrix}\right)}{\longrightarrow} & p(X_1) \oplus p(X_2) \to 0 \\ \\ \downarrow \mu_K & & \downarrow \mu_X & & \downarrow \mu_C \\ \\ 0 \to \quad K & \stackrel{i}{\to} & X & \stackrel{p}{\to} & C \qquad \to 0 \end{array} \right)$$

where μ_K, μ_X, μ_C are given by the appropriate additions $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \mapsto x_1 + x_2$. With this it is straightforward to show that

$$\begin{aligned} \mu : \mathcal{E}_1 \oplus \mathcal{E}_2 \to \mathcal{E} \text{ is an isomorphism of extensions} \\ \iff p(X_1) \cap p(X_2) = 0. \end{aligned} \tag{7.1}$$

For the rest of this section we shall write $\mathcal{T} = \mathcal{T}_q(A, \pi)$ and $\mathcal{Q} = \mathbb{Z}[C_q]$ so that a generalized Swan module E is given by an extension

$$0 \to \mathcal{T} \to E \to \mathcal{Q} \to 0.$$

PROPOSITION 7.1

Let $\mathcal{E} = (0 \to \mathcal{T} \oplus \mathcal{T} \stackrel{i}{\to} X \stackrel{p}{\to} \mathcal{Q} \oplus \mathcal{Q} \to 0)$ *be an extension and suppose* X *is the internal direct sum* $X = X_1 \dot{+} X_2$ *of submodules* X_1, X_2. *If* X_1 *is a generalized Swan module, then* $p(X_1) \cap p(X_2) = 0$.

Proof

Let $\mathbb{F}$ be a field of characteristic zero and put $\mathcal{T}_{\mathbb{F}} = \mathcal{T} \otimes_{\mathbb{Z}} \mathbb{F}$, $\mathcal{Q}_{\mathbb{F}} = \mathcal{Q} \otimes_{\mathbb{Z}} \mathbb{F}$, $Y = X \otimes_{\mathbb{Z}} \mathbb{F}$, $Y_i = X_i \otimes_{\mathbb{Z}} \mathbb{F}$. By Maschke's Theorem, $Y \cong \mathcal{T}_{\mathbb{F}} \oplus \mathcal{T}_{\mathbb{F}} \oplus \mathcal{Q}_{\mathbb{F}} \oplus \mathcal{Q}_{\mathbb{F}}$. By hypothesis, X_1 occurs as an extension $(0 \to \mathcal{T} \to X_1 \to \mathcal{Q} \to 0)$ so that, again by Maschke's Theorem $Y_1 \cong \mathcal{T}_{\mathbb{F}} \oplus \mathcal{Q}_{\mathbb{F}}$. As $Y = Y_1 \dot{+} Y_2$, it follows from Wedderburn's Theorem that $Y_2 \cong \mathcal{T}_{\mathbb{F}} \oplus \mathcal{Q}_{\mathbb{F}}$. As $\mathrm{Hom}_\Lambda(\mathcal{T}, \mathcal{Q}) = 0$, then the restrictions $p : Y_i \to \mathcal{Q}_{\mathbb{F}} \oplus \mathcal{Q}_{\mathbb{F}}$ vanish on $\mathcal{T}_{\mathbb{F}}$ and hence $\dim_{\mathbb{F}}(p(Y_i) \le \dim_{\mathbb{F}}(\mathcal{Q}_{\mathbb{F}}) = q$. However, $\dim_{\mathbb{F}}(p(Y_1) + p(Y_2)) = \dim_{\mathbb{F}}(p(Y)) = \dim_{\mathbb{F}}(\mathcal{Q}_{\mathbb{F}} \oplus \mathcal{Q}_{\mathbb{F}}) = 2q$. Thus,

$$\dim_{\mathbb{F}}(p(Y_1) + \dim_{\mathbb{F}}(p(Y_2) \le \dim_{\mathbb{F}}\big(p(Y_1) + p(Y_2)\big).$$

and so $\dim_{\mathbb{F}}(p(Y_1) \cap p(Y_2)) = 0$. Hence, $p(X_1) \cap p(X_2) = 0$. $\square$

Suppose given an extension $\mathcal{E} = (0 \to \mathcal{T} \oplus \mathcal{T} \stackrel{i}{\to} X \stackrel{p}{\to} \mathcal{Q} \oplus \mathcal{Q} \to 0)$ where X is the internal direct sum $X = X_1 \dot{+} X_2$ of submodules X_1, X_2, and consider the extensions $\mathcal{E}_1, \mathcal{E}_2$ as defined above. With this notation we have

PROPOSITION 7.2

If X_1 *is a generalized Swan module, then* X_2 *is also a generalized Swan module.*

Proof

As X_1 is a generalized Swan module, then $p(X_1) \cap p(X_2) = 0$ by Proposition 7.1

above. Consider the extension which defines X_1 as a generalized Swan module

$$\mathcal{E}' = (0 \to \mathcal{T} \xrightarrow{i'} X_1 \xrightarrow{p'} \mathcal{Q} \to 0)$$

and compare this with the exact sequence $\mathcal{E}_1 = (0 \to K_1 \xrightarrow{i_1} X_1 \xrightarrow{p_1} p(X_1) \to 0)$ via the diagram

$$\mathcal{D}_+ \quad \begin{cases} 0 \to K_1 \overset{i_1}{\hookrightarrow} X_1 \overset{p}{\to} p(X_1) \to 0 \\ \qquad\qquad \downarrow \mathrm{Id} \\ 0 \to \mathcal{T} \overset{i'}{\hookrightarrow} X_1 \overset{p'}{\to} \mathcal{Q} \to 0 \end{cases}$$

As K_1 is a submodule of $\mathcal{T} \oplus \mathcal{T}$, then $p' \circ i_1 = 0$ by (2.15). Thus, we may complete $\mathcal{D}_+$ to a commutative diagram

$$\widetilde{\mathcal{D}}_+ \quad \begin{cases} 0 \to & K_1 & \overset{i_1}{\hookrightarrow} & X_1 & \overset{p}{\to} & p(X_1) & \to 0 \\ & \downarrow h_- & & \downarrow \mathrm{Id} & & \downarrow h_+ & \\ 0 \to & \mathcal{T} & \overset{i'}{\hookrightarrow} & X_1 & \overset{p'}{\to} & \mathcal{Q} & \to 0 \end{cases}$$

in which h_+ is necessarily surjective. Note that $\mathrm{Hom}_\Lambda(\mathcal{T}, \mathcal{Q}) = 0$ by (2.14). As $p(X_1) \subset \mathcal{Q} \oplus \mathcal{Q}$, then $\mathrm{Hom}_\Lambda(\mathcal{T}, p(X_1)) = 0$. In particular, $p \circ i' = 0$ so that, in similar fashion to the above, we obtain a diagram

$$\widetilde{\mathcal{D}}_- \quad \begin{cases} 0 \to & \mathcal{T} & \overset{i'}{\hookrightarrow} & X_1 & \overset{p'}{\to} & \mathcal{Q} & \to 0 \\ & \downarrow g_- & & \downarrow \mathrm{Id} & & \downarrow g_+ & \\ 0 \to & K_1 & \overset{i_1}{\hookrightarrow} & X_1 & \overset{p}{\to} & p(X_1) & \to 0 \end{cases}$$

Composing $\widetilde{\mathcal{D}}_- \circ \widetilde{\mathcal{D}}_+$ we obtain a commutative diagram

$$\begin{cases} 0 \to & K_1 & \overset{i_1}{\hookrightarrow} & X_1 & \overset{p}{\to} & p(X_1) & \to 0 \\ & \downarrow g_- \circ h_- & & \downarrow \mathrm{Id} & & \downarrow g_+ \circ h_+ & \\ 0 \to & K_1 & \overset{i_1}{\hookrightarrow} & X_1 & \overset{p}{\to} & p(X_1) & \to 0 \end{cases}$$

from which it follows that $g_+ \circ h_+ = \mathrm{Id}$. Thus, h_+ is also injective and so gives an isomorphism $h_+ : p(X_1) \xrightarrow{\simeq} Q$. Extending $\widetilde{\mathcal{D}}_+$ one place to the left by zeroes and applying the Five Lemma, we see that $h_- : K_1 \to \mathcal{T}$ is also an isomorphism. Now consider the exact sequence

$$\mathcal{E}_2 = \big(0 \to K_2 \xrightarrow{i_2} X_2 \xrightarrow{p_2} p(X_2) \to 0\big).$$

We have $K_1 \dot{+} K_2 \cong \mathcal{T} \oplus \mathcal{T}$ and $K_1 \cong \mathcal{T}$. Hence, $K_2 \cong \mathcal{T}$ by (3.4). Also, $p(X_1) \dot{+} p(X_2) \cong \mathcal{Q} \oplus \mathcal{Q}$ and $p(X_1) \cong \mathcal{Q}$. Hence, $p(X_2) \cong \mathcal{Q}$ by (3.6). Thus, X_2 occurs in an exact sequence $0 \to \mathcal{T} \to X_2 \to \mathcal{Q} \to 0$ and so X_2 is a generalized Swan module. □

COROLLARY 7.1

Let X, S be Λ-lattices such that $X \oplus \Lambda \cong S \oplus \Lambda$. If X is a generalized Swan module, then so also is S and $\delta(S) = \delta(X)$.

Proof

As both X and Λ are generalized Swan modules, then there is an extension

$$0 \to \mathcal{T} \oplus \mathcal{T} \to X \oplus \Lambda \to Q \oplus Q \to 0.$$

As $S \oplus \Lambda \cong X \oplus \Lambda$, then there is an extension

$$0 \to \mathcal{T} \oplus \mathcal{T} \to S \oplus \Lambda \to Q \oplus Q \to 0.$$

Again, as Λ is a generalized Swan module, it follows from Proposition 7.2 that S is a generalized Swan module. As $X \sim S$, the conclusion $\delta(S) = \delta(X)$ follows from Corollary 5.1. □

We come to the following which is Theorem A of the Introduction.

THEOREM 7.1

Let X be a generalized Swan module. If X' is a Λ-lattice such that $X' \oplus \Lambda \cong X \oplus \Lambda$, then $X' \cong X$.

Proof

First suppose that $\delta(X) = \emptyset$. Then X is projective by Proposition 5.1. Hence, X' is also projective and the conclusion follows from the Swan–Jacobinski Theorem as Λ satisfies the Eichler condition. In the general case, $\delta(X) \neq \emptyset$. Then by Corollary 7.1, X' is also a generalized Swan module and $\delta(X') = \delta(X)$. It now follows from Theorem 6.1 that $X' \cong X$. □

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