A non-geometric representation of the Dirac equation in curved spacetime

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We write the Dirac equation in curved 4-dimensional Lorentzian spacetime using concepts from the analysis of partial differential equations as opposed to geometric concepts.

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1. Playing Field

Let M be a connected 4-manifold without boundary. We will work with 2-columns $v : M \to \mathbb{C}^2$ of complex-valued half-densities (a half-density is a quantity which transforms as the square root of a density under changes of local coordinates). The inner product on such 2-columns is defined as $\langle v, w\rangle := \int_M w^* v \, dx$, where $x = (x^1, x^2, x^3, x^4)$ are local coordinates, $dx = dx^1 dx^2 dx^3 dx^4$ and the star stands for Hermitian conjugation.

Let L be a formally self-adjoint first order linear differential operator acting on 2-columns of complex-valued half-densities. Our initial objective will be to examine the geometric content of the operator L. In order to pursue this objective we first need to provide an invariant analytic description of the operator.

In local coordinates our operator reads

$$L = F^\alpha(x) \frac{\partial}{\partial x^\alpha} + G(x), \tag{1}$$

where $F^\alpha(x)$, $\alpha = 1, 2, 3, 4$, and $G(x)$ are some 2×2 matrix-functions. The principal and subprincipal symbols of the operator L are defined as

$$L_{\mathrm{prin}}(x, p) := iF^\alpha(x)\, p_\alpha\,, \tag{2}$$

$$L_{\mathrm{sub}}(x) := G(x) + \frac{i}{2}(L_{\mathrm{prin}})_{x^\alpha p_\alpha}(x)\,, \tag{3}$$

where $p = (p_1, p_2, p_3, p_4)$ is the dual variable (momentum); see Ref. 3. The principal and subprincipal symbols are invariantly defined 2×2 Hermitian matrix-functions on T^*M and M respectively which uniquely determine the operator L.

Further on in this paper we assume that the principal symbol of our operator satisfies the following non-degeneracy condition:

$$L_{\mathrm{prin}}(x, p) \neq 0, \qquad \forall (x, p) \in T^*M \setminus \{0\}. \tag{4}$$

Condition (4) means that the elements of the 2×2 matrix-function $L_{\mathrm{prin}}(x, p)$ do not vanish simultaneously for any $x \in M$ and any nonzero momentum p.

2. Lorentzian Metric and Orthonormal Frame

Observe that the determinant of the principal symbol is a quadratic form in the dual variable (momentum) p:

$$\det L_{\text{prin}}(x,p) = -g^{\alpha\beta}(x)\, p_\alpha p_\beta\,. \tag{5}$$

We interpret the real coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha,\beta = 1,2,3,4$, appearing in formula (5) as components of a (contravariant) metric tensor.

The following result was established in Ref. 2.

Lemma 2.1. *Our metric is Lorentzian, i.e. it has three positive eigenvalues and one negative eigenvalue.*

Furthermore, the principal symbol of our operator defines an orthonormal frame ${e_j}^\alpha(x)$. Here the Latin index $j = 1,2,3,4$ enumerates the vector fields, the Greek index $\alpha = 1,2,3,4$ enumerates the components of a given vector e_j and orthonormality is understood in the Lorentzian sense:

$$g_{\alpha\beta}\, {e_j}^\alpha {e_k}^\beta = \begin{cases} 0 & \text{if} \quad j \neq k, \\ 1 & \text{if} \quad j = k \neq 4, \\ -1 & \text{if} \quad j = k = 4. \end{cases} \tag{6}$$

The orthonormal frame is recovered from the principal symbol as follows. Decomposing the principal symbol with respect to the standard basis

$$s^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the real vector space of 2×2 Hermitian matrices, we get $L_{\text{prin}}(x,p) = s^j c_j(x,p)$. Each coefficient $c_j(x,p)$ is linear in momentum p, so $c_j(x,p) = {e_j}^\alpha(x)\, p_\alpha$.

The existence of an orthonormal frame implies that our manifold M is parallelizable. We see that our analytic non-degeneracy condition (4) has far reaching geometric consequences.

3. Gauge Transformations and Covariant Subprincipal Symbol

Let us consider the action (variational functional) $\int_M v^*(Lv)\, dx$ associated with our operator. Take an arbitrary smooth matrix-function

$$R : M \to \mathrm{SL}(2,\mathbb{C}) \tag{7}$$

and consider the following transformation of our 2-column of unknowns:

$$v \mapsto Rv. \tag{8}$$

We interpret (8) as a gauge transformation because we are looking here at a change of basis in our vector space of unknowns $v : M \to \mathbb{C}^2$.

The transformation (8) of the 2-column v induces the following transformation of the action: $\int_M v^*(Lv)\,dx \;\mapsto\; \int_M v^*(R^*LRv)\,dx$. This means that our 2×2 differential operator L experiences the transformation

$$L \mapsto R^*LR\,. \tag{9}$$

This section is dedicated to the analysis of the transformation (9).

Remark 3.1. We chose to restrict our analysis to matrix-functions $R(x)$ of determinant one, see formula (7), because we want to preserve our Lorentzian metric defined in accordance with formula (5).

Remark 3.2. In non-relativistic theory one normally looks at the transformation

$$L \mapsto R^{-1}LR \tag{10}$$

rather than at (9). The reason we chose to go along with (9) is that we are thinking in terms of actions and corresponding Euler–Lagrange equations rather than operators as such. We believe that this point of view makes more sense in the relativistic setting. If one were consistent in promoting such a point of view, then one would have had to deal with actions throughout the paper rather than with operators. We did not adopt this 'consistent' approach because this would have made the paper difficult to read. Therefore, throughout the paper we use the concept of an operator, having in mind that we are really interested in the action and corresponding Euler–Lagrange equation.

Remark 3.3. The transformations (9) and (10) coincide if the matrix-function $R(x)$ is special unitary. Applying special unitary transformations is natural in the non-relativistic 3-dimensional setting when dealing with an elliptic system, see Ref. 1, but in the relativistic 4-dimensional setting when dealing with a hyperbolic system special unitary transformations are too restrictive.

The transformation (9) of the differential operator L induces the following transformations of its principal (2) and subprincipal (3) symbols:

$$L_{\mathrm{prin}} \mapsto R^*L_{\mathrm{prin}}R\,, \tag{11}$$

$$L_{\mathrm{sub}} \mapsto R^*L_{\mathrm{sub}}R + \frac{i}{2}\left(R^*_{x^\alpha}(L_{\mathrm{prin}})_{p_\alpha}R - R^*(L_{\mathrm{prin}})_{p_\alpha}R_{x^\alpha}\right). \tag{12}$$

Comparing formulae (11) and (12) we see that, unlike the principal symbol, the subprincipal symbol does not transform in a covariant fashion due to the appearance of terms with the gradient of the matrix-function $R(x)$.

It turns out that one can overcome the non-covariance in (12) by introducing the *covariant subprincipal symbol* $L_{\mathrm{csub}}(x)$ in accordance with formula

$$L_{\mathrm{csub}} := L_{\mathrm{sub}} + \frac{i}{16}\,g_{\alpha\beta}\{L_{\mathrm{prin}}, \operatorname{adj} L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta}\,, \tag{13}$$

where $\{F, G, H\} := F_{x^\alpha} G H_{p_\alpha} - F_{p_\alpha} G H_{x^\alpha}$ is the generalised Poisson bracket on matrix-functions and adj is the operator of matrix adjugation

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =: \operatorname{adj} F \tag{14}$$

from elementary linear algebra.

The following result was established in Ref. 2.

Lemma 3.1. *The transformation* (9) *of the differential operator induces the transformation* $L_{\text{csub}} \mapsto R^* L_{\text{csub}} R$ *of its covariant subprincipal symbol.*

Comparing formulae (3) and (13) we see that the standard subprincipal symbol and covariant subprincipal symbol have the same structure, only the covariant subprincipal symbol has a second correction term designed to 'take care' of special linear transformations in the vector space of unknowns $v : M \to \mathbb{C}^2$. The standard subprincipal symbol (3) is invariant under changes of local coordinates (its elements behave as scalars), whereas the covariant subprincipal symbol (13) retains this feature but gains an extra $\mathrm{SL}(2, \mathbb{C})$ covariance property. In other words, the covariant subprincipal symbol (13) behaves 'nicely' under a wider group of transformations.

4. Electromagnetic Covector Potential

The covariant subprincipal symbol can be uniquely represented in the form

$$L_{\text{csub}}(x) = L_{\text{prin}}(x, A(x)), \tag{15}$$

where $A = (A_1, A_2, A_3, A_4)$ is some real-valued covector field. We interpret this covector field as the electromagnetic covector potential.

Lemma 3.1 and formulae (11) and (15) tell us that the electromagnetic covector potential is invariant under gauge transformations (9).

5. Adjugate Operator

Definition 5.1. The adjugate of a formally self-adjoint non-degenerate first order 2×2 linear differential operator L is the formally self-adjoint non-degenerate first order 2×2 linear differential operator $\operatorname{Adj} L$ whose principal and covariant subprincipal symbols are matrix adjugates of those of the operator L.

We denote matrix adjugation by adj, see formula (14), and operator adjugation by Adj. Of course, the coefficients of the adjugate operator can be written down explicitly in local coordinates via the coefficients of the original operator (1), see Ref. 2 for details.

Applying the analysis from Sections 2–4 to the differential operator $\operatorname{Adj} L$ it is easy to see that the metric and electromagnetic covector potential encoded within the operator $\operatorname{Adj} L$ are the same as in the original operator L. Thus, the metric and electromagnetic covector potential are invariant under operator adjugation.

It also easy to see that $\operatorname{Adj} \operatorname{Adj} L = L$, so operator adjugation is an involution.

6. Main Result

We define the Dirac operator as the differential operator

$$D := \begin{pmatrix} L & mI \\ mI & \operatorname{Adj} L \end{pmatrix} \tag{16}$$

acting on 4-columns $\psi = \begin{pmatrix} v_1 & v_2 & w_1 & w_2 \end{pmatrix}^T$ of complex-valued half-densities. Here m is the electron mass and I is the 2×2 identity matrix.

The 'traditional' Dirac operator D_{trad} is written down in Appendix A of Ref. 2 and acts on bispinor fields $\psi_{\text{trad}} = \begin{pmatrix} \xi^1 & \xi^2 & \eta_{\dot{1}} & \eta_{\dot{2}} \end{pmatrix}^T$. Here we assume, without loss of generality, that the orthonormal frame used in the construction of the operator D_{trad} is the one from Section 2.

Our main result is the following theorem established in Ref. 2.

Theorem 6.1. *The two operators, our analytically defined Dirac operator* (16) *and geometrically defined Dirac operator* D_{trad}*, are related by the formula*

$$D = |\det g_{\kappa\lambda}|^{1/4} \, D_{\text{trad}} \, |\det g_{\mu\nu}|^{-1/4} \,. \tag{17}$$

Consider now the two Dirac equations

$$D\psi = 0, \tag{18}$$

$$D_{\text{trad}} \psi_{\text{trad}} = 0. \tag{19}$$

Formula (17) implies that the solutions of equations (18) and (19) differ only by a prescribed scaling factor: $\psi = |\det g_{\mu\nu}|^{1/4} \, \psi_{\text{trad}}$. This means that for all practical purposes equations (18) and (19) are equivalent.

7. Spin Structure

Let us consider all possible formally self-adjoint non-degenerate first order 2×2 linear differential operators L corresponding, in the sense of formula (5), to the prescribed Lorentzian metric. In this section our aim is to classify all such operators L.

Let us fix a reference operator $\mathbf{L}$ and let $\mathbf{e}_j$ be the corresponding orthonormal frame (see Section 2). Let L be another operator and let e_j be the corresponding orthonormal frame. We define the following two real-valued scalar fields

$$\mathbf{c}(L) := -\frac{1}{4!} \, (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4)_{\kappa\lambda\mu\nu} \, (e_1 \wedge e_2 \wedge e_3 \wedge e_4)^{\kappa\lambda\mu\nu} \,, \quad \mathbf{t}(L) := -\,\mathbf{e}_{4\alpha} \, {e_4}^{\alpha} \,.$$

Observe that these scalar fields do not vanish; in fact, $\mathbf{c}(L)$ can take only two values, $+1$ or -1. This observation gives us a primary classification of operators L into four classes determined by the signs of $\mathbf{c}(L)$ and $\mathbf{t}(L)$. The four classes correspond to the four connected components of the Lorentz group.

Note that

$$\mathbf{c}(-L) = \mathbf{c}(L), \qquad \mathbf{t}(-L) = -\mathbf{t}(L),$$
$$\mathbf{c}(\operatorname{Adj} L) = -\mathbf{c}(L), \qquad \mathbf{t}(\operatorname{Adj} L) = \mathbf{t}(L),$$

which means that by applying the transformations $L \mapsto -L$ and $L \mapsto \operatorname{Adj} L$ to a given operator L one can reach all four classes of our primary classification.

Further on we work with operators L such that $\mathbf{c}(L) > 0$ and $\mathbf{t}(L) > 0$.

We say that the operators L and $\tilde{L}$ are equivalent if there exists a smooth matrix-function (7) such that $\tilde{L}_{\mathrm{prin}} = R^* L_{\mathrm{prin}} R$. The equivalence classes of operators obtained this way are called *spin structures*.

The above 4-dimensional Lorentzian definition of spin structure is an extension of the 3-dimensional Riemannian definition from Ref. 1. The difference is that we have now dropped the condition $\operatorname{tr} L_{\mathrm{prin}}(x,p) = 0$, replaced the ellipticity condition by the weaker non-degeneracy condition (4) and extended our group of transformations from special unitary to special linear.

One would hope that for a connected Lorentzian 4-manifold admitting a global orthonormal frame (see (6) for definition of orthonormality) our analytic definition of spin structure would be equivalent to the traditional geometric one. Unfortunately, we do not currently have a rigorous proof of equivalence in the 4-dimensional Lorentzian setting.

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References

1. Z. Avetisyan, Y.-L. Fang and D. Vassiliev, `arXiv:1512.06281` (2015).
2. Y.-L. Fang and D. Vassiliev, *J. Phys. A* **48**, 165203 (2015).
3. Yu. Safarov and D. Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators* (American Mathematical Society, Providence, 1997, 1998).