BOUNDARY ANGULAR DERIVATIVES OF GENERALIZED SCHUR FUNCTIONS

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Abstract. Characterization of generalized Schur functions in terms of their Taylor coefficients was established by M. G. Krein and H. Langer in [14]. We establich a boundary analog of this characterization.

1. Introduction

Generalized Schur functions are the functions of the form

$$f(z)=\frac{s(z)}{b(z)}, \tag{1.1}$$

where the numerator s is an analytic function mapping the open unit disk $\mathbb{D}$ into the closed unit disk (i.e., s is a Schur function) and where the denominator b is a finite Blaschke product. Such functions appeared first in [16] in the interpolation context and were studied later in [13, 14]. In what follows, we will write $\mathcal{S}$, $\mathcal{GS}$ and $\mathcal{FB}$ for the set of Schur functions, the set of generalized Schur functions and the set of finite Blaschke products, respectively. Formula (1.1) is called the Krein-Langer representation of a generalized Schur function f (see [13]); the entries s and b are defined by f uniquely up to a unimodular constant provided they have no common zeroes. Via nontangential boundary limits, the $\mathcal{GS}$-functions can be identified with the functions from the closed unit ball of $L^{\infty}(\mathbb{T})$ which admit meromorphic continuation inside the unit disk with a finite total pole multiplicity. The class $\mathcal{GS}$ can be alternatively defined as the class of functions f meromorphic on $\mathbb{D}$ and such that the associated kernel

$$K_f(z,\zeta):=\frac{1-f(z)\overline{f(\zeta)}}{1-z\bar{\zeta}}$$

has finitely many negative squares on $\rho(f)$, the domain of analyticity of f. A consequence of this characterization is that there exists an integer $\kappa\geq 0$ such that for every choice of an integer $n>0$ and a point $z\in\rho(f)$, the

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Hermitian matrix

$$\mathbf{P}_n^f(z) = \left[\frac{1}{\ell!r!}\frac{\partial^{\ell+r}}{\partial z^\ell \partial \bar{z}^r}\frac{1-|f(z)|^2}{1-|z|^2}\bigg|_{z=\zeta}\right]_{\ell,r=0}^{n-1} \tag{1.2}$$

which will be referred to as to the *Schwarz-Pick matrix*, has at most κ negative eigenvalues counted with multiplicities. This number κ turns out to be equal to the total pole multiplicity of f, i.e., to the degree of the denominator b in the coprime Krein-Langer representation (1.1). In what follows, we will denote by $\pi(P)$, $\nu(P)$ and $\delta(P)$ respectively the numbers of positive, negative and zero eigenvalues, counted with multiplicities, of a Hermitian matrix P. Straightforward differentiation in (1.2) gives explicit formuals

$$\begin{aligned}\Big[\mathbf{P}_n^f(z)\Big]_{\ell,r} = & \sum_{s=0}^{\min\{\ell,r\}} \frac{(\ell+r-s)!}{(\ell-s)!s!(r-s)!}\frac{z^{r-s}\bar{z}^{\ell-s}}{(1-|z|^2)^{\ell+r-s+1}} \\ & -\sum_{\alpha=0}^{\ell}\sum_{\beta=0}^{r}\sum_{s=0}^{\min\{\alpha,\beta\}}\frac{(\alpha+\beta-s)!}{(\alpha-s)!s!(\beta-s)!}\frac{z_i^{\beta-s}\bar{z}^{\alpha-s}f_{\ell-\alpha}(z)\overline{f_{r-\beta}(z)}}{(1-|z|^2)^{\alpha+\beta-s+1}}\end{aligned} \tag{1.3}$$

for the entries of $\mathbf{P}_n^f(z)$ in terms of Taylor coefficients $f_j(z) := f^{(j)}(z)/j!$ and the uniform bound $\nu(\mathbf{P}_n^f(z)) \le \kappa$ (with actual equality $\nu(\mathbf{P}_n^f(z)) = \kappa$ if n is large enough) eventually leads to a characterization of generalized Schur functions in terms of their Taylor coefficients (see [14]). The objective of this paper is to establish a similar characterization in the boundary context where the ambient point z is moved to units circle $\mathbb{T}$, the boundary of $\mathbb{D}$, and where the Taylor coefficients at z are replaced by the boundary limits $f_j = \lim_{z\to t_0}\frac{f^{(j)}(z)}{j!}$. In contrast to the interior case, the boundary limits f_j's may not exist; however, if the limit f_j exists, the limits f_k also exists for all $k = 0,\dots,j-1$. We therefore, distinguish two cases: the finite (truncated) problem $\mathbf{P}_N$ and the infinite problem $\mathbf{P}_\infty$.

Problem $\mathbf{P}_N$: *Given a point $t_0 \in \mathbb{T}$ and given $N < \infty$ complex numbers $f_0,\dots,f_N$, find a function $f \in \mathcal{GS}$ which admits the asymptotic expansion*

$$f(z) = f_0 + f_1(z-t_0) + \ldots + f_N(z-t_0)^N + o(|z-t_0|^N) \tag{1.4}$$

as z tends to t_0 nontangentially.

Problem $\mathbf{P}_\infty$: *Given a point $t_0 \in \mathbb{T}$ and given a complex sequence $\{f_i\}_{i\ge 0}$, find a function $f \in \mathcal{GS}$ which admits asymptotic expansions* (1.4) *for every $N \ge 0$.*

Remark 1.1. It is known that condition (1.4) holds if and only if the first $N+1$ nontangential derivatives of f exist at t_0 with values

$$\lim_{z \widehat{\to} t_0}\frac{f^{(j)}(z)}{j!} = f_j, \quad \text{for } j = 0,1,\dots,N. \tag{1.5}$$

Here and in what follows we write $z\widehat{\rightarrow}t_0$ if a point z tends to a boundary point t_0 nontangentially.

To present the answers to the above problems we first introduce some needed definitions and notation. Given a sequence $\mathbf{f} = \{f_i\}_{i=0}^N$ (with $N \leq \infty$), we define the lower triangular toeplitz matrix $\mathbb{U}_n^{\mathbf{f}}$ and the Hankel matrix $\mathbb{H}_n^{\mathbf{f}}$ by

$$\mathbb{U}_n^{\mathbf{f}} = \begin{bmatrix} f_0 & 0 & \dots & 0 \\ f_1 & f_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ f_{n-1} & \dots & f_1 & f_0 \end{bmatrix}, \quad \mathbb{H}_n^{\mathbf{f}} = \begin{bmatrix} f_1 & f_2 & \dots & f_n \\ f_2 & f_3 & \dots & f_{n+1} \\ \vdots & \vdots & & \vdots \\ f_n & f_{n+1} & \dots & f_{2n-1} \end{bmatrix} \tag{1.6}$$

for every appropriate integer $n \geq 1$ (i.e., for every $n \leq N+1$ in the first formula and for every $n \leq (N+1)/2$ in the second). Given a point $t_0 \in \mathbb{T}$, we introduce the upper-triangular matrix

$$\mathbf{\Psi}_n(t_0) = \begin{bmatrix} t_0 & -t_0^2 & \cdots & (-1)^{n-1}\binom{n-1}{0}t_0^n \\ 0 & -t_0^3 & \cdots & (-1)^{n-1}\binom{n-1}{1}t_0^{n+1} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & (-1)^{n-1}\binom{n-1}{n-1}t_0^{2n-1} \end{bmatrix} \tag{1.7}$$

with the entries

$$\Psi_{j\ell} = \begin{cases} 0, & \text{if } \ j > \ell, \\ (-1)^{\ell-1}\binom{\ell-1}{j-1}t_0^{\ell+j-1}, & \text{if } \ j \leq \ell, \end{cases} \qquad (j,\ell = 1,\dots,n), \tag{1.8}$$

and finally, for every $n \leq (N+1)/2$, we introduce the structured matrix

$$\mathbb{P}_n^{\mathbf{f}} = \left[p_{ij}^{\mathbf{f}}\right]_{i,j=1}^n = \mathbb{H}_n^{\mathbf{f}}\mathbf{\Psi}_n(t_0)\mathbb{U}_n^{\mathbf{f}*} \tag{1.9}$$

with the entries (as it follows from (1.6)–(1.8))

$$p_{ij}^{\mathbf{f}} = \sum_{r=1}^{j}\left(\sum_{\ell=1}^{r} f_{i+\ell-1}\Psi_{\ell r}\right)\overline{f_{j-r}}. \tag{1.10}$$

Since the factors $\mathbf{\Psi}_n(t_0)$ and $\mathbb{U}_n^{\mathbf{f}*}$ in (1.9) are upper-triangular, it follows that $\mathbb{P}_k^{\mathbf{f}}$ is the leading submatrix of $\mathbb{P}_n^{\mathbf{f}}$ for every $k < n$. Although the matrix $\mathbb{P}_n^{\mathbf{f}}$ defined in (1.9) does not have to be Hermitian, it then follows that if it is Hermitian, then the matrices $\mathbb{P}_k^{\mathbf{f}}$ are Hermitian for all $k < n$. We thus may introduce the quantity $n_0 \in \mathbb{N} \cup \{\infty\}$ (the size of the maximal Hermitan matrix $\mathbb{P}_n^{\mathbf{f}}$) by

$$n_0 = \begin{cases} 0, & \text{if } \mathbb{P}_1^{\mathbf{f}} = f_1 t_0 \overline{f_0} \notin \mathbb{R}, \\ \sup\limits_{1 \leq k \leq (N+1)/2} \left\{k : \ \mathbb{P}_k^{\mathbf{f}} = (\mathbb{P}_k^{\mathbf{f}})^*\right\}, & \text{otherwise}, \end{cases} \tag{1.11}$$

with the convention that $n_0=\infty$ if the matrices $\mathbb{P}_k^{\mathbf{f}}$ are Hermitian for all $k\ge 1$. We also observe that formula (1.10) defines the numbers $p_{ij}^{\mathbf{f}}$ in terms of $\mathbf{f}=\{f_0,\dots,f_N\}$ for every pair of indices (i,j) subject to $i+j\le N+1$. In particular, if $n\le N/2$, one can define via this formula the numbers $p_{n+1,n}^{\mathbf{f}}$ and $p_{n,n+1}^{\mathbf{f}}$ and therefore, the number

$$\gamma_n := t_0\cdot\left(p_{n+1,n}^{\mathbf{f}}-\overline{p_{n,n+1}^{\mathbf{f}}}\right). \tag{1.12}$$

Two theorems below are the main results of the paper.

Theorem 1.2. *Let $t_0\in\mathbb{T}$ and $\mathbf{f}=\{f_0,\dots,f_N\}$ ($1\le N<\infty$) be given. Let the integer n_0 be defined as in* (1.11) *and, in case $0<n_0\le N/2$, let γ_{n_0} be given by* (1.12). *The problem $\mathbf{P}_N$ has a solution if and only if one of the following holds:*

(1) $|f_0|<1$;
(2) $|f_0|=1, \quad n_0=(N+1)/2$;
(3) $|f_0|=1, \quad n_0=N/2, \quad \gamma_{n_0}\ge 0$:
(4) $|f_0|=1, \quad 0<n_0<N/2, \quad \gamma_{n_0}>0$.

Whenever the problem is solvable, it has infinitely many rational solutions.

Theorem 1.3. *Let $t_0\in\mathbb{T}$ and $\mathbf{f}=\{f_i\}_{i\ge 0}$ be given. Let n_0 be defined as in* (1.11) *and, in case $0<n_0<\infty$, let γ_{n_0} be the number given by* (1.12). *The problem $\mathbf{P}_\infty$ has a solution if and only if one of the following holds:*

(1) $|f_0|<1$;
(2) $|f_0|=1, \quad n_0<\infty, \quad \gamma_{n_0}>0$.
(3) $|f_0|=1, \quad n_0=\infty, \quad \nu(\mathbb{P}_n^{\mathbf{f}})=\kappa$ *for all large n and some $\kappa<\infty$.*

The problem may have a unique solution only in case (3).

The paper is organized as follows. In Section 2 we present the proof of Theorem 1.2 based on recent results [5] on Schur-class interpolation. The proof of Theorem 1.3 is given in Section 3, at the end of which we also discuss the possible determinacy of the problem.

2. The truncated problem

Since the boundary values of generalized Schur functions cannot exceed one in modulus, the condition $|f_0|\le 1$ is necessary for the problem $\mathbf{P}_N$ to have a solution. On the other hand, the condition $|f_0|<1$ is sufficient: in this case there are infinitely many Schur functions solving the problem (see e.g., [2]). It remains to consider a more subtle case where f_0 is unimodular.

Since every function $f\in\mathcal{GS}$ can be written in the form (1.1) and since the denominator $b\in\mathcal{FB}$ is analytic on $\mathbb{D}$, it is readily seen that the limits in (1.5) exist if and only if the similar limits for the numerator s exist and satisfy the convolution equalities

$$\lim_{z\widehat{\to}t_0}\frac{s^{(j)}(z)}{j!}=s_j:=\sum_{\ell=0}^{j}b_\ell f_{j-\ell}\quad\text{for}\quad j=0,\dots,N. \tag{2.1}$$

Here we have set

$$b_j := \frac{b^{(j)}(t_0)}{j!} \tag{2.2}$$

to be the Taylor coefficients of product $b \in \mathcal{FB}$ at the given boundary point $t_0 \in \mathbb{T}$. For any fixed $b \in \mathcal{FB}$ we can calculate the sequence $\mathbf{s} = \{s_0, \dots, s_N\}$ via the second equality in (2.1), and if this sequence satisfies the first equality in (2.1) for some $s \in \mathcal{S}$, then the problem $\mathbf{P}_n$ has a solution: namely, $f = s/b$. On the other hand, if $\mathbf{f}$ is such that for *every* $b \in \mathcal{FB}$, the interpolation conditions (2.1) are satisfied by no Schur function, then the problem $\mathbf{P}_N$ has no solutions. This simple idea allows us to reduce the problem $\mathbf{P}_N$ to a similar problem for Schur function the answer for which is known [5].

With any $b \in \mathcal{FB}$ we may associate the matrices $\mathbb{U}_n^b$, $\mathbb{H}_n^b$ and $\mathbb{P}_n^b$ constructed via formulas (1.6) and (1.9) from the Taylor coefficients (2.3). On the other hand, for the sequence $\mathbf{s} = \{s_0, \dots, s_N\}$ obtained from the given $\mathbf{f}$ and a fixed $b \in \mathcal{FB}$ via convolution formulas (2.1), we may define the structured matrices

$$\mathbb{P}_n^{\mathbf{s}} = \left[p_{ij}^{\mathbf{s}}\right]_{i,j=1}^n = \mathbb{H}_n^{\mathbf{s}} \mathbf{\Psi}_n(t_0) \mathbb{U}_n^{\mathbf{s}*} \tag{2.3}$$

as in (1.6)–(1.9), with the entries $p_{ij}^{\mathbf{s}}$ defined in the same way as in (1.10). We also may define the numbers

$$\gamma_n^{\mathbf{s}} := t_0 \cdot \left(p_{n+1,n}^{\mathbf{s}} - \overline{p_{n,n+1}^{\mathbf{s}}}\right). \tag{2.4}$$

for every $n \le N/2$ and the integer

$$n_0^{\mathbf{s}} = \begin{cases} 0, & \text{if } \mathbb{P}_1^{\mathbf{s}} = s_1 t_0 \overline{s_0} \notin \mathbb{R}, \\ \max\limits_{1 \le k \le (N+1)/2} \{k : \ \mathbb{P}_k^{\mathbf{s}} = (\mathbb{P}_k^{\mathbf{s}})^*\}, & \text{otherwise.} \end{cases} \tag{2.5}$$

Lemma 2.1. *Let $b \in \mathcal{FB}$ and let us assume that the two sequences $\mathbf{f} = \{f_0, \dots, f_N\}$ ($|f_0| = 1$) and $\mathbf{s} = \{s_0, \dots, s_N\}$ are related as in (2.1), Then:*

1. *For every $n \ge 1$,*
$$\mathbb{P}_n^b = \mathbb{H}_n^b \mathbf{\Psi}_n(t_0) \mathbb{U}_n^{b*} \ge 0 \quad \text{and} \quad \mathbb{U}_n^{b\top} \Psi_n(t_0) \mathbb{U}_n^{b*} = \Psi_n(t_0), \tag{2.6}$$
where $\mathbb{U}_n^{b\top}$ is the transpose of $\mathbb{U}_n^b$.
2. *For every $n \le (N+1)/2$,*
$$\mathbb{P}_n^{\mathbf{s}} := \mathbb{H}_n^{\mathbf{s}} \mathbf{\Psi}_n(t_0) \mathbb{U}_n^{\mathbf{s}*} = \mathbb{U}_n^{\mathbf{f}} \mathbb{P}_n^b \mathbb{U}_n^{\mathbf{f}*} + \mathbb{P}_n^{\mathbf{f}}. \tag{2.7}$$
3. *The integers n_0 and $n_0^{\mathbf{s}}$ defined in (1.11) and (2.5) are equal.*
4. *The numbers γ_{n_0} and $\gamma_{n_0}^{\mathbf{s}}$ defined in (1.12) and (2.4) are equal.*
5. *If $b(z) = z^m$, then $\mathbb{P}_{n_0}^{\mathbf{s}}$ is positive definite for m large enough.*

Proof. The proof of the inequality in (2.6) can be found in [7]. The second equality in (2.6) is a consequence of the identity $b(z)\overline{b(1/\bar{z})} \equiv 1$ (see[9, Theorem 2.5] for details). To prove (2.7) we first observe that the convolution equalities (2.1) are equivalent to the matrix equality $\mathbb{U}_n^{\mathbf{s}}=\mathbb{U}_n^{\mathbf{f}}\mathbb{U}_n^{b}$ and imply that

$$\mathbb{H}_n^{\mathbf{s}}=\begin{bmatrix}0 & & f_0 & f_1 & \cdots & f_n\\ & \iddots & \vdots & \vdots & \iddots & \vdots\\ f_0 & \cdots & f_{n-1} & f_n & \cdots & f_{2n-1}\end{bmatrix}\begin{bmatrix}b_n & \cdots & b_{2n-1}\\ \vdots & & \vdots\\ b_1 & \cdots & b_n\\ b_0 & \cdots & b_{n-1}\\ & \ddots & \vdots\\ 0 & & b_0\end{bmatrix}$$
$$=\mathbb{U}_n^{\mathbf{f}}\mathbb{H}_n^{b}+\mathbb{H}_n^{\mathbf{f}}\mathbb{U}_n^{b\top}.$$

Making use of the last two identities and of equality (2.6), we get (2.7):

$$\begin{aligned}\mathbb{P}_n^{\mathbf{s}}=\mathbb{H}_n^{\mathbf{s}}\Psi_n(t_0)\mathbb{U}_n^{\mathbf{s}*}&=(\mathbb{U}_n^{\mathbf{f}}\mathbb{H}_n^{b}+\mathbb{H}_n^{\mathbf{f}}\mathbb{U}_n^{b\top})\Psi_n(t_0)\mathbb{U}_n^{b*}\mathbb{U}_n^{\mathbf{f}*}\\ &=\mathbb{U}_n^{\mathbf{f}}\mathbb{P}_n^{b}\mathbb{U}_n^{\mathbf{f}*}+\mathbb{H}_n^{\mathbf{f}}\Psi_n(t_0)\mathbb{U}_n^{\mathbf{f}*}=\mathbb{U}_n^{\mathbf{f}}\mathbb{P}_n^{b}\mathbb{U}_n^{\mathbf{f}*}+\mathbb{P}_n^{\mathbf{f}}\end{aligned}$$

Since $\mathbb{P}_n^{b}$ is Hermitian (by the first relation in (2.6)), it follows from (2.7) that $\mathbb{P}_n^{\mathbf{s}}-\mathbb{P}_n^{\mathbf{f}}$ is Hermitian for all $n\geq 1$. Statements (3) and (4) are now immediate.

Since $|f_0|=1$, the triangular toeplitz matrix $\mathbb{U}_{n_0}^{\mathbf{f}}$ is invertible, which allows us write (2.7) (for $n=n_0$) equivalently as

$$(\mathbb{U}_{n_0}^{\mathbf{f}})^{-1}\mathbb{P}_{n_0}^{\mathbf{s}}(\mathbb{U}_{n_0}^{\mathbf{f}})^{-*}=\mathbb{P}_{n_0}^{b}+(\mathbb{U}_{n_0}^{\mathbf{f}})^{-1}\mathbb{P}_{n_0}^{\mathbf{f}}(\mathbb{U}_{n_0}^{\mathbf{f}})^{-*}. \tag{2.8}$$

The second term on the right is completely determined by the given $\mathbf{f}$. Theorem 4.1 below gives asymptotics of all eigenvalues of the matrix $\mathbb{P}_{n_0}^{b}$ for the particular choice of $b(z)=z^m$. These asymptotics show in particular, that the minimum eigenvalue of $\mathbb{P}_{n_0}^{b}$ tends to infinity as $m\to\infty$. Hence, for sufficiently large m, the matrix on the left hand side of (2.8) is positive definite and so is $\mathbb{P}_{n_0}^{\mathbf{s}}$ as desired. $\square$

Proof of Theorem 1.2: As we mentioned at the beginning of this section, the problem $\mathbf{P}_n$ has infinitely many rational Schur function solutions if $|f_0|<1$ and has no solutions if $|f_0|>1$.

A Carthéodory-Julia type theorem for generalized Schur functions (see [9, Theorem 4.2]) asserts that whenever a function $f\in\mathcal{GS}$ admits the boundary limits

$$f_0=\lim_{z\widehat{\rightarrow}t_0}f(z)\quad\text{and}\quad f_1=\lim_{z\widehat{\rightarrow}t_0}f'(z)$$

and $|f_0|=1$, then necessarily $t_0f_1\overline{f}_0\in\mathbb{R}$. Therefore, in case $|f_0|=1$ and $n_0=0$ (that is , if $t_0f_1\overline{f}_0\notin\mathbb{R}$), the problem $\mathbf{P}_n$ has no solutions.

In case $|f_0|=1$ and $n_0>0$, we fix a finite Blaschke product b and construct the sequence $\mathbf{s}=\{s_0,\ldots,s_N\}$ by the convolution formula (2.1). No matter what b we take, we will have $|s_0|=1$ (since $s_0=b_0f_0$), $n_0^{\mathbf{s}}=n_0$ and $\gamma_{n_0}^{\mathbf{s}}=\gamma_{n_0}$. By Theorem 2.3 in [5], if $\gamma_{n_0}^{\mathbf{s}}<0$ (in case $n_0=N/2$) or if $\gamma_{n_0}^{\mathbf{s}}\leq 0$ (in case $0<n_0<N/2$), *there is no* Schur function s subject to

equalities (2.1). Thus, there is no function f of the form (1.1) with $s \in \mathcal{S}$ and $b \in \mathcal{FB}$ satisfying conditions (1.5). In other words, the problem $\mathbf{P}_N$ has no solutions in the two following cases:

$$(1)\ n_0 = N/2 \quad \text{and} \quad \gamma_{n_0} < 0; \qquad (2)\ 0 < n_0 < N/2 \quad \text{and} \quad \gamma_{n_0} \le 0.$$

On the other hand, upon choosing $b(z) = z^m$ with m sufficiently large, we can guarantee that the structured matrix $\mathbb{P}^{\mathbf{s}}_{n_0}$ associated with the sequence $\mathbf{s} = \{s_0, \dots, s_N\}$ constructed as in (2.1) is positive definite. In case $n_0 = (N+1)/2$, this is enough to guarantee the existence of infinitely many rational functions $s \in \mathcal{S}$ satisfying conditions (2.1) (see [2] or [8]). The existence of such functions in two remaining cases where $n_0 = N/2$ and $\gamma_{n_0} \ge 0$ or where $0 < n_0 < N/2$ and $\gamma_{n_0} > 0$ is guaranteed by Theorem 2.3 in [5]. For every such s, the function $f(z) = s(z)/z^m$ solves the problem $\mathbf{P}_N$. This proves the sufficiency of the case (2)–(4) in Theorem 1.2 which together with the sufficiency of the first case completes the proof of the "if" part of the theorem. Since we have examined *all* possible cases and shown that in all other cases the proble has no solutions, the "only if" part follows. □

Remark 2.2. We showed that whenever the problem $\mathbf{P}_N$ has a solution, it has a solution of the form $f(z) = s(z)/z^m$ (in case $|f_0| < 1$, we can let $m = 0$).

3. The Infinite Case

Comparing the formulations of Theorems 1.2 and 1.3 we can see that only the third case in Theorem 1.3 is essentially infinite. The proof of its sufficiency requires some preliminary work which will be done below. First we will prove the rest in Theorem 1.3.

3.1. **Beginning of the proof of Theorem 1.3.** As in the finite case, a necessary condition for problem $\mathbf{P}_\infty$ to have a solution is that $|f_0| \le 1$. If $|f_0| < 1$, then there are infintely many functions $f \in \mathcal{S}$ subject to conditions (1.5); see [11, Theorem 2.2] for the proof.

It is obvious that if the truncated problem $\mathbf{P}_N$ has no solutions for some $N < \infty$, the infinite problem $\mathbf{P}_\infty$ has no solutions either. Thus, the absense of solutions in some cases follows from Theorem 1.2. In particular, the problem $\mathbf{P}_\infty$ has no solutions if $|f_0| = 1$ and $n_0 = 0$. Also, the problem $\mathbf{P}_\infty$ has no solutions if $0 < n_0 < \infty$ and $\gamma_{n_0} \le 0$ (recall that n_0 and γ_n are defined in (1.11) and (1.12), respectively). On the other hand, if

$$|f_0| = 1, \quad 0 < n_0 < \infty \quad \text{and} \quad \gamma_{n_0} > 0, \tag{3.1}$$

then the problem $\mathbf{P}_\infty$ has infinitely many solutions of the form $f(z) = s(z)/z^m$. Indeed, under assumptions (3.1), we may use the function $b(z) = z^m$ to define the infinite sequence $\mathbf{s} = \{s_j\}_{j \ge 0}$ via convolution equalities (2.1). We then have

$$|s_0| = |t_0^m f_0| = 1, \quad n_0^{\mathbf{s}} = n_0 < \infty, \quad \gamma^{\mathbf{s}}_{n_0} = \gamma_{n_0} > 0, \quad \mathbb{P}^{\mathbf{s}}_{n_0} > 0, \tag{3.2}$$

where the first equality is obvious and the next two equalities and the positivity of the structured matrix $\mathbb{P}^{\mathbf{s}}_{n_0}$ for a sufficiently large m follow from Lemma 2.1. By Theorem 1.2 in [11], conditions (3.2) are sufficient for the existence of infinitely many fucntions $s \in \mathcal{S}$ such that

$$\lim_{z \widehat{\to} t_0} \frac{s^{(j)}(z)}{j!} = s_j := \sum_{\ell=0}^{j} b_\ell f_{j-\ell} \quad \text{for} \quad j = 0, 1, \ldots$$

For each such s, the function $f(z) = s(z)/z^m$ solves the problem $\mathbf{P}_\infty$.

It remains to consider the case where $|f_0| = 1$ and $n_0 = \infty$; the latter means that the structured matrices $\mathbb{P}^{\mathbf{f}}_n$ are Hermitian for all $n \geq 1$. Let us show that in this case, uniform boundedness of the negative inertia of matrices $\mathbb{P}^{\mathbf{f}}_n$ is necessary for the problem $\mathbf{P}_N$ to have solution. To this end, we first recall a result from [9] (see Theorem 1.5 there).

Theorem 3.1. *Let f be analytic in a neighborhood $\{z \in \mathbb{D} : |z - t_0| < \varepsilon\}$ of $t_0 \in \mathbb{T}$ and let us assume that the nontangential boundary limits*

$$f_j = \lim_{z \widehat{\to} t_0} \frac{f^{(j)}(z)}{j!} \quad \textit{exist for } j = 0, \ldots 2n-1$$

and are such that $|f_0| = 1$ and the structured matrix $\mathbb{P}^{\mathbf{f}}_n$ constructed from these limits as in (1.9)*, is Hermitian. Then the Schwarz-Pick matrix $\mathbf{P}^f_n(z)$ (see* (1.2)*) converges as $z \widehat{\to} t_0$ and moreover $\lim_{z \widehat{\to} t_0} \mathbf{P}^f_n(z) = \mathbb{P}^{\mathbf{f}}_n$.*

Note that in Theorem 3.1 the function f is not assumed to be in $\mathcal{GS}$. We now assume that f belongs to $\mathcal{GS}$ and satisfies conditions (1.5) for all $j \geq 0$. Then all the assumptions in Theorem 3.1 are met and we conclude that

$$\lim_{z \widehat{\to} t_0} \mathbf{P}^f_n(z) = \mathbb{P}^{\mathbf{f}}_n \quad \text{for all} \quad n \geq 1. \tag{3.3}$$

Let us denote by κ the total pole mutiplicity of f, i.e., the degree of the Blaschke product b in the coprime Krein-Langer representation (1.1) for f. By a result of Krein and Langer, the Schwarz-Pick matrix $\mathbf{P}^f_n(z)$ given by formula (1.2) has at most κ negative eigenvalues for every $z \in \rho(f)$ and for all $n \geq 1$. Then we conclude from (3.3) that $\nu(\mathbb{P}^{\mathbf{f}}_n) \leq \nu(\mathbf{P}^f_n(z)) \leq \kappa$ for all $n \geq 1$.

To complete the proof of Theorem 1.3 it remains to justify the sufficiency of the case (3), which will be done at the end of this section, after some needed preliminaries. For the three next subsections we assume that

$$|f_0| = 1, \quad \mathbb{P}^{\mathbf{f}}_n = \mathbb{P}^{\mathbf{f}*}_n \quad \text{and} \quad \nu(\mathbb{P}^{\mathbf{f}}_n) \leq \kappa \quad \text{for all } n \geq 1. \tag{3.4}$$

By the third condition in (3.4), we may assume without loss of generality that $\mathbb{P}^{\mathbf{f}}_n$ has *exactly* κ negative eigenvalues if n is large enough:

$$\nu(\mathbb{P}^{\mathbf{f}}_n) = \kappa \quad \text{for all } n \geq n_1. \tag{3.5}$$

For the technical convenience we assume that $t_0 \neq 1$ which we also can do without loss of generality.

3.2. **Schur complements and Stein identities.** If $|f_0| = 1$ and the structured matricx $\mathbb{P}_n^{\mathbf{f}}$ is Hermitian, then (see [8, Section 3]) it satisfies the Stein identity

$$\mathbb{P}_n^{\mathbf{f}} - T_n \mathbb{P}_n^{\mathbf{f}} T_n^* = E_n E_n^* - M_n M_n^*, \tag{3.6}$$

where $T_n \in \mathbb{C}^{n\times n}$ and E_n, $M_n \in \mathbb{C}^{n\times 1}$ are given by

$$T_n = \begin{bmatrix} t_0 & 0 & \cdots & 0 \\ 1 & t_0 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & 1 & t_0 \end{bmatrix}, \quad E_n = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad M_n = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{bmatrix}. \tag{3.7}$$

For every positive integer $d < n$ we write conformal block decompositions

$$\mathbb{P}_n^{\mathbf{f}} = \begin{bmatrix} \mathbb{P}_d^{\mathbf{f}} & B^* \\ B & C \end{bmatrix}, \quad T_n = \begin{bmatrix} T_d & 0 \\ R & T_{n-d} \end{bmatrix}, \quad E_n = \begin{bmatrix} E_d \\ 0 \end{bmatrix}, \quad M_n = \begin{bmatrix} M_d \\ \widetilde{M} \end{bmatrix}, \tag{3.8}$$

where T_d, T_{n-d}, E_d, M_d are defined accordingly to (3.7) and where

$$R = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{C}^{(n-d)\times d}, \qquad \widetilde{M} = \begin{bmatrix} f_d \\ \vdots \\ f_{n-1} \end{bmatrix} \in \mathbb{C}^{(n-d)\times 1}.$$

Substituting block decompositions (3.8) into (3.6) and comparing the corresponding blocks we get the following three equalities:

$$\begin{aligned} &\mathbb{P}_d^{\mathbf{f}} - T_d \mathbb{P}_d^{\mathbf{f}} T_d^* = E_d E_d^* - M_d M_d^*, \\ &B - T_{n-d} B T_d^* - R P_d T_d^* = -\widetilde{M} M_d^*, \\ &C - T_{n-d} C T_{n-d}^* - R \mathbb{P}_d^{\mathbf{f}} R^* - T_{n-d} B R^* - R B^* T_{n-d}^* = -\widetilde{M}\widetilde{M}^*. \end{aligned} \tag{3.9}$$

Assuming that $\det(\mathbb{P}_d^{\mathbf{f}}) \neq 0$, we define the Schur complement of $\mathbb{P}_d^{\mathbf{f}}$ in $\mathbb{P}_n^{\mathbf{f}}$ as

$$\mathbb{S}_{n-d} = C - B(\mathbb{P}_d^{\mathbf{f}})^{-1} B^*.$$

Proposition 3.2. *Let $\mathbb{P}_d^{\mathbf{f}}$ be an invertible leading submatrix of $\mathbb{P}_n^{\mathbf{f}}$. Then its Schur complement $\mathbb{S}_{n-d}$ satisfies the Stein identity*

$$\mathbb{S}_{n-d} - T_{n-d} \mathbb{S}_{n-d} T_{n-d}^* = G_{n-d} G_{n-d}^* - Y_{n-d} Y_{n-d}^*, \tag{3.10}$$

where G_{n-d} and Y_{n-d} are defined in terms of the decomposition (3.8) *as*

$$\begin{aligned} G_{n-d} &= (R - (I - T_{n-d}) B (\mathbb{P}_d^{\mathbf{f}})^{-1})(I - T_d)^{-1} E_d, \\ Y_{n-d} &= \widetilde{M} + (R - (I - T_{n-d}) B (\mathbb{P}_d^{\mathbf{f}})^{-1})(I - T_d)^{-1} M_d. \end{aligned} \tag{3.11}$$

For the proof, it suffices to multiply both sides of (3.6) by $\begin{bmatrix} -B(\mathbb{P}_d^{\mathbf{f}})^{-1} & I \end{bmatrix}$ on the left and its adjoint on the right and then to invoke equalities (3.9). Equality (3.10) was proved in [10] for the interior case $t_0 \in \mathbb{D}$, but the proof does not rely on this assumption, so we refer to [10, Theorem 2.5] for computational details.

Let us denote by g_j and y_j the entries in the columns (3.11) so that $G_{n-d} = \begin{bmatrix} g_0 & \cdots & g_{n-d-1} \end{bmatrix}^\top$ and $Y_{n-d} = \begin{bmatrix} y_0 & \cdots & y_{n-d-1} \end{bmatrix}^\top$. Explicit formulas for g_j and y_j are easily derived from (3.10):

$$\begin{aligned} g_0 &= (\mathbf{e}_d - (1-t_0)B(\mathbb{P}_d^{\mathbf{f}})^{-1})(I-T_d)^{-1}E_d, \\ y_0 &= f_d + (\mathbf{e}_d - (1-t_0)B(\mathbb{P}_d^{\mathbf{f}})^{-1})(I-T_d)^{-1}M_d, \end{aligned} \tag{3.12}$$

where $\mathbf{e}_d = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{C}^{1\times d}$, and

$$\begin{aligned} g_j &= U_j(\mathbb{P}_d^{\mathbf{f}})^{-1}(I-T_d)^{-1}E_d, \\ y_j &= f_{j+d} + U_j(\mathbb{P}_d^{\mathbf{f}})^{-1})(I-T_d)^{-1}M_d \qquad \text{for } j \ge 1. \end{aligned} \tag{3.13}$$

where U_j is defined in terms of numbers (1.10) as follows:

$$U_j = (t_0-1)\cdot\begin{bmatrix} p^{\mathbf{f}}_{j+d+1,1} & \cdots & p^{\mathbf{f}}_{j+d+1,d} \end{bmatrix} + \begin{bmatrix} p^{\mathbf{f}}_{j+d,1} & \cdots & p^{\mathbf{f}}_{j+d,d} \end{bmatrix}.$$

Formulas (3.13) enable us to define g_j and y_j for every $j \ge 1$. Being combined with (1.10), they also show that the numbers x_j are completely determined by t_0 and $f_0, f_1, \ldots, f_{2d+j}$ for every fixed $j \ge 0$. Observe that the term R in (3.11) affects only the top entries g_0 and y_0 making formulas (3.12) and (3.13) slightly different.

Lemma 3.3. *For g_0 and y_0 defined as in* (3.12), $|g_0| = |y_0| \neq 0$.

Proof: Since $|t_0| = 1$, by examining the top-left entries on both sides of (3.10), we get $0 = |g_0|^2 - |y_0|^2$ so that $|g_0| = |y_0|$. It will be shown below that

$$g_0E^* - y_0M^* = \Big(\mathbf{e}_d + B(\mathbb{P}_d^{\mathbf{f}})^{-1}(t_0I - T_d)\Big)(I-T_d)^{-1}\mathbb{P}_d^{\mathbf{f}}(I-T_d^*). \tag{3.14}$$

Since the matrix $(I-T_d)^{-1}\mathbb{P}_d^{\mathbf{f}}(I-T_d^*)$ is invertible (recall that $t_0 \neq 1$) and since the rightmost entry in the row-vector $\mathbf{e}_d + B(\mathbb{P}_d^{\mathbf{f}})^{-1}(t_0I-T_d)$ is 1, it follows that the row-vector on the right hand side of (3.14) is not the zero and thus g_0 and y_0 cannot both be zero.

It remains to verify (3.14). We have from (3.12)

$$\begin{aligned} g_0E_d^* - y_0M_d^* = &-f_dM_d^* \\ &+ (\mathbf{e}_d - (1-t_0)B(\mathbb{P}_d^{\mathbf{f}})^{-1})(I-T_d)^{-1}(E_dE_d^* - M_dM_d^*). \end{aligned} \tag{3.15}$$

Due to the first equation in (3.9) we have

$$(I-T_d)^{-1}(E_dE_d^* - M_dM_d^*) = (I-T_d)^{-1}\mathbb{P}_d^{\mathbf{f}}(1-T_d^*) + \mathbb{P}_d^{\mathbf{f}}T_d^*, \tag{3.16}$$

while by equating the top rows on both sides of (3.9)) we get

$$-f_dM_d^* = B - \mathbf{e}_d\mathbb{P}_d^{\mathbf{f}}T_d^* - t_0BT_d^*.$$

Substituting the two last identities into the right hand side of (3.15) gives

$$\begin{aligned} g_0E_d^* - y_0M_d^* &= B - \mathbf{e}_d\mathbb{P}_d^{\mathbf{f}}T_d^* - t_0BT_d^* \\ &\quad + (\mathbf{e}_d - (1-t_0)B(\mathbb{P}_d^{\mathbf{f}})^{-1})\left((I-T_d)^{-1}\mathbb{P}_d^{\mathbf{f}}(1-T_d^*) + \mathbb{P}_d^{\mathbf{f}}T_d^*\right) \\ &= B(I-T_d^*) + \mathbf{e}_d(\mathbb{P}_d^{\mathbf{f}})^{-1}(I-T_d)^{-1}\mathbb{P}_d^{\mathbf{f}}(1-T_d^*) \\ &\quad - (1-t_0)B(\mathbb{P}_d^{\mathbf{f}})^{-1}(I-T_d)^{-1}\mathbb{P}_d^{\mathbf{f}}(1-T_d^*) \end{aligned}$$

which is clearly equivalent to (3.14). This completes the proof. □

The next theorem is the main result of this subsection.

Theorem 3.4. *Given* $\mathbf{f} = \{f_j\}_{j\geq 0}$ *let us assume that conditions* (3.4) *are met and thet the matrix* $\mathbb{P}_d^{\mathbf{f}}$ *is invertible. Define the sequence* $\mathbf{x} = \{x_j\}_{j\geq 0}$ *as a (unique) solution of the infinite linear system*

$$\sum_{k=0}^{j} x_k g_{j-k} = y_k \qquad j = 0, 1, \ldots, \tag{3.17}$$

where g_j *and* y_j *are defined in* (3.12), (3.13). *Then*

$$|x_0| = 1, \quad \mathbb{P}_n^{\mathbf{x}} = \mathbb{P}_n^{\mathbf{x}*} \quad \text{and} \quad \nu(\mathbb{P}_n^{\mathbf{x}}) \leq \kappa - \nu(\mathbb{P}_d^{\mathbf{f}}) \quad \text{for all } \; n \geq 1. \tag{3.18}$$

Proof: Letting $j = 0$ in (3.17) we get $x_0g_0 = y_0$; therefore, $|x_0| = 1$, by Lemma 3.3. For every fixed $n > d$, let $\mathbb{G}_{n-d}$ denote the lower triangular toeplitz $(n-d)\times(n-d)$ matrix with the leftmost column equal G_{n-d}, so that $\mathbb{G}_{n-d}E_{n-d} = G_{n-d}$. By Lemma 3.3, $g_0 \neq 0$, hence the matrix $\mathbb{G}_{n-d}$ is invertible. The column $X_{n-d} = \begin{bmatrix} x_0 & x_1 & \cdots & x_{n-d-1}\end{bmatrix}^\top$ satisfies

$$X_{n-d} = \begin{bmatrix} x_0 & x_1 & \cdots & x_{n-d-1}\end{bmatrix}^\top = \mathbb{G}_{n-d}^{-1}Y_{n-d} \tag{3.19}$$

due to (3.17). Let us introduce the matrix

$$\widetilde{P}_{n-d} = \mathbb{G}_{n-d}^{-1}\mathbb{S}_{n-d}\mathbb{G}_{n-d}^{-*} = \mathbb{G}_{n-d}^{-1}(C - B(\mathbb{P}_d^{\mathbf{f}})^{-1}B^*)\mathbb{G}_{n-d}^{-*}. \tag{3.20}$$

By a well-known property of the Schur complement, $\nu(\mathbb{P}_n^{\mathbf{f}}) = \nu(\mathbb{P}_d^{\mathbf{f}}) + \nu(\mathbb{S}_{n-d})$. Since $\widetilde{P}_{n-d}$ is congruent to $\mathbb{S}_{n-d}$, it follows that

$$\nu(\widetilde{P}_{n-d}) = \nu(\mathbb{S}_{n-d}) = \nu(\mathbb{P}_n^{\mathbf{f}}) - \nu(\mathbb{P}_d^{\mathbf{f}}). \tag{3.21}$$

Since the matrix $\mathbb{G}_{n-d}^{-1}$ is lower triangular, it also follows from (3.20) that $\widetilde{P}_k$ is a leading principal submatrix of $\widetilde{P}_n$ for every $k < n$.

Multiplying both sides of (3.10) by $\mathbb{G}_{n-d}^{-1}$ on the left, by its adjoint on the right, commuting $\mathbb{G}_{n-d}^{-1}$ and T_{n-d} and making use of (3.19), (3.20) we obtain the Stein identity

$$\widetilde{P}_{n-d} - T_{n-d}\widetilde{P}_{n-d}T_{n-d}^* = E_{n-d}E_{n-d}^* - X_{n-d}X_{n-d}^*.$$

Since the latter identity holds for every $n > d$, we conclude that

$$\widetilde{P}_n - T_n\widetilde{P}_nT_n^* = E_nE_n^* - X_nX_n^* \quad \text{for all } \; n \geq 1. \tag{3.22}$$

By Theorem 10.5 in [6], a necessary and sufficient condition for the Stein equation

$$A - T_n A T_n^* = E_n E_n^* - X_n X_n^* \tag{3.23}$$

to have a solution $A \in \mathbb{C}^{n\times n}$ is that

$$\mathbb{U}_n^{\mathbf{x}\top} \Psi_n(t_0) \mathbb{U}_n^{\mathbf{x}*} = \Psi_n(t_0) \tag{3.24}$$

where $\mathbb{U}_n^{\mathbf{x}}$ and $\Psi_n(t_0)$ are defined via formulas (1.6) and (1.7). Thus, equality (3.24) holds for all $n \geq 1$. By Theorem 2.5 in [9], the double-sized equality

$$\mathbb{U}_{2n}^{\mathbf{x}\top} \Psi_{2n}(t_0) \mathbb{U}_{2n}^{\mathbf{x}*} = \Psi_{2n}(t_0)$$

guarantees that the structured matrix $\mathbb{P}_n^{\mathbf{x}}$ is Hermitian (which proves the second equality in (3.18)) and therefore, it satisfies the same Stein equation (3.23) as $\widetilde{P}_n$. It is known that the Stein equation (3.23) uniquely determines the entries a_{ij} (for $2 \leq i+j \leq n$) of its solution $A = [a_{ij}]_{i,j=1}^n$ (see [6, [p. 77]). Therefore, the (i,j)-entry in $\widetilde{P}_n$ is equal to the corresponding entry in $\mathbb{P}_n^{\mathbf{x}}$ for all (i,j) subject to $2 \leq i+j \leq n$. Since $\widetilde{P}_n$ and $\mathbb{P}_n^{\mathbf{x}}$ are leading submatrices of respectively $\widetilde{P}_m$ and $\mathbb{P}_m^{\mathbf{x}}$ for all $m > n$, we may increase n to conclude that $\mathbb{P}_n^{\mathbf{x}} = \widetilde{P}_n$ for all $n \geq 1$. Now the last relation in (3.18) follows from (3.21). $\square$

By our assumption (3.5), $\nu(\mathbb{P}_d^{\mathbf{f}}) = \kappa$ if d is large enough. For such d we would conclude from the third relation in (3.18) that $\nu(\mathbb{P}_n^{\mathbf{x}}) = 0$, that is, that the matrix $\mathbb{P}_n^{\mathbf{x}}$ is positive semidefinite for all $n \geq 1$. The question is whether exists an *invertible* matrix $\mathbb{P}_d^{\mathbf{f}}$ which captures the maximally possible negative inertia. The next lemma shows that such d always exists.

Lemma 3.5. *Let us assume that conditions* (3.4) *and* (3.5) *are in force. Then there exists an integer $d \geq 1$ such that $\mathbb{P}_d^{\mathbf{f}}$ is invertible and $\nu(\mathbb{P}_d^{\mathbf{f}}) = \kappa$.*

Proof: Let us define $\widetilde{n} = \sup\{n \in \mathbb{N} : \ \det \mathbb{P}_n^{\mathbf{f}} \neq 0\}$. If $\widetilde{n} = \infty$, the statement of the lemma is obvious due to (3.5). If $\widetilde{n} < \infty$, we can take $d = \widetilde{n}$, since for this choice of d, as we will show below, $\nu(\mathbb{P}_d^{\mathbf{f}}) = \kappa$. Indeed, since $\mathbb{P}_d^{\mathbf{f}}$ is invertible, we can define the sequence $\mathbf{x} = \{x_j\}_{j\geq 0}$ as in (3.17). By the proof of Theorem 3.4, the structured matrix $\mathbb{P}_m^{\mathbf{x}}$ is congruent to the Schur complement $\mathbb{S}_m$ of $\mathbb{P}_d^{\mathbf{f}}$ in $\mathbb{P}_{d+m}^{\mathbf{f}}$ for every $m \geq 1$. The definition of $d = \widetilde{n}$ tells us that $\det \mathbb{P}_{d+m}^{\mathbf{f}} = 0$ for every $m \geq 1$ so that $\mathbb{P}_m^{\mathbf{x}}$ is singular for every $m \geq 1$. Due to the structure (1.9) of $\mathbb{P}_m^{\mathbf{x}} = \mathbb{H}_m^{\mathbf{x}} \mathbf{\Psi}_m(t_0) \mathbb{U}_m^{\mathbf{x}*}$ and since the matrices $\mathbf{\Psi}_m(t_0)$ and $\mathbb{U}_m^{\mathbf{x}*}$ are invertible (recall that $|x_0| = 1$, by Theorem 3.4), it follows that the Hankel matrix $\mathbb{H}_m^{\mathbf{x}} = [x_{i+j-1}]_{i,j=1}^m$ is singular for every $m \geq 1$. The latter implies that $x_j = 0$ for every $j \geq 1$. Therefore $\mathbb{P}_m^{\mathbf{x}} = 0$ and hence $\nu(\mathbb{P}_{d+m}^{\mathbf{f}}) = \nu(\mathbb{P}_d^{\mathbf{f}}) + \nu(\mathbb{P}_m^{\mathbf{x}}) = \nu(\mathbb{P}_d^{\mathbf{f}})$ for all $m \geq 1$. Combining this with (3.5) leads us to $\nu(\mathbb{P}_d^{\mathbf{f}}) = \kappa$ which completes the proof of the lemma. $\square$

Remark 3.6. Let us assume that $\mathbb{P}_d^{\mathbf{f}}$ is invertible and $\nu(\mathbb{P}_d^{\mathbf{f}}) = \kappa$. Then either $\operatorname{rank} \mathbb{P}_m^{\mathbf{f}} = d$ for all $m \geq d$ or $\mathbb{P}_m^{\mathbf{f}}$ is invertible for all $m \geq d$.

Proof: The Schur complement of $\mathbb{P}^{\mathbf{f}}_d$ in $\mathbb{P}^{\mathbf{f}}_n$ is congruent to $\mathbb{P}^{\mathbf{x}}_{n-d}$. Hence

$$\operatorname{rank}\mathbb{P}^{\mathbf{f}}_n = \operatorname{rank}\mathbb{P}^{\mathbf{f}}_d + \operatorname{rank}\mathbb{P}^{\mathbf{x}}_{n-d}, \tag{3.25}$$

and the matrix $\mathbb{P}^{\mathbf{x}}_{n-d}$ is positive semidefinite, by the rightmost inequality in (3.18). It is known for positive semidefinite structured matrices (1.9) that either they are invertible for all n or their rank stabilizes as n goes to infinity. If the rank of $\mathbb{P}^{\mathbf{x}}_{n-d}$ stabilizes, then the rank of $\mathbb{P}^{\mathbf{f}}_n$ stabilizes as well and thus, there exists the *maximal* invertible matrix $\mathbb{P}^{\mathbf{x}}_d$. By the proof of Lemma 3.5, $\mathbb{P}^{\mathbf{x}}_n = 0$ which along with (3.25) implies $\operatorname{rank}\mathbb{P}^{\mathbf{f}}_n = \operatorname{rank}\mathbb{P}^{\mathbf{f}}_d$. Otherwise, the matrix $\mathbb{P}^{\mathbf{x}}_{n-d}$ is invertible for every $n > d$ and (3.25) implies that $\mathbb{P}^{\mathbf{f}}_n$ is invertible as well. □

3.3. **The matrix-function Θ.** Still assuming that $t_0 \neq 1$, $|f_0| = 1$ and the matrix $\mathbb{P}^{\mathbf{f}}_d$ is Hermitian and invertible, let us introduce the 2×2 matrix-valued function

$$\Theta(z) = I + (z-1)\begin{bmatrix} E_d^* \\ M_d^* \end{bmatrix}(I - zT_d^*)^{-1}(\mathbb{P}^{\mathbf{f}}_d)^{-1}(I - T_d)^{-1}\begin{bmatrix} E_d & -M_d \end{bmatrix}. \tag{3.26}$$

and let

$$\widetilde{\Theta}(z) = \begin{bmatrix} \widetilde{\theta}_{11}(z) & \widetilde{\theta}_{12}(z) \\ \widetilde{\theta}_{21}(z) & \widetilde{\theta}_{22}(z) \end{bmatrix} := (z - t_0)^d \cdot \Theta(z). \tag{3.27}$$

Since $(I - zT_d^*)^{-1}$ the upper-triangular toeplitz matrix with the top row equal to $\begin{bmatrix}(1 - z\bar{t}_0)^{-1} & z(1 - z\bar{t}_0)^{-2} & \ldots & z^{d-1}(1 - z\bar{t}_0)^{-d}\end{bmatrix}$; therefore Θ is a rational function with the only pole of multiplicity d at t_0 whereas $\widetilde{\Theta}$ is a matrix polynomial. It is not hard to see from (3.26), (3.27) and (3.7) that

$$\widetilde{\Theta}(t_0) = (-1)^d t_0^{2d-1}(t_0 - 1)\begin{bmatrix} 1 \\ \bar{f}_0 \end{bmatrix}\mathbf{e}_d(\mathbb{P}^{\mathbf{f}}_d)^{-1}(I - T_d)^{-1}\begin{bmatrix} E_d & -M_d \end{bmatrix}, \tag{3.28}$$

where $\mathbf{e}_d$ is the row-vector introduced just before Lemma 3.3. The next lemma establishes several equalities needed for the subsequent analysis.

Lemma 3.7. *Let Θ and $\widetilde{\Theta}$ be defined as in* (3.26), (3.27) *and let G_{n-d}, Y_{n-d} be the columns given in* (3.11) *with the top entries g_0, y_0 displayed in* (3.12). *Then*

$$\widetilde{\theta}_{21}(t_0)\bar{g}_0 + \widetilde{\theta}_{22}(t_0)\bar{y}_0 = \frac{(-1)^d t_0^{2d-1}(t_0 - 1)\bar{f}_0}{1 - \bar{t}_0}. \tag{3.29}$$

Furthermore, if for some $n > d$, the numbers $f_{2d}, \ldots, f_{2n-1}$ are such that the matrix $\mathbb{P}^{\mathbf{f}}_n$ is Hermitian, then

$$(zI - T_n)^{-1}\begin{bmatrix} E_n & -M_n \end{bmatrix}\Theta(z) = \begin{bmatrix} 0 \\ (zI - T_{n-d})^{-1}\begin{bmatrix} G_{n-d} & -Y_{n-d} \end{bmatrix} \end{bmatrix} + \Phi(z), \tag{3.30}$$

where $\Phi(z)$ is defined in terms of decompositions (3.8) *as follows:*

$$\Phi(z) = \begin{bmatrix} \mathbb{P}^{\mathbf{f}}_d \\ B \end{bmatrix}(I - zT_d^*)^{-1}(\mathbb{P}^{\mathbf{f}}_d)^{-1}(I - T_d)^{-1}\begin{bmatrix} E_d & -M_d \end{bmatrix}. \tag{3.31}$$

Finally, $\det \widetilde{\Theta}(z) = (z-t_0)^{2d}$ *for all* $z \in \mathbb{C}$ *and* $|\widetilde{\theta}_{21}(t_0)| = |\widetilde{\theta}_{22}(t_0)| \neq 0$.

Proof: Tt follows from (3.28) that

$$\widetilde{\theta}_{21}(t_0)\bar{g}_0 + \widetilde{\theta}_{22}(t_0)\bar{y}_0 = (-1)^d t_0^{2d-1}(t_0-1)\bar{f}_0 \mathbf{e}_d (\mathbb{P}_d^{\mathbf{f}})^{-1}(I-T_d)^{-1} \times (E_d\bar{g}_0 - M_d\bar{y}_0). \tag{3.32}$$

Taking adjoints on both sides of (3.9) we see that

$$\begin{aligned}\mathbf{e}_d(\mathbb{P}_d^{\mathbf{f}})^{-1}&(I-T_d)^{-1}(E_d\bar{g}_0 - M_d\bar{y}_0)\\ &= \mathbf{e}_d(I-T_d^*)^{-1}\left\{\mathbf{e}_d^* + ((\bar{t}_0 - T_d^*)\mathbb{P}_d^{\mathbf{f}})^{-1}B^*\right\} = \frac{1}{1-\bar{t}_0},\end{aligned} \tag{3.33}$$

where the last equality holds true since $\mathbf{e}_d(I-T_d^*)^{-1}(\bar{t}_0 - T_d^*) = 0$. Substituting (3.33) into (3.32) gives (3.29).

Since $|f_0| = 1$ and since we assume that $P_n^{\mathbf{f}}$ is Hermitian, it follows that the Stein identity (3.6) holds (we again refer to [8] for the proof) which is equivalent to three identities in (3.9). To verify (3.30), it suffices to plug in the formula (3.26) for Θ, decompositions (3.8) and the conformal decomposition

$$(zI-T_n)^{-1} = \begin{bmatrix} (zI-T_d)^{-1} & 0 \\ (zI-T_{n-d})^{-1}R(zI-T_d)^{-1} & (zI-T_{n-d})^{-1}\end{bmatrix}$$

into the left side of (3.30) and then to invoke the two top identities in (3.9). The calculations are straightforward and will be omitted.

To prove the formula for $\det\widetilde{\Theta}(z)$ we use a well known determinantal equality $\det(I+AB) = \det(I+BA)$ along with the explicit formula (3.26) and the Stein identity from (3.9):

$$\begin{aligned}\det\Theta(z) &= \det\left(I + (z-1)(I-zT_d^*)^{-1}(\mathbb{P}_d^{\mathbf{f}})^{-1}(I-T_d)^{-1}\begin{bmatrix}E_d & -M_d\end{bmatrix}\begin{bmatrix}E_d^* \\ M_d^*\end{bmatrix}\right)\\ &= \det\left(I + (z-1)(I-zT_d^*)^{-1}(\mathbb{P}_d^{\mathbf{f}})^{-1}(I-T_d)^{-1}\left(\mathbb{P}_d^{\mathbf{f}} - T_d\mathbb{P}_d^{\mathbf{f}}T_d^*\right)\right)\\ &= \det\left((I-zT_d^*)^{-1}(\mathbb{P}_d^{\mathbf{f}})^{-1}(I-T_d)^{-1}(zI-T_d)\mathbb{P}_d^{\mathbf{f}}(I-T_d^*)\right)\\ &= \frac{\det(zI-T_d)\cdot\det(I-T_d^*)}{\det(I-zT_d^*)\cdot\det(I-T_d)} = 1 \qquad (z \neq t_0)\end{aligned}$$

where the last equality follows from the special structure (3.7) of T_d. The desired formula $\det\widetilde{\Theta}(z) = (z-t_0)^{2d}$ follows from (3.27).

Finally, we have from (3.28) and (3.16)

$$\begin{aligned}|\widetilde{\theta}_{21}(t_0)|^2 &- |\widetilde{\theta}_{21}(t_0)|^2\\ &= |t_0-1|^2\cdot\mathbf{e}_d(\mathbb{P}_d^{\mathbf{f}})^{-1}(I-T_d)^{-1}\left(E_dE_d^* - M_dM_d^*\right)(I-T_d^*)^{-1}(\mathbb{P}_d^{\mathbf{f}})^{-1}\mathbf{e}_d^*\\ &= |t_0-1|^2\mathbf{e}_d\left((\mathbb{P}_d^{\mathbf{f}})^{-1}(I-T_d)^{-1} + T_d^*(I-T_d^*)^{-1}(\mathbb{P}_d^{\mathbf{f}})^{-1}\right)\mathbf{e}_d^*,\end{aligned} \tag{3.34}$$

where the first equality follows from (3.28) and the second equality is a consequence of (3.16). Due to the special form of T_d and $\mathbf{e}_d$, we have

$$(I-T_d)^{-1}\mathbf{e}_d^* = \frac{1}{1-t_0}\mathbf{e}_d^*. \qquad \mathbf{e}_d T_d^*(I-T_d^*)^{-1} = \frac{\bar{t}_0}{1-\bar{t}_0}\mathbf{e}_d = \frac{1}{t_0-1}\mathbf{e}_d,$$

which being substituted in (3.34) gives $|\widetilde{\theta}_{21}(t_0)|^2 - |\widetilde{\theta}_{21}(t_0)|^2 = 0$. Since by (3.29), $\widetilde{\theta}_{21}(t_0)$ and $\widetilde{\theta}_{22}(t_0)$ cannot be both equal zero, it follows that $|\widetilde{\theta}_{21}(t_0)| = |\widetilde{\theta}_{22}(t_0)| \neq 0$ which completes the proof. □

3.4. **The Schur reduction.** The idea going back to I. Schur [15] is to reduce a given interpolation problem to a similar one but with fewer interpolation conditions.

Theorem 3.8. *Let us assume that $t_0 \neq 1$, $|f_0| = 1$ and that the matrix $\mathbb{P}_d^{\mathbf{f}} = \mathbb{P}_d^{\mathbf{f}*}$ is invertible. Let $\widetilde{\Theta}$ be defined as in* (3.27), (3.26). *Then a function f belongs to $\mathcal{GS}$, has $\nu(\mathbb{P}_d^{\mathbf{f}})$ poles inside $\mathbb{D}$ and admits the boundary asymptotic*

$$f(z) = f_0 + f_1(z-t_0) + \ldots + f_{2d-1}(z-t_0)^{2d-1} + O(|z-t_0|^{2d}) \tag{3.35}$$

as $z \widehat{\rightarrow} t_0$ if and only if it is of the form

$$f = \frac{\widetilde{\theta}_{11}h + \widetilde{\theta}_{12}}{\widetilde{\theta}_{21}h + \widetilde{\theta}_{22}} \tag{3.36}$$

for some $h \in \mathcal{S}$ such that the boundary limit $h_0 = \lim\limits_{z \widehat{\rightarrow} t_0} h(z)$ either does not exist or satisfies

$$\widetilde{\theta}_{21}(t_0)h_0 + \widetilde{\theta}_{22}(t_0) \neq 0. \tag{3.37}$$

The proof is given in [2] (see also [1], [3]) for *rational* functions (in which case $h_0 = h(t_0)$ always exists and the nontangential approach to the boundary can be replaced by evaluation at t_9), but all the arguments go through in the general meromorphic setting. We remark, however, that in the general setting, condition (3.35) *is not* equivalent to the condition with the additional term of the form $o(|z-t_0|^{2d-1})$ as in the problem $\mathbf{P}_{2d-1}$.

We now use Theorem 3.8 to carry out the Schur reduction.

Theorem 3.9. *Given $\mathbf{f} = \{f_j\}_{j\geq 0}$, let us assume that conditions* (3.4), (3.5) *are met and thet the matrix $\mathbb{P}_d^{\mathbf{f}}$ is invertible. Let $\mathbf{x} = \{x_j\}_{j\geq 0}$ be the sequence defined in* (3.17). *A function f is a solution to the problem $\mathbf{P}_\infty$ and has κ poles inside $\mathbb{D}$ if and only if it is of the form* (3.36) *for some $h \in \mathcal{S}$ such that*

$$\lim_{z \widehat{\rightarrow} t_0} \frac{h^{(j)}(z)}{j!} = x_j \quad \textit{for all } j \geq 0. \tag{3.38}$$

Proof: Let us assume f is a solution to the problem $\mathbf{P}_\infty$ and has κ poles inside $\mathbb{D}$. In particular, f satisfies condition (3.35) and therefore, it can be represented in the form (3.36) for some Schur function $h \in \mathcal{S}$. Since

$\widetilde{\theta}_{22}(t_0) \neq 0$ (by Lemma 3.7), the Schur function $s \equiv 0$ meets condition (3.37) and therefore. the rational function $\mathbf{a} = \frac{\widetilde{\theta}_{11}s + \widetilde{\theta}_{12}}{\widetilde{\theta}_{21}s + \widetilde{\theta}_{22}} = \frac{\widetilde{\theta}_{12}}{\widetilde{\theta}_{22}}$ satisfies condition (3.35), by Theorem 3.8. We have

$$f = \frac{\widetilde{\theta}_{11}h + \widetilde{\theta}_{12}}{\widetilde{\theta}_{21}h + \widetilde{\theta}_{22}} = \mathbf{a} + \frac{\det \widetilde{\Theta} \cdot h}{\widetilde{\theta}_{22}(\widetilde{\theta}_{21}h + \widetilde{\theta}_{22})}.$$

Since both f and $\mathbf{a}$ satisfy the same asymptotic equality (3.35) and since f solves in addition the problem $\mathbf{P}_\infty$, it follows that

$$f(z) - \mathbf{a}(z) = f_{2d} - \frac{\mathbf{a}^{(2d)}(t_0)}{(2d)!} + O(|z-t_0|^{2d}).$$

Since $\det \widetilde{\Theta}(z) = (z-t_0)^{2d}$ (by Lemma 3.7), we may conclude from the two latter equalities that the limit $h_0 = \lim_{z \widehat{\to} t_0} h(z)$ exists and satisfies the equality

$$f_{2d} - \frac{\mathbf{a}^{(2d)}(t_0)}{(2d)!} = \frac{h_0}{\widetilde{\theta}_{22}(t_0)(\widetilde{\theta}_{21}h_0 + \widetilde{\theta}_{22}(t_0))}. \tag{3.39}$$

By Theorem 3.8, h_0 satisfies inequality (3.37), although this inequality can be derived directly from (3.39). Observe the equality

$$f(z) - \sum_{j=0}^{n-1} f_j(z-t_0)^j = (z-t_0)^n \cdot \mathbf{e}_n(zI - T_n)^{-1} \begin{bmatrix} E_n & -M_n \end{bmatrix} \begin{bmatrix} f(z) \\ 1 \end{bmatrix} \tag{3.40}$$

where $\mathbf{e}_n = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}^\top \in \mathbb{C}^{1\times n}$. Also observe that equality (3.36) can be written as

$$\begin{bmatrix} f \\ 1 \end{bmatrix} = \widetilde{\Theta} \begin{bmatrix} h \\ 1 \end{bmatrix} \cdot \mathbf{q}, \quad \text{where} \quad \mathbf{q} = \widetilde{\theta}_{21}h + \widetilde{\theta}_{22} \tag{3.41}$$

Substituting (3.41) into (3.40) and making use of (3.31) gives

$$\begin{aligned} f(z) &- \sum_{j=0}^{n-1} f_j(z-t_0)^j \\ &= (z-t_0)^n \cdot \mathbf{e}_n(zI - T_n)^{-1} \begin{bmatrix} E_n & -M_n \end{bmatrix} \widetilde{\Theta}(z) \begin{bmatrix} h(z) \\ 1 \end{bmatrix} \mathbf{q}(z) \\ &= (z-t_0)^{n+d} \cdot \mathbf{e}_n \begin{bmatrix} 0 \\ (zI - T_{n-d})^{-1} \begin{bmatrix} G_{n-d} & -Y_{n-d} \end{bmatrix} \end{bmatrix} \begin{bmatrix} h(z) \\ 1 \end{bmatrix} \mathbf{q}(z) \\ &\quad + (z-t_0)^{n+d} \cdot \mathbf{e}_n \Phi(z) \begin{bmatrix} h(z) \\ 1 \end{bmatrix} \mathbf{q}(z) \\ &= (z-t_0)^{n+d} \cdot \mathbf{e}_{n-d}(zI - T_{n-d})^{-1} \left(G_{n-d}h(z) - Y_{n-d}\right) \mathbf{q}(z) \\ &\quad + (z-t_0)^{n+d} \cdot \mathbf{e}_n \Phi(z) \begin{bmatrix} h(z) \\ 1 \end{bmatrix} \mathbf{q}(z). \end{aligned} \tag{3.42}$$

Since f solves the problem $\mathbf{P}_\infty$, we have

$$f(z) - \sum_{j=0}^{n-1} f_j(z-t_0)^j = O(|z-t_0|^n) \tag{3.43}$$

for all $n \geq 1$ as $z \widehat{\to} t_0$. Since h is uniformly bounded on $\mathbb{D}$, it follows from (3.37) and formula (3.31) for Φ that

$$(z-t_0)^{n+d} \cdot \mathbf{e}_n \Phi(z) \begin{bmatrix} h(z) \\ 1 \end{bmatrix} \mathbf{q}(z) = O(|z-t_0|^n) \quad \text{for all } \; n \geq 1.$$

Now we conclude from (3.42) that

$$(z-t_0)^d \cdot \mathbf{e}_{n-d}(zI - T_{n-d})^{-1} \left(G_{n-d}h(z) - Y_{n-d}\right) = O(1) \tag{3.44}$$

for all $n > d$. Of course, the latter relation is trivial and contains no information for $n = d+1, \ldots, 2d$. For $n > 2d$, let us multiply both sides in (3.44) by $(z-t_0)^{n-2d}$ and take into account the structure of T_{n-d} to get

$$\begin{bmatrix} 1 & z-t_0 & \ldots & (z-t_0)^{n-2d-1} \end{bmatrix} \left(G_{n-d}h(z) - Y_{n-d}\right) = O(|z-t_0|^{n-2d})$$

or equivalently (in terms of the entries g_j and y_j defined in (3.12), (3.13)) as

$$h(z) \cdot \sum_{j=0}^{n-2d-1} g_j(z-t_0)^j - \sum_{j=0}^{n-2d-1} y_j(z-t_0)^j = O(|z-t_0|^{n-2d}).$$

Due to convolution relations (3.17) and since $g_0 \neq 0$, the latter equality is equaivalent to

$$h(z) - \sum_{j=0}^{n-2d-1} x_j(z-t_0)^j = O(|z-t_0|^{n-2d})$$

which in turn, implies equalities (3.38) for $j = 0, \ldots, n-2d-1$. Since n can be chosen arbitrarily large, we get equalities (3.38) for all $j \geq 0$.

Conversely, let us assume that h is a Schur function satisfying conditions (3.38). Since $x_0 = y_0/g_0$ by the first equation in (3.17) and since x_0 is unimodular (by Lemma 3.3), it follows from (3.29) that

$$\widetilde{\theta}_{21}(t_0)x_0 + \widetilde{\theta}_{22}(t_0) = \frac{1}{\overline{y}_0}\left(\widetilde{\theta}_{21}(t_0)\overline{g}_0 + \widetilde{\theta}_{22}(t_0)\overline{y}_0\right) = \frac{(-1)^d t_0^{2d-1}(t_0-1)\overline{f}_0}{\overline{y}_0(1-\overline{t}_0)}$$

and thus, condition (3.37) is satisfied. By Theorem 3.8, the function f constructed from h by formula (3.36) has κ poles inside $\mathbb{D}$ and satisfies (3.35), that is the requested boundary derivatives f_n at t_0 for $n = 0, \ldots, 2d$. For $n > 2d$ we use calculation (3.42) to conlude that equalities (3.38) for $j = 0, \ldots, n-2d-1$ for h imply the asymptotic equality (3.43) for f. Letting n go to infinity we then conclude that f is a solution to the problem $\mathbf{P}_\infty$. $\square$

3.5. **Completion of the proof of Theorem 1.3.** We now complete the proof of Theorem 1.3 by demonstrating the sufficiency of conditions (3.4), (3.5). By Lemma 3.5, we can find $d \geq 1$ such that $\mathbb{P}_d^{\mathbf{f}}$ is invertible and $\nu(\mathbb{P}_d^{\mathbf{f}}) = \kappa$. Let $\mathbf{x}$ be the sequence defined in (3.17). By Theorem 3.9, the problem $\mathbf{P}_\infty$ has a solution $f \in \mathcal{GS}$ with κ poles inside $\mathbb{D}$ if ana only if there exists a Schur function $h \in \mathcal{S}$ subject to interpolation conditions (3.38). Since the structured matrices $\mathbb{P}_n^{\mathbf{x}}$ are positive semidefinite for all $n \geq 0$ (by Theorem 3.4 and since $\nu(\mathbb{P}_d^{\mathbf{f}}) = \kappa$), such a function h does exist (see e.g., [12]). Substituting this h into (3.36) results in a solution f to the problem $\mathbf{P}_\infty$. □

3.6. **Concluding remarks.** Since in cases (1) and (2), the problem $\mathbf{P}_\infty$ is indeterminate, the last statement in Theorem 1.3 need not be proved. However, we will show that the problem $\mathbf{P}_\infty$ indeed may be detrminate.

Let us consider the subcase of (3) where there exists the *maximal invertible* structured matrix $\mathbb{P}_d^{\mathbf{x}}$. Then for the associated sequence $\mathbf{x} = \{x_j\}$ we have $x_j = 0$ for all $j \geq 1$ and the only Schur function h satisfying conditions (3.38) is a unimodular constant function $h \equiv x_0$. Substituting this h into (3.36) we get a solution f to the problem $\mathbf{P}_\infty$. This f is rational and has κ poles inside $\mathbb{D}$ (by Theorem 3.8). It is not hard to show that $\deg f = d$ and that f is unimodular on $\mathbb{T}$. Therefore, f is the ratio of two Blaschke products of respective degrees $d - \kappa$ and κ. So far, we have shown that $\mathbf{P}_\infty$ has a unique solution $f \in \mathcal{GS}$ with κ poles inside $\mathbb{D}$. Let us assume that $\widetilde{f} \in \mathcal{GS}$ is another solution to the problem $\mathbf{P}_\infty$. Take it in the form (1.1), i.e.. $\widetilde{f} = s/b$ for some $s \in \mathcal{S}$ and $b \in \mathcal{FB}$. Then the associated structured matrices $\mathbb{P}_n^{\mathbf{f}}$, $\mathbb{P}_n^s$ and $\mathbb{P}_n^b$ are related as in (2.7) for all $n \geq 1$. Since $\operatorname{rank} \mathbb{P}_n^b = \max\{n, \deg b\}$ and since $\operatorname{rank} \mathbb{P}_n^{\mathbf{f}} = d$ for all $n \geq d$, it follows from (2.7) that the rank of $\mathbb{P}_n^s$ stabilizes for large n. Therefore, s is a finite Blaschke product so that the function $\widetilde{f}$ is rational and therefore, it is analytic at t_0. Thus, f and $\widetilde{f}$ are two rational functions with the same Taylor coefficients at t_0. Therefore $f \equiv \widetilde{f}$ which means that the problem $\mathbf{P}_\infty$ has only one solution in $\mathcal{GS}$.

In the complementary subcase of (3) where $\nu(\mathbb{P}_d^{\mathbf{f}}) = \kappa$ and $\det \mathbb{P}_d^{\mathbf{f}} \neq 0$ for all $n \geq d$ (and therefore, all structured matrices $\mathbb{P}_n^{\mathbf{x}}$ are positive definite), the problem (3.38) may be indeterminate or determinate depending on the convergence or divergence of certain positive series (see [12]). In the first case the problem $\mathbf{P}_\infty$ is indereminate, since every $h \in \mathcal{S}$ subject to (3.38) leads via formula (3.36) to a solution f to the problem $\mathbf{P}_\infty$ and since the transformation (3.36) is one-to-one. In the second case, it follows that the problem $\mathbf{P}_\infty$ has a unique solution $f \in \mathcal{GS}$ with κ poles inside $\mathbb{D}$; however we do not know if it may or may not have solutions in $\mathcal{GS}$ with a larger pole multiplicity. A separate topic in interpolation theory for generalized Schur functions is to characterize all possible pole multiplicities for solutions of the problem and to find the minimally possible one. This issue will be addressed on a separate occasion.

4. Appendix: Asymptotics for eigenvalues of the matrix $\mathbb{P}_n^b$

Theorem 4.1. *Let $b(z)=z^m$ and let $\lambda_1\le\lambda_2\le\cdots\le\lambda_n$ be the eigenvalues of the matrix $\mathbb{P}_n^b$. Then*

$$\lambda_r=\frac{(n-r)!^2(2r-1)!(2r-2)!}{(r-1)!^2(n+r-1)!^2}m^{2r-1}+O(m^{2r-2}).$$

In particular, all eigenvalues tend to infinity as $m\to\infty$.

Proof: Recall that $\mathbb{P}_n^b$ is constructed via formulas (1.6)–(1.9) from the parameters

$$b_j=\frac{b^{(j)}(t_0)}{j!}=\binom{m}{j}t_0^{m-j}.$$

Thus, $\mathbb{P}_n^b=\left[p_{ij}^b\right]_{i,j=1}^n=\mathbb{H}_n^b\Psi_n(t_0)\mathbb{U}_n^{\mathbf{b}*}$ and we can compute its entries using the formula (1.10) (with b_j instead of f_j) and the explicit formula (1.8) for the numbers $\Psi_{\ell r}$:

$$\begin{aligned}
p_{ij}^{\mathbf{b}}&=\sum_{r=1}^{j}\left(\sum_{\ell=1}^{r}b_{i+\ell-1}\Psi_{\ell r}\right)\overline{b}_{j-r}\\
&=\sum_{r=1}^{j}\sum_{\ell=1}^{r}(-1)^{r-1}\binom{m}{i+\ell-1}\binom{r-1}{\ell-1}\binom{m}{j-r}t_0^{j-i}\\
&=t_0^{j-i}\sum_{r=1}^{j}(-1)^{r-1}\binom{m}{i+r-1}\binom{m-r}{j-r}\\
&=t_0^{j-i}m^{i+j-1}\sum_{r=1}^{j}\frac{(-1)^{r-1}}{(i+r-1)!(j-r)!}+O(m^{i+j-2})\\
&=\frac{t_0^{j-i}m^{i+j-1}}{(i-1)!(j-1)!(i+j-1)}+O(m^{i+j-2}).
\end{aligned}\tag{4.1}$$

For $1\le i_1<i_2<\cdots<i_r\le n$, denote $M_{i_1i_2\cdots i_r}$ the principal minor of the matrix $\mathbb{P}_n^b$ with rows and columns $i_1,\ldots,i_r$. Then

$$\sum_{1\le i_1<i_2<\cdots<i_r\le n}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r}=\sum_{1\le i_1<i_2<\cdots<i_r\le n}M_{i_1i_2\cdots i_r},\tag{4.2}$$

If $\{i_1,\ldots,i_r\}\ne\{n-r+1,\ldots,n\}$, then

$$M_{i_1i_2\cdots i_r}=O(m^{(2i_1-1)+\cdots+(2i_r-1)})=O(m^{r(2n-r)-1}).\tag{4.3}$$

If $\{i_1, \dots, i_r\} = \{n-r+1, \dots, n\}$, we have

$$
\begin{aligned}
M_{i_1 i_2 \cdots i_r} &= \det \left[\frac{m^{i+j-1} t_0^{j-i}}{(i+j-1)(i-1)!(j-1)!} + O(m^{i+j-2}) \right]_{i,j=n-r+1}^{n} \\
&= \det \left[\frac{m^{i+j-1} t_0^{j-i}}{(i+j-1)(i-1)!(j-1)!} \right]_{i,j=n-r+1}^{n} + O(m^{r(2n-r)-1}) \\
&= \det \left[mD\, \mathcal{H}_{n-r+1}^{n} D^* \right] + O(m^{r(2n-r)-1}), \qquad (4.4)
\end{aligned}
$$

where

$$
D = \operatorname{diag} \left(\frac{t_0^{-i+1} m^{i-1}}{(i-1)!} \right)_{i=n-r+1}^{n} \quad \text{and} \quad \mathcal{H}_\ell^n = \left[\frac{1}{i+j-1} \right]_{i,j=\ell}^{n}.
$$

The matrix $\mathcal{H}_\ell^n$ is a Hilbert-type matrix and it is known that

$$
\det \mathcal{H}_\ell^n = \frac{(n-\ell)!!^2 (n+\ell-2)!!^2}{(2n-1)!!(2\ell-3)!!}, \qquad (4.5)
$$

where we use the notation $n!! := n!(n-1)! \cdots 1!$. Hence, if $\{i_1, \dots, i_r\} = \{n-r+1, \dots, n\}$, we have from (4.4)

$$
M_{i_1 i_2 \cdots i_r} = c_{n,r} m^{r(2n-r)} + O(m^{r(2n-r)-1}), \qquad (4.6)
$$

where

$$
c_{n,r} = \frac{(n-r-1)!!^2 (r-1)!!^2 (2n-r-1)!!^2}{(n-1)!!^2 (2n-1)!!(2n-2r-1)!!}.
$$

We now claim that for $r = 1, \dots, n$,

$$
\lambda_{n-r+1} = \frac{c_{n,r}}{c_{n,r-1}} m^{2n-2r+1} + O(m^{2n-2r}). \qquad (4.7)
$$

We prove (4.7) by double induction, first on n, then on r. For all $n \geq 1$, if $r = 1$, the claim is about the asymptotics of the largest eigenvalue of $\mathbb{P}_n^b$. From (4.1), the bottom-right entry of $\mathbb{P}_n^b$ is $p_{n,n}^{\mathbf{b}} = \frac{m^{2n-1}}{(n-1)!^2(2n-1)} + O(m^{2n-2})$, which dominates all other entries. Therefore, the largest eigenvalue of $\mathbb{P}_n^b$

$$
\lambda_n = \frac{1}{(n-1)!^2(2n-1)} m^{2n-1} + O(m^{2n-2}) = \frac{c_{n,1}}{c_{n,0}} m^{2n-1} + O(m^{2n-2}).
$$

Suppose $n \geq 2$ and we have proven the claim for all eigenvalues of $\mathbb{P}_{n-1}^b$. Denote the eigenvalues of $\mathbb{P}_{n-1}^b$ in increasing order by $\lambda_0', \lambda_1', \dots, \lambda_{n-1}'$, then we have the asymptotics $\lambda_{n-r+1}' = \frac{c_{n-1,r}}{c_{n-1,r-1}} m^{2n-2r+1} + O(m^{2n-2r})$. Assume also that the $r-1$ largest eigenvalues ($r \geq 1$) of $\mathbb{P}_n^b$, namely $\lambda_n, \lambda_{n-1}, \dots, \lambda_{n-r+2}$, all have asymptotics as described in (4.5). Note that $\mathbb{P}_{n-1}^b$ is the leading submatrix of $\mathbb{P}_n^b$. So by Interlacing Theorem,

$$
\lambda_1 \leq \lambda_1' \leq \lambda_2 \leq \cdots \leq \lambda_{n-1}' \leq \lambda_n.
$$

Then for $\{i_1, \dots, i_r\} \neq \{n-r+1, \dots, n\}$ we have,

$$\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r} \le \lambda_n\cdots\lambda_{n-r+2}\lambda_{n-r} \le \lambda_n\cdots\lambda_{n-r+2}\lambda'_{n-r} = O(m^{r(2n-r)-2}). \tag{4.8}$$

Substituting the estimates (4.8), (4.3) and (4.6) into the identity (4.2), we have

$$\lambda_n\lambda_{n-1}\cdots\lambda_{n-r+1} + O(m^{r(2n-r)-2}) = c_r m^{r(2n-r)} + O(m^{r(2n-r)-1}).$$

Divide the above identity through by the asymptotics for $\lambda_n, \dots, \lambda_{n-r+2}$ we obtain

$$\begin{aligned}\lambda_{n-r+1} &= \frac{c_{n,r}}{c_{n,r-1}} m^{2n-2r+1} + O(m^{2n-2r}) \\ &= \frac{(r-1)!^2(2n-2r)!(2n-2r+1)!}{(n-r)!^2(2n-r)!^2} m^{2n-2r+1} + O(m^{2n-2r}).\end{aligned}$$

which can be rewritten as

$$\lambda_r = \frac{(n-r)!^2(2r-1)!(2r-2)!}{(r-1)!^2(n+r-1)!^2}\, m^{2r-1} + O(m^{2r-2}).$$

This completes the proof. □

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