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# Generating the Fukaya categories of Hamiltonian $G$-manifolds

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Let $G$ be a compact Lie group and $k$ be a field of characteristic $p \geq 0$ such that $H^*(G)$ has no $p$-torsion if $p > 0$. We show that a free Lagrangian orbit of a Hamiltonian $G$-action on a compact, monotone, symplectic manifold $X$ split-generates an idempotent summand of the monotone Fukaya category $\mathcal{F}(X;k)$ if and only if it represents a non-zero object of that summand (slightly more general results are also provided). Our result is based on: an explicit understanding of the wrapped Fukaya category $\mathcal{W}(T^*G;k)$ through Koszul twisted complexes involving the zero-section and a cotangent fibre; and a functor $D^b\mathcal{W}(T^*G;k) \to D^b\mathcal{F}(X^- \times X;k)$ canonically associated to the Hamiltonian $G$-action on $X$. We explore several examples which can be studied in a uniform manner including toric Fano varieties and certain Grassmannians.

## 1 Introduction

### 1.1 Background

In this paper we develop a technique for studying monotone Fukaya categories of symplectic manifolds admitting a Hamiltonian action of a compact Lie group. The Fukaya category of a symplectic manifold is a triangulated $A_\infty$-category which organises information about Lagrangian submanifolds and their intersections. It is well-known for its appearance in Kontsevich's homological mirror symmetry conjecture, however a more basic motivation for studying the triangulated $A_\infty$-structure is that it makes information about Lagrangian intersections accessible to computation. For example, one can make sense of the notion of a *split-generator*, a Lagrangian submanifold from which (representatives of) all objects of the category can be obtained by forming iterated cones and taking direct summands. This has geometric ramifications: if $K$

split-generates then $K$ is not displaceable from any Lagrangian submanifold $L$ with $HF(L,L) \neq 0$. Finding a split-generator is also the first step in many modern proofs of homological mirror symmetry (for example [7, 57, 59]).

The aim of this paper is to determine when a free Lagrangian orbit of a Hamiltonian $G$-action split-generates a summand[1] of the Fukaya category. We show in Theorem 6.2.4 that, when the characteristic of the coefficient field is not a torsion prime for $G$, the orbit generates a summand whenever it represents a non-zero object in that summand. In the final section of the paper, we give a number of examples where this yields explicit split-generators, including toric Fano manifolds and certain Grassmannians.

## 1.2 The main result

Fix a compact Lie group $G$ and a monotone symplectic manifold $X$. Suppose that there is a Hamiltonian $G$-action on $X$ with equivariant moment map $\mu\colon X \to \mathfrak{g}^*$. There is a natural monotone Lagrangian correspondence $C \subset (T^*G)^- \times X^- \times X$ called the *moment Lagrangian* defined as follows:

$$C := \{(g, v, x, y) \in (T^*G)^- \times X^- \times X \ : \ v = \mu(gx), \ y = gx\}.$$

This was noticed by Weinstein [69] and Guillemin-Sternberg [32]. The authors first learned of it from Teleman's paper [63].

Fix a field $k$. We will use the correspondence $C$ to define a triangulated $A_\infty$-functor

$$\mathfrak{C} : D^b\mathcal{W}(T^*G; k) \to D^b\mathcal{F}(X^- \times X; k)$$

from the derived wrapped Fukaya category of $T^*G$ to the derived Fukaya category of monotone Lagrangians in $X^- \times X$ (Section 6.1).

Abouzaid [2] proves that $D^b\mathcal{W}(T^*G; k)$ is generated by the cotangent fibre $T^*_1G$ as a triangulated category, so to define a triangulated $A_\infty$-functor on $D^b\mathcal{W}(T^*G; k)$, it suffices to define it on the subcategory with a single object $T^*_1G$. Our functor is defined using the quilt formalism of [43] and the geometric composition theorem [40, 66]; we review this formalism in Section 5 and

[1] In Section 4, we explain how the quantum cohomology splits as a direct sum of local rings and the Fukaya category correspondingly splits into summands. This is a finer splitting than the usual splitting by eigenvalues of quantum product with the first Chern class.

explain how to modify it in our setting. The key point in defining this functor is the fact that the cotangent fibre $T_1^*G$ at the identity in $G$ can be composed geometrically with the correspondence $C$, and the composition is the diagonal $\Delta \subset X^- \times X$ (Lemma 6.1.2). In particular, the functor $\mathfrak{C}$ gives a map

$$\mathfrak{C}^1 : CW^*(T_1^*G, T_1^*G; k) \to CF^*(\Delta, \Delta; k). \tag{1}$$

On cohomology, this sends the wrapped Floer cohomology of the cotangent fibre of $G$ (infinite-dimensional) to quantum cohomology of $X$ (finite-dimensional). In general, the functor $\mathfrak{C}$ is far from being full or faithful.

**Remark 1.2.1** It is conjectured in [63] that the morphism from (1) can be upgraded to an $E_2$-algebra morphism.

Next, let us assume that $\mu$ is transverse to $0 \in \mathfrak{g}^*$ and that $G$ acts freely on $\mu^{-1}(0)$. In this situation, we can prove that the image $\mathfrak{C}(G)$ of the zero-section $G \subset T^*G$ is represented in $D^b\mathcal{F}(X^- \times X; k)$ by the Lagrangian submanifold:

$$\mathfrak{C}(G) \cong \left\{(x, y) \in X^- \times X \ : \ x, y \in \mu^{-1}(0), \ y = gx \text{ for some } g \in G\right\}. \tag{2}$$

In particular, when $\mu^{-1}(0)$ is a free Lagrangian orbit $L$, we have $\mathfrak{C}(G) \cong L \times L$.

To obtain more information about $\mathfrak{C}(G)$, we first study the domain of our functor $\mathfrak{C}$, namely the wrapped Fukaya category $\mathcal{W}(T^*G; k)$. Wrapped Fukaya categories of cotangent bundles have been studied extensively in the past few years. For example, the papers of Fukaya-Seidel-Smith [29, 30] and Abouzaid [2, 4] provide concrete ways of understanding the wrapped Fukaya categories of cotangent bundles: there is a full and faithful embedding of $\mathcal{W}(T^*G; k)$ into the category of $A_\infty$-bimodules over the dg-algebra $C_{-*}(\Omega G; k)$ of cubical chains on the based loop space, equipped with the Pontryagin product. Under this embedding, the zero-section $G \subset T^*G$ is identified with the trivial $A_\infty$-bimodule $k$.

In Section 2 we prove that if $p = \mathrm{char}(k)$ and $H^*(G; \mathbf{Z})$ has no $p$-torsion (if $p > 0$), then there is a quasi-isomorphism of $A_\infty$-algebras (Theorem 2.2.7) :

$$H_{-*}(\Omega G; k) \to C_{-*}(\Omega G; k)$$

While this follows from classical arguments, we could not find a proof in the literature, so we have explained this in some detail here. Since $H_{-*}(\Omega G; k)$ is a polynomial algebra, this allows us to argue that the standard Koszul complex resolving $k$ as a $H_{-*}(\Omega G; k)$-bimodule defines a *Koszul twisted complex* (Definition 3.2.1) representing $k$ in the category of $A_\infty$-bimodules over $C_{-*}(\Omega G; k) \cong$

$H_{-*}(\Omega G;k)$. In $\mathcal{W}(T^*G;k)$, this tells us that $G$ is a Koszul twisted complex built out of $T_1^*G$.

In Section 3, we show that Koszul twisted complexes are pushed forward to Koszul twisted complexes under $A_\infty$-functors. In particular, applying our functor $\mathfrak{C}$, we see that $\mathfrak{C}(G)$ can be expressed as a Koszul twisted complex built out of $\Delta$ in $\mathcal{F}(X^- \times X;k)$. A Koszul twisted complex $K$ built out of an object $K'$ has the crucial property that, if certain morphisms involved in the differential are nilpotent, then $K$ *split-generates* $K'$ (Corollary 3.3.2). This can be seen as a form of Koszul duality; analogous nilpotence conditions play a decisive role in establishing the convergence of Eilenberg-Moore spectral sequence [20]; see also [21]. For a recent study of Koszul duality in the context of Fukaya categories see [23], which has partially inspired our work.

The complex representing $\mathfrak{C}(G)$ is built out of $\Delta$, so these morphisms are elements of $CF(\Delta,\Delta;k)$ which, by the homological perturbation lemma, can be identified with the quantum cohomology $QH(X;k)$ as an $A_\infty$-algebra with vanishing differential but possibly non-trivial higher products. In particular, our main result exploits the algebra structure of $QH(X;k)$ to determine whether $\mathfrak{C}(G)$ split-generates $\Delta$.

To state our main theorem, recall that we have a block decomposition of the quantum cohomology $QH(X;k) = \bigoplus_\alpha QH(X;k)_\alpha$ into local rings (Section 4), a corresponding splitting of $\Delta$ into idempotent summands, $\bigoplus_{\alpha\in W}\Delta_\alpha$ and a decomposition of the Fukaya category into summands $D^\pi\mathcal{F}(X;k) = \bigoplus_\alpha D^\pi\mathcal{F}(X;k)_\alpha$. This is a refinement of the usual splitting of quantum cohomology into eigenspaces of quantum product with $c_1(X)$.

In Section 6, we prove our main theorem:

**Theorem 1.2.2** *Suppose $G$ is a compact Lie group and $k$ is a field of characteristic $p \geq 0$ such that $H^*(G)$ has no $p$-torsion (if $p > 0$). Let $X$ be a compact, monotone, Hamiltonian $G$-manifold such that the associated moment map $\mu : X \to \mathfrak{g}^*$ has $0$ as a regular value and that $G$ acts freely on $\mu^{-1}(0)$.*

*Then, the Lagrangian submanifold $\mathfrak{C}(G) \subset X^- \times X$, defined by (2), split-generates $\Delta_\alpha$ in $\mathcal{F}(X^- \times X)$ if and only if $HF(\mathfrak{C}(G),\Delta_\alpha;k) \neq 0$.*

*In particular, if $\mu^{-1}(0) = L$ is a free Lagrangian orbit which represents a non-zero object in the summand $D^\pi\mathcal{F}(X;k)_\alpha$, then $L$ split-generates the Fukaya category $D^\pi\mathcal{F}(X;k)_\alpha$.*

The last part of our theorem states if $\mu^{-1}(0) = L$ is a free Lagrangian orbit, then either $L$ projects to a zero object in the summand $D^\pi\mathcal{F}(X;k)_\alpha$ or it split-generates the summand. We shall see in Section 7 that both situations may arise. Determining the summands to which $L$ projects non-trivially boils down to computing $HF(L,L)$ as a module over the ring of semi-simple elements in quantum cohomology (see Remark 4.2.1).

We expect that there will be straightforward generalisations of our theorem to the case where $G$ acts on $\mu^{-1}(0)$ with finite stabilisers; in this case the immersed Lagrangian $\mathfrak{C}(G)$ should be equipped with a bounding cochain in the sense of [27, Ch. 3]. It would also be interesting to study the functor $\mathfrak{C}$ when $X$ is non-compact. However, in this case the correspondence $C$ would be non-compact and one has to arrange that holomorphic curve theory is well-behaved.

**Remark 1.2.3** Another curious consequence of our theorems is the fact that if $L$ is a free Lagrangian orbit which split-generates as in Theorem 1.2.2 then any other non-zero object in $D^\pi\mathcal{F}(X;k)_\alpha$ split-generates $D^\pi\mathcal{F}(X;k)_\alpha$. See Corollary 6.2.6.

**Remark 1.2.4** Abouzaid, Fukaya, Oh, Ohta and Ono [5] prove that a Lagrangian $L$ split-generates a summand $D^\pi\mathcal{F}(X;k)_\alpha$ of the Fukaya category if the closed-open map

$$\mathcal{CO}\colon QH^\bullet(X;k)_\alpha \to \mathrm{HH}^\bullet(CF(L_\alpha,L_\alpha;k))$$

is injective. For a proof of this in the monotone case see [59, Corollary 2.18]. Our proof does not appeal to this result, however, we observe that our generation result implies that $QH^\bullet(X;k)_\alpha \cong \mathrm{HH}^\bullet(CF(L_\alpha,L_\alpha;k))$ (see Corollary 6.2.8).

## 1.3 Examples and applications

We now present a selection of examples where Theorem 6.2.4 can be applied. We defer the study of more examples to future work in the context of mirror symmetry.

The following is an immediate corollary our main theorem coupled with of computations of Floer cohomology due to Cho-Oh [16] and Fukaya-Oh-Ohta-Ono [28]:

**Corollary 1.3.1** (Corollary 7.2.1) *Let $X$ be a toric Fano variety and $k$ be an algebraically closed field of arbitrary characteristic. Any summand $D^\pi \mathcal{F}(X;k)_\alpha$ of the Fukaya category is generated by the barycentric (monotone) torus fibre $L$ equipped with an appropriate flat $k$-line bundle $\xi_\alpha$.*

Note that, since $\mathcal{CO}$ is a unital ring homomorphism, in view of Remark 1.2.4, it is immediate [59, Corollary 2.18] that if $QH(X;k)_\alpha$ is isomorphic to a field (i.e. a finite extension of $k$) and $L$ represents a non-zero object in $D^\pi \mathcal{F}(X;k)_\alpha$, then $L$ split-generates $D^\pi \mathcal{F}(X;k)_\alpha$. The more interesting and difficult case for proving split-generation is therefore when the quantum cohomology is not semi-simple, that is, it does not split as a direct sum of fields. The above corollary proves split-generation in all cases. For example, it applies in the case of the Ostrover-Tyomkin toric Fano four-fold whose quantum cohomology was demonstrated to be non-semi-simple over $\mathbf{C}$ [53].

Since we work with monotone symplectic manifolds, the quantum cohomology can be defined over $\mathbf{Z}$ and we can choose to specialise to any coefficient field. The resulting ring $QH(X;k)$ may be semi-simple for some values of $\mathrm{char}(k)$ and not for others. This is the analogue of varieties over $\mathrm{Spec}(\mathbf{Z})$ having ramification at $(p)$:

**Example 1.3.2** The quantum cohomology of $\mathbf{CP}^n$ is the $\mathbf{Z}/(2n+2)$-graded ring

$$QH^\bullet(\mathbf{CP}^n;\mathbf{Z}) = \mathbf{Z}[H]/(H^{n+1}-1), \quad |H| = 2.$$

If $k$ is an algebraically closed field of characteristic $p$, and $n+1 = p^s q$ ($\gcd(p,q) = 1$) then

$$QH(\mathbf{CP}^n;k) \cong \bigoplus_{\mu\,:\,\mu^q=1} QH(\mathbf{CP}^n;k)_{e_\mu}$$

where $e_\mu$ is the idempotent

$$\frac{1}{q}\sum_{i=0}^{q-1} \mu^i H^{p^s i}.$$

We see that this is semisimple if and only if $p$ does not divide $n+1$ (so that there are $n+1$ roots of unity, and hence $n+1$ field summands).

At first sight, this seems like an artificial way of making the quantum cohomology non-semi-simple. However, we emphasize that Fukaya categories over a field of characteristic $p$ may contain geometric information about Lagrangian submanifolds which is not available if we work over other fields:

**Example 1.3.3** Recall that if $L$ is a Lagrangian submanifold with nonzero self-Floer cohomology and $\mathfrak{m}_0(L)$ is the count of Maslov 2 discs passing through a generic point of $L$ then $2\mathfrak{m}_0(L)$ is an eigenvalue of quantum product with $2c_1(X)$. There are many Lagrangian submanifolds in $\mathbf{CP}^n$ whose minimal Maslov number is strictly bigger than 2. For example, $\mathbf{RP}^n \subset \mathbf{CP}^n$ $(n \geq 2)$, $PSU(n) \subset \mathbf{CP}^{n^2-1}$, and many more (see [11] for some further examples). For these Lagrangians, we have $\mathfrak{m}_0(L) = 0$. The first Chern class of $\mathbf{CP}^n$ is $(n+1)H$, so its eigenvalues over $k$ are $\{(n+1)\mu \ : \mu^{n+1} = 1\}$. Zero is an eigenvalue precisely when $n+1 \equiv 0 \in k$, that is when $p = \mathrm{char}(k)$ divides $n+1$. We see that these Lagrangian submanifolds can only define non-zero objects of the Fukaya category in characteristic $p$.

**Remark 1.3.4** In [24], the authors studied an example of a Lagrangian submanifold in $\mathbf{CP}^3$ whose Floer cohomology was non-vanishing only in characteristic 5. However, there the relevant Fukaya category was semi-simple.

We will use our technology to prove the following split-generation results which crucially rely on working in non-zero characteristic:

**Proposition 1.3.5** *(a) If the real part of a toric Fano manifold is orientable, then it split-generates the Fukaya category in characteristic 2 (Example 7.2.4).*

*(b) There is a Hamiltonian $U(n)$-action on the Grassmannian $\mathrm{Gr}(n, 2n)$ with a free Lagrangian orbit $L$. If $n$ is a power of 2 then $L$ split-generates the Fukaya category in characteristic 2 (Corollary 7.4.2).*

*(c) There is a Lagrangian $PSU(n) \subset \mathbf{CP}^{n^2-1}$. If $n$ is a power of $p$ then $PSU(n)$ split-generates the Fukaya category of $\mathbf{CP}^{n^2-1}$ in characteristic $p$ (Example 7.2.3).*

These results follow directly from our split-generation criterion and computations of Floer cohomology (due respectively to Haug [33], Oh [52] and the authors (example 7.2.3)).

Another interesting consequence of our results is a non-formality result for the quantum cohomology ring when it fails to be semi-simple. Recall that we can equip $QH(X;k)$ with an $A_\infty$-structure by applying the homological perturbation lemma to $CF(\Delta, \Delta; k)$. This $A_\infty$-algebra, which we denote by $\mathcal{QH}(X;k)$, seems only to have been computed in cases where it is intrinsically formal.

There were early expectations [55] that it should be formal when $X$ is Kähler, at least over $\mathbf{C}$. However, we show:

**Proposition 1.3.6** (Corollary 7.3.5) *If $X$ is a toric Fano and $k$ is an algebraically closed field such that $QH(X;k)$ is not semisimple as a $\mathbf{Z}/2$-graded algebra, then $\mathcal{QH}(X;k)$ is not quasi-isomorphic to $QH(X;k)$.*

For example, the quantum cohomology $A_\infty$-algebra of the Ostrover-Tyomkin toric Fano four-fold [53] is not formal over $\mathbf{C}$.

## 1.4 Outline of paper

In Section 2, we collect together results on the formality of compact Lie groups and their based loop spaces over a field of arbitrary characteristic; in particular, we show that Lie groups are formal in characteristic $p$ if their cohomology ring has no $p$-torsion.

In Section 3, we develop the notion of a Koszul twisted complex and show that, in the derived wrapped Fukaya category of the cotangent bundle of a compact Lie group with no $p$-torsion, the zero-section can be written as a Koszul twisted complex built out of the cotangent fibre.

In Section 4, we review the decomposition of the quantum cohomology into idempotent summands and the corresponding decomposition of the Fukaya category.

In Section 5, we review the standard quilt theory for constructing $A_\infty$-functors from Lagrangian correspondences and adapt it to the situation where some of the Lagrangians are non-compact (but the correspondences are still compact).

In Section 6, we state and prove all the main results on generation.

Finally, in Section 7, we find split-generators in some explicit examples and give further applications.

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# 2 Formality results for compact Lie groups

In this section, we collect together formality results for $C^*(G)$ and $C_{-*}(\Omega G)$ as $A_\infty$-algebras, over a field of arbitrary characteristic, where $G$ is a compact, connected Lie group.

## 2.1 Lie groups and products of spheres

Let $G$ be a compact, connected Lie group (over $\mathbf{R}$). It is well-known that any such $G$ is a finite quotient of a product of a torus and a simple non-abelian Lie group by a central subgroup. Recall that a compact connected Lie group $G$ is called *simple* if it is non-abelian and has no closed proper normal subgroup of positive dimension. For convenience, let us also recall the classification theorem of connected, compact, simple Lie groups as given by the following table:

| | $\dim G$ | Linear group | Universal cover | Centre | $\frac{\dim G}{\operatorname{rank} G} - 1$ |
|---|---|---|---|---|---|
| $A_n (n \geq 1)$ | $n(n+2)$ | $SU(n+1)$ | | $\mathbf{Z}_{n+1}$ | $n+1$ |
| $B_n (n \geq 2)$ | $n(2n+1)$ | $SO(2n+1)$ | $Spin(2n+1)$ | $\mathbf{Z}_2$ | $2n$ |
| $C_n (n \geq 3)$ | $n(2n+1)$ | $Sp(n)$ | | $\mathbf{Z}_2$ | $2n$ |
| $D_n (n \geq 4)$ | $n(2n-1)$ | $SO(2n)$ | $Spin(2n)$ | $\mathbf{Z}_2 \ltimes \mathbf{Z}_2$ | $2n-2$ |
| $G_2$ | 14 | | | 1 | 6 |
| $F_4$ | 52 | | | 1 | 12 |
| $E_6$ | 78 | | | $\mathbf{Z}_3$ | 12 |
| $E_7$ | 133 | | | $\mathbf{Z}_2$ | 18 |
| $E_8$ | 248 | | | 1 | 30 |

Figure 1: Classification of connected, compact, simple Lie groups (The centre of $SO(2n)$ is $\mathbf{Z}_4$ if $n$ is odd, $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ if $n$ is even)

There is an extensive literature on the homotopy theory of Lie groups. We recommend the excellent survey article [46] by Mimura. In particular, let us recall the classical theorem of Hopf [34] which states that, over $\mathbf{Q}$, the singular cohomology of a compact, simple group $G$ is given by:

$$H^*(G; \mathbf{Q}) = \Lambda(x_{2n_1-1}, x_{2n_2-1}, \ldots x_{2n_l-1}) \text{ with } |x_{2n_i-1}| = 2n_i - 1$$

where rank $G = l$, dim $G = \sum_{i=1}^{l}(2n_i - 1)$, and $(n_1, \ldots, n_l)$ is called the type of $G$. In fact, $G$ has the rational homotopy type of a product of odd-dimensional spheres:

$$G \underset{\mathbf{Q}}{\cong} \prod_{i=1}^{l} S^{2n_i-1}$$

Indeed, in [58], Serre constructs a map $f : G \to \prod S^{2n_i-1}$ such that the induced map of dg-algebras:

$$f^* : C^*\left(\prod S^{2n_i-1}; \mathbf{Q}\right) \to C^*(G; \mathbf{Q})$$

is a quasi-isomorphism.

In characteristic $p > 0$, the story is more complicated. First of all, in general, cohomology groups of compact, connected, simple Lie groups can have torsion (see [46]). Nonetheless, the classical Lie groups $U(n), SU(n), Sp(n)$ have no torsion:

$$\begin{aligned} H^*(U(n); \mathbf{Z}) &= \Lambda(x_1, x_3, \ldots, x_{2n-1}) \\ H^*(SU(n); \mathbf{Z}) &= \Lambda(x_3, x_5, \ldots, x_{2n-1}) \\ H^*(Sp(n); \mathbf{Z}) &= \Lambda(x_3, x_7, \ldots, x_{4n-1}) \end{aligned}$$

However, even for these groups, the mod $p$ homotopy type is not the same as the mod $p$ homotopy type of a product of odd-dimensional spheres for all $p$ as can be detected by non-triviality of Steenrod operations. For example, $SU(3)$ is not homotopy equivalent to $S^3 \times S^5$ mod 2.

In general, for $p$ sufficiently large, Serre [58] (for classical groups) and Kumpel [39] (for exceptional groups), proved that the mod $p$ homotopy type of a compact connected simple Lie group $G$ is the same as a product of odd-spheres:

**Theorem 2.1.1** *(Serre [58], Kumpel [39]) Let $G$ be a compact connected simple Lie group of type $(n_1, \ldots, n_l)$, and $p$ be a prime then*

$$G \underset{p}{\cong} \prod S^{2n_i-1}$$

*if and only if $p \geq (\dim G/\text{rank } G) - 1$.*

In particular, for $p \geq (\dim G/\text{rank } G) - 1$ there exists a map $f : G \to \prod S^{2n_i-1}$ such that the induced map of dg-algebras:

$$f^* : C^*\left(\prod S^{2n_i-1}; \mathbf{F}_p\right) \to C^*(G; \mathbf{F}_p).$$

is a quasi-isomorphism.

## 2.2 Formality

Throughout, we will treat a dg-algebra as a special case of an $A_\infty$-algebra which has vanishing higher products. In this paper, the following notion of formality for $A_\infty$-algebras will play a key role:

**Definition 2.2.1** Let $C^*$ be a graded $A_\infty$-algebra defined over $\mathbf{Z}$. Let us write $C^*_{\mathbf{Q}}$ and $C^*_p$ for $p$ prime, the graded $A_\infty$-algebras obtained from $C^*$ by tensoring it (in the derived sense) with $\mathbf{Q}$ and $\mathbf{F}_p$ respectively. We say that $C^*$ is formal mod $p$ if there exists an $A_\infty$-quasi-isomorphism:

$$H(C^*_p) \to C^*_p$$

where we view $H(C^*)$ as an $A_\infty$-algebra with vanishing differential and higher products. Similarly, we say that $C^*$ is formal over $\mathbf{Q}$ if the same condition holds over $\mathbf{Q}$.

For a topological space $X$, we say that $X$ is formal mod $p$ (respectively over $\mathbf{Q}$) if the singular cochain complex $C^*(X)$ (viewed as an $A_\infty$-algebra) is formal mod $p$ (respectively over $\mathbf{Q}$).

**Remark 2.2.2** We note that over $\mathbf{Q}$ the singular cochain complex $C^*(X)$ for a topological space can in fact be modelled by a $C_\infty$-algebra $X$ and for simply connected spaces of finite-type the $C_\infty$ quasi-isomorphism type of this is a complete invariant of the rational homotopy type of $X$ (see [37]). On the other hand, we do not know if two $C_\infty$-algebras over $\mathbf{Q}$ that are quasi-isomorphic as $A_\infty$-algebras are also quasi-isomorphic as $C_\infty$-algebras (most probably, this is not true).

**Example 2.2.3** A sphere $S^n$ is formal mod $p$ for all $p$, and over $\mathbf{Q}$. Picking cochain representative for the fundamental class gives us a linear map that is a quasi-isomorphism:

$$\Lambda(x) = H^*(S^n) \to C^*(S^n), \text{ with } |x| = n$$

Furthermore, using a normalized simplicial chain model for $C^*(S^n)$ one sees that $x^2 = 0$ at the chain level. Hence, this map can be taken to be a dg-algebra map.

**Example 2.2.4** If $X$ and $Y$ are formal mod $p$ (respectively over $\mathbf{Q}$), then $X \times Y$ is formal mod $p$ (respectively over $\mathbf{Q}$). This follows from the Eilenberg-Zilber theorem which gives a dg-algebra quasi-isomorphism over $\mathbf{Z}$ [22, Section 17]:

$$C^*(X \times Y) \to C^*(X) \otimes C^*(Y)$$

and the Künneth theorem, which identifies $H^*(X \times Y)$ with $H^*(X) \otimes H^*(Y)$ over a field. We remark that the Eilenberg-Zilber theorem establishes quasi-isomorphisms between $C^*(X) \otimes C^*(Y)$ and $C^*(X \times Y)$ in each direction but only one of these can be taken to be a dg-algebra quasi-isomorphism. Indeed, there is an $A_\infty$-algebra quasi-isomorphism from $C^*(X) \otimes C^*(Y) \to C^*(X \times Y)$ but not a dg-algebra quasi-isomorphism (cf. Munkholm [47, Theorem 2.6]).

As a particular example, let us note that $G = T^n$ is formal mod $p$ for all $p$, and over $\mathbf{Q}$.

As a consequence of the previous discussion, we immediately conclude the following:

**Corollary 2.2.5** *A connected, compact, simple Lie group $G$ is formal over $\mathbf{Q}$, and it is formal mod $p$ for $p \geq \dim G/\text{rank } G - 1$.*

**Remark 2.2.6** Another way to see that Lie groups are formal in characteristic 0 is to consider the dg subalgebra of the de Rham complex consisting of invariant forms. These are precisely the harmonic forms, with respect to an invariant metric, therefore the subspace of harmonic forms is closed under multiplication. By the Hodge theorem, the inclusion of the subspace of harmonic forms is a quasi-isomorphic embedding of the cohomology into the de Rham cochains, proving formality. (The de Rham complex is *geometrically formal*.)

Note that Corollary 2.2.5 is obtained via the characterization of mod $p$ homotopy type of the group $G$. By a theorem of Mandell [41], the mod $p$ homotopy type of a simply-connected finite-type space $X$ can be recovered from the $E_\infty$-algebra structure on $C^*(X; \mathbf{F}_p)$. However, our interest is in a much coarser invariant of $X$, namely the $A_\infty$-algebra (i.e $E_1$-algebra) structure of $C^*(X; \mathbf{F}_p)$ up to quasi-isomorphism of $A_\infty$-algebras. In particular, note that the dg-algebra structure of $C^*(X; \mathbf{F}_p)$ is not enough to construct Steenrod operations. Therefore, it is possible for a space $X$ that $C^*(X; \mathbf{F}_p)$ is formal as an $A_\infty$-algebra mod $p$ without being formal as an $E_\infty$-algebra mod $p$. We next address this delicate formality question for the connected, compact, simple Lie groups $G$ and primes $p$ such that $H^*(G)$ does not have $p$-torsion. In view of the above corollary, our result is of particular interest when $p < \dim G/\text{rank G} - 1$.

We recall that $\Omega G$ is the based loop space of $G$. We denote by $C_*(\Omega G)$ the normalized singular chain complex. To be consistent, we prefer to use instead

the cohomologically graded complex $C_{-*}(\Omega G)$ which is concentrated in non-positive degrees. This has a dg-algebra structure coming from the Pontryagin product on $\Omega G$ given by concatenation of loops.

**Theorem 2.2.7** *Let $G$ be a connected, compact, simple Lie group and $p$ be a prime such that $H^*(G)$ has no $p$-torsion:*

*(i) The $A_\infty$-algebra $C_{-*}(\Omega G)$ is formal mod $p$.*

*(ii) The $A_\infty$-algebra $C^*(G)$ is formal mod $p$.*

We first give a proof of (i) which holds in slightly more general setting. Recall that if $H^*(G)$ has no $p$-torsion, then over a field $k$ of characteristic $p$, $H_{-*}(\Omega G; k)$ is isomorphic to a polynomial algebra.

**Proposition 2.2.8** *Let $k$ be a field. Let $G$ be an H-space such that*

$$H_{-*}(\Omega G) \cong k[y_1, y_2, \dots y_n]$$

*Then, the differential graded algebra $C_{-*}(\Omega G)$ is formal over $k$.*

**Proof** The crucial ingredient is that $C_{-*}(\Omega G)$ is more than just an $A_\infty$-algebra: it comes equipped with an $A_\infty$-morphism

$$\Phi\colon C_{-*}(\Omega G) \otimes_k C_{-*}(\Omega G) \to C_{-*}(\Omega G)$$

which makes it into a strongly homotopy commutative algebra (see Clark [17, Corollaries 1.7 and 3.5] and [62, Theorem 4.3]). In particular, one has $\Phi(x \otimes 1) = \Phi(1 \otimes x) = x$ for all $x$ where $1$ denotes the 0-chain at the identity element of the group $\Omega G$. Conceptually, the existence of this $A_\infty$-morphism is due to the fact that $\Omega G$ is homotopy equivalent to the double-loop space $\Omega^2 BG$ (cf. [8, Chapter 2]). One can use this iteratively to define an $A_\infty$-morphism

$$\Phi^{[n]} :: C_{-*}(\Omega G)^{\otimes n} \to C_{-*}(\Omega G)$$

for any $n \geq 2$ (see [47, Section 4.4]) defined by

$$\Phi^{[n]} = \Phi \circ (\Phi^{[n-1]} \otimes \mathrm{id}).$$

It follows that for all $x$, we have

$$\Phi^{[n]}(1 \otimes \dots \otimes x \otimes \dots \otimes 1) = x.$$

We next borrow an argument from [47, Section 7.2]. Let $c_i \in C_{-*}(\Omega G)$ be chains representing the cohomology classes $y_i = [c_i]$. Consider the dg-algebra map

$$H_{-*}(\Omega G) \to C_{-*}(\Omega G)^{\otimes n}$$

given by

$$y_i \to 1 \otimes \ldots \otimes c_i \otimes \ldots \otimes 1$$

where $c_i$ is inserted at the $i^{th}$ entry. This is clearly a dg-algebra homomorphism because $H_{-*}(\Omega G)$ is a polynomial algebra by hypothesis. Composing this $\Phi^{[n]}$ gives us an $A_\infty$-morphism:

$$H_{-*}(\Omega G) \to C_{-*}(\Omega G)^{\otimes n} \to C_{-*}(\Omega G)$$

which is a quasi-isomorphism since it sends $y_i$ to $c_i$. □

**Proof of Theorem 2.2.7** First, we observe that we can restrict our attention to simply-connected Lie groups. Indeed, if $G$ is any compact, connected, simple Lie group, there exists a simply-connected finite cover $\pi : \tilde{G} \to G$ such that $G$ is the quotient of $\tilde{G}$ by a subgroup of the centre of $\tilde{G}$. Hence, for $p$ not dividing the order of $H_1(G)$, the covering map is a homotopy equivalence mod $p$.

Working over a field $k$ of characteristic $p$, let us first observe that since $B\Omega G \cong G$, we have

$$C^*(G) = C^*(B\Omega G) = \mathrm{B}(C_{-*}(\Omega G))^{\#}$$

where B is the bar complex and # denotes the $k$-linear dual. Now, this is the standard bar resolution that computes $\mathrm{Ext}_{C_{-*}(\Omega G)}(k,k)$, thus we have the Eilenberg-Moore equivalence of dg-algebras:

$$\mathrm{Rhom}_{C_{-*}(\Omega G)}(k,k) \cong C^*(G)$$

On the other hand, $H_{-*}(\Omega G)$ is a symmetric algebra hence is formal by Proposition 2.2.8. Therefore, we have:

$$\mathrm{Rhom}_{H_{-*}(\Omega G)}(k,k) \cong C^*(G)$$

The left hand side has a bigrading given by the cohomological grading and the internal grading with respect to which $d$ has degree $(1,0)$. Applying the homological perturbation lemma, we get a quasi-isomorphic $A_\infty$-algebra on the homology such that the products $\mu^k : H^*(G)^{\otimes k} \to H^*(G)$ for $k \geq 2$ has bidegree $(2-k, 0)$.

On the other hand, by Koszul duality between the symmetric algebra $H_{-*}(\Omega G)$ and the exterior algebra $H^*(G)$ (cf. [12]), we know that the bigrading collapses to a single grading at the level of homology $H^*(G) \cong \mathrm{Ext}_{H_{-*}(\Omega G)}(k,k)$. Hence, for grading reasons all the higher products $\mu^k$ for $k > 2$ have to vanish. It follows that $C^*(G)$ is formal as required. □

**Remark 2.2.9** It is an interesting question whether the formality result from Theorem 2.2.7 holds in characteristic $p$ for which $H^*(G)$ has $p$-torsion. We note that $SO(3) \cong \mathbf{RP}^3$ can be shown to be formal mod 2 similar to the argument given in Example 2.2.3. Namely, recall that $H^*(\mathbf{RP}^3; \mathbf{F}_2) = \mathbf{F}_2[x]/(x^4)$ where $|x| = 1$. Using a simplicial decomposition of $\mathbf{RP}^3$, we can consider the normalized simplicial cochain complex $C^*(\mathbf{RP}^3; \mathbf{F}_2)$ which is non-zero only over $* = 0, 1, 2, 3$ and can be equipped with a product structure by dualizing the Alexander-Whitney diagonal approximation. Therefore, we can construct a dg-algebra map $\mathbf{F}_2[x]/(x^4) = H^*(\mathbf{RP}^3; \mathbf{F}_2) \to C^*(\mathbf{RP}^3; \mathbf{F}_2)$ by sending $x$ to a cochain level generator and extending it to a dg-algebra map. By construction, this is a quasi-isomorphism, thus we deduce that $C^*(\mathbf{RP}^3; \mathbf{F}_2)$ is formal.

# 3 Twisted complexes

## 3.1 Cones and nilpotence

We will use the conventions of [56, Section (3p)]. In particular, the shift functor, which we denote by $[1]$, will conceal a multitude of signs. Note that, with these conventions, if $x \in \hom^n(X, Y)$ then $x[k] \in \hom^n(X[k], Y[k])$. Recall that, given a closed morphism $\alpha \in \hom^0(X, Y)$, the cone on $\alpha$ is represented by the twisted complex

$$X[1] \overset{-\alpha[1]}{\dashrightarrow} Y.$$

To avoid confusion, we will use solid arrows to denote morphisms of degree zero and dashed arrows to denote morphisms of degree one.

Recall from [56, Section 3l] that a twisted complex $(\bigoplus_i L_i[s_i], \delta = \{\delta_{ij}\})$ is a direct sum of shifts of objects $L_i[s_i]$ together with a collection of morphisms $\delta_{ij} \colon L_i[s_i] \to L_j[s_j]$ of degree one satisfying some conditions (a lower-triangularity condition and a closedness condition [56, Equation 3.19]). When we draw a collection of objects and (dashed, degree one) arrows and call it a twisted complex, this is shorthand for the twisted complex obtained by summing the objects and taking the indicated morphisms as $\delta$. For example, the twisted complex

$$X[1] \overset{-\alpha[1]}{\dashrightarrow} Y \overset{-1[1]}{\dashleftarrow} Y[1] \overset{-\beta[1]}{\dashrightarrow} Z$$

means we take $(\bigoplus_i L_i[s_i], \delta)$ with

$$L_1[s_1] = X[1],\ L_2[s_2] = Y[1],\ L_3[s_3] = Y,\ L_4[s_4] = Z$$

and

$$\delta_{13} = -\alpha[1],\ \delta_{23} = -1[1],\ \delta_{24} = -\beta[1]$$

(all other $\delta_{ij} = 0$). Despite the fact that some of the arrows in the diagram go backwards, this is still lower-triangular ($\delta_{ij} = 0$ for $j < i$) for the labelling of objects we have picked.

A *morphism of twisted complexes* from $(\bigoplus_i L_i[s_i], \delta)$ to $(\bigoplus_j M_j[t_j], \epsilon)$ comprises a collection of morphisms $L_i[s_i] \to M_j[t_j]$. There is a differential $\mu^1_{\mathrm{Tw}}$ and a composition $\mu^2_{\mathrm{Tw}}$ (indeed, an $A_\infty$-structure $\mu^d_{\mathrm{Tw}}$) on the space of morphisms defined by [56, Equation 3.20]. For clarity, we will always write our twisted complexes (and their differentials) horizontally and morphisms between twisted complexes vertically downwards. With this convention, the operations $\mu^d_{\mathrm{Tw}}$ are defined by stacking morphisms vertically and summing over all possible $A_\infty$-compositions of morphisms (including the internal differentials of the twisted complexes) with suitable signs.

**Lemma 3.1.1** *Let $X, Y, Z$ be objects of a strictly unital $A_\infty$-category and*

$$\alpha \in \hom^0(X, Y), \quad \beta \in \hom^0(Y, Z)$$

*be two $\mu^1_{\mathrm{Tw}}$-closed morphisms. Then the twisted complex*

$$X[1] \xrightarrow{-\alpha[1]} Y \xleftarrow{-1[1]} Y[1] \xrightarrow{-\beta[1]} Z \tag{3}$$

*is quasi-isomorphic to the twisted complex*

$$\mathrm{Cone}\left(\mu^2(\beta, \alpha)\right) = \left(X[1] \xrightarrow{-\mu^2(\beta,\alpha)[1]} Z\right). \tag{4}$$

*Since the twisted complex in Equation (3) is the cone on a $\mu^1_{\mathrm{Tw}}$-closed morphism*

$$\mathrm{Cone}\,(\beta) \to \mathrm{Cone}\,(\alpha)\,,$$

*this implies that* $\mathrm{Cone}\,(\beta)$ *and* $\mathrm{Cone}\,(\alpha)$ *generate* $\mathrm{Cone}\left(\mu^2(\beta, \alpha)\right)$.

**Proof** A quasi-isomorphism is given by

$$\begin{tikzcd}
X[1] \arrow[r, dashed, "-\alpha[1]"] \arrow[d, "1[1]"'] & Y \arrow[dr, "-\beta"] & Y[1] \arrow[l, dashed, "-1[1]"'] \arrow[r, dashed, "-\beta[1]"] & Z \arrow[d, "1"] \\
X[1] \arrow[rrr, dashed, "-\mu^2(\beta,\alpha)[1]"'] & & & Z
\end{tikzcd}$$

with quasi-inverse

$$\begin{tikzcd}
X[1] \arrow[rr, dashed, "-\mu^2(\beta,\alpha)[1]"] \arrow[dd, "1[1]"'] \arrow[drr, "-\alpha[1]"] & & Z \arrow[dd, "1"] \\
& & \\
X[1] \arrow[r, dashed, "-\alpha[1]"'] & Y & Y[1] \arrow[l, dashed, "-1[1]"] \arrow[r, dashed, "-\beta[1]"'] & Z
\end{tikzcd}$$

□

**Definition 3.1.2** Given a morphism $a \in \hom^0(X[s], X)$, we set $a^1 := a$ and define $a^k \in \hom^0(X[ks], X)$, $k \geq 1$, inductively by $a^{k+1} := \mu^2(a, a^k[s])$. For simplicity of notation we will define

$$a^{m_1 \odot m_2 \odot \cdots \odot m_n} := (\ldots((a^{m_1})^{m_2})\ldots)^{m_n}$$

As we are working with $A_\infty$-algebras, which are not associative unless $\mu^1 = 0$, the precise bracketing we have chosen matters. We will say that $a$ is *nilpotent* if there exists a sequence $m_1, \ldots, m_n$ such that $a^{m_1 \odot m_2 \odot \cdots \odot m_n} = 0$.

**Corollary 3.1.3** *For any morphism $a \in \hom^0(X[s], X)$ and any sequence of integers $m_1, \ldots, m_n$,* $\mathrm{Cone}\,(a)$ *generates* $\mathrm{Cone}\left(a^{m_1 \odot m_2 \odot \cdots \odot m_n}\right)$*. In particular, if $a$ is nilpotent then* $\mathrm{Cone}\,(a)$ *split-generates $X$.*

**Proof** Lemma 3.1.1 tells us that $\mathrm{Cone}\,(a)$ and $\mathrm{Cone}\left(a^k\right)$ generate $\mathrm{Cone}\left(a^{k+1}\right)$. Inductively, we see that $\mathrm{Cone}\,(a)$ generates $\mathrm{Cone}\,(a^{m_1})$ for any $m_1 \geq 1$. Applying this argument again with $a$ replaced by $a^{m_1 \odot m_2 \odot \cdots \odot m_{n-1}}$, we see that $\mathrm{Cone}\,(a)$ and $\mathrm{Cone}\left(a^{m_1 \odot m_2 \odot \cdots \odot m_{n-1}}\right)$ generate $\mathrm{Cone}\left(a^{m_1 \odot m_2 \odot \cdots \odot m_n}\right)$. By induction, $\mathrm{Cone}\,(a)$ generates $\mathrm{Cone}\left(a^{m_1 \odot m_2 \odot \cdots \odot m_n}\right)$. If $a^{m_1 \odot m_2 \odot \cdots \odot m_n} = 0$ then

$$\mathrm{Cone}\left(a^{m_1 \odot m_2 \odot \cdots \odot m_n}\right) = X \oplus X[sm].$$

Therefore, if $a$ is nilpotent we see that $\mathrm{Cone}\,(a)$ split-generates $X$. □

The following lemma will be useful for proving nilpotence of certain morphisms.

**Lemma 3.1.4** *Suppose that $P$ is an object in an $A_\infty$-category and that $x \in \hom^0(P[s], P)$ is a closed morphism. Suppose moreover that $a \in \hom^0(P[t], P)$ and $c \in \hom^0(P[s+t+1], P)$ are such that*

$$\begin{tikzcd}
P[s+t+1] \arrow[r, dashed, "-x[t+1]"] \arrow[d, "a[s+1]"'] \arrow[dr, "c"] & P[t] \arrow[d, "a"] \\
P[s+1] \arrow[r, dashed, "-x[1]"'] & P
\end{tikzcd}$$

*defines a* $\mu^1_{\mathrm{Tw}}$*-closed morphism* $b\colon \mathrm{Cone}\left(P[s] \xrightarrow{x} P\right)[t] \to \mathrm{Cone}\left(P[s] \xrightarrow{x} P\right)$. *Then* $b^k$ *has the form*

$$\begin{tikzcd}
P[s+kt+1] \arrow[r, dashed, "-x[kt+1]"] \arrow[d, "a^k[s+1]"'] \arrow[dr, "c'"] & P[kt] \arrow[d, "a^k"] \\
P[s+1] \arrow[r, dashed, "-x[1]"'] & P
\end{tikzcd}$$

*for some* $c' \in \hom^0(P[s+kt+1], P)$. *Moreover, if* $a^m = 0$ *then* $(b^m)^2 = 0$.

**Proof** We will verify the claim about the form of $b^k$ by induction. It is clearly true for $k = 1$. We need to check that $b^{k+1}$ has no component connecting $P[(k+1)t]$ to $P[s+1]$. By definition of $\mu^2_{\mathrm{Tw}}$ (the composition for morphisms of twisted complexes [56, Equation 3.20]) this component is a sum of terms of the form $\mu^m(a, \delta_{m-1}, \ldots, \delta_2, a^k)$ where $\delta_i$ stands for some $\delta \in \hom^0(P[t], P[s+t+1])$ occuring as part of the differential in the twisted complex $\mathrm{Cone}\,(x[t])$. However, the differential in $\mathrm{Cone}\,(x[t])$ has no component connecting $P[t]$ and $P[s+t+1]$ (this would point left in our picture, where the only differential is the right-pointing arrow $-x[t+1]$).

We now show that $a^m = 0$ implies $(b^m)^2 = 0$. If $a^m = 0$ then the first part of the lemma tells us that $b^m$ has the form

$$\begin{tikzcd}
P[s+mt+1] \arrow[r, dashed, "-x[mt+1]"] \arrow[d, "0"'] \arrow[dr, "c'"] & P[mt] \arrow[d, "0"] \\
P[s+1] \arrow[r, dashed, "-x[1]"'] & P
\end{tikzcd}$$

We find $(b^m)^2$ (that is $\mu^2_{\mathrm{Tw}}(b^m, b^m)$ by stacking $b^m$ on top of itself and summing over all possible compositions:

$$\begin{tikzcd}
P[s+2mt+1] \arrow[r, dashed, "-x[2mt+1]"] \arrow[d, "0"'] \arrow[dr, "c'"] & P[2mt] \arrow[d, "0"] \\
P[s+mt+1] \arrow[r, dashed, "-x[mt+1]"] \arrow[d, "0"'] \arrow[dr, "c'"] & P[mt] \arrow[d, "0"] \\
P[s+1] \arrow[r, dashed, "-x[1]"'] & P
\end{tikzcd}$$

In this picture there are no non-zero compositions because no non-zero arrow ends where another begins, therefore $(b^m)^2 = 0$. $\square$

## 3.2 Koszul twisted complexes

We are interested in a certain class of iterated cones, a class which we will see is preserved by triangulated $A_\infty$-functors.

**Definition 3.2.1** Let $\mathcal{A}$ be an $A_\infty$-category and let $K$ be a twisted complex in $\mathrm{Tw}(\mathcal{A})$. We say $K$ is a *Koszul twisted complex built out of* $K'$ if the following conditions hold.

I. $K$ can be expressed as an iterated cone of the form:

$$\begin{aligned} K = K_n &= \mathrm{Cone}\left(K_{n-1}[s_{n-1}] \xrightarrow{x_{n-1}} K_{n-1}\right) \\ K_{n-1} &= \mathrm{Cone}\left(K_{n-2}[s_{n-2}] \xrightarrow{x_{n-2}} K_{n-2}\right) \\ \vdots \quad \vdots \quad \vdots \\ K_1 &= \mathrm{Cone}\left(K_0[s_0] \xrightarrow{x_0} K_0\right) \\ K_0 &= K' \end{aligned}$$

for some sequence of objects $K_0, \ldots, K_{n-1}$, shifts $s_0, \ldots, s_{n-1}$ and degree zero, $\mu^1_{\mathrm{Tw}}$-closed morphisms $x_0, \ldots, x_{n-1}$.

II. Moreover, for $0 \leq i \leq n-1$, each $x_i$ must be expressible via the following recursive construction. For each $i$, there exist morphisms $a_{i,j} \in \hom^0\left(K_j[s_i], K_j\right)$ and $c_{i,j} \in \hom^0\left(K_j[s_i + s_j + 1], K_j\right)$ for $0 \leq j < i$, such that $\mu^1_{\mathrm{Tw}}(a_{i,j}) = 0$ and the following diagram

$$\begin{tikzcd} K_j[s_i+s_j+1] \arrow[r, dashed, "-x_j[s_i+1]"'] \arrow[d, "a_{i,j}[s_j+1]"'] \arrow[dr, "c_{i,j}"] & K_j[s_i] \arrow[d, "a_{i,j}"] \\ K_j[s_j+1] \arrow[r, dashed, "-x_j[1]"'] & K_j \end{tikzcd}$$

viewed as a map $\mathrm{Cone}\left(x_j\right)[s_i] \to \mathrm{Cone}\left(x_j\right)$ gives $a_{i,j+1}$ for $0 \leq j < i-1$, and finally it gives $x_i$ for $j = i-1$.

If $K$ is a Koszul twisted complex as above, we call the morphisms $a_{i,0}$, $i = 0, \ldots, n-1$, its *edges*.

**Lemma 3.2.2** *Suppose that $K$ is a Koszul twisted complex in $\mathrm{Tw}(\mathcal{A})$ and that $\mathcal{F}\colon \mathrm{Tw}(\mathcal{A}) \to \mathrm{Tw}(\mathcal{B})$ is the triangulated $A_\infty$-functor $\mathrm{Tw}(\mathcal{G})$ induced by an $A_\infty$-functor $\mathcal{G}\colon \mathcal{A} \to \mathcal{B}$. Then $\mathcal{F}(K)$ is a Koszul twisted complex in $\mathrm{Tw}(\mathcal{B})$.*

**Proof** By [56, Lem. 3.30], we have $\mathrm{Cone}\left(\mathcal{F}^1(m)\right) = \mathcal{F}(\mathrm{Cone}\,(m))$ for any morphism $m$, so we only need to check that Condition II of the definition of Koszul twisted complexes remains true for $\mathcal{F}^1(a_{i,j+1})$. The way we define $\mathcal{F}$ for morphisms of twisted complexes means that applying $\mathcal{F}$ to the diagram in Condition II yields

$$\begin{tikzcd}
\mathcal{F}(K_j)[s_i+s_j+1] \arrow[r, dashed, "-\mathcal{F}^1(x_j)[s_i+1]"] \arrow[d, "\mathcal{F}^1(a_{i,j})[s_j+1]"'] \arrow[dr, "C"] & \mathcal{F}(K_j)[s_i] \arrow[d, "\mathcal{F}^1(a_{i,j})"] \\
\mathcal{F}(K_j)[s_j+1] \arrow[r, dashed, "-\mathcal{F}^1(x_j)[1]"'] & \mathcal{F}(K_j)
\end{tikzcd}$$

where

$$C = \mathcal{F}^1(c_{i,j}) \pm \mathcal{F}^2(a_{i,j}, x_j[s_i+1]) \pm \mathcal{F}^2(x_j[1], a_{i,j}[s_j+1]).$$

This diagram now represents $\mathcal{F}^1(a_{i,j+1})$ and this has the required form. □

### 3.3 Extreme cases

There are two extreme cases which are useful to consider: the case when the edges of a Koszul twisted complex are nilpotent, and the case when one of the morphisms $a_{i,j}$ is a quasi-isomorphism. In the first case, we will see that $K$ split-generates $K'$. In the second case, we will see that $K$ is quasi-isomorphic to the zero object.

**Lemma 3.3.1** *Suppose $K$ is a Koszul twisted complex built out of $K'$, for which $(a_{i,j})^{m\odot 2^{\odot q}} = 0$ for some $i,j$. Then $(a_{i,j+1})^{m\odot 2^{\odot(q+1)}} = 0$. In particular, if $(a_{i,0})^m = 0$ then $(a_{i,i})^{m\odot 2^{\odot i}} = 0$.*

**Proof** This is immediate from Lemma 3.1.4. □

**Corollary 3.3.2** *If $K$ is a Koszul twisted complex built out of $K'$, such that there exist integers $m_0, \ldots, m_{n-1}$ for which $(a_{i,0})^{m_i} = 0$ for each $i = 0, 1, \ldots, n-1$ then $K$ split generates $K'$.*

**Proof** Lemma 3.3.1 tells us that $x_i = a_{i,i}$ is nilpotent. Corollary 3.1.3 then implies that $K_{i+1}$ split-generates $K_i$. Since this holds for each $i$, this implies that $K = K_n$ split-generates $K_0 = K'$. □

**Lemma 3.3.3** *Suppose that $K$ is a Koszul twisted complex in which $a_{i,j}$ is a quasi-isomorphism for some $0 \leq j \leq i \leq n-1$. Then $a_{i,j+1}$ is also a quasi-isomorphism*

**Proof** Let $J$ be an object of the given $A_\infty$-category. We will show that the induced map

$$H(\hom(J, K_{j+1}[s_i])) \xrightarrow{(a_{i,j+1})_*} H(\hom(J, K_{j+1}))$$

is an isomorphism. Since $K$ is a Koszul twisted complex, $\hom(J, K_{j+1})$ has the form

$$\left(\hom(J, K_j[1]) \oplus \hom(J, K_j[s_j]), \begin{pmatrix} \partial_{J,K_j}[1] & 0 \\ -x_j[1] & \partial_{J,K_j}[s_i] \end{pmatrix}\right)$$

This differential is lower triangular, hence there is a two-step spectral sequence whose $E_1$-page is $H(\hom(J, K_j[1])) \oplus H(\hom(J, K_j[s_i]))$ and which converges to $H(\hom(J, K_{j+1}))$. The map $\left(a_{i,j+1}\right)_*$ has the form

$$\begin{pmatrix} a_{i,j}[s_j+1] & 0 \\ c_{i,j} & a_{i,j} \end{pmatrix}$$

and therefore induces a map of filtered complexes between $\hom(J, K_j[s_i])$ and $\hom(J, K_j)$. This map is an isomorphism on the $E_1$-page since $a_{i,j}$ is a quasi-isomorphism. Therefore $(a_{i,j+1})_*$ is an isomorphism as desired. A similar argument proves that

$$H(\hom(K_{j+1}, J)) \xrightarrow{(a_{i,j+1})^*} H(\hom(K_{j+1}[s_i], J))$$

is an isomorphism for any test object $J$. □

**Corollary 3.3.4** *Suppose that $K$ is a Koszul twisted complex such that $a_{i,0}$ is a quasi-isomorphism for some $i$. Then $K$ is quasi-isomorphic to zero.*

**Proof** By induction from Lemma 3.3.3, it follows that $a_{i,j}$ is a quasi-isomorphism for all $0 \leq j \leq i$. In particular, $x_i = a_{i,i}$ is a quasi-isomorphism, hence its cone, $K_{i+1}$, is quasi-isomorphic to zero. Since $K = K_n$ is generated by $K_{i+1}$ (by definition), $K$ must also be quasi-isomorphic to zero. □

### 3.4 Koszul twisted complexes in cotangent bundles of groups

Let $k$ be a field and let $e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+n}$ be a collection of graded variables with gradings in negative even degrees $-s_1, \ldots, -s_m$ and $s_{m+1} = \ldots = s_{m+n} = 0$. Let $A$ denote the $\mathbf{Z}$-graded algebra

$$k[e_1, \ldots, e_m] \otimes k[e_{m+1}^{\pm}, \ldots, e_{m+n}^{\pm}]$$

generated by these variables, considered as a $\mathbf{Z}$-graded $A_\infty$-algebra with vanishing higher products. The algebra $A$ is the prototype of $H_{-*}(\Omega G)$ for $G$ a compact connected Lie group that is a product of a simple Lie group and a torus.

Let $(A, A)$-bimod denote the category of $A_\infty$-bimodules over $A$. We summarise some basic results about Koszul complexes (in the usual sense).

**Lemma 3.4.1** *(a) There is a unique $A_\infty$ $A$-bimodule structure on the field $k$, where the $e_i$ for $i = 1, \ldots m$ and $1 - e_i$ for $i = m+1, \ldots m+n$ annihilate the module and the higher $A_\infty$-operations are zero (for grading reasons).*

*(b) Let $K(e_i)$ denote the complex $A[-s_i + 1] \xrightarrow{-e_i[1]} A$ for $i = 1, \ldots, m$ and the complex $A[1] \xrightarrow{-1[1]+e_i[1]} A$ for $i = m+1, \ldots m+n$, and let $K(\underline{e})$ be the tensor product $K(e_1) \otimes \cdots \otimes K(e_{m+n})$. Let $k$ be the ground field considered as an $A_\infty$-bimodule with trivial higher products. The complex $K(\underline{e})$ is the standard Koszul resolution of $k$.*

**Lemma 3.4.2** *In the $A_\infty$-category* $\mathrm{Tw}((A, A)\text{-bimod})$*, the object $K(\underline{e})$ is a Koszul twisted complex built out of $A$, with shifts $s_i$, edges $a_{i,0} = e_i$ for $i = 1, \ldots, m$ and $a_{i,0} = 1 - e_i$ for $i = m+1, \ldots m+n$, and diagonal morphisms $c_{i,j} = 0$.*

**Proof** Let $C$ be a complex of ordinary $A$-modules let $e_i\star$ denote the left module action of $e_i \in A$. The total tensor product complex for $C \otimes K(e_i)$ exhibits $C \otimes K(e_i)$ as a cone on the morphism $e_i\star\colon C[s_i] \to C$. If $C$ is itself a cone $\mathrm{Cone}\left(C_1 \xrightarrow{\gamma} C_2\right)$ then this module action has the form

$$\begin{tikzcd}
C_1[s_i+1] \arrow[r, dashed, "-\gamma[s_i+1]"] \arrow[d, "(e_i\star)[1]"'] \arrow[dr, "0"] & C_2[s_i] \arrow[d, "e_i\star"] \\
C_1[1] \arrow[r, dashed, "-\gamma[1]"'] & C_2
\end{tikzcd}$$

If we take $C_k = K(e_1)\otimes\cdots\otimes K(e_k)$ then we deduce that $C_{k+1} = \mathrm{Cone}\left(e_{k+1}\star\right)$. □

**Corollary 3.4.3** *Let $G$ be a compact, connected Lie group and let $k$ be a field of characteristic $p \geq 0$ such that $p$ is not a torsion prime for $G$. Then, in the wrapped Fukaya category of $T^*G$, the zero-section is quasi-isomorphic to a Koszul twisted complex built out of $T^*_1G$.*

**Proof** As usual, by passing to a finite cover (which is a homotopy equivalence away from torsion primes for $G$), we can restrict our attention to the case of a product of a simple non-abelian Lie group and a torus. By Theorem 2.2.7, the dga $C_{-*}(\Omega G;k)$ of cubical chains on the based loop space of $G$, equipped with the Pontryagin product, is quasi-isomorphic to its homology, $H_{-*}(\Omega G;k)$. This homology group is tensor product of a polynomial ring on generators of negative even degree and a Laurent polynomial ring in degree 0 so Lemma 3.4.2 applies.

There is a full and faithful embedding of the wrapped Fukaya category of $T^*G$ into $(C_{-*}(\Omega G;k), C_{-*}(\Omega G;k))$-bimod [2] which takes a Lagrangian brane to its Floer $A_\infty$-bimodule with the cotangent fibre. Under this embedding, the $A_\infty$-bimodule $k$ represents the zero-section, since the zero-section and the cotangent fibre intersect at precisely one point. Lemma 3.4.2 tells us that there is a Koszul twisted complex $K$ quasi-isomorphic to $k$, built out of $C_{-*}(\Omega G;k)$. Since the wrapped Fukaya category maps fully faithfully into the bimodule category, this tells us that there is a Koszul twisted complex in $\mathcal{W}(T^*G)$ quasi-isomorphic to the zero-section, built out of $T^*_1G$. □

### 3.5 Twisted complexes of functors

Recall that an $A_\infty$-functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ comprises an assignment of objects $\mathcal{F}X \in \mathcal{B}$ to every object $X \in \mathcal{A}$ and a sequence of multilinear maps

$$\mathcal{F}^k\colon \hom_{\mathcal{A}}(X_k, X_{k+1}) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_1, X_2) \to \hom_{\mathcal{B}}(FX_1, FX_2)[1-k]$$

satisfying a sequence of equations (see [56, Eq (1.6)]). The non-unital $A_\infty$-functors $\mathcal{A} \to \mathcal{B}$ form an $A_\infty$-category $\mathfrak{nu}\text{-}\mathfrak{fun}(\mathcal{A}, \mathcal{B})$. Given $F, G \in \mathfrak{nu}\text{-}\mathfrak{fun}(\mathcal{A}, \mathcal{B})$, an element $T \in \hom^g_{\mathfrak{nu}\text{-}\mathfrak{fun}}(F, G)$ is given by a *pre-natural transformation* $T^0, T^1, T^2, \ldots$ where $T^0$ assigns an element $T^0_X \in \hom^g_{\mathcal{B}}(FX, GX)$ for each $X$ in $\mathcal{A}$ and where $T^k\colon \hom(X_k, X_{k+1}) \otimes \cdots \otimes \hom(X_1, X_2) \to \hom(X_1, X_{k+1})[g-k]$ are multilinear maps.

There is an $A_\infty$-structure $\lambda$ on $\mathfrak{nu}\text{-}\mathfrak{fun}(\mathcal{A}, \mathcal{B})$, described in [56, Eq. 1.9]. The following property of this $A_\infty$-structure will be important to us: if $S, T \in \hom_{\mathfrak{nu}\text{-}\mathfrak{fun}}(\mathrm{Id}_{\mathcal{A}}, \mathrm{Id}_{\mathcal{A}})$ then we have

$$\left(\lambda^k(T_k, \ldots, T_1)\right)^0_X = \mu^k_{\mathcal{A}}((T^0_k)_X, \ldots, (T^0_1)_X). \tag{5}$$

Let $(\bigoplus_i F_i, \{T_{ij}\})$ be a twisted complex of functors in $\mathrm{Tw}(\mathfrak{nu}\text{-}\mathfrak{fun}(\mathcal{A}, \mathcal{B}))$. Using Equation (5), the Maurer-Cartan equation

$$\sum \lambda^k(T, \ldots, T) = 0$$

implies the Maurer-Cartan equation

$$\sum \mu^k(T^0_X, \ldots, T^0_X) = 0$$

for the twisted complex $(\bigoplus_i F_i X, \{(T^0_{ij})_X\})$ in $\mathcal{B}$. The following lemma is easy to verify.

**Lemma 3.5.1** *Let $\mathcal{A}$ and $\mathcal{B}$ be strictly unital $A_\infty$-categories. Suppose that $F = (\bigoplus_i F_i, \{S_{ij}\})$ and $G = (\bigoplus_i G_i, \{T_{ij}\})$ are quasi-isomorphic twisted complexes in $\mathrm{Tw}(\mathfrak{nu}\text{-}\mathfrak{fun}(\mathcal{A}, \mathcal{B}))$. Then for any $X \in \mathcal{A}$, the objects $(\bigoplus_i F_i X, \{S^0_{ij} X\})$ and $(\bigoplus_i G_i X, \{T^0_{ij} X\})$ are quasi-isomorphic in $\mathrm{Tw}(\mathcal{B})$.*

## 4 Quantum cohomology and generation of the diagonal

### 4.1 Block decomposition of quantum cohomology

The quantum cohomology of a monotone symplectic manifold decomposes as a direct sum of local rings. It is not special to quantum cohomology, and the

proof cited works for any (super) commutative $k$-algebra, finite-dimensional over $k$.

**Lemma 4.1.1** *Let $X$ be a compact, monotone symplectic manifold and $k$ be an arbitrary field. Then there exists a finite set $W$ and idempotent elements*

$$\{e_\alpha \in QH^{ev}(X;k)\}_{\alpha \in W}$$

*such that $1 = \sum_{\alpha \in W} e_\alpha$ and $e_\alpha e_\beta = 0$ if $\alpha \neq \beta$. Moreover, each summand*

$$QH(X;k)_\alpha = e_\alpha QH(X;k)$$

*is a subalgebra which is a local ring with nilpotent maximal ideal. In other words, for every element $x \in QH(X;k)_\alpha$, either $x^m = 0$ for some $m$ or else there exists $y \in QH(X;k)_\alpha$ such that $xy = e_\alpha$.*

**Proof** A finite-dimensional algebra over a field is an Artinian ring. Now, a commutative Artinian ring is uniquely (up to isomorphism) a finite direct product of commutative Artininan local rings (see for example [9, Theorem 8.7]). The assertions follow easily from this. □

**Remark 4.1.2** The property of a commutative ring splitting as a finite direct sum of local rings with nilpotent maximal ideal is equivalent to being the ring of functions on a zero-dimensional Noetherian scheme. In the case of quantum cohomology of a toric Fano variety, which is supported in even degree and hence commutative, the zero-dimensional scheme in question is the critical locus of the mirror superpotential, which consists of isolated critical points.

**Remark 4.1.3** For each $\alpha$, let $\mathfrak{m}_\alpha$ be the maximal ideal of the local ring $QH(X;k)_\alpha$, then it follows by the nullstellensatz that the residue field $k_\alpha = QH(X;k)_\alpha/\mathfrak{m}_\alpha$ is a finite extension of $k$.

## 4.2 Lagrangians in $X^- \times X$

Recall that the quantum cohomology is isomorphic, via the PSS map, to the self-Floer cohomology of the diagonal $\Delta \subset (X^- \times X, (-\omega) \oplus \omega)$. It therefore admits a chain-level description as an $A_\infty$-algebra $CF(\Delta, \Delta; k)$. We can transfer this $A_\infty$-structure to the quantum cohomology via homological perturbation. Seidel [56, Lem. 4.2] tells us that the idempotents $e_\alpha$ in $QH(X;k)$ occur as the first part $\mathcal{E}^1_\alpha$ of an *idempotent up to homotopy*, $\mathcal{E}_\alpha$, defining an object in the

Fukaya category of $X^- \times X$ which behaves like a summand of $\Delta$ and which we will write as $\Delta_\alpha$. Note that

$$HF(\Delta_\alpha, \Delta_\beta; k) = e_\alpha QH(X;k) e_\beta = \begin{cases} QH(X;k)_\alpha & \text{if } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$$

Let $L \subset X$ be a Lagrangian submanifold. Under the zeroth closed-open map $CO^0 \colon QH(X;k) \to HF(L,L;k)$, the idempotent $e_\alpha$ is sent to an idempotent (possibly zero) in $HF(L,L;k)$. Again, this lifts to an idempotent up to homotopy and defines an object $L_\alpha$ in the Fukaya category $D^\pi \mathcal{F}(X;k)$. Since the identity element splits as a sum of orthogonal idempotents, the Fukaya category splits as a disjoint union of subcategories $D^\pi \mathcal{F}(X;k)_\alpha$ comprising precisely those summands in which $CO^0(e_\alpha)$ hits the identity element.

**Remark 4.2.1** We emphasise that $L_\alpha \neq 0$ if and only if $CO^0(e_\alpha) \neq 0 \in HF(L,L;k)$. In Section 7 below, much of the challenge will be to decide, given a Lagrangian $L$ with $HF(L,L;k) \neq 0$, for which $\alpha$ is $HF(L_\alpha, L_\alpha; k) \neq 0$? By [59, Proposition 1.2], we have $CO^0(2c_1(X)) = 2\mathfrak{m}_0(L)1_L$, so, in the happy circumstance that the local summands of $QH(X;k)$ are generalised eigenspaces of $2c_1(X)$ with distinct eigenvalues, one knows that $L$ belongs to the summand corresponding to the eigenvalue $2\mathfrak{m}_0(L)$. In many interesting examples, the local summands refine the splitting into generalised eigenspaces, so one needs more information than $\mathfrak{m}_0(L)$ to decide for which summands $\alpha$ we have $L_\alpha \neq 0$. See Remark 7.1.2.

**Lemma 4.2.2** *Suppose $M$ is an object in the Fukaya category of $X^- \times X$ which is quasi-isomorphic to a Koszul twisted complex built out of $\Delta$. Then $M$ split-generates the subcategory $D^\pi \mathcal{F}(X;k)_\alpha$ if and only if $HF(M, \Delta_\alpha) \neq 0$.*

**Proof** Since $\Delta = \bigoplus_{\alpha \in W} \Delta_\alpha$ and the objects $\Delta_\alpha$ are orthogonal in the Fukaya category, the Koszul twisted complex representing the object $M$ splits as a direct sum $\bigoplus M_\alpha$ of objects each of which is quasi-isomorphic to a Koszul twisted complex $K_\alpha$ built out of $\Delta_\alpha$. We will analyse these independently of one another, so for the remainder of the proof we will fix $\alpha \in W$ and omit the $\alpha$-decorations from the morphisms in the Koszul twisted complex.

By Lemma 4.1.1, the morphisms $a_{i,0}$ in the Koszul twisted complex $K_\alpha$ are all either nilpotent or invertible in the ring $QH(X;k)_\alpha$. In case one of them is invertible, it gives a quasi-isomorphism $\Delta_\alpha \to \Delta_\alpha$ and hence, by Corollary 3.3.4, $M_\alpha = 0$. In case none of the $a_{i,0}$ is invertible, they are all nilpotent,

and Corollary 3.3.2 implies that $M_\alpha$ split-generates $\Delta_\alpha$. Therefore $M_\alpha$ split-generates $\Delta_\alpha$ if and only if $M_\alpha \neq 0$.

Since $M_\alpha$ is itself generated by $\Delta_\alpha$, and since $HF(M, \Delta_\alpha; k) = HF(M_\alpha, \Delta_\alpha; k)$, we see that $M_\alpha \neq 0$ if and only if $HF(M, \Delta_\alpha; k) \neq 0$; hence $M_\alpha$ split-generates $\Delta_\alpha$ if and only if $HF(M, \Delta_\alpha; k) \neq 0$. □

# 5 Quilted Floer theory

In Section 5.1, we will briefly review the idea of quilted Floer theory in the form it appears in [43]. We will need this theory to work in a slightly more general setting, involving flat line bundles (Section 5.2) and some mildly noncompact Lagrangians in exact manifolds with non-trivial fundamental group (Section 5.3). In each case, we will indicate what technical modifications need to be made. We expect that the construction given here can be generalised vastly and carried out in a more natural setting by appealing to the recent work of Fukaya [25].

## 5.1 Standard theory

Given a symplectic manifold $Z$ with symplectic form $\omega$, we will use $Z^-$ to denote the same manifold equipped with the symplectic form $-\omega$. A *Lagrangian correspondence* between $Y$ and $Z$ is a Lagrangian submanifold $C \subset Y^- \times Z$. We will write $C^-$ for the same submanifold considered as a correspondence in $Z^- \times Y$.

A *generalised Lagrangian correspondence from $Y$ to $Z$* is a sequence $\underline{L} = \{L_{i,i+1} \subset A_i^- \times A_{i+1}\}_{i=0}^{k-1}$ of Lagrangian correspondences with $A_0 = Y$ and $A_k = Z$. A brane structure on a generalised correspondence comprises a choice of orientation, relative spin structure and grading on each $L_{i,i+1}$ (the grading may only live in $\mathbf{Z}/2$). We denote by $\underline{L}^T$ the generalised Lagrangian correspondence $\left(L_{k-1,k}^-, \ldots, L_{01}^-\right)$, where the minus sign denotes a reversal of orientation and associated brane data.

Given two generalised correspondences $\underline{K}$ from $Z_1$ to $Z_2$ and $\underline{L}$ from $Z_2$ to $Z_3$, there is a concatenated generalised correspondence $(\underline{K}, \underline{L})$ from $Z_1$ to $Z_3$. A

generalised Lagrangian correspondence from $\{pt\}$ to $Z$ is called a *generalised Lagrangian submanifold* of $Z$.

Given

$$\underline{K} = \{K_{i,i+1} \subset A_i^- \times A_{i+1}\}_{i=0}^{k-1} \qquad \underline{L} = \{L_{i,i+1} \subset B_i^- \times B_{i+1}\}_{i=0}^{\ell-1} \tag{6}$$

generalised Lagrangian submanifolds of $Z = A_k = B_\ell$, a *generalised intersection point* is a tuple

$$(x_0, x_1, \ldots, x_{k+\ell}) \in A_0 \times \cdots \times A_{k-1} \times Z \times B_{\ell-1} \times \cdots \times B_0$$

such that $(x_i, x_{i+1}) \in K_{i,i+1}$ for $i < k$ and $(x_i, x_{i+1}) \in L^-_{\ell-i,\ell-i+1}$ for $i \geq k$. Let us denote by $\underline{K} \cap \underline{L}$ the set of generalised intersection points of $(\underline{K}, \underline{L}^T)$, which one can arrange to be a finite set after a Hamiltonian perturbation.

Given $\underline{K}$ and $\underline{L}$ as in Equation (6), one defines the Floer cochain complex $CF(\underline{K}, \underline{L})$ to be

$$\bigoplus_{\underline{x} \in \underline{K} \cap \underline{L}} k\langle \underline{x} \rangle.$$

We explain the construction of the differential, which counts pseudoholomorphic quilted strips. First let $\underline{M} = (\underline{K}, \underline{L}^T) = \{M_{i,i+1} \subset C_i^- \times C_{i+1}\}_{i=0}^{m-1}$ $(m = k+\ell)$. The domain of a pseudoholomorphic quilted strip is $\mathbf{R} \times [0,1]$, but it has a number of *seams* $\sigma_i = \mathbf{R} \times \{p_i\}$, $i = 1, \ldots, m-2$. The seams cut the domain into *patches* $S_i = \mathbf{R} \times [p_{i-1}, p_i]$, $i = 1, \ldots, m-1$, where we define $p_0 = 0$ and $p_{m-1} = 1$. A *quilted strip* consists of an $(m-1)$-tuple $\underline{u} = (u_1, \ldots, u_{m-1})$ of maps $u_i \colon S_i \to C_i$ for $i = 1, \ldots, m-1$, satisfying the boundary and seam conditions

$$u_1(s,0) \in M_{01}, \quad (u_i(s,p_i), u_{i+1}(s,p_i)) \in M_{i,i+1}, \quad u_{m-1}(s,1) \in M_{m-1,m}$$

for all $s \in \mathbf{R}$. A *pseudoholomorphic quilted strip* is a finite-energy quilted strip such that each $u_i$ satisfies the Floer equation with respect to a choice of translation invariant time-dependent almost complex structure $J_i$ on $C_i$ and a suitable choice of Hamiltonian perturbation.

More generally, one can define a *quilted Riemann surface* [65, Definition 3.1], which is a Riemann surface $S$ with strip-like ends, separated into patches $S_i$ by a collection of oriented, properly embedded, disjoint real analytic arcs. We write $\sigma_{ij}$ for an oriented seam which has $S_i$ on its right and $S_j$ on its left; if $b$ is a component of $\partial S$ then we write $S_{i(b)}$ for the patch containing $b$.

**Definition 5.1.1** A *labelled, quilted Riemann surface* is a quilted Riemann surface $S$ as in the previous paragraph together with labels: we label each patch

$S_i$ with a target manifold $Z_i$, each seam $\sigma_{ij}$ with a Lagrangian $L_{ij} \subset Z_i^- \times Z_j$ and each component $b$ of $\partial S$ with a Lagrangian $L_b \subset Z_{i(b)}$. A *quilted map* modelled on a labelled, quilted Riemann surface is a collection of maps $u_i \colon S_i \to Z_i$ satisfying the boundary and seam conditions

$$u_{i(b)}(z) \in L_{i(b)} \text{ for } z \in b, \qquad (u_i(z), u_j(z)) \in L_{ij} \text{ for } z \in \sigma_{ij}.$$

One can define higher $A_\infty$-operations for collections of generalised Lagrangian submanifolds, in a similar way to the differential, by counting pseudoholomorphic quilted Riemann surfaces modelled on punctured discs with strip-like ends asymptotic to generalised intersection points. The underlying moduli spaces of quilted Riemann surfaces which one uses have natural compactifications by allowing nodal degenerations. The $A_\infty$-equations follow from the fact that these compactifications are homeomorphic to associahedra [42, 43, 44].

**Remark 5.1.2** In fact, with this notion of pseudoholomorphic quilt, it is not possible to achieve transversality for moduli spaces when some but not all of the maps $u_i$ are constant. We must allow *folded perturbations* of the Floer equation. To define these perturbations, we pick a collection of squares $(-\delta, \delta) \times (p_i - \epsilon, p_i + \epsilon)$ in $\mathbf{R} \times [0, 1]$ centred on the seams. For each square, let $\tau_i$ be the reflection in the seam. Consider the map $(-\delta, \delta) \times [p_i, p_i + \epsilon] \to C_i^- \times C_{i+1}$ defined by $z \mapsto (u_i(\tau(z)), u_{i+1}(z))$. We require that this solves Floer's equation in $C_i^- \times C_{i+1}$ for a $z$-dependent almost complex structure $J$ which has the split form $(-J_i) \oplus J_{i+1}$ except in a half-ball in $(-\delta, \delta) \times [p_i, p_i + \epsilon]$ centred at $(0, p_i)$. In more general quilted domains, discussed above, the seams are required to be real analytic; this allows one to fold locally along seams and, hence, make sense of folded perturbations on balls centred on seams. See also [68].

In order that the pseudoholomorphic quilt moduli spaces above define an $A_\infty$-structure, we will require some monotonicity conditions to hold. First, there is the requirement that each generalised Lagrangian submanifold is monotone:

**Definition 5.1.3** Let $Z$ be a compact, monotone, symplectic manifold. A generalized Lagrangian submanifold $\underline{L}$ of $Z$ is monotone if all the $A_i$ are monotone with the same monotonicity constant and if all the $L_{i,i+1}$ are monotone Lagrangian branes with minimal Maslov number $N_{L_{i,i+1}} \geq 2$. Note that if $Z_{i(b)}$ is exact and $c_1(Z_{i(b)}) = 0$ and $L_b$ is an exact, Maslov zero Lagrangian, then it is $\tau$-monotone for any $\tau$, since both area and index are zero.

To ensure that the Floer differential squares to zero, we also need control on Maslov 2 disc bubbling:

**Definition 5.1.4** If $\underline{M} = \{M_{i,i+1} \subset C_i^- \times C_{i+1}\}_{i=0}^{m-1}$ is a monotone Lagrangian correspondence from $\{pt\}$ to $\{pt\}$ then define

$$\mathfrak{m}_0(\underline{M}) = \sum_{i=0}^{m-1} \mathfrak{m}_0(M_{i,i+1}),$$

where $\mathfrak{m}_0(M_{i,i+1})$ is the count of Maslov 2 discs in $C_i^- \times C_{i+1}$ with boundary on $M_{i,i+1}$ and one point constraint on the boundary.

**Remark 5.1.5** If $\underline{K}$ and $\underline{L}$ are monotone generalised Lagrangian submanifolds of $Z$, and $\partial$ denotes the differential on $CF(\underline{K}, \underline{L}; k)$ then $\partial^2 = \mathfrak{m}_0((\underline{K}, \underline{L}^T))\,\mathrm{Id}$ (proved as in [50, 51]).

Finally, to ensure that counts of pseudoholomorphic quilts defining the $A_\infty$-structure are finite, and to control bubbling, one requires a *simultaneous* monotonicity condition to hold [65, Definition 3.6]. We summarise these conditions in the following definition:

**Definition 5.1.6** A (possibly infinite) set of generalized Lagrangian submanifolds $\{\underline{K}_i\}_{i\in I}$ of $Z$ is called *an admissible set* if there exists a $\tau \geq 0$ such that:

- each $\underline{K}_i$ is a $\tau$-monotone generalised Lagrangian submanifold;
- for all $i, j \in I$, we have $\mathfrak{m}_0(\underline{K}_i, \underline{K}_j^T) = 0$; and
- for all continuous quilted maps modelled on a compact, genus zero quilted Riemann surface, where each seam is labelled by a component of one of the $\underline{K}_i$, we have

$$\mathrm{area}(\underline{u}) = \tau\,\mathrm{ind}(\underline{u}). \tag{7}$$

Equation (7) is automatically satisfied if the generalised Lagrangian submanifolds are monotone with the same monotonicity constant and all the manifolds $Z_i$ are simply-connected [65, Remark 3.7].

In [43], Mau, Wehrheim and Woodward defined an $A_\infty$-category $\mathcal{F}^\#(Z; k)$ which they call the *extended Fukaya category*, whose objects are given by an admissible set of generalised Lagrangian submanifolds of $Z$.

Given a monotone Lagrangian correspondence $L \subset Y^- \times Z$ with a brane structure, [43, Theorem 1.1] tells us that there is an $A_\infty$-functor

$$\Phi(L)\colon \mathcal{F}^\#(Y;k) \to \mathcal{F}^\#(Z;k)$$
$$\Phi(L)(L_{01},\dots,L_{k-1,k}) = (L_{01},\dots,L_{k-1,k},L)$$

if one can ensure that this assignment preserves the requirements of admissibility as given in Definition 5.1.6. Moreover, the assignment $\Phi\colon \mathcal{F}(Y^- \times Z;k) \to \mathfrak{nu}\text{-}\mathfrak{fun}(\mathcal{F}^\#(Y;k),\mathcal{F}^\#(Z;k))$ is an $A_\infty$-functor.

Below, whenever we describe a general result about the functors $\Phi(L)$, we implicitly assume that the functor preserves admissibility (a condition that we check when we make an explicit use of these functors).

**Definition 5.1.7** Let $L \subset Z_1^- \times Z_2$ and $M \subset Z_2^- \times Z_3$ be Lagrangian correspondences. We say the pair $(L,M)$ is composable if:

- The intersection
$$L \times_{Z_2} M := (L \times M) \cap \left(Z_1^- \times \Delta_{Z_2}^- \times Z_3\right)$$
is transverse.
- The projection $\mathrm{pr}_{Z_1^- \times Z_3}\colon Z_1^- \times Z_2 \times Z_2^- \times Z_3 \to Z_1^- \times Z_3$ restricted to $L \times_{Z_2} M$ is an embedding.

we define $L \circ M = \mathrm{pr}_{Z_1^- \times Z_3}(L \times_{Z_2} M)$ and we call this the *geometric composition* of $L$ and $M$.

**Theorem 5.1.8** ([43, Theorem 1.2]) *If $Z_1, Z_2, Z_3$ are monotone symplectic manifolds with the same monotonicity constants and $L \subset Z_1^- \times Z_2$ and $M \subset Z_2^- \times Z_3$ are composable monotone generalised Lagrangian correspondences with brane structures then*

$$\Phi(L) \circ \Phi(M) \simeq \Phi(L \circ M)$$

*where $\simeq$ denotes quasi-isomorphism of $A_\infty$-functors.*

Mostly, we will use this result only in the case $Z_1 = \{pt\}$, where it takes the slightly simpler form

$$(L,M) \simeq L \circ M,$$

where $\simeq$ denotes a quasi-isomorphism in $\mathcal{F}^\#(Z_3;k)$. As explained in [43, Proposition 7.2.5], this simpler result is a reinterpretation of the cohomology-level

geometric composition theorem [40, 67]: given a test object $N$, the quasi-isomorphism (Y-map)

$$CF(L \circ M, N) \to CF((L, M), N)$$

constructed in [40] is chain homotopic, by a deformation of the quilt domain, to a map of the form $\mu^2(\eta, -)$ for a Floer cocycle $\eta \in CF((L, M), L \circ M)$. Similarly, the inverse Y-map has the form $\mu^2(-, \eta')$ for a cocycle $\eta' \in CF(L \circ M, (L, M))$. The cocycles $\eta$ and $\eta'$ provide the desired quasi-isomorphism.

**Remark 5.1.9** We remark that to define the cocycle $\eta$ using the technology of [40], one must first establish monotonicity in the sense of Equation (7) for annuli in $Z_2^- \times Z_3$ with boundary on $L \times (L \circ M)$ and $M$ [40, Corrigendum]. For example, this is automatic if the image of $\pi_1(L \times (L \circ M))$ in $\pi_1(Z_2^- \times Z_3)$ is torsion. Later (Section 6.2.3) we will see an example where monotonicity holds for different reasons.

Only in Corollary 6.2.10 will we use the geometric composition theorem in the more general situation.

**Example 5.1.10** If $\Delta \subset X^- \times X$ denotes the diagonal Lagrangian then the associated functor $\Phi(\Delta)$ is quasi-isomorphic to the identity functor. Moreover, recall that $\Phi$ is itself functorial, so we get a map on morphisms

$$\Phi^1 \colon \hom(\Delta, \Delta) \to \hom(\mathrm{id}, \mathrm{id}).$$

The self-morphisms of the identity functor are the pre-natural transformations $(T^0, T^1, \ldots)$, where $T^0$ in particular assigns to each element $L$ a morphism $T^0_L \in \hom(L, L)$. The composition of $\Phi^1$ with this gives a map $\hom(\Delta, \Delta) \to \hom(L, L)$ for each object $L$. On cohomology this map is the closed-open map $CO^0 \colon QH(X; k) \to HF(L, L; k)$ [66, Remark 7.3].

**Example 5.1.11** If $L_1$ and $L_2$ are monotone Lagrangian branes then the functor $\Phi(L_1 \times L_2)$ is quasi-isomorphic to a *projection functor* $\mathcal{I}_{L_1, L_2}$ which acts in the following way on other monotone Lagrangian branes:

$$\mathcal{I}_{L_1, L_2}(L) = CF(L_1, L) \otimes L_2.$$

See [7, Lem. 7.4] for a proof of this fact and [56, Section 3c] for the definition of the tensor product of an object and a cochain complex.

**Remark 5.1.12** In Example 5.1.11, $CF(L_1, L) \otimes L_2$ is the tensor product of an object $L_2$ and a cochain complex $CF(L_1, L)$. We note that this contains $L_2$ as a direct summand provided $HF(L_1, L) \neq 0$. To see this, suppose that $\eta \in CF(L_1, L)$ is a closed element not contained in the image of the differential. Let $S$ be a subspace of $CF(L_1, L)$ which (a) contains the image of the differential, and (b) is complementary to the span of $\eta$. Since this contains the image of the differential, it is a subcomplex. The splitting $CF(L_1, L) = S \oplus \langle \eta \rangle$ allows us to define an idempotent chain map $CF(L_1, L) \to CF(L_1, L)$ whose image is $\langle \eta \rangle$. This gives an idempotent summand of $CF(L_1, L) \otimes L_2$ quasi-isomorphic to $L_2$.

One important consequence of the theory is:

**Corollary 5.1.13** *To prove split-generation of $D^\pi \mathcal{F}(X; k)$ by a collection of Lagrangians $L_i$ it suffices to prove that $\Delta$ is split-generated in $D^\pi \mathcal{F}(X^- \times X; k)$ by product Lagrangians $L_i \times L_j$.*

**Proof** If this holds then, upon applying the functor $\Phi$, we see that the identity functor is split-generated by the functors $\Phi(L_i \times L_j)$. Therefore, by Lemma 3.5.1, $P$ is quasi-isomorphic to an object split-generated by the objects $\Phi(L_i \times L_j)(P) = CF(L_i, P) \otimes L_j$, which is clearly in the subcategory split-generated by $L_j$. □

## 5.2 Flat line bundles

Let $k$ be a field. A *flat $k$-line bundle* on a Lagrangian $L$ is a rank 1 $k$-vector bundle over $L$ equipped with parallel transport maps along paths which are independent of the path up to homotopy relative to its endpoints. We will drop the $k$ from the notation in what follows.

The definition of the Fukaya category $\mathcal{F}(Z; k)$ can be extended so that its objects are pairs $(L, E)$ of Lagrangians with flat line bundles[2]. We explain how to define an extended Fukaya category of generalised Lagrangian submanifolds equipped with flat line bundles.

**Definition 5.2.1** Let $\underline{L} = \{L_{i,i+1}\}_{i=0}^{k-1}$ be a generalised Lagrangian submanifold of $Z$ and let $k$ be a field. A flat line bundle on $\underline{L}$ is a $k$-tuple $\{E_{i,i+1}\}_{i=0}^{k-1}$ of flat

[2]One could also use higher rank local systems, as in [3], but this can cause issues in the presence of Maslov 2 discs, as exploited in the work of Damian [19]; see [38] for a fuller discussion of these issues.

line bundles $E_{i,i+1}$ on $L_{i,i+1}$. Given a flat line bundle $\underline{E}$ on $\underline{L}$, we define $\underline{E}^T$ to be the flat line bundle on $\underline{L}^T$ given by $E^\vee_{i,i+1}$ on $L^-_{i,i+1}$ for $i = k-1, \dots, 1$.

If $\underline{E}$ is a flat line bundle on $\underline{K}$ then we get a flat line bundle

$$E_{01} \boxtimes \cdots \boxtimes E_{k-1,k} := \left(p^*_{01} E_{01}\right) \otimes \cdots \otimes \left(p^*_{k-1,k} E_{k-1,k}\right)$$

on $K_{01} \times \cdots \times K_{k-1,k}$, where $p_{i,i+1}$ denotes the projection to the factor $K_{i,i+1}$. Given flat line bundles $\underline{E}$ and $\underline{F}$ on $\underline{K}$ and $\underline{L}$ respectively, we get a bundle

$$\underline{E} \boxtimes \underline{F}^\vee := E_{01} \boxtimes \cdots \boxtimes E_{k-1,k} \boxtimes F^\vee_{\ell-1,\ell} \boxtimes \cdots \boxtimes F^\vee_{0,1}$$

on $K_{01} \times \cdots \times K_{k-1,k} \times L^-_{\ell-1,\ell} \times \cdots \times L^-_{01}$. Given a generalised intersection point $\underline{x} \in \underline{K} \cap \underline{L}$, the point $(x_0, x_1, x_1, x_2, \dots, x_{k+\ell-1}, x_{k+\ell-1}, x_{k+\ell})$ lives in $K_{01} \times \cdots \times K_{k-1,k} \times L^-_{\ell-1,\ell} \times \cdots \times L^-_{01}$ and we can take the fibre of $\underline{E} \boxtimes \underline{F}^T$ at this point. We write $\underline{E} \boxtimes \underline{F}^T(\underline{x})$ for this vector space.

**Definition 5.2.2** Given $(\underline{K}, \underline{E})$ and $(\underline{L}, \underline{F})$, generalised Lagrangian correspondences equipped with flat line bundles, define

$$\hom((\underline{K}, \underline{E}), (\underline{L}, \underline{F})) = \bigoplus_{\underline{x} \in \underline{K} \cap \underline{L}} \underline{E} \otimes \underline{F}^T(\underline{x}).$$

We can define a differential on this morphism space by counting pseudoholomorphic quilted strips as before. Each quilted strip $\underline{u} = (u_1, \dots, u_{k+\ell-1})$, asymptotic at $-\infty$ to $\underline{x}$ and at $+\infty$ to $\underline{y}$, defines a path

$$(u_1(0,t), u_1(p_1,t), u_2(p_1,t), u_2(p_2,t), \dots, u_{k+\ell-1}(p_{k+\ell-1},t), u_{k+\ell}(1,t))$$

in $K_{01} \times \cdots \times K_{k-1,k} \times L^-_{\ell-1,\ell} \times \cdots \times L^-_{01}$ and, hence, a monodromy map

$$T_{\underline{u}} \colon \underline{E} \otimes \underline{F}^T(\underline{x}) \to \underline{E} \otimes \underline{F}^T(\underline{y}).$$

The contribution of the quilted strip $\underline{u}$ to the differential is precisely this linear map.

Higher $A_\infty$-products are handled similarly. If a seam $\sigma$ connects an incoming and an outgoing end then the parallel transport along $\sigma$ is used to define the contribution of the quilt in the same way as in the definition of the differential. There is an additional complication when a seam connects two incoming strip-like ends (not necessarily distinct) or an outgoing end to itself.

In the case of the incoming ends, this is resolved as follows. Suppose that we have a quilt with several incoming strip-like ends, two of which ($p$ and $q$) are

labelled with Floer cochain groups $\hom((\underline{K},\underline{E}),(\underline{L},\underline{F}))$ and $\hom((\underline{K}',\underline{E}'),(\underline{L}',\underline{F}'))$. Suppose moreover that there is a seam whose ends connect these punctures. This seam will be labelled with $(L,E)$ at the $p$-end and $(L^-,E^\vee)$ at the $q$-end. Suppose that $x$ is the asymptote at the $p$-end and $y$ is the asymptote at the $q$-end. The contribution of this quilt $A_\infty$-operation will be a linear map of the form

$$Q\colon H_1 \otimes \hom((\underline{K},\underline{E}),(\underline{L},\underline{F}) \otimes H_2 \otimes \hom((\underline{K}',\underline{E}'),(\underline{L}',\underline{F}')) \otimes H_3 \to H_4$$

where the vector spaces $H_i$ are tensor products of morphism spaces. The tensor product on the left-hand side includes $E(x)\otimes E^\vee(y)$. Parallel transport along the seam allows us to identify this with $E(x)\otimes E^\vee(x)$, which has a canonical contraction $E(x)\boxtimes E^\vee(x)\to k$. We require that the map $Q$ factors through this contraction.

In the case of a seam connecting an outgoing end to itself, we make a similar argument using the canonical map $k\to E\otimes E^\vee$ which sends 1 to the identity. This is sufficient to determine all quilt contributions.

**Definition 5.2.3** Suppose that $(K\subset Y, L\subset Y^-\times Z)$ is a composable generalised Lagrangian correspondence. Let $K$ and $L$ be equipped with flat line bundles, $E$ and $F$. Note that there is a natural map

$$p\colon K\times_Y L\to K\times L$$

as $K\times_Y L$ includes into $L$ and admits a projection to $K$. Let

$$E\boxtimes F := \left(\mathrm{pr}_K^* E\right)\otimes\left(\mathrm{pr}_L^* F\right).$$

We can define a flat line bundle $E\circ F$ on $K\circ L$ by $p^*(E\boxtimes F)$.

As before, it is easy to associate to any pair $(L,F)$ comprising a Lagrangian correspondence with a flat line bundle, an $A_\infty$-functor $\Phi(L,F)$ between these enlarged categories. The proof of the geometric composition theorem generalises readily to show:

$$\Phi(L,F)(K,E) = (K\circ L, E\circ F).$$

## 5.3 Non-compact Lagrangians

Let $(W,\theta)$ be a Liouville manifold with $2c_1(W)=0$. Suppose there is a compact Liouville subdomain $W^{in}$ whose complement is isomorphic to

$$\left([1,\infty)\times\partial W^{in}, r\left(\theta|_{\partial W^{in}}\right)\right)$$

where $r$ is the coordinate on $[1, \infty)$. The wrapped Fukaya category of $W$ is an $A_\infty$-category whose objects are exact, Maslov zero, Lagrangian submanifolds, equipped with orientations, grading, relative spin structures and functions $f_L$ such that $\theta|_L = df_L$. Additionally, and importantly, we require that outside a compact set, $\theta|_L = 0$. This constrains the behaviour of $L$ at infinity: it must have the form $[R, \infty) \times \Lambda$ for some Legendrian submanifold $\Lambda \subset \partial W^{in}$.

We will work with the model for the wrapped Fukaya category explained in [1] (an alternative model appears in [6], which is known to agree with the model in [1]). Consider the class of Hamiltonian functions $\mathcal{H}(W)$ which agree with $r^2$ outside a compact set. Given two Lagrangians $L_1, L_2$ and a Hamiltonian $H \in \mathcal{H}(W)$, an $X_H$-chord is an integral curve $x$ of the Hamiltonian vector field $X_H$ with $x(0) \in L_1$ and $x(1) \in L_2$. Define $CW(L_1, L_2, H; k)$ to be the $k$-vector space generated by the chords. The $A_\infty$-operations are defined using moduli spaces of solutions to Floer's equation in a standard way. There is a subtlety: for example, the natural composition is $CW(L_2, L_3, H; k) \otimes CW(L_1, L_2, H; k) \to CW(L_1, L_3, 2H; k)$; to define an $A_\infty$-structure on $CW(\cdot, \cdot, H; k)$ itself requires a careful choice of Hamiltonian perturbations depending on the domain and the use of the Liouville flow to "rescale" from $kH$ to $H$ (see [1]).

We now fix:

- a Liouville manifold $(W, \theta)$ with $2c_1(W) = 0$;
- a simply-connected, compact, monotone symplectic manifold $Z$ with monotonicity constant $\tau$; and
- a compact, $\tau$-monotone Lagrangian brane $C \subset W^- \times Z$.

We define $\mathcal{F}_C(Z; k)$, an extended category of generalised Lagrangians in $Z$ having a very special form:

**Definition 5.3.1** (Objects of $\mathcal{F}_C(Z; k)$) The objects of $\mathcal{F}_C(Z; k)$ come in two flavours:

(A) compact $\tau$-monotone Lagrangian branes $M \subset Z$.

(B) generalised Lagrangian correspondences $(L, C)$ where $L \subset W$ is an exact, Maslov zero Lagrangian brane in $\mathcal{W}(W)$.

We also require the objects to form an admissible set (Definition 5.1.6).

**Remark 5.3.2** Checking admissibility when there is non-trivial fundamental group can be difficult. However, in the situation of Definition 5.3.1, there is a

sequence of modifications to a general genus zero quilted map which allow us to reduce to checking monotonicity for annuli with boundary on $L \times M$ and $C$ where $(L, C)$ is an object of type (B) and $M$ is some fixed object of type (A) (it does not matter which). These modifications have the form described in the following lemma, where we first homotope the quilted map to be patchwise-constant in some region and then perform an excision to change the map in that region. The lemma guarantees that, at each stage in the sequence, the modifications preserve monotonicity of the quilted map:

**Lemma 5.3.3** (Excision lemma) *Let $T_{12}$ and $T_{34}$ be two labelled, quilted Riemann surfaces (see Definition 5.1.1). Pick two simple closed curves $\gamma \subset T_{12}$ and $\gamma' \subset T_{24}$ in their interiors intersecting the seams transversally. Let us write $T_{12} = T_1 \cup_\gamma T_2$ and $T_{34} = T_3 \cup_{\gamma'} T_4$. Suppose that the quilt and label data along $\gamma$ and $\gamma'$ match, so that $T_{13} = T_1 \cup T_3$ and $T_{24} = T_2 \cup T_4$ define new quilted domains. Suppose also that, for each $i \in \{1, 2, 3, 4\}$, there is a quilted map $v_i$ modelled on $T_i$ such that $v_{ij} := v_i \cup v_j$ defines a continuous quilted map on $T_{ij}$ for $ij \in \{12, 34, 13, 24\}$. Then $v_{12}$ satisfies Equation (7) if $v_{34}, v_{13}, v_{24}$ all satisfy Equation (7).*

**Proof** This is an immediate consequence of additivity for area and the excision theorem for index. See [40, Section 2.4] for an explanation of excision in the quilted context. □

**Definition 5.3.4** (Morphism spaces of $\mathcal{F}_C(Z; k)$) The morphism spaces come in four flavours. Let $M_i \subset Z$, $i = 1, 2$ be objects of type (A) and $(L_i, C)$ be objects of type (B). Define

$$\begin{aligned}
\hom(M_1, M_2) &= CF(M_1, M_2; k) \text{ in } Z \\
\hom(M_1, (L_2, C)) &= CW(L_2 \times M_1, C; k) \text{ in } W^- \times Z \\
\hom((L_1, C), M_2) &= CW(M_2 \times L_1, C^-; k) \text{ in } Z^- \times W \\
\hom((L_1, C), (L_2, C)) &= CW(L_1 \times \Delta_Z \times L_2, C \times C^-; k) \text{ in } W \times Z^- \times Z \times W^-.
\end{aligned}$$

Here, $CW$ indicates wrapping with respect to a Hamiltonian $\underline{H}$ defined on the target manifold $A_1 \times \cdots \times A_k$ which has the form $\sum_{i=1}^k H_i$ where $H_i \colon A_i \to \mathbf{R}$ is in $\mathcal{H}(W)$ whenever $A_i = W$ or $W^-$. We will assume that our Lagrangians have been perturbed and our Hamiltonians chosen so that all Hamiltonian chords are non-degenerate.

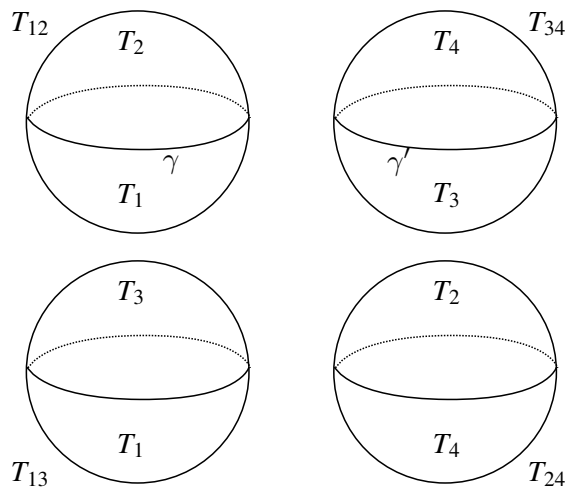


Figure 2: Excision.

**Remark 5.3.5** Since our correspondence is compact, we never have to take Floer cohomology between two non-compact Lagrangians (only between a compact and a non-compact Lagrangian). For this reason, although our Floer complexes are wrapped, they are finite-dimensional.

The $A_\infty$-structure can be defined for $\mathcal{F}_C(Z;k)$ in the same way that Mau, Wehrheim and Woodward [43] define the extended Fukaya category in their setting. For example, Figure 3 illustrates a quilt contributing to a $\mu^3$-product in the category (in fact, the $\mu^3$-product is defined by counting pseudoholomorphic quilts modelled on this picture with a suitably chosen family of perturbation data, as explained in [43]).

There are two points where one must be careful with this construction: in proving transversality and in proving compactness. We deal with these considerations in the next two subsections.

### 5.3.1 Transversality

As before, we must make domain-dependent folded perturbations in balls in the domain centred on the seams. We will need to use non-split almost complex structures on $Z^- \times W$ or $W^- \times Z$, but we will require that the almost complex structure is split outside the compact region $Z^- \times W^{in}$ or $(W^{in})^- \times Z$.

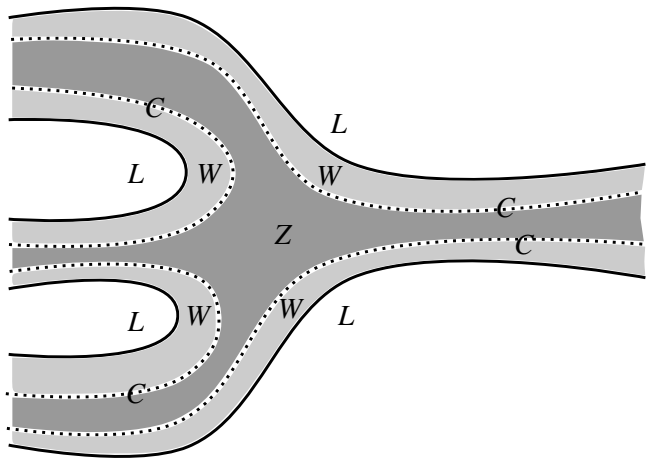


Figure 3: A quilt contributing to the $A_\infty$-operation $\mu^3$ on $\hom_{\mathcal{F}_C(Z;k)}((L,C),(L,C))$. The light patches map to $W$; the dark patch maps to $Z$; the edges map to the Lagrangian $L \subset W$; the dotted lines are seams which map to the Lagrangian correspondence $C \subset W^- \times Z$.

This is enough to prove transversality because the seam condition is given by a compact Lagrangian correspondence contained in $Z^- \times W^{in}$ or $(W^{in})^- \times Z$, so the pseudoholomorphic curve must pass through this region.

### 5.3.2 Compactness

To prove compactness, we need to show that patches of pseudoholomorphic quilts which map to $W$ with given asymptotic behaviour at the punctures stay in a given compact region of $W$. Suppose $S_i$ is a patch in a quilted Riemann surface and $u$ is a pseudoholomorphic quilt for which $u_i \colon S_i \to W$ lands in $W$. The part of $u_i$ which escapes from $W^{in}$ is a genuine solution to Floer's equation, as we require our domain-dependent almost complex structures to be split outside of regions which project to $W^{in}$ (see the remark on transversality). For fixed asymptotics, solutions to Floer's equation are forced to remain in a compact region of $W$, and for fixed input asymptotics, there can only be finitely many output asymptotics; this is explained in detail for $\mu^1$ in [1, Lem. 2.4 and 2.5].

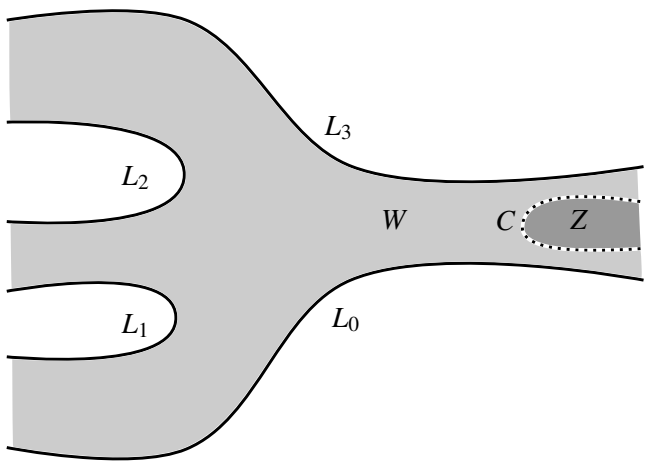


Figure 4: A quilt contributing to the $A_\infty$-functor $\Phi(C)\colon \mathcal{W}(W) \to \mathcal{F}_C(Z;k)$, specifically to $\Phi(C)^3\colon \hom(L_2,L_3)\otimes\hom(L_1,L_2)\otimes\hom(L_0,L_1) \to \hom_{\widetilde{\mathcal{F}}(Z)}((L_0,C),(L_3,C))$.

### 5.3.3 $A_\infty$-functor

Exactly as in [43], the correspondence $C$ can be used to define an $A_\infty$-functor $\Phi(C)\colon \mathcal{B} \to \mathcal{F}_C(Z;k)$, defined on the subcategory $\mathcal{B}$ of $\mathcal{W}(L)$ consisting of objects $L$ such that $(L,C)$ is a brane of type (B) in $\mathcal{F}_C(Z;k)$. This functor acts on objects by $\Phi(C)(L) = (L,C)$. The component maps in the $A_\infty$-functor are defined by counts of pseudoholomorphic quilts, see Figure 4. Transversality, compactness and monotonicity issues can be dealt with in the same way as in setting up the $A_\infty$-category $\mathcal{F}_C(Z;k)$.

If $(L,C)$ is composable and $L\circ C$ is monotone then geometric composition works as explained in Section 5.1; the monotonicity condition explained in Remark 5.1.9 is built into the assumption of admissibility in Definition 5.3.1.

We summarise this discussion in the following statement:

**Theorem 5.3.6** *Let $\mathcal{B}$ denote the subcategory of $\mathcal{W}(W)$ consisting of exact Lagrangian branes $L$ such that $(L,C)$ is a brane of type (B) in $\mathcal{F}_C(Z;k)$. There is an $A_\infty$-functor*

$$\Phi(C)\colon \mathcal{B} \to \mathcal{F}_C(Z;k)$$

*which acts on objects as*

$$\Phi(C)(L) = (L,C).$$

*Moreover, if $(L, C)$ is a composable correspondence in $\mathcal{F}_C(Z;k)$ then*

$$(L, C) \simeq L \circ C.$$

# 6 Hamiltonian group actions

## 6.1 Moment correspondence functor

Let $G$ be a compact Lie group, let $(X, \omega)$ be a simply-connected monotone symplectic manifold, and suppose that $X$ admits a Hamiltonian action of $G$ with equivariant moment map $\mu\colon X \to \mathfrak{g}^*$. The *moment correspondence* is the Lagrangian submanifold

$$C := \{(g, v, x, y) \in (T^*G)^- \times X^- \times X \ : \ v = \mu(gx),\ y = gx\}.$$

Here, we are using left-multiplication to trivialise the cotangent bundle $T^*G \cong G \times \mathfrak{g}^*$ and write $(g, v)$ for a point in $G \times \mathfrak{g}^*$.

**Lemma 6.1.1** *The moment correspondence is a monotone Lagrangian submanifold.*

**Proof** The relative homotopy exact sequence is

$$\cdots \to \pi_2(C) \to \pi_2(T^*G \times X \times X) \to \pi_2(T^*G \times X \times X, C) \to$$
$$\to \pi_1(C) \to \pi_1(T^*G \times X \times X) \to \cdots$$

We know that $\pi_2(G) = 0$ and that the map $\pi_1(G) = \pi_1(C) \to \pi_1(T^*G \times X \times X)$ is induced by the map $(g, x) \mapsto (g, \mu(gx), x, gx)$ and is therefore injective. This implies $\pi_2(T^*G \times X \times X, C)$ is a quotient of $\pi_2(X \times X)$ and monotonicity follows from monotonicity of $X$. □

As the moment correspondence is compact and monotone, the quilt machinery from Section 5.3 defines for us an $A_\infty$-functor

$$\Phi(C)\colon \mathcal{B} \to \mathcal{F}_C(X^- \times X; k)$$

where $\mathcal{B}$ is the subcategory of $\mathcal{W}(T^*G;k)$ consisting of exact Lagrangian branes $L$ such that $(L, C)$ is a brane of type (B) in Definition 5.3.1. The following observation is crucial to our paper.

**Lemma 6.1.2** *Let $T_1^*G$ denote the cotangent fibre at the identity element in $G$. The generalised correspondence $(T_1^*G, C)$ is composable and its geometric composition is the diagonal Lagrangian $\Delta \subset X^- \times X$.*

**Proof** We have a transverse intersection $(T_1^*G \times X^- \times X) \cap C = \{(1, \mu(x), x, x)\}$ and the projection of this to $X^- \times X$ is the embedding of the diagonal. □

Since $T_1^*G$ is simply-connected, the object $(T_1^*G, C)$ satisfies monotonicity for annuli with boundaries on $T_1^*G \times \Delta$ and $C$ and hence, by Remark 5.3.2, can be incorporated as an object of type (B) into $\mathcal{F}_C(X^- \times X; k)$. This implies:

**Definition 6.1.3** (Moment correspondence functor) If we restrict $\Phi(C)$ to the subcategory $\langle T_1^*G \rangle$ generated by $T_1^*G$, then, after a quasi-isomorphism, we can assume that it lands in the usual Fukaya category $\mathcal{F}(X^- \times X)$. The derived wrapped Fukaya category $D^b\mathcal{W}(T^*G)$ is generated by this cotangent fibre so, by passing to triangulated closures, we obtain a functor

$$\mathfrak{C}\colon D^b\mathcal{W}(T^*G) \to D^b\mathcal{F}(X^- \times X).$$

We call this the *moment correspondence functor*.

## 6.2 Main results

We now describe the situation we are interested in analysing.

**Setup 6.2.1** Let $G$ be a compact Lie group, let $(X, \omega)$ be a monotone symplectic manifold, and suppose that $X$ admits a Hamiltonian action of $G$ with equivariant moment map $\mu\colon X \to \mathfrak{g}^*$. Let $k$ be an arbitrary field of characteristic $p \geq 0$, where $G$ has no $p$-torsion if $p > 0$. We will assume that $\mu$ vanishes transversely along $\mu^{-1}(0)$ and that $G$ acts freely on $\mu^{-1}(0)$. We will also assume that the Lagrangian submanifold

$$M := \{(x, y) \in X^- \times X \ : \ x, y \in \mu^{-1}(0),\ y = gx \text{ for some } g \in G\} \tag{8}$$

is monotone.

It is easy to check the following:

**Lemma 6.2.2** *Let $G$ denote the zero-section in $T^*G$. In the situation of Setup 6.2.1, the generalised correspondence $(G, C)$ is composable and the geometric composition is $M$.*

**Lemma 6.2.3** *If $u\colon S^1 \times [0,1] \to (T^*G)^- \times X^- \times X$ is an annulus with $u(s,0) \in G \times (G \circ C)$ and $u(s,1) \in C$ then*

$$\operatorname{area}(u) = \tau \operatorname{index}(u),$$

*where $\tau$ is the monotonicity constant for $X$.*

**Proof** We write $u(s,t) = (w(s,t), z(s,t))$ where $w(s,t) \in (T^*G)^-$ and $z(s,t) \in X^- \times X$. Let $K := C \cap (G \times (G \circ C)) = \{(g,0,x,gx) \ : \ x \in \mu^{-1}(0)\}$. Note that the map $\pi_1(K) \to \pi_1(C)$ is surjective and that $K \cong G \circ C$ is connected, so $\pi_1(C,K) = \{1\}$. This means that the loop $u(s,1)$ in $C$ is homotopic in $C$ to a loop contained in $K$. Since we are only interested in $u$ up to homotopy, we can therefore assume that $u(s,1)$ is contained in $K$. Thus $w(s,t)$ is an annulus in $T^*G$ with boundary on $G$. Since $T^*G$ retracts onto $G$, this is homotopic, through annuli in $T^*G$ with boundary on $G$, to the annulus $w'(s,t) := w(s,1)$. Therefore $u(s,t)$ is homotopic to an annulus $u'(s,t) = (w(s,1), z(s,t))$ with boundary on $G \times (G \circ C)$ and $K \subset C$. Note that $z$ is an annulus with boundary on $G \circ C$. The area and index of $u'$ agree with the area and index of $z$ [67, Lemma 2.1.2]. The boundary of $z$ can be capped off in $X^- \times X$ since $\pi_1(X) = \{1\}$. Therefore monotonicity for $z$ follows from monotonicity for discs with boundary on $G \circ C \subset X^- \times X$. □

This monotonicity result allows us to incorporate $(G, C)$ as an object of type (B) into the category $\mathcal{F}_C(X^- \times X; k)$. As a consequence of Lemmas 6.2.2 and 6.2.3, we can apply the geometric composition theorem in quilted Floer theory to deduce that

$$\mathfrak{C}(G) \simeq M.$$

**Theorem 6.2.4** *In the situation of Setup 6.2.1, the Lagrangian $M$ in $\mathcal{F}(X^- \times X)$ split-generates $\Delta_\alpha$ over $k$ if and only if $HF(M, \Delta_\alpha; k) \neq 0$.*

**Proof** By Corollary 3.4.3, we know that, in $\mathcal{W}(T^*G)$, the zero-section $G \subset T^*G$ can be written as a Koszul twisted complex built out of $T_1^*G$. By Lemma 3.2.2, the image of the Koszul twisted complex under the moment correspondence functor $\mathfrak{C}$ is a Koszul twisted complex representing $\mathfrak{C}(G)$ built out of $\mathfrak{C}(T_1^*G)$. By Lemma 6.2.2 and Theorem 2.2.7(i) we see that $\mathfrak{C}(T_1^*G) \simeq \Delta$ and $\mathfrak{C}(G) \simeq M$. Therefore $M$ is quasi-isomorphic to a Koszul twisted complex built out of $\Delta$. Lemma 4.2.2 immediately implies the desired result. □

We first work out the implications of this theorem in the situation where the $G$-action has a free Lagrangian orbit.

**Corollary 6.2.5** *In the situation of Setup 6.2.1, if $L = \mu^{-1}(0)$ is a free Lagrangian $G$-orbit with $HF(L_\alpha, L_\alpha; k) \neq 0$ then $L_\alpha$ split-generates $D^\pi \mathcal{F}(X;k)_\alpha$.*

**Proof** In the situation that $\mu^{-1}(0)$ is a Lagrangian orbit, the image of the zero-section under the moment correspondence functor is $L \times L$. We have

$$HF(L \times L, \Delta; k) = HF(L, L; k)$$

and each side decomposes under $1 = \sum_{\alpha \in W} e_\alpha$ as

$$HF(L \times L, \Delta_\alpha; k) = HF(L_\alpha, L_\alpha; k).$$

Therefore the statement follows from Theorem 6.2.4. □

The next corollary shows that, if a summand of the Fukaya category contains a nonzero object $L_\alpha$ arising as a free Lagrangian $G$-orbit, then any other object in that summand split-generates the summand. It also gives an upper bound on the rank of Floer cohomology of $L_\alpha$ with any other object.

**Corollary 6.2.6** *In the situation of Setup 6.2.1, if $L = \mu^{-1}(0)$ is a free Lagrangian $G$-orbit with $HF(L_\alpha, L_\alpha; k) \neq 0$ and $L' \in D^\pi \mathcal{F}(X;k)_\alpha$ is another nonzero object then $L'$ split-generates $D^\pi \mathcal{F}(X;k)_\alpha$ and the inequality*

$$\mathrm{hf}(L_\alpha, L') \leq 2^{n/2} \sqrt{\mathrm{hf}(L', L')}.$$

*holds. Here,* hf *denotes the rank of $HF$.*

**Proof** As in the proof of Theorem 6.2.4, $L_\alpha \times L_\alpha$ is quasi-isomorphic to a Koszul twisted complex built out of $\Delta_\alpha$. Under the functor $\Phi$, this says that $\Phi(L_\alpha \times L_\alpha)$ is quasi-isomorphic to a Koszul twisted complex $\mathcal{T}$ of functors built out of the identity functor. Since $\Phi(L_\alpha \times L_\alpha)(L') \simeq CF(L_\alpha, L') \otimes L_\alpha$ and $\Phi(\Delta)(L') \simeq L'$, Lemma 3.5.1 implies that $CF(L_\alpha, L') \otimes L_\alpha$ is quasi-isomorphic to a Koszul twisted complex $K$ built out of $L'$. This contains $L_\alpha$ as a summand (see Remark 5.1.12) and is generated by $L'$, hence $L'$ split-generates $L_\alpha$. Since $L_\alpha$ itself split-generates $D^\pi \mathcal{F}(X;k)_\alpha$, we deduce that $L'$ split-generates $D^\pi \mathcal{F}(X;k)_\alpha$.

Computing the $\mu^1_{\mathrm{Tw}}$-cohomology of the space of morphisms from $K$ to $L'$ gives

$$HF(L', L_\alpha) \otimes HF(L_\alpha, L') \cong H(\hom(K, L'), \mu^1_{\mathrm{Tw}}).$$

Expanding all the cones in the Koszul twisted complex $K$, we see that it is built out of $2^n$ copies of $L'$. To see this, using the notation of Definition 3.2.1, we

see that $K = \mathrm{Cone}\left(K_{n-1}[s_{n-1}] \xrightarrow{x_{n-1}} K_{n-1}\right)$ is quasi-isomorphic to the twisted complex

$$K_{n-1}[s_{n-1}+1] \overset{-x_{n-1}[1]}{\dashrightarrow} K_{n-1} \tag{9}$$

built out of two copies of $K_{n-1}$. Similarly $K_{n-1}$ is quasi-isomorphic to

$$K_{n-2}[s_{n-2}+1] \overset{-x_{n-2}[1]}{\dashrightarrow} K_{n-2}, \tag{10}$$

so replacing each $K_{n-1}$ in (9) by (10), we get a twisted complex for $K$ built out of four copies of $K_{n-2}$. Continuing in this manner, we find a twisted complex for $K$ built out of $2^n$ copies of $K_0 = L'$.

By definition of the category of twisted complexes, the chain group $\hom(K, L')$ is therefore a direct sum of $2^n$ shifted copies of $CF(L', L')$. The differential $\mu^1_{\mathrm{Tw}}$ preserves the length filtration on this chain group, so we get a spectral sequence whose $E_1$-term is the direct sum of $2^n$ shifted copies of $HF(L', L')$. The rank of $HF(L', L_\alpha) \otimes HF(L_\alpha, L')$, which equals $\mathrm{hf}(L', L_\alpha)^2$, is therefore bounded above by $2^n \mathrm{hf}(L', L')$, which implies the stated inequality. □

**Remark 6.2.7** This inequality generalises to the case when $G$ is an arbitrary compact Lie group. In that situation, you can still represent the zero-section as a twisted complex built out of the cotangent fibre [4]. Abouzaid's construction produces a twisted complex out of any Morse function $f\colon G \to \mathbf{R}$ with one copy of $T^*_1 G$ for every critical point. The argument in the corollary shows that if $HF(L_\alpha, L') \neq 0$ then $L'$ split-generates $L_\alpha$ and that $\mathrm{hf}(L_\alpha, L') \leq \sqrt{m(G)\mathrm{hf}(L', L')}$ where $m(G)$ is the minimal number of critical points of a Morse function on $G$.

**Corollary 6.2.8** *In the situation of Setup 6.2.1, if $L = \mu^{-1}(0)$ is a free Lagrangian $G$-orbit with $HF(L_\alpha, L_\alpha; k) \neq 0$ then*

$$\mathrm{HH}^\bullet(D^\pi \mathcal{F}(X;k)_\alpha) \cong QH^\bullet(X;k)_\alpha.$$

**Proof** The final part of the proof of [60, Corollary 3.11] shows that, if $\Delta_\alpha$ is split-generated by product Lagrangians, then $\mathrm{HH}^\bullet(D^\pi \mathcal{F}(X;k)_\alpha) \cong QH(X;k)_\alpha$. The proof of Theorem 6.2.4 implies that $\Delta_\alpha$ is split-generated by $M_\alpha = L_\alpha \times L_\alpha$. This proves the corollary. □

**Remark 6.2.9** The proofs also carry over if $M$ is equipped with a local system pushed forward along the moment correspondence from a local system on the

zero-section. In the case when $M = L \times L$, and $L$ is a free $G$-orbit, this is easy to understand: if $E$ is a local system on $G$ and $\underline{k}$ is the trivial local system on $C$ then $E \circ \underline{k}$ is the local system on $L \times L$ isomorphic to $E^* \boxtimes E$. In other words, if $\rho\colon \pi_1(L) \to k^\times$ is the monodromy of $E$ then the monodromy of $E \circ \underline{k}$ around a loop $(\gamma, \delta) \in \pi_1(L \times L)$ is $\rho(\gamma)^{-1}\rho(\delta)$.

The next corollary, which explains the full implications of Theorem 6.2.4, uses the full strength of quilt theory, in particular the existence of an $A_\infty$-functor $\Phi\colon \mathcal{F}(X^- \times Y) \to \mathfrak{nu}\text{-}\mathfrak{fun}(\mathcal{F}^\#(X), \mathcal{F}^\#(Y))$ with the property that the functors $\Phi(J \circ K)$ and $\Phi(J) \circ \Phi(K)$ are quasi-isomorphic when the geometric composition is transverse and embedded.

**Corollary 6.2.10** *In the situation of Setup 6.2.1, suppose that (a) $\mu^{-1}(0)/G$ has the property that its derived extended Fukaya category is quasi-equivalent to its derived Fukaya category, and that (b) $HF(M, \Delta_\alpha; k) \neq 0$. Then the $\alpha$-summand of the monotone Fukaya category $D^\pi\mathcal{F}(X;k)_\alpha$ is split-generated by $G$-invariant Lagrangian submanifolds.*

**Proof** By Theorem 6.2.4, $M_\alpha$ split-generates $\Delta_\alpha$. Therefore the image of the corresponding functor $\Phi(M_\alpha)$ split-generates $D^\pi\mathcal{F}(X;k)_\alpha$. We will show that $\Phi(M)$ factors through another functor whose image is split-generated by the $G$-invariant Lagrangian submanifolds.

The subset $\mu^{-1}(0)$ is a fibred coisotropic (i.e. the symplectic reduction is a manifold and the isotropic leaves are the fibres of a fibre bundle over the symplectic reduction). Indeed, the isotropic leaves are precisely the $G$-orbits. Let $B$ denote the symplectic reduction $\mu^{-1}(0)/G$. Let $\varpi\colon \mu^{-1}(0) \to B$ denote the projection map. Let

$$\Gamma := \{(x, y) \in B^- \times X \ : \ \varpi(y) = x \in B\}$$

denote the associated Lagrangian correspondence. This correspondence gives us a quilt functor $\Phi(\Gamma)\colon \mathcal{F}(B;k) \to \mathcal{F}^\#(X;k)$. It is easy to see that $M = \Gamma^T \circ \Delta_B^T \circ \Gamma \subset X^- \times X$. Therefore, by the geometric composition result for functors, the functor $\Phi(M)$ factors through $\Phi(\Gamma)$. The functor $\Phi(\Gamma)$ is defined on the full extended category $\mathcal{F}^\#(B;k)$, but by assumption if we pass to the derived category we can consider just the objects in the image of $\Phi(\Gamma)$ applied to $\mathcal{F}(B;k)$. The geometric composition of such an object with $\Gamma$ is a $G$-invariant Lagrangian submanifold: if $Q \subset B$ is a Lagrangian then $\Phi(\Gamma)(Q) = \varpi^{-1}(Q)$. This gives the corollary. □

# 7 Examples and applications

## 7.1 The quadric 3-fold

Let $X$ be the quadric 3-fold. The fact that a real ellipsoid in $X$ split-generates the zero-summand of $\mathcal{F}(X;\mathbf{C})$ was proved in [60]. We begin by recovering this result, as we found this example very instructive in formulating the general theory.

The quadric 3-fold admits an $SU(2)$-action with a free Lagrangian $SU(2)$-orbit $L$. The Lagrangian $L$ has minimal Maslov number 6, since it is simply-connected and the ambient minimal Chern number is 3. The number 6 is twice the dimension of $L$, so there cannot be any quantum contributions to the Floer differential. Therefore we have $HF^*(L, L\ k) \cong H^*(S^3;k)$ for any field. This certainly implies that $M = L \times L$ is nonzero in $\mathcal{F}(X^- \times X)$, so split-generates some of the summands of $\Delta$.

Let $k$ be a field. The quantum cohomology $QH(X;k)$ is

$$k[H,E]/(H^2 - 2E, E^2 - H)$$

and the first Chern class is $c_1(X) = 3H$. The block decomposition of this ring depends on the characteristic of $k$ and on whether the characteristic polynomial of $H$ splits over $k$.

- If $\operatorname{char} k = 2$ then the nilradical of $QH(X;k)$ is the ideal $I$ generated by $H$ and $E$; therefore $QH(X;k)$ is a local ring with maximal ideal $I$ and residue field $k$. In this case, the Fukaya category has only one summand and the quantum cohomology is not semisimple.
- If $\operatorname{char} k \neq 2$ then the element $e_1 = H^3/4$ is a nontrivial idempotent and the identity splits as a sum of orthogonal idempotents $1 = e_0 + e_1$, with $e_0 = 1 - H^3/4$. (If $\operatorname{char} k = 3$ or if $\lambda^3 - 2$ is irreducible over $k$ then this is a maximal splitting of the identity into idempotents; otherwise $H^3/4$ can be further decomposed as a sum of idempotents.) The summand $QH(X;k)_1$ associated to $e_1$ is the ideal $(H,E)$ and the summand $QH(X;k)_0$ associated to $e_0$ is the ideal $(1 - H^3/4)$.

Note that, if $\operatorname{char}(k) \neq 2$, the element $E = H^2/2$ acts invertibly on the summand $QH(X;k)_1$ and annihilates the summand $QH(X;k)_0$.

**Proposition 7.1.1** *Let $M \subset X^- \times X$ be the Lagrangian submanifold defined in Equation (8). Then $HF(M, \Delta_\alpha; k) \neq 0$ if and only if $\alpha = 0$. Equivalently, by*

*Theorem 6.2.4, the Lagrangian orbit $L$ split-generates $D^\pi\mathcal{F}(X)_0$ and represents the zero object in the other summands.*

**Proof** Since the minimal Maslov number of $L$ is 6, the grading on $HF^*(L, L; k)$ can be taken in $\mathbf{Z}/6$. Moreover, the moment correspondence is itself simply-connected so has minimal Maslov number 6. This means that the moment correspondence functor $\mathfrak{C}$ is naturally a functor between $\mathbf{Z}/6$-graded categories. The Koszul twisted complex representing the zero-section of $SU(2)$ has the form

$$T_1^*SU(2)[3] \overset{-x[1]}{\dashrightarrow} T_1^*SU(2)$$

where $x \in C_2(\Omega SU(2); k) = CW^{-2}(T_1^*SU(2), T_1^*SU(2))$. The Koszul twisted complex representing $M = L \times L$ is

$$\Delta[3] \overset{-\mathfrak{C}^1(x)[1]}{\dashrightarrow} \Delta$$

As the moment correspondence functor preserves the $\mathbf{Z}/6$-grading, $\mathfrak{C}^1(x) \in QH^{-2}(X; k) = QH^4(X; k)$, so $\mathfrak{C}^1(x)$ is a multiple of $E \in QH^4(X; k)$. Suppose that $\mathfrak{C}^1(x) = NE$ for some $N \in k$; since the whole theory makes sense over $\mathbf{Z}$, we know that $N$ is the reduction to $k$ of some fixed integer $N$. We claim that $N = \pm 1$.

If we accept this for a moment, then we see the Koszul twisted complex splits as a direct sum of twisted complexes

$$\Delta_\alpha[3] \overset{-Ee_\alpha[1]}{\dashrightarrow} \Delta_\alpha$$

where $e_\alpha$ is the idempotent corresponding to the $\alpha$-summand. As we observed above, $Ee_0 = 0$ and $Ee_1$ is invertible in $QH(X; k)_1$. Therefore we see that $HF(M, \Delta_\alpha) = 0$ unless $\alpha = 0$, and $M$ split-generates $\Delta_0$ by Corollary 3.3.2, so $HF(M, \Delta_0; k) \neq 0$.

It remains to prove that $N = \pm 1$. Suppose this were not the case, and let $p$ be a prime dividing $N$ and $k'$ be a field of characteristic $p$. Over this field, the Koszul twisted complex is

$$\Delta[3] \overset{0}{\dashrightarrow} \Delta$$

and $L \times L \simeq \Delta[3] \oplus \Delta$. Taking Floer cohomology of this twisted complex with $\Delta$ gives

$$HF(L \times L, \Delta; k') \cong HF(\Delta, \Delta; k')[-3] \oplus HF(\Delta, \Delta; k'). \tag{11}$$

We know that $HF(L \times L, \Delta; k') \cong HF(L, L; k')$ which has rank 2. We also know that $HF(\Delta, \Delta; k') = QH(X; k')$ has rank 2. This gives a contradiction to Equation (11). □

**Remark 7.1.2** Note that the first Chern class of $X$ is $3H$. In the literature on Fukaya categories, it is common to decompose the Fukaya category into summands according to the eigenvalues of quantum product with $c_1$. This is a coarser decomposition than the one we use, which separates out summands according to the simultaneous eigenspaces of quantum multiplication by all semisimple elements of quantum cohomology. In the case of the quadric 3-fold and a field $k$ of characteristic 3, if one only uses eigenvalues of $c_1$, then there is only one piece in the decomposition, corresponding to the repeated eigenvalue zero. The Lagrangian $SU(2)$-orbit does not split-generate the whole of this Fukaya category, only the summand corresponding to the idempotent $e_0 = 1 - H^3/4$.

**Remark 7.1.3** Note that there is a monotone Lagrangian torus $T$ which is (a) disjoint from $L$ and (b) nonzero in the Fukaya category provided the characteristic of the ground field is not equal to 2. This torus is obtained by taking the standard nodal degeneration of $Q$ and observing that it is toric; symplectically parallel-transporting the barycentric fibre from the nodal quadric to a nearby smooth quadric gives the monotone Lagrangian torus $T$, whose superpotential is

$$x + y + z + \frac{1}{xy} + \frac{1}{xz}.$$

This follows from the general formula in [48, Theorem 1] for the superpotential of a Lagrangian torus arising from a toric degeneration, together with the explicit calculation of this superpotential for polytope PALP 3 in the Fanosearch table of 3-dimensional reflective polytopes [18]. This torus is clearly disjoint from $L$, as $L$ is the vanishing cycle of the nodal degeneration, and its behaviour in the Fukaya category is determined by the critical points of its superpotential. In characteristic not equal to 2, it inhabits the summands of the Fukaya category away from $\alpha = 0$.

**Remark 7.1.4** Smith proves split-generation of the 0-summand (in the weaker sense of $c_1$-eigenvalues) over $\mathbf{C}$. For comparison, he uses Seidel's exact triangle to show that $L \times L$ generates the following twisted complex

$$\Delta \overset{C}{\dashrightarrow} \mathrm{Graph}(\tau_L)^5 \simeq \Delta[-5].$$

The second term is the graph of fifth power of the Dehn twist in $L$, which is Hamiltonian isotopic to the identity in this case. The morphism $C$ is the "cycle class", defined by counting sections of a quadric Lefschetz fibration with five nodal fibres; he shows it is a multiple of the first Chern class. Of course, this cannot be true in characteristic 3, where $c_1 = 0$ and where we know that there are summands $\Delta_\alpha$ not split-generated by $L$ (so $C \neq 0$).

## 7.2 Toric Fanos

Let $X$ be a smooth toric Fano complex projective $n$-fold, and let $L$ denote the monotone (barycentric) torus fibre. Let $k$ be an algebraically-closed field and $W\colon H^1(L;k^\times) \to k$ be the superpotential of $L$. Each point in $H^1(L;k^\times)$ defines a $k^\times$-local system on $L$. If $\xi$ is a critical point of $W$ then $HF((L,\xi),(L,\xi);k) \cong H^*(T^n;k)$ [16, Proposition 13.2], [28, Theorem 3.9], [14, Proposition 3.3.1].

The critical set of $W$ is a 0-dimensional scheme whose ring of functions (the *Jacobian ring*) is known to be isomorphic to quantum cohomology. For a proof, see [28, Theorem 1.3] for the statement over $\mathbf{C}$ and [61] for the statement in arbitrary characteristic; we restrict to algebraically closed fields here only to get an easy correspondence between $k$-points of the critical scheme and elements of $H^1(L;k^\times)$. Each $k$-point $\xi$ of $\mathrm{Crit}(W)$ therefore gives us (a) a local system $\xi$ on $L$ with non-vanishing Floer cohomology, and (b) a local summand $QH(X;k)_\xi \subset QH(X;k)$, the local ring at $\xi$. Indeed, $(L,\xi)$ is non-zero as an object in $D^\pi\mathcal{F}(X;k)_\xi$. Since $L$ is a free Lagrangian $T^n$-orbit, $(L,\xi)$ split-generates $D^\pi\mathcal{F}(X;k)_\xi$ by Theorem 6.2.4.

**Corollary 7.2.1** *For each nonzero summand of the Fukaya category of a smooth toric Fano variety, there exists a local system $\xi$ on the barycentric torus fibre $L$ such that $(L,\xi)$ split-generates this summand. In fact, by Corollary 6.2.6, any Lagrangian which is nonzero in the summand $D^\pi\mathcal{F}(X;k)_\xi$ split-generates this summand.*

**Remark 7.2.2** Such generation results have been established when the superpotential has non-degenerate critical points in characteristic zero [54] and for Morse or $A_2$-singularities in characteristic $\neq 2,3$ [64]. Note that in Corollary 7.2.1 there is no assumption on nondegeneracy of the critical points of the superpotential or on the characteristic of the ground field.

**Example 7.2.3** Projective spaces $\mathbf{CP}^n$ are toric Fano, so Corollary 7.2.1 applies to them. If $n = p^k - 1$ and $k$ is an algebraically closed field of characteristic $p$ then $QH(\mathbf{CP}^{p^k-1}; k)$ has no proper idempotent summands, so any Lagrangian with $HF(L, L; k) \neq 0$ split-generates the whole Fukaya category. In $\mathbf{CP}^{m^2-1}$ there is a Lagrangian copy of $PSU(m)$. This Lagrangian has minimal Maslov number $2m$ and, when $m = p^k$ for some prime $p \geq 3$, its cohomology over $k$ is [15, Théorème 11.4]

$$k[y]/(y^{p^k}) \otimes \Lambda(x_1, x_3, \ldots, x_{2p^k-3})$$

where $|x_i| = i$, $|y| = 2$. Since this is generated by cohomology classes of degree strictly less than $2p^k - 2$ and the minimal Maslov number is $2p^k$, [13, Theorem 1.2.2(i)] implies that $HF(L, L; k) \neq 0$, hence this Lagrangian split-generates the Fukaya category over $k$. See also Iriyeh's paper [35] where these examples are discussed.

**Example 7.2.4** In a similar vein, we can prove that the real part $L$ of a toric Fano variety $X$ split-generates the Fukaya category when $\mathrm{char}(k) = 2$, provided $L$ is orientable. This was proved in certain cases by Tonkonog [64], including $\mathbf{RP}^{2n+1} \subset \mathbf{CP}^{2n+1}$ (his methods also enable him to study some non-orientable cases). The real part $L$ is known to have nonzero Floer cohomology [33]. To show that $L$ generates every summand, we need an observation of Tonkonog [64, Theorem 1.12]: if $CO^0 \colon QH(X; k) \to HF(L, L; k)$ is the closed-open map then

$$\ker(CO^0) = \ker(\mathrm{Frob})$$

where $\mathrm{Frob} \colon QH(X; k) \to QH(X; k)$ is the Frobenius map $x \mapsto x^2$. Since $e_\alpha^2 = e_\alpha$ for all idempotents $e_\alpha$ in $QH(X; k)$, none of them live in the kernel of $CO^0$. Therefore $L_\alpha \neq 0$ for every summand $\alpha$, and $L$ split-generates the whole Fukaya category by Corollary 7.2.1.

The reason we need to assume orientability despite working in characteristic 2 is that for a non-orientable Lagrangian $L$, the value of $\mathfrak{m}_0(L_\alpha)$ is not determined by $\alpha$: we only know that $2\mathfrak{m}_0(L_\alpha)$ is equal to $2\alpha(c_1(X)_s)$, which is an empty statement in characteristic 2. For example $\mathbf{RP}^2$ has $\mathfrak{m}_0(\mathbf{RP}^2) = 0$ while $\mathfrak{m}_0(T^2) = 1 \mod 2$ for the Clifford torus $T^2$. We thank Dmitry Tonkonog for drawing attention to this. By contrast, for an orientable Lagrangian, the Maslov class $\mu \in H^2(X, L; \mathbf{Z})$ is divisible by 2, so one can always find a pseudo-cycle in $X \setminus L$ representing this relative class and the proof of [59, Lemma 2.7] shows that $\mathfrak{m}_0(L)$ is an eigenvalue of $c_1(X)$ without factors of 2.

## 7.3 Nonformality of quantum cohomology

The diagonal Lagrangian $\Delta \subset X^- \times X$ is relatively spin [26, Proof of Theorem 1.9(1)], and monotone if $X$ is monotone, so for any field $k$ there is a Fukaya-Floer $A_\infty$-algebra $CF(\Delta, \Delta; k)$. Since $HF(\Delta, \Delta; k) \cong QH(X; k)$, we can transfer this to an $A_\infty$-structure on $QH(X; k)$ by homological perturbation. We will denote this $A_\infty$-algebra by $\mathcal{QH}(X; k)$.

Our arguments for split-generation used formality of $C_*(\Omega G; k)$ in a very serious way, to express the zero-section as a Koszul twisted complex built out of the cotangent fibre. However, we did not require $\mathcal{QH}(X; k)$ to be formal, and indeed when the underlying algebra $QH(X; k)$ is not semisimple, formality often fails. Recall the following result:

**Theorem 7.3.1** ([10]) *Let $k$ be a field and $A$ a commutative $k$-algebra, finite-dimensional as a $k$-vector space. If $\mathrm{HH}^n(A) = 0$ for two nonnegative values of $n$ of different parity (for example if $\mathrm{HH}^\bullet(A)$ is finite-dimensional over $k$) then $A$ is a product of separable field extensions of $k$. In particular, it is semisimple.*

**Remark 7.3.2** Note that this theorem is about ungraded algebras. It is possible for a non-semisimple graded algebra to become semisimple when the grading is collapsed (if the idempotents have mixed degree). The quantum cohomology is naturally $\mathbf{Z}/2$-graded; it sometimes admits a more refined $\mathbf{Z}/2N$-grading, but we are only interested in the $\mathbf{Z}/2$-grading. In fact, we are only interested in $\mathbf{Z}/2$-graded algebras supported in even degree, which are semisimple as $\mathbf{Z}/2$-graded algebras if and only if they are semisimple as ungraded algebras.

**Corollary 7.3.3** *Let $\mathcal{A}$ be a $\mathbf{Z}/2$-graded $A_\infty$-algebra over $k$ supported in even degree whose cohomology $A := H(\mathcal{A})$ is commutative and finite-dimensional as a $k$-vector space. If $A$ is not semisimple as a $\mathbf{Z}/2$-graded algebra then $\mathrm{HH}^\bullet(\mathcal{A})$ has infinite rank.*

**Proof** Recall that the Hochschild cohomology $\mathrm{HH}^r(A)^s$ of a $\mathbf{Z}/2$-graded algebra $A$ admits a $(\mathbf{Z}, \mathbf{Z}/2)$-bigrading $(r, s)$; the $r$-grading is the length of a Hochschild cochain. Given the hypotheses of the corollary, $A = A^0$ is finite-dimensional, commutative and not semisimple. By Theorem 7.3.1, the Hochschild cohomology group $\mathrm{HH}^r(A)^0$ is nonzero for infinitely many $r$.

By contrast, while the Hochschild differential for the $A_\infty$-algebra $\mathcal{A}$ still respects the length filtration, it might not have degree one. Only the combined grading $r + s \in \mathbf{Z}/2$ survives on $\mathrm{HH}^\bullet(\mathcal{A})$. If the $A_\infty$-algebra $\mathcal{A}$ is $A_\infty$-quasi-isomorphic to its cohomology algebra $A$ then the Hochschild cohomology $\mathrm{HH}^\bullet(\mathcal{A})$ is simply $\bigoplus_{\bullet = r+s \in \mathbf{Z}/2} \mathrm{HH}^r(A)^s$. Since this sum is taken over infinitely many values of $r$, we see that $\mathrm{HH}^\bullet(\mathcal{A})$ has infinite rank. □

**Proposition 7.3.4** *If $X$ is a toric Fano manifold and $k$ is an algebraically closed field then $\mathrm{HH}^\bullet(\mathcal{QH}(X;k))$ is finite-dimensional over $k$.*

**Proof** Suppose $X$ is a toric Fano. Then $X^- \times X$ is also a toric Fano and, by Corollary 7.2.1, its Fukaya category is split-generated by the product of monotone toric fibres $T \times T$, equipped with various different local systems. The diagonal $\Delta$ is a nonzero object in $\mathcal{F}(X^- \times X; k)$ for any field and hence it split-generates a subset of the summands of the Fukaya category, say $\langle \Delta \rangle = \bigoplus_{\alpha \in A} D^\pi \mathcal{F}(X^- \times X; k)_\alpha$ (where $\langle \Delta \rangle$ denotes the subcategory split-generated by $\Delta$). By Corollary 6.2.8, the Hochschild cohomology of $\langle \Delta \rangle$ is equal to a subspace of the quantum cohomology of $X^- \times X$. Since $QH(X^- \times X; k)$ is finite-dimensional over $k$, this implies that $\mathrm{HH}^\bullet(\langle \Delta \rangle)$ is finite-dimensional over $k$. Since the endomorphism $A_\infty$-algebra of $\Delta$ in $\langle \Delta \rangle$ is $\mathcal{QH}(X;k)$, we have $\mathrm{HH}^\bullet(\langle \Delta \rangle) = \mathrm{HH}^\bullet(\mathcal{QH}(X;k))$, which must therefore be finite-dimensional over $k$ as claimed. □

If $X$ is a toric Fano manifold then its quantum cohomology is supported in even degrees. Therefore Corollary 7.3.3 and Proposition 7.3.4 together imply the following result.

**Corollary 7.3.5** *If $X$ is a toric Fano and $k$ is an algebraically closed field such that $QH(X;k)$ is not semisimple as a $\mathbf{Z}/2$-graded algebra, then $\mathcal{QH}(X;k)$ is not quasi-isomorphic to $QH(X;k)$.*

This highlights a sense in which the quantum cohomology of a toric Fano still behaves "semisimply" provided you remember its $A_\infty$-structure, even when it is not semisimple as a ring. Remark 1.2.3 is another hint in this direction.

**Example 7.3.6** The simplest example is $\mathcal{QH}(\mathbf{CP}^1; \mathbf{F}_2) = \mathbf{F}_2[H]/(H^2 + 1)$ with $|H| = 2$. If one passes to the strictly unital $A_\infty$ model for this, it is not hard to show that the first non-vanishing higher product is $\mu^4(H, H, H, H) = H$.

## 7.4 Grassmannians

The Grassmannian $\mathrm{Gr}(n, 2n)$ admits a $U(n)$-action with a free Lagrangian orbit [49, Proposition 2.7] with minimal Maslov number $2n$. We will show that, when $n$ is a power of 2, this Lagrangian split-generates the Fukaya category over a field of characteristic 2.

**Proposition 7.4.1** *If $n = 2^s$ and $k$ is a field of characteristic 2 then $QH(\mathrm{Gr}(n, 2n); k)$ is a local ring; in other words, it only has one summand in its block decomposition.*

**Proof** If $\mathrm{char}(k)$ does not divide $r + 1$, then the quantum cohomology ring of $\mathrm{Gr}(r, N)$ has a description as the Jacobian ring of a potential function $W\colon k^r \to k$. This description is due to Gepner [31] and the potential function is defined as follows. Let $\sigma_i(\underline{\lambda})$ denote the $i$th elementary symmetric polynomial in $r$ variables $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$. Consider the map

$$\Phi\colon k^r \to k^r, \quad \Phi(\underline{\lambda}) = (\sigma_1(\underline{\lambda}), \dots, \sigma_r(\underline{\lambda})).$$

Define the function $\tilde{W}\colon k^r \to k$ by

$$\sum_{i=1}^{r} \left( \frac{\lambda_i^{N+1}}{N+1} + (-1)^r \lambda_i \right).$$

Since this is a symmetric polynomial, it factors as $\tilde{W} = W \circ \Phi$ for some function $W\colon k^r \to k$. This $W$ is Gepner's potential function (see also [45, Section 11.3]). The spectrum of the Jacobian ring is a 0-dimensional scheme whose underlying variety is the set of critical points of $W$. The preimages under $\Phi$ of critical points of $W$ are critical points of $\tilde{W}$. The critical points of $\tilde{W}$ are

$$\{\underline{\lambda} \ : \ \lambda_i^N = 1, \ i = 1, \dots, r\}.$$

Suppose that $r = 2^s$, $N = 2^{s+1}$ and $\mathrm{char}(k) = 2$. There is only one $N$th root of unity in $k$, namely 1. Therefore $\tilde{W}$ has a unique critical point $\underline{\lambda} = (1, \dots, 1)$. The image of this under $\Phi$ is $(0, \dots, 0, 1)$, because

$$\sigma_i(1, \dots, 1) = \binom{2^s}{i} \equiv \begin{cases} 0 \mod 2 & \text{if } i \neq 0, 2^s \\ 1 \mod 2 & \text{if } i = 0, 2^s. \end{cases}$$

Therefore the spectrum of $QH(\mathrm{Gr}(2^s, 2^{s+1}); k)$ is a 0-dimensional scheme whose underlying variety is a single point, $(0, \dots, 0, 1)$. This means that the quantum cohomology $QH(\mathrm{Gr}(2^s, 2^{s+1}); k)$ is a local ring. □

**Corollary 7.4.2** *The homogeneous Lagrangian $U(2^s) \subset \mathrm{Gr}(2^s, 2^{s+1})$ split-generates the Fukaya category in characteristic 2.*

**Proof** The Lagrangian $U(n)$ is the fixed point set of an antisymplectic involution on a Hermitian symmetric space [36, Remark 7]. Oh [52] tells us that this Lagrangian, equipped with the trivial local system, has nonzero self-Floer cohomology over a field of characteristic 2. By Corollary 6.2.5, it split-generates whichever summand of the Fukaya category it inhabits. Lemma 7.4.1 tells us that there is only one summand. □

**Remark 7.4.3** The Lagrangian $U(n) \subset \mathrm{Gr}(n, 2n)$ has minimal Maslov number $2n$ and the minimal holomorphic discs with boundary on $U(n)$ have been explicitly described [49, Proposition 4.1]. It should be possible, using this information, to work out for all $n$ precisely which summands of the Fukaya category of $\mathrm{Gr}(n, 2n)$ are split-generated by $U(n)$, equipped with different local systems, in different characteristics.

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