Transforming the mathematical practices of learners and teachers through digital technology[1]

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Abstract

This paper argues that mathematical knowledge, and its related pedagogy, is inextricably linked to the tools in which the knowledge is expressed. The focus is on digital tools and the different roles they play in shaping mathematical meanings and in transforming the mathematical practices of learners and teachers. Six categories of digital tool-use that distinguish their differing potential are presented: i. dynamic and graphical tools, ii. tools that outsource processing power, iii. tools that offer new representational infrastructures for mathematics, iv. tools that help to bridge the gap between school mathematics and the students' world; v. tools that exploit high-bandwidth connectivity to support mathematics learning; and vi. tools that offer intelligent support for the teacher when their students engage in exploratory learning with digital technologies Following exemplification of each category, the paper ends with some reflections on the progress of research in this area and identifies some remaining challenges.

Keywords: digital technology; transforming mathematical practices

Introduction

Mathematics has a dual nature, with procedures and calculations on the one hand, and concepts logically connected in structures on the other. For most people, whatever their age, the former view of mathematics dominates: mathematics is simply a set of disparate rules for calculation and students attempt to master this 'mathematical machinery' without seeing its purpose; using another metaphor, they 'practice the scales without playing a tune or even recognising that there is a tune to be played' (for a more elaborated argument, see Hoyles, 2015b).

Schools have access to a multiplicity of digital tools of varying sophistication and function, some specifically designed for mathematics and others more general. Some digital tools simply replicate mathematics as expressed in paper and pencil. By contrast, this paper attempts to tease out from the corpus of research and practice, a vision for the use of those digital technologies with the potential to transform the teaching and learning of mathematics and, in particular, enhance conceptual engagement. There are two inevitable and interrelated tensions in such use of digital technologies. The first is that learners and teachers need to be able to cope with the syntax and semantics of the digital technologies: to find out how they work, what they afford, and how they might be employed. The second tension is that students will tend to use the power of the technology to avoid the cognitive load of 'mathematical thinking'. Thus, it is important to distinguish the needs of mathematical *learners* from the needs of mathematical *users* – learners need to search for and appreciate generality and structure, while users want simply to get a particular job done or a problem solved (for an elaboration of this argument, see Hoyles and Noss, 2003). The interest in this paper is in digital technologies with at least some claim to transformative potential for mathematics learning. They might fall into two overlapping, cateogories: technologies that can catalyse shifts in the how mathematics is 'known', and technologies that can open windows on students' conceptions and practices, an idea elaborated in Noss and Hoyles (1996). I summarise the theoretical background underpinning, of what might promise to be transformational use of digital tools in mathematics teaching and learning, before outlining six categories of digital tools that distinguish their differing potential. I note from the outset that the potential for transformational change depends utterly on how the digital tools are used and the support offered by teachers on their use: the availability of hardware or software is a necessary, but far from sufficient, condition for transformational mathematics teaching.

Theoretical background

There are several theories that underpin research into the use of digital technologies in mathematics education (for an early overview, see, Drijvers, Doorman, Boon, Reed & Gravemeijer, 2010). Many, including the author, have remained committed to constructionism as a way of thinking about using computers for mathematics learning that will truly revolutionise access to mathematical ways of thinking. Seymour Papert launched the notion of constructionism in the mid-1980s, arguing that a powerful way for learners to build knowledge structures in their mind, is to build with external representations, to construct physical or virtual entities that can be reflected on, edited and shared:

> Constructionism […] shares constructivism's connotation of learning as "building knowledge structures" irrespective of the circumstances of the learning. It then adds the idea that this happens especially felicitously in a context where the learner is consciously engaged in constructing a public entity, whether it's a sand castle on the beach or a theory of the universe. (Harel & Papert, 1991, p.1).

It is useful to try to describe the characteristics of a constructionist environment. It embeds a compelling medium in which to explore 'powerful ideas' or intellectual nuggets, often while ostensibly constructing something else. It allows learners to take some ownership of the construction process which, potentially at least, leads to their engagement, confidence and empowerment. This framework underpins the design of what are termed microworlds, where a successful microworld is both an epistemological and an emotional universe, a place where powerful (mathematical, but also scientific, musical or artistic) ideas can be explored; but explored 'in safety', acting as an incubator both in the sense of fostering conceptual growth, and a place where it is safe to make mistakes and show ignorance. And, of course, crucially these days, a place where ideas can be effortlessly shared, remixed and improved (for an earlier discussion of all these aspects, see Noss & Hoyles, 2006).

It is important to emphasise that, as Papert was at pains to point out, constructionism was as much a theory of epistemology as of pedagogy. In fact in the ICMI Study on Technology Revisited (Hoyles & Lagrange, 2010), all participants were encouraged to reflect on the 10% of knowledge that would need to be rethought given the use of new digital tools. When using a scientific calculator, or a spreadsheet to engage with mathematics what is changed, in what the user needs to know mathematically? This was later abbreviated to 'Papert's 10%' and proved to be a worthwhile but challenging task followed by later research and development as part of constructionism and beyond.

Constructionism continues to attract innovative ways of designing tools and working with learners from across the world (see, for example, the proceedings of a series of conferences originally called Eurologo, and, since 2012, held under the banner of Constructionism). Constructionist research has tried to pin down more precisely what kind of 'thing' constructionism is. While constructionism might 'act' like a theory, it is perhaps best thought of as a "framework for action", providing "focus and direction to the design of learning environments with much left implicit and open to diverse interpretation" (Cobb & diSessa, 2004). The point is to acknowledge that digital technology can shape and be shaped by mathematical knowledge and its expression, and mathematical abstraction is scaffolded within computational media. Our way of thinking about this problem has centred around understanding how mathematical expression can at once appear as decontextualised and yet – notably in the context of suitably-crafted digital technology – remain connected to and situated in the linguistic and conceptual web of resources afforded by the medium and the activity system (see Noss & Hoyles, 1996; Hoyles, Noss and Kent, 2004).

In summary, this paper seeks to distinguish the different categories of digital tools that support an agenda for research into transformational change in mathematics teaching and learning, largely but not exclusively underpinned by constructionism. Some categories are well established with a considerable evidence base, while others are at a more embryonic stage. In what follows, six categories are outlined, each including an illustrative example. They are:

- *dynamic and graphical tools* that allow mathematics to be explored in diverse ways, from different perspectives;
- *tools that outsource processing power* that previously could only be undertaken by humans, to change the focus of attention;
- *tools that offer new representational infrastructures for mathematics* to change what can be learned and by whom;
- *tools that offer connections between school mathematics and learners' agendas and culture*, bridging the gap between school mathematics and the students' world;
- *tools that exploit high-bandwidth connectivity* to *support mathematics learning* opening new opportunities for students to share knowledge construction and their evolving solutions both within and between classrooms;
- *intelligent support for the teacher* to reduce the burden of scaffolding the diverse solutions generated when students engage in exploratory learning with digital technologies.

These categories are neither mutually exclusive nor exhaustive. Rather they serve as lenses through which to investigate the potential of using different digital tools to transform mathematical practices and make it possible to identify some challenges for future research which are discussed at the end of this paper.

1. Dynamic and graphical tools

Digital technology can provide tools that are dynamic, graphical and interactive. Using these tools, learners can explore mathematical objects from different but interlinked perspectives, where the relationships that are key for mathematical understanding are highlighted, made more tangible and manipulable. This category of digital tool use has a robust research base, mainly arising from arguments around the semiotic mediation of the tools, which can focus the learner's attention on the things that matter while simultaneously giving them some agency in this process. The computer screen affords the opportunity for teachers and learners to make explicit that which is implicit, and draw attention to that which is often left unnoticed (Noss & Hoyles, 1996, Falcade, Laborde & Mariotti, 2007).

By using digital technologies, students can produce an accurate sketch of the solution to a problem, where an accurate sketch is to be interpreted in a technical sense: it is accurate in that it meets the requirements of the problem situation but it is a sketch in that the necessary invariants of the mathematical structure of the problem are not formalised. However, the accuracy of the sketch means that by reflection on and manipulation of the sketch, the students can more easily come to notice what varies and what does not, and thus are more likely to become aware of what to focus on (Mason, 1996). Thus, during the process of dragging their sketch, students can test, by eye, if the constraints of the problem they had hoped to satisfy are indeed satisfied, and become aware of invariants and possible relationships between the elements under dragging. Without the dynamic aspect, expressed through dragging, this would be difficult, since the accuracy of the sketch as well as its interactivity (through hand/eye coordination) is essential to the process of noticing such relationships. The key factor is the interplay between dynamic (while dragging key points) and static (stopping when some relationship seems evident), and crucially, the management of this interplay is in the control of learners, so they can pause, reflect, go back and test in the light of feedback from the graphical image: a constructionist approach.

It is noteworthy that increasingly, dynamic sketches are built by teachers or curriculum designers and introduced to students as part of their schemes of work, in student worksheets, or more frequently, as teacher demonstrations presented on interactive whiteboards. This latter mode bypasses the need for students to be fluent with the software, having acquired the skill to build the sketches for themselves. Clearly it is possible for students to learn by watching moving diagrams, especially if accompanied with text or spoken words. But care needs to be taken to guide students in ways that interweave the pragmatic and the epistemic; so they first *notice* the impact of the changes made, and second have some appreciation as to *why* the changes are significant mathematically. (See Dagienė and Jasute (2012) for examples of different guided approaches to learning with dynamic sketches).

The dynamic and graphical functionality of digital tools can be illustrated in an episode from design research 2 using dynamic geometry undertaken as part of a research project[3] that set out to build bridges between informal argumentation and formal proof through participation in carefully designed activities involving construction and experimentation using digital tools, The aim of this task was that students using dynamic geometry tools would come up with conjectures about the properties of a quadrilateral whose adjacent angle bisectors crossed at right angles, properties that were then used as the starting points of a proof (for more details see, Healy & Hoyles, 2001). We noticed that some students, while dragging the vertices of a quadrilateral until the angle between the two angle bisectors measured 90°, noticed that one pair of sides of the quadrilateral were parallel. The constraints of the task created 'by eye' provided an object on which the students could reflect and make conjectures as to its structure, and then explore the generality of their conjectures by constructing and dragging. Our activities were designed for students to use with a teacher to scaffold and orchestrate the interactions, and specifically, to point to how the tools might be manipulated to highlight key points for mathematical learning.

The process of exploiting digital tools that represent variation and functional dependence has provided the basis for taking a Vygotskian perspective of semiotic mediation on student tool use in mathematics (see for example Bartolini Bussi, & Mariotti, 2008). It has also been theorised as part of instrumental genesis, with its constituent twin sub-processes of instrumentation and instrumentalisation (for early work, see Vérillon & Rabardel, 1995). Drijvers et al. (2010) put it thus when talking about the instrumental approach:

> According to this approach [the instrumental approach], the use of a technological tool involves a process of instrumental genesis, during which the object or artefact is turned into an instrument. This instrument is a psychological construct, which combines the artefact and the schemes (in the sense of Vergnaud, 1996) the user develops to use it for specific types of tasks. In such instrumentation schemes, technical knowledge about the artifact and domain-specific knowledge (in this case, mathematical knowledge) are intertwined. Instrumental genesis, therefore, is essentially the co-emergence of schemes and techniques for using the artifact. (pp. 214)

In using dynamic and graphical tools in the way illustrated above in dynamic geometry, the digital tool does not provide a language of description that 'captures' the moves undertaken. More recent work suggests that there might be a need to promote explicit discussion or demonstration of how digital technology controls the movement, the mathematical purpose served by the variation and a suitable language to describe it (see Clark-Wilson & Hoyles, under review).

From a constructionist viewpoint, in order for users to engage with the sketch in the ways anticipated by the designer, it is important that some aspect of the constructionist agenda is implemented – the black box of the digital tool is opened 'just a little' so the students can take some control of the solution process by building or adding something. They are able, in the spirit of constructionism, to explore with the digital tools at different 'layers' according to the student's goals and expertise: by, for example, watching a simulation to notice patterns and regularities, making some alterations in some values of the variables, exploring how the variables relate to each other, or even editing these relationships (for elaboration of the idea of layered learning, see Kahn & Noble, 2010). When these openings are on offer, the dynamic sketch would more likely serve as a 'boundary object' and facilitate the communication of meanings across communities, in this case between the designer and the student. The notion of the boundary object is used in the sense of Star and Griesemer (1989), in that it can serve as a focal point to trigger the coordination of perspectives of the different communities of practice engaged in a common problem, and, crucially facilitate communication between them. Research that introduces students to microworlds that are intentionally improvable, with students challenged to find and fix faults or bugs, shows much promise as a way of ensuring that students engage with the code, and open the black box just a little (see Healy & Kynigos, 2010).

There is new momentum behind the constructionist agenda with the widespread popularity of coding and programming, where young people are put in the role of designing and creating with digital media and not simply playing and searching online (see Resnick, 2012). Although coding initiatives tend to take place outside formal education, there are some promising initiatives in school. For example, the *ScratchMaths* project[4] seeks to harness the enthusiasm and energy for coding in the interests of mathematics learning (Benton, Hoyles, Kalas and Noss, 2017). *ScratchMaths* uses the programming language Scratch, (an online descendant of Logo) to promote mathematical learning. The project encompasses more than 100 schools, some 200 teachers, and 3,000 pupils of all abilities. It is a carefully designed intervention for pupils aged 9-11 and includes the provision of detailed curriculum materials (for 20+ lessons in each year), as well as carefully crafted teacher professional development - both necessary conditions for successful implementation in schools. The resources support teachers in planning and delivering their lessons, and in adapting the lessons for different pupils. (see https://www.ucl.ac.uk/ioe/research/projects/scratchmaths). The *ScratchMaths* project takes programming to scale within an education system (in contrast to taking place in out-of-school clubs). It offers effective pedagogy for learning mathematics, for all pupils and has broadened understanding of the potential benefits of learning programming as a tool to foster model-based reasoning across disciplines. Schools that utilise *ScratchMaths* talk of its transformative impact as a “breath of fresh air” that gives teachers the knowledge, skills and understanding to engage pupils in the mathematics curriculum more effectively. There will no doubt be more design research projects that follow in this path.

2. Tools that outsource processing power

There are a range of tools that outsource processing power, whether it is a simple pocket calculator to ‘do arithmetic’, a graphics calculator, a computer algebra system or, more recently, finely tuned applets for particular tasks. The goal of calculator use was originally articulated as ‘setting students free’ from technical procedures in order to focus on interpretation and problem solving. Heid, a leading researcher in this area, suggested that while using Computer Algebra Systems (CAS) mathematics teachers can focus on reasoning, and help students engage in problem solving, see for example, Heid & Blume (2008). Artigue (2002) stressed the need to attain a fine balance between the ‘pragmatic’ and ‘theoretical’ (or ‘epistemic’) roles of techniques, exemplifying her argument with research using CAS, undertaken within the framework of mathematical didactics. With the development of more high-level languages (computer algebra systems, dynamic geometry and statistical software), the debate about the potential and consequences of outsourcing calculation has intensified. Wolfram (2015), for example, has argued that "mathematics" is the wrong word for much of what is taught in schools. Rather, students learn to "calculate", and he goes on to assert that calculation "is an obsolete skill, since almost all calculating is done by computer these days."

The research evidence about the 'costs' of outsourcing to a computer all mathematical calculations and manipulations is mixed. It could be argued that relying on computers for calculation might lead to a loss in computational or algebraic fluency, handicapping future problem solving. Will students be able to spot a 'trig identity' within expressions that need to be simplified, without considerable prior practice? How is this 'skill' to be replaced if similar simplifications are automated? What is the balance of amplification, or mental reorganisation, from using tools with dependency on them (for a discussion of mental reorganisation, see Pea, 1985)? Does mathematics education research throw light on the role of routine tasks more generally whose outcomes are 'needed' in problem solving? What are the long term effects of outsourcing calculation to digital tools? These remain major questions for research.

An example is presented below of an attempt to address this dilemma from a research project [5] that investigated how mathematics was used in workplaces, where digital tools were all-pervasive and expected to do all necessary calculations. The users of the tools were not engaged in learning; rather they sought to reach correct outcomes efficiently. In our studies of the shopfloor, we noted how mathematical symbols generated by computers in the form of numbers, charts, tables and graphs were ubiquitous with displays put up to trigger the coordination of perspectives of the different communities of practice involved in the work, and facilitate communication between them (see Hoyles, Noss, Kent & Bakker, 2010). Yet, the language of communication in the work-based training around these symbolic artefacts was generally algebra, which proved a barrier to sense-making for most workers and a barrier to communication. As one trainer remarked: "The response to the formula is laughter. … Eyes glaze over … really lose it.". Another trainer told us that there was: "lots of information for the shopfloor to use in the form of graphical data but the shopfloor did not use it partly as it was quite obscure and partly because they did not really have a connection in their minds between what they were doing, what the [assembly] line was doing, and what the graphs were showing" (Hoyles, Noss et al., 2010). The symbolic charts and tables on the shopfloor simply did not achieve their purpose in supporting communication between different groups and tended to be ignored: "expensive wallpaper," as one worker remarked.

In the subsequent design research phase of the project arising from our constructionist framework, we developed what we called 'technology enhanced boundary objects' (TEBOs), through which different layers of structure could be revealed in the control of the user. TEBOs are highly constrained microworlds designed so that interaction in these worlds provides users with a glimpse of part of the underlying mathematical system. Thus the TEBOs offered dynamic, graphical and visual tools *alongside* and simultaneously with the calculations. The tool certainly took on the calculation role, but also enabled the user to observe elements of the structure that served as the basis for the calculation, thus promoting more meaningful conceptual engagement. As the first trainer above later remarked: "the web-based tool [the TEBO] helped us get across this concept … move to 'visualisation' and away from 'calculation'". And as the second trainer explained: "we could encourage them [through using the TEBO] to look at the information on the graph and see it as a story that was telling them something would be of benefit so that they could troubleshoot the process themselves". And even more powerfully, the second trainer went on to say "there are so many things we learnt from the tool [the TEBO] that we did not anticipate … it provided a window for me as to what the guys on the shopfloor thought - not only about the process but also about trouble shooting and problem solving as well" (Hoyles, Noss et al., 2010, pp. 60-67). One TEBO supported the interpretation of data presented in a variety of forms; another was designed to develop some overall appreciation of the production system and its underlying model (see Hoyles, Noss et al., 2010).

The point of this example is that the digital tool, the TEBO, certainly performed 'the calculations' but did much more: it was designed so users could glimpse the 'other side of mathematics' through dynamic and visual manipulation of key variables in a context that was meaningful to them. This trainer put into her own context my vision for using digital technologies in mathematics more generally: to build meaning (in her words, "stories") by opening windows on learners' ideas and conceptions, which can serve as starting points for the design of activities and teaching. The message from this work was that the mathematics-related skills needed in a variety of workplaces cannot be described only in terms of academic mathematics, but as mathematics framed by the work situation and context and it is such descriptions, which provide the language of communication between different communities in the workplace, for eample, workers on the line, senior management and sales reps. Rasmussen and Keene (2015) used of the notion of TEBOs in advanced school mathematics. They provided students with minimalist applets as a 'step before' working with CAS. Thus the use of the applets supported the students in *creating* concepts and methods and this experience later helped them to interpret the output from CAS. This promises to be a fruitful area of future research.

3. Tools that offer new representational infrastructures for mathematics

The difficulty of a mathematical idea often inheres in the system with which it is expressed. As Hoyles and Noss, 2008 argued "[I]magine just how difficult it would be to remember the procedural rules of calculus (like the chain rule) without Leibniz's elegant notation". Similarly, Papert's thought experiment about the transition from Roman to Arabic numerals serves as a provocative example of the opportunities and challenges that follow from changes in the representational infrastructure of mathematics (see Wilensky, 2010b). Clearly with the emergence of Arabic numerals, students would conceptualise numbers and perform calculations differently.

The hypothesis underlying this category of digital tools is that digital technology can offer such a new representational infrastructure, and thus change and shape a learner's language of interaction with mathematics and the ways mathematical ideas are expressed; that is, promote transformational change, not least as this shift in representational infrastructure could provoke discussion of what is important in mathematics, resonating with the idea of Papert's 10% mentioned earlier. The research studies investigating this hypothesis are limited so below is an illustrative example.

The example concerns a thought experiment triggered by an unexpected (for the author) result from the Longitudinal Proof project [6]. Surveys were conducted each year for 3 years, with the same students followed up and tested each year. In one question, shown in Figure 1, a geometric diagram was presented which lent support to a false conjecture, and students were asked whether or not they agreed with the conjecture and to explain their decision. We found that 40% of students (age 13) in our large national sample of top-set students answered incorrectly with the percentage only dropping to 26% at age 15.

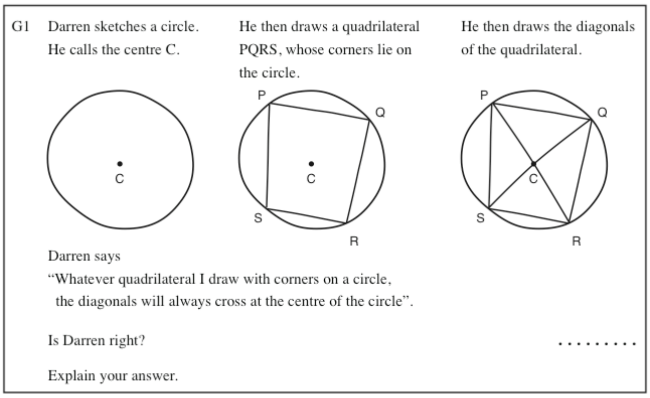

Figure 1. Geometry item from Proof survey [7]

But suppose dynamic geometry systems were infrastructural, and students were just as familiar using these systems as they are using a pencil. Would they not "naturally drag a point in their heads" - any vertex would do - leading to a rather different profile of responses? And would we then not need to rethink our learning goals, learning hierarchies – and of course our teaching approaches? The following question was submitted by a secondary mathematics teacher as part of the online discussion on the teaching and learning of proof organised by the National Centre for Excellence in the Teaching of Mathematics (NCETM), in June 2008: "I now use a dynamic geometry package on my interactive whiteboard to illustrate the circle theorems. This shows that these theorems always work. Why should my students still have to learn how to write down formal geometric proofs?" (See online panel discussion on proof, https://www.ncetm.org.uk/).

This question is by no means uncommon now and represents a general challenge for teachers and for researchers that needs to be addressed. If certain digital tools become infrastructural so that students 'think with them', what are the consequences for mathematics teaching and learning, and for school mathematics more broadly? What are the evidence-based frameworks that we could turn to for advice and support? There is in fact some advice available in this particular case arising from the considerable research into the teaching of proof, which suggests that it is important to make a shift in teaching following the use of dynamic geometry systems from illustrating that a particular theorem 'always works' to discussing why this is the case: or put another way a shift to prioritise in school "the explanatory power of proof" (see Laborde, Kynigos, Hollebrands & Strasser, 2006). More research is needed to investigate how a similar shift in teaching with digital technology that promotes explanation of the computer output among peers, orchestrated by the teacher might be fruitful.

4. Tools that help to bridge the gap between school mathematics and the students' world

Mathematics is central to the school curriculum, yet, as argued earlier, all too often mathematics does not engage learners who do not discern the point of the mathematics they are forced to learn. The technology-based 'information society' needs model-based reasoners who can use mathematical thinking as a way of making sense of the world. Thus the importance of tapping into youth culture must not be underestimated, through the design of 'engaging environments' in which some mathematical thinking and application is actually *needed* for students to achieve the goals that they find compelling. One way to achieve this is for learners to take the role, to some extent at least, of producers rather than consumers of digital tools and while engaging in collaborative design of these tools, they develop the ability to construct, describe, and explain the effects of the tools; that is how the system works. There is some research that provides indications of this potential but there more research could usefully be conducted.

Two case studies, along with their underlying theoretical principles of this approach are reported in Hoyles and Lagrange (2010). Both studies comprised sets of interactive modelling activities, microworlds, targeted at 11-14 year old students, with the following underlying rationale:

> Students are typically told that they must study mathematics in order to keep open their options to pursue quantitatively-oriented careers in mathematics, science, technology, or engineering. For most of them, this is a very distant and abstract motivation, especially for students whose familial network does not include members who currently engage in such work. (See Confrey et al., 2009, p. 19)

The emphasis in the design allowed the exploration of mathematical ideas in different 'layers', embedding increasing problem-solving complexity into the software. The first case study concerned designing a Space Travel Games Construction Kit, in which specially designed programme fragments were provided that the students could customise, assemble and re-assemble to build a computer game. The student could 'dig down' and edit as far as they wished even to the programming code, or stay at the level of changing variables with sliders, or changing backgrounds by selecting different images. The second case study concerned the design of an animation microworld to strengthen and connect students' numerical and geometric knowledge, with the aim of teaching students a range of mathematical ideas, including integer and rational number operations, similarity and scaling, coordinate graphing and tables, basic geometric concepts, transformations, and ratio reasoning (for more detail, see Confrey et al., 2009).

Many studies outside mathematics education research similarly suggest that digital technology devices hold the potential to tap into the culture of young people to support their 'STEM learning'. To take one quite provocative quote:

> As core education is increasingly distributed in out of school contexts, it is time to start considering how everyday things might lend themselves to teaching the fundamentals students need to know. After all, they will be using the science and mathematics to invent the next, newest everyday things. (Lewis & Ju, 2013, pp.295)

Design research for mathematics learning with this explicit affective dimension as a goal - along with the aim of building connections between 'school' and 'outside school' - is rather less well developed than the other categories above. Such research will be of increasing importance at least in England, as more and more students are required to study mathematics for longer but may not know why. Research and development to ensure their mathematical studies provide some intellectual reward as well as important qualifications would be timely.

5. Tools that exploit high-bandwidth connectivity to support mathematics learning

With the massive increase in the power and reach of web infrastructure, both in homes and schools, there is little doubt that much has changed in terms of access to connectivity. But the question remains as to how far affordances arising from this enhanced connectivity have been exploited in the interests of mathematics education. Connectivity has the potential to lead to sharing solutions, problem solving strategies and knowledge construction. In practice, this potential is rarely achieved. At the ICMI Study Conference (Hoyles & Lagrange, 2010), contributions on this topic of connectivity and virtual networks for learning mathematics were called for, but very few proposals or papers were received. A few experts were therefore invited to be part of a plenary panel on this topic. This field is still at quite an early stage but their contributions, summarised below, give an idea for some fruitful avenues for research.

First it is possible to develop microworlds specifically for on-line collaborative learning with opportunities for communication and working in common learning spaces, physically or virtually (Kalas, 2010). Key components of designing for connectivity were outlined by Noss and Hoyles (derived from two large-scale projects: the Playground project and the WebLabs project, both funded by the European Union), where students built and shared their own models, so had developed ownership of the models, and in the process of writing a joint report the students (face-to-face or online) developed a shared language of communication. On the basis of research in 'ordinary classrooms', Trouche and Hivon (2010) noted how the teacher's behaviour was significantly modified when designing to exploit connectivity for mathematics learning: the teacher not only had to create the conditions for students to build a mathematical object but to take on board that this object would be built by the *community* of students. To quote: "A student does not play only the music written by the conductor anymore. He is writing part of the music. The question is now to know how the teacher can create conditions to make the music not too different from what s/he wants it to be!" Wilensky (2010b) asserted that concurrent connectivity was a rather neglected affordance of connectivity. His examples of classroom participatory interactive simulations and their outcomes indicated the potential in the classroom contexts. Finally, Noss and Hoyles (2010) reported the findings of two European projects, the Playground Project and the WebLabs project that set out to investigate ways that students could be motivated to collaborate while physically separated. In the Playground young children (aged 4 to 8 years) built and shared their own videogames during out-of-school clubs, while in WebLabs, students engaged in mathematics or science lessons, with a web-based collaboration system to share the models they had built to solve a particular problem and to discuss their assumptions and evolving solutions. All the presentations reported positive outcomes but also mentioned technical, linguistic and cultural obstacles that needed to be addressed (for more detail, see Hoyles et al., 2010).

So what new insights for mathematics education can be gleaned from research that takes account of the ubiquitous connectivity now on offer? There is a large and growing body of research in computer-supported collaborative learning that includes some research in learning mathematics (see, for example, Stahl, Cress, Law, and Ludvigsen, 2014). I wonder if this rich strand of research permeates more 'mainstream' mathematics education research? If, as I conjecture, the answer is 'rather little', is this yet another example of the 'isolation of technology-related research' that Hoyles and Noss (2003) noted. One example that seeks to build bridges between research with digital tools including exploiting connectivity is the ScratchMaths project, mentioned earlier, that includes tasks to be performed collaboratively on and off the computer. But there is much more to be done, not least to research in detail the impact of such tasks.

6. Intelligent support for the teacher

In 2008, Richard Noss and I led a research project, Migen,[8] which set out to design some intelligent support for the teacher when students are engaged with specific tasks. The aim of the project was to reduce the burden of scaffolding the diverse responses generated when students engage in exploratory learning with digital technologies. We know that the teacher's role with a class of students using digital technologies in exploratory ways is certain to be challenging, due to the sheer number and variety of student outputs that need appropriate response. The research set out to design scaffolding as part of our microworld to 'address' at least some of the predictable responses originating from known conceptual obstacles. In Migen, we took as our focus algebra and generalisation in early secondary school, an area that has attracted a considerable volume of research on student errors, approaches and alternative conceptions. The project then designed a pedagogical and technical system that allowed students to interact with mathematical objects and relationships (in the form of figural patterns and the rules that governed them), and, as a result of their interactions with carefully designed tasks, could 'see' the general case unfold. We aimed to create a situation where, by engagement with the microworld, the following three key 'algebraic ways of thinking' would be developed: perceiving structure and exploiting its power; seeing the general in the particular, including identifying variants and invariants; and recognising and articulating generalisations, including expressing them symbolically. (For more details see, for example, Mavrikis, Noss, Hoyles & Geraniou, 2012).

The microworld provided informative feedback from its dynamic functionality, but also from the 'intelligent support' designed to be triggered when responses pointed to a 'typical' misconception – that is, a misconception we could predict. This is the point where interdisciplinary input is essential to include those who could conjecture about the nature of these challenges (as gleaned from the mathematics education research literature), those who could conjecture what an expert teacher might do in these situations, and those with technical expertise who could implement the required prompts. A major challenge – arguably, the major challenge – was to design support in ways that provided students with 'enough' freedom so they could actively engage in their construction task, yet with adequate constraints so as to be able to generate feedback that assists students in achieving the planned mathematical goals – an example of the play paradox described in Noss and Hoyles (1996). The system has been developed further and integrated into larger platforms with additional teacher support (Dragon et al., 2013) and colleagues have developed e-books based on these ideas as part of the Mathematical Creativity Squared project[9]. The design ideas of the microworld have also formed the basis of one unit of Cornerstone Mathematics (see below). Finally there have been interesting developments associated with intelligent support in natural language designed for tertiary education students, as reported in Rojano and Garcia-Campos (2016). But again this area is rich in opportunities for more research in different mathematical domains and with different students.

Challenges for the mathematics education research community

This outline and exemplification of the six categories of digital tools that might support an agenda for research into transformational change in mathematics teaching and learning indicates that much remains to be done to fulfil the potential promised. In some areas, research is at an early stage; in others, outcomes are more established but are yet to have widespread impact. What new directions for research have opened up and what challenges are still to be faced? Below are some broader issues to that it would be fruitful to investigate further.

Embedding digital tools to support epistemological transformation through design research

The volumes of RME reveal rather little research addressing this challenge. I mention one article here: Sinclair and Yurita (2008) discussed how using dynamic geometry changed classroom discourse with significant differences in the ways in which teachers talked about geometric objects, made use of visual artifacts and modelled geometric reasoning. Is it worth the mathematics education research community revisiting or catalysing more research into how mathematical processes and objects are transformed with the use of digital tools and how this impacts teaching and learning? This research would, in my view, require an explicit focus on design – the design of tasks, digital tools, feedback and evaluation – as well as the investigation and identification of what might be different epistemological goals for school mathematics in the light of the available technological infrastructure, or put another way, work on 'Papert's 10%'. I wonder if the move in general *towards* researching mathematical classroom practice that embeds digital tools has had the unintended consequence of moving *away* from design research and considerations of epistemology and curriculum?

In her editorial for the ICMI book (Watson & Ohtani, 2015), Watson makes a perceptive point, arguing that: "Few studies justify task choice or identify what features of a task are essential and what features are irrelevant to the study" (p.12). Is it the case that much of mathematics education research tends to engage with systematic analyses of what happens in mathematics classrooms with tasks taken as givens? This echoes what Papert might have had in mind when he criticised the general field of mathematics education research for not allocating sufficient energy to consider the 'what' rather than merely the 'how' of teaching. By contrast, in the much smaller, sub-domain of mathematics education research around 'design research with digital tools', the task, its design and the software are all at the forefront of the collective research effort: the design of digital tools is predicated on identifying and expressing mathematical concepts in novel ways to promote learning mathematics.

So there seems a good case for more research into the transformational potential of using digital technology in mathematical practice, and to move beyond the documentation of new mathematical meanings and discourses, to proposals and evaluations of 'mini' curricular systems that deliberately exploit the expressive potential and functionalities of digital tools and identify 'new hierarchies' of learning. Well-researched notions such as that of "hypothetical learning trajectory" (HLT), originally conceived by Simon (1995), recognise that learning activities and the learning processes are interdependent and co-emergent, rather than setting out universal steps of learning. Can and should design research embedding digital technologies usefully contribute to this corpus of research, for different age-ranges and for different mathematical topics? I refer here to a volume (Monaghan, Trouche & Borwein, 2016), which presents an excellent starting foundation with its multiple examples of 'doing mathematics with tools' and the associated issues around teaching, teachers and the curriculum.

We have contributed to the agenda of transformational change through the design and implementation of Cornerstone Mathematics (CM) (See Hoyles, Noss, Roschelle & Vahey, 2013; Clark-Wilson, Hoyles, Noss, Vahey & Roschelle, 2015). CM encompasses four elements, each of which have been extensively researched and developed over many years: digital technology for mathematics learning, professional development and teachers' skills and knowledge, developing curriculum replacements activities, along with scaling and sustainability of research-based interventions. The novelty of the Cornerstone approach lies in the particular combination of the elements, seeking to exploit the positive potentials of each, and overcoming some of the limitations. CM has designed activities embedding digital technology that focus on a selection of topics, where we predicted the mobilisation of digital tools would make a difference to student engagement, student learning and progress: our contribution to Papert's 10%. In order to simplify the methodological challenge of evaluating CM in practice, we provided a common focus for the analyses of classroom interactions and teacher moves by identifying what we termed 'landmark activities', designed so that students, through their explorations with the digital tools, would come up against unexpected outcomes. (see also Rowland, 2015).

However, unsurprisingly all innovations that embed digital technologies rely fundamentally on appropriate teacher support, which takes me to my second challenge.

Progress in supporting teachers and teaching in using digital technologies?

It was recognised in the ICMI study (Sinclair & Yurita, 2008, section 2), that the teacher had been a relatively neglected player in digitally oriented research, where the focus had tended to be on individual(s) 'doing mathematics with software' and the mutual effects of their interactions. Early design research with computers reveals rather little detail of the role of the researchers and the teachers, although teacher scaffolding of mathematics learning was certainly recognised as critical. For digital technologies to move from the periphery to centre stage in mathematics teaching and learning and for its potential for transforming mathematical practice to be realised, I would argue that teachers must be part of the transformative process as co-designers and teacher researchers. But the design process is challenging: the dialectical influence of tools on mathematical expression and communication must be taken into account, the diverse foci of design and analyses (software and activity design, interactions in classrooms, teacher scaffolding moves) each demand different expertise, methodologies and resources.

Nonetheless, many mathematics education researchers are now investigating explicitly the role of the teacher in classrooms where digital technology is used (see for example, Clark-Wilson, Robuttie & Sinclair, 2014). In addition, turning the research gaze onto teachers has raised the question of the nature and extent of 'necessary' continuing professional development and learning. One way to support such professional learning could be through a national infrastructure for Continuing Professional development (CPD) such as the National Centre for Excellence in the Teaching of Mathematics (NCETM) in England. Such networks and communities could support the use of digital technologies in classrooms, share good practice and ways to overcome obstacles. Is such a national infrastructure a necessary backdrop for embedding digital technologies at scale? Is it possible to maintain a balance of 'top-down' and 'bottom-up' initiatives? The initiatives mentioned above might be important components in this next step (see Sinclair et al., 2010, for an earlier discussion of implementing digital technologies at a national scale) and insights can be gleaned from the research on scaling CPD as reported in Roesken-Winter, Hoyles & Blömeke (2015) but this is just the start of a much longer research journey.

Managing methodological complexity

Artigue (2002) complained there was no framework to underpin analyses of the use digital technologies in mathematics classrooms. There are now many: some general for analyses of teacher, student and classroom interactions and teacher expertise, and some specifically concerned with practice embedding digital technologies. (For a description of analyses of learning using the documentational approach, see Gueudet, Buteau, Mesa, & Misfeldt, 2014).

This multiplicity of frameworks brings new challenges not least to compare and contrast research and to build a cumulative picture of results. Ruthven (2014) provides a helpful summary of a subset of 'contemporary' frameworks as he calls them, and concludes his chapter by suggesting that: 'More intensive research at the concrete level could serve better to operationalize the existing frameworks or to fuel the development of a single more powerful one' (p. 392). It would seem important for the research community in mathematics education to take steps in this direction?

Despite this focus, that research task is complex, with multiple sources of data to be collected, analysed and synthesised. In Hoyles and Noss (2016) it was conjectured that a potentially rich new strand of design research methodology might usefully include more complementarity between qualitative and quantitative data analyses, thus harnessing further the emerging techniques of big data and learning analytics. There are already initiatives in school mathematics that are working to capitalise on the ease with which data can be collected and analysed, Several of these are initiated and managed by experienced practicing teachers (see for example Hegarty Maths, (https://hegartymaths.com, a massive online resource with explanations of carefully modelled examples, and a tracking system that allows teachers to focus on pupils' mistakes). My point here is there is rather little evidence of any role of the mathematics education research community in such initiatives that reach so many teacher, and there might be a fruitful role for more research into their design, data analyses and interpretation, and into the form of the feedback communicated to teachers and schools?

Final remarks

I end by reiterating what might be read as a rather obvious point. Clearly, the mere presence of digital technology or even the ready access to data makes little difference to student learning outcomes. Outcomes depend on how all these resources are used separately or in combination. This point is worthy of emphasis, however, as I still hear of projects looking at the 'effects of, say, a technology on X or Y' (see, for example, a study reporting on 'The Impact of Computer Usage on Academic Performance' Carter, Greenberg & Walker, 2016). As early as 1985, an article by Pea conveyed the following crucial message: "The urgency of updating education's goals and methods recommends an activist research paradigm: to simultaneously create and study changes in processes and outcomes of human learning with new cognitive and educational tools." (Pea, 1985). Can the mathematics education research community embrace such an activist paradigm and take steps to overcome any remaining "marginalization of technology", and make progress in elaborating evidence-based ways to exploit the use of digital tools to enhance mathematics learning for all? All too frequently, the costs and accessibility of digital technologies are given as reasons why impact has not reached expectations. It could be argued that this is no longer the case, in many countries including the U.K. With ever increasing knowledge, a more robust theoretical basis, along with systematic evidence from the research community to underpin the necessary teacher support (see, for example, Clark-Wilson & Hoyles, 2017), it seems a promising time for research to move forward with practice and to support teachers so students progress along trajectories of learning with digital tools, with the ultimate goal that more students reach a broader view of mathematics – one that is so much more than calculation and one that they judge to be personally empowering and fulfilling.

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1. This paper derives from the keynote presentation of the author to the International Congress of Mathematics Education (ICME), delivered in Mexico in 2008. The text is inevitably in part retrospective. It has been updated in relevant parts without the pretence of producing an exhaustive review of the research in the intervening years. Some parts of this paper were elaborated in Hoyles & Noss (2009). Earlier versions of the first part of the paper appeared in Hoyles (2012) and in Spanish in Hoyles (2015).
2. A framework for design research was proposed in the seminal paper by Cobb et al. (2003), and was the subject of a special issue of ZDM, Prediger, S. Gravemeijer K., & Confrey, J (2015)
3. “Justifying and Proving in School Mathematics”, Hoyles & Healy, funded by the Economic and Social Research Council, 1999-2003.
4. ScratchMaths: supporting computational and mathematical thinking through programming, Noss and Hoyles, funded by the Education Endowment Fund, 2014–18.
5. Techno-mathematical Literacies in the Workplace Noss & Hoyles 2003-07 funded by the Teaching and Learning Research Programme, TLRP, ESRC.
6. Longitudinal Study of Mathematics Reasoning, Hoyles & Küchemann funded by the Economic and Social Research Council, 1999-2003.
7. This item is taken from the Year 8 and Year 10 versions of the longitudinal proof test: the Yr 9 test was slightly different.
8. The MiGen project funded by EPSRC/ESRC, TLRP-TEL, (Technology Enhanced Learning programme) 2007- 2011, Grant number RES-139-25-0381
9. See http://mc2-project.eu/