The use of singularity structure to find special solutions of differential equations: an approach from Nevanlinna theory

by

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I, Khadija , confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

For a certain family of ordinary differential equations, Nevanlinna theory is used to find all solutions in a special class. A differential equation is said to possess the (strong) Painlevé property if all solutions have only poles as movable singularities. The solutions of such equations are particularly well behaved and the Painlevé property is closely associated with integrability. In this thesis, we extend the idea of using singularity structure to find all special solutions with good singularity structure, even when the general solution is badly behaved. We begin by finding solutions that are meromorphic in the complex plane and more complicated than the coefficients in the equation in a sense made precise by Nevanlinna theory. Such meromorphic solutions are called admissible and include all non-rational meromorphic solutions of an equation with rational coefficients. The use of Nevanlinna theory in the entire complex plane does not allow the solutions to be branched at fixed singularities, which seems more natural from the perspective of the Painlevé property. Motivated by this, we consider an extension of Nevanlinna theory to a large sector-like region with a deleted disc to allow for such branching and apply this theory to differential equations.

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Contents

Chapter 1

Introduction

When one encounters a particular differential equation, it is natural to ask whether it has any explicit solutions. There are some ad hoc methods that work for very specific equations, such as looking for solutions of a very special form, however, there are only two widely-used general methods for identifying equations that can be solved in some sense explicitly. The first is Lie's symmetry method. Effectively, if an ordinary differential equation has a sufficiently rich group of symmetries, then it can be solved by quadrature. The second method uses the singularity structure of solutions in the complex domain as an indicator of whether the equation is integrable. The work reported in this thesis is an extension of the second approach.

Kowalevskaya [21] was the first to use singularity structure to identify a new integrable class of equations. The equations of motion for a spinning top contain a number of parameters. Kowalevskaya noticed that for each of the previously known choices of parameters for which the equations could be solved, the general solution was meromorphic, so she went on to determine which choices of parameters led to a meromorphic general solution. She found one new case which she subsequently solved using theta functions. This is the last case of these equations to be solved in 128 years. Subsequently Picard posed the problem of classifying all differential equations of a certain form such that the solution is single-valued about all singularities apart from certain singularities that are fixed by the form of the equation. Such equations are now said to possess the Painlevé property. Painlevé and his colleagues [35] classified all equations with the Painlevé property of the form $y'' = R(z, y, y')$, where R is rational in y and y' and analytic in z on some open set. Six new equations, the Painlevé equations, were found. All other equations in the class could either be solved in terms of previously known functions or in terms of solutions of the Painlevé equations. The solutions of the Painlevé equations are now considered to be important functions of mathematical physics in their own right and have many remarkable properties.

For the purposes of this thesis, we will refer to a slightly stronger version of the property. An ordinary differential equation is said to possess the (strong) Painlevé property if all movable singularities are poles. A movable singularity is one that occurs at a value of the independent variable for which the equation is not singular in some sense for generic values of the dependent variable. Such singularities are called movable because their locations change as we vary the initial conditions.

The standard so called Painlevé test, which is actually closer to the procedure originally used by Kowalevskaya rather than that used by Painlevé, involves the substitution into the equation of a formal Laurent series expansion of a solution about a movable singularity. In order to have a sufficiently rich class of such solutions one demands that a sufficiently large number of the coefficients in the expansion are arbitrary. This results in a number of resonance conditions that relate the values of various coefficients in the equation to their derivatives evaluated at the movable singularity. We want these conditions to hold at all movable singularities of all solutions, so these conditions ultimately become differential equations that are solved to determine the final form of the equation.

The main idea behind the present thesis is to extend the idea of using the singularity structure of solutions to find differential equations that are integrable, i.e., equations for which we can in some sense nicely characterise the general solution, to the use of singularity structure to find well behaved solutions, regardless of whether the general solution is well behaved. To this extent a natural problem would be to find all solutions of a given equation such that the only movable singularities of the solution are poles. At the moment this problem is too difficult. In particular, we are not able to prove that we have found all such solutions. However, if we know that in some sense we have enough movable singularities then we can still use series type methods such as those used in Painlevé analysis to answer the question. This leads us to Nevanlinna's theory on the value distribution of meromorphic functions.

In this thesis, Nevanlinna theory will be used to show that we have sufficiently many movable singularities to be able to use series methods. These methods go well beyond ideas in Painlevé analysis as in many cases we do not have resonance conditions at all or we do not know if we have resonance conditions (without doing a lot more analysis). Nevertheless, in order to use Nevanlinna theory we add two assumptions to the original statement of our problem. First, we require the solutions to be meromorphic. This means that not only are the movable singularities poles, but the fixed ones are as well. This is somewhat artificial from the point of view of the Painlevé property. Also, in order to extract information about the coefficients we assume that they are simpler than the solution we are considering in a sense made precise by Nevanlinna theory. In particular, if the coefficients are rational functions, we can only perform our analysis on non-rational meromorphic solutions. In the case of more general meromorphic coefficients, we talk about admissible solutions, a concept that often arises in the application of Nevanlinna theory to differential equations.

Chapter 2 contains a brief introduction to classical Nevanlinna theory, which describes the complexity and value distribution of a function meromorphic on the entire complex plane in terms of the growth of several real-valued functions. The chapter also contains an overview of the Painlevé property, fixed and movable singularities and a standard example of Painlevé analysis.

Chapter 3 discusses some applications of Nevanlinna theory to differential equations. The study of differential equations in the complex domain, with the aid of Nevanlinna theory, is an active field of research. The first such application was due to the Japanese mathematician Yosida [52] who used Nevanlinna theory to present an alternative proof of Malmquist's theorem: if R is a rational function of its arguments, and $y' = R(z, y)$ is a differential equation which has a transcendental meromorphic solution, then R must be a polynomial in y of degree at most 2. If R has degree 1 then the equation is linear and solvable by quadrature. If the degree of R is two then the differential equation is called a Riccati equation. It is well know that any Riccati equation can be solved in terms of an appropriate second-order linear differential equation. This theorem shows that if a solution has simple singularity structure (i.e. that the solution be meromorphic) then it can be expressed in terms of solutions of a linear differential equation and is in some sense integrable.

Many studies still attempt to generalise Malmquist's theorem to characterising the forms of equations of higher order. Otherwise, many studies in the field of complex differential equations, which use Nevanlinna theory, have focussed on the rates of growth of meromorphic solutions of linear or non-linear differential equations (see [24, and references therein]).

Apart from the previous studies cited, there is still a lack of research that investigates the impact of Nevanlinna theory on the existence of meromorphic solutions to differential equations. So far, however, a few studies have discussed this problem (see [5], [12], [42], [43] and [44]). Halburd and Wang [12] used local series analysis with the aid of Nevanlinna theory to obtain all admissible meromorphic solutions of an ordinary differential equation, even when the general solution is not meromorphic.

These methods are extended in chapter 3 to give an explicit characterisation of all admissible meromorphic solutions of equations of the form

$$w'' = \sum_{j=1}^{N} \kappa_j \frac{(w')^2}{w - a_j} + \alpha(z),$$

where α is a non-zero meromorphic function and $a_1, \ldots, a_N$ are distinct constants. All such solutions are shown to be either polynomials, elliptic functions, or expressible in terms of solutions of the first or second Painlevé equations. This is the main result of this thesis and is to be published in [3]. A fundamental difference between this work and that of Halburd and Wang is that we must consider an arbitrary number of different types of movable singularities for which $w(z_0) = a_j$ for some j as well as the fact that some solutions can have poles.

As was the case in the work of Halburd and Wang, as part of our proof we find a number of explicit "well behaved" solutions that we must discard because they do not satisfy either the assumption that the solution is meromorphic or that it is admissible. In the cases where a solution fails to be meromorphic it is because it is branched at one or more fixed singularities. This underlines the fact that demanding that the solutions be meromorphic at fixed singularities is artificial.

Chapter 4 contains an initial attempt to address the problem of the possible branching of solutions at fixed singularities by extending Nevanlinna theory to a large sector-like region outside a disc centred at the origin. This is based directly on Tsuji's version of Nevanlinna theory for functions meromorphic on the half-plane. This chapter contains a self-contained introduction to the Tsuji characteristic. We follow standard derivations but we use the right half-plane instead of the upper half-plane as the formulas are more symmetrical when we transform to sectors. This derivation is merely a rewriting of the standard derivation included for completeness. The standard formulas can be recovered by a rotation in the complex plane by $\pi/2$.

After some well-known introduction of Tsuji theory, we establish analogous tools of Nevanlinna theory for meromorphic functions in a sectorial domain type using Tsuji's theory to the extent that we will need it for applications to differential equations. This allows us to extend various results about meromorphic solutions of differential equations to solutions whose only movable singularities are poles. However, in this context admissibility becomes more problematic.

Chapter 2

Preliminaries

This chapter contains some essential background material on Nevanlinna theory as well as a brief introduction to the Painlevé property and the Painlevé equations. A key idea in Nevanlinna theory is to consider certain quantities of a meromorphic function on a disc of radius r, such as the number of poles, and then study the asymptotic behaviour of these quantities, which are real-valued functions of r, as r tends to infinity. These real-valued functions will be introduced in section 2.1 together with a number of useful inequalities. Beyond these inequalities, the main tools from Nevanlinna theory that we will use in this thesis are the first main theorem and the lemma on the logarithmic derivative. We will introduce the Painlevé property in section 2.2 and give a standard example of a series-based procedure, known as the Painlevé test, for checking necessary conditions for an equation to have this property. The distinction between fixed and movable singularities plays an essential role here and will be referred to repeatedly throughout the thesis.

2.1 Nevanlinna theory

Nevanlinna theory was created to provide a quantitative measure of the value distribution of a meromorphic function. This theory originated over ninety years ago [14]. It has many applications in different areas of mathematics, such as differential equations [6, 9, 17, 24], difference equations [38, 49] and number theory [37, 47]. The literature on Nevanlinna theory is very large. Comprehensive surveys of Nevanlinna theory can be found in [13] and [25]. There are also good accounts on the development of the theory in [4, 8, 14, 24].

The origins of the value distribution theory of entire functions cover many classical theorems, for instance the fundamental theorem of algebra and Picard's theorem. In the late nineteenth and early twenteth centuries, the theory of the value distribution of entire functions was developed by various French mathematical schools such as those of Hadamard, Borel and Valiron. Studies therein characterised the order of entire functions by

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log\log M(r,f)}{\log r}, \tag{2.1}$$

where $M(r,f) = \max_{|z|=r}|f(z)|$. However, this does not work for meromorphic functions. Since a meromorphic function can have a pole on the circle $|z|= r$, where the maximum modulus $M(r,f)$ would be infinite, the growth of the function f cannot be defined using $M(r,f)$. Studies related to the behaviour of a meromorphic function with regard to the distribution of its zeros and poles was built in the 1920s by the Finnish mathematician Rolf Nevanlinna, partly in collaboration with his brother Frithiof, through a series of publications in 1919-1980 [14, and references therein]. These publications provided an insight into the value distribution of meromorphic functions.

Nevanlinna used three auxiliary real valued functions related to a meromorphic function f which are defined on $[0, \infty)$, and he investigated how these functions characterise the growth and the behaviour of the function f. In the following, precise definitions [8, 13, 24] of these functions will be given.

For any meromorphic function f in the complex plane $\mathbb{C}$, the proximity function $m(r, f)$ is defined as the integral

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta,$$

where $\log^+ x = \max(0, \log x)$ for all $x > 0$. The proximity function measures how close f is, on average, to infinity on the circle of radius r centred at the origin. In addition, the growth of the function $m(r, a) = m(r, a, f) = m(r, 1/(f - a))$ relates to the closeness of the values of $f(z)$ to the point $a \in \mathbb{C}$: the closer the values of f are to the point a on average on the circle $|z| = r$, the larger the function $m(r, a)$ is.

The counting function $N(r, f)$ is defined by

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

where $n(t, f)$ denotes the number of poles of $f(z)$ inside the closed disc of radius t centred at the origin, each pole counted with its multiplicity. In particular, $n(0, f)$ is the order of the pole of the function f at $z = 0$. In addition, consider the number of a-points, i.e. the number of roots of the equation $f(z) = a$, counting multiplicity in $|z| \leq t$ which is denoted by $n(r, a)$, then the counting function $N(r, 1/(f - a))$ counts the number of a-points of $f(z)$ in the closed disc $|z| \leq r$. For brevity we write $N(r, 1/(f - a)) = N(r, a)$.

The Nevanlinna characteristic function of $f(z)$ is defined by

$$T(r, f) = m(r, f) + N(r, f).$$

The function $T(r, f)$ plays a significant role in the growth of meromorphic functions. Nevanlinna presented [34] a natural approach to determining the order of a meromorphic function f as

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r}.$$

It is noted that the order $\rho(f)$ is defined by replacing $\log M(r, f)$ in (2.1) by $T(r, f)$.

One of the most important results in Nevanlinna theory is the first main theorem [13]. Nevanlinna derived this theorem from Jensen's formula, namely

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta + \sum_{|a_i|<r} \log\frac{|a_i|}{r} - \sum_{|b_j|<r} \log\frac{|b_j|}{r}, \tag{2.2}$$

where f is a meromorphic function such that $f(0) \neq 0$ or ∞, and $a_1, a_2, ...$ ($b_1, b_2, ...$) denote its zeros (poles, respectively), each taken into account according to its multiplicity. With the above notation, Jensen's formula (2.2) becomes

$$\log|f(0)| = m(r, f) - m\Big(r, \frac{1}{f}\Big) + N(r, f) - N\Big(r, \frac{1}{f}\Big).$$

On applying this result to $f(z) - a$ instead of $f(z)$, we see that $N(r, f)$ is unchanged and $m(r, f)$ differs by at most $\log^+|a| + \log 2$. Hence we have

$$m(r, a) + N(r, a) = m(r, f) + N(r, f) + \log|f(0) - a| - \epsilon(a),$$

where $|\epsilon(a)| \leq \log^+|a| + \log 2$, and $f(a) \neq 0$ or ∞. If r varies, then for every value a, finite or infinite, the first main theorem can be written simply as follows.

Theorem 2.1.1 *Let f be a meromorphic function and let $a \in \widehat{\mathbb{C}}$. Then*

$$T(r, a) = m(r, f) + N(r, f) + O(1). \tag{2.3}$$

From relation (2.3), it can be observed that $T(r,f)$ has almost the same sum of terms $m(r,a)$ and $N(r,a)$ for any value $a \in \widehat{\mathbb{C}}$, except for a bounded term independent of r. In loose terms, the first fundamental theorem implies that if a function f takes on a certain point a relatively fewer times than the point b, then the values of $f(z)$ are 'closer' to the point a on a large part of the complex plane.

Let us now consider the following example where $f(z) = e^z$ to illustrate the above point.

Example 2.1.1

For the function e^z, we have

$$m(r,0) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|e^{-re^{i\theta}}|\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log^+ e^{-r\cos\theta} d\theta = -\frac{1}{2\pi}\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} r\cos\theta d\theta = \frac{r}{\pi}.$$

Furthermore, $N(r,0) = 0$ since the function e^z never assumes the value 0. Likewise, $m(r,f) = r/\pi$ and $N(r,f) = 0$. On the other hand, if $a \neq 0$ or ∞ and z_0 is a root of the equation $e^{z_0} = a$, then it can be seen that all other roots are of the form $z_0 + 2k\pi i$, for all $k \in \mathbb{Z}$. This means that the function e^z attains the value a regularly. A more delicate analysis will show that if $a \neq 0$ or ∞, then

$$m(r,a) = O(1), \text{ and } N(r,a) = \frac{r}{\pi} + O(1).$$

In this instance, the first fundamental theorem says that the values of the function e^z are close to 0 or ∞ on large part of the circle $|z| = r$ while the values of the function e^z approach to the value $a \neq 0$ or ∞ on a very small arc of each large circle.

We now give some elementary properties for the functions $m(r,f)$, $N(r,f)$ and $T(r,f)$, which can be found in [13] and [24].

Proposition 2.1.2 *Let $f_1,\ f_2,\ldots,\ f_n$ be meromorphic functions on $\mathbb{C}$. Then*

$$\begin{aligned}
N\left(r,\sum_{i=1}^{n} f_i\right) &\leq \sum_{i=1}^{n} N(r,f_i),\\
N\left(r,\prod_{i=1}^{n} f_i\right) &\leq \sum_{i=1}^{n} N(r,f_i),\\
m\left(r,\sum_{i=1}^{n} f_i\right) &\leq \sum_{i=1}^{n} m(r,f_i)+\log n,\\
m\left(r,\prod_{i=1}^{n} f_i\right) &\leq \sum_{i=1}^{n} m(r,f_i),\\
T\left(r,\sum_{i=1}^{n} f_i\right) &\leq \sum_{i=1}^{n} T(r,f_i)+\log n,\\
T\left(r,\prod_{i=1}^{n} f_i\right) &\leq \sum_{i=1}^{n} T(r,f_i),\\
T(r,f^m) &= mT(r,f), \quad m\in\mathbb{N}.
\end{aligned}$$

In Nevanlinna theory we often deal with quantities that grow slower than the Nevanlinna characteristic of a meromorphic function f at the rate $o(T(r,f))$ as r tends to infinity outside of a possible exceptional set E of real values satisfying $\int_E dt<\infty$, i.e. E has finite linear measure.

Definition 2.1.1 Let f and g be meromorphic functions. Then it is said that g is small compared to f if

$$T(r,g)=o(T(r,f)), \quad r\to\infty,$$

possibly outside of an exceptional set of r-values with finite measure, and we use the notation

$$T(r,g)=S(r,f).$$

There is a significant property in Nevanlinna theory which is an estimate of the proximity function of the logarithmic derivative, which we give below.

Lemma 2.1.3 *Let f be a meromorphic function and $k \geq 1$ be an integer. If f is of infinite order of growth then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f) \tag{2.4}$$

and if f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r). \tag{2.5}$$

For $k = 1$, this lemma is called the lemma of the logarithmic derivative. Using this lemma and the properties of m above, it is straightforward [24] to show that

$$m(r, f^{(k)}) \leq m(r, f) + S(r, f). \tag{2.6}$$

Furthermore, if a meromorphic function f has a pole of order s at a point z_0, say, then the function $f^{(k)}$ has a pole of order $s + k \leq (k + 1)s$ at z_0, and hence

$$N(r, f^{(k)}) \leq (k + 1)N(r, f). \tag{2.7}$$

The following result for the rational function $R(z, f)$ is due to Valiron [46] and generalised by Mohon'ko [31] which has many applications in the analytic theory of differential and difference equations [11].

Theorem 2.1.4 *Let f be a meromorphic function and $R(z, f)$ be an irreducible rational function of f with meromorphic coefficients $a_\lambda(z)$ such that $T(r, a_\lambda) = S(r, f)$, for all λ. Then*

$$T(r, R(z, f)) = \deg_f(R(z, f))T(r, f) + S(r, f),$$

where $\deg_f$ *the degree of* $R(z, f)$ *as a rational function in* f.

The second fundamental theorem is widely regarded as the most significant result of the value distribution theory of meromorphic functions. In particular, it implies Picard's Theorem.

Theorem 2.1.5 (*Second Main Theorem*) *Let* f *be a non-constant meromorphic function in* $|z| \leq r$, *and* $a_1, a_2, ..., a_q \in \mathbb{C}$ *be distinct points, where* $q \geq 2$. *Then*

$$m(r, f) + \sum_{i=1}^{q} m(r, a_i) \leq 2T(r, f) - N_1(r, f) + S(r, f), \tag{2.8}$$

where $N_1(r, f) = 2N(r, f) - N(r, f') + N(r, 1/f')$.

The quantity $N_1(r, f)$ is a non-negative quantity measuring the total number of a-points of the function f in the disc of radius r and centred at the origin, where an a-point of order n is counted $n-1$ times.

Adding $N(r, f) + \sum_{i=1}^{q} N(r, a_i)$ to both side of inequality (2.8) we obtain

$$\sum_{i=1}^{q} T(r, a_i) \leq N(r, f) + \sum_{i=1}^{q} N\left(r, \frac{1}{f - a_i}\right) + T(r, f) - N_1(r, f) + S(r, f).$$

Using the first fundamental Theorem 2.1.1 we have $T(r, a_i) = T(r, f) + O(1)$, and hence the above equation becomes

$$(q-1)T(r, f) \leq N(r, f) + \sum_{i=1}^{q} N\left(r, \frac{1}{f - a_i}\right) - N_1(r, f) + S(r, f), \tag{2.9}$$

which gives another version of the second fundamental Theorem 2.1.5. It can be noted that the expression (2.8) of the second fundamental theorem shows that in general the term $m(r, a)$ is small compared with $T(r, f)$ and so $N(r, a)$ comes close to $T(r, f)$.

In summary, Nevanlinna's first fundamental theorem implies that the equality

(2.3) is satisfied for each value a, so the sum $m(r,a) + N(r,a)$ does not depend heavily on the value a. On the other hand, the second fundamental theorem shows that the contribution to this sum from each of the terms $m(r,a)$ and $N(r,a)$ may depend explicity on the value a, and for the 'majority' of the values of a, the term $N(r,a)$ is dominant.

In order to determine the deviation from regular value distribution more precisely, Nevanlinna defined the deficiency of the value a as

$$\delta(a) = \delta(a,f) = \varliminf_{r\to\infty} \frac{m(r,a)}{T(r,f)} = 1 - \varlimsup_{r\to\infty} \frac{N(r,a)}{T(r,f)}.$$

It is clear that $0 \leq \delta(a) \leq 1$ for all $a \in \widehat{\mathbb{C}}$, and the value a is called a deficient value of the function f if $\delta(a) > 0$. The quantity $\delta(a)$ is positive only if there are relatively few a-points.

Nevanlinna also defined the ramification index

$$\theta(a) = \theta(a,f) = \varliminf_{r\to\infty} \frac{N(t,a) - \overline{N}(t,a)}{T(r,f)},$$

where $\overline{N}(r,a)$ is the integrated counting function for the a-points of $f(z)$, ignoring multiplicities which is given as

$$\overline{N}(r,a) = \overline{N}\left(r, \frac{1}{f(z)-a}\right) = \int_0^r \frac{\overline{n}(t,a) - \overline{n}(0,a)}{r} dt + \overline{n}(0,a)\log r,$$

and every a-point in $\overline{n}(t,a)$ is counted only once independently of the multiplicity. The point $a \in \mathbb{C}$ is called a ramified value of the function f if the quantity $\theta(a)$ is positive. Clearly, $\theta(a) > 0$ if there are relatively many multiple roots at the point a.

From the second fundamental theorem, we have the following result of Nevanlinna on deficient values [13]. For any meromorphic function f the set of the value a such that $\delta(a, f) > 0$ or $\theta(a, f) > 0$ is countable, and moreover

$$\sum_{a\in\widehat{\mathbb{C}}}(\delta(a, f) + \theta(a, f)) \leq 2.$$

The point $a \in \mathbb{C}$ is a totally ramified value of f if all a-points of f have multiplicity two or higher. Relation (2.9) implies that a non-constant meromorphic function f admits at most four completely ramified values.

2.2 The Painlevé property

Singularities of a differential equation are of two kinds: fixed and movable. The term 'fixed' refers to the independence of their location on the differential equation's initial conditions, while a singularity of a solution to a differential equation is movable if its position depends on the initial conditions. Extensive discussions on these kind of singularities can be found in [1], [2] and [17]. To provide a more precise mathematically rigorous definition involves the introduction of a lot of mathematical machinery that is not needed for the examples that we consider but can be found in [20] and [32]. Essentially, fixed singularities are singularities of a solution that occur at a point where the equation is in some sense singular.

The first-order nonlinear differential equation

$$\frac{dw}{dz} = -\frac{1}{z}w^2, \tag{2.10}$$

has a singular point at $z = 0$. The general solution to (2.10),

$$w = \frac{1}{\log(z/c)},$$

has both fixed and movable singularities. $z = 0$ is a fixed logarithmic branch point and $z = c$ is a movable pole since its position depends on c, where c is a constant of integration, which depends on the initial condition.

The second-order nonlinear differential equation

$$ww'' - w' + 1 = 0 \tag{2.11}$$

has general solution

$$w = (z - a)\log(z - a) + b(z - a),$$

which has a logarithmic branch point at $z = a$; the location of this point varies with the initial conditions, so it is a movable singularity.

The work in this thesis is motivated by the following property.

Definition 2.2.1 A differential equation is said to possess the (strong) Painlevé property if all movable singularities of all solutions are poles.

This is slightly stronger than the now more standard definition which only requires that all solutions are single-valued around all movable singularities. The stronger definition does not permit movable essential singularities.

The Painlevé test is one of two procedures which are widely used to determine whether a differential equation possesses the Painlevé property based on the singularity structure of the general solutions of the equation. A more detailed explanation can be found in [1], [2] and [22].

Consider the second order differential equation

$$\frac{d^2w}{dz^2} = F\left(w, \frac{dw}{dz}, z\right), \tag{2.12}$$

where F is rational in w and dw/dz and its coefficients are locally analytic. Painlevé and his coworkers identified equations of the form (2.12) possessing the Painlevé property. They found that each such equation could be transformed to one of fifty canonical forms [18]. There are six equations known as Painlevé equations. The first two are

$$P_I \qquad y'' = 6y^2 + z, \tag{2.13}$$

$$P_{II} \qquad y'' = 2y^3 + zy + \gamma, \tag{2.14}$$

where γ is a constant. The remaining forty-four equations can be either solved in terms of previously known functions or solutions of second-order linear equations or reducible to one of the six Painlevé equations.

The Painlevé test is a method that provides necessary conditions for the Painlevé property to hold. However, it is well known that this does not imply that the equation possesses the Painlevé property. Here we present an illustrative example to describe this technique.

Example 2.2.1

Consider the differential equation

$$y'' = 6y^2 + f(z), \tag{2.15}$$

where $f(z)$ is an analytic function.

Since the right hand side of (2.15) will be well behaved, analytic, at the initial condition when y and y' are finite, then Cauchy's theorem guarantees the existence of a unique solution of (2.15), i.e. analytic in a neighbourhood of the initial point. Hence checking the existence of local series expansions will only provide information regarding expansions about the pole. Therefore, we look for a Laurent series expansion of a solution of equation (2.15) about a movable singularity z_0:

$$y(z) = a_0(z - z_0)^p + O((z - z_0)^{p+1}), \tag{2.16}$$

where $a_0 \neq 0$ and $p < 0$. Substituting (2.16) into (2.15), we have

$$p(p-1)a_0(z-z_0)^{p-2} + O((z-z_0)^{p-1}) = 6a_0^2(z-z_0)^{2p} + O((z-z_0)^{2p+1}). \quad (2.17)$$

Equation (2.17) is balanced if $p = -2$ and hence $a_0 = 1$, for which the series expansion of the solution is necessarily of the form

$$y(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^{k-2}, \qquad a_0 = 1. \quad (2.18)$$

Substituting equation (2.18) in the differential equation (2.15) and equating the coefficients shows that $a_1 = a_2 = a_3 = 0$, and the recurrence relation is

$$(k+1)(k-6)a_k = 6\sum_{m=1}^{k-1} a_m a_{k-m} + \frac{1}{(k-4)!} f^{(k-4)}(z_0). \quad (2.19)$$

If $k = 6$ in the above equation, then the right side of this equation must vanish, which is a nontrivial constraint on f. This implies that a_6 is an arbitrary coefficient. The index of the free coefficient, a_6, is called a resonance. Thus $k = 6$ is the resonance. Therefore, from equation (2.19), we have $f''(z_0) = 0$. Since z_0 is arbitrary, we get the resonance condition $f'' \equiv 0$. Hence the function f must be linear and equation (2.15) has the form

$$y'' = 6y^2 + az + b, \quad (2.20)$$

where a and b are constants. The general solution of equation (2.20) can be written in terms of the Weierstrass elliptic function or its degenerations, provided $a = 0$. Otherwise, a rescaling of y and z along with a translation in z shows that the general solution of equation (2.20) is given in terms of solutions of the first Painlevé equation (2.13).

The Painlevé test provides very strong constraints on an equation. In practice these are often enough to reduce the equation to one of Painlevé type. However, as in the previous example, if the equation cannot be solved explicitly it then requires another argument to show that it does indeed have the Painlevé property. The Painlevé test is based on the fact that if the resonance condition is not satisfied then the Laurent series expansion can be modified to include logarithmic terms and one has found a solution with a movable branch point. Initially, the resonance condition holds at just one point, z_0, but then one argues, as above, that in order to avoid a movable branch point in any solution, the resonance condition must hold for all z_0.

In this thesis, we will be looking for special solutions with movable poles. In this case we cannot use the fact that there may be other solutions with movable branch points in order to discard an equation. Instead we argue that a given, sufficiently nontrivial (e.g. admissible meromorphic) solution must have sufficiently many movable singularities that the resonance condition must hold everywhere. In the example above, it would be sufficient if we restrict f to be a rational function and if we know that y has an infinite number of poles. It is in concluding the latter statement that Nevanlinna theory proves to be a very power tool.

Chapter 3

Application of Nevanlinna theory to differential equations

We present a brief survey of some applications of Nevanlinna theory to differential equations in section 3.1. Then we introduce a certain class of non-linear second-order ordinary differential equations in section 3.2. Initially, we shall find all complicated meromorphic solutions of this class by considering the singularity structure with the aid of Nevanlinna theory. We show all solutions can be found in terms of classical special functions such as elliptic functions and two remaining cases are transformed to the first and second Painlevé equations. The results of section 3.2 will appear in [3].

3.1 Nevanlinna theory and differential equations

A considerable amount of literature has described the properties of solutions of complex differential equations using Nevanlinna theory. On the other hand, a limited amount of literature has been concerned with the existence of solutions of a differential equation with the aid of Nevanlinna theory. This theory provides many of the concepts to deal with meromorphic functions. We proceed by presenting an overview of some applications of Nevanlinna theory to meromorphic solutions of differential equations.

The first important application was made by Yosida [52]. In 1933, he [52] gave an alternative proof of Malmquist's theorem via Nevanlinna theory. Indeed, Malmquist's theorem was published in 1913 [30] and is presented here (Theorem 3.1.1) for completeness.

Theorem 3.1.1 *Let $R(z, y)$ be a rational function in y with coefficients $a_\lambda(z)$ which are rational functions in z. If the equation*

$$y' = R(z, y) \tag{3.1}$$

admits a non-rational meromorphic solution $y(z)$, then equation (3.1) has the form

$$y' = c_0(z) + c_1(z)y + c_2(z)y^2 \tag{3.2}$$

with rational coefficients.

If $c_2 = 0$ equation (3.2) is a linear equation. If $c_2 \neq 0$ the general solution can be given in terms of the general solution of a second order linear equation. We will refer to equations of the form (3.2) as Riccati equations, even when $c_2 = 0$.

Proof. Let $R(z, y)$ be an irreducible rational function, then equation (3.1), can be written as

$$y' = R(z, y) = \frac{P(z, y)}{Q(z, y)}, \tag{3.3}$$

where $P(z, y)$ and $Q(z, y)$ are polynomials in y with rational coefficients. Then by

Theorem 2.1.4, we have

$$T(r, y') = \deg_y(R(z, y))T(r, y) + S(r, y). \tag{3.4}$$

Applying the lemma of logarithmic derivative (2.4) and using inequality (2.7), we obtain

$$\begin{aligned} T(r, y') &= N(r, y') + m(r, y') \\ &\leq N(r, y') + m(r, y) + m(r, \frac{y'}{y}) \\ &\leq N(r, y') + m(r, y) + S(r, y) \\ &\leq 2N(r, y) + m(r, y) + S(r, y) \\ &\leq 2T(r, y) + S(r, y). \end{aligned} \tag{3.5}$$

Therefore by combining (3.4) and (3.5), we have

$$\deg_y(R(z, y)) \leq 2.$$

Hence both of the polynomials $P(z, y)$ and $Q(z, y)$ have degree less than or equal to two. Therefore, these polynomials can be written as

$$P(z, y) = a_0(z) + a_1(z)y(z) + a_2(z)y^2(z),$$

$$Q(z, y) = b_0(z) + b_1(z)y(z) + b_2(z)y^2(z).$$

Without loss of generality, we assume that $a_0(z) \neq 0$. Now the function $\widetilde{y} = 1/y$ is a solution of the equation

$$\widetilde{y}' = -\frac{\widetilde{y}^2\Big(a_0(z)\widetilde{y}^2 + a_1(z)\widetilde{y} + a_2(z)\Big)}{b_0(z)\widetilde{y}^2 + b_1(z)\widetilde{y} + b_2(z)} = \widetilde{R}(z, \widetilde{y}) = \frac{\widetilde{P}(z, \widetilde{y})}{\widetilde{Q}(z, \widetilde{y})}, \tag{3.6}$$

where $\widetilde{P}(z,\widetilde{y}) = -\widetilde{y}^2\Big(a_0(z)\widetilde{y}^2+a_1(z)\widetilde{y}+a_2(z)\Big) = -\widetilde{y}^4P(z,\frac{1}{\widetilde{y}})$, and $\widetilde{Q}(z,\widetilde{y}) = b_0(z)\widetilde{y}^2+b_1(z)\widetilde{y}+b_2(z) = \widetilde{y}^2Q(z,\frac{1}{\widetilde{y}})$. Using the first fundamental theorem in equation (2.3), we have $T(r,a_i) = T(r,b_i) = S(r,\widetilde{y})$. Likewise, equation (3.6) has the same general form as equation (3.3), and so $\deg_{\widetilde{y}}(\widetilde{R}(z,\widetilde{y})) \leq 2$. Since $\deg_{\widetilde{y}}(\widetilde{P}(z,\widetilde{y})) = 4$ and $\deg_{\widetilde{y}}(\widetilde{Q}(z,\widetilde{y})) = 2$, then $\widetilde{Q}$ must divide $\widetilde{P}$. Since P and Q are co-prime, it follows that $\widetilde{P}(z,\widetilde{y}) = -\widetilde{y}^4P(z,1/\widetilde{y})$ and $\widetilde{Q}(z,\widetilde{y}) = \widetilde{y}^2Q(z,1/\widetilde{y})$ are also co-prime. Therefore $Q(z,\widetilde{y})$ must divide $\widetilde{y}^2$, which is only possible if $b_1 = b_2 = 0$. □

Malmquist's theorem has been extended to the case of rational functions $R(z,y)$ with meromorphic coefficients, and several forms of non-linear algebraic differential equations of the first order (see [17, 24, and references therein], and [6]). Steinmetz [42] identified, in his thesis, all birational cases of

$$(y')^n = R(z,y) \tag{3.7}$$

that admit transcendental meromorphic solutions. A number of papers due to Laine [23], Rieth [36] and Yuzan and Laine [16] were published, which generalised the case (3.7) to meromorphic coefficients (see also Laine [24]).

Theorem 3.1.2 *[24]* (Malmquist-Yosida) *Let $R(z,y)$ be a rational function in y with coefficients $a_\lambda(z)$ which are meromorphic functions in z. If the equation (3.7) admits a transcendental meromorphic solution $y(z)$ such that $T(r,a_\lambda) = S(r,y)$ for all λ, then $R(z,y)$ reduces to a polynomial in y of degree at most $2n$.*

On the other hand, many systematic studies used Nevanlinna theory to reduce second order algebraic differential equations which have a class of transcendental meromorphic solutions to standard forms of differential equations. Actually Laine [24, p. 251] conjectured that if there exists a transcendental meromorphic solution for a differential equation

$$y'' = R(z, y, y'), \tag{3.8}$$

where R is rational in z, y and y' then (3.8) has the form

$$y'' = L(z, y)\ (y')^2 + M(z, y)\ y' + N(z, y), \tag{3.9}$$

where L, M, and N are birational functions.

Several studies have determined necessary conditions for some classes of equations of the form (3.9) to admit transcendental meromorphic solutions (see [24, and references therein], [19, 28, 29, 43]). For example, Steinmetz [43] considered the equation

$$y'' = M(z, y)\ y' + N(z, y), \tag{3.10}$$

where M, and N are polynomials in y with rational coefficients. He showed that if y is a transcendental meromorphic solution of (3.10), then

1. either y satisfies a Riccati differential equation with rational coefficients or,
2. $\deg_y(M(z, y)) \leq 1$, and $\deg_y(N(z, y)) \leq 3$.

In the second case, equation (3.10) reduces to

$$y'' = p_0(z) + p_1(z)y + p_2(z)y^2 + p_3(z)y^3 + q_0(z)y' + q_1(z)y\ y',$$

with rational coefficients. Steinmetz also [44] considered all second-order differential equations

$$y'' = L(z, y)\ (y')^2, \tag{3.11}$$

where L is rational in all its arguments. If y is a transcendental meromorphic solution of equation (3.11), then either y satisfies a Riccati differential equation with rational coefficients, or a differential equation of the form $(y')^m = R(y)$ where R is rational and $m \in \{2,\ 3,\ 4,\ 6\}$.

In fact, Laine's expectation was proved if the transcendental meromorphic solutions of (3.9) have infinite order, by Liao et al. [29].

Theorem 3.1.3 *If the algebraic differential equation (3.8) admits a meromorphic solution y of infinite order, then y satisfies a second order algebraic differential equation of the form (3.9), where L, M, and N are rational coefficients.*

However, this expectation is not true if the meromorphic solution has a finite order, for example, $y = 1/(z - z_0)$ is a solution of the equation

$$y'' + 2yy' = (y' + y^2)^3.$$

Another direction is taken by many studies on Nevanlinna theory that have focused on whether the solutions of particular non-linear differential equations have finite or infinite order (see [24] and references therein). On the other hand, few attempts have been made to find explicit meromorphic solutions of a differential equation (see e.g. [5, 12]).

Hayman [15] considered the differential equation

$$ff'' - f'^2 = k_0 + k_1 f + k_2 f' + k_3 f'' \tag{3.12}$$

and conjectured that all entire solutions of (3.12) have a finite order where the k_j are rational functions. Chiang and Halburd [5] considered the constant coefficients case and used the transformation $f = w + k_3$ with (3.12) which gives

$$ww'' - w'^2 = \alpha w + \beta w' + \gamma, \tag{3.13}$$

where $\alpha = k_1$, $\beta = k_2$, and $\gamma = k_0 + k_1 k_3$ are constants. Chiang and Halburd found all meromorphic solutions of equation (3.13) and showed that they are either polynomials or linear combinations of exponential functions and constants. Moreover, they proved that Hayman's conjecture is indeed correct in the constant coefficients case.

An interesting method was used by Halburd and Wang [12] to find all admissible meromorphic solutions, via Nevanlinna theory, of the Hayman's differential equation (3.13), where α, β, and γ are meromorphic functions. Intuitively, a meromorphic solution is admissible [12] if it is more complicated than the coefficients that appear in the equation. In particular, if the coefficients are constants, then any non-constant meromorphic solution is admissible. If the coefficients are rational functions, then any transcendental (i.e. non-rational) meromorphic solution is admissible. More precisely, w is an admissible meromorphic solution of (3.13) if it satisfies

$$T(r, \alpha) + T(r, \beta) + T(r, \gamma) = S(r, w).$$

Halburd and Wang used local series analysis with Nevanlinna tools to obtain all admissible meromorphic solutions of (3.13), regardless of whether the general solution is meromorphic. In summary, a series expansion of the solution w of the equation (3.13) is considered on a region Ω which does not contain the zeros and poles of the coefficients α, β, and γ. Next, they use the first two terms of the series expansion to construct a meromorphic function, f, in terms of w, w', the coefficients and their derivatives that is analytic on Ω and satisfies $T(r, f) = S(r, w)$. Then differentiating f and using equation (3.13) to eliminate the derivative w'', the function f is calculated. In this way, w is shown to be an admissible solution of a first order differential equation. This method will be used and discussed in more detail for a certain class of differential equations in the next section.

3.2 Admissible meromorphic solutions of a differential equation in the complex plane

The main purpose of this section is to find or at least identify all admissible meromorphic solutions of the differential equation

$$w'' - \sum_{j=1}^{N} \frac{\kappa_j}{w - a_j} w'^2 = \alpha(z), \tag{3.14}$$

where the constants κ_j are non-zero, the constants a_j are distinct and α is a non-zero meromorphic function satisfying

$$T(r, \alpha) = S(r, w). \tag{3.15}$$

The method to derive all these solutions of equation (3.14) is two-fold. We first use an approach based on the first few terms of the series expansion of the solution combined with Nevanlinna theory to construct a small function (in sense of Nevanlinna) in terms of w and w' with small coefficients. In so doing, we show that the solution w is an admissible meromorphic solution of a first-order polynomial differential equation. This approach was used by Halburd and Wang in [12] in their work on equation (3.13), however the situation is more complicated in our case. The only movable singularities of (3.13) are zeros of w. However, equation (3.14) is singular when $w = a_i$ for $i = 1, \ldots, N$ and may also have movable poles. Ultimately we show that each such solution can be expressed in terms of an admissible solution of either a Riccati equation or an equation of the form

$$(u')^2 = P(u), \tag{3.16}$$

possibly after a change of independent variable, where P is a polynomial of degree at most four with constant coefficients. If P has degree zero or one then u is a polynomial. If P has degree two then u is an exponential or trigonometric function. If P has degree three or four then u is an elliptic function or one of their degenerations (trigonometric, exponential or rational). We do not explicitly write down the forms of u in our theorems as there are many subclasses depending on the structure of P but it is elementary to do so in any given case.

The second step is to use resonance conditions (as in Painlevé analysis) for the remaining cases of (3.14) to identify admissible meromorphic solutions by obtaining necessary conditions on the coefficient α. We subsequently show that these remaining cases can be transformed to either the first or the second Painlevé equations, however the question of whether these solutions are ultimately admissible meromorphic is left open. The Painlevé transcendents (the solutions of the Painlevé equations) are now generally considered to be "special functions" (indeed they appear in the Digital Library of Mathematical Functions) and it is with this in mind that we claim to have found a list containing all admissible meromorphic solutions.

The case $\alpha \equiv 0$ in (3.14) is a special case of the equation (3.11), where $L(z,w) = \sum_{j=1}^{N} \kappa_j/(w-a_j)$, which was studied by Steinmetz [44].

The main result of this section is the following.

Theorem 3.2.1 *Let $w(z)$ be a meromorphic solution of (3.14), where $\sum_{i=1}^{N} \kappa_i \neq 2$ for $N>1$ and the meromorphic coefficient α satisfies (3.15). Then $\kappa_i \neq 1/2$ for all $i = \{1,\dots,N\}$ and one of the following statements is true. In the following, $\alpha_0 \neq 0$ and d_1 are constants.*

(i) $N=1$, $\alpha=\alpha_0$ and

$$(w')^2 + \frac{2\alpha_0}{2\kappa_1 - 1}(w-a_1) = 0. \tag{3.17}$$

(ii) $N = 1$, $\kappa_1 = 1$, $\alpha = \alpha_0$ *and*

$$(w')^2 = d_1(w - a_1)^2 - 2\alpha_0(w - a_1). \tag{3.18}$$

(iii) $N = 1$, $\kappa_1 = 5/4$, $\alpha = \alpha_0$ *and* $w = a_1 - d_1/u^2$, *where* u *satisfies*

$$(u')^2 = \frac{\alpha_0}{3d_1}u^4 - \frac{d_1}{4}u.$$

(iv) $N = 1$, $\kappa_1 = 3/4$, $\alpha = \alpha_0$ *and* $w = a_1 + u^2$, *where* u *satisfies*

$$(u')^2 = d_1 u - \alpha_0.$$

(v) $N = 1$, $\kappa_1 = 3/2$ *and*

$$112\alpha'^4 - 192\alpha\alpha'^2\alpha'' + 36\alpha^2\alpha''^2 + 54\alpha^2\alpha'\alpha^{(3)} - 9\alpha^3\alpha^{(4)} = 0. \tag{3.19}$$

(vi) $N = 1$, $\kappa_1 = 2$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}z}\Big\{\beta^2\frac{\mathrm{d}}{\mathrm{d}z}(\beta^3\Big(2(\beta^4\beta'')'' - \beta^3(\beta'')^2)\Big)\Big\} = 0, \tag{3.20}$$

where $\alpha = 6/\beta^5$.

(vii) $N = 2$, $\kappa_1 + \kappa_2 = 1$, $\alpha = \alpha_0$ *and*

$$(w')^2 + \frac{2\alpha_0(w - a_1)(w - a_2)}{(1 - 2\kappa_1)(a_2 - a_1)} = 0. \tag{3.21}$$

(viii) $N = 2$, $\kappa_1 = \kappa_2 = 3/4$, $\alpha = \alpha_0$ *and either*

1- w is a solution of

$$w'^2 = \frac{2\alpha_0}{(a_2-a_1)^2}(w-a_1)(w-a_2)(2w-(a_1+a_2)),$$

or

2- $w = (a_2 - a_1u^2)/(1-u^2)$ and u is a solution of

$$4u'^2 = (a_2-a_1)\ (1-u^2)\Big[d_1\ u + \frac{2\alpha_0}{(a_2-a_1)^2}\left(1+u^2\right)\Big],$$

where $d_1 \neq 0$.

(ix) $N = 2$, $\kappa_1 + \kappa_2 = 3/2$, $\alpha = \alpha_0$ *and*

$$(w')^2 + \frac{2\alpha_0}{2\kappa_1 - 1}\left((w-a_1) - \frac{2\kappa_2}{(2\kappa_2-1)\widetilde{a}_2}(w-a_1)^2 + \frac{1}{(2\kappa_2-1)\widetilde{a}_2}(w-a_1)^3\right) = 0,$$

where $\widetilde{a}_2 = a_2 - a_1$.

(x) $N = 2$, $\kappa_1 = 7/4$, $\kappa_2 = -1/4$, $\alpha = \alpha_0$ *and* $w = a_1 - d_1/u^2$, *where* $d_1 \neq 0$ *and* u *is a solution of*

$$4(u')^2 = \frac{4\alpha_0}{5d_1}u^4 + \frac{4\alpha_0}{15(a_2-a_1)}u^2 - d_1u - \frac{8d_1\alpha_0}{15(a_2-a_1)^2}.$$

(xi) $N = 3$, $\kappa_1 + \kappa_2 + \kappa_3 = 3/2$, $(2\kappa_2 - 1)(a_2 - a_1) + (2\kappa_3 - 1)(a_3 - a_1) = 0$, $\alpha = \alpha_0$ *and*

$$(w')^2 + \frac{2\alpha_0}{(2\kappa-1)(a_2-a_1)(a_3-a_1)}(w-a_1)(w-a_2)(w-a_3) = 0.$$

(xii) $w - a_i = -\delta_1/(\delta_2 - u^2)$ *where* $\delta_1 \neq 0$ *and* δ_2 *are meromorphic functions satisfying* $T(r, \delta_k) = S(r, w)$ *for* $k = 1, 2$ *and* u *is an admissible solution of a Riccati equation.*

The case $\kappa = 1$ was solved as a special case of the equation considered by Halburd and Wang in [12]. Before we present the proof of main result, we put forth required definitions and lemmas for completeness.

Let Φ be the set of all zeros and poles of the function α and $\Omega = \mathbb{C} \setminus \Phi$. Any singularity of w in Φ will be fixed and any singularity of w in Ω will be movable.

The counting functions which count the poles of any meromorphic function f in the set Φ are denoted by $N_\Phi(r, f)$ and $\overline{N}_\Phi(r, f)$, with and without multiplicities, respectively. In particular, if w is analytic on Ω then $N(r, w) = N_\Phi(r, w)$. Furthermore, for any meromorphic function f, $\overline{N}_\Phi(r, f) \leq \overline{N}_\Phi(r, \alpha) + \overline{N}_\Phi(r, 1/\alpha) = S(r, w)$. Also, if α is a rational function, then Φ is a finite set, and hence $N_\Phi(r, w) = S(r, w)$. We denote by $N_\Omega(r, f)$ the counting function of the poles of f in Ω with multiplicities.

Equation (3.14) is singular when $w = a_i$ for some $i \in \{1, \ldots, N\}$ or $w = \infty$. Suppose that $w(z_0) = a_i$ for some $z_0 \in \Omega$ and some $i \in \{1, \ldots, N\}$. Then w has a Taylor series expansion of the form

$$w(z) = a_i + \sum_{m=0}^{\infty} c_m \zeta^{m+q}, \tag{3.22}$$

where $\zeta = z - z_0$, q is a positive integer and $c_0 \neq 0$. Substituting (3.22) in (3.14) gives

$$q([1 - \kappa_i]q - 1)c_0 \zeta^{q-2} + \cdots = \alpha(z_0) + \cdots. \tag{3.23}$$

If $[1 - \kappa_i]q - 1 = 0$ then the leading order term on the left side of (3.23) is proportional to $\zeta^{q-2+\nu}$ where ν is a positive integer. This must balance the leading order term $\alpha(z_0)$ (which is non-zero since $z_0 \in \Omega$) on the right side, giving $q = 2 - \nu$. Since q and ν are both positive integers we must have $q = 1$. However, this contradicts the assumption $[1 - \kappa_i]q - 1 = 0$ since $\kappa_i \neq 0$. Therefore $[1 - \kappa_i]q - 1 \neq 0$ and the leading order term on the left side is proportional to ζ^{q-2}. Equating this term with $\alpha(z_0)$ gives $q = 2$, $\kappa_i \neq 1/2$ and $c_0 = \alpha(z_0)/[2(1 - 2\kappa_i)]$. So all a_i-points of w in Ω

are double.

So around any a_i-point $z_0 \in \Omega$, we have

$$w = a_i + c_0\zeta^2 + \cdots + c_m\zeta^{m+2} + \cdots.$$

It follows that

$$w' = 2c_0\zeta + \cdots + (m+2)c_m\zeta^{m+1} + \cdots$$

and hence

$$w'^2 = 4c_0^2\zeta^2 + \cdots + [4(m+2)c_0c_m + p_m(c_0, \ldots, c_{m-1})]\zeta^{m+2} + \cdots,$$

where p_m is a polynomial in its arguments. Also

$$\begin{aligned}(w - a_i)^{-1} &= c_0^{-1}\zeta^{-2}\Big(1 + \frac{c_1}{c_0}\zeta + \cdots + \frac{c_m}{c_0}\zeta^{m-2} + \ldots\Big)^{-1} \\ &= c_0^{-1}\zeta^{-2}\Big(1 + \cdots + \Big[-\frac{c_m}{c_0} + q_m(c_0, \ldots, c_{m-1})\Big]\zeta^{m-2} + \ldots\Big) \\ &= c_0^{-1}\zeta^{-2} + \cdots + [-c_0^{-2}c_m + r_m(c_0, \ldots, c_{m-1})]\zeta^{m-2} + \cdots,\end{aligned}$$

where q_m and r_m are polynomials in their arguments divided by a power of c_0. Therefore

$$\frac{w'^2}{w - a_i} = 4c_0 + \cdots + [4(m+1)c_m + g_m(c_0, \ldots, c_{m-1})]\zeta^m + \cdots,$$

and similarly for $j \neq i$

$$\frac{w'^2}{w - a_j} = \frac{4{c_0}^2}{(a_i - a_j)}\zeta^2 + \cdots + h_m(c_0, \ldots, c_{m-1})\zeta^m + \cdots,$$

where g_m and h_m are polynomials in their arguments divided by a power of c_0. We also have the Taylor series

$$\alpha(z) = \alpha(z_0) + \cdots + \frac{\alpha(z_0)^{(m)}}{m!}\zeta^m + \cdots.$$

Substituting these expressions in equation (3.14) and equating the coefficients of ζ^m gives

$$(m+1)(m+2)c_m = \kappa_i[4(m+1)c_m + g_m(c_0,\ldots,c_{m-1})] + \sum_{j\neq i}\kappa_j h_m(c_0,\ldots,c_{m-1}) + \frac{\alpha^{(m)}(z_0)}{m!},$$

which can be rearranged to give the recurrence relation

$$(m+1)(m+2-4\kappa_i)c_m = G_m(c_0,\ldots,c_{m-1}) + \frac{\alpha^{(m)}(z_0)}{m!},$$

where $G_m = \kappa_i g_m + \sum_{j\neq i}\kappa_j h_m$. In particular, for $m = 1$ we have

$$2(3-4\kappa_i)c_1 = \alpha'(z_0).$$

Therefore, if $\kappa_i = 3/4$ then $\alpha'(z_0) = 0$. Otherwise if $\kappa_i \neq 3/4$ then $c_1 = \alpha'(z_0)/(2(3-4k_i))$. More generally, if for some positive integer M, $\kappa_i \neq (m+2)/4$ for all $m \in \{0,\ldots,M\}$, then the coefficients $c_0,\ldots,c_M$ are uniquely determined. If $\kappa_i = (m+2)/4$ then the resonance condition $G_m(c_0,\ldots,c_{m-1}) + \frac{\alpha^{(m)}(z_0)}{m!} = 0$ must be satisfied.

We have proved the following two lemmas.

Lemma 3.2.2 *Let w be a solution of equation (3.14) analytic in a neighbourhood of $z_0 \in \Omega$ such that $w(z_0) = a_i$ for some $i \in \{1,\ldots,N\}$. Then*

1. $\kappa_i \neq 1/2$.

2. *If* $\kappa_i = 3/4$ *then* $\alpha'(z_0) = 0$.

Lemma 3.2.3 *Let $M \geq 0$ and $N > 0$ be integers and suppose that for some $i \in \{1, \ldots, N\}$, $\kappa_i \neq \frac{m+2}{4}$ for all $m \in \{0, \ldots, M\}$. Then there are unique functions*

$$c_m(z) := C_m(\alpha(z), \ldots, \alpha^{(m)}(z)),$$

where each C_m is a polynomial in its arguments divided by a power of $\alpha(z)$, such that if w is a solution of equation (3.14) analytic in a neighbourhood of $z_0 \in \Omega$ where $w(z_0) = a_i$ then

$$w(z) = a_i + c_0(z_0)(z - z_0)^2 + \cdots + c_M(z_0)(z - z_0)^{M+2} + O\left((z - z_0)^{M+3}\right). \quad (3.24)$$

In particular,

$$c_0(z) = \frac{\alpha(z)}{2(1 - 2\kappa_i)} \quad \text{and if} \quad \kappa_i \neq 3/4 \quad \text{then} \quad c_1(z) = \frac{\alpha'(z)}{2(3 - 4\kappa_i)}. \quad (3.25)$$

The main idea behind the proof of Theorem 3.2.1 is to show that, in all but a small number of cases, we can use the first few terms in the series expansions around singularities to prove that w necessarily satisfies a first-order equation. In this way we avoid the explicit consideration of cases in which the resonances occur for large positive integers. Resonance conditions are used to characterise solutions of the few remaining equations.

We begin with a lemma that guarantees that there are a lot of a_i-points in Ω.

Lemma 3.2.4 *Let w be a meromorphic solution of equation (3.14) satisfying $T(r, \alpha) = S(r, w)$. Then*

1. $m\left(r, (w - a_j)^{-1}\right) = S(r, w)$ *for all* $j \in \{1, \ldots, N\}$.

2. $N_\Phi(r, (w - a_j)^{-1}) = S(r, w)$ *for all* $j \in \{1, \ldots, N\}$.

3. $\kappa_j \neq 1/2$ *for all* $j \in \{1, \dots, N\}$.

Proof.

1. Choose $i \in \{1, \dots, N\}$. Equation (3.14) can be written as

$$\frac{1}{w - a_i} = \frac{1}{\alpha(z)} \left\{ \left(\frac{w'}{w - a_i} \right)' + \left(\frac{w'}{w - a_i} \right)^2 - \sum_{j=1}^{N} \frac{\kappa_j}{w - a_j} \frac{w'^2}{w - a_i} \right\}. \tag{3.26}$$

Taking the proximity function of both sides and then employing relations (2.6), (3.15) and using elementary properties as well as the lemma on the logarithmic derivative yields

$$\begin{aligned} m\Big(r, \frac{1}{w - a_i}\Big) &\leq m\Big(r, \frac{1}{\alpha}\Big) + m\Big(r, \frac{w'}{w - a_i}\Big) + m\Big(r, \Big(\frac{w'}{w - a_i}\Big)'\Big) \\ &\quad + m\Big(r, \sum_{j=1}^{N} \frac{\kappa_j}{w - a_j} \frac{w'^2}{w - a_i}\Big) + S(r, w) \\ &\leq m\Big(r, \frac{1}{\alpha}\Big) + 3m\Big(r, \frac{w'}{w - a_i}\Big) + m\Big(r, \sum_{j=1}^{N} \frac{\kappa_j}{w - a_j} \frac{w'^2}{w - a_i}\Big) + S(r, w) \\ &\leq 3m\Big(r, \frac{w'}{w - a_i}\Big) + \sum_{j=1}^{N} m\Big(r, \frac{\kappa_j}{w - a_j} \frac{w'^2}{w - a_i}\Big) + S(r, w) \\ &\leq 3m\Big(r, \frac{w'}{w - a_i}\Big) + \sum_{j=1}^{N} \Big\{ m\Big(r, \frac{w'}{w - a_j}\Big) + m\Big(r, \frac{w'}{w - a_i}\Big) \Big\} + S(r, w) \\ &= S(r, w). \end{aligned}$$

2. Taking the counting function of both sides of (3.26) and using elementary properties with relations (2.7), (3.15) we get

$$
\begin{aligned}
N_\Phi\Big(r, \frac{1}{w-a_i}\Big) &\le N_\Phi\Big(r, \frac{1}{\alpha}\Big) + 4N_\Phi\Big(r, \frac{w'}{w-a_i}\Big) + N_\Phi\Big(r, \sum_{j=1}^{N} \frac{\kappa_j}{w-a_j}\frac{w'^2}{w-a_i}\Big) \\
&\le 4N_\Phi\Big(r, \frac{w'}{w-a_i}\Big) + \sum_{j=1}^{N} N_\Phi\Big(r, \frac{\kappa_j}{w-a_j}\frac{w'^2}{w-a_i}\Big) + S(r,w) \\
&\le 4N_\Phi\Big(r, \frac{w'}{w-a_i}\Big) + \sum_{j=1}^{N} \Big\{N_\Phi\Big(r, \frac{w'}{w-a_j}\Big) + N_\Phi\Big(r, \frac{w'}{w-a_i}\Big)\Big\} + S(r,w) \\
&\le (4+N)N_\Phi\Big(r, \frac{w'}{w-a_i}\Big) + \sum_{j=1}^{N} N_\Phi\Big(r, \frac{w'}{w-a_j}\Big) + S(r,w) \\
&\le (4+N)\Big\{\overline{N}_\Phi(r, w-a_i) + \overline{N}_\Phi\Big(r, \frac{1}{w-a_i}\Big)\Big\} \\
&+ \sum_{j=1}^{N} \Big\{\overline{N}_\Phi(r, w-a_j) + \overline{N}_\Phi\Big(r, \frac{1}{w-a_j}\Big)\Big\} + S(r,w) \\
&\le 2(5+N)\{N_\Phi(r,\alpha) + N_\Phi(r, 1/\alpha)\} + S(r,w) \\
&\le 4(5+N)T(r,\alpha) + S(r,w) \\
&= S(r,w).
\end{aligned}
$$

3. It follows that $N(r,(w-a_j)^{-1}) \ne N_\Phi(r,(w-a_j)^{-1})$, since otherwise

$$
\begin{aligned}
T\Big(r, \frac{1}{w-a_j}\Big) &= m\Big(r, \frac{1}{w-a_j}\Big) + N\Big(r, \frac{1}{w-a_j}\Big) \\
&= m\Big(r, \frac{1}{w-a_j}\Big) + N_\Phi\Big(r, \frac{1}{w-a_j}\Big) \\
&= S(r,w), \qquad (3.27)
\end{aligned}
$$

which is impossible, so there exists $z_0 \in \Omega$ such that $w(z_0) = a_j$. It follows from Lemma 3.2.2 that $\kappa_j \ne 1/2$. □

The main idea in showing that, in most cases, w is an admissible solution of a first-order polynomial differential equation is to find a rational function of w and w'

with small coefficients (in the sense of Nevanlinna) that is itself small. To this end we begin by constructing a meromorphic function F_i satisfying $m(r, F_i) = S(r, w)$ that is analytic in Ω apart from at poles of w. For any $i \in \{1, \dots, N\}$, we know that $m\left(r, w'/(w-a_i)\right) = S(r,w)$ and $m\left(r, 1/(w-a_i)\right) = S(r,w)$. Furthermore apart from at poles of w, the functions $w'/(w-a_i)$ and $1/(w-a_i)$ only have poles at $z_0 \in \Omega$ if $w(z_0) = a_i$, in which case $w'/(w-a_i)$ has a simple pole and $1/(w-a_i)$ has a double pole. From Lemma 3.2.4 we have seen that $\kappa_i \neq 1/2$. If furthermore $\kappa_i \neq 3/4$ then we have

$$w(z) - a_i = c_0(z_0)\zeta^2 + c_1(z_0)\zeta^3 + O\left(\zeta^4\right),$$

where c_0 and c_1 are given by (3.25). This means that we can calculate the principal parts of the Laurent expansions of $1/(w-a)$ and $(w')^2/(w-a)^2$, so we can find a linear combination such that the double poles cancel, leaving at most a simple pole of known residue, which we can remove by subtracting a multiple of $w'/(w-a_i)$. We have

$$\begin{aligned}\frac{1}{w(z)-a_i} &= \frac{1}{c_0(z_0)}\zeta^{-2} - \frac{c_1(z_0)}{c_0(z_0)^2}\zeta^{-1} + O(1)\\ &= \frac{2(1-2\kappa_i)}{\alpha(z_0)}\zeta^{-2} - \frac{2(1-2\kappa_i)^2}{3-4\kappa_i}\frac{\alpha'(z_0)}{\alpha(z_0)^2}\zeta^{-1} + O(1),\end{aligned}$$

$$\begin{aligned}\frac{w'(z)}{w(z)-a_i} &= 2\zeta^{-1} + \frac{c_1(z_0)}{c_0(z_0)} + O(\zeta)\\ &= 2\zeta^{-1} + \frac{1-2\kappa_i}{3-4\kappa_i}\frac{\alpha'(z_0)}{\alpha(z_0)} + O(\zeta)\end{aligned}$$

and

$$\left(\frac{w'(z)}{w(z)-a_i}\right)^2 = 4\zeta^{-2} + \frac{4(1-2\kappa_i)}{3-4\kappa_i}\frac{\alpha'(z_0)}{\alpha(z_0)}\zeta^{-1} + O(1).$$

Hence

$$\left(\frac{w'(z)}{w(z)-a_i}\right)^2 - \frac{2\alpha(z)}{1-2\kappa_i}\frac{1}{w(z)-a_i} = -\frac{4}{3-4\kappa_i}\frac{\alpha'(z_0)}{\alpha(z_0)}\zeta^{-1} + O(1).$$

Which leads us to the following.

Lemma 3.2.5 *Let*

$$F_i(z) := \left(\frac{w'}{w-a_i}\right)^2 + \frac{2}{3-4\kappa_i}\frac{\alpha'}{\alpha}\frac{w'}{w-a_i} - \frac{2\alpha}{1-2\kappa_i}\frac{1}{w-a_i}, \tag{3.28}$$

where w *is a meromorphic solution of equation (3.14) satisfying* $T(r,\alpha) = S(r,w)$ *and* $\kappa_i \neq 3/4$ *for some* $i \in \{1,\ldots,N\}$*. Then*

1. $m(r,F_i) = S(r,w)$.
2. *If* F_i *has a pole at some* $z_0 \in \Omega$ *then it is a double pole and* w *also has a pole at* z_0.
3. $N_\Phi(r,F_i) = S(r,w)$.

Proof.

1. Taking the proximity function of (3.28) and using relation (3.15) with Lemmas 2.1.3 and 3.2.4 , we obtain

$$\begin{aligned} m(r,F_i) &= 3m\left(r,\frac{w'}{w-a_i}\right) + m\left(r,\frac{1}{w-a_i}\right) + m\left(r,\frac{\alpha'}{\alpha}\right) + m\left(r,\alpha\right) + O(1) \\ &= S(r,w). \end{aligned}$$

2. It follows from the definition (3.28) that F_i can only have a pole at $z_0 \in \Omega$ if z_0 is either a pole or an a_i-point of w. However, the calculation before the

statement of the lemma shows that, by construction, F_i is regular at the a_i-points of w in Ω. Furthermore, if w has a pole at $z_0 \in \Omega$ then $w'/(w-a_i)$ has a simple pole and $1/(w-a_i)$ has a zero at z_0.

3. Taking the counting function of both sides of (3.28) we obtain

$$\begin{aligned} N_\Phi(r, F_i) &\leq 3N_\Phi\left(r, \frac{w'}{w-a_i}\right) + N_\Phi\left(r, \frac{\alpha'}{\alpha}\right) + N_\Phi(r, \alpha) + N_\Phi\left(r, \frac{1}{w-a_i}\right) \\ &\leq 3N_\Phi\left(r, \frac{w'}{w-a_i}\right) + S(r,w) \\ &\leq 3\Big\{\overline{N}_\Phi(r, w-a_j) + \overline{N}_\Phi\Big(r, \frac{1}{w-a_j}\Big)\Big\} + S(r,w) \end{aligned}$$

where we have used (3.15) and Lemma 3.2.4. Hence

$$\begin{aligned} N_\Phi(r, F_i) &\leq 6\Big\{\overline{N}_\Phi(r,\alpha) + \overline{N}_\Phi\Big(r, 1/\alpha\Big)\Big\} + S(r,w) \\ &\leq 12T(r,\alpha) + S(r,w) \\ &= S(r,w). \end{aligned}$$

The proof of Lemma 3.2.5 is completed. □

Lemma 3.2.6 *Let w be a solution of equation (3.14) with a pole of order n at $z_0 \in \Omega$. Then*

$$\kappa_\infty := \sum_{j=1}^{N} \kappa_j = \frac{n+1}{n}.$$

Proof. Substituting $w(z) = \sum_{m=0}^{\infty} b_m \zeta^{m-n}$, where $\zeta = z - z_0$, $b_0 \neq 0$ and $n > 0$, in (3.14) gives

$$n(1 + [1-\kappa_\infty]n)b_0\zeta^{-(n+2)} + \cdots = \alpha_0(z_0) + \cdots$$

Now $\zeta^{-(n+2)}$ cannot be the leading order term on the left side as it cannot balance with the leading order on the right side, so $1 + [1-\kappa_\infty]n = 0$. □

In what follow we will show that if there is a small function g compared to the solution of equation (3.14) (in the sense of Nevanlinna) and it is equal to zero at a_i-points, then the function g vanishes identically.

Lemma 3.2.7 *Let w be a meromorphic solution of equation (3.14) satisfying $T(r, \alpha) = S(r, w)$. Then if there is a meromorphic function g such that $T(r, g) = S(r, w)$ and that at every point $z_0 \in \Omega$ such that $w(z_0) = a_i$, we have $g(z_0) = 0$, then $g(z) \equiv 0$.*

Proof. Assume that there is a non-zero meromorphic function g such that $T(r, g) = S(r, w)$ and that at every point $z_0 \in \Omega$ such that $w(z_0) = a_i$, we have $g(z_0) = 0$. Applying the first fundamental theorem (2.3) and Lemma 3.2.4 (1) and (2), we obtain

$$\begin{aligned}
T(r, w) &= T\Big(r, \frac{1}{w - a_i}\Big) + O(1) \\
&= m\Big(r, \frac{1}{w - a_i}\Big) + N\Big(r, \frac{1}{w - a_i}\Big) + O(1) \\
&= N\Big(r, \frac{1}{w - a_i}\Big) + S(r, w) \\
&= N_\Omega\Big(r, \frac{1}{w - a_i}\Big) + N_\Phi\Big(r, \frac{1}{w - a_i}\Big) + S(r, w) \\
&= N_\Omega\Big(r, \frac{1}{w - a_i}\Big) + S(r, w) \\
&= 2\overline{N}_\Omega\Big(r, \frac{1}{w - a_i}\Big) + S(r, w) \\
&\le 2\overline{N}_\Omega\Big(r, \frac{1}{g}\Big) + S(r, w) \\
&\le 2N_\Omega\Big(r, \frac{1}{g}\Big) + S(r, w) \\
&\le 2T\Big(r, \frac{1}{g}\Big) + S(r, w) \\
&= 2T(r, g) + S(r, w) \\
&= S(r, w),
\end{aligned}$$

which is a contradiction. Therefore, $g(z) \equiv 0$. □

In the following lemma, we use the function F_i defined by (3.28) to construct small meromorphic functions of z that are rational functions of w and w' with small coefficients. We can calculate the small coefficient functions explicitly from the series expansion (3.24) of w at a_i-points however this is not necessary as we only require the general forms of the equations at this stage. It is sufficient to keep track of the restrictions on κ_i to ensure that we have enough terms in the series expansion (3.24) that are uniquely determined by the evaluation at z_0 of some functions of $c_1(z), \ldots, c_n(z)$, $\alpha(z)$ and their derivatives.

Lemma 3.2.8 *Let w be a meromorphic solution of equation (3.14) satisfying $T(r,\alpha) = S(r,w)$ and let F_i be given by equation (3.28).*

(i) If $\kappa_\infty \neq (n+1)/n$, for all positive integer n and $\kappa_i \neq 3/4$ for some $i \in \{1,\ldots,N\}$, or if $\kappa_\infty = (n+1)/n$ for some integer $n \geq 5$ and $\kappa_i \notin \{3/4, 1\}$ for some $i \in \{1,\ldots,N\}$, then $T(r,F_i) = S(r,w)$.

(ii) If $\kappa_\infty = 5/4$ and $\kappa_i \notin \{3/4, 1\}$ for some $i \in \{1,\ldots,N\}$ then there exist meromorphic functions β_1 and β_2 satisfying $T(r,\beta_k) = S(r,w)$ for $k = 1,2$, such that

$$(F_i(z) - \beta_1(z))^2 = \beta_2(z)\left(w(z) - a_i\right). \tag{3.29}$$

(iii) If $\kappa_\infty = 4/3$ or $\kappa_\infty = 3/2$ and furthermore $\kappa_i \notin \{3/4, 1, 5/4, 3/2\}$ for some $i \in \{1,\ldots,N\}$ then either there exist meromorphic functions β_1 and β_2 satisfying $T(r,\beta_k) = S(r,w)$ for $k = 1,2$, such that

$$F_i(z) - \beta_1(z) = \beta_2(z)(w(z) - a_i) \tag{3.30}$$

or there exist meromorphic functions $\gamma_1,\ldots,\gamma_4$ satisfying $T(r,\gamma_k) = S(r,w)$ for $k = 1,\ldots,4$, with $\gamma_3 \neq 0$, such that

$$\left(\frac{F_i(z) - \gamma_1(z)}{w(z) - a_i}\right)^2 + \gamma_2(z)\left(\frac{F_i(z) - \gamma_1(z)}{w(z) - a_i}\right) + \frac{\gamma_3(z)}{w(z) - a_i} + \gamma_4(z) = 0. \tag{3.31}$$

Proof.

(i) If $\kappa_\infty \neq (n+1)/n$ for all positive integer n then it follows from Lemma 3.2.6 that w is analytic on Ω and therefore by Lemma 3.2.5, F_i is analytic on Ω, so $N_\Omega(r, F_i) = 0$. Therefore, from Lemma 3.2.5,

$$\begin{aligned} T(r, F_i) &= m(r, F_i) + N_\Phi(r, F_i) + N_\Omega(r, F_i) \\ &= S(r, w). \end{aligned}$$

Now suppose that $\kappa_i \notin \{3/4, 1\}$ for some $i \in \{1, \ldots, N\}$. Then from Lemma 3.2.3 there are functions c_0, c_1 and c_2 such $T(r, c_j) = S(r, w)$, $j = 0, 1, 2$, $c_0 \not\equiv 0$ and that at every point $z_0 \in \Omega$ such that $w(z_0) = a_i$, we have

$$w(z) = a_i + c_0(z_0)\zeta^2 + c_1(z_0)\zeta^3 + c_2(z_0)\zeta^4 + O\left(\zeta^5\right),$$

where $\zeta = z - z_0$. We saw in the calculation preceding Lemma 3.2.5 that knowledge of c_0 and c_1 alone was enough to show that $F_i(z) = O(\zeta)$ as $z \to z_0$ (i.e. as $\zeta \to 0$). Therefore, since we know w to the next order in its series expansion, we see that there is a meromorphic function $\beta_1(z)$ that is a polynomial in c_0, c_1 and c_2 divided by a power of c_0, which in turn means that $\beta_1(z)$ is a polynomial in α, α' and α'' divided by a power of α, such that $F_i(z) = \beta_1(z_0) + O(\zeta)$ at all a_i-points of w in Ω. It follows that $\beta_1(z)$ is analytic on Ω and $T(r, \beta_1) = S(r, w)$. If $\kappa_\infty = (n+1)/n$ for some integer $n \geq 5$ then if F_i has a pole at $z_0 \in \Omega$ then by Lemmas 3.2.5 and 3.2.6 it must be of order 2 and w must have a pole of order n at z_0, respectively. Now consider the function

$$g_1(z) = \frac{(F_i(z) - \beta_1(z))^5}{(w - a_i)^2}.$$

From Lemmas 3.2.4 and 3.2.5, we see that $m(r, g_1) = S(r, w)$ and $N_\Phi(r, g_1) =$

$S(r,w)$. The only possible ways in which g_1 can have a pole at $z_0 \in \Omega$ is if either $w(z_0) = a_i$ or w has a pole (of order n) at z_0. If $w(z_0) = a_i$ then $(F_i(z) - \beta_1(z))^5$ has a zero of order at least five while $(w(z) - a_i)^2$ has a zero of order exactly four. Therefore $g_1(z_0) = 0$ at all a_i-points of w. If w has a pole at $z_0 \in \Omega$, then F_i has a pole of order exactly two at z_0, so $(F_i(z) - \beta_1(z))^5$ has a pole of order 10, on the other hand, $(w(z) - a_i)^2$ will have a pole of order $2n \geq 10$ at z_0, so g_1 will be analytic there. Therefore $N_\Omega(r, g_1) = 0$ and so $T(r, g_1) = S(r, w)$. However, g_1 vanishes at all the a_i-points of w in Ω, so by Lemma 3.2.7 $g_1 \equiv 0$, giving $F_i(z) = \beta_1(z)$, which in turn shows that $T(r, F_i) = T(r, \beta_1) = S(r, w)$.

(ii) Suppose that $\kappa_\infty = 5/4$, $\kappa_i \notin \{3/4, 1\}$ for some $i \in \{1, \dots, N\}$ and let β_1 be defined as above. Consider the function

$$g_2(z) = \frac{(F_i(z) - \beta_1(z))^2}{w - a_i}.$$

Clearly $m(r, g_2) = S(r, w)$ and $N_\Phi(r, g_2) = S(r, w)$. The only possible poles of g_2 in Ω must occur at either a_i-points or poles of w. However, if $z_0 \in \Omega$ is an a_i-point of w then $w - a_i$ has a double zero and $(F_i(z) - \beta_1(z))^2$ has a zero of multiplicity at least two at z_0, so z_0 is a regular point of g_2. On the other hand, if $z_0 \in \Omega$ is a pole of w then from Lemma 3.2.6 it is a pole of order 4 (since $\kappa_\infty = (n+1)/n$, where $n = 4$). Therefore $w - a_i$ and $(F_i(z) - \beta_1(z))^2$ both have poles of order 4. It follows g_2 is analytic on Ω, so $T(r, g_2) = S(r, w)$. Writing $\beta_2 = g_2$ gives the desired result.

(iii) Since $\kappa_i \notin \{3/4, 1, 5/4, 3/2\}$ for some $i \in \{1, \dots, N\}$, we know two more functions $c_3(z)$ and $c_4(z)$ such that near any a_i-point $z_0 \in \Omega$ of w, we have

$$w(z) = a_i + c_0(z_0)\zeta^2 + c_1(z_0)\zeta^3 + c_2(z_0)\zeta^4 + c_3(z_0)\zeta^5 + c_4(z_0)\zeta^6 + O\left(\zeta^7\right).$$

The fact that c_3 is a fixed function satisfying $T(r, c_3) = S(r, w)$ means that there is a meromorphic function g_3, which is a polynomial in α, α', α'' and α''' divided by a power of α, such that

$$F_i(z) - \beta_1(z) = g_3(z_0)(z - z_0) + O\left((z - z_0)^2\right).$$

Again we have that g_3 is analytic on Ω and $T(r, g_3) = S(r, w)$.

Let

$$g_4(z) = \frac{F_i(z) - \beta_1(z)}{w - a_i}.$$

Now if $g_3 \equiv 0$ then at any a_i-point of w in Ω, the function $F_i(z) - \beta_1(z)$ has a zero of multiplicity at least two and $w - a_i$ has a zero of multiplicity exactly two, so g_4 is analytic there. If $\kappa_\infty = 4/3$ or $\kappa_\infty = 3/2$ then any pole of w in Ω has order three or two respectively, while F_i has a double pole. Therefore we see that g_4 is analytic on Ω and so $T(r, g_4) = S(r, w)$. Setting $\beta_2 = g_4$ gives equation (3.30).

Now suppose that $g_3 \not\equiv 0$. It follows that

$$\begin{aligned} g_4(z) &= \frac{F_i(z) - \beta_1(z)}{w - a_i} \\ &= \frac{g_3(z_0)}{c_0(z_0)}(z - z_0)^{-1} + O(1). \end{aligned}$$

Once again it is clear that $m(r, g_4) = S(r, w)$. Recall that the coefficients in the expansion of $w(z)$ at an a_i-point $z_0 \in \Omega$ are fixed up to $c_4(z_0)$, which is the coefficient of $(z - z_0)^6$. This means that we can determine the next term in the expansion of g_4 at z_0. So there is a meromorphic function $g_5(z)$ satisfying $T(r, g_5) = S(r, w)$ and analytic on Ω such that

$$g_4(z) = \frac{F_i(z) - \beta_1(z)}{w - a_i}$$

$$= \frac{g_3(z_0)}{c_0(z_0)}(z-z_0)^{-1} + g_5(z_0) + O\left((z-z_0)\right).$$

It follows that

$$g_4(z)^2 - \frac{g_3(z)^2}{c_0(z)}\frac{1}{w(z)-a_i} = \frac{g_6(z_0)}{z-z_0} + O(1),$$

where the meromorphic function g_6 is analytic on Ω and satisfies $T(r, g_6) = S(r, w)$. Let

$$g_7(z) = g_4(z)^2 - \frac{g_3(z)^2}{c_0(z)}\frac{1}{w(z)-a_i} - \frac{g_6(z)c_0(z)}{g_3(z)}g_4(z). \tag{3.32}$$

Then $m(r, g_7) = S(r, w)$ and $N_\Omega(r, g_7) = 0$, so $T(r, g_7) = S(r, w)$. Equation (3.32) is an equation of the form (3.31). □

Lemma 3.2.9 *If w is a meromorphic solution of equation (3.14) satisfying $T(r, \alpha) = S(r, w)$ then one of the following is true.*

(i) For some $i \in \{1, \ldots, N\}$, the function $W = w - a_i$ satisfies an equation of the form

$$(W')^2 + \mu(z)WW' + \nu(z)W + \rho(z)W^2 + \sigma(z)W^3 = 0, \tag{3.33}$$

where

$$\mu(z) = \frac{2}{3-4\kappa_i}\frac{\alpha'}{\alpha}, \quad \nu(z) = -\frac{2\alpha}{1-2\kappa_i}, \quad T(r, \rho) = S(r, W) \tag{3.34}$$

and $T(r, \sigma) = S(r, W)$.

(ii) For some $i \in \{1, \ldots, N\}$, the function w satisfies (3.31) which corresponds to an equation of the form (3.33) where μ, ν and ρ satisfy (3.34), $\sigma(z) =$

$\gamma_2(z)/2 - u(z)$ and u is a meromorphic function such that

$$W = \frac{\gamma_3}{\gamma_5 - u^2}, \tag{3.35}$$

where $\gamma_3 \not\equiv 0$, $T(r,\gamma_2) = S(r,W)$, $T(r,\gamma_3) = S(r,W)$ and $T(r,\gamma_5) = S(r,W)$.

(iii) $\alpha = \alpha_0$ is a a non-zero constant and either

1- $N = 1$, $\kappa_1 = 3/4$ and $W = w - a_1$ satisfies

$$\{W'^2 + 4\alpha_0 W\}^2 W^{-3} = d^2, \tag{3.36}$$

where d is a constant or

2- $N = 2$, $\kappa_1 = \kappa_2 = 3/4$ and w satisfies

$$\left\{\frac{(w')^2}{(w-a_1)(w-a_2)} - \frac{4\alpha_0}{(a_1-a_2)^2}\left(w - \frac{a_1+a_2}{2}\right)\right\}^2 = d_1^2(w-a_1)(w-a_2), \tag{3.37}$$

where d_1 is a constant

(iv) N=1 and $W = w - a_1$ satisfies

$$W'' = \frac{3}{2}\frac{(W')^2}{W} + \alpha(z). \tag{3.38}$$

(v) $\kappa_\infty = 2$.

Proof. The conclusion of each part of Lemma 3.2.8 is that w satisfies a first-order differential equation. We will show that each of these equations is a special case of the equations listed in parts *(i)* and *(ii)* of Lemma 3.2.9. We will then consider the equations that do not satisfy the assumptions of any of the parts of Lemma 3.2.8 and show that such equations either have no admissible meromorphic solutions or that w

still satisfies an equation of the form (3.33) or that one of the remaining conclusions *(iii)*–*(v)* of Lemma 3.2.9 holds.

Referring to Lemma 3.2.8, we note that the conclusion in part *(i)* and the first conclusion of part *(iii)* (i.e., equation (3.30)) is that $W = w - a_i$ satisfies an equation of the form (3.33), where μ, ν and ρ satisfy (3.34) and $T(r, \sigma) = S(r, w)$. Next we consider equation (3.29) in part *(ii)* of Lemma 3.2.8. If $\beta_2 = 0$ then the conclusion is the same as that of part *(i)* of Lemma 3.2.8, which we have just discussed. If $\beta_2 \neq 0$ then equation (3.29) is a special case of (3.31) in part *(iii)* corresponding to $\gamma_1 = \beta_1$, $\gamma_3 = -\beta_2$ and $\gamma_2 = 0 = \gamma_4$. So it only remains to analyse equation (3.31), which can be written as

$$\left(\frac{F_i(z) - \gamma_1(z)}{w(z) - a_i} + \frac{\gamma_2}{2} \right)^2 = \gamma_5(z) - \frac{\gamma_3(z)}{w(z) - a_i}, \tag{3.39}$$

where $\gamma_5 = (\gamma_2/2)^2 - \gamma_4$. Let the meromorphic function u be defined by

$$u = \frac{F_i(z) - \gamma_1(z)}{w(z) - a_i} + \frac{\gamma_2}{2}. \tag{3.40}$$

Then equation (3.39) becomes (3.35) and so (3.40) becomes (3.33) where μ and ν are given by (3.34) and $\rho = -\gamma_1$ and $\sigma = (\gamma_2/2) - u$.

Next we consider all equations of the form (3.14) that are not considered in the assumptions of Lemma 3.2.8. None of the cases in which $\kappa_j = 3/4$ for all $j \in \{1, \ldots, N\}$ are considered in Lemma 3.2.8. It follows by Lemma 3.2.2 that $\alpha'(z_0) = 0$ at all a_i-points of w in Ω, and then from Lemma 3.2.7 it follows that $\alpha = \alpha_0$, a non-zero constant and $\Omega = \mathbb{C}$. Each a_j-point of w is totally ramified. A meromorphic function can have at most four totally ramified points, so $N \leq 4$. For $N = 3$ and $N = 4$ we have $\kappa_\infty = 9/4$ and $\kappa_\infty = 3$ respectively. Since neither of these values is of the form $(n+1)/n$ for some positive integer n it follows from Lemma 3.2.6 that any meromorphic solution is in fact entire and can therefore have at most two totally ramified values, which is a contradiction. So we are left with the equation

$$w'' = \frac{3}{4}\sum_{j=1}^{N} \frac{(w')^2}{w-a_j} + \alpha_0, \tag{3.41}$$

with $N = 1$ or $N = 2$. For $N = 1$, W satisfies

$$W'' = \frac{3}{4}\frac{W'^2}{W} + \alpha_0$$

where $W = w - a_1$. The above equation can be written as

$$2(W'W^{-3/4})(W'W^{-3/4})' = 2\alpha_0(W'W^{-3/2})$$

and the first integral of the above equation yields

$$(W'W^{-3/4})^2 = -4\alpha_0 W^{-1/2} + d.$$

which is equivalent to equation (3.36).

For $N = 2$, we have $\kappa_1 = \kappa_2 = 3/4$ and equation (3.41) can be written as

$$\left(w' \prod_{j=1}^{2}(w-a_j)^{-(3/4)}\right)' = \alpha_0 \prod_{j=1}^{2}(w-a_j)^{-3/4},$$

and the first integral is

$$\left(w' \prod_{j=1}^{2}(w-a_j)^{-3/4}\right)^2 = 2\alpha_0 \int \prod_{j=1}^{2}(w-a_j)^{-3/2}\, w'\, dz.$$

Therefore, integrating the right side of the above equation gives the first-order equation (3.37).

The other case to consider from parts *(i)* and *(ii)* of Lemma 3.2.8 is when $\kappa_\infty = (n+1)/n$ for some integer $n \geq 4$ and for all $j \in \{1, \ldots, N\}$, $\kappa_j = 3/4$ or $\kappa_j = 1$.

Clearly $N \neq 1$. If $N > 1$ then $\kappa_\infty = (n+1)/n \leq 5/4 < 3/2 = 3/4 + 3/4$ for all $n \geq 4$. So there are no cases to consider here.

If $\kappa_\infty = (n+1)/n$ for some positive integer $n < 4$ then $\kappa_\infty = 4/3$, $\kappa_\infty = 3/2$ or $\kappa_\infty = 2$. If $\kappa_\infty = 4/3$ or $\kappa_\infty = 3/2$ and for all $j \in \{1, \ldots, N\}$ we have $\kappa_j \in \{3/4, 1, 5/4, 3/2\}$, then either $N = 1$ and $\kappa_1 = 3/2$, which corresponds to equation (3.38), or $N = 2$ and $\kappa_1 = \kappa_2 = 3/4$, which corresponds to equation (3.37) with $N = 2$. Otherwise, if $\kappa_\infty = 4/3$ or $\kappa_\infty = 3/2$, then w solves an equation of the form (3.31) which corresponds to the part *(ii)* of Lemma 3.2.9. □

The following lemma shows that if a solution (not necessary admissible) of a second-order differential equation satisfies a differential equation of first-order and second degree, then again it satisfies a first-order linear differential equation.

Lemma 3.2.10 *Let w be a solution of the second-order equation*

$$w'' = L(z,w)(w')^2 + M(z,w)w' + N(z,w) \tag{3.42}$$

and also the first-order equation

$$(w')^2 + A(z,w)w' + B(z,w) = 0, \tag{3.43}$$

where L, M, N, A and B are meromorphic functions of there arguments. Then

$$G(z,w)w' + H(z,w) = 0, \tag{3.44}$$

where

$$G(z,w) = 2N + \{LA - M - A_w\} A + A_z - 2LB + B_w$$

and

$$H(z,w) = NA + \{LA - 2M - A_w\}\, B + B_z.$$

Proof. Let w be a solution of equations (3.42) and (3.43). Differentiating (3.43) and then using the derivative of equation (3.42) to eliminate the second derivative term w'' we obtain

$$2L{w'}^3 + (2M + AL + A_w){w'}^2 + (2N + AM + A_z + B_w)w' + AN + B_z = 0.$$

Using (3.43) again to eliminate powers of w' greater than one leads to equation (3.44). □

Proof of Theorem 3.2.1. Let W be a (not necessarily admissible) solution of (3.33) such that $w = a_i + W$ solves (3.14). From Lemma 3.2.10, then W also satisfies equation (3.44), where

$$\begin{aligned} G(z,X) = \quad & (2\alpha + \nu) + (\mu' - \mu^2 + 2\rho)X + 3\sigma X^2 \\ & -X(2\nu + [2\rho - \mu^2]X + 2\sigma X^2) \textstyle\sum_{j=1}^{N} \frac{\kappa_j}{X - \widetilde{a}_j} \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} H(z,X) = \; & (\nu' + [\alpha - \nu]\mu)X + (\rho' - \mu\rho)X^2 + (\sigma' - \mu\sigma)X^3 \\ & +\mu X^2(\nu + \rho X + \sigma X^2) \textstyle\sum_{j=1}^{N} \frac{\kappa_j}{X - \widetilde{a}_j}, \end{aligned} \tag{3.46}$$

where $\widetilde{a}_j = a_j - a_i$.

(I) **Equation (3.33)**

Consider the case, described in part *(i)* of Lemma 3.2.9, in which W is an admissible solution of equation (3.33). Now either $G(z,W) = 0 = H(z,W)$ or

W is an admissible solution of the first-order first-degree differential equation $W' = -H(z,W)/G(z,W)$. Now if W is an admissible solution of a first-order first-degree equation, then by the Malmquist-Yosida Theorem 3.1.2, $W' = P(z) + Q(z)W + R(z)W^2$, where P, Q and R are small compared to W. Furthermore, if $R \not\equiv 0$ then $m(r,W) = S(r,W)$. But in this case any pole of W in Ω must be simple, which according to Lemma 3.2.6 means that $\kappa_\infty = 2$. So if $\kappa_\infty \neq 2$ then $R \equiv 0$ and hence $W' = P(z) + Q(z)W$. However, a simple calculation shows that $w = a_i + W$ cannot simultaneously be be an admissible solution of (3.14) and $W' = P(z) + Q(z)W$.

We will now study the cases in which $G(z,W(z))$ and $H(z,W(z))$ vanish identically. Now G and H are rational functions of W with coefficients that are $S(r,W)$. So if $G(z,W(z))$ and $H(z,W(z))$ vanish identically, then $G(z,X)$ and $H(z,X)$ vanish identically as functions of the two independent variables z and X.

Lemma 3.2.11 *Let G and H be defined by (3.45–3.46), where $\kappa_j \neq 1/2$ for all $j \in \{1,\ldots,N\}$. Furthermore, suppose that if $N=1$ then $\kappa_\infty \neq 3/2$. Then both $G(z,X)$ and $H(z,X)$ vanish identically as rational functions of X if and only if one of the following holds:*

(i) $N = 1$, $\sigma = 0$, α_0 *and* z_∞ *are constants,* $\nu(z) = 2\alpha(z)/(2\kappa - 1)$ *where* $\kappa = \kappa_1 = \kappa_\infty$ *and*

- *(a)* $\kappa = 1$, $\alpha = \alpha_0 \exp(-\mu_0 z/2)$, $\mu = \mu_0$, $\rho = \rho_0$, *where* μ_0 *and* ρ_0 *are constants.*
- *(b)* $\kappa \neq 1$, $\alpha = \alpha_0$, $\mu = 0$, $\rho = 0$.
- *(c)* $\kappa \neq 1$, $\alpha = \alpha_0(z - z_\infty)^{-(4\kappa-3)/(2[\kappa-1])}$, $\mu = \frac{1}{\kappa-1}\frac{1}{z-z_\infty}$, $\rho = 0$.
- *(d)* $\kappa \neq 1$, $\alpha = \alpha_0([z-z_\infty]^2 - d^2)^{-(4\kappa-3)/(2[\kappa-1])}$, $\mu = \frac{1}{\kappa-1}\frac{2(z-z_\infty)}{(z-z_\infty)^2-d^2}$, $\rho = \frac{1}{(\kappa-1)^2}\frac{1}{(z-z_\infty)^2-d^2}$.

(ii) $N > 1$. *Without loss of generality we take* $i = 1$. *Then* $\alpha = \alpha_0$ *is a constant,* $\mu = 0$, $\nu = 2\alpha_0/(2\kappa_1 - 1)$ *and*

(a) $N = 2$, $\kappa_\infty = 1$, $\rho = -\nu/\widetilde{a}_2$, $\sigma = 0$.

(b) $N = 2$, $\kappa_\infty = 3/2$, $\rho = -\frac{2\kappa_2\nu}{(2\kappa_2-1)\widetilde{a}_2}$, $\sigma = \frac{\nu}{(2\kappa_2-1)\widetilde{a}_2^2}$.

(c) $N = 3$, $\kappa_\infty = 3/2$, $\rho = -\frac{(\widetilde{a}_2+\widetilde{a}_3)}{\widetilde{a}_2\widetilde{a}_3}\nu$, $\sigma = \nu/(\widetilde{a}_2\widetilde{a}_3)$ *and*

$$(2\kappa_2 - 1)\widetilde{a}_2 + (2\kappa_3 - 1)\widetilde{a}_3 = 0.$$

Note that $N = 1$, $\kappa = 3/2$ corresponds to part *(iv)* of Lemma 3.2.9.

Proof.

<u>The case $N = 1$.</u> Now

$$G(z, X) = (2\alpha + [1 - 2\kappa]\nu) + (\mu' + [2\rho - \mu^2][1 - \kappa])X + \sigma(3 - 2\kappa)X^2$$

and

$$H(z, X) = (\nu' + \mu(\alpha + [\kappa - 1]\nu)X + (\rho' - (1 - \kappa)\mu\rho)X^2 + (\sigma' - [1 - \kappa]\mu\sigma)X^3.$$

Setting the coefficient of X^0 in G and the coefficient of X^1 in H to zero, we recover (3.34). The remaining equations are

$$\sigma(3 - 2\kappa) = 0, \quad (3.47)$$

$$\sigma' - [1 - \kappa]\mu\sigma = 0, \quad (3.48)$$

$$\mu' + [2\rho - \mu^2][1 - \kappa] = 0, \quad (3.49)$$

$$\rho' - (1 - \kappa)\mu\rho = 0. \quad (3.50)$$

Recall that $\kappa \neq 3/2$ so (3.47) shows that $\sigma = 0$ and (3.48) is satisfied identically.

- If $\kappa = 1$ then by (3.49) and (3.50) we obtain $\mu' = 0$ and $\rho' = 0$, respectively. Hence, $\mu = \mu_0$ and $\rho = \rho_0$ are constants. Therefore equation (3.34) gives $-2(\alpha'/\alpha) = \mu_0$ and hence $\alpha = \alpha_0 \exp(-\mu_0 z/2)$.

- If $\kappa \neq 1$ and $\rho = 0$ then by (3.49) either $\mu = 0$ or $\mu \neq 0$. In the case $\mu \neq 0$, equation (3.49) gives $\mu'/\mu^2 = (1-\kappa)$ and hence $\mu = [(\kappa-1)(z-z_\infty)]^{-1}$, where z_∞ is a constant. Therefore, by equation (3.34) either

$$\alpha' = 0 \quad \text{or} \quad \frac{2}{3-4\kappa}\frac{\alpha'}{\alpha} = \frac{1}{(\kappa-1)(z-z_\infty)}.$$

  Solving the above equations we obtain $\alpha = \alpha_0$ when $\mu = 0$ which corresponds to the part *(i-b)* of Lemma 3.2.11 and $\alpha = \alpha_0(z-z_\infty)^{-(4\kappa-3)/(2[\kappa-1])}$ when $\mu \neq 0$ which corresponds to the part *(i-c)* of Lemma 3.2.11.

- If $\kappa \neq 1$ and $\rho \neq 0$ then (3.50) gives

$$\mu = (1-\kappa)^{-1}(\rho'/\rho) \tag{3.51}$$

  and (3.49) becomes $(\rho^{-2}\rho')' + 2(\kappa-1)^2 = 0$, which yields $\rho = (\kappa-1)^{-2}[(z-z_\infty)^2 - d^2]^{-1}$. Using (3.51) and the function ρ we obtain μ corresponds to the part *(i-d)*. Finally equation (3.34) and the function μ imply that

$$\frac{\alpha'}{\alpha} = -\frac{4\kappa-3}{2(\kappa-1)}\frac{2(z-z_\infty)}{(z-z_\infty)^2-d^2}.$$

  Hence solving the above differential equation gives α in part *(i-d)* of Lemma 3.2.11.

<u>The case $N > 1$.</u> Note that $\widetilde{a}_j = 0$ if and only if $j = i$. Setting the residues of the (apparent) simple poles of G and H, viewed as rational functions of X, to zero shows that for all $j \in \{1, \ldots, N\}$, $j \neq i$, we have $\mu(\nu + \rho\widetilde{a}_j + \sigma\widetilde{a}_j^2) = 0$ and

$2(\nu+\rho\widetilde{a}_j+\sigma\widetilde{a}_j^2)=\mu^2\widetilde{a}_j$. It follows that $\mu\equiv 0$ (so α is a nonzero constant) and

$$\nu+\rho\widetilde{a}_j+\sigma\widetilde{a}_j^2=0. \tag{3.52}$$

Since the $\widetilde{a}_j$s are distinct, we see that $N\leq 3$. Using (3.52), the expressions for G and H simplify to

$$G(z,X)=2\left([1-\kappa_\infty]\rho-\sigma\sum_{j\neq i}\kappa_j\widetilde{a}_j\right)X+(3-2\kappa_\infty)\sigma X^2$$

and

$$H(z,X)=\rho' X^2+\sigma' X^3,$$

where we have used the definition of ν. From H we see that ρ and σ are constants (as are μ and α). From G we see that either $\sigma=0$ or $\kappa_\infty=3/2$.

- If $\sigma=0$. then we see from (3.52) that $\rho\neq 0$, $N=2$ and $\nu=-\widetilde{a}_j\rho$. The final constraint comes from setting the coefficient of X in G to zero, which yields $\kappa_\infty=1$.
- If $\sigma\neq 0$ then $\kappa_\infty=3/2$. Setting the coefficient of X in G to zero gives

$$\rho=-2\sigma_0\sum_{j\neq i}\kappa_j\widetilde{a}_j. \tag{3.53}$$

  If $N=2$ then $\rho=-2\kappa_2\widetilde{a}_2\sigma_0$. So (3.52) gives $(2\kappa_2-1)\widetilde{a}_2{}^2\sigma_0=\nu$. If $N=3$ then $\widetilde{a}_2$ and $\widetilde{a}_3$ are distinct roots of (3.52), so $\rho=-(\widetilde{a}_2+\widetilde{a}_3)\sigma$ and $\nu=\widetilde{a}_2\widetilde{a}_3\sigma$. Equation (3.53) becomes $2(\kappa_2\widetilde{a}_2+\kappa_3\widetilde{a}_3)=\widetilde{a}_2+\widetilde{a}_3$. □

We now consider the first-order equations (3.33) corresponding to each of the cases in Lemma 3.2.11.

(a) $N=1$

In all of the following cases we have $\sigma = 0$.

i. $\kappa = 1$, $\alpha = \alpha_0 \exp(-\mu_0 z/2)$, $\mu = \mu_0$, $\rho = \rho_0$ and W satisfies

$$(W')^2 + \mu_0 W W' + 2\ \alpha(z) W + \rho_0 W^2 = 0.$$

Letting $W = \exp(-\mu_0 z/2) v(z)$, then the above equation becomes

$$(v')^2 = \left(\frac{\mu_0^2}{4} - \rho_0\right) v^2 - 2\alpha_0 v. \tag{3.54}$$

ii. $\kappa \neq 1$, $\alpha = \alpha_0$, $\mu = 0$, $\rho = 0$ and W satisfies

$$(W')^2 + \frac{2\alpha_0}{2\kappa - 1} W = 0. \tag{3.55}$$

iii. $\kappa \neq 1$, $\alpha = \alpha_0 (z - z_\infty)^{-(4\kappa-3)/(2[\kappa-1])}$, $\mu = \frac{1}{\kappa-1}\frac{1}{z-z_\infty}$, $\rho = 0$ and W satisfies

$$(W')^2 + \mu(z) W W' + \frac{2\alpha(z)}{2\kappa - 1} W = 0.$$

Letting $W = (z - z_\infty)^{-1/(2[\kappa-1])} v$, then the above equation becomes

$$(v')^2 + (z - z_\infty)^{-2} \left(\frac{2\alpha_0}{2\kappa - 1} v - \frac{1}{4(\kappa - 1)^2} v^2 \right) = 0. \tag{3.56}$$

iv. $\kappa \neq 1$, $\alpha = \alpha_0([z - z_\infty]^2 - d^2)^{-(4\kappa-3)/(2[\kappa-1])}$, $\mu = \frac{1}{\kappa-1}\frac{2(z-z_\infty)}{(z-z_\infty)^2-d^2}$, $\rho = \frac{1}{(\kappa-1)^2}\frac{1}{(z-z_\infty)^2-d^2}$ and W satisfies

$$(W')^2 + \mu(z) W W' + \frac{2\alpha(z)}{2\kappa - 1} W + \rho(z) W^2 = 0.$$

Letting $W = \left\{(z - z_\infty)^2 - d^2\right\}^{-1/(2[\kappa-1])} v$, then the above equation becomes

$$(v')^2 + \left\{(z-z_\infty)^2 - d^2\right\}^{-2}\left(\frac{2\alpha_0}{2\kappa-1}v - \frac{d^2}{(\kappa-1)^2}v^2\right) = 0. \qquad (3.57)$$

(b) $N > 1$.

In all of the following cases we have $\alpha = \alpha_0$ is a constant, $\mu = 0$ and $\nu = 2\alpha_0/(2k_i - 1)$.

i. $N = 2$, $\kappa_\infty = 1$, $\rho = -\nu/\widetilde{a}_j$, $\sigma = 0$. Setting $\kappa = \kappa_i$ and $\widetilde{a} = \widetilde{a}_j \neq 0$, $j \neq i$, we see that non-constant solutions of the equation

$$(W')^2 + \frac{2\alpha_0}{(1-2\kappa)\widetilde{a}}\left(W^2 - \widetilde{a}W\right) = 0, \qquad (3.58)$$

are also solutions of the equation

$$W'' = \left(\frac{\kappa}{W} + \frac{1-\kappa}{W-\widetilde{a}}\right)(W')^2 + \alpha_0.$$

ii. If $N = 2$, $\kappa_\infty = 3/2$, $\rho = -\frac{2\kappa_2\nu}{(2\kappa_2-1)\widetilde{a}_2}$, $\sigma = \frac{\nu}{(2\kappa_2-1)\widetilde{a}_2^2}$, we see that any non-constant solution of the equation

$$(W')^2 + \frac{2\alpha_0}{2\kappa_1 - 1}\left(W - \frac{2\kappa_2}{(2\kappa_2-1)\widetilde{a}_2}W^2 + \frac{1}{(2\kappa_2-1)\widetilde{a}_2}W^3\right) = 0, \qquad (3.59)$$

where $\kappa_1 + \kappa_2 = 3/2$, is a also a solution of

$$W'' = \left(\frac{\kappa_1}{W} + \frac{\kappa_2}{W-\widetilde{a}_2}\right)(W')^2 + \alpha_0.$$

iii. If $N = 3$, $\kappa_\infty = 3/2$, $\rho = -(\widetilde{a}_2 + \widetilde{a}_3)\nu$, $\sigma = \widetilde{a}_2\widetilde{a}_3\nu$ we see that all non-constant solutions of the equation

$$(W')^2 + \frac{2\alpha_0}{(2\kappa-1)\widetilde{a}_2\widetilde{a}_3}W(W-\widetilde{a}_2)(W-\widetilde{a}_3) = 0, \qquad (3.60)$$

where $(2\kappa_2-1)\widetilde{a}_2+(2\kappa_3-1)\widetilde{a}_3=0$, are also solutions of the equation

$$W''=\left(\frac{\kappa}{W}+\frac{\kappa_2}{W-\widetilde{a}_2}+\frac{\kappa_3}{W-\widetilde{a}_3}\right)(W')^2+\alpha_0.$$

(II) **Equation (3.31)**

Now we consider the conclusion of part *(ii)* of Lemma 3.2.9, which corresponds to the case in which w is an admissible meromorphic solution of an equation of the form (3.31). Note that in this case, W is a meromorphic solution of the equation (3.33), which has meromorphic coefficients, but it is not an admissible solution because the u-dependence of σ implies that $T(r,\sigma)=(1/2)T(r,W)+S(r,W)$.

Nevertheless, we still have that $G(z,W)W'+H(z,W)=0$, where G and H are given by (3.45) and (3.46). Rewriting this to make the u-dependence more explicit, we have

$$(G_0(z,W)+uG_1(z,W))\,u'+(H_0(z,W)+uH_1(z,W))=0, \qquad (3.61)$$

where G_0, G_1, H_0 and H_1 are rational functions in W (with small coefficients) given by

$$\begin{aligned}
G_0 &= W^3\left\{2\left(1-\frac{\gamma_5}{\gamma_3}W\right)\left((3-2\kappa_\infty)+\sum_{j=1}^{N}\frac{2\kappa_j\widetilde{a}_j}{W-\widetilde{a}_j}\right)-1\right\},\\
G_1 &= \frac{2W^2}{\gamma_3}\left\{(2\alpha+\nu)+(\mu'-\mu^2-2\gamma_1)W+\frac{3}{2}\gamma_2W^2\right.\\
&\quad\left.-W\left(2\nu-(2\gamma_1+\mu^2)W+\gamma_2W^2\right)\sum_{j=1}^{N}\frac{\kappa_j}{W-\widetilde{a}_j}\right\},\\
H_0 &= (\nu'+\mu(\alpha-\nu))W+(\mu\gamma_1-\gamma_1')W^2+\frac{1}{2}(\gamma_2'-\mu\gamma_2)W^3\\
&\quad+\mu W^2\left(\nu-\gamma_1W+\frac{\gamma_2}{2}W^2\right)\sum_{j=1}^{N}\frac{\kappa_j}{W-\widetilde{a}_j}+\frac{1}{2W}\left(\gamma_3'-\gamma_5'W\right)G_1,
\end{aligned}$$

$$H_1 = W^3 \left\{ \frac{\gamma_5' W - \gamma_3'}{\gamma_3} \left(3 - 2\sum_{j=1}^{N} \frac{\kappa_j W}{W - \widetilde{a}_j} \right) + \mu \left(1 - \sum_{j=1}^{N} \frac{\kappa_j W}{W - \widetilde{a}_j} \right) \right\}.$$

Now the condition that $G_1 \equiv 0 \equiv H_0$ is exactly the same as the condition that $G \equiv 0 \equiv H$ using equations (3.45) and (3.46) with $\rho = -\gamma_1$ and $\sigma = \gamma_2/2$.

The case $N = 1$. In this case

$$G_0 = W^3 \left\{ 2 \left(1 - \frac{\gamma_5}{\gamma_3} W \right) (3 - 2\kappa_\infty) - 1 \right\}$$

and again we assume that $\kappa_\infty \neq 3/2$. Setting $G_0 = 0$ we see that if $\gamma_5 \neq 0$, then $W = (\gamma_3/\gamma_5)(1 - 1/(6 - 4\kappa_\infty))$ and hence we have $T(r, W) = T(r, \gamma_3) + T(r, \gamma_5) + S(r, W) = S(r, W)$, which is a contradiction. It follows that $\gamma_5 = 0$ and $\kappa_\infty = 5/4$. Therefore

$$H_1 = -\frac{W^3}{4} \left(2\frac{\gamma_3'}{\gamma_3} + \mu \right).$$

Setting $H_1 = 0$ gives

$$\gamma_3' + \frac{\mu}{2}\gamma_3 = 0. \tag{3.62}$$

Substituting $W = -\gamma_3/u^2$ in (3.40) and using (3.62) shows that u satisfies

$$4(u')^2 = \frac{\nu}{\gamma_3} u^4 + \left(\gamma_1 + \frac{\mu^2}{4} \right) u^2 - \gamma_3 u + \frac{\gamma_2 \gamma_3}{2}. \tag{3.63}$$

Setting G_1 and H_0 to zero and using Lemma 3.2.11 as described above, we see that there are three possibilities corresponding to $\kappa = 5/4$, namely

- $\alpha = \alpha_0$, $\mu = 0$, $\nu = 4\alpha_0/3$, $\gamma_1 = 0$, $\gamma_2 = 0$. Equation (3.62) shows that γ_3 is a nonzero constant. Then $W = -\gamma_3/u^2$, where u satisfies

$$(u')^2 = \frac{\alpha_0}{3\gamma_3} u^4 - \frac{\gamma_3}{4} u. \tag{3.64}$$

- $\alpha = \alpha_0(z - z_\infty)^{-4}$, $\mu = 4(z - z_\infty)^{-1}$, $\nu = (4\alpha_0/3)(z - z_\infty)^{-4}$, $\gamma_1 = 0$, $\gamma_2 = 0$. $\gamma_3 = C(z - z_\infty)^{-2} \neq 0$. Then

$$W = -C/\{(z - z_\infty)u(z)\}^2,$$

where u satisfies

$$4(u')^2 = (z - z_\infty)^{-2}\left(\frac{4\alpha_0}{3C}u^4 + 4u^2 - Cu\right). \tag{3.65}$$

- $\alpha = \alpha_0([z - z_\infty]^2 - d^2)^{-4}$, $\mu = \frac{8(z-z_\infty)}{(z-z_\infty)^2-d^2}$, $\nu = (4\alpha_0/3)([z - z_\infty]^2 - d^2)^{-4}$, $\gamma_1 = -\frac{16}{(z-z_\infty)^2-d^2}$, $\gamma_2 = 0$. $\gamma_3 = C\{(z - z_\infty)^2 - d^2\}^{-2} \neq 0$. Then

$$W = -C/\{[(z - z_\infty)^2 - d^2]u(z)\}^2,$$

where u satisfies

$$4(u')^2 = ([z - z_\infty]^2 - d^2)^{-2}\left(\frac{4\alpha_0}{3C}u^4 + 16d^2u^2 - Cu\right). \tag{3.66}$$

<u>The case $N > 1$.</u> On setting $G_0 = 0$ and considering the residue of the apparent pole at $W = \widetilde{a}_2$ we see that $N = 2$ and $\gamma_3 = \widetilde{a}_2\gamma_5$. G_0 then takes the form

$$G_0 = W^3\left\{2\left(1 - \widetilde{a}_2^{-1}W\right)(3 - 2\kappa_\infty) - (4\kappa_2 + 1)\right\},$$

giving $\kappa_\infty = 3/2$, $\kappa_2 = -1/4$ and therefore $\kappa_1 = 7/4$. It follows from Lemma 3.2.11 that $\mu = 0$ and therefore

$$H_1 = \frac{-2\kappa_2\widetilde{a}_2}{\gamma_3}\frac{(\gamma_5'W - \gamma_3')W^3}{W - \widetilde{a}_2},$$

so γ_3 and γ_5 are non-zero constants. From Lemma 3.2.11 with $N = 2$ and

$\kappa_\infty = 3/2$ we see that $\nu = 4\alpha_0/5$, $\gamma_1 = 4\alpha_0/(15\widetilde{a}_2)$ and $\gamma_2 = -16\alpha_0/(15\widetilde{a}_2^2)$. Hence the conclusion in this case is that $W = \gamma_3/u^2$, where γ_3 is a non-zero constant and u is a solution of

$$4(u')^2 = \frac{4\alpha_0}{5\gamma_3}u^4 + \frac{4\alpha_0}{15\widetilde{a}_2}u^2 - \gamma_3 u - \frac{8\gamma_3\alpha_0}{15\widetilde{a}_2^2}. \tag{3.67}$$

Now if $G_0(z, W) + uG_1(z, W) \neq 0$, then (3.61) becomes

$$u' = \frac{H_0(z, W) + uH_1(z, W)}{G_0(z, W) + uG_1(z, W)}.$$

Using (3.35) we can rewrite the above equation as $u' = R(z, u)$, where $R(z, u)$ is a rational function in u and the coefficients are small compared to u since $T(r, u) = (1/2)T(r, w) + S(r, w)$. If u is an admissible solution of the above equation then by Malmquist-Yosida Theorem 3.1.2 u must be a solution of a Riccati differential equation

$$u' = p_0(z) + p_1(z)u + p_2(z)u^2,$$

where $T(r, p_i) = S(r, w)$ for $i = 1, 2, 3$. Setting $\delta_1 = \gamma_3$, $\delta_2 = \gamma_5$ and $W = \frac{\delta_1}{\delta_2 - u^2}$, then the conclusion in this case corresponds to **the part** *(xii)* **of Theorem 3.2.1**.

(III) **The case $\kappa_j = 3/4$ for all $j = 1, \dots, N$ and $N = 1$ or 2**

This case is the conclusion of part *(iii)* of Lemma 3.2.9, which corresponds to the case in which $\alpha = \alpha_0$ is a constant and w is an admissible meromorphic solution of the equations of the form (3.36) for $N = 1$ and (3.37) for $N = 2$.

For $N = 2$, setting $W = w - a_1$, $\widetilde{a} = a_2 - a_1$, $\beta = 2\alpha_0/\widetilde{a}^2$ and $\gamma = d_1$, equation

(3.37) becomes

$$\left\{ \frac{(W')^2}{W(W-\widetilde{a})} - \beta\,(2W-\widetilde{a}) \right\}^2 = \gamma^2 W(W-\widetilde{a}),$$

which is equivalent to

$$\left\{ \left(\frac{W'}{W}\right)^2 \frac{1}{(W-\widetilde{a})} - \beta\left(2-\frac{\widetilde{a}}{W}\right) \right\}^2 = \gamma^2 \frac{(W-\widetilde{a})}{W}, \tag{3.68}$$

If $\gamma = 0$, then W satisfies

$$W'^2 = \beta W(W-\widetilde{a})\,(2W-\widetilde{a})\,. \tag{3.69}$$

If $\gamma \neq 0$, then let

$$\gamma\ u = \left(\frac{W'}{W}\right)^2 \frac{1}{(W-\widetilde{a})} - \beta\left(2-\frac{\widetilde{a}}{W}\right). \tag{3.70}$$

Then equation (3.68) becomes $u^2 = (W-\widetilde{a})/W$ and hence

$$W = \frac{\widetilde{a}}{1-u^2}, \qquad 2uu' = \widetilde{a}\ \frac{W'}{W^2}. \tag{3.71}$$

Equation (3.71) implies that

$$\frac{W'}{W} = \frac{2uu'}{1-u^2}. \tag{3.72}$$

Furthermore, using equations (3.70) and (3.71) we obtain

$$\begin{aligned} \left(\frac{W'}{W}\right)^2 &= \frac{(W-\widetilde{a})}{W}\Big[\gamma\ u\ W + \beta\,(2W-\widetilde{a})\Big] \\ &= \frac{\widetilde{a}\ u^2}{1-u^2}\Big[\gamma\ u + \beta\left(1+u^2\right)\Big]. \end{aligned} \tag{3.73}$$

Therefore, using equations (3.72) and (3.73) we get

$$4u'^2 = \widetilde{a}\ (1-u^2)\Big[\gamma\ u + \beta\left(1+u^2\right)\Big]. \tag{3.74}$$

(IV) **Admissible meromorphic solution**

In all cases covered by parts **(I)**, **(II)** and **(III)** the conclusion is that the solutions of equation (3.14) can be written in terms of an admissible meromorphic solution of either a Riccati equation or a first-order differential equation of the form $u'^2 = Q(z,u)$, where Q is a polynomial of degree at most four and the coefficients are not necessarily constants. In what follows we will determine whether the solutions so obtained really are meromorphic and admissible.

We first consider the first-order differential equations (3.54-3.60) corresponding to each of the cases in part **(I)**.

The case $N = 1$

Looking back at the equation (3.54), we see that if $(\mu_0^2/4) - \rho_0 = 0$, this gives $(v')^2 = -2\alpha_0 v$, so the solution v is a polynomial of degree 2. If $(\mu_0^2/4) - \rho_0 \neq 0$, then the solution v of (3.54) is a trigonometric function. When $\mu_0 \neq 0$, we have

$$\alpha = \alpha_0 \exp(-\mu_0 z/2)$$

and $T(r,W) = T(r,\alpha) + S(r,W) = S(r,W)$, which is a contradiction. Therefore, $\mu_0 = 0$, so $\alpha = \alpha_0$ is a constant and $W = v$ is admissible meromorphic solution of (3.54), which corresponds to **part *(ii)* of Theorem 3.2.1**. This case was solved by Halburd and Wang [12].

For the equation (3.55) the solution W is a polynomial of degree 2. Since $\alpha = \alpha_0$ is a constant, this gives $W = w - a_1$ is an admissible meromorphic solution of (3.55), which corresponds to **part***(i)* **of Theorem 3.2.1**.

For the equation (3.56), we have $\alpha = \alpha_0(z - z_\infty)^{-(4\kappa-3)/(2[\kappa-1])}$. In this instance equation (3.56) can be written as

$$\frac{dv}{\sqrt{\Big((Av - B)^2 - B^2\Big)}} = (z - z_\infty)^{-1}dz,$$

where $A = 1/(2(\kappa - 1))$ and $B = (2(k-1)\alpha_0)/(2k-1)$. Therefore, the solution of the above equation is given by

$$\begin{aligned} v &= \frac{4(\kappa-1)^2\alpha_0}{2\kappa-1}\{\cosh\log\left[d_1(z-z_\infty)^{1/(2[\kappa-1])}\right]+1\}, \\ &= \frac{4(\kappa-1)^2\alpha_0}{2\kappa-1}\Big\{\frac{(d_1(z-z_\infty)^{1/(2[\kappa-1])})+(d_1(z-z_\infty)^{1/(2[\kappa-1])})^{-1}}{2}+1\Big\}, \\ &= \frac{2(\kappa-1)^2\alpha_0}{d_1(2\kappa-1)}(z-z_\infty)^{-1/(2[\kappa-1])}(d_1(z-z_\infty)^{1/(2[\kappa-1])}+1)^2, \end{aligned}$$

where $d_1 \neq 0$ is an arbitrary constant. Furthermore, the function $\alpha(z)$ is meromorphic only if $\kappa = (n+1)/n$ and $|n|$ is an even number excluding the values $n = -4$ or $n = -2$ since $\kappa \neq 3/4$ or $1/2$, respectively. Hence, the solution w is given by

$$\begin{aligned} w &= a_1 + (z-z_\infty)^{-1/(2[\kappa-1])}v \\ &= a_1 + \frac{2(\kappa-1)^2\alpha_0}{d_1(2\kappa-1)}(z-z_\infty)^{-1/([\kappa-1])}(d_1(z-z_\infty)^{1/(2[\kappa-1])}+1)^2 \\ &= a_1 + \frac{2\alpha_0}{n(n+2)d_1}(z-z_\infty)^{-n}(d_1(z-z_\infty)^{(n/2)}+1)^2, \end{aligned} \tag{3.75}$$

which is a meromorphic function but it is not admissible since $T(r,w) = O(T(r,\alpha))$.

Recall that in equation (3.57) we have $\alpha(z) = \alpha_0([z-z_\infty]^2 - d^2)^{-(4\kappa-3)/(2[\kappa-1])}$. In a similar manner to the previous case the solution of equation (3.57) is given by

$$
\begin{aligned}
v &= \frac{\alpha_0(\kappa-1)^2}{d^2(1-2\kappa)}\left\{1+\cosh\log e_1\left(\frac{[z-z_\infty]^2-d^2}{[z-z_\infty]^2+d^2}\right)^{1/(2[\kappa-1])}\right\} \\
&= \frac{\alpha_0(\kappa-1)^2}{2e_2(1-2\kappa)}\left(\frac{[z-z_\infty]^2-d^2}{[z-z_\infty]^2+d^2}\right)^{-1/(2[\kappa-1])}\left(e_1\left(\frac{[z-z_\infty]^2-d^2}{[z-z_\infty]^2+d^2}\right)^{1/(2[\kappa-1])}+1\right)^2
\end{aligned}
$$

where $e_2 = d^2 e_1$ and e_1 is a constant. Furthermore, α is meromorphic function only if $\kappa = (n+1)/n$ and $|n|$ is an even number excluding the values $n=-4$ or $n=-2$. Therefore, the solution w is given by

$$
\begin{aligned}
w &= a_1 + \left\{(z-z_\infty)^2-d^2\right\}^{-1/(2[\kappa-1])} v \\
&= a_1 - \frac{\alpha_0}{2e_2 n(n+2)}\frac{\left([z-z_\infty]^2+d^2\right)^{(n/2)}}{\left([z-z_\infty]^2-d^2\right)^n}\left(e_1\left(\frac{[z-z_\infty]^2-d^2}{[z-z_\infty]^2+d^2}\right)^{(n/2)}+1\right)^2,
\end{aligned}
$$

which is not admissible solution.

The case $N > 1$

In this case we have the first-order differential equations (3.58-3.60) with constant coefficients and $\alpha = \alpha_0$ is a constant. Since the degree of W is two in equation (3.58), then this equation can be solved in terms of an exponential or trigonometric function, so the solution W is admissible. This gives **part** *(vii)* **of Theorem 3.2.1**. As for the equations (3.59) and (3.60), they can be solved in terms of elliptic functions or one of their degenerations, so the solution W is admissible, which corresponds to **parts** *(ix)* **and** *(xi)* **of Theorem 3.2.1**, respectively.

Next we consider the first-order equations (3.64-3.66) for $N = 1$ and (3.67) for $N > 1$ corresponding to each of the cases in part **(II)**.

The case $N = 1$

Notice that equations (3.64), (3.65) and (3.66) can all be solved in terms of elliptic functions or their degenerations. In the case of equation (3.64) the coefficients are constant and $\alpha = \alpha_0$ is a constant, so that $w = a_1 - d_1/u^2$, where $d_1 = \gamma_3$ is an admissible meromorphic solution which corresponds to **part** *(iii)* **of Theorem 3.2.1**. In the case of equations (3.65) and (3.66) however, a change of independent variable is required first to make the equation constant coefficient. In the case of equation (3.65) a suitable new independent variable is $Z = \log(z - z_\infty)$, in terms of which (3.65) can be written as

$$4\Big(\frac{\mathrm{d}u}{\mathrm{d}Z}\Big)^2 = \frac{4\alpha_0}{3C} u(Z)^4 + 4u(Z)^2 - Cu(Z).$$

For equation (3.66) a suitable variable is $Z = (1/2d) \log \frac{([z-z_\infty]/d)-1}{([z-z_\infty]/d+1)}$ if $d \neq 0$ and $Z = 1/(z - z_\infty)$ if $d = 0$ imply that (3.66) can be written as

$$4\Big(\frac{\mathrm{d}u}{\mathrm{d}Z}\Big)^2 = \frac{4\alpha_0}{3C} u^4 + 16d^2u^2 - Cu.$$

The solution of the above equation is an elliptic function or one of their degenerations. Unfortunately in these cases w fails to be a meromorphic function because of the transformations, so these solutions do not appear in the final theorem. It is important to observe however that these solutions fail to be meromorphic only at fixed singularities of the solutions (because of the forms of α).

The case $N > 1$

In this case $N = 2$ and we have the first-order differential equation (3.67) with constant coefficients and $\alpha = \alpha_0$ is a constant. This equation can be solved in terms of elliptic functions or their degenerations, so $w = a_1 + d_1/u^2$, where $d_1 = \gamma_3$ is a non-zero constant, is an admissible meromorphic solution, which corresponds to **part** *(x)* **of Theorem 3.2.1**.

Recall that in part **(III)** we have for $N = 1$ the first-order differential equation (3.36) and $\alpha = \alpha_0$ is a constant. Taking $u = W^{1/2}$, then this equation becomes

$$u'^2 = d_1 u - \alpha_0. \tag{3.76}$$

where $d_1 = -d/(4)$ is a constant. If $d_1 = 0$, then $u'^2 = -\alpha_0$. If $d_1 \neq 0$, then integrating the above equation we obtain

$$u = -\frac{1}{d_1}\Big(\frac{d_1^2\alpha_0}{4}(z+d_2)^2 + 1\Big).$$

In both cases $w = a_1 + u^2$ is a polynomial which is an admissible solution of equation (3.76), which corresponds to **the part** *(iv)* **of Theorem 3.2.1**. Furthermore, when $N = 2$ we have the equations (3.69) if $\gamma = 0$ and (3.74) if $\gamma \neq 0$, which can be solved in terms of elliptic functions or their degenerations. Since $\alpha = \alpha_0$ and the coefficients of (3.69) and (3.74) are constants, then w is admissible meromorphic solution, which corresponds to **the part** *(viii)* **of Theorem 3.2.1**.

(V) **The case $N = 1$ and $\kappa = 3/2$ or $\kappa = 2$**

Now, we consider the cases $\kappa = 3/2$ or 2 for $N = 1$, described in part *(iv)* and part *(v)* of Lemma 3.2.9, in which W is an admissible solution of equation

$$WW'' - \kappa W'^2 = \alpha(z)W \tag{3.77}$$

where $W = w - a_1$. In these cases we use the resonance conditions (as in Painlevé analysis) to obtain necessary conditions on the form of equation (3.77). Let $z_0 \in \Omega$ be a double zero of an admissible non-zero meromorphic solution

W of the equation (3.77) which has series expansion of the form

$$W(z) = a_0\,\zeta^2 + \cdots + a_j\,\zeta^{j+2} + \ldots, \qquad \zeta = z - z_0. \tag{3.78}$$

Substituting the expansion (3.78) into (3.77) we obtain $a_0 = \frac{\alpha(z_0)}{2(1-2\kappa)}$ and a recurrence relation of the form

$$(j+1)(j+2-4\kappa)a_0 a_j = P_j(a_0,\ a_1, \ldots,\ a_{j-1}, \alpha^{(j)}),$$

where P_j is rational function in its arguments. The left side of the above equation vanishes when $j = -1$ or $j = 4\kappa - 2$. When $\kappa = 3/2$ there is a positive integer resonance at $j = 4$ and when $\kappa = 2$ there is a positive integer resonance at $j = 6$.

For the values $\kappa = 3/2$ or 2, we proceed by obtaining the resonance conditions which are in terms of the coefficient α and its derivatives to identify admissible meromorphic solutions of the equation (3.77).

Lemma 3.2.12 *Let w be a meromorphic solution of equation (3.77) satisfying (3.15) for the values $\kappa = 3/2$ or 2. Then the coefficient α satisfies the differential equations (3.19) or (3.20), respectively.*

Proof. Consider the case $\kappa = 3/2$. Then the resonance at $j = 4$ implies that the coefficient a_4 is free, provided the resonance condition is satisfied. Substituting the expansion (3.78) into (3.77), and then equating coefficients of like powers of ζ, we get $a_0 = -\frac{\alpha(z_0)}{4}$, $a_1 = -\frac{\alpha'(z_0)}{6}$, $a_2 = \frac{-\alpha'^2(z_0)+3\alpha(z_0)\alpha''(z_0)}{36\alpha(z_0)}$, $a_3 = -\frac{4\alpha'^3(z_0)-6\alpha(z_0)\alpha'(z_0)\alpha''(z_0)+3\alpha^2(z_0)\alpha^{(3)}(z_0)}{72\alpha^2(z_0)}$ and

$$\begin{aligned} 0 \times a_4 &= \frac{3}{2}\frac{a_1^4}{a_0^3} + \frac{15a_1^2 a_2}{2a_0^2} - \frac{6a_2^2}{a_0(z_0)} - \frac{1}{24}\alpha^{(4)}(z_0) \\ &= -\frac{14\alpha'^4(z_0)}{27\alpha^3(z_0)} - \frac{8\alpha'^2(z_0)\alpha''(z_0)}{9\alpha^2(z_0)} + \frac{\alpha''^2(z_0)}{6\alpha(z_0)} + \frac{\alpha'(z_0)\alpha^{(3)}(z_0)}{4\alpha(z_0)} - \frac{1}{24}\alpha^{(4)}(z_0) \end{aligned}$$

Since the right hand side of the above equation vanishes at all the a_i-points of w in Ω and $T(r, \alpha) = S(r, w)$, so by Lemma 3.2.7 the condition (3.19) is verified. Similarly, we obtain the conditions (3.20) for $\kappa = 2$. □

The results of **the parts** *(v)* **and** *(vi)* **of Theorem 3.2.1** follow immediately from Lemma 3.2.12.

The proof of Theorem 3.2.1 is completed. □

In what follow we will show that in the cases $\kappa = 3/2$ or 2, the existence of an admissible meromorphic solution implies that equation (3.14) with $N = 1$ can be transformed to a special case of the second or first Painlevé equation or their autonomous version, respectively, by using the conditions (3.19) or (3.20).

Theorem 3.2.13 *Equation (3.77) with $\kappa = 3/2$ can be transformed to the second Painlevé equation if the coefficient α satisfies the condition (3.19).*

Proof. Define a function $Z(z)$ by the equation

$$\frac{\mathrm{d}Z}{\mathrm{d}z} = V(Z)^2,$$

where $\alpha(z) = -4V(Z)^6$. Note that Z and V are not necessarily meromorphic. The condition (3.19) becomes

$$\left(\frac{V_{ZZ}}{V}\right)_{ZZ} = 0,$$

which is equivalent to

$$V_{ZZ} = (aZ + b)V, \tag{3.79}$$

where a and b are arbitrary constants. Now define $U(Z)$ such that $W(z) = V(Z)^2/U(Z)^2$. Then W solves equation (3.77), $\kappa = 3/2$ subject to (3.19) if and only if U satisfies

$$U_{ZZ} = 2U^3 + (aZ + b)U. \tag{3.80}$$

Note that if $a \neq 0$ then by rescaling variables equation (3.79) and equation (3.80) can be transformed to the Airy equation $V_{ZZ} = ZV$ and a special case of the second Painlevé equation $U_{ZZ} = 2U^3 + ZU$. If $a = 0$ then equation (3.79) is a constant coefficient equation solved by exponentials or an affine function of Z and equation (3.80) is solved by elliptic functions. If $a = b = 0$, then solving the equation (3.79) gives $V = c_1 Z + c_2$ and equation (3.80) can be solved in terms of elliptic functions.

Remark 3.2.14 *Notice that from the proof of Theorem 3.2.13, if $a = b = 0$ then from equation (3.80) we have $U_{ZZ} = 2U^3$, so either $U = \pm 1/(z - C)$ for some constant C or U is an elliptic function and $\alpha(z) = -4(c_1 Z + c_2)^6$. Furthermore, we have*

$$\frac{\mathrm{d}Z}{\mathrm{d}z} = (c_1 Z + c_2)^2.$$

If $c_1 = 0$, then $\alpha = \alpha_0$ is a constant while if $c_1 \neq 0$, then solving the above equation gives $Z = (1 + c_1 c_2 z)/(c_1^2 z)$, so $\alpha(z)$ is a rational function. Therefore, in the case $\alpha = \alpha_0$ and U is an elliptic function the solution $w = a_1 + V(Z)^2/U(Z)^2$ is an admissible meromorphic solution of (3.77).

Theorem 3.2.15 *Equation (3.77) with $\kappa = 2$ can be transformed to the first Painlevé equation (2.13) if the coefficient α satisfies the condition (3.20).*

Proof. In this case equation (3.77) with $\kappa = 2$ can be written as

$$\left(\frac{W'}{W^2}\right)' = \frac{\alpha}{W^2}. \tag{3.81}$$

Let $u = 1/W$, then the above equation becomes

$$u'' = -\alpha u^2. \tag{3.82}$$

The above equation has been considered previously by Wyman [51] and Halburd [10].

Since the coefficient α satisfies condition (3.20), then the general solution u is given in terms of solutions of the first Painlevé equation or its autonomous versions. □

Remark 3.2.16 *If $\alpha = \alpha_0$ is a non-zero constant, then equation (3.82) can be integrated to*

$$u'^2 = -\frac{2}{3}\alpha_0 u^3 + c_1,$$

which has elliptic function solutions if $c_1 \neq 0$. Therefore, $w = a_1 + 1/u$ is an admissible meromorphic solution of (3.77).

The results obtained in this section were proved by using singularity structure with the concepts of Nevanlinna theory. We extended the approach of Halburd and Wang [12] to characterise all admissible meromorphic solutions of equation (3.14). However, we discarded many nice solutions because these solutions grow like the coefficients e.g. the solution (3.75) or they are not meromorphic e.g. (3.65). Indeed, many solutions of the Hayman equation (3.13) have the form $\int q(z)e^z dz$, where $q(z)$ is a rational function, were also discarded in [12] since they are branched at fixed singularities when $q(z)$ is not polynomial.

Motivated by this, in the next chapter these techniques will be extended to allow solutions to be branched around fixed singularities.

Chapter 4

Application of Nevanlinna theory in a sectorial domain to differential equations

In section 4.1 we present a brief introduction of Tsuji's approach to the value distribution theory of meromorphic functions in the half-plane [8]. We introduce Levin's formula which is used to derive analogues of the counting, proximity and characteristic functions in the half plane. Some of their properties are presented as well as some analogous results of Nevanlinna theory for the half plane, e.g. the first main theorem and the lemma on the logarithmic derivative. In section 4.2 we define the characteristic function of a sectorial domain by the use of value distribution theory in the half-plane as well as analogous results of these in the half plane to the extent that we will need it for applications to complex differential equations in section 4.3. We also present a proof of Malmquist-Yosida and Wittich's theorems in the sectorial domain and employ the technique presented in the previous chapter to find all solutions of a differential equation for which all movable singularities are poles, i.e. the solutions can be branched at fixed singularities.

4.1 Nevanlinna theory in the half-plane

Nevanlinna theory in the half-plane has two independent origins due to Nevanlinna [33] and Tsuji [45]. In this section we will follow the second approach since the central result: the lemma on the logarithmic derivative, is not true in general in the first approach. In fact, Nevanlinna [33] stated a conjecture regarding this lemma in the context of the first approach but Goldberg [7] later disproved it by a counterexample. A concrete treatment and development of Tsuji's approach has been offered by Levin, Ostrovskii and Goldberg (see, e.g., [8], [26], and [27]). The basic definitions and relevant proof of the results presented in this section can be found in [8]. For convenience, we follow Tsuji's approach to study meromorphic functions in the right half-plane instead of the upper half-plane which was presented in [8].

Nevanlinna theory in the right half-plane is based on Levin's formula [48], given in Theorem 4.1.1 below.

Theorem 4.1.1 *Let f be a meromorphic function in the closed half plane $\operatorname{Re} z \geq 0$. Then for $0 < R_0 < R$, we have*

$$\sum_m \Big(\frac{\cos\alpha_m}{r_m} - \frac{1}{R}\Big) - \sum_n \Big(\frac{\cos\beta_n}{\rho_n} - \frac{1}{R}\Big) = \frac{1}{2\pi}\int_{-\arccos(R_0R^{-1})}^{\arccos(R_0R^{-1})} \log|f(R\cos\theta e^{i\theta})| \frac{d\theta}{R\cos^2\theta} + O(1), \quad (4.1)$$

where $r_m e^{i\alpha_m}$ are zeros and $\rho_n e^{i\beta_n}$ are poles of the function $f(z)$ in the domain $D = \{|z - \frac{R}{2}| < \frac{R}{2},\ |z| > R_0\}$, listed according to multiplicity.

Remark 4.1.2 *Levin's formula was first proved by Levin [26] (see also [8]) and an alternative, stronger proof was given by Wang [48]. Indeed, Wang assumes that f is meromorphic in the half plane $\operatorname{Re} z > 0$, while Levin requires $f(z)$ to be meromorphic in the disc $\{|z| < R_0\}$ as well as the half plane since his proof is based on Green's* *formula. Wang's proof on the other hand relies on Cauchy's integral formula applied to the function* $\log f(z)/z^2$ *on the contour*

$$C = \{z : |z - \frac{R}{2}| = \frac{R}{2},\ |z| \geq R_0\} \cup \left\{z : |z| = R_0,\ -\arccos\left(\frac{R_0}{R}\right) < \arg z < \arccos\left(\frac{R_0}{R}\right)\right\},$$

where f *is meromorphic in the half plane* $\operatorname{Re} z > 0$*. He proved that the integral along the arc*

$$\left\{z : |z| = R_0,\ -\arccos\left(\frac{R_0}{R}\right) < \arg z < \arccos\left(\frac{R_0}{R}\right)\right\}$$

is bounded but this is not necessarily true if we have an accumulation point on the imaginary line $\operatorname{Re} z = 0$*. However, we shall assume that* f *is meromorphic in the closed half plane* $\operatorname{Re} z \geq 0$*.*

Let $f(z)$ be a non-constant meromorphic function in the closed half-plane $H = \{z : \operatorname{Re} z \geq 0\}$. We introduce several real-valued functions which characterise the behaviour of $f(z)$ in H. The number of poles of $f(z)$ in $D = \{|z - \frac{r}{2}| \leq \frac{r}{2},\ |z| > 1\}$ will be denoted by $\mathfrak{n}(r, f)$ where each pole is counted according to its multiplicity. The counting function in the right half-plane is defined by

$$\mathfrak{N}(r, f) = \int_1^r \frac{\mathfrak{n}(t, f)}{t^2} dt, \qquad 1 < r < \infty.$$

Let $z_n = \rho_n e^{i\theta_n}$ denote the poles of the function $f(z)$ in H. It is to be noted that the function $\mathfrak{n}(t, f)$ is a step function with steps at the points $t = \rho_n / \cos\theta_n$ and the values of the jumps are equal to the number of poles of $f(z)$ lying on the circular arc

$$\left\{\left|z - \frac{t}{2}\right| = \frac{t}{2},\ |z| > 1\right\}.$$

Thus, we have

$$\begin{aligned}\mathfrak{N}(r,f) &= \int_1^r \frac{\mathfrak{n}(t,f)}{t^2}dt\\ &= \int_1^{\rho_1/\cos\theta_1} \frac{0}{t^2}dt + \int_{\rho_1/\cos\theta_1}^{\rho_2/\cos\theta_2} \frac{1}{t^2}dt + \int_{\rho_2/\cos\theta_2}^{\rho_3/\cos\theta_3} \frac{2}{t^2}dt + \cdots + \int_{\rho_n/\cos\theta_n}^{r} \frac{n}{t^2}dt\\ &= \sum_{1<\rho_n<r\cos\theta} \Big(\frac{\cos\theta_n}{\rho_n} - \frac{1}{r}\Big).\end{aligned}$$

Hence, Levin's formula given by (4.1) can be written in the form

$$\mathfrak{N}\Big(r,\frac{1}{f}\Big) - \mathfrak{N}(r,f) = \frac{1}{2\pi}\int_{-\kappa(r)}^{\kappa(r)} \log|f(r\cos\theta e^{i\theta})| \,\frac{d\theta}{r\cos^2\theta} + O(1), \tag{4.2}$$

where $R_0 = 1$ and $\kappa(r) = \arccos(1/r)$.

The analogue of the proximity function is given by

$$\mathfrak{m}(r,f) = \frac{1}{2\pi}\int_{-\kappa(r)}^{\kappa(r)} \log^+|f(r\cos\theta e^{i\theta})| \,\frac{d\theta}{r\cos^2\theta},$$

and the Tsuji characteristic function is defined by

$$\mathfrak{T}(r,f) = \mathfrak{m}(r,f) + \mathfrak{N}(r,f).$$

For any point $a \in \mathbb{C}$, the functions $\mathfrak{m}(r,1/(f-a))$, $\mathfrak{n}(r,1/(f-a))$, $\mathfrak{N}(r,1/(f-a))$ and $\mathfrak{T}(r,1/(f-a))$ will be denoted by $\mathfrak{m}(r,a)$, $\mathfrak{n}(r,a)$, $\mathfrak{N}(r,a)$ and $\mathfrak{T}(r,a)$, respectively.

The analogue of the first main theorem follows from equation (4.2).

Theorem 4.1.3 *Let $f(z)$ be a non-constant meromorphic function in the right half-plane, H. Then for $a \neq \infty$*

$$\mathfrak{T}(r,a) = \mathfrak{T}(r,f) + O(1), \qquad 1 < r < \infty. \tag{4.3}$$

Proof. Applying Levin's formula in the form (4.2) to $f(z)-a$, we get

$$\mathfrak{N}\Big(r,\frac{1}{f-a}\Big)-\mathfrak{N}(r,f-a)=\frac{1}{2\pi}\int_{-\kappa(r)}^{\kappa(r)}\log|f(r\cos\theta e^{i\theta})-a|\,\frac{d\theta}{r\cos^2\theta}+O(1). \quad (4.4)$$

Using the equality $\log|x|=\log^+|x|-\log^+|1/x|$ yields

$$\frac{1}{2\pi}\int_{-\kappa(r)}^{\kappa(r)}\log|f(r\cos\theta e^{i\theta})|\,\frac{d\theta}{r\cos^2\theta}=\mathfrak{m}(r,f-a)-\mathfrak{m}\Big(r,\frac{1}{f-a}\Big)$$

and hence relation (4.4) can be rewritten in the form

$$\mathfrak{m}\Big(r,\frac{1}{f-a}\Big)+\mathfrak{N}\Big(r,\frac{1}{f-a}\Big)=\mathfrak{m}(r,f)+\mathfrak{N}(r,f)+\mathfrak{m}(r,f-a)-\mathfrak{m}(r,f)+O(1). \quad (4.5)$$

Using the inequality [8, p. 14]

$$|\log^+|x_1|-\log^+|x_2||\leq\log^+|x_1-x_2|+\log 2,$$

we obtain

$$|\mathfrak{m}(r,f-a)-\mathfrak{m}(r,f)|\leq\frac{(\log^+|a|+\log 2)}{2\pi}\int_{-\kappa(r)}^{\kappa(r)}\frac{d\theta}{r\cos^2\theta}=(\log^+|a|+\log 2)\frac{\sqrt{r^2-1}}{\pi r}. \quad (4.6)$$

Hence, using equations (4.5) and (4.6) we get (4.3) with

$$|\mathfrak{m}(r,f-a)-\mathfrak{m}(r,f)|\leq\frac{(\log^+|a|+\log 2)}{\pi}=O(1)$$

as $r\to\infty$. □

We proceed with some basic, elementary relations.

Proposition 4.1.4 *Let* $f_1,\ f_2,\ldots,\ f_q$ *be meromorphic functions in the half-plane*

H. Then

$$\begin{aligned}
\mathfrak{N}\Big(r, \prod_{i=1}^{q} f_i\Big) &\leq \sum_{i=1}^{q} \mathfrak{N}(r, f_i),\\
\mathfrak{N}\Big(r, \sum_{i=1}^{q} f_i\Big) &\leq \sum_{i=1}^{q} \mathfrak{N}(r, f_i),\\
\mathfrak{m}\Big(r, \prod_{i=1}^{q} f_i\Big) &\leq \sum_{i=1}^{q} \mathfrak{m}(r, f_i),\\
\mathfrak{m}\Big(r, \sum_{i=1}^{q} f_i\Big) &\leq \sum_{i=1}^{q} \mathfrak{m}(r, f_i) + O(1),\\
\mathfrak{T}\Big(r, \prod_{i=1}^{q} f_i\Big) &\leq \sum_{i=1}^{q} \mathfrak{T}(r, f_i),\\
\mathfrak{T}\Big(r, \sum_{i=1}^{q} f_i\Big) &\leq \sum_{i=1}^{n} \mathfrak{T}(r, f_i) + O(1),\\
\mathfrak{T}(r, f^m) &= m\mathfrak{T}(r, f), \quad m \in \mathbb{N}.
\end{aligned}$$

Remark 4.1.5 *Goldberg and Ostrovskii [8] proved that there exists a continuous non-decreasing function* $\mathring{\mathfrak{T}}(r, f)$ *such that*

$$\mathfrak{T}(r, f) = \mathring{\mathfrak{T}}(r, f) + O(1), \tag{4.7}$$

where

$$\mathring{\mathfrak{T}}(r, f) = \frac{1}{\pi}\int\int_{|z-\frac{r}{2}|\leq\frac{r}{2},\ |z|\geq 1} \Big(\frac{\cos\theta}{t} - \frac{1}{r}\Big)\Big(\frac{|f'(te^{i\theta})|}{1+|f(te^{i\theta}|^2)}\Big)^2 t\ dt\ d\theta.$$

This function plays a pivotal role in the proof of the lemma on the logarithmic derivative.

Definition 4.1.1 Let f be a meromorphic function in H. The order of function f is given by

$$\mathfrak{L}(f) = \overline{\lim_{r\to\infty}} \frac{\log \mathring{\mathfrak{T}}(r, f)}{\log r}.$$

Definition 4.1.2 If f and g are meromorphic functions in H, then we say g is small compared to f if

$$\mathring{\mathfrak{T}}(r, g) = o(\mathring{\mathfrak{T}}(r, f)) \text{ as } r \to \infty,$$

outside of a possible exceptional set of finite linear measure, and we use the notation

$$\mathring{\mathfrak{T}}(r, g) = \mathfrak{Q}(r, f).$$

We get the following analogue of the Valiron Mohon'ko theorem.

Theorem 4.1.6 *If $R(z, f)$ is a rational function in f and of degree d with meromorphic coefficients $a_i(z)$ in H such that $\mathfrak{T}(r, a_i) = \mathfrak{Q}(r, f)$, and if f is a meromorphic function in H, then the characteristic function of $R(z, f)$ satisfies*

$$\mathfrak{T}(r, R(z, f)) = d\mathfrak{T}(r, f) + \mathfrak{Q}(r, f).$$

The proof of Theorem 4.1.6 can be deduced by similar arguments to those used in the proof of the Valiron Mohon'ko theorem in the complex plane in [24, p. 29].

We now turn to analogues of the lemma on the logarithmic derivative for meromorphic functions in H.

Theorem 4.1.7 *Let f be a meromorphic function in H. Then*

$$\mathfrak{m}\left(r, \frac{f'}{f}\right) = O(\log r)$$

if f is of finite order and

$$\mathfrak{m}\left(r, \frac{f'}{f}\right) = \mathfrak{Q}(r, f) \tag{4.8}$$

if f is of infinite order.

We need the following lemmas to prove Theorem 4.1.7.

Lemma 4.1.8 *[8, p. 7] Let f(z) be a non-identically zero meromorphic function in the disc* $\{|z|\leq R\}$*. Then*

$$\frac{f'(z)}{f(z)} = \int_0^{2\pi} \frac{2Re^{i\theta}}{(Re^{i\theta}-z)^2} \log|f(Re^{i\theta})| \frac{d\theta}{2\pi} + \sum_{|a_m|<R} \left(\frac{\bar{a}_m}{R^2 - \bar{a}_m z} + \frac{1}{z-a_m} \right) - \sum_{|b_n|<R} \left(\frac{\bar{b}_n}{R^2 - \bar{b}_n z} + \frac{1}{z-b_n} \right), \tag{4.9}$$

where $a_1, \ldots, a_m$ *the zeros of* $f(z)$ *and* $b_1, \ldots, b_m$ *are the poles.*

Lemma 4.1.9 *[8, p. 87] Let* $f(x)$ *and* $g(x)$ *be non-negative measurable functions on* $[a,b]$*, and let*

$$A = \int_a^b g(x)dx > 0.$$

Then

$$\frac{1}{A}\int_a^b \{\log^+ f(x)\}\ g(x)\ dx \leq \log^+ \left\{ \frac{1}{A}\int_a^b f(x)g(x)dx \right\} + \log 2.$$

Lemma 4.1.10 *[8, p. 90] Let* $u(r)$ *be a continuous, non-decreasing function on* $[r_0, \infty]$*, tending to* $+\infty$ *as* $r \to \infty$*. Let* $\psi(u)$ *be a continuous positive non-increasing function on* $[u_0, \infty)$*,* $u_0 = u(r_0)$*, having zero limit as* $u \to \infty$ *and satisfying*

$$\int_{u_0}^{\infty} \psi(u)du < \infty.$$

Then for all $r \geq r_0$ *except, possibly, a set of finite measure,*

$$u\{r + \psi(u(r)\} < u(r) + 1.$$

Proof of Theorem 4.1.7 First we use Lemma 4.1.8 for any z in the disc $|z|< \frac{s}{2}$

and the function $f(z+s/2)$, and then setting $\zeta = z + s/2$ we get

$$\begin{aligned}\frac{f'(\zeta)}{f(\zeta)} &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(\frac{s}{2}e^{i\varphi}+\frac{s}{2}\right)\right| \frac{s\ e^{i\varphi}}{\left(\frac{s}{2}e^{i\varphi}-\zeta+\frac{s}{2}\right)^2}\, d\varphi \\ &+ \sum_{|a_m-\frac{s}{2}|<\frac{s}{2}} \frac{\frac{s^2}{4}-|a_m-\frac{s}{2}|^2}{\left(\frac{s^2}{4}-\overline{[a_m-\frac{s}{2}]}[\zeta-\frac{s}{2}]\right)(\zeta-a_n)} \\ &- \sum_{|b_n-\frac{s}{2}|<\frac{s}{2}} \frac{\frac{s^2}{4}-|b_n-\frac{s}{2}|^2}{\left(\frac{s^2}{4}-\overline{[b_n-\frac{s}{2}]}[\zeta-\frac{s}{2}]\right)(\zeta-b_n)}, \qquad \left|\zeta-\frac{s}{2}\right|<\frac{s}{2},\end{aligned} \tag{4.10}$$

where a_m are the zeros of $f(z)$ and b_n are its poles. In what follows, let $\zeta = r\cos\theta e^{i\theta}$, $-\kappa(r) \le \theta \le \kappa(r)$, $\kappa(r) = \arccos\frac{1}{r}$ and $s = \frac{1}{2}(R+r)$ where $2 \le r < R < \infty$. These assumptions imply that

$$r^2\cos^2\theta\Big(1-\frac{s}{r}\Big) < \Big(1-\frac{s}{r}\Big),$$

and hence, we get the inequality

$$\left|\zeta-\frac{s}{2}\right| \le \left|r\cos\kappa(r)e^{i\kappa(r)}-\frac{s}{2}\right| = \left|e^{i\kappa(r)}-\frac{s}{2}\right| = \sqrt{1-\frac{s}{r}+\frac{s^2}{4}}.$$

Thus we have

$$\frac{s}{2}-\left|\zeta-\frac{s}{2}\right| > \frac{s}{2}-\sqrt{1-\frac{s}{r}+\frac{s^2}{4}} > \frac{(s-r)}{sr}. \tag{4.11}$$

Writing $\{c_q\} = \{a_m\} \cup \{b_n\}$, then inequality (4.11) implies

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_0^{2\pi} \log\left|f\Big(\frac{s}{2}e^{i\varphi}+\frac{s}{2}\Big)\right| \frac{s\ e^{i\varphi}}{\left(\frac{s}{2}e^{i\varphi}-\zeta+\frac{s}{2}\right)^2}\,d\varphi\right| \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \left|\log\left|f\Big(\frac{s}{2}e^{i\varphi}+\frac{s}{2}\Big)\right|\right| \left|\frac{s\ e^{i\varphi}}{\left(\frac{s}{2}e^{i\varphi}-\zeta+\frac{s}{2}\right)^2}\right| d\varphi \\
&\le \frac{1}{2\pi}\int_0^{2\pi} \left|\log\left|f\Big(\frac{s}{2}e^{i\varphi}+\frac{s}{2}\Big)\right|\right| \frac{s}{\left(\frac{s}{2}-\left|\zeta-\frac{s}{2}\right|\right)^2}\,d\varphi \\
&\le \frac{s^3\ r^2}{(s-r)^2}\frac{1}{2\pi}\int_0^{2\pi}\left|\log|f(\frac{s}{2}e^{i\varphi}+\frac{s}{2})|\right|d\varphi \\
&=: \Phi_1(\zeta)
\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}
&\left|\sum_{|c_m-\frac{s}{2}|<\frac{s}{2}} \frac{\frac{s^2}{4}-|c_m-\frac{s}{2}|^2}{\left(\frac{s^2}{4}-\overline{[c_m-\frac{s}{2}]}[\zeta-\frac{s}{2}]\right)(\zeta-c_n)}\right| \\
&\le \sum_{|c_m-\frac{s}{2}|<\frac{s}{2}} \frac{s^2/4}{\left|\left(\frac{s^2}{4}-\overline{[c_m-\frac{s}{2}]}[\zeta-\frac{s}{2}]\right)(\zeta-c_n)\right|} \\
&\le \sum_{|c_m-\frac{s}{2}|<\frac{s}{2}} \frac{s^2/4}{\left(\frac{s^2}{4}-|c_m-\frac{s}{2}||\zeta-\frac{s}{2}|\right)|\zeta-c_n|} \\
&\le \sum_{|c_m-\frac{s}{2}|<\frac{s}{2}} \frac{s^2/4}{\frac{s}{2}\left(\frac{s}{2}-|\zeta-\frac{s}{2}|\right)(\zeta-c_n)} \\
&\le \frac{s^2 r}{s-r}\sum_{|c_q-\frac{s}{2}|<\frac{s}{2}} \frac{1}{|\zeta-c_q|} \\
&=: \Phi_2(\zeta).
\end{aligned} \tag{4.13}$$

Using (4.12) and (4.13), then (4.10) implies

$$\left|\frac{f'(\zeta)}{f(\zeta)}\right| = \left|\frac{f'(r\cos\theta e^{i\theta})}{f(r\cos\theta e^{i\theta})}\right| \le \Phi_1(\zeta)+\Phi_2(\zeta). \tag{4.14}$$

Therefore

$$\begin{aligned}\mathfrak{m}\Big(r,\frac{f'}{f}\Big) &\leq \frac{1}{2\pi}\int_{-\kappa(r)}^{\kappa(r)} \log^+[\Phi_1(\zeta)+\Phi_2(\zeta)]\frac{d\theta}{r\cos^2\theta}\\
&\leq \frac{1}{2\pi}\int_{-\kappa(r)}^{\kappa(r)} \log^+\Phi_1(\zeta)\frac{d\theta}{r\cos^2\theta}+\frac{1}{2\pi}\int_{-\kappa(r)}^{\kappa(r)} \log^+\Phi_2(\zeta)\frac{d\theta}{r\cos^2\theta}\\
&+\frac{\log 2}{2\pi}\int_{-\kappa(r)}^{\kappa(r)}\frac{d\theta}{r\cos^2\theta}\\
&= I_1+I_2+\frac{\log 2}{\pi r}\tan\kappa(r)\\
&= I_1+I_2+\frac{\log 2}{\pi}\sqrt{1-\frac{1}{r^2}},\end{aligned}$$

where

$$I_1=\frac{1}{2\pi}\int_{-\kappa(r)}^{\kappa(r)} \log^+\Phi_1(\zeta)\frac{d\theta}{r\cos^2\theta},$$

and

$$I_2=\frac{1}{2\pi}\int_{-\kappa(r)}^{\kappa(r)} \log^+\Phi_2(\zeta)\frac{d\theta}{r\cos^2\theta}.$$

Now we estimate the integrals I_1 and I_2 in the above inequality. By K we denote positive constants independent of r and R. For Φ_1 we take $[-\pi,\pi]$ as the interval of integration instead of $[0,2\pi]$ and let $\varphi=2\tau$. Thus we have

$$\begin{aligned}\Phi_1(\zeta) &= \frac{s^3\ r^2}{(s-r)^2}\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big|\log|f(\frac{s}{2}e^{i2\tau}+\frac{s}{2})|\Big|d\tau\\
&= \frac{s^3\ r^2}{(s-r)^2}\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big|\log|f(s(\cos^2\tau+i\cos\tau\sin\tau))|\Big|d\tau\\
&= \frac{s^3\ r^2}{(s-r)^2}\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big|\log|f(s\cos\tau e^{i\tau})|\Big|d\tau\\
&= \frac{s^3\ r^2}{(s-r)^2}\Bigg\{\frac{1}{\pi}\int_{-\kappa(s)}^{\kappa(s)} s\ \cos^2\tau\ \Big|\log|f(s\cos\tau e^{i\tau})|\Big|\frac{d\tau}{s\ \cos^2\tau}\\
&\qquad +\frac{1}{\pi}\Bigg[\int_{-\frac{\pi}{2}}^{-\kappa(s)}+\int_{\kappa(s)}^{\frac{\pi}{2}}\Bigg]\Big|\log|f(s\cos\tau e^{i\tau})|\Big|d\tau\Bigg\}\\
&\leq \frac{s^3\ r^2}{(s-r)^2}\Big\{\Big[2\ s\Big(\mathfrak{m}(s,0)+\mathfrak{m}(s,\infty)\Big)\Big]+K\Big\}.\end{aligned}$$

Using Theorem 4.1.3 and relation (4.7), we obtain

$$\Phi_1(\zeta) \leq K \frac{R^6}{(R-r)^2} [\, \mathring{\mathfrak{T}}(R,f) + 1 \,],$$

and thus we have

$$I_1 \leq K \Big\{ \log^+ \mathring{\mathfrak{T}}(R,f) + \log^+ \frac{1}{R-r} + \log R + 1 \Big\}. \tag{4.15}$$

Following a similar procedure for I_2 we have

$$\begin{aligned} I_2 &= \frac{1}{2\pi} \int_{-\kappa(r)}^{\kappa(r)} \log^+ \Phi_2(\zeta) \frac{d\theta}{r\cos^2\theta} \\ &= \frac{1}{2\pi} \int_{-\kappa(r)}^{\kappa(r)} \log^+ \left(\frac{s^2 r}{s-r} \sum_{|c_q - \frac{s}{2}| < \frac{s}{2}} \frac{1}{|\zeta - c_q|} \right) \frac{d\theta}{r\cos^2\theta} \\ &\leq \frac{1}{2\pi} \int_{-\kappa(r)}^{\kappa(r)} \log^+ \left(\frac{R^3}{R-r} \sum_{|c_q - \frac{s}{2}| < \frac{s}{2}} \frac{1}{|\zeta - c_q|} \right) \frac{d\theta}{r\cos^2\theta} \\ &\leq \bar{I}_2 + K \Big(\log^+ \frac{1}{R-r} + \log R \Big), \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} \bar{I}_2 &= \frac{1}{2\pi} \int_{-\kappa(r)}^{\kappa(r)} \log^+ \left(\sum_{|c_q - \frac{s}{2}| < \frac{s}{2}} \frac{1}{|\zeta - c_q|} \right) \frac{d\theta}{r\cos^2\theta} \\ &\leq \frac{2}{\pi} \int_{-\kappa(r)}^{\kappa(r)} \log^+ \left(\sum_{|c_q - \frac{s}{2}| < \frac{s}{2}} \frac{1}{|\zeta - c_q|^{\frac{1}{4}}} \right) \frac{d\theta}{r\cos^2\theta}. \end{aligned}$$

Applying Lemma 4.1.9 for the non-negative function $f(\theta) = \sum_{|c_q - \frac{s}{2}| < \frac{s}{2}} \frac{1}{|\zeta - c_q|^{\frac{1}{4}}}$ and $g(\theta) = \frac{2}{\pi r \cos^2\theta}$ on $[-\kappa(r), \kappa(r)]$ with $A = \int_{-\kappa(r)}^{\kappa(r)} g(\theta) d\theta = \frac{4\tan\kappa(r)}{\pi r}$, we obtain

$$\bar{I}_2 \leq \frac{4\tan\kappa}{\pi r} \int_{-\kappa(r)}^{\kappa(r)} \frac{\pi r}{2\tan\kappa(r)} \log^+ \left(\sum_{|c_q - \frac{s}{2}| < \frac{s}{2}} \frac{1}{|\zeta - c_q|^{\frac{1}{4}}} \right) \frac{1}{\pi r \cos^2\theta} d\theta$$

$$
\begin{aligned}
&\leq \frac{4\tan\kappa}{\pi r}\left\{\log^+\left[\frac{r}{2\tan\kappa(r)}\int_{-\kappa(r)}^{\kappa(r)}\left(\sum_{|c_q-\frac{s}{2}|<\frac{s}{2}}\frac{1}{|\zeta-c_q|^{\frac{1}{4}}}\right)\frac{d\theta}{r\cos^2\theta}\right]+\log 2\right\}\\
&\leq \frac{4\tan\kappa}{\pi r}\left\{\log^+\left[\int_{-\kappa(r)}^{\kappa(r)}\left(\sum_{|c_q-\frac{s}{2}|<\frac{s}{2}}\frac{1}{|\zeta-c_q|^{\frac{1}{4}}}\right)\frac{d\theta}{r\cos^2\theta}\right]+\log^+\frac{r}{2\tan\kappa(r)}+\log 2\right\}\\
&\leq K\left\{\log^+\left[\int_{-\kappa(r)}^{\kappa(r)}\left(\sum_{|c_q-\frac{s}{2}|<\frac{s}{2}}\frac{1}{|\zeta-c_q|^{\frac{1}{4}}}\right)\frac{d\theta}{r\cos^2\theta}\right]+1\right\}\\
&\leq K\left\{\log^+\left[r\sum_{|c_q-\frac{s}{2}|<\frac{s}{2}}\int_{-\kappa(r)}^{\kappa(r)}\frac{1}{|\zeta-c_q|^{\frac{1}{4}}}\,d\theta\right]+1\right\}\\
&\leq K\left\{\log r+\log^+\left[\sum_{|c_q-\frac{s}{2}|<\frac{s}{2}}\int_{-\kappa(r)}^{\kappa(r)}\frac{1}{|\zeta-c_q|^{\frac{1}{4}}}\,d\theta\right]+1\right\}.
\end{aligned}
$$

We now estimate the integral in the square bracket above. If $\psi_q = \arg c_q$, then

$$
\begin{aligned}
\int_{-\kappa(r)}^{\kappa(r)}\frac{1}{|\zeta-c_q|^{\frac{1}{4}}}d\theta &= \int_{-\kappa(r)}^{\kappa(r)}\frac{1}{\left|r\cos\theta e^{i(\theta-\psi_q)}-|c_q|\right|^{\frac{1}{4}}}\,d\theta\\
&\leq \int_{-\kappa(r)}^{\kappa(r)}\frac{1}{\left|r\cos\theta\sin(\theta-\psi_q)\right|^{\frac{1}{4}}}\,d\theta\\
&\leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\left|r\cos\theta\sin(\theta-\psi_q)\right|^{\frac{1}{4}}}\,d\theta\\
&\leq \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\left|\cos\theta\right|^{\frac{1}{2}}}\,d\theta\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{\left|\sin(\theta-\psi_q)\right|^{\frac{1}{2}}}\,d\theta\right)^{1/2}\\
&\leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta}{(\cos\theta)^{\frac{1}{2}}}.
\end{aligned}
$$

Using the above we have

$$
\begin{aligned}
\bar{I}_2 &\leq K\left\{\log r+\log^+\sum_{|c_q-\frac{s}{2}|<\frac{s}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\theta}{(\cos\theta)^{\frac{1}{2}}}+1\right\}\\
&\leq K\left\{\log r+\log^+\sum_{|c_q-\frac{s}{2}|<\frac{s}{2}}1+1\right\}\\
&= K\left\{\log r+\log^+\Big(\mathfrak{n}(s,0)+\mathfrak{n}(s,f)\Big)+1\right\}. \qquad (4.17)
\end{aligned}
$$

Since $s < R$, then

$$\begin{aligned}
\mathfrak{N}(R,a) &= \int_1^R \frac{\mathfrak{n}(t,a)}{t^2}dt \\
&= \int_1^s \frac{\mathfrak{n}(t,a)}{t^2}dt + \int_s^R \frac{\mathfrak{n}(t,a)}{t^2}dt \\
&\geq \int_s^R \frac{\mathfrak{n}(t,a)}{t^2}dt \\
&\geq \mathfrak{n}(s,a)\int_s^R \frac{1}{t^2}dt \\
&= \frac{R-s}{R^2}\mathfrak{n}(s,a).
\end{aligned}$$

Thus

$$\mathfrak{n}(s,a) \leq \frac{R^2}{R-s}\mathfrak{N}(R,a). \tag{4.18}$$

In view of the first main theorem and relations (4.7) and (4.18), we have

$$\begin{aligned}
\mathfrak{n}(s,0) + \mathfrak{n}(s,f) &\leq \frac{R^2}{R-s}\Big\{\mathfrak{N}(R,0) + \mathfrak{N}(R,f)\Big\} \\
&\leq \frac{R^2}{R-s}\ \Big(2\ \mathfrak{T}(R,f) + O(1)\Big) \\
&= \frac{R^2}{R-s}\ \Big(2\ \mathring{\mathfrak{T}}(R,f) + O(1)\Big) \\
&\leq K\frac{R^2}{R-r}\ \Big(2\ \mathring{\mathfrak{T}}(R,f) + O(1)\Big).
\end{aligned}$$

From relations (4.16) and (4.17) with the above , we have

$$I_2 \leq K\Big\{\ \log^+ \mathring{\mathfrak{T}}(R,f) + \log^+ \frac{1}{R-r} + \log R + 1\Big\}. \tag{4.19}$$

Now, (4.15) and (4.19) yield

$$\mathfrak{m}\Big(r, \frac{f'}{f}\Big) \leq K\Big(\ \log^+ \mathring{\mathfrak{T}}(R,f) + \log^+ \frac{1}{R-r} + \log R + 1\Big). \tag{4.20}$$

To complete the proof we distinguish two cases.

Case (i). If $\mathring{\mathfrak{T}}(r,f)$ is of finite order, then $\log \mathring{\mathfrak{T}}(r,f) = O(\log r)$ and taking $R = 2r$ in (4.20) we obtain

$$\mathfrak{m}\Big(r, \frac{f'}{f}\Big) \leq K\Big(\log 2r + \log^{+}\frac{1}{r} + \log 2r + 1\Big) = O(\log r).$$

Case (ii). If $\mathring{\mathfrak{T}}(r,f)$ is of infinite order, then taking $R = r + \{\mathring{\mathfrak{T}}(r,f)\}^{-2} \leq 2r$ in (4.20), we get, by Lemma 4.1.10, that the inequality

$$\mathring{\mathfrak{T}}(R,f) = \mathring{\mathfrak{T}}(r + \{\mathring{\mathfrak{T}}(r,f)\}^{-2}, f) < \mathring{\mathfrak{T}}(r,f) + 1$$

holds everywhere except, possibly, a set of finite measure. From (4.20) and the above inequality, (4.21) is satisfied. □

Theorem 4.1.7 and the elementary properties of the proximity function lead immediately to the following corollary.

Corollary 4.1.11 *Let $f(z)$ be a meromorphic function in H and let $k \geq 1$ be an integer. Then*

$$\mathfrak{m}\Big(r, \frac{f^{(k)}}{f}\Big) = O(\log r)$$

if f is of finite order and

$$\mathfrak{m}\Big(r, \frac{f^{(k)}}{f}\Big) = \mathfrak{Q}(r,f) \tag{4.21}$$

if f is of infinite order.

Many analogous results of Nevanlinna theory, e.g. Clunie's theorem [39] is valid also for the version of this theory in the half plane (see also [8], and [53]). Tsuji's approach has been modified for sectors by [39], [40], [41] and [48].

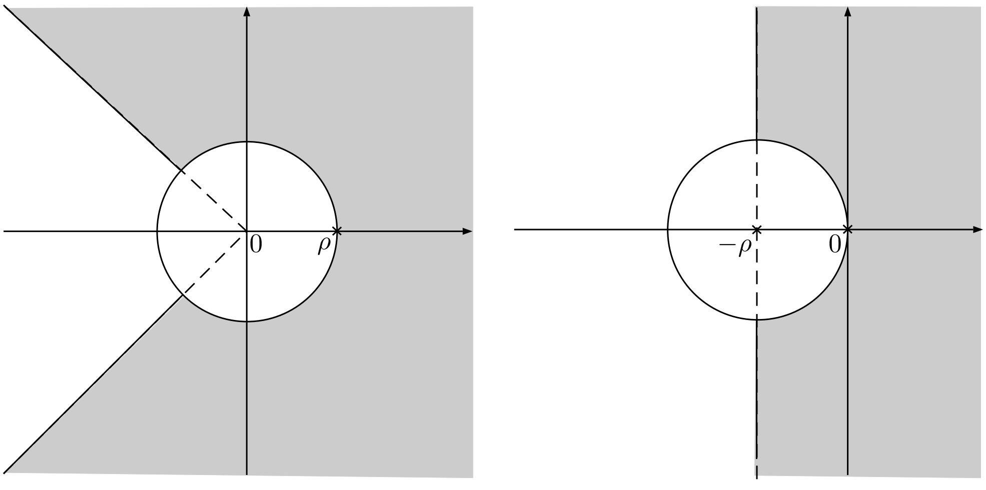
4.2 Characteristic function of a sectorial domain

In this section, we will set up an analogous machinery of Nevanlinna tools in a sectorial domain. We are essentially motivated by properties of a differential equation, extending the technique that was used in section 3.2 to allow solutions to be branched at fixed singularities, i.e. we want to develop similar tools as introduced in Nevanlinna theory when we have a solution which is branched at a finite number of values. The main idea here is to take all these values inside a disc, D, of fixed radius, and then investigate solutions in a domain outside D which can be branched in this domain. Hence, this domain would be modified to the universal cover, which leads us to consider a sectorial domain Ω^* outside the disc D on the universal covering of $\mathbb{C}\backslash\{0\}$ (see Figure 4.1a). Thereafter we use a transformation which maps Ω^* to a domain that contains the closed real half plane H (see Figure 4.1b). Then we use this transformation along with Nevanlinna theory in the half plane to obtain analogous tools of Nevanlinna theory in a sectorial domain contained in Ω^*. In that instance, we will apply a similar method that we used in section 3.2 with some modification so as to allow for branching at fixed singularities.

For given real numbers $\mu > 0$ and $0 < \rho < 1$, let $f(z)$ be a meromorphic function in the sector

$$\Omega^*(\mu, \rho) = \{z : \ |\arg z| < \mu, |z| \geq \rho\}$$

on the universal covering of $\mathbb{C}\backslash\{0\}$, which is the image of a domain containing the closed real half plane H under the conformal mapping $z = (\zeta + \rho^\lambda)^{\frac{1}{\lambda}}$, where $\lambda = \pi/2\mu$ and we take a suitable branch cut in the left half-plane. Under this transformation



(a) (b)

Figure 4.1

we also consider the image of $\{\zeta : \text{Re}\,\zeta \geq 0,\ |\zeta| > 1\}$ as

$$\Omega(\lambda, \rho) = \{z : z = (\zeta + \rho^{\lambda})^{\frac{1}{\lambda}},\ \text{Re}\,\zeta \geq 0,\ |\zeta| > 1\}$$

on the universal covering of $\mathbb{C}\backslash\{0\}$. Note that for each $\varepsilon > 0$, there exists a number $\rho_\varepsilon > 1$ satisfying

$$\{z :\ |\arg z| < \mu - \epsilon, |z| > \rho_\epsilon\} \subset \Omega(\lambda, \rho) \subset \{z :\ |\arg z| < \mu, |z| \geq \rho\}.$$

Let $f(z)$ be a meromorphic function in $\Omega^*(\mu, \rho)$. Then let $F_\lambda(\zeta) = f((\zeta + \rho^\lambda)^{\frac{1}{\lambda}})$ represents $f(z)$ in $\Omega(\lambda, \rho)$. Note that since the function $F_\lambda(\zeta)$ is meromorphic in H, then it is sufficient to define the Tsuji functions for meromorphic functions in $\Omega(\lambda, \rho)$ by employing value distribution theory in the half plane. Furthermore, since the function F_λ is meromorphic at $\zeta = 0$, then the lemma on the logarithmic derivative can be deduced for any meromorphic function f in $\Omega(\lambda, \rho)$ by way of the transformed function F_λ. Broadly speaking, we can derive (by means of this transformation) a version of Nevanlinna theory for functions meromorphic in $\Omega(\lambda, \rho)$.

We are motivated by the wish to study solutions of a differential equation that are branched at fixed singularities. Thus we take all these singularities in a disc of radius ρ centred at the origin, and then the solutions whose only movable singularities are poles will be meromorphic in $\Omega(\lambda,\rho)$. One could use a shifted version of the half plane to be away from the disc but for our purposes that is not good enough. For example, if the solution is an exponential function, then the characteristic function of the solution does not grow fast enough to get a useful estimate for the lemma on the logarithmic derivative. Indeed, the proximity function of the logarithmic derivative of the exponential function has the same order of as the growth of the function itself in the half plane. Note that the sectorial domain $\Omega(\lambda,\rho)$ has the advantage that there is no restriction imposed on the opening angle μ which implies that we can have an extra growth for functions as the exponential function at least.

Consider the domain

$$\Omega(\lambda,\rho,r)=\{z:z=(\zeta+\rho^{\lambda})^{\frac{1}{\lambda}},\ |\zeta-\frac{r^{\lambda}}{2}|\leq\frac{r^{\lambda}}{2},\ |\zeta|>1\},$$

which is the image of $\{|\zeta-r^{\lambda}/2|\leq r^{\lambda}/2,\ |\zeta|>1\}$ under the transformation $z=(\zeta+\rho^{\lambda})^{\frac{1}{\lambda}}$ and contained in $\Omega(\lambda,\rho)$. Taking these facts into consideration we introduce, for $r>1$, the sectorial counting function in $\Omega(\lambda,\rho)$ given by

$$\mathfrak{N}_{\lambda}(r,f)=\mathfrak{N}(r^{\lambda},F_{\lambda})=\int_{1}^{r^{\lambda}}\frac{\mathfrak{n}(t,F_{\lambda})}{t^{2}}dt=\lambda\int_{1}^{r}\frac{\mathfrak{n}_{\lambda}(t,f)}{t^{1+\lambda}}dt,$$

where $\mathfrak{n}_{\lambda}(t,f)$ denotes the number of poles of $f(z)$ in $\Omega(\lambda,\rho,t)$, each counted according to its multiplicity. Furthermore, we define the sectorial proximity function

$$\begin{aligned}\mathfrak{m}_{\lambda}(r,f)&=\mathfrak{m}(r^{\lambda},F_{\lambda})\\&=\frac{1}{2\pi}\int_{-\kappa(r^{\lambda})}^{\kappa(r^{\lambda})}\log^{+}|F_{\lambda}(r^{\lambda}\cos\theta e^{i\theta})|\ \frac{d\theta}{r^{\lambda}\cos^{2}\theta}\\&=\frac{1}{2\pi}\int_{-\kappa(r^{\lambda})}^{\kappa(r^{\lambda})}\log^{+}\left|f\Big((r^{\lambda}\cos\theta e^{i\theta}+\rho^{\lambda})^{\frac{1}{\lambda}}\Big)\right|\ \frac{d\theta}{r^{\lambda}\cos^{2}\theta},\end{aligned}$$

and the sectorial characteristic function

$$\mathfrak{T}_\lambda(r,f) = \mathfrak{T}(r^\lambda, F_\lambda) = \mathfrak{m}_\lambda(r,f) + \mathfrak{N}_\lambda(r,f).$$

Remark 4.2.1 *Shimomura defined a sectorial domain type in [39]. Indeed, our domain $\Omega(\lambda,\rho)$ is similar to that of Shimomura since there is no restriction imposed on the opening angle of these sectorial domains.*

Proposition 4.2.2 *Let $f_1,\ f_2,\dots,\ f_q$ be meromorphic functions in $\Omega(\lambda,\rho)$. Then*

$$\begin{aligned} D\Bigg(r,\sum_{i=1}^{q} f_i\Bigg) &\le \sum_{i=1}^{q} D(r,f_i) + O(1),\\ D\Bigg(r,\prod_{i=1}^{q} f_i\Bigg) &\le \sum_{i=1}^{q} D(r,f_i),\\ D(r,f^m) &= mD(r,f), \quad m\in\mathbb{N}, \end{aligned}$$

where $D(r,f)$ represents the sectorial functions $\mathfrak{m}_\lambda(r,f)$, $\mathfrak{N}_\lambda(r,f)$ and $\mathfrak{T}_\lambda(r,f)$.

Similarly, by (4.7) and the function $F_\lambda(z)$, the sectorial Tsuji characteristic $\mathfrak{T}_\lambda(r,f)$ differs from a non-decreasing continuous function $\mathring{\mathfrak{T}}_\lambda(r,f)$ by a bounded additive term

$$\mathfrak{T}_\lambda(r,f) = \mathring{\mathfrak{T}}_\lambda(r,f) + O(1).$$

With the customary convention for the meaning of the symbols $\mathfrak{m}_\lambda(t,a)$, $\mathfrak{N}_\lambda(t,a)$, $\mathfrak{T}_\lambda(t,a)$, $\mathfrak{L}_\lambda(f)$, and $\mathfrak{Q}_\lambda$, the following results on the sectorial domain $\Omega(\lambda,\rho)$ can be deduced from section 4.1.

For a non-constant meromorphic function f in $\Omega(\lambda,\rho)$ and any $a\neq\infty$ the first fundamental theorem is given by

$$\mathfrak{T}_\lambda(r,a) = \mathfrak{T}_\lambda(r,f) + O(1), \qquad 1 < r < \infty. \tag{4.22}$$

We can also obtain a Valiron Mohon'ko type result for the sectorial characteristic.

Theorem 4.2.3 *Let f be a meromorphic function in $\Omega(\lambda,\rho)$ and $R(z,f)$ be a rational function in f of order d with meromorphic coefficients $a_i(z)$ such that $\mathfrak{T}_\lambda(r,a_i) = \mathfrak{Q}_\lambda(r,f)$. Then*

$$\mathfrak{T}_\lambda(r,R(z,f)) = d\mathfrak{T}_\lambda(r,f) + \mathfrak{Q}_\lambda(r,f).$$

Example 4.2.1

For $f(z) = z^n$, we have

$$\begin{aligned}
\mathfrak{m}_\lambda(r,z^n) &= \frac{1}{2\pi}\int_{-\kappa(r^\lambda)}^{\kappa(r^\lambda)} \log^+|f(r^\lambda\cos\theta e^{i\theta} + \rho^\lambda)| \, \frac{d\theta}{r^\lambda\cos^2\theta} \\
&= \frac{1}{2\pi}\int_{-\kappa(r^\lambda)}^{\kappa(r^\lambda)} \log^+|(r^\lambda\cos\theta e^{i\theta} + \rho^\lambda)^{\frac{n}{\lambda}}| \, \frac{d\theta}{r^\lambda\cos^2\theta} \\
&= \frac{1}{2\pi}\int_{-\kappa(r^\lambda)}^{\kappa(r^\lambda)} \log^+\left(r^n|\cos\theta|^{\frac{n}{\lambda}}\left(\left(1+\frac{\rho^\lambda}{r^\lambda}\right)^2 + \frac{\rho^{2\lambda}}{r^{2\lambda}}\tan^2\theta\right)^{\frac{n}{2\lambda}}\right) \frac{d\theta}{r^\lambda\cos^2\theta}.
\end{aligned}$$

Setting $\eta = (1+\frac{\rho^\lambda}{r^\lambda}) - i\frac{\rho^\lambda}{r^\lambda}\tan\theta$ and noting that since $|\theta| < \kappa(r^\lambda)$, it follows that

$$|\cos\theta| > \frac{1}{r^\lambda}, \quad \text{and} \quad |\tan\theta| < r^\lambda\sqrt{1-\frac{1}{r^{2\lambda}}}.$$

Thus

$$\begin{aligned}
1 < \left(1+\frac{\rho^\lambda}{r^\lambda}\right)^{\frac{n}{\lambda}} < r^n|\cos\theta|^{\frac{n}{\lambda}}\,|\eta|^{\frac{n}{\lambda}} &< r^n\left(\left(1+\frac{\rho^\lambda}{r^\lambda}\right)^2 + \rho^{2\lambda}\left(1-\frac{1}{r^{2\lambda}}\right)\right)^{\frac{n}{2\lambda}} \\
&= r^n\left(1+2\frac{\rho^\lambda}{r^\lambda} + \rho^{2\lambda}\right)^{\frac{n}{2\lambda}} \\
&< r^n\left(1+\rho^{2\lambda}\right)^{\frac{n}{2\lambda}} < r^n,
\end{aligned} \tag{4.23}$$

whence

$$\mathfrak{m}_\lambda(r,z^n)=\frac{n}{2\pi}\int_{-\kappa(r^\lambda)}^{\kappa(r^\lambda)}\log\left(r|\cos\theta|^{\frac{1}{\lambda}}\left(\left(1+\frac{\rho^\lambda}{r^\lambda}\right)^2+\frac{\rho^{2\lambda}}{r^{2\lambda}}\tan^2\theta\right)^{\frac{1}{2\lambda}}\right)\frac{d\theta}{r^\lambda\cos^2\theta}$$

and

$$0<\frac{n}{\pi\lambda}\sqrt{1-\frac{1}{r^{2\lambda}}}\,\log\left(1+\frac{\rho^\lambda}{r^\lambda}\right)<\mathfrak{m}_\lambda(r,z^n)<\frac{n}{\pi}\sqrt{1-\frac{1}{r^{2\lambda}}}\,\log r.$$

Therefore, $\mathfrak{m}_\lambda(r,z^n)=O(\log r)$. Since f has no poles in $\Omega(\lambda,\rho,r)$, then $\mathfrak{T}_\lambda(r,f)=O(\log r)$. However, using (4.23) implies that $1/(r^n|\cos\theta|^{\frac{n}{\lambda}}\,|\eta|^{\frac{n}{\lambda}})<1$, we have $\mathfrak{m}_\lambda(r,1/f)=0$. Thus, using (4.22) we obtain $\mathfrak{N}_\lambda(r,1/f)=\mathfrak{T}_\lambda(r,1/f)=\mathfrak{T}_\lambda(r,f)+O(1)=O(\log r)$.

Based on Theorems 4.1.7 and 4.1.11 the lemma on logarithmic derivative for the meromorphic functions in the sectorial domain is given as

Theorem 4.2.4 *Suppose that $f(z)$ is a meromorphic function in $\Omega(\lambda,\rho)$. If $k\geq 1$ is an integer and f is of finite order, then*

$$\mathfrak{m}_\lambda\left(r,\frac{f^{(k)}}{f}\right)=O(\log r),$$

and if f is of infinite order, then

$$\mathfrak{m}_\lambda\left(r,\frac{f^{(k)}}{f}\right)=\mathfrak{Q}_\lambda(r,f).$$

where function meromorphic in $\Omega(\lambda,\rho)$.

Example 4.2.2

In order to estimate $\mathfrak{T}_\lambda(r,e^z)$, we will consider $\mathfrak{T}_\lambda(r,f)$, where $f(z)=1/(e^z-1)$. Note that the poles of f are all simple and occur at the points $z=2k\pi i$, where k is an integer. So $\mathfrak{n}_\lambda(t,f)$ is the number of integers k satisfying $2k\pi i=(\zeta+\rho^\lambda)^{1/\lambda}$ such that $|\zeta|> 1$ and

$$\left|\zeta - \frac{t^\lambda}{2}\right| \leq \frac{t^\lambda}{2}. \tag{4.24}$$

We begin by considering the case $k > 0$. In this case we have

$$\zeta = (2k\pi)^\lambda e^{i\lambda\pi/2} - \rho^\lambda$$

and equation (4.24) becomes

$$\left(\frac{2k\pi}{t}\right)^\lambda \leq \cos\frac{\lambda\pi}{2} + E_k(t), \tag{4.25}$$

where

$$E_k(t) = \rho^\lambda \left(2\cos\frac{\lambda\pi}{2} - \left(\frac{\rho}{2k\pi}\right)^\lambda\right) t^{-\lambda} - \left(\frac{\rho}{2k\pi}\right)^\lambda.$$

Let us estimate the number of integer values of k satisfying (4.25) such that $k > t^\alpha$ for some fixed $\alpha > 0$. First, note that for $k > t^\alpha$,

$$|E_k(t)|\leq \rho^\lambda \left(2\cos\frac{\lambda\pi}{2} + \left(\frac{\rho}{2\pi t^\alpha}\right)^\lambda\right) t^{-\lambda} + \left(\frac{\rho}{2\pi t^\alpha}\right)^\lambda = O\left(t^{-\lambda}\right).$$

Also, the number of integer $k > 0$ satisfying $k \leq t^\alpha$ is clearly at most t^α, therefore, the number of positive integers k satisfying (4.25) is

$$\frac{t}{2\pi}\left(\cos\frac{\lambda\pi}{2} + O\left(t^{-\lambda}\right)\right)^{1/\lambda} + O(t^\alpha) + O(1) = \frac{t}{2\pi}\left(\cos\frac{\lambda\pi}{2}\right)^{1/\lambda} + O(t^{1-\lambda}) + O(t^\alpha).$$

Choosing $\lambda < 1$ and $\alpha \leq 1-\lambda$, this becomes

$$\frac{t}{2\pi}\left(\cos\frac{\lambda\pi}{2}\right)^{1/\lambda} + O\left(t^{1-\lambda}\right).$$

There is only a finite number of k corresponding to the condition $|\zeta|> 1$, so noting that $\mathfrak{n}_\lambda(t,f)$ counts negative as well as positive k, we have

$$\mathfrak{n}_\lambda(t,f) = \frac{t}{\pi}\left(\cos\frac{\lambda\pi}{2}\right)^{1/\lambda} + O\left(t^{1-\lambda}\right).$$

So we have

$$\mathfrak{N}_\lambda(r,f) = \lambda\int_1^r \frac{\mathfrak{n}_\lambda(t,f)}{t^{\lambda+1}}dt = \lambda\int_1^r \left\{\left(\cos\frac{\lambda\pi}{2}\right)^{\frac{1}{\lambda}} + o(1)\right\}\frac{t^{-\lambda}}{\pi}dt = \frac{\lambda}{\pi(1-\lambda)}\cos\frac{\lambda\pi}{2}r^{1-\lambda}+o(r^{1-\lambda}).$$

Also note that $f'/f = \mathrm{e}^z/(\mathrm{e}^z-1) = f+1$, so by the sectorial lemma on the logarithmic derivative we have

$$\mathfrak{m}_\lambda(r,f) = \mathfrak{m}_\lambda\left(r, \frac{f'}{f}-1\right) \le \mathfrak{m}_\lambda\left(r,\frac{f'}{f}\right) + O(1) = O(\log r).$$

Therefore

$$\mathfrak{T}_\lambda(r,\mathrm{e}^z) = \mathfrak{T}_\lambda(r,f) + O(1) = \frac{\lambda}{\pi(1-\lambda)}\cos\frac{\lambda\pi}{2}r^{1-\lambda} + o(r^{1-\lambda}).$$

So when $\lambda < 1$ (i.e., when the sectorial region is wider than a half-plane), the characteristic function of the exponential grows like a power of r and is therefore large compared to the error term in the Lemma on the Logarithmic Derivative.

4.3 Applications of differential equations in the sectorial domain $\Omega(\lambda,\rho)$

The aim of this section is to present a number of applications of differential equations in the sectorial domain $\Omega(\lambda,\rho)$. We will prove Malmquist-Yosida and Wittich's theorems in $\Omega(\lambda,\rho)$. As noted earlier, we are interested in finding all solutions whose movable singularities are poles. We will explain how we use the singularity structure and $\Omega(\lambda, \rho)$ to allow the solutions to be branched at fixed singularities.

For a differential equation in y we set the disc

$$D_\rho = \{z : \quad |z| < \rho\}, \tag{4.26}$$

which contains the zeros and poles of all meromorphic coefficients a_σ in $\Omega(\lambda, \rho)$ satisfying $\mathfrak{T}_\lambda(r, a_\sigma) = O(\log r)$ as $r \to \infty$. If $\rho > 1$, then rescaling z gives radius less than one. For convenience, from now on we take $\rho < 1$. Consider also the domain $\Omega(\lambda, \rho)$ that we defined in section 4.2. In this way, we take all fixed singularities of a differential equation outside the domain $\Omega(\lambda, \rho)$. Indeed, it follows that $\mathfrak{N}_\lambda(r, a_\sigma) = 0$ for all coefficients a_σ, and hence $\mathfrak{T}_\lambda(r, a_\sigma) = \mathfrak{m}_\lambda(r, a_\sigma)$.

Suppose that a meromorphic function f in $\Omega(\lambda, \rho)$ has finite order. If f is a rational function

$$f = K \frac{\prod_{i=0}^{n}(z - a_i)^i}{\prod_{j=0}^{m}(z - b_j)^j},$$

where a_i and b_j are constants, then $\mathfrak{m}_\lambda(r, f'/f)$ is bounded because

$$\frac{f'}{f} = \sum_{i=0}^{n} \frac{1}{z - a_i} - \sum_{j=0}^{m} \frac{1}{z - b_j} \to 0$$

as $z \to \infty$ for any rational function f. Note that by Theorem 4.2.3 and Example 4.2.1 we obtain

$$\mathfrak{T}_\lambda(r, f) = \deg_f(f(z))\mathfrak{T}_\lambda(r, z) + \mathfrak{Q}_\lambda(r, z) = O(\log r).$$

Thus

$$\frac{\mathfrak{m}_\lambda(r, f'/f)}{\mathfrak{T}_\lambda(r, f)} \to 0 \text{ as } \; r \to \infty. \tag{4.27}$$

On the other hand, if f is a non-rational function, then $\mathfrak{m}_\lambda(r, f'/f) = O(\log r)$ by Theoreom 4.2.4 and the characteristic function of f could be close to $\log r$, i.e the growth of f is not big enough to obtain a useful estimate for $\mathfrak{m}_\lambda(r, f'/f)$, which is not sufficient in our applications to differential equation. However, if we assume that

$$\frac{\log r}{\mathfrak{T}_\lambda(r,f)} \to 0 \text{ as } \ r \to \infty, \tag{4.28}$$

then the relation (4.27) is satisfied by Theorem 4.2.4. Broadly speaking, suppose that a meromorphic function f in $\Omega(\lambda, \rho)$ has finite or infinite order. If f is a rational or non-rational function and satisfies (4.28), then $\mathfrak{m}_\lambda(r, f'/f) = o(\mathfrak{T}_\lambda(r, f))$, i.e.

$$\mathfrak{m}_\lambda\Big(r, \frac{f'}{f}\Big) = \mathfrak{Q}_\lambda(r,f). \tag{4.29}$$

We will now proceed to present and prove an analogue of the Malmquist-Yosida theorem in $\Omega(\lambda, \rho)$. The proof of this theorem is somewhat similar to classical case in the complex plane [24] with some modifications.

Theorem 4.3.1 *Let $R(z, y)$ be a rational function in y with coefficients a_σ which are meromorphic in $\Omega(\lambda, \rho)$ and satisfy $\mathfrak{T}_\lambda(r, a_\sigma) = O(\log r)$ as $r \to \infty$. If the differential equation of the form*

$$(y')^n = R(z,y) \tag{4.30}$$

admits a meromorphic solution y in $\Omega(\lambda, \rho)$ such that $\log r/\mathfrak{T}_\lambda(r, y) \to 0$ as $r \to \infty$, then equation (4.30) reduces into

$$(y')^n = \sum_{i=0}^{2n} c_i(z)\ y^i, \tag{4.31}$$

where at least one of the coefficients $c_i(z)$ does not vanish and $\mathfrak{T}_\lambda(r, c_i) = O(\log r)$.

Proof. Let $R(z, y)$ be an irreducible rational function of degree d in y, then equation

(4.30), can be written as

$$(y')^n = R(z,y) = \frac{P(z,y)}{Q(z,y)}, \tag{4.32}$$

where $P(z,y)$ and $Q(z,y)$ are polynomials in y with meromorphic coefficients in $\Omega(\lambda,\rho)$. Since $\log r/\mathfrak{T}_\lambda(r,y) \to 0$, and a_σ is meromorphic function for all σ satisfying $\mathfrak{T}_\lambda(r,a_\sigma) = O(\log r)$, then $\mathfrak{T}_\lambda(r,a_\sigma) = \mathfrak{Q}_\lambda(r,y)$. Taking the sectorial characteristic function of both sides of equation (4.32) and then by Theorem 4.2.3 and the lemma on the logarithmic derivative with (4.29) we have

$$\begin{aligned}
d\mathfrak{T}_\lambda(r,y) + \mathfrak{Q}_\lambda(r,y) &= \mathfrak{T}_\lambda(r,(y')^n)\\
&= n\ \mathfrak{T}_\lambda(r,y')\\
&= n\ \mathfrak{m}_\lambda(r,y') + n\ \mathfrak{N}_\lambda(r,y')\\
&\le n\ \mathfrak{m}_\lambda\Big(r,\frac{y'}{y}\Big) + n\ \mathfrak{m}_\lambda(r,y) + 2n\ \mathfrak{N}_\lambda(r,y)\\
&\le n\ \mathfrak{m}_\lambda(r,y) + 2n\ \mathfrak{N}_\lambda(r,y) + \mathfrak{Q}_\lambda(r,y)\\
&\le 2n\ \mathfrak{T}_\lambda(r,y) + \mathfrak{Q}_\lambda(r,y).
\end{aligned}$$

Therefore , we have $d \le 2n$. Hence both of the polynomials $P(z,y)$ and $Q(z,y)$ have degree less than or equal to $2n$ and these polynomials can be written as

$$P(z,y) = \sum_{i=0}^{p} a_i(z)\ y^i,$$

$$Q(z,y) = \sum_{j=0}^{q} b_j(z)\ y^j,$$

where $d = \deg_y(R(z,y))$ is at most $2n$. Let $\eta \in \mathbb{C}$ such that

$$\begin{cases} \sum_{i=0}^{p} a_i(z)\ \eta^i \neq 0, \\ \sum_{i=0}^{p} b_i(z)\ \eta^i \neq 0. \end{cases} \tag{4.33}$$

Substituting $\widetilde{y} = (y-\eta)^{-1}$ into (4.32), we have

$$(-1)^n(\widetilde{y}')^n = \frac{\widetilde{y}^{2n}\sum_{i=0}^{p} a_i(z)\ (\eta + 1/\widetilde{y})^i}{\sum_{j=0}^{q} b_j(z)\ (\eta + 1/\widetilde{y})^j}. \tag{4.34}$$

Case 1: If $p - 2n \geq q$, then the above equation can be written as

$$\begin{aligned}(-1)^n(\widetilde{y}')^n &= \frac{\sum_{i=0}^{p} a_i(z)\ \widetilde{y}^{p-i}\ (\eta\widetilde{y}+1)^i}{\sum_{j=0}^{q} b_j(z)\ \widetilde{y}^{p-2n-j}\ (\eta\widetilde{y}+1)^j} \\ &= \frac{\widetilde{P}(z,\widetilde{y})}{\widetilde{Q}(z,\widetilde{y})},\end{aligned} \tag{4.35}$$

where $\widetilde{P}(z,\widetilde{y})$ and $\widetilde{Q}(z,\widetilde{y})$ are polynomials in $\widetilde{y}$. Using the first fundamental theorem, we have $\mathfrak{T}_\lambda(r,a_i) = \mathfrak{T}_\lambda(r,b_i) = \mathfrak{Q}_\lambda(r,\widetilde{y})$. Likewise, equation (4.35) has the same general form as equation (4.32), i.e. $\widetilde{R}(z,\widetilde{y}) = \widetilde{P}(z,\widetilde{y})/\widetilde{Q}(z,\widetilde{y})$. By (4.33), we see that the polynomials $\widetilde{P}(z,\widetilde{y})$ and $\widetilde{Q}(z,\widetilde{y})$ are of degree p and $p-2n$ in $\widetilde{y}$, respectively. The rational function $\widetilde{R}(z,\widetilde{y})$ is irreducible, since otherwise $R(z,y)$ would be reducible. Hence we get $\deg_{\widetilde{y}}(\widetilde{R}(z,\widetilde{y})) \leq 2n$ and

$$q + 2n \leq p \leq 2n,$$

hence $q = 0$.

Case 2: If $p - 2n \leq q$, then (4.34) can be written as

$$(-1)^n(\widetilde{y}')^n = \frac{\sum_{i=0}^{p} a_i(z)\ \widetilde{y}^{q+2n-i}\ (\eta\widetilde{y}+1)^i}{\sum_{j=0}^{q} b_j(z)\ \widetilde{y}^{q-j}\ (\eta\widetilde{y}+1)^j}$$

and again we get $q + 2n \leq 2n$, hence $q = 0$.

Thus both cases imply that Q is of degree zero and hence the rational function $R(z, y)$ must be a polynomial of the form (4.31). □

The following theorem is regarded as version in the sectorial domain $\Omega(\lambda, \rho)$ of result due to Wittich for the whole complex plane [50].

Theorem 4.3.2 *Let y be a meromorphic solution in $\Omega(\lambda, \rho)$ of equation*

$$y'' = 6y^2 + f, \tag{4.36}$$

such that $\log r / \mathfrak{T}_\lambda(r, y) \to 0$ as $r \to \infty$ and the coefficient f is meromorphic in $\Omega(\lambda, \rho)$ satisfying $\mathfrak{T}_\lambda(r, f) = O(\log r)$ as $r \to \infty$. Then $\mathfrak{m}_\lambda(r, y) = \mathfrak{Q}_\lambda(r, y)$ and furthermore $f = az + b$, where a and b are constants.

Proof. Equation (4.36) can be written as

$$y^2 = \frac{1}{6}(y'' - f). \tag{4.37}$$

Since $\log r / \mathfrak{T}_\lambda(r, y) \to 0$ and f satisfies $\mathfrak{T}_\lambda(r, f) = O(\log r)$ as $r \to \infty$, then

$$\mathfrak{T}_\lambda(r, f) = \mathfrak{Q}_\lambda(r, y).$$

Taking the sectorial proximity function of both sides of (4.37), and then using the above equation, the lemma on the logarithmic derivative and (4.29) we find that

$$\begin{aligned} 2\mathfrak{m}_\lambda(r, y) = \mathfrak{m}_\lambda(r, y^2) &= \mathfrak{m}_\lambda\Big(r, \frac{1}{6}(y'' - f)\Big) \\ &\leq \mathfrak{m}_\lambda\Big(r, y\frac{y''}{y}\Big) + \mathfrak{m}_\lambda(r, f) + \mathfrak{Q}_\lambda(r, y) \\ &\leq \mathfrak{m}_\lambda(r, y) + \mathfrak{m}_\lambda\Big(r, \frac{y''}{y}\Big) + \mathfrak{Q}_\lambda(r, y) \end{aligned}$$

$$= \mathfrak{m}_\lambda(r, y) + \mathfrak{Q}_\lambda(r, y),$$

so $\mathfrak{m}_\lambda(r, y) = \mathfrak{Q}_\lambda(r, y)$. Now, suppose that the solution has a pole at $z_0 \in \Omega(\lambda, \rho)$. Thus a Laurent series expansion of the solution about the movable singularity $z = z_0$ is necessarily of the form

$$y(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^{k-2}, \qquad a_0 = 1. \tag{4.38}$$

Following the calculation in Example 2.2.1, we obtain the recurrence relation (2.19). Therefore there is a resonance at $k = 6$ and the corresponding resonance condition is $f''(z_0) = 0$. Since $\mathfrak{T}_\lambda(r, f) = \mathfrak{Q}_\lambda(r, y)$ and f vanishes at all poles z_0 of y in $\Omega(\lambda, \rho)$, this implies $f'' \equiv 0$. That is, $f = az + b$, where a and b are constants. As we illustrated in Example 2.2.1, if $a = 0$, then the solution of (4.36) can be written in terms of the Weierstrass elliptic function or its degenerations. Otherwise, a rescaling of y and z along with a translation in z shows that the solution of (4.36) is given in terms of solutions of the first Painlevé equation (2.13). □

Having studied a certain class of second order differential equations in section 3.2 in the complex plane using Nevanlinna theory we now continue with some special cases of this class in $\Omega(\lambda, \rho)$ under some considerations for which we show that the solutions can be branched at fixed singularities.

If we look back at what we did before in section 3.2, we considered equation (3.14) and used Nevanlinna theory to show that a certain rational function, defined in terms of the solution and its first derivative, is small. Basically we used the lemma on the logarithmic derivative and some elementary properties. In the sectorial domain those arguments run essentially the same with some modifications. In addition to Nevanlinna tools, we use series expansion of the solutions around movable singularities, which is exactly the same as before.

Firstly, we explain how we can apply this method in $\Omega(\lambda,\rho)$ without repeating all steps for the differential equation

$$WW'' - \kappa W'^2 = \alpha(z)W, \tag{4.39}$$

for the case $\kappa \neq 3/4,\ 1$ such that $W = w - a_1$ is a meromorphic solution in $\Omega(\lambda,\rho)$ satisfying

$$\log r/\mathfrak{T}_\lambda(r,W) \to 0 \quad \text{as} \quad r \to \infty \tag{4.40}$$

and the coefficient α is meromorphic in $\Omega(\lambda,\rho)$ satisfying

$$\mathfrak{T}_\lambda(r,\alpha) = O(\log r) \quad \text{as} \quad r \to \infty. \tag{4.41}$$

Consider the series expansion of the solution W in (3.22) with $q = 2$. Since $\kappa \neq 3/4$ we construct the function $F = F_i$ in (3.28) as follows

$$F(z) := \left(\frac{W'}{W}\right)^2 + \frac{2}{3-4\kappa}\frac{\alpha'}{\alpha}\frac{W'}{W} - \frac{2\alpha}{1-2\kappa}\frac{1}{W}.$$

Furthermore, $\kappa \neq (n+1)/n$ implies that F is analytic in $\Omega(\lambda,\rho)$, and hence, $\mathfrak{N}_\lambda(r,f) = 0$. Using our assumptions on W in (4.40) with Theorem 4.2.4 and α in (4.41) we obtain $\mathfrak{m}_\lambda(r,W'/W) = \mathfrak{Q}_\lambda(r,W)$ and $\mathfrak{T}_\lambda(r,\alpha) = \mathfrak{Q}_\lambda(r,W)$, respectively. Now taking the sectorial proximity function of both sides of the above equation and then using (3.26) with $N = 1$ we get

$$\begin{aligned}
\mathfrak{m}_\lambda(r,F) &\leq 3\mathfrak{m}_\lambda\Big(r,\frac{W'}{W}\Big) + \mathfrak{m}_\lambda(r,\frac{\alpha'}{\alpha}) + \mathfrak{m}_\lambda(r,\alpha) + \mathfrak{m}_\lambda(r,\frac{1}{W}) + O(1)\\
&\leq 5\mathfrak{m}_\lambda\Big(r,\frac{W'}{W}\Big) + \mathfrak{m}_\lambda\Big(r,\Big(\frac{W'}{W}\Big)'\Big) + \mathfrak{m}_\lambda\Big(r,\frac{1}{\alpha}\Big) + \mathfrak{Q}_\lambda(r,W)\\
&\leq 6\mathfrak{m}_\lambda\Big(r,\frac{W'}{W}\Big) + \mathfrak{Q}_\lambda(r,W)
\end{aligned}$$

$$= \mathfrak{Q}_\lambda(r, W).$$

Therefore $\mathfrak{T}_\lambda(r, F) = \mathfrak{Q}_\lambda(r, W)$. Hence, in this case W satisfies the first order differential equation

$$(W')^2 + \mu(z)WW' + \nu(z)W + \rho(z)W^2 = 0, \tag{4.42}$$

where

$$\mu(z) = \frac{2}{3-4\kappa}\frac{\alpha'}{\alpha}, \quad \nu(z) = -\frac{2\alpha}{1-2\kappa}, \quad \mathfrak{T}_\lambda(r, \rho) = \mathfrak{Q}_\lambda(r, W).$$

In a similar manner of the proof of Theorem 3.2.1 we will analyse equation (4.42). Hence we now consider the first-order equations (4.42) corresponding to each of the cases in Lemma 3.2.11 for $N = 1$, $\kappa \neq 1$.

(1) $\alpha = \alpha_0$ is a constant, $\mu = 0$, $\rho = 0$ and W satisfies (3.55) where the solution W is a polynomial of degree two which is admissible in $\Omega(\lambda, \rho)$.

(2) $\alpha = \alpha_0(z - z_\infty)^{-(4\kappa-3)/(2[\kappa-1])}$ is meromorphic in $\Omega(\lambda, \rho)$, $\mu = \frac{1}{\kappa-1}\frac{1}{z-z_\infty}$, $\rho = 0$ and $W = (z - z_\infty)^{-1/(2[\kappa-1])}v$ satisfies (3.56) where the solution w is given by

$$w = a_1 + \frac{2(\kappa-1)^2\alpha_0}{d_1(2\kappa-1)}(z - z_\infty)^{-1/([\kappa-1])}(d_1(z - z_\infty)^{1/(2[\kappa-1])} + 1)^2, \tag{4.43}$$

which is meromorphic function in $\Omega(\lambda, \rho)$ but it is not admissible since $T(r, w) = O(T(r, \alpha))$.

(3) $\alpha = \alpha_0([z-z_\infty]^2-d^2)^{(4\kappa-3)/(2[\kappa-1])}$ is meromorphic in $\Omega(\lambda, \rho)$, $\mu = \frac{1}{\kappa-1}\frac{2(z-z_\infty)}{(z-z_\infty)^2-d^2}$, $\rho = \frac{1}{(\kappa-1)^2}\frac{1}{(z-z_\infty)^2-d^2}$ and $W = \{(z - z_\infty)^2 - d^2\}^{-1/(2[\kappa-1])}v$ satisfies (3.57) where the solution w is given by

$$w = a_1 + \frac{\alpha_0(\kappa-1)^2}{2e_2(1-2\kappa)}\frac{\left([z - z_\infty]^2 - d^2\right)^{-1/([\kappa-1])}}{\left([z - z_\infty]^2 + d^2\right)^{-1/(2[\kappa-1])}}\left(e_1\left(\frac{[z - z_\infty]^2 - d^2}{[z - z_\infty]^2 + d^2}\right)^{1/(2[\kappa-1])} + 1\right)^2,$$

which is again meromorphic solution in $\Omega(\lambda, \rho)$ but it is not admissible.

So the solutions in the cases (1-3) do not appear in the final statement of the theorem because the condition of the admissibility. Observe that the solutions in the cases (2) and (3) are branched at fixed singularities.

We now turn to the Hayman equation (3.13). The following theorem is somewhat similar to the results of Halburd and Wang [12] in the complex plane, the proof, however, requires some modifications.

Theorem 4.3.3 *Let w be a meromorphic solution in $\Omega(\lambda, \rho)$ of equation (3.13) satisfying the condition*

$$\frac{\log r}{\mathfrak{T}_\lambda(r, w)} \to 0 \ \textit{as} \ \ r \to \infty, \tag{4.44}$$

and the coefficients α, β and γ are meromorphic in $\Omega(\lambda, \rho)$ and satisfy

$$\mathfrak{T}_\lambda(r, \alpha) + \mathfrak{T}_\lambda(r, \beta) + \mathfrak{T}_\lambda(r, \gamma) = O(\log r). \tag{4.45}$$

Then w is one of the solutions described in the following list, where c_1 and c_2 are constants.

1. *If $\alpha = \beta = \gamma = 0$, then $w(z) = c_2 \mathrm{e}^{c_1 z}$.*
2. *If $\beta = \gamma = 0$, then $d_1 = \alpha \neq 0$ is a constant and $w = \frac{d_1}{c_1^2}\{1 + \cosh(c_1 z + c_2)\}$ or $w = -\frac{d_1}{2}(z + c_2)^2$.*
3. *If $\gamma = 0$, $\beta \neq 0$ and $d_1 = -\alpha/\beta$ is a constant, then $w(z) = c_1 \mathrm{e}^{d_1 z}$.*
4. *If $\gamma = 0$ and $\alpha + \beta' = 0$, then*

$$w(z) = \mathrm{e}^{c_1 z}\left\{c_2 - \int \beta(z)\mathrm{e}^{-c_1 z}\mathrm{d}z\right\}. \tag{4.46}$$

5. *If* $\gamma \neq 0$ *and there is a constant* d_1 *and a meromorphic function* h *satisfying* $h^2 + \beta h + \gamma = 0$ *and* $h' - d_1 h = \alpha + d_1\beta$*, then*

$$w = \mathrm{e}^{d_1 z}\left(c_1 + \int h(z)\mathrm{e}^{-d_1 z}\mathrm{d}z\right). \tag{4.47}$$

6. *Suppose that* $\gamma \neq 0$ *and* $A = \frac{\beta(\alpha+\beta')-\gamma'}{\gamma}$ *is a constant.*

   (a) If $A = 0$ *and there are nonzero constants* d_1 *and* d_2 *such that*

$$d_2^2 = \frac{1}{d_1^2}\left\{\frac{1}{4d_1^2}\left(\beta' + 2\alpha\right)^2 + \left(\gamma - \frac{\beta^2}{4}\right)\right\},$$

   then $w = \pm d_2 \cosh(d_1 z + c_1) + \frac{\beta'+2\alpha}{2d_1^2}$.

   (b) If $d_1^2 = \frac{\left(\frac{\beta}{2}A-\beta'-2\alpha\right)^2}{\beta^2-4\gamma}$ *is a nonzero constant then*

$$w = c_1\mathrm{e}^{\left(-\frac{A}{2}\pm d_1\right)z} - \frac{1}{2d_1^2}\left(\frac{\beta}{2}A - \beta' - 2\alpha\right).$$

   (c) If $\frac{\beta}{2}A - \beta' - 2\alpha = 0$*, then*

$$w = \mathrm{e}^{-Az/2}\left\{d_1 c_1 - \int \frac{\beta}{2}\mathrm{e}^{Az/2}\mathrm{d}z\right\}, \tag{4.48}$$

   where $\beta^2/4 - \gamma = 0$

Proof. A straightforward calculation shows that for $\alpha = \beta = \gamma = 0$, we have $w(z) = c_2\mathrm{e}^{c_1 z}$. From now we take at least one of $\alpha,\ \beta,\ \gamma$ to be nonzero.

Consider a Laurent series expansion of the solution w around $z_0 \in \Omega(\lambda,\rho)$, which is either a zero or a pole of w

$$w(z) = \sum_{i=0}^{\infty} a_i\,\zeta^{i+p}, \qquad \zeta = z - z_0, \tag{4.49}$$

where $a_0 \neq 0,\ z_0 \in \Omega$ and $p \in \mathbb{Z}^+$ is the leading power that needs to be found.

Substituting the expansion (4.49) into the equation (3.13) gives

$$-p\, a_0^2\, \zeta^{2p-2} + \cdots = \alpha(z_0)(a_0\, \zeta^p + \dots) + \beta(z_0)(a_0\, p\, \zeta^{p-1} + \dots) + (\gamma(z_0) + \dots). \quad (4.50)$$

Hence, if $\beta = \gamma = 0$, then $p = 2$. Otherwise $p = 1$.

Recall that all zeros and the poles of the coefficients α, β and γ are in the disc D_ρ given by (4.26), i.e. they are outside the domain $\Omega(\lambda, \rho)$. The main ideas for the proof are taken from the paper by Halburd and Wang paper [12], but the arguments due to Nevanlinna theory regarding the growth of the solutions are replaced by sectorial theoretic arguments with slight modifications. It suffices to write down a brief proof of the cases $\beta \neq 0$.

The case $\beta \neq 0$, $\gamma = 0$:

Substituting $w(z) = a_0\zeta + a_1\zeta^2 + O(\zeta^3)$ in equation (3.13) we get $a_0 = -\beta$ and

$$\alpha(z_0) + \beta'(z_0) = 0.$$

Hence, $\alpha(z_0) + \beta'(z_0) = 0$ for every zero z_0 of w in $\Omega(\lambda, \rho)$. Now we have two cases:

Case 1: $\alpha + \beta' \neq 0$.

Consider $f = w'/w$. If z_0 is a pole of w then $z_0 \in D_\rho$. If z_0 is a zero of w, then either $z_0 \in D_\rho$ or $\alpha(z_0) + \beta'(z_0) = 0$ Thus the function f has a pole in $\Omega(\lambda, \rho)$ only if z_0 is a zero of w and $\alpha(z_0) + \beta'(z_0) = 0$. Hence, it follows that

$$\begin{aligned}
\mathfrak{N}_\lambda(r, f) &= \mathfrak{N}_\lambda\Big(r, \frac{w'}{w}\Big) \\
&= \overline{\mathfrak{N}}_\lambda(r, \frac{1}{\alpha + \beta'}) \\
&\leq \mathfrak{T}_\lambda\Big(r, \alpha + \beta'\Big) + \mathfrak{Q}_\lambda(r, w) \\
&\leq \mathfrak{T}_\lambda(r, \alpha) + \mathfrak{T}_\lambda(r, \beta') + \mathfrak{Q}_\lambda(r, w) \\
&= \mathfrak{Q}_\lambda(r, w),
\end{aligned}$$

and by the lemma on the logarithmic derivative and (4.44) we have

$$\mathfrak{m}_\lambda(r,f) = \mathfrak{m}_\lambda\Big(r, \frac{w'}{w}\Big) = \mathfrak{Q}_\lambda(r,f).$$

So $\mathfrak{T}_\lambda(r,f) = \mathfrak{Q}_\lambda(r,f)$. Substituting $w' = fw$ and $w'' = (f'+f^2)w$ in equation (3.13) with $\gamma = 0$ yields

$$f'w = \alpha + f\beta.$$

Since the coefficients of the different powers of w are all $\mathfrak{Q}_\lambda(r,f)$, we must have $f' = \alpha + f\beta = 0$. Hence, $f(z) = d_1$ and $\alpha(z) = -d_1\beta(z)$. Therefore, $w(z) = c_1 e^{d_1 z}$

Case 2: $\alpha + \beta' = 0$.

Equation (3.13) can be written as $((w' + \beta)/w)' = 0$. Thus we conclude that the general solution as given by (4.46). □

Remark 4.3.4 *Indeed, the solutions given by (4.46) when $\beta \neq 0$ and (4.47), (4.48) when $\gamma \neq 0$ were obtained in [12] using Nevanlinaa theory. If β or γ or h are non-polynomial rational functions, then in general these solutions are not meromorphic in the complex plane because they are branched at fixed singularities while they are meromorphic in $\Omega(\lambda,\rho)$ as well as admissible if $\lambda < 1$. In other words, we could allow for the solutions to be branched at fixed singularities.*

Chapter 5

Conclusion

This thesis has been motivated by the desire to extend the idea of using the simple singularity structure of solutions as a way of finding integrable equations, to the use of singularity structure as a tool for finding all particular solutions with simple singularity structure, even for non-integrable equations. Based on the success of the Painlevé property, perhaps the first guess for an appropriate class of solutions that one might reasonably expect to be able to find is the class of solutions with only poles as movable singularities. The main difficulty in working with this class is in proving that all such solutions have been found. In the case of the Painlevé property it is enough to show that an equation has a single solution with branching around a movable singularity in order to remove the equation from further consideration. This can often be done using series methods alone. When dealing with individual solutions the situation is much more delicate. In this case, in order to use series methods, we need to address the question of whether a solution takes a particular value often or has many poles. This kind of question is naturally addressed in Nevanlinna's theory of the value distribution of meromorphic functions. In order to use the classical theory we have to make two further assumptions about our solutions: they need to be globally meromorphic (i.e., not only are their movable singularities poles, but so are any fixed singularities) and they must be admissible in the sense that the Nevanlinna characteristic of the solution must grow faster than that of the the coefficients.

The original work in chapter 3 of this thesis was directed at finding all admissible meromorphic solutions of equation (3.14). In most cases we were able to show that any such solution must also solve a first-order equation that ultimately could be transformed to either a linear equation, a Riccati equation or an equation of the form $(u')^2 = P(u)$, where P is a polynomial of degree at most four, possible after a change of variables. The remaining cases were studied using resonance conditions and were related to the first and second Painlevé equations. Some of the solutions that we were led to in our proof Theorem 3.2.1 failed to be admissible or even meromorphic. Interestingly, all of the solutions that failed to be meromorphic did so purely because of non-pole fixed singularities. This further confirms the original intuition that one should allow solutions to be branched at fixed singularities.

Chapter 4 was a first attempt at constructing a theory that would allow us to consider solutions that are branched at fixed singularities. We started with a self-contained introduction to the Tsuji version of Nevanlinna theory for functions meromorphic in the half-plane. This was then extended to functions meromorphic in a larger sector outside a disc centred at the origin. We then applied this theory to differential equations in which all fixed singularities were contained in the deleted disc. In this way, we were able to consider solutions with branching at the fixed singularities. After deriving some simple analogues of some classical theorems, we returned to our main equation as well as some solutions to equation (3.13) considered in the work of Halburd and Wang. We were able to keep some, but not all, of the previously discovered solutions as some solutions did not satisfy the required admissibility assumptions. We hope to address this problem in future work.

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