Investing in Time-to-Build Projects with Uncertain Revenues and Costs: a Real Options Approach

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Abstract

Lagging public-sector investment in infrastructure and the deregulation of many industries mean that the private sector has to make decisions under multiple sources of uncertainty. We analyse such investment decisions by accounting for both multiple sources of uncertainty and the time-to-build aspect. The latter feature arises in the energy and transportation sectors because investors can decide the rate at which the project is completed. Furthermore, two explicit sources of uncertainty represent the discounted cash inflows and outflows of the completed project. We use a finite-difference scheme to solve numerically the option value and the optimal investment threshold. Somewhat counterintuitively, with a relatively long time to build, a reduction in the growth rate of the discounted operating cost may actually lower the investment threshold. This is contrary to the outcome when the stepwise aspect is ignored in a model with uncertain price and cost. Hence, research and development (R&D) efforts to enhance emerging technologies may be more relevant for infrastructure projects with long lead times.

Managerial Relevance Statement

The deregulation of energy and transportation sectors over the past three decades was intended to foster technology adoption. Indeed, most industrialised countries experienced a lag in public-sector investment at precisely the same time that infrastructure upgrades were required. In this context, the private sector could have a role in catalysing both research and development (R&D) as well as adoption of new technologies, e.g., electric vehicles. However, launching new technologies is confounded by multiple sources of uncertainty, e.g., both in revenues and costs, along with a non-negligible time to build associated with the underlying infrastructure. We take the perspective of a plug-in electric vehicle aggregator constructing charging infrastructure to focus on how these two features interact in determining the optimal investment timing for the new technology. In particular, when the discounted operating cost decreases, the investment threshold for launching the new technology actually

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decreases under a relatively long time to build. This is in contrast to the outcome under both (i) a negligible time to build and (ii) a high growth rate for discounted revenues in a time-to-build model without discounted operating costs. Hence, our enhancement to the real options framework supports policymakers and practitioners in assessing their R&D strategies for emerging infrastructure-based technologies.

1 Introduction

Public investment in infrastructure, such as power grids, telecommunications, and transport, in OECD countries has languished since the 1990s [1], dropping from a mean of over 4% of GDP in 1990 to 3% in 2007. In conjunction with a transition towards service-based economies in many OECD countries, the resulting "infrastructure gap" could have serious consequences for their competitiveness. While newly industrialising countries have the comparative luxury of developing their infrastructure now, infrastructure in OECD countries is decades old in many sectors and faces a lack of public funding. For example, spending on roads, rail, and inland waterways in the G7 had averaged less than 1% of GDP in each year over the past decade [2]. In effect, the trend towards deregulation has in the past thirty years put more emphasis on private provision of infrastructure investment, which is confounded by the exposure to uncertain revenues and costs in decision making. While public-private partnerships in emerging economies may be brokered to incentivise investment without direct exposure to market risk, investors, nevertheless, face other uncertainties, viz., political risk and exchange rate fluctuations. The former, in particular, could deter private investors if it can lead to re-possession of divested assets or re-negotiation of contracts due to evolving political considerations. For example, five utilities in Tanzania were opened up to private vendors in 2003 but were affected by re-possession by the government during 2010-2011 [3].

Due to the private sector's greater role in handling infrastructure investments, concerns about managing uncertainty in the context of maximising profit have become more important. For example, John Laing PLC, a British private equity firm that develops and operates public infrastructure, has recently announced its intention to raise capital through a flotation on the London Stock Exchange to finance a fund for environmental infrastructure and is aiming to provide annual returns of 8% [4]. Thus, the introduction of private incentives into the public sphere necessitates the development and application of appropriate decision-making methods, which consider uncertainty in cash flows, managerial flexibility, and salient features of infrastructure projects. The analyses may also provide insights to regulators in designing mechanisms that elicit desired outcomes, e.g., environmental or social, from a private sector that has profit maximisation as its principal motivation.

In this paper, we use the real options approach [5] to examine an infrastructure project that is to be carried out by a private firm. Similar to [6], we assume that the firm has the discretion not only to decide when to launch the project but also to determine the rate at which its construction proceeds. Indeed, large infrastructure projects can take years or even decades to build, and once construction is initiated, the cash flows may fluctuate to the point where it is optimal at times for the firm to suspend construction temporarily. Thus, an optimal decision rule is characterised by a threshold that indicates the minimum revenues from the completed project for every possible realisation of discounted operating costs and remaining investment. If the discounted revenue level is above this threshold, then the next tranche of investment is undertaken; otherwise, it is optimal to suspend investment.

Motivated by the fact that next-generation infrastructure projects, e.g., for smart grids or plug-in electric vehicles (PEVs), may have both uncertain and non-cointegrated discounted revenues and operating costs in both industrialised and emerging economies, we extend [6] to the case of two sources of uncertainty. We find that this feature has a non-monotonic effect on the optimal investment threshold for investment when the discounted operating cost is high and its growth rate decreases. Intuitively, with almost no time to build, a reduction in this growth rate increases the investment threshold monotonically because the marginal benefit of waiting (related to a higher expected net present value) increases by more than its marginal cost (stemming from the value of forgone cash flows in the interim period). However, with a relatively long time to build, a reduction in the growth rate of the discounted operating cost may actually lower the investment threshold as the marginal benefit from a higher expected net present value (NPV) is discounted more heavily and the marginal cost of waiting is the option value to continue with a staged project. This effect–which is especially pronounced when the discounted operating cost is high–is contrary to the finding in [6] that a higher growth rate for the discounted revenues increases the investment threshold.

2 Literature Review

In contrast to the now-or-never NPV approach, the real options framework reflects the value of managerial flexibility in response to unfolding uncertainties. For example, [7] examine the value of the deferral option in which a firm waits for the optimal time to invest when both revenues and investment costs are uncertain. Embedded options, such as the discretion to suspend and resume operations [8], expand or modify the project after initial investment [9], and determine the capacity of the project [10, 11, 12, 13], may also be handled. The framework can also be extended to tackle technological uncertainty [14] as well as the management of portfolios of R&D projects [15]. Such flexibility is often present in real projects and can affect the initial investment decision regarding selection of efficient vehicles [16]. Unlike the now-or-never NPV approach, the real options framework helps assess effects of these features on the value of the investment opportunity and optimal adoption thresholds [5].

A simplifying assumption in most of the real options literature is that the project is constructed immediately after the investment decision has been taken. In other words, the rate of investment is infinite, which is defensible only if lead times are low relative to the lifetime of the project. However, this assumption does not hold in most infrastructure projects: for instance, transmission lines for electricity may take several years to construct with several stages encompassing the securing of planning permission to assembling the towers to restoring the land [17].

Relaxing the assumption of an infinite investment rate, [6] consider a firm that faces an uncertain value of a completed project and has discretion over not only initiation of the investment cycle but also suspension and resumption of the investment process as each stage is completed. Thus, the optimal decision rule involves a trigger value for each stage that depends on the remaining investment until project completion. In effect, they have embedded options to manage the time to build and show that this additional flexibility reinforces the standard real options result that higher volatility and the effective growth rate of the revenues delay action. More concretely, the state variable in [6], V_t, is defined as the market value at time t of a *completed* factory that evolves exogenously and stochastically throughout the building phase. Consequently, even though the factory may still be under construction, the investor can observe the market value of a typical finished factory and decides at each point in time whether to continue with construction or to suspend it. Once construction is complete, the investor receives a finished factory at its market value. [6] define K_t as the investment amount (in \$, for example) remaining at time t, where K_t decreases at rate I_t, which is controlled by the investor. Now, if it is optimal for investment to proceed to the next stage, then the investor will simply proceed at the maximum rate k, i.e., $I_t = k$. The decision to continue or to suspend the construction given the current level of remaining investment outlay, K, depends on a threshold $V^*(K)$: at each point in time, the investor checks the market value of a finished factory to see if it is above this threshold. Once $K_t = 0$ at, say, $t = T$, the building phase is completed, and the investor receives an active factory worth V_T.

By contrast, [18] present a model with investment lags and an embedded option to abandon the project costlessly after its completion. Because the marginal costs of waiting, i.e., the forgone revenues from not investing, are higher due to the lag until cash flows are received and the abandonment option that puts a lower bound on the value of those cash flows, the standard real options result is weakened or even reversed: higher uncertainty may reduce the investment trigger. [19] extends the issue of investment lags to include competition.

[20] develop quasi-analytical solutions to problems when both revenues and operating costs are uncertain. In contrast to [7], they relax the assumption that the project's payoff is homogenous in the revenues and costs because it is operating costs rather than the investment cost that may be more prone to uncertainty. Consequently, the dimension-reducing step from [7] of turning the partial differential equation (PDE) into an ordinary differential equation (ODE) no longer holds, and the optimal investment trigger is not a linear relationship between revenues and costs. [20] motivate their work in the context of renewal assets, while [21] apply a similar model to the case of commodity switching in a production plant. An important consideration of this strand of the literature is that often revenues and

operating costs cannot be modelled together as a single stochastic process that describes the profit flow because the two processes are not cointegrated. Indeed, for infrastructure projects concerning new technologies, e.g., smart grids or PEVs, there is not even a time series of the relevant revenues and operating costs from which to detect the presence of cointegration. Taking this point of view, our work also makes a methodological enhancement to real options by tackling the time-to-build attribute together with multiple uncertain factors.

3 Analytical Model

3.1 Assumptions

We extend [6] by considering two stochastic variables that determine the value of the project. Likewise, our work could also be thought of as adding investment lags to the two-factor model of [20]. As a specific example, we may consider the problem of a PEV aggregator that controls charging decisions via direct load control (DLC). In [22], such an entity must make trading decisions in various electricity markets, ranging from forward contracts to day-ahead to balancing. Although the bulk of electricity trading occurs day-ahead, the balancing market is becoming increasingly important due to the penetration of intermittent variable renewable energy sources. Thus, by using PEVs as mobile storage devices, the PEV aggregator is able to profit from arbitrage between trading platforms. However, it must first plan to build the charging infrastructure for PEVs, which takes time to build and over which the PEV aggregator has discretion regarding investment progress. Hence, we characterise its discounted revenues as arbitrage profits and discounted operating costs as expenses related to maintaining the infrastructure and the ICT services necessary to manage the PEV fleet.

The dynamics in our model are similar to those of [6] except that instead of a single variable like V_t (which cannot become negative due to the geometric Brownian motion (GBM) assumption) that reflects the market value of the finished project, we have both (i) the market value of the discounted revenue from the finished project, V_t (arbitrage profit from PEV aggregation), and (ii) the market value of the discounted operating costs of the finished project, C_t (expenditure of managing the charging infrastructure and providing ICT services to implement the trading strategy with the PEVs). Consequently, in our model, the market value of the completed project is $V_t - C_t$, which may, indeed, be negative. As in [6], our PEV aggregator continuously monitors the market values of the discounted revenues and costs associated with the completed infrastructure to decide whether or not to proceed to the next phase of construction.[1] However, unlike [6], our PEV aggregator's optimal decision rule given the current discounted operating cost, C, and the current remaining investment required, K, is a free boundary, $V^*(C,K)$, rather than a single threshold, $V^*(K)$. More important, although $V_t - C_t$ may become negative at any point, our PEV aggregator could still decide to proceed with construction provided that V_t exceeds $V^*(C,K)$. This is a key difference from [6] in terms of optimal decision making.

We assume that the option to invest in the project is perpetual, i.e., without any expiration date.[2] We also assume that V_t and C_t follow the GBMs

$$dV_t = \alpha_V V_t dt + \sigma_V V_t dz_t \tag{1a}$$

$$dC_t = \alpha_C C_t dt + \sigma_C C_t dw_t \tag{1b}$$

where α_V and α_C are drift rates, $\sigma_V \geq 0$ and $\sigma_C \geq 0$ are volatilities, and dz_t and dw_t are increments of uncorrelated Wiener processes. We take the increments of the GBMs to be uncorrelated as they are assumed to arise from non-cointegrated time series. However, instantaneous correlation between the two increments could arise, e.g., due to higher ICT/infrastructure maintenance costs resulting from more frequent charging/discharging patterns by PEVs triggered by more volatile electricity prices. Such correlation would be straightforward to implement.

[1] Admittedly, V_t and C_t will have to be estimated using proxies: in our example with the PEV aggregator, the discounted revenues and operating costs can be adequately approximated by arbitrage profits in electricity markets and ICT/infrastructure maintenance costs. However, if the project involves launching a fundamentally different product, e.g., smartphones in the year 2007, where market value cannot be readily approximated because of the lack of close substitutes, then another layer of uncertainty is involved because the value of the underlying asset cannot be observed.

[2] The benefit of assuming a perpetual option is that it relieves us from making the option value an explicit function of time.

The assumption of constant drift and volatility parameters of GBMs in (1a) and (1b) implies that the investor cannot affect the evolution of V_t and C_t. In the case of V_t, this is essentially a perfect market assumption, i.e., the investor takes the market value of the output of the project as given. In the case of C_t, the interpretation depends on the situation. If $\alpha_C = 0$, then the interpretation is simply that the evolution of the discounted operating cost variable is stochastic yet without a trend. When $\alpha_C > 0$, the interpretation is that the discounted operating costs are expected to increase in the long run. For example, if the main cost determinant of the finished project is a diminishing natural resource, then the costs of production may rise because the price of this resource will increase in the future.

By contrast, if $\alpha_C < 0$, then we can consider the case of new technology adoption, e.g., PEVs and a charging infrastructure. If the adoption of PEVs is in line with the goals of policymakers, then they might support the R&D required to initiate private-sector investment and further accelerate the adoption process. In this case, it is also feasible for policymakers to make their information and progress available to the public so that the private sector can capitalise on the evolving technology, thereby fulfilling the goals behind the public investments.[3] This implies that the private investor, in our case, experiences an exogenous learning curve effect that decreases the discounted operating costs of the finished project over time. Hence, by considering the case $\alpha_C < 0$, we can examine how an exogenous learning curve effect described above affects the actions of rational investors.

Following [6], we model the investment process so that the capital investment left at time t is K_t, the investment rate is I_t, and the maximum investment rate is $k \geq 0$. Thus, the dynamics of K_t are as follows:

$$\frac{dK_t}{dt} = -I_t, \; 0 \leq I_t \leq k \tag{2}$$

We assume that the PEV aggregator can continuously adapt the rate at which she invests

[3] According to the Joint Research Centre of the European Commission [23], about 65% of the outstanding total European PEV RD&D budget of €1.9 billion is from public funding. The report also finds that an increased exchange of information between the projects would result in a better societal return for the investments due to the exogenous learning effects described above.

as new information about the expected profitability of the finished project arrives. This implies that our framework is most relevant in modelling situations in which the investment is made in multiple stages and the investor can suspend the investment between the stages.[4] If the investor has an opportunity to suspend the process during the stages, then our model is even more relevant. In fact, the more irreversible the investment process becomes, the less appropriate our model is in describing the optimal investment behaviour. With such irreversibility, the model of [18] should be employed, which will lead to different results. Irreversibility in our context means that any sunk investment costs, i.e., in terms of decreasing K, cannot be recovered. However, there is some flexibility for the PEV aggregator since it can suspend construction indefinitely between stages. In effect, once construction is started, it is not necessarily seen through to completion at the highest possible rate, e.g., as in [18]. Thus, our model provides the decision maker with more discretion.

Since we use the dynamic programming approach to value the investment option, we denote the firm's required rate of return with $\rho \geq 0$. As is typical with dynamic programming, ρ is interpreted as an exogenous parameter that represents the cost of maintaining the investment opportunity. We assume $\rho > \alpha_V$ in order to rule out the case that it would be never optimal to exercise the option to invest.

3.2 Problem Formulation

Given initial values $V \equiv V_0$, $C \equiv C_0$, and $K \equiv K_0$, we denote the value of the option to invest as $F(V, C, K)$.[5] The option value in $(V, C, K) \in \mathcal{X} \equiv (0, \infty) \times (0, \infty) \times (0, \infty)$ given the investment policy $I^*(V, C, K) \equiv I$ can be obtained from the following Bellman equation:

$$\rho F = \max_{I \in [0,k]} \left(\frac{\mathbb{E}[dF]}{dt} - I \right) \quad (3)$$

[4]The state variables' time indices are omitted since we consider the return equilibrium at the current time, i.e., $V = V_0$, $C = C_0$, and $K = K_0$. Likewise, the time index for I_t is omitted without loss of generality since [6] point out that it will be either 0 or k, which is the maximum rate at which investment can proceed: if we did not have the time-to-build issue, then k would simply be infinity, which is the case in most real options papers.

[5]The option valuation can be implemented for any starting point and not only $t = 0$. Without loss of generality, we start the valuation at time zero.

Note that dF is a function of I, and dt in the denominator means that the expression in the nominator is divided by the increment of time and not differentiated with respect to time. Intuitively, Eq. (3) states that the instantaneous return on the investment opportunity is equal to its net appreciation if it were managed optimally. By expanding dF using Itô's lemma and taking the expected value, we obtain:

$$\rho F = \max_{I \in [0,k]} \left(\frac{1}{2}\sigma_V^2 V^2 F_{VV} + \frac{1}{2}\sigma_C^2 C^2 F_{CC} + \alpha_V V F_V + \alpha_C C F_C - I F_K - I \right) \tag{4}$$

Since the expression to be maximised with respect to I is linear in I, if it is optimal to invest at all, then it is also optimal to invest at the maximum rate k. Therefore, the optimal investment policy is "bang-bang" control as in [6].

Following [20], we use backward induction to obtain first the value of the option to invest when it is optimal to continue the investment program. This is separated from the option value in the waiting region by a continuous surface $V^*(C, K)$ in $\mathcal{X}$ so that it is optimal to invest if $V \geq V^*(C, K)$ and to wait otherwise. This assumption is based on the intuition that the option value is increasing in V. Thus, we denote the option value in the investment region $\mathcal{R} \equiv \mathcal{X} \cap \{V \geq V^*(C, K)\}$ with F and in the waiting region $\mathcal{W} \equiv \mathcal{X} \setminus \mathcal{R}$ with f. Under this assumption, the option value functions in the two regions are given by PDEs obtained by re-arranging Eq. (4):

$$\tfrac{1}{2}\sigma_V^2 V^2 F_{VV} + \tfrac{1}{2}\sigma_C^2 C^2 F_{CC} + \alpha_V V F_V + \alpha_C C F_C - k F_K - \rho F - k = 0 \text{ in } \mathcal{R} \tag{5a}$$

$$\tfrac{1}{2}\sigma_V^2 V^2 f_{VV} + \tfrac{1}{2}\sigma_C^2 C^2 f_{CC} + \alpha_V V f_V + \alpha_C C f_C - \rho f = 0 \text{ in } \mathcal{W} \tag{5b}$$

Note that only Eq. (5a) contains partial derivatives with respect to K as no investment occurs in $\mathcal{W}$.

The appropriate boundary conditions for the problem are:

$$F(V, C, 0) = \max\{V - C, 0\} \tag{6a}$$

$$\lim_{V\to 0} f(V, C, K) = 0 \tag{6b}$$

$$\lim_{C\to\infty} f(V, C, K) = 0 \tag{6c}$$

$$F(V^*(C, K), C, K) = f(V^*(C, K), C, K) \tag{6d}$$

$$F_V(V^*(C, K), C, K) = f_V(V^*(C, K), C, K) \tag{6e}$$

$$F_C(V^*(C, K), C, K) = f_C(V^*(C, K), C, K) \tag{6f}$$

Eq. (6a) is simply the payoff of the option when there is no investment requirement remaining,[6] which holds at the point in time, say $t = T$, when the building phase is complete, i.e., $K_T = 0$, and the PEV aggregator receives the completed infrastructure: if it turns out that $V_T - C_T < 0$, then the infrastructure will not be taken into use. Eq. (6b) states that when V reaches zero, the option becomes worthless because zero is an absorbing barrier to the GBM given by Eq. (1a). Eq. (6c) means that the option value converges to zero as the discounted operating costs of the finished project grow arbitrarily large. Eq. (6d) is the value-matching condition stitching together the two option values along the free boundary, $V^*(C, K)$. Eqs. (6e) and (6f) are the smooth-pasting conditions, which are first-order conditions for making optimal transitions across the free boundary. Note that now there are two smooth-pasting conditions as there are two stochastic variables.[7]

[6] By considering $\max\{V - C, 0\}$, we also take care of the boundary conditions $\lim_{V\to\infty} F_V(V, C, K) = e^{-(\rho-\alpha_V)\frac{K}{k}}$ and $\lim_{C\to 0} F_C(V, C, K) = -e^{-(\rho-\alpha_C)\frac{K}{k}}$ because the NPV rule is to invest only if $e^{-\rho\frac{K}{k}}\left(Ve^{\alpha_V\frac{K}{k}} - Ce^{\alpha_C\frac{K}{k}}\right) - \frac{k}{\rho}\left(1 - e^{-\rho\frac{K}{k}}\right) \geq 0$.

[7] [24] comment that accounting for smooth pasting with respect to K omitted by [6] will result in a lower free boundary. Although they demonstrate that the discrepancy is larger for higher volatility and growth rate, the qualitative findings about the impact of the time to build on option value from [6] are unchanged. In order to have directly comparable results, we build on the results of [6].

3.3 Quasi-Analytical Solution

A general solution to Eq. (5b) is of the form

$$f(V, C, K) = A(K)V^{\beta(K)}C^{\eta(K)} \tag{7}$$

where $A(K)$ is an endogenous constant to be determined and coefficients $\beta(K)$ and $\eta(K)$ must satisfy the condition

$$\frac{1}{2}\sigma_V^2\beta(\beta-1) + \frac{1}{2}\sigma_C^2\eta(\eta-1) + \alpha_V\beta + \alpha_C\eta - \rho = 0 \tag{8}$$

for each value of K. We use short-hand notation for $\beta(K)$ and $\eta(K)$ here. By "general solution," we mean that any linear combination of functions of the form given by Eq. (7) satisfies the PDE given by Eq. (5b). Eq. (8) has solutions in all four quadrants of the (β, η)-plane [20]. However, we can rule out three of the four quadrants by using the boundary conditions given by Eqs. (6b) and (6c). Doing so, we obtain that $\beta(K) > 0$ and $\eta(K) < 0$, i.e., the option value increases (decreases) with discounted revenues (operating costs) in line with economic intuition. From now on, we will assume that the solution to PDE (5b) is $f(V, C, K) = A(K)V^{\beta(K)}C^{\eta(K)}$, where $(\beta(K), \eta(K)) \in (0, \infty) \times (-\infty, 0)\ \forall\ K \in (0, \infty)$ so that Eq. (8) holds. $A(K)$ must be solved for by using the other boundary conditions and the option value in $\mathcal{R}$.

Since the PDE in the investment region has no analytical solutions, we use a numerical approach based on an explicit finite-difference method to solve the rest of the investor's problem. However, now that we know the form of the analytical solution in the waiting region, we can write boundary conditions (6d)-(6f) in a more convenient form. By inserting the quasi-analytical solution given by Eq. (7) into the conditions mentioned above, we find that the following conditions must be met at the free boundary:

$$\frac{F(V^*(C,K),C,K)}{F_V(V^*(C,K),C,K)} = \frac{V^*(C,K)}{\beta(K)} \tag{9a}$$

$$\frac{F(V^*(C,K),C,K)}{F_C(V^*(C,K),C,K)} = \frac{C}{\eta(K)} \tag{9b}$$

where $\beta(K)$ and $\eta(K)$ satisfy Eq. (8). We will utilise conditions (9a) and (9b) to determine the free boundary numerically. Once the free boundary is obtained, we can solve for the values of $A(K)$, $\beta(K)$, and $\eta(K)$ for each discrete value of K. The numerical solution method is in further detail in Appendix A.

4 Numerical Examples

We present the results of the model in two parts. First, we consider a base case and provide a discussion of the results in general. Next, we present the most interesting results by performing comparative statics to isolate the effects of individual parameters on the investor's optimal investment policy.

4.1 Base Case

For the base case, we assume that the total investment required to finish the investment program is $K = 6$ (M€) and the maximum investment rate is $k = 1$ (M€/year). This implies that the minimum time to complete the investment program is six years and that the unit of time is years. We set $\alpha_V = 0.04$ and $\sigma_V = 0.14$ in this section and consider a case in which the drift and volatility of C_t are the same as those of V_t ($\alpha_C = 0.04$ and $\sigma_C = 0.14$),[8] where V and C represent the current discounted revenues and operating costs of the completed project. Finally, we assume that the discount rate is $\rho = 0.08$.

[8]If we were considering an all-equity firm that consisted only of the investment opportunity studied here, then the base case values would imply that the volatility of the firm's stock is approximately $\sqrt{0.14^2 + 0.14^2} = 19.8\%$. Considering that the implied volatility of the S&P 500 index options sold on the Chicago Board Options Exchange is usually around 20%, the assumptions made on the volatilities of the processes are fairly realistic.

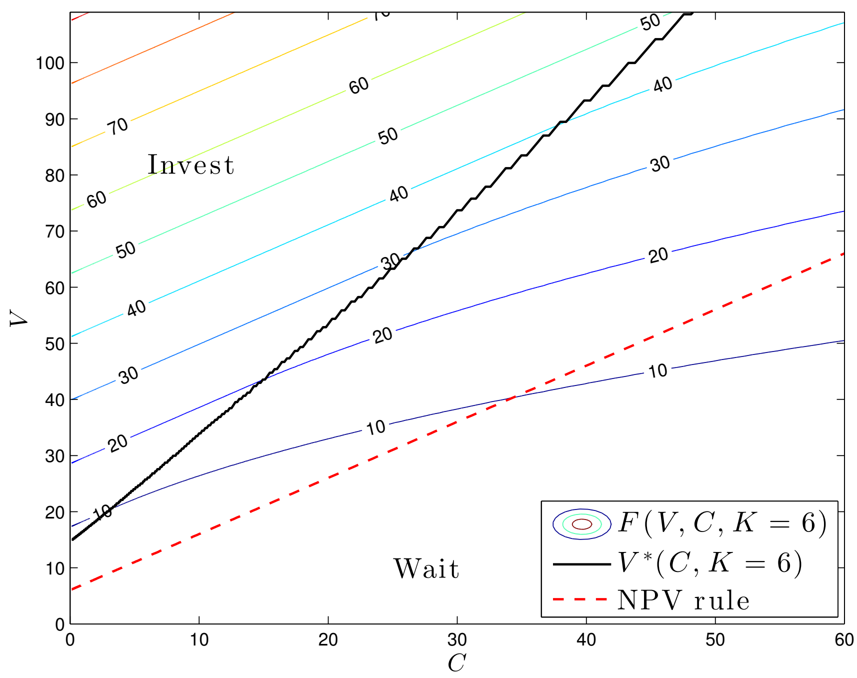

Figure 1: Option value, free boundary, and NPV threshold when $K = 6$

Figure 1 shows the level sets of the option value (contour curves with numbers indicating the option value for every combination of V and C when $K = 6$) and the free boundary (solid black line) in the base case when $K = 6$. The option value is increasing in V and decreasing in C as intuition suggests, which is the case for other values of K as well. The black line indicating the position of the free boundary, $V^*(C, K = 6)$, tells the PEV aggregator when to proceed with construction if $K = 6$, e.g., the minimum V required to continue construction when $C = 15$ is approximately 40.[9] At $(V, C) = (40, 15)$, the option value is approximately 20 as indicated by the contour curve passing through that point. As expected, the investment threshold increases in C. Note also that the free boundary is not a level set of the option value. Therefore, we cannot, in general, draw a straight connection between the option value and the location of the investment threshold.

The red dashed line in Figure 1 shows the NPV investment threshold assuming that the entire investment is finished at the full rate if it is optimal to invest, i.e., the PEV aggregator

[9]The free boundary does not appear smooth because of the numerical finite-difference method used to solve the problem.

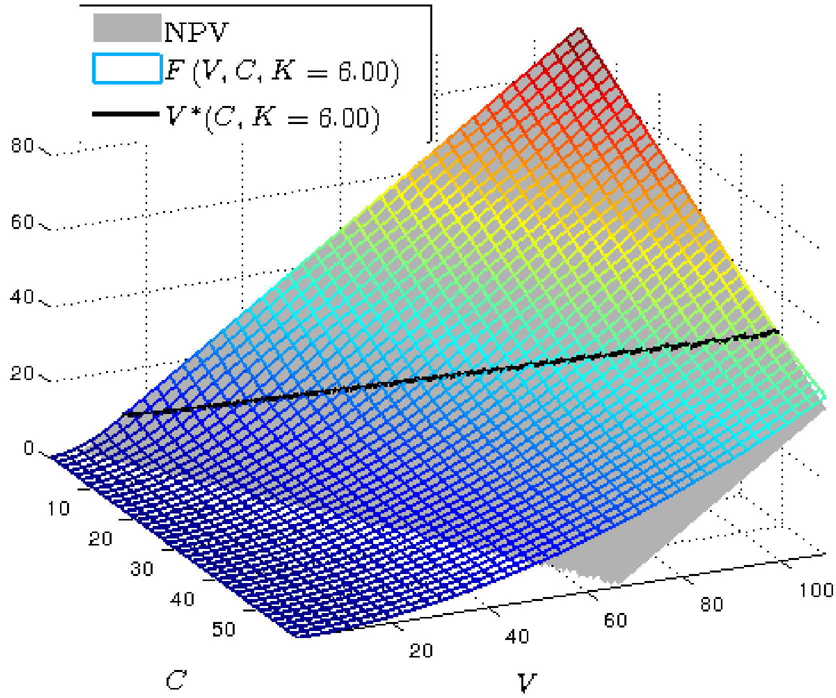

Figure 2: Option value surface, now-or-never NPV, and free boundary when $K = 6$ (z-axis measures option value and now-or-never NPV)

ignores the discretion to suspend construction. Since the NPV rule, by definition, is obtained by calculating the expected cash flows of the completed project net of the initial investment costs, its use leads to investment in cases when it is optimal to wait according to the real options rule. For example, considering again the case when $C = 15$, the NPV rule tells the PEV aggregator to proceed with construction when V is at least 20. Therefore, there must be other reasons than the initial investment cost for the free boundary, $V^*(C, K = 6)$, to be above the NPV threshold. First, since both V_t and C_t evolve stochastically in time, there is a chance that the investment opportunity might increase in value over time. This implies that there are benefits to waiting that are not present in the NPV analysis. Second, as there is uncertainty in the value of the finished project due to the time-to-build aspect, it is optimal to wait longer than the NPV rule suggests in order to cover this uncertainty by waiting for the expected value of the finished project to rise well above the NPV rule.

Figure 2 shows the option value surface (measured on the z-axis), the now-or-never NPV, and the projection of the free boundary onto the option value surface when $K = 6$. The

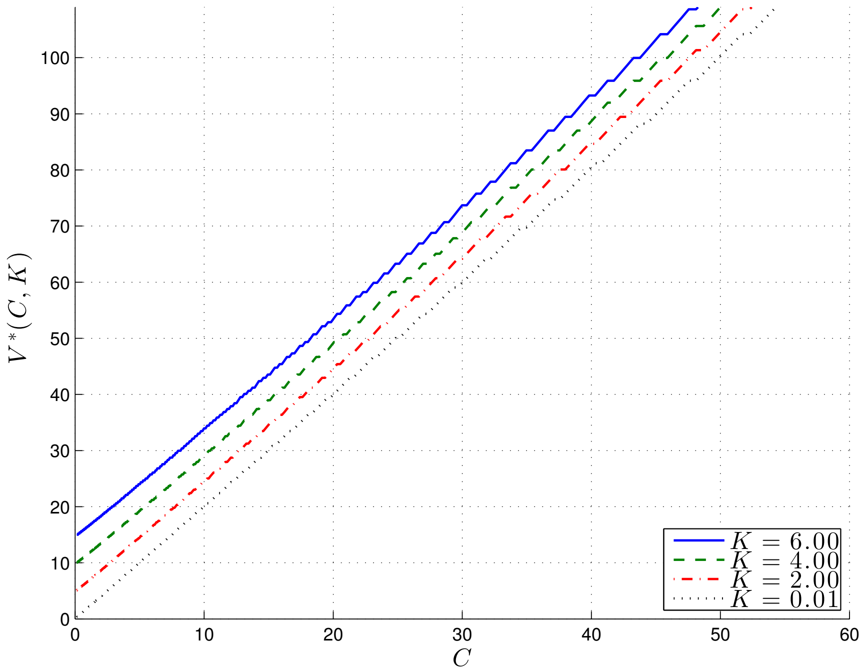

Figure 3: Free boundaries in the base case for different values of K

value-matching and smooth-pasting conditions are satisfied by the numerical solution as the option values in the investing and waiting regions meet smoothly at the free boundary. Also, the option value is non-negative for all values of (V, C). A comparison of the option value and the NPV indicates that the option value is greater than the NPV for all values of (V, C), thereby reflecting the fact that real options analysis considers also the value of waiting and the possibility to vary the investment rate. By contrast, the now-or-never NPV approach ignores this managerial flexibility. As in Figure 1, we can examine in Figure 2 how much the option value and now-or-never NPV are worth for a given combination, e.g., $(V, C) = (40, 15)$, along with the respective investment thresholds. We also notice that the difference between the option value and the NPV converges to zero as V increases and C decreases. This happens because then the investment program will be completed almost certainly at full pace yielding, on average, a total payoff that equals the NPV.

Figure 3 shows the investment thresholds for various values of K in the base case. The threshold curves are increasing in C for each value of K because the value of the finished project is decreasing in C. Also, in the base case, the investment thresholds increase in

K. This is due to two reasons. First, the remaining initial investment cost increases in K. Second, the uncertainty over the value of the payoff when the investment program is completed is increasing in K because a large value of K indicates that the minimum time-to-build is large as well. We discuss this feature further in Section 4.2.2. In summary, Figure 3 can be used as a decision rule for the PEV aggregator: since we have implicitly assumed that the investor can observe V, C, and K at each point in time, she may use the investment thresholds at different values of K as a guide on how to proceed optimally with the construction of the charging infrastructure.

4.2 Comparative Statics

4.2.1 Sensitivity with Respect to the Growth Rate

As our main motivation is to gain insight into how the inclusion of uncertain discounted operating costs of the completed project affects the investor's choices, we first discuss the mechanics behind the effects of α_C on the investor's optimal behaviour in detail. Figure 4 illustrates the investment thresholds at different values of K for various values of α_C while holding the other parameters the same as in the base case. For the smaller values of K, i.e., when the remaining time to build is negligible, the effect of α_C on the results is monotonic: a decrease in α_C shifts the investment threshold up and, thus, increases the incentive to wait. Intuitively, lowering α_C increases the value of the option to wait as the expected discounted operating cost upon completion of the project will be lower. This is precisely the finding of [7], i.e., a model without time to build and an NPV that depends homogeneously on two stochastic processes.[10]

When $K \gg 0$, the time-to-build aspect is non-negligible and may have a subtle effect on how the growth rate affects the optimal investment threshold. In particular, Figure 4 illustrates that when α_C decreases from 0.08 to -0.10, $V^*(C, K = 6)$ increases, but as α_C decreases further, $V^*(C, K = 6)$ actually decreases. This non-monotonic impact of α_C on $V^*(C, K = 6)$ is more pronounced when the discounted operating costs are higher. In

[10]It can be shown numerically that $V^*(C, K)$ converges to the analytical results of [7] when $K \to 0$.

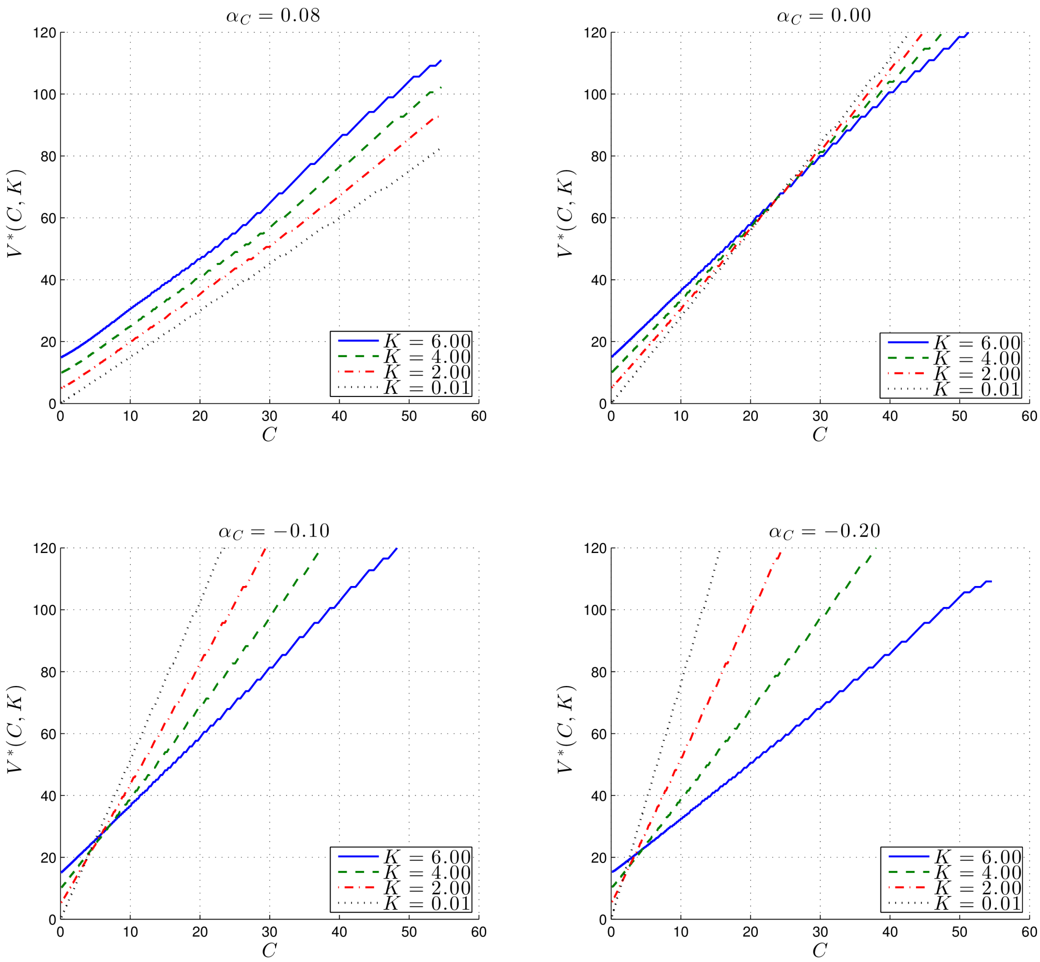

Figure 4: Sensitivity of the free boundary with respect to α_C

order to explain this seemingly counterintuitive result, we first examine the cases with low discounted operating costs: in Figure 4, as α_C is decreased, $V^*(C, K = 6)$ increases for low values of C. This effectively recovers the result of [6] in which an increase in α_V (equivalent to a decrease in α_C here) increases $V^*(C, K)$. Indeed, an increase in the growth rate of the discounted revenues increases the incentive to wait both due to (i) a higher expected NPV of a completed project and (ii) a lower value of flexibility related to managing the rate of construction during the time to build. Likewise, in our model, for low values of C, a decrease in α_C will increase $V^*(C, K)$ due to the same two dynamics.

However, for high values of C, the value of flexibility over the construction rate increases with decreases in α_C. Thus, if α_C is low enough, then the increase in the value of flexibility during the time to build will more than offset the increased incentive to wait for a higher expected NPV, thereby reversing the results of [6]. This is seen in the bottom two panels of Figure 4 as $V^*(C, K = 6)$ becomes lower when α_C decreases for high C.

The results in Figure 4 can also be viewed in terms of the impact of K on the investment threshold. First, for low C, the threshold increases with K regardless of the value of α_C. Intuitively, a higher K increases the incentive to wait because it increases the required net discounted revenues to cover the investment cost. Moreover, since low C results in a negligible value of flexibility over managing the investment program optimally, it is unaffected by an increase in K. Therefore, an increase in K increases the investment threshold for low C for any given α_C.

When C is relatively high, the impact of K on the investment threshold depends on the growth rate of the discounted operating costs: for a high (low) α_C, the threshold increases (decreases) with K. As with low C, a higher K leads to an increase in the threshold for high α_C as there is an incentive to wait longer to cover the higher investment costs without a substantial increase in the value of flexibility over managing the rate of construction. Indeed, when α_C is high, the expected payoff from exercising the option to obtain the project at time $t+s$ given information at time t, $\mathcal{F}_t$, is $\mathbb{E}[(V-C)_{t+s}|\mathcal{F}_t] = V_t e^{\alpha_V s} - C_t e^{\alpha_C s}$, which is "out of the money" for high α_C. By contrast, this option becomes "in the money" for low α_C when C is high. Thus, an increase in K increases the duration during which the PEV aggregator has flexibility over managing the rate of construction.

The effect of α_V on the results is similar to that of α_C. An increase in α_V increases the benefits of waiting and shifts the investment threshold $V^*(C, K = 0.01)$ upwards. Again, the investment thresholds at larger values of K are located in a way that once the first initial investment is made, the investor will, on average, be able to invest continuously at the maximum rate up to the end of the investment program. We should also note that the investment thresholds grow without boundaries as $\alpha_V \to \rho$ (assuming that $\alpha_V > \alpha_C$) since then the long-term capital rate of return of the payoff converges to ρ and the cost of waiting diminishes. Finally, as ρ represents the cost of waiting in our model, the effect of an increase in ρ is to shift the investment threshold down for all values of K and, thus, hasten investment.

4.2.2 Sensitivity with Respect to the Time to Build

As our explanation for the results above relies on the logic that the investor holding the option considers both the expected evolution of $V - C$ during the investment period and the optimal investment policy at smaller values of K when making decisions on whether to invest or wait, we would assume that the results of the comparative statics above would be amplified for smaller values of k since this would imply a longer time to build. Consider, for example, the case where $\alpha_C < 0$ and C is expected to decrease while V is expected to increase. Now, if we decrease the maximum investment rate, then we expect that it is optimal to start investing at even higher values of C given a value of V since the minimum time to build is longer, thereby implying that the expected decrease of C during the investment period is larger as well. By generalising the logic above, we would assume that a decrease in k would amplify the results of the comparative statics above. Motivated by this, we will next analyse the results of the same comparative statics as above using a smaller maximum investment rate, $k = 0.5$. This doubles the minimum time-to-build for every value of K in comparison to the value $k = 1$ used above.

Figure 5 shows the results of the comparative statics with respect to α_C when $k = 0.5$ and the other parameters are the same as in the base case. We note that our intuition is correct as the smaller value of k amplifies the effects of α_C on the investment thresholds. Note that the investment thresholds are not affected by the change in the value of k when $K = 0.01$ since then the payoff can be received almost instantly. The reason why the other thresholds react more dramatically to changes in α_C than in the case above is that now the investor needs to look further ahead in time when making decisions for larger values of K because the minimum time-to-build is longer.

An interesting result occurs in the lower right case of Figure 5 where $\alpha_C = -0.20$. For

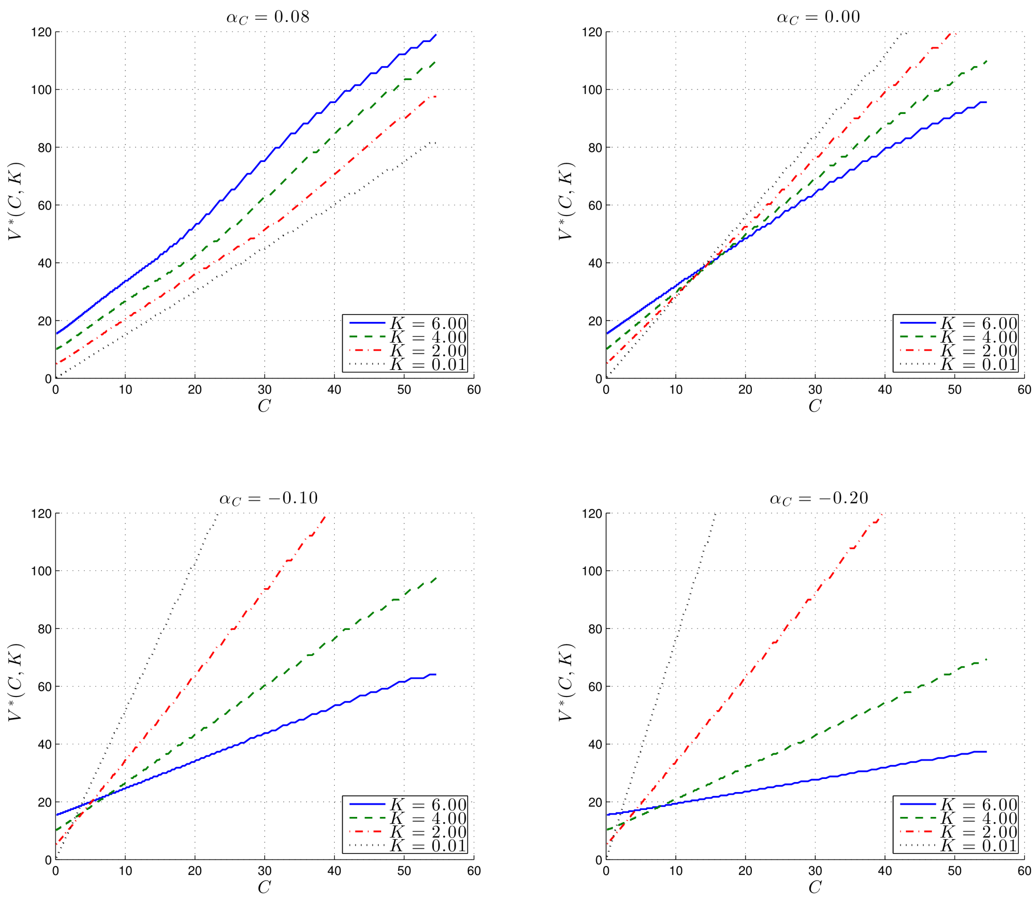

Figure 5: Sensitivity of the investment threshold with respect to α_C when $k = 0.5$

large values of C and K, it is optimal to invest even if $V - C < 0$. However, this is well explained by the expected increase of $V - C$ during the investment program. This does not contradict the boundary condition (6a) because that refers to whether or not the infrastructure is accepted upon completion of construction. The result in Figure 5 pertains to proceeding or not with construction at some intermediate phase. Indeed, it may be profitable to do so when (i) sufficient time remains to build and (ii) discounted operating costs are expected to decrease. Also, for each value of C, the NPV rule in this extreme situation is to invest at a smaller value of V than the real options rule suggests. The observation applies generally: the real options investment threshold is always larger than the now-or-never NPV threshold. This strengthens our explanation for why it might be optimal to invest even if the current value of the payoff is negative since the fact that the real options threshold is larger than the NPV threshold in all situations ensures that the average value of the investment program executed by the real options rule is positive in all cases.

4.2.3 Sensitivity with Respect to the Volatility

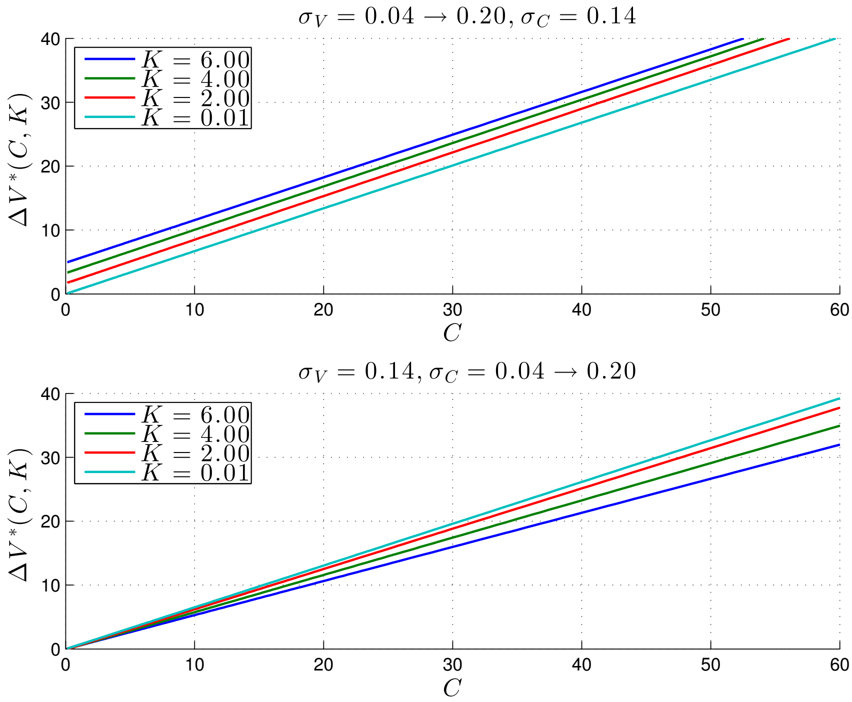

Figure 6: Sensitivity of the investment threshold with respect to σ_V and σ_C when $\alpha_V = \alpha_C = 0.04$, $k = 1.00$, and $\rho = 0.08$

Figure 6 indicates the sensitivity of the investment threshold with respect to the volatilities when the other parameters are as in the base case.[11] The upper graph shows how much the threshold changes given a value of (C, K) as σ_V changes from 0.04 to 0.20 and $\sigma_C = 0.14$, and the lower graph shows how much the threshold changes as σ_C changes from 0.04 to 0.20 and $\sigma_V = 0.14$. Both graphs reveal that the change in the threshold, i.e., $\Delta V^*(C, K)$, is positive for all values of (C, K). In fact, the observation holds for all tested parameter values: $V^*(C, K)$ is increasing in both σ_V and σ_C in all situations. This reflects the well-known

[11] The lines in the graphs of this subsection are linearised to smooth out noise due to the numerical method used.

property of options with convex payoffs: since the payoff $\max\{V - C, 0\}$ is bounded from below, an increase in the volatility of $V - C$ increases the benefits of waiting and, thus, increases the investment threshold [6, 7].

A second observation common to both of the graphs in Figure 6 is that $\Delta V^*(C, K)$ is increasing in C for all values of K. Our interpretation is that this is due to the assumption that V and C follow GBMs. This assumption implies that the standard deviations of dV and dC are increasing in σ_V and σ_C by Eqs. (1a)-(1b). Therefore, the spread of the future values of the payoff $\max\{V - C, 0\}$ is more sensitive to the volatilities of V and C when the values of V and C are large. Thus, the sensitivity of the threshold with respect to the volatilities is increasing in C given a value of K since for a large value of C the value of V needs to be large as well in order to investment to occur as $V^*(C, K)$ is always increasing in C.

In the upper graph of Figure 6, $\Delta V^*(C, K)$ is increasing in K given a value of C, while in the lower graph, $\Delta V^*(C, K)$ is slightly decreasing with respect to K given any positive value of C. The evolution of $\Delta V^*(C, K)$ as a function of K given a value of C depends on other parameters than the volatilities as well. For example, Figure 7 displays $\Delta V^*(C, K)$ when $\alpha_V = 0.04$ and $\alpha_C = -0.20$. In this case, $\Delta V^*(C, K)$ is decreasing in K for all values of C when σ_C increases. Also, when σ_V increases, $\Delta V^*(C, K)$ is decreasing in K for large values of C.

Next, we consider how the distribution of the total volatility of $V - C$ among the two variables affects the investment threshold. By total volatility, we mean the value of $\sigma_{total} = \sqrt{\sigma_V^2 + \sigma_C^2}$. Although σ_{total} is not an accurate measure of the standard deviation of $d(V - C) = dV - dC$ as this depends on the value of (V, C), we will use σ_{total} as a useful approximation of the volatility of $V - C$. Then, the question is that how does the investment threshold change as the value of (σ_V, σ_C) is varied so that σ_{total} remains constant. Figure 8 depicts such a sensitivity analysis when (σ_V, σ_C) evolves according to the chain $(0.20, 0.00) \to (0.14, 0.14) \to (0.00, 0.20)$, during which $\sigma_{total} = 0.20$, and $\alpha_V = \alpha_V = 0.04$, $k = 1.00$, and $\rho = 0.08$. The figure shows that for large values of K, $V^*(C, K)$ shifts down

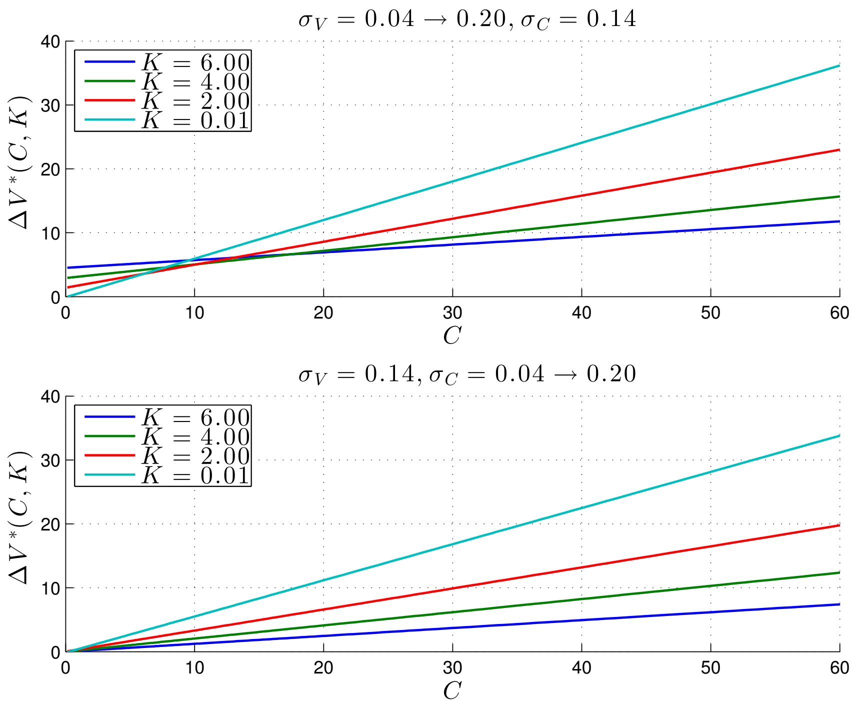

Figure 7: Sensitivity of the investment threshold with respect to σ_V and σ_C when $\alpha_V = 0.04$, $\alpha_C = -0.20$, $k = 1.00$, and $\rho = 0.08$

as σ_V decreases and σ_C increases. However, when $K = 0.01$, the investment threshold does not change visibly as (σ_V, σ_C) is varied so that σ_{total} remains constant, which is consistent with the result of [7]. The behaviour of $V^*(C, K)$ with respect to the distribution of the volatilities described above is common to all tested parameter values. Hence, we conclude that when $K >> 0$, a situation in which most of the uncertainty is due to σ_V leads to higher investment thresholds than a situation in which most of the uncertainty stems from σ_C. Still, the threshold $V^*(C, K \to 0)$, which governs the completion of the investment program, depends only on σ_{total}.

5 Conclusions

Given the trend towards deregulation of many infrastructure industries in OECD countries as well as the involvement of the private sector in infrastructure investment in emerging economies, policymakers and the private sector need appropriate valuation methods to sup-

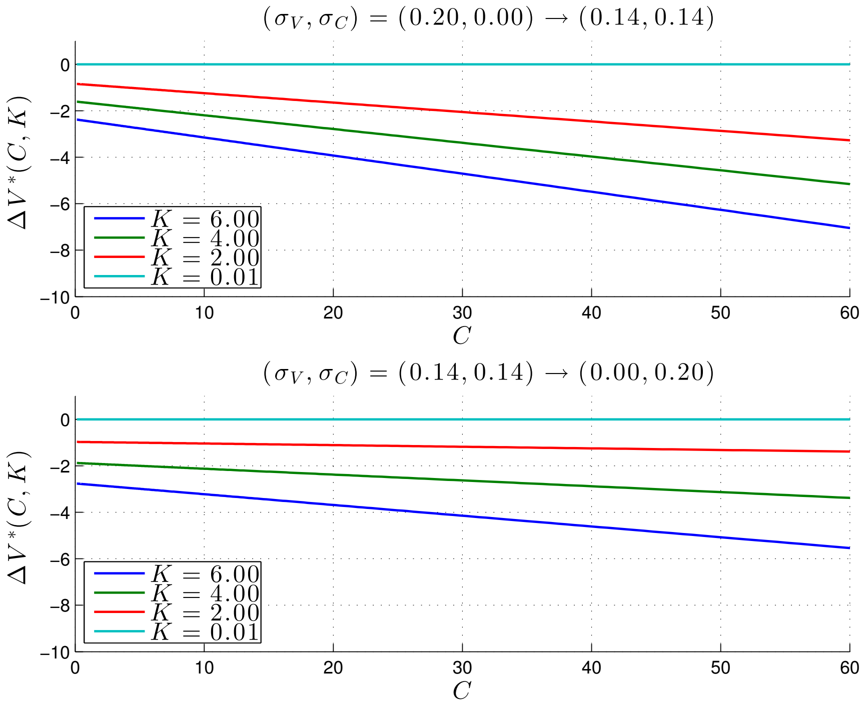

Figure 8: Sensitivity of the investment threshold with respect to the distribution of σ_{total} when $\alpha_V = 0.04$, $\alpha_C = 0.04$, $k = 1.00$, and $\rho = 0.08$

port decision making. In particular, investors in infrastructure projects with long lead times often face multiple sources of uncertainty and have the flexibility to control the rate of their investment program. In this paper, we propose a method to compute the option value and the investment thresholds for an investor who sequentially invests in project opportunity whose payoff is a function of two stochastic variables. The sequential nature of the investment process is modelled by allowing the investor to choose the rate at which to invest continuously in time. As the investment rate is assumed to be bounded between zero and a positive constant, the investor cannot obtain the payoff instantly but has to wait for at least a minimum time to build.

By enhancing the time-to-build model of [6], we develop an approach to handle infrastructure investment in a deregulated paradigm using the example of a PEV aggregator building charging infrastructure. In addition to the methodological innovation of accounting for multiple sources of uncertainty, we also demonstrate how conventional results from the real options literature are partially reversed:

1. A lower growth rate for the discounted value of the completed project's operating costs reduces the investment threshold in contrast to the analogous result from [6].

2. The investment threshold may decrease with the remaining required investment when the discounted value of the completed project's operating costs are high.

3. It may be optimal to proceed with investment even when the net discounted cash flows of the completed project are negative.

We choose V_t and C_t to represent the discounted cash in- and outflows of the completed project, respectively. We also assume that the value of the completed project at time t is $\max\{V_t - C_t, 0\}$. However, the stochastic variables could have other interpretations depending on which particular investment situation is of interest. Also, the payoff could be generally any function of V_t and C_t in our framework.[12] In this sense, our model is general and can be used to analyse multiple investment situations that meet the assumptions about the nature of the investment process and the stochastic variables.

One of the assumptions of the model is that the investor can decide continuously on whether to invest or wait in time. As discussed above, this assumption might be an appropriate approximation in some situations. However, if the initial investment decision is completely irreversible, then our model does not apply. Therefore, the exercise of building and solving a two-factor model in which the investment decision is modelled following the lead of [18] could be interesting. In this case, the effect of the volatilities of the processes that V_t and C_t follow on the results could be in contrast to that in our model. In addition, it would be interesting to see whether the effect of the drift rates would be similar to that in our model: for even if the initial investment decision is made completely irreversible, the investment lag implies that a rational investor considers how the payoff is expected to evolve during the lag when making investment decisions. Finally, our model could be extended to incorporate other types of managerial flexibilities, e.g., options to size the capacity of the completed project or to abandon it during the construction phase, and regime-switching or

[12] We do not take a stance on which conditions the payoff function should meet in order for the problem to have a solution.

stochastic-volatility processes instead of GBMs to reflect the fact that discounted revenues and operating costs of the completed project may not be adequately approximated by the market values of any existing assets.

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Appendix A Numerical Solution Method

We first apply the transformation $F(V,C,K) = e^{-\rho\frac{K}{k}}G(X,Y,K)$, where $X = \ln V$ and $Y = \ln C$, to the PDE given by Eq. (5a) in order to modify the PDE to a simpler form and to ensure numerical stability. After the transformation, the PDE in $\mathcal{R}$ is:

$$\frac{1}{2}\sigma_V^2 G_{XX} + \frac{1}{2}\sigma_C^2 G_{YY} + \left(\alpha_V - \frac{1}{2}\sigma_V^2\right) G_X + \left(\alpha_C - \frac{1}{2}\sigma_C^2\right) G_Y - kG_K - ke^{\rho\frac{K}{k}} = 0 \qquad (10)$$

Note that the coefficients of the PDE are now constant. After the transformation, the boundary conditions that the solution for Eq. (10) must satisfy are:

$$G(X,Y,0) = e^{XY}, \qquad (11a)$$

$$\frac{G(X^*(Y,K),Y,K)}{G_X(X^*(Y,K),Y,K)} = \frac{1}{\beta(K)} \qquad (11b)$$

$$\frac{G(X^*(Y,K),Y,K)}{G_Y(X^*(Y,K),Y,K)} = \frac{1}{\eta(K)} \qquad (11c)$$

where $\beta(K)$ and $\eta(K)$ solve Eq. (8) for each value of K.

Since we will solve the PDE numerically in a cubic grid, we need some additional boundary conditions that apply at the boundaries of the grid. For this purpose, we assume the following second-order boundary conditions:

$$\lim_{X\to\infty} G_{XX} = 0 \qquad (12a)$$

$$\lim_{X\to-\infty} G_{XX} = 0 \qquad (12b)$$

$$\lim_{Y\to\infty} G_{YY} = 0 \qquad (12c)$$

$$\lim_{Y\to-\infty} G_{YY} = 0 \qquad (12d)$$

These boundary conditions are chosen since they are known to work well with many financial options [25] as well as for our model. We will from now require that these conditions are approximately met at the boundaries of the lattice.

Let us denote $G(i\Delta X, j\Delta Y, \ell\Delta K) = G^{\ell}_{i,j}$, where $i \in \{i_{min}, i_{min}+1, ..., i_{max}\}$, $j \in$

$\{j_{min}, j_{min}+1, ..., j_{max}\}$, and $\ell \in \{\ell_{min}, \ell_{min}+1, ..., \ell_{max}\}$. ΔX, ΔY, ΔK, and the minimum and maximum indices are predetermined constants that govern the dimensions of the lattice.[13] We use the following finite-difference approximations for the partial derivatives of G:

$$G_X(i\Delta X, j\Delta Y, \ell\Delta K) = \frac{G^{\ell}_{i+1,j} - G^{\ell}_{i-1,j}}{2\Delta X} \tag{13a}$$

$$G_Y(i\Delta X, j\Delta Y, \ell\Delta K) = \frac{G^{\ell}_{i,j+1} - G^{\ell}_{i,j-1}}{2\Delta Y} \tag{13b}$$

$$G_{XX}(i\Delta X, j\Delta Y, \ell\Delta K) = \frac{G^{\ell}_{i+1,j} - 2G^{\ell}_{i,j} + G^{\ell}_{i-1,j}}{(\Delta X)^2} \tag{13c}$$

$$G_{YY}(i\Delta X, j\Delta Y, \ell\Delta K) = \frac{G^{\ell}_{i,j+1} - 2G^{\ell}_{i,j} + G^{\ell}_{i,j-1}}{(\Delta Y)^2} \tag{13d}$$

$$G_K(i\Delta X, j\Delta Y, \ell\Delta K) = \frac{G^{\ell+1}_{i,j} - G^{\ell}_{i,j}}{\Delta K} \tag{13e}$$

By inserting the approximations above in the transformed PDE given by Eq. (10), we obtain the following difference equation:

$$G^{\ell+1}_{i,j} = a_+ G^{\ell}_{i+1,j} + a_- G^{\ell}_{i-1,j} + b_+ G^{\ell}_{i,j+1} + b_- G^{\ell}_{i,j-1} + cG^{\ell}_{i,j} - n_\ell \tag{14}$$

where

$$a_+ = \frac{\Delta K}{2k\Delta X}\left(\frac{\sigma_V^2}{\Delta X} + \alpha_V - \frac{\sigma_V^2}{2}\right) \tag{15a}$$

$$a_- = \frac{\Delta K}{2k\Delta X}\left(\frac{\sigma_V^2}{\Delta X} - \alpha_V + \frac{\sigma_V^2}{2}\right) \tag{15b}$$

$$b_+ = \frac{\Delta K}{2k\Delta Y}\left(\frac{\sigma_C^2}{\Delta Y} + \alpha_C - \frac{\sigma_C^2}{2}\right) \tag{15c}$$

$$b_- = \frac{\Delta K}{2k\Delta Y}\left(\frac{\sigma_C^2}{\Delta Y} - \alpha_C + \frac{\sigma_C^2}{2}\right) \tag{15d}$$

$$c = 1 - \frac{\sigma_V^2 \Delta K}{k(\Delta X)^2} - \frac{\sigma_C^2 \Delta K}{k(\Delta Y)^2} \tag{15e}$$

$$n_\ell = \Delta K e^{\rho \frac{\ell \Delta K}{k}} \tag{15f}$$

If the lattice point considered is on the lattice boundary, then we discretise the boundary conditions given by Eqs. (13a)–(13e). Subsequently, the discretised boundary conditions can

[13] i_{min} and j_{min} will be negative in order to obtain option values near the zero border in the (V, C)-world. The value of ℓ_{min} will be zero.

be inserted into Eq. (14) to compute the option value at the lattice point.

In terms of the computational method, first we calculate the values of G when $\ell = 0$ using Eq. (11a). Next, we calculate the values of option when $\ell = 1$ using Eq. (14) and the discretised versions of boundary conditions (13a)–(13e). Now that we know the preliminary option values at $\ell = 1$, the next task is to find the investment threshold. For this, we use boundary conditions (11b), (11c), and (8). By combining these conditions, the following equation must be met on the investment threshold:

$$\frac{1}{2}\sigma_V^2 \frac{G_X}{G}\left(\frac{G_X}{G} - 1\right) + \frac{1}{2}\sigma_C^2 \frac{G_Y}{G}\left(\frac{G_Y}{G} - 1\right) + \alpha_V \frac{G_X}{G} + \alpha_C \frac{G_Y}{G} - \rho = 0 \tag{16}$$

Our strategy is then to evaluate the left-hand side of this equation at every lattice point for $\ell = 1$ by using the finite-difference approximations in Eqs. (13a) and (13b).[14] The location of the investment threshold given a value of j is then the pair (i, j), for which the absolute value of the left-hand side of Eq. (16) is the smallest in $i \in \{i_{min} + 1, i_{min} + 1, ..., i_{max} - 1\}$.[15] After we have numerically solved for the free boundary for $\ell = 1$, we can obtain the values of constants $A(\Delta K)$, $\beta(\Delta K)$, and $\eta(\Delta K)$ using Eqs. (11b) and (11c), the value-matching condition in Eq. (6d), the functional transformation, and the form of the analytical solution in the waiting region given by Eq. (7). We solve for the values of these constants at each investment threshold for $\ell = 1$ and take the averages of these values to determine the final values.

After having calculated the initial option values, the placement of the investment threshold, and the constants of the analytical solution in the lower region, we should fill the waiting region for $\ell = 1$ with the values given by the analytical solution before repeating the procedure above for $\ell = 2$. However, as this proves to cause numerical instability, we update the option values after the initial option values and the investment thresholds have been determined for all values of ℓ. Once the iteration above has been completed for all values of ℓ and the option values in the waiting region are updated, the final solution for the investor's

[14]The locations of the investment threshold at $\ell = \ell_{min}$ and $\ell = \ell_{max}$ are extrapolated.

[15]Here, we implicitly assume that the investment threshold is not at i_{min} or i_{max} for any value of Y or K. This assumption is met if the lattice dimensions are chosen properly.

problem is obtained by using the functional and variable transformations in the opposite direction than what was initially done.