Residual-Based A Posteriori Error Estimate for Interface Problems: Nonconforming Linear Elements*

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Abstract. In this paper, we study the residual-based a posteriori error estimation for the nonconforming linear finite element approximation to the interface problem. We introduce a new and direct approach, without using the Helmholtz decomposition, to analyze the reliability of the estimator. It is proved that a slightly modified estimator is reliable with the constant independent of the jump of the interfaces, without the assumption that the diffusion coefficient is quasi-monotone. Numerical results for a test problem with intersecting interfaces are also presented.

1 Introduction.

During the past decade, the construction, analysis, and implementation of robust a posteriori error estimators for various finite element approximations to partial differential equations with parameters have been one of the focuses of research in the field of the a posteriori error estimation. For the elliptic interface problem, various robust estimators have been constructed, analyzed, and implemented (see, e.g., [4, 23, 22, 8, 9, 11, 26, 12, 13] for conforming elements, [1, 20, 10] for nonconforming elements, [10] for mixed elements, and [7] for discontinuous elements). The robustness for residual based estimator in the reliability bound is established theoretically under the assumption of the quasi-monotone distribution of the diffusion coefficients, see [4] for more details. However, numerical results by many researchers including ours strongly suggest that those estimators are robust even when the diffusion coefficients are not quasi-monotone. In this paper, we provide a theoretical evidence for the nonconforming linear element without the quasi-monotone assumption.

One of the key steps in obtaining the robust reliability bound of classical residual based estimator is to construct a modified Clément-type interpolation operator such that it satisfies specific approximation and stability properties in the energy norm (see [4] for details). For the conforming element, the degrees of freedom are the nodal values at vertices of triangles. The nodal value of the modified Clément-type interpolation is defined by the average value of the function over connected elements whose corresponding

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diffusion coefficients are the greatest. Under the quasi-monotone assumption, Bernardi and Verfürth [4] are able to establish the required properties of the interpolation operator to guarantee the robust reliability bound. A key advantage for the nonconforming linear element is that its degrees of freedom are nodal values at the middle points of edges of triangles and that each middle point is shared by at most two triangles. Hence, we are able to construct a modified Clément-type interpolation satisfying the desired properties (see Section 4).

The a posteriori error estimation for the nonconforming elements has been studied by many researchers. Due to the lack of the error equation, Dari, Duran, Padra, and Vampa [15] established the reliability bound of the residual-based error estimator for the Poisson equation through the Helmholtz decomposition of the true error. Their analysis is widely used by other researchers (see, e.g., [14, 5, 1, 6]), and the Helmholtz decomposition becomes a necessary tool for obtaining the reliability bound for the nonconforming elements. This approach has also been applied to the mixed finite element method [21] and discontinuous Garlerkin finite element method [3, 2, 7]. It is obvious that application of their analysis to the interface problem will lead to the same distribution assumption as the conforming elements in [4].

Ainsworth [1] constructed an equilibrated estimator without using the Clément type interpolation but the error bounds depend on the jump of diffusion constants. Despite the main trend of using Helmholtz decomposition in the nonconforming finite element analysis, there are several other interesting papers that approached differently. Hoppe and Wohlmuth [18] constructed two a posteriori error estimators by using the hierarchical basis under the saturation assumption. Schieweck [24] constructed a two-sided bound of the energy error using the analysis of conforming case with some simple additional arguments. Nevertheless, conforming Clément type interpolation was applied in this paper hence again impose the assumption of quasi-monotonicity.

In this paper, we present a new and direct analysis, which does not involve the Helmholtz decomposition, for estimating the reliability bound with the aim of removing the quasi-monotone assumption. To do so, our analysis makes use of (a) our newly developed the error equation for the nonconforming finite element approximation in [7] and (b) the fundamental orthogonality of the nonconforming elements. Combining with our observation on the modified Clément-type interpolation for the nonconforming elements, we are able to bound both the element residuals and the numerical flux jumps uniformly without the quasi-monotonicity. This is also done for a slightly modified numerical solution jumps of the a posteriori error estimator.

The outline of the paper is as follows. The interface problem and its nonconforming finite element approximation are introduced in Section 2 as well as the L^2 representation of the true error in the (broken) energy norm. The indicator and the estimators are presented in Section 3. The modified Clément-type interpolation operator is defined and its approximation properties are proved in Section 4. Robust local efficiency and global reliability bounds are established in Sections 5 and 6, respectively. Finally, we provide some numerical results in Section 7.

2 Nonconforming linear element approximation to interface problem.

2.1 Interface problem.

For simplicity of the presentation, we consider only two dimensions. Extension of the results in this paper to three dimensions is straightforward. Let Ω be a bounded, open, connected subset of $\Re^2$ with a Lipschitz continuous boundary $\partial\Omega$. Denote by $\mathbf{n} = (n_1, n_2)^t$ the outward unit vector normal to the boundary. We partition the boundary of the domain Ω into two open subsets Γ_D and Γ_N such that $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and that $\Gamma_D \cap \Gamma_N = \emptyset$. For simplicity, we assume that Γ_D is not empty (i.e., $\text{mes}\,(\Gamma_D) \neq 0$). Consider the following elliptic interface problem

$$-\nabla \cdot\, (\alpha(x)\nabla\, u) = f \quad \text{in } \Omega \tag{2.1}$$

with boundary conditions

$$u = g_D \quad \text{on } \Gamma_D \quad \text{and} \quad \mathbf{n} \cdot (\alpha \nabla\, u) = g_N \quad \text{on } \Gamma_N, \tag{2.2}$$

where $\nabla\cdot$ and ∇ are the divergence and gradient operators, respectively; f, g_D, and g_N are given scalar-valued functions; and the diffusion coefficient $\alpha > 0$ is piecewise constant with respect to a partition of the domain $\bar{\Omega} = \bigcup_{i=1}^{n} \bar{\Omega}_i$. Here the subdomain Ω_i is open and polygonal. The jump of the α across interfaces (subdomain boundaries) are possibly very large. For simplicity, assume that f, g_D, and g_N are piecewise linear functions.

We use the standard notations and definitions for the Sobolev spaces $H^s(\Omega)$ and $H^s(\partial\Omega)$ for $s \geq 0$. The standard associated inner products are denoted by $(\cdot,\, \cdot)_{s,\Omega}$ and $(\cdot,\, \cdot)_{s,\partial\Omega}$, and their respective norms are denoted by $\|\cdot\|_{s,\Omega}$ and $\|\cdot\|_{s,\partial\Omega}$. (We omit the subscript Ω from the inner product and norm designation when there is no risk of confusion.) For $s = 0$, $H^s(\Omega)$ coincides with $L^2(\Omega)$. In this case, the inner product and norm will be denoted by $\|\cdot\|$ and $(\cdot,\, \cdot)$, respectively. Let

$$H^1_{g,D}(\Omega) := \{v \in H^1(\,\Omega\,) \,:\, v = g_D \text{ on } \Gamma_D\}.$$

The corresponding variational formulation of problem (2.1)-(2.2) is to find $u \in H^1_{g,D}(\Omega)$ such that

$$a(u, v) = f(v), \quad \forall\, v \in H^1_{0,D}(\Omega), \tag{2.3}$$

where the bilinear and linear forms are defined by

$$a(u, v) = (\alpha(x)\nabla u, \nabla v)_\Omega \quad \text{and} \quad f(v) = (f, v)_\Omega + (g_N, v)_{\Gamma_N}.$$

2.2 Nonconforming finite element approximation.

Let $\mathcal{T}_h$ be a triangulation of the domain Ω. Assume that $\mathcal{T}_h$ is regular; i.e., for all $K \in \mathcal{T}_h$, there exist a positive constant κ such that

$$h_K \leq \kappa\, \rho_K,$$

where h_K denotes the diameter of the element K and ρ_K the diameter of the largest circle that may be inscribed in K. Note that the assumption of the mesh regularity does not exclude highly, locally refined meshes. Let

$$\mathcal{N}_h = \mathcal{N}_I^h \cup \mathcal{N}_D^h \cup \mathcal{N}_N^h \quad \text{and} \quad \mathcal{E}_h = \mathcal{E}_I^h \cup \mathcal{E}_D^h \cup \mathcal{E}_N^h,$$

where $\mathcal{N}_I^h$ ($\mathcal{E}_I^h$) is the set of all interior vertices (edges) in $\mathcal{T}_h$, and $\mathcal{N}_D^h$ ($\mathcal{E}_D^h$) and $\mathcal{N}_N^h$ ($\mathcal{E}_N^h$) are the respective sets of all vertices (edges) on Γ_D and Γ_N. For each $e \in \mathcal{E}_h$, denote by m_e the mid-point of the edge e. Furthermore, assume that interfaces

$$F = \{\partial\Omega_i \cup \partial\Omega_j \, : \, i, j = 1, \cdots n\}$$

do not cut through any element $K \in \mathcal{T}_h$. Denote by $\mathcal{N}_F$ the set of all interface intersecting points. Then the assumption implies

$$\mathcal{N}_F \subset \mathcal{N}_h.$$

Let $P_k(K)$ be the space of polynomials of degree less than or equal to k on the element K. Denote the conforming piecewise linear finite element space associated with the triangulation $\mathcal{T}_h$ by

$$\mathcal{U}^c = \{v \in H^1(\Omega) : \, v|_K \in P_1(K), \; \forall\, K \in \mathcal{T}_h\}$$

and its subset by

$$\mathcal{U}^c_{g,D} = \{v \in \mathcal{U}^c : v = g_D \,\text{on}\, \Gamma_D\}.$$

Denote the nonconforming piecewise linear finite element space, the Crouzeix-Raviart element [17], associated with the triangulation $\mathcal{T}_h$ by

$$\mathcal{U}^{nc} = \{v \in L^2(\Omega) : v|_K \in P_1(K), \; \forall\, K \in \mathcal{T}_h, \text{ and } v \text{ is continuous at } m_e \text{ for all } e \in \mathcal{E}_I^h\}$$

and its subset by

$$\mathcal{U}^{nc}_{g,D} = \{v \in \mathcal{U}^{nc} \, : \, v(m_e) = g_D(m_e), \; \forall\, e \in \mathcal{E}_D^h\}.$$

Let

$$H^1(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_K \in H^1(K), \quad \forall\, K \in \mathcal{T}_h\}.$$

For any $v,\, w \in H^1(\mathcal{T}_h)$, denote the (broken) bilinear form by

$$a_h(v,\, w) = \sum_{K \in \mathcal{T}_h} (\alpha \nabla v,\, \nabla w)_K$$

and the (broken) energy norm by

$$|\!|\!| v |\!|\!|_\Omega = \sqrt{a_h(v,\, v)} = \left(\sum_{K \in \mathcal{T}_h} \|\alpha^{1/2} \nabla v\|_{0,K}^2 \right)^{1/2}.$$

The nonconforming finite element approximation is to find $u_h \in \mathcal{U}^{nc}_{g,D}$ such that

$$a_h(u_h, v) = f(v), \; \forall\, v \in \mathcal{U}^{nc}_{0,D}. \tag{2.4}$$

2.3 L^2 represetation of the error.

For each edge $e \in \mathcal{E}_h$, denote by h_e the length of e; denote by $\mathbf{n}_e$ a unit vector normal to e. When $e \in \mathcal{E}_D^h \cup \mathcal{E}_N^h$, denote by K_+^e the boundary element with the edge e, and assume that $\mathbf{n}_e$ is the unit outward normal vector of K_+^e. For any $e \in \mathcal{E}_I^h$, let K_+^e and K_-^e be the two elements sharing the common edge e assuming that

$$\alpha_e^+ \equiv \alpha_{K_+^e} \geq \alpha_{K_-^e} \equiv \alpha_e^-,$$

and that $\mathbf{n}_e$ coincides with the unit outward normal vector of K_+^e. Denote by $v|_+^e$ and $v|_-^e$, respectively, the traces of the double valued function v over e restricted on K_+^e and K_-^e. For any $v \in H^1(\mathcal{T}_h)$, denote the normal flux jump over edge $e \in \mathcal{E}_h$ by

$$[\![\alpha \frac{\partial v}{\partial n}]\!]_e := \begin{cases} (\alpha \nabla v \cdot \mathbf{n}_e)|_+^e - (\alpha \nabla v \cdot \mathbf{n}_e)|_-^e, & e \in \mathcal{E}_I^h, \\ 0, & e \in \mathcal{E}_D^h, \\ (\alpha \nabla v \cdot \mathbf{n}_e)|_+^e - g_N, & e \in \mathcal{E}_N^h, \end{cases}$$

and the value jump over edge $e \in \mathcal{E}_h$ by

$$[\![v]\!]_e = \begin{cases} v|_+^e - v|_-^e, & e \in \mathcal{E}_I^h, \\ v|_+^e - g_D, & e \in \mathcal{E}_D^h, \\ 0, & e \in \mathcal{E}_N^h. \end{cases}$$

The arithmetic average over edge $e \in \mathcal{E}_h$ is denoted by

$$\{v\}_e = \begin{cases} \dfrac{v|_+^e + v|_-^e}{2}, & e \in \mathcal{E}_I^h, \\ v|_+^e, & e \in \mathcal{E}_D^h \cup \mathcal{E}_N^h. \end{cases}$$

A simple calculation leads to the following identity:

$$[\![uv]\!]_e = \{u\}_e [\![v]\!]_e + [\![u]\!]_e \{v\}_e, \quad \forall\, e \in \mathcal{E}_I^h. \tag{2.5}$$

For any $v \in \mathcal{U}_{0,D}^{nc}$, it is well known that the following orthogonality property holds

$$\int_e [\![v]\!]\, ds = 0, \ \forall\, e \in \mathcal{E}_I^h \cup \mathcal{E}_N^h \quad \text{and} \quad \int_e v\, ds = 0, \ \forall\, e \in \mathcal{E}_D^h. \tag{2.6}$$

Let u and u_h be the solutions of (2.3) and (2.4), respectively. It is shown in [7] that

$$a_h(u, v_h) = f(v) + \sum_{e \in \mathcal{E}_I^h} \int_e \alpha \frac{\partial u}{\partial n} [\![v_h]\!]\, ds + \sum_{e \in \mathcal{E}_D^h} \int_e \alpha \frac{\partial u}{\partial n} v_h\, ds, \quad \forall\, v_h \in \mathcal{U}_{0,D}^{nc}. \tag{2.7}$$

Denote the true error by

$$E = u - u_h.$$

Difference of (2.7) and (2.4) yields the following error equation:

$$a_h(E, v_h) = \sum_{e\in\mathcal{E}_I^h} \int_e \alpha \frac{\partial u}{\partial n} [\![v_h]\!]\, ds + \sum_{e\in\mathcal{E}_D^h} \int_e \alpha \frac{\partial u}{\partial n} v_h\, ds, \quad \forall\, v_h \in \mathcal{U}_{0,D}^{nc}. \tag{2.8}$$

Introducing the element residual, the numerical flux jump, and the numerical solution jump

$$r_K = \big(f + \nabla\cdot(\alpha\nabla u_h)\big)\big|_K, \quad \forall\, K \in \mathcal{T}_h,$$

$$j_{\sigma,e} = [\![\alpha \frac{\partial u_h}{\partial n}]\!]_e \quad \text{and} \quad j_{u,e} = [\![u_h]\!]_e, \quad \forall\, e \in \mathcal{E}_h,$$

respectively, then the true error in the (broken) energy norm may be expressed in terms of those quantities.

Lemma 2.1. *Let $E_h \in \mathcal{U}_{0,D}^{nc}$ be an interpolation of E, then we have the following L^2 representation of the error E in the (broken) energy norm:*

$$a_h(E, E) = \sum_{K\in\mathcal{T}_h} (r_K, E - E_h)_K - \sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_N^h} \int_e j_{\sigma,e}\, \{E - E_h\}\, ds - \sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_D^h} \int_e \{\alpha \frac{\partial E}{\partial n}\}\, j_{u,e}\, ds. \tag{2.9}$$

Proof. First note that $\alpha \frac{\partial u_h}{\partial n}\big|_e$ is a constant for every $e \in \mathcal{E}_h$ and that $E_h \in \mathcal{U}_{0,D}^{nc}$. The orthogonality in (2.6) leads to

$$\int_e \{\alpha \frac{\partial u_h}{\partial n}\}[\![E_h]\!]\, ds = 0, \quad \forall\, e \in \mathcal{E}_I^h \quad \text{and} \quad \int_e \alpha \frac{\partial u_h}{\partial n} E_h\, ds = 0, \quad \forall\, e \in \mathcal{E}_D^h. \tag{2.10}$$

It follows from integration by parts, (2.5), the continuities of the normal component of the flux $\boldsymbol{\sigma} = -\alpha\nabla u$ and the solution u, and (2.10) that

$$\begin{aligned}
a_h(E,\, E - E_h) &= \sum_{K\in\mathcal{T}_h} (\alpha\nabla E, \nabla(E - E_h)) \\
&= \sum_{K\in\mathcal{T}_h} (r_K,\, E - E_h)_K + \sum_{e\in\mathcal{E}_I^h} \int_e [\![\alpha \frac{\partial E}{\partial n} (E - E_h)]\!]\, ds + \sum_{e\in\mathcal{E}_D^h\cup\mathcal{E}_N^h} \int_e \alpha \frac{\partial E}{\partial n} (E - E_h)\, ds \\
&= \sum_{K\in\mathcal{T}_h} (r_K,\, E - E_h)_K + \sum_{e\in\mathcal{E}_I^h} \int_e [\![\alpha \frac{\partial E}{\partial n}]\!]\, \{E - E_h\}\, ds + \sum_{e\in\mathcal{E}_I^h} \int_e \{\alpha \frac{\partial E}{\partial n}\}\, \big([\![E]\!] - [\![E_h]\!]\big)\, ds \\
&\quad + \sum_{e\in\mathcal{E}_D^h\cup\mathcal{E}_N^h} \int_e \alpha \frac{\partial E}{\partial n} (E - E_h)\, ds \\
&= \sum_{K\in\mathcal{T}_h} (r_K,\, E - E_h)_K - \sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_N^h} \int_e j_{\sigma,e}\, \{E - E_h\}\, ds - \sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_D^h} \int_e \{\alpha \frac{\partial E}{\partial n}\}\, j_{u,e}\, ds \\
&\quad - \sum_{e\in\mathcal{E}_I^h} \int_e \alpha \frac{\partial u}{\partial n} [\![E_h]\!]\, ds - \sum_{e\in\mathcal{E}_D^h} \int_e \alpha \frac{\partial u}{\partial n} E_h\, ds,
\end{aligned}$$

which, together with the error equation in (2.8) with $v_h = E_h$, yields

$$\begin{aligned} a_h(E,E) &= a_h(E,\, E - E_h) + a_h(E, E_h) \\ &= \sum_{K\in\mathcal{T}_h} (r_K,\, E - E_h)_K - \sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_N^h} \int_e j_{\sigma,e}\,\{E - E_h\}\, ds - \sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_D^h} \int_e \{\alpha \frac{\partial E}{\partial n}\}\, j_{u,e}\, ds. \end{aligned}$$

This completes the proof of the lemma. □

3 Indicator and estimators.

In this section, based on Lemma 2.1, we first introduce the standard indicator and estimator. Since the reliability bound of this estimator was established under the quasi-monotonicity assumption on the distribution of the coefficient, to avoid such an assumption we introduce a new estimator which is slightly bigger than the standard estimator.

For any $K \in \mathcal{T}_h$, denote by $\mathcal{N}_K^h$ and $\mathcal{E}_K^h$, respectively, the sets of three vertices and three edges of K. Let

$$\eta_{R_f,K}^2 = \frac{h_K^2}{\alpha_K}\,\|r_K\|_{0,K}^2,$$

$$\eta_{J_\sigma,K}^2 = \sum_{e\in\mathcal{E}_K^h\cap\mathcal{E}_I^h} \frac{h_e}{2\,\alpha_+^e}\,\|j_{\sigma,e}\|_{0,e}^2 + \sum_{e\in\mathcal{E}_K^h\cap\mathcal{E}_N^h} \frac{h_e}{\alpha_e}\,\|j_{\sigma,e}\|_{0,e}^2, \quad \text{and}$$

$$\eta_{J_u,K}^2 = \sum_{e\in\mathcal{E}_K^h\cap\mathcal{E}_I^h} \frac{\alpha_-^e}{2\,h_e}\,\|j_{u,e}\|_{0,e}^2 + \sum_{e\in\mathcal{E}_K^h\cap\mathcal{E}_D^h} \frac{\alpha_e}{h_e}\,\|j_{u,e}\|_{0,e}^2.$$

Then the indicator associated with $K \in \mathcal{T}_h$ is defined by

$$\eta_K = \left(\eta_{R_f,K}^2 + \eta_{J_\sigma,K}^2 + \eta_{J_u,K}^2\right)^{1/2}, \tag{3.1}$$

and the estimator by

$$\eta = \left(\sum_{K\in\mathcal{T}_h} \eta_K^2\right)^{1/2}. \tag{3.2}$$

By the standard argument [4], it is shown in section 5 that the indicator η_K is efficient uniformly with respect to the jump of the diffusion coefficient. By using the Helmholtz decomposition and the modified Clément-type interpolation, one can also prove that the estimator η is reliable. Moreover, the reliability constant is independent of the jump of $\alpha(x)$ provided that the distribution of $\alpha(x)$ is quasi-monotone [23]. In order to remove this assumption, we present a new analysis for estimating the reliability bound without using the Helmholtz decomposition. The analysis will make use of the structure of the nonconforming element in two-dimensions, and it enables us to bound both the element residual and the numerical flux jump uniformly without the quasi-monotonicity. Unfortunately, we are unable to do the same for the numerical solution jump, and hence it needs to be modified.

To this end, for each vertex $z \in \mathcal{N}_h$, denote by ω_z^h and $\mathcal{E}_z^h$, respectively, the sets of all elements $K \in \mathcal{T}_h$ and all edges $e \in \mathcal{E}_h$ having z as a common vertex. Let

$$\hat{\omega}_z^h = \{K \in \omega_z^h \, : \, \alpha_K = \max_{K' \in \omega_z^h} \alpha_{K'}\} \subset \omega_z^h$$

be the set of all elements in ω_z^h such that the corresponding diffusion coefficients are the greatest. For any interface intersecting point $z \in \mathcal{N}_F \subset \mathcal{N}_h$, the vertex patch ω_z^h is called quasi-monotone (see [23]) if for each $K \in \omega_z^h$, there exists a subset $\hat{w}_{z,K}^h$ of ω_z^h such that the union of elements in $\hat{w}_{z,K}^h$ is a Lipschitz domain and that

- If $z \in \mathcal{N}_h \setminus \mathcal{N}_D^h$, then $\{K\} \cup \hat{\omega}_z^h \subset \hat{w}_{z,K}^h$ and $\alpha_K \le \alpha_{K'}, \forall\, K' \in \hat{w}_{z,K}^h$;
- if $z \in \mathcal{N}_D^h$, then $K \in \hat{w}_{z,K}^h$, $\partial(\cup_{K' \in \hat{w}_{z,K}^h} K') \cap \Gamma_D \neq \emptyset$ and $\alpha_K \le \alpha_{K'}, \forall\, K' \in \hat{w}_{z,K}^h$.

Denote by

$$\mathcal{N}_M = \{z \in \mathcal{N}_F : \omega_z^h \text{ is not quasi-monotone }\} \subset \mathcal{N}_F \subset \mathcal{N}_h$$

the set of all interface intersecting points whose vertex patches are not quasi-monotone.

For each element $K \in \mathcal{T}_h$, subdivide it into four sub-triangles by connecting three mid-points of edges of K, and denote by $\mathcal{T}_{h/2}$ the refined triangulation. Let

$$\mathcal{N}_{h/2} = \mathcal{N}_I^{h/2} \cup \mathcal{N}_D^{h/2} \cup \mathcal{N}_N^{h/2} \quad \text{and} \quad \mathcal{E}_{h/2} = \mathcal{E}_I^{h/2} \cup \mathcal{E}_D^{h/2} \cup \mathcal{E}_N^{h/2}$$

where $\mathcal{N}_I^{h/2}(\mathcal{E}_I^{h/2})$, $\mathcal{N}_D^{h/2}(\mathcal{E}_D^{h/2})$, and $\mathcal{N}_N^{h/2}(\mathcal{N}_N^{h/2})$ are the sets of all interior vertices (edges) of $\mathcal{T}_{h/2}$, all boundary vertices (edges) on Γ_D and Γ_N , respectively. Let

$$\mathcal{U}_{g,D}^{h/2,c} = \left\{ v \in H^1(\Omega) : \quad v|_K \in P^1(K), \ \forall\, K \in \mathcal{T}_{h/2} \text{ and } v|_{\Gamma_D} = g_D \right\},$$

which is the continuous piecewise linear finite element space associated with the triangulation $\mathcal{T}_{h/2}$.

Next, we introduce an interpolation operator, $I_{h/2} : \mathcal{U}_{g,D}^{nc} \to \mathcal{U}_{g,D}^{h/2,c}$, from the nonconforming finite element space on $\mathcal{T}_h$ to the conforming finite element space on $\mathcal{T}_{h/2}$. For a given $v \in \mathcal{U}_{g,D}^{nc}$, the nodal values of $I_{h/2} v \in \mathcal{U}_{g,D}^{h/2,c}$ are defined as follows:

(i) set

$$(I_{h/2})(z) = g_D(z), \quad \forall\, z \in \mathcal{N}_D^h.$$

(ii) set

$$(I_{h/2} v)(m_e) = v(m_e), \quad \forall\ e \in \mathcal{E}_h;$$

(iii) for $z \in (\mathcal{N}_I^h \cup \mathcal{N}_N^h) \setminus \mathcal{N}_M$, set

$$(I_{h/2} v)(z) = v|_{K_z}(z),$$

where K_z is chosen to be one element in $\hat{\omega}_z^h$;

(iv) for $z \in \mathcal{N}_M \setminus \mathcal{N}_D^h$, set

$$(I_{h/2}v)(z) = \frac{1}{n_z} \sum_{K \in \omega_z^h} v|_K(z),$$

or

$$(I_{h/2}v)(z) = \frac{\sum\limits_{K \in \omega_z^h} \alpha_K \, v|_K(z)}{\sum\limits_{K \in \omega_z^h} \alpha_K},$$

where n_z is the number of triangles in ω_z^h.

For each vertex $z \in \mathcal{N}_h$, denote by $\omega_z^{h/2}$ and $\mathcal{E}_z^{h/2}$ the sets of all elements $K \in \mathcal{T}_{h/2}$ and all edges $e \in \mathcal{E}_{h/2}$ having z as a common vertex. For each $K \in \mathcal{E}_h$, denote by $\mathcal{E}_K^{h/2}$ the set of all sub-edges on $\mathcal{E}_K^h$. For the element $K \in \mathcal{T}_h$ with at least one vertex in $\mathcal{N}_M$, the numerical solution jump $\eta_{J_u,K}$ is modified as follows:

$$\begin{aligned} \tilde{\eta}_{J_u,K}^2 \quad = \quad & \sum_{z \in \mathcal{N}_K^h \setminus \mathcal{N}_M} \left(\sum_{e \in \mathcal{E}_z^{h/2} \cap \mathcal{E}_K^{h/2} \cap \mathcal{E}_I^{h/2}} \frac{\alpha_-^e}{2\, h_e} \|j_{u,e}\|_{0,e}^2 + \sum_{e \in \mathcal{E}_z^{h/2} \cap \mathcal{E}_K^{h/2} \cap \mathcal{E}_D^{h/2}} \frac{\alpha_e}{h_e} \|j_{u,e}\|_{0,e}^2 \right) \\ & + \sum_{z \in \mathcal{N}_K^h \cap \mathcal{N}_M} \frac{\alpha_K}{h_{T_{K,z}}} \left\| I_{h/2} u_h - u_h \right\|_{0,\partial T_{K,z}}^2, \end{aligned} \tag{3.3}$$

where $T_{K,z} = \omega_z^{h/2} \cap K$. The modified estimator is then given by

$$\tilde{\eta} = \left(\sum_{K \in \mathcal{T}_h} \tilde{\eta}_K^2 \right)^{1/2}.$$

In adaptive local mesh refinement algorithms, the indicator η_K is used for local mesh refinement, and the estimator $\tilde{\eta}$ is used for global error control.

Remark 3.1. *In the case that $\mathcal{N}_M = \emptyset$; i.e., the distribution of the diffusion coefficient is quasi-monotone, then $\eta_{J_u,K} = \tilde{\eta}_{J_u,K}$ for all $K \in \mathcal{T}_h$.*

4 The modified Clément-type interpolation.

In this section, following the idea in [4, 16], we introduce the modified Clément-type interpolation operator for the nonconforming linear element and establish its approximation property.

Denote by

$$\fint_\omega v \, dx = \frac{1}{\text{meas}(\omega)} \int_\omega v \, dx$$

the mean value of a given function v on a given measurable set ω in $\mathcal{R}^2$ with positive 2-dimensional Lebesgue measure meas(ω). With this convention, set

$$\pi_e(v) = \fint_{K_+^e} v\,dx, \quad \forall\, e \in \mathcal{E}_h.$$

The modified Clément interpolation operator $\mathcal{I}_h : L^2(\Omega) \to \mathcal{U}^{nc}$ is defined by

$$\mathcal{I}_h(v) = \sum_{e\in\mathcal{E}_h} (\pi_e v)\phi_e, \tag{4.1}$$

where ϕ_e is the nodal basis function of $\mathcal{U}^{nc}$ which takes value 1 at m_e and takes 0 at mid-points of other edges.

For any $K \in \mathcal{T}_h$, let $\triangle_K$ be the union of elements in $\mathcal{T}_h$ sharing an edge with K. For any $e \in \mathcal{E}_h$, let $\triangle_e$ be the union of elements in $\mathcal{T}_h$ having the common edge e.

Lemma 4.1. *For any function $v \in H^1(\mathcal{T}_h)$, then the modified Clément interpolation satisfies the following approximation properties:*

$$\|v - \mathcal{I}_h v\|_{0,K} \lesssim \frac{h_K}{\alpha_K^{1/2}} \left(\sum_{K'\in\triangle_K} |\!|\!| v |\!|\!|_{K'} + \sum_{e\in\mathcal{E}_K^h} \left(\frac{\alpha_-^e}{h_e}\right)^{1/2} \big\| [\![v]\!] \big\|_{0,e} \right) \quad \forall\, K \in \mathcal{T}_h \tag{4.2}$$

and

$$\big\| v|_+^e - \pi_e v \big\|_{0,e} \lesssim \left(\frac{h_e}{\alpha_+^e}\right)^{1/2} |\!|\!| v |\!|\!|_{0,K_+^e}, \quad \forall\, e \in \mathcal{E}_h. \tag{4.3}$$

Here and thereafter, we use the $a \lesssim b$ notation to indicate that $a \le c\,b$ for a further not speficified constant c, which depends only on the shape regularity of $\mathcal{T}_h$ but not on the data of the underlying problems, in particular, the jump of the diffusion coefficient. Unlike the modified Clément interpolation for the conforming elements, there is an extra jump term in the approximation property in (4.2) which is due to the discontinuity of the function v across the edges of K.

Proof. For any $K \in \mathcal{T}_h$, since the nodal basis functions form a partition of the unity, the triangle inequality gives

$$\|v - \mathcal{I}_h v\|_{0,K} = \Big\| \sum_{e\in\mathcal{E}_K^h} \phi_e(v - \pi_e v) \Big\|_{0,K} \le \sum_{e\in\mathcal{E}_K^h} \big\|\phi_e(v - \pi_e v)\big\|_{0,K} \le \sum_{e\in\mathcal{E}_K^h} \|v - \pi_e v\|_{0,K}.$$

Hence, to show the validity of (4.2), it suffices to prove that

$$\|v - \pi_e v\|_{0,K} \lesssim \frac{h_K}{\alpha_K^{1/2}} \left(\sum_{K'\in\triangle_e} |\!|\!| v |\!|\!|_{K'} + \left(\frac{\alpha_-^e}{h_e}\right)^{1/2} \big\| [\![v]\!] \big\|_{0,e} \right), \quad \forall\, e \in \mathcal{E}_K^h. \tag{4.4}$$

Since the set $\triangle_e$ contains at most two elements, it is obvious that $K = K_+^e$ or K_-^e. If $K = K_+^e$, then (4.4) is a direct consequence of the Poincaré inequality:

$$\|v - \pi_e v\|_{0,K} = \big\|v - \fint_K v\,dx\big\|_{0,K} \lesssim h_K \alpha_K^{-1/2} |\!|\!| v |\!|\!|_K.$$

In the case that $K = K_-^e$, the triangle and the Poincaré inequalities imply

$$\begin{aligned}\|v - \pi_e v\|_{0,K} &\le \|v - \fint_K v\,dx\|_{0,K} + \|\fint_K v\,dx - \fint_{K_+^e} v\,dx\|_{0,K} \\ &\lesssim h_K \alpha_K^{-1/2} |||v|||_K + h_K^{1/2} \| \left(\fint_K v\,dx - v|_-^e\right) + \left(v|_+^e - \fint_{K_+^e} v\,dx\right) - [\![v]\!] \|_{0,e} \\ &\le h_K \alpha_K^{-1/2} |||v|||_K + h_K^{1/2} \left(\| \fint_K v\,dx - v|_-^e \|_{0,e} + \| \fint_{K_+^e} v\,dx - v|_+^e \|_{0,e} + \| [\![v]\!] \|_{0,e} \right).\end{aligned}$$

Next, we bound the three terms above. It follows from the trace theorem and the Poincaré inequality that

$$h_K^{1/2} \| \fint_K v\,dx - v|_-^e \|_{0,e} \lesssim \| \fint_K v\,dx - v \|_{0,K} + h_K \left| \fint_K v\,dx - v \right|_{1,K} \lesssim h_K \alpha_K^{-1/2} |||v|||_K.$$

Similarly, we have

$$h_K^{1/2} \| \fint_{K_+^e} v\,dx - v|_+^e \|_{0,e} \lesssim h_K \alpha_{K_+^e}^{-1/2} |||v|||_{K_+^e}. \tag{4.5}$$

Note that $\alpha_-^e = \alpha_K \le \alpha_{K_+^e}$, combining above three inequalities gives

$$\|v - \pi_e v\|_{0,K} \lesssim \frac{h_K}{\alpha_K^{1/2}} \left(\sum_{K' \in \triangle_e} |||v|||_{K'} + \left(\frac{\alpha_-^e}{h_e}\right)^{1/2} \| [\![v]\!] \|_{0,e} \right),$$

which proves the validity of (4.4) when $K = K_-^e$. (4.3) is a direct consequence of (4.5). This completes the proof of the lemma. □

5 Local efficiency bound.

It is standard to obtain the local efficiency bound for the residual-based a posteriori error estimator by using local edge and element bubble functions, ψ_e and ψ_K (see [25] for their definitions and properties). By properly weighting terms in the indicator by the diffusion coefficient, one can show that the local efficiency bound is robust (see [4]). For the convenience of readers, we only sketch the proof in this section.

Theorem 5.1. (Local Efficiency) *Assuming that $u \in H^{1+\epsilon}(\Omega)$ and u_h are the solutions of* (2.3) *and* (2.4)*, respectively, then the indicator η_K satisfies the following local efficiency bound:*

$$\eta_K \lesssim \sum_{K' \in \triangle_K} |||E|||_{K'}, \quad \forall\, K \in \mathcal{T}_h. \tag{5.1}$$

Proof. For any $K \in \mathcal{T}_h$, it follows from the properties of ψ_K, integration by parts, and the Cauchy-Schwarz inequality that

$$\begin{aligned} \|r_K\|_{0,K}^2 &\lesssim \int_K (f + \nabla \cdot (\alpha \nabla u_h))\, r_K \psi_K \, dx = \int_K \alpha \nabla (u - u_h) \cdot \nabla (r_K \psi_K)\, dx \\ &\lesssim \alpha_K^{1/2} \|\!|u - u_h|\!\|_K \, |r_K \psi_K|_{1,K} \lesssim \alpha_K^{1/2} h_K^{-1} \|\!|u - u_h|\!\|_K \, \|r_K\|_{0,K}, \end{aligned}$$

which implies

$$\|r_K\|_{0,K} \lesssim \frac{\alpha_K^{1/2}}{h_K} \|\!|u - u_h|\!\|_K, \quad \forall\, K \in \mathcal{T}_h. \tag{5.2}$$

For any $e \in \mathcal{E}_I^h$, by using the properties of ψ_e, integration by parts, the Cauchy-Schwartz inequality, and (5.2), we have

$$\|j_{\sigma,e}\|_{0,e}^2 \lesssim \int_e [\![\alpha \frac{\partial u_h}{\partial n}]\!]\, j_{\sigma,e} \psi_e \, ds = - \sum_{K \in \triangle_e} \int_{\partial K} \alpha \frac{\partial (u - u_h)}{\partial n}\, j_{\sigma,e} \psi_e \, ds$$

$$= \sum_{K \in \triangle_e} \left(- \int_K \alpha \nabla (u - u_h) \cdot \nabla (j_{\sigma,e} \psi_e)\, dx + \int_K r_K \, j_{\sigma,e} \psi_e \, dx \right) \lesssim \left(\frac{\alpha_+^e}{h_e} \right)^{1/2} \|\!|E|\!\|_{\triangle_e} \|j_{\sigma,e}\|_{0,e},$$

where $\|\!|E|\!\|_{\triangle_e} = \left(\sum_{K \in \triangle_e} \|\!|E|\!\|_K^2 \right)^{1/2}$. Together with a similar bound for $e \in \mathcal{E}_D^h \cup \mathcal{E}_N^h$, it implies

$$\|j_{\sigma,e}\|_{0,e} \lesssim \left(\frac{\alpha_+^e}{h_e} \right)^{1/2} \|\!|E|\!\|_{\triangle_e}, \quad \forall\, e \in \mathcal{E}_h. \tag{5.3}$$

For any $e \in \mathcal{E}_h$, let $\mathbf{n}_e = (n_1, n_2)$, then $\boldsymbol{\tau}_e = (-n_2, n_1)$ is the unit vector tangent to the edge e. Denote by $j_{\tau,e} = [\![\frac{\partial u_h}{\partial \tau}]\!]_e$ the jump of the tangential derivative of the numerical solution u_h along the edge e. By the continuity of u_h at the midpoint m_e, we have

$$\|j_{u,e}\|_{0,e} = \frac{1}{\sqrt{12}} h_e \, \|j_{\tau,e}\|_{0,e}, \quad \forall\, e \in \mathcal{E}_I^h. \tag{5.4}$$

For any $e \in \mathcal{E}_I^h$, it follows from the properties of ψ_e, integration by parts, and the Cauchy-Schwartz inequality that

$$\begin{aligned} \|j_{\tau,e}\|_{0,e}^2 &\lesssim \int_e [\![\frac{\partial u_h}{\partial \tau}]\!]\, j_{\tau,e} \psi_e \, ds = - \sum_{K \in \triangle_e} \int_{\partial K} \frac{\partial (u - u_h)}{\partial \tau}\, j_{\tau,e} \psi_e \, ds \\ &= - \sum_{K \in \triangle_e} \int_K \nabla (u - u_h) \cdot \nabla^{\perp} (j_{\tau,e} \psi_e)\, dx \lesssim \sum_{K \in \triangle_e} \alpha_K^{-1/2} \|\!|u - u_h|\!\|_K \, |j_{\tau,e} \psi_e|_{1,K} \\ &\lesssim \sum_{K \in \triangle_e} \alpha_K^{-1/2} \|\!|u - u_h|\!\|_K \, h_e^{-1/2} \|j_{\tau,e}\|_{0,e} \lesssim \left(\alpha_-^e h_e \right)^{-1/2} \|\!|E|\!\|_{\triangle_e} \|j_{\tau,e}\|_{0,e}, \end{aligned}$$

which, tegoether with (5.4) and a similar bound for $e \in \mathcal{E}_D^h \cup \mathcal{E}_N^h$, yields

$$\|j_{u,e}\|_{0,e} \lesssim \left(\frac{h_e}{\alpha_-^e} \right)^{1/2} \|\!|E|\!\|_{\triangle_e}, \quad \forall\, e \in \mathcal{E}_h. \tag{5.5}$$

Now, the efficiency bound in (5.1) is a direct consequence of the bounds in (5.2), (5.3), and (5.5). This completes the proof of the theorem. □

6 Global reliability bound.

Let

$$\hat{\eta}_{J_u,K} = \frac{\alpha_K^{1/2}}{h_K^{1/2}} \|I_{h/2}u_h - u_h\|_{\partial K} \quad \text{and} \quad \hat{\eta}_{J_u} = \left(\sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{h_K} \|I_{h/2}u_h - u_h\|_{\partial K}^2 \right)^{1/2}.$$

Lemma 6.1. *Let u_h be the solution of* (2.4) *and $I_{h/2}$ be the interpolation operator defined in* Section 3, *then the jump of the numerical solution has the following upper bound:*

$$\sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_D^h} \int_e \{\alpha\frac{\partial E}{\partial n}\}\, j_{u,e}\, ds \lesssim \hat{\eta}_{J_u}\, \|\!|E|\!\|_\Omega. \tag{6.1}$$

Proof. Since the integral over edge segment $e \in \partial K$ on the left-hand side of inequality (6.1) may be only regarded as the duality pair between $H^{\delta-1/2}(\partial K)$ and $H^{1/2-\delta}(\partial K)$ for an arbitrarily small $\delta > 0$, we are not able to bound this integral directly. To overcome this difficulty, we express them in terms of integrals along the boundary of elements. To this end, first note that

$$[\![I_{h/2}u_h]\!]_e = 0 \quad \text{and} \quad [\![\alpha\nabla u\cdot\mathbf{n}_e]\!]_e = 0, \quad \forall\, e\in\mathcal{E}_I^h.$$

By (2.5) and the fact that $I_{h/2}u_h = g_D$ on Γ_D, we have

$$\begin{aligned}
&-\sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_D^h} \int_e \{\alpha\frac{\partial E}{\partial n}\} j_{u,e}\, ds \\
=\;& \sum_{e\in\mathcal{E}_I^h} \int_e \{\alpha\frac{\partial E}{\partial n}\}\, [\![I_{h/2}u_h - u_h]\!]\, ds + \sum_{e\in\mathcal{E}_D^h} \int_e \alpha\frac{\partial E}{\partial n}\,(g_D - u_h)\, ds \\
=\;& \sum_{K\in\mathcal{T}_h} \int_{\partial K} \alpha\frac{\partial E}{\partial n}\,(I_{h/2}u_h - u_h)\, ds - \sum_{e\in\mathcal{E}_I^h} \int_e [\![\alpha\frac{\partial E}{\partial n}]\!]\, \{I_{h/2}u_h - u_h\}\, ds \\
&- \sum_{e\in\mathcal{E}_D^h\cup\mathcal{E}_N^h} \int_e \alpha\frac{\partial E}{\partial n}\,(I_{h/2}u_h - u_h)\, ds + \sum_{e\in\mathcal{E}_D^h} \int_e \alpha\frac{\partial E}{\partial n}\,(g_D - u_h)\, ds \\
=\;& \sum_{K\in\mathcal{T}_h} \int_{\partial K} \alpha\frac{\partial E}{\partial n}\,(I_{h/2}u_h - u_h)\, ds + \sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_N^h} \int_e j_{\sigma,e}\{I_{h/2}u_h - u_h\}\, ds \equiv I_1 + I_2.
\end{aligned}$$

The I_1 may be bounded above by using the definition of the dual norm, the trace theorem (see, e.g., [7]), the inverse inequality, and (5.2) as follows:

$$
\begin{aligned}
I_1 &\le \sum_{K\in\mathcal{T}_h} \big\|\alpha\frac{\partial E}{\partial n}\big\|_{-1/2,\partial K}\big\|I_{h/2}u_h - u_h\big\|_{1/2,\partial K} \\
&\lesssim \sum_{K\in\mathcal{T}_h} \alpha_K^{-1/2}\Big(\|\alpha\nabla E\|_{0,K} + h_K\,\|r_K\|_{0,K}\Big)\,\hat\eta_{J_u,K} \lesssim \hat\eta_{J_u}\,|\!|\!|E|\!|\!|_{\Omega}. \qquad (6.2)
\end{aligned}
$$

To bound the I_2, first note that

$$
\int_e j_{\sigma,e}\,[\![I_{h/2}u_h - u_h]\!]\,ds = 0, \quad \forall\, e\in\mathcal{E}_I^h,
$$

which is a consequence of the orthogonality property in (2.6) and the facts that $j_{\sigma,e}$ is a constant and that $[\![I_{h/2}u_h]\!]_e = 0$ for all $e\in\mathcal{E}_I^h$. Hence,

$$
\begin{aligned}
\int_e j_{\sigma,e}\{I_{h/2}u_h - u_h\}\,ds &= \int_e j_{\sigma,e}\{I_{h/2}u_h - u_h\}\,ds + \frac{1}{2}\int_e j_{\sigma,e}[\![I_{h/2}u_h - u_h]\!]\,ds \\
&= \int_e j_{\sigma,e}(I_{h/2}u_h - u_h|_+^e)\,ds, \quad \forall e\in\mathcal{E}_I^h.
\end{aligned}
$$

Now, it follows from the Cauchy-Schwartz inequality and (5.3) that

$$
\begin{aligned}
I_2 &= \sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_N^h} \int_e j_{\sigma,e}(I_{h/2}u_h - u_h|_+^e)\,ds \\
&\le \left(\sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_N^h} \frac{\alpha_+^e}{h_e}\,\|I_{h/2}u_h - u_h|_+^e\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_N^h} \frac{h_e}{\alpha_+^e}\,\|j_{\sigma,e}\|_{0,e}^2\right)^{1/2} \\
&\lesssim \left(\sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_N^h} \frac{\alpha_+^e}{h_e}\|I_{h/2}u_h - u_h|_+^e\|_{0,e}^2\right)^{1/2} |\!|\!|E|\!|\!|_{\Omega} \lesssim \hat\eta_{J_u}\,|\!|\!|E|\!|\!|_{\Omega}. \qquad (6.3)
\end{aligned}
$$

(6.1) is then a consequence of (6.2) and (6.3). This completes the proof of the lemma. □

Since $u_h - I_{h/2}u_h$ vanishes on all boundary edges of $w_z^{h/2}$ for all $z\in\mathcal{N}_h$, we have

$$
\sum_{K\in\mathcal{T}_h} \frac{\alpha_K}{h_K}\|I_{h/2}u_h - u_h\|_{0,\partial K}^2 = \sum_{z\in\mathcal{N}_h}\sum_{T\in w_z^{h/2}} \frac{\alpha_T}{h_T}\|I_{h/2}u_h - u_h\|_{0,\partial T}^2.
$$

Lemma 6.2. *Let u_h be the solution of* (2.4) *and $I_{h/2}$ be the interpolation operator defined in* Section 3. *For any vertex $z\in\mathcal{N}_h$, if the vertex patch $\omega_z^{h/2}$ is quasi-monotone, then*

$$
\sum_{T\in\omega_z^{h/2}} \frac{\alpha_T}{h_T}\,\|u_h - I_{h/2}u_h\|_{0,\partial T}^2 \lesssim \sum_{e\in\mathcal{E}_z^{h/2}\cap\mathcal{E}_I^{h/2}} \frac{\alpha_-^e}{h_e}\,\big\|[\![u_h]\!]\big\|_{0,e}^2. \qquad (6.4)
$$

Proof. To show the validity of (6.4), it suffices to prove that

$$\frac{\alpha_T}{h_T}\,\|u_h - I_{h/2}u_h\|^2_{0,\partial T} \lesssim \sum_{e\in\mathcal{E}_z^{h/2}\cap\mathcal{E}_I^{h/2}} \frac{\alpha_-^e}{h_e}\,\left\|[\![u_h]\!]\right\|^2_{0,e}, \quad \forall T\in\omega_z^{h/2}. \tag{6.5}$$

To this end, let $K_z \in \hat{\omega}_z^h$ be the element such that $I_{h/2}u_h(z) = u_h|_{K_z}(z)$, and let

$$T_z = K_z \cap \omega_z^{h/2}.$$

For any $T\in\omega_z^{h/2}$, if $T = T_z$, then $u_h - I_{h/2}u_h = 0$ in T_z. Hence, (6.5) holds.

In the case that T is adjacent to T_z, let $e = \partial T\cap\partial T_z$, then $T = T_-^e$ and $T_z = T_+^e$. For simplicity, assume that all edges in $\mathcal{E}_z^{h/2}$ have the same length. A direct calculation gives

$$\frac{\alpha_T}{h_T}\,\|u_h - I_{h/2}u_h\|^2_{0,\partial T} = \frac{2\alpha_T h_e}{3h_T}\,[\![u_h]\!]_e^2(z) = \frac{2\alpha_T}{h_T}\,\|[\![u_h]\!]\|^2_{0,e},$$

which, together with the fact that $\alpha_T \le \alpha_{T_z}$, yields (6.5).

In the case that T and T_z are not adjacent, by the quasi-monotonicity, there exists a connected path from T to T_z such that the diffusion coefficient α is monotonically increasing. Then (6.5) follows from the same argument above applying to each pair of two adjacent elements in the path and the triangle inequality. This completes the proof of (6.5) and, hence, the lemma. □

Theorem 6.1. (Global Reliability) *Let u and u_h be the solutions of* (2.3) *and* (2.4)*, respectively. Then the estimator $\tilde{\eta}$ satisfies the following global reliability bound:*

$$|\!|\!| E |\!|\!|_\Omega \lesssim \tilde{\eta}. \tag{6.6}$$

Proof. Let $\mathcal{I}_h$ be the modified Clément interpolation operator defined in Section 4. Then (2.9) with $E_h = \mathcal{I}_h E$ becomes

$$\begin{aligned} a_h(E,E) &= \sum_{K\in\mathcal{T}_h}(r_K, E-\mathcal{I}_hE)_K - \sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_N^h}\int_e j_{\sigma,e}\,\{E-\mathcal{I}_hE\}ds - \sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_D^h}\int_e\{\alpha\frac{\partial E}{\partial n}\}\,j_{u,e}ds \\ &\equiv I_1 + I_2 + I_3 \end{aligned} \tag{6.7}$$

The first term in (6.7) may be bounded by the Cauchy-Schwartz inequality, Lemma 4.1, and (5.5) as follows:

$$\begin{aligned} I_1 &\le \sum_{K\in\mathcal{T}_h}\|r_K\|_{0,K}\,\|E-\mathcal{I}_hE\|_{0,K} \lesssim \sum_{K\in\mathcal{T}_h}\eta_{R_f,K}\left(|\!|\!|E|\!|\!|_{0,\triangle_K} + \sum_{e\in\mathcal{E}_K^h}\left(\frac{\alpha_-^e}{h_e}\right)^{1/2}\|[\![u_h]\!]\|_{0,e}\right) \\ &\lesssim \left(\sum_{K\in\mathcal{T}_h}\eta^2_{R_f,K}\right)^{1/2}|\!|\!|E|\!|\!|_\Omega. \end{aligned} \tag{6.8}$$

To bound the second term in (6.7), first notice that

$$[\![E-\mathcal{I}_hE]\!]_e = -[\![u_h+\mathcal{I}_hE]\!]_e, \quad \forall e\in\mathcal{E}_I^h.$$

Since $u_h+\mathcal{I}_hE\in\mathcal{U}^{nc}$ and the the fact that $j_{\sigma,e}$ is a constant for all $e\in\mathcal{E}_h$, (2.6) yields

$$\int_e j_{\sigma,e}\,[\![E-E_h]\!]\,ds = 0, \quad \forall e\in\mathcal{E}_I^h.$$

Hence,

$$\int_e \{E-I_hE\}_e\,ds+\frac{1}{2}\int_e [\![E-I_hE]\!]_e\,ds = \int_e (E-I_hE)|_+^e\,ds = \int_e (E|_+^e-\pi_eE)\,ds, \ \ \forall\, e\in\mathcal{E}_I^h. \quad (6.9)$$

The last equality comes from the property of the nonconforming nodal basis functions: $⨍_{e_i}\phi_{e_j}=\delta_{ij}$. It then follows from (6.9), the Cauchy-Schwartz inequality, and Lemma 4.1 that

$$\begin{aligned} I_2 &= \sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_N^h}\int_e j_{\sigma,e}\,(E|_+^e-\pi_eE)ds \le \sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_N^h}\|j_{\sigma,e}\|_{0,e}\left\|E|_+^e-\pi_eE\right\|_{0,e} \\ &\lesssim \sum_{e\in\mathcal{E}_I^h\cup\mathcal{E}_N^h}\left(\frac{h_e}{\alpha_+^e}\right)^{1/2}\|j_{\sigma,e}\|_{0,e}\,|\!|\!|E|\!|\!|_{0,K_+^e} \lesssim \left(\sum_{K\in\mathcal{T}_h}\eta_{J_N,K}^2\right)^{1/2}|\!|\!|E|\!|\!|_{\Omega}. \end{aligned} \quad (6.10)$$

Now, (6.6) is a direct consequence of (6.7), (6.8), (6.10), Lemma 6.1, and Lemma 6.2. This completes the proof of the theorem. □

7 Numerical experiments.

In this section, we report some numerical results for an interface problem with intersecting interfaces used by many authors (see, e.g., [19, 8, 10, 7, 12, 13]), which is considered as a benchmark test problem. Let $\Omega=(-1,1)^2$ and

$$u(r,\theta)=r^\beta\mu(\theta)$$

in the polar coordinates at the origin with

$$\mu(\theta)=\begin{cases} \cos((\pi/2-\sigma)\beta)\cdot\cos((\theta-\pi/2+\rho)\beta) & \text{if } \ 0\le\theta\le\pi/2,\\ \cos(\rho\beta)\cdot\cos((\theta-\pi+\sigma)\beta) & \text{if } \ \pi/2\le\theta\le\pi,\\ \cos(\sigma\beta)\cdot\cos((\theta-\pi-\rho)\beta) & \text{if } \ \pi\le\theta\le 3\pi/2,\\ \cos((\pi/2-\rho)\beta)\cdot\cos((\theta-3\pi/2-\sigma)\beta) & \text{if } \ 3\pi/2\le\theta\le 2\pi, \end{cases}$$

where σ and ρ are numbers. The function $u(r,\theta)$ satisfies the interface problem in (2.1) with $\Gamma_N=\emptyset$, $f=0$, and

$$\alpha(x)=\begin{cases} R & \text{in } \ (0,1)^2\cup(-1,0)^2,\\ 1 & \text{in } \ \Omega\setminus([0,1]^2\cup[-1,0]^2). \end{cases}$$

The numbers β, R, σ, and ρ satisfy some nonlinear relations. For example, when $\beta = 0.1$, then

$$R \approx 161.4476387975881, \quad \rho = \pi/4, \quad \text{and} \quad \sigma \approx -14.92256510455152.$$

Note that when $\beta = 0.1$, this is a difficult problem for computation.

Remark 7.1. *This problem does not satisfy* Hypothesis 2.7 *in* [4] *as the quasi-monotonicity is not satisfied about the origin.*

Started with a coarse triangulation, a sequence of mesh is generated by using a standard adaptive meshing algorithm that adopts the L^2 strategy: (i) mark the elements such that η_K is among the first 20 percent of L^2 norm of the total error, and (ii) refine the marked triangles by bisection. The stopping criteria is

$$\text{rel-err} := \frac{\|u - u_u\|_\Omega}{\|u\|_\Omega} \le tol$$

is used, and numerical results with $tol = 0.1$ is reported. Mesh generated by $tol = 0.1$ is in Figure 1. Refinements are centered at origin as expected.

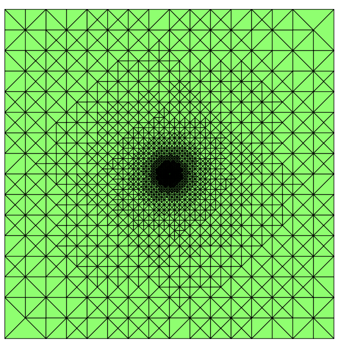

Figure 1: mesh generated with relative error ¡ 0.1

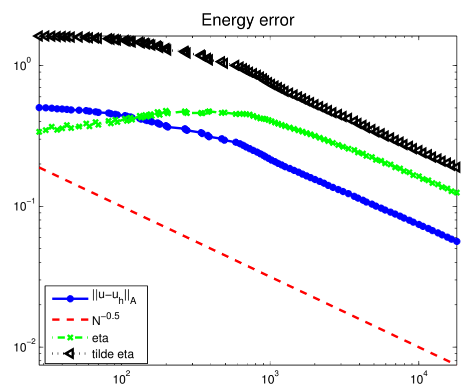

Figure 2: $\tilde{\eta}$ vs η.

As shown in Figure 2, the slope for the log (number of unknowns)-log (energy error) is about $-1/2$ which indicates the optimal decay of the error with respect to the number of unknowns. The efficiency index,

$$\text{eff-index} := \frac{\text{estimator}}{\|u - u_h\|_\Omega},$$

for η is about 2.2, and for $\tilde{\eta}$ is about 3.3.

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