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Doctoral Thesis

Group algebras of metacyclic type

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Declaration of Authorship

I, John Derek Peter Evans, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

Let $\Lambda = \mathbf{Z}[G]$ denote the integral group ring of a finite group G. In the first part of this thesis we consider the stable syzygies $\Omega_r(\mathbf{Z})$ over Λ. These are defined to be the stable classes of the intermediate modules in a free Λ-resolution of the trivial module $\mathbf{Z}$. If we let p be an odd prime, then the groups of concern to us will be $G_1 = D_{2p}$ which has free period 4, and $G_2 = C_p \rtimes C_3$ which has free period 6. Along the way it will also be necessary to consider the syzygies of the cyclic group C_n which has free period 2, the smallest possible nontrivial periodic resolution.

The key point of note in each of these cases is that the augmentation ideal splits, thereby allowing us to show the existence of a diagonalised resolution. Moreover, there exist two strands corresponding to the action of the generators of C_p, and of either C_2 or C_3. For each strand we show there exists a group structure within the stable class generated by part of the first syzygy $\Omega_1(\mathbf{Z})$, and in which part of the zeroth syzygy $\Omega_0(\mathbf{Z})$ is the identity.

In the second part of this thesis we make the jump to infinite groups. By setting $G = C_p \rtimes C_q$ where p, q are prime such that $q|p-1$, we discuss the stably free modules over $\mathbf{Z}[G \times F_n]$, where F_n denotes the free group of rank n. As we shall see, the stably free modules over this group ring are necessarily trivial; that is, they are free.

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On a more personal note, I would like to express my deepest gratitude for the support of my family. My mother in particular has sacrificed so much to make me the man that I am today. It is my hope that this PhD is but a first step in showing her that her sacrifices were not a total waste. I would also like to thank Vanisha Hirani from the bottom of my heart for her continued support and understanding. Not only has she put up with me, but she has also become increasingly adept at feigning interest when I talk about syzygies.

Lastly, I would like to thank Michael Burrells for being nearly always willing to go to the pub when I'd had enough of metacyclic groups.

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For Mum.

Part I

Finite metacyclic groups

Chapter 1

Overview

1.1 Motivation

We can view this thesis as roughly being divided into two distinct parts: finite and infinite. Consider first the finite case in which we are concerned with the integral group rings $\mathbf{Z}[G]$ for G a finite fundamental group. Specifically, we look at metacyclic groups of the form,

$$G(p,\, q) = C_p \rtimes C_q =< x,\, y \mid x^p = y^q = 1,\, yx = \theta(x)y >$$

where p, q are primes such that $q|p-1$, and $\theta \in Aut(C_p)$ has order exactly q. In this way, C_q acts on C_p via the natural embedding $C_q \hookrightarrow Aut(C_p)$. In Chapter 4 we set $q = 2$, and in Chapter 5 we set $q = 3$. In the second part of this thesis we shall extend the discussion of such metacyclic groups to an infinite setting. We therefore say G is a *metacyclic group of infinite type* if G has the form $G = G(p,\, q) \times \Phi$, where Φ is some free group. In this thesis, we will primarily be concerned with the case $\Phi = F_n$, the free group of rank $n \geq 1$. In particular, the case $n = 1$ corresponds to the infinite cyclic group C_∞.

The motivation for the above can be seen as both algebraic and topological, although we shall primarily be interested in the algebraic considerations in this thesis. Nevertheless, it is beneficial to briefly discuss the topological nature, if only to put the algebraic treatment into perspective. For a detailed exposition, the reader is directed to [20].

Algebraically, we are concerned with the explicit calculation of the interaction of syzygies under the tensor product. For G finite, consider a free resolution over $\mathbf{Z}[G]$ of the trivial module $\mathbf{Z}$,

$$\cdots \overset{\partial_{n+2}}{\to} \mathcal{F}_{n+1} \overset{\partial_{n+1}}{\to} \mathcal{F}_n \overset{\partial_n}{\to} \cdots \overset{\partial_2}{\to} \mathcal{F}_1 \overset{\partial_1}{\to} \mathcal{F}_0 \overset{\partial_0}{\to} \mathbf{Z} \to 0$$

in which each $\mathcal{F}_i$ is a finitely generated free module over $\mathbf{Z}[G]$. We then define the syzygies $(J_r)_{r\geq 1}$ to be the intermediate modules $J_r = Im(\partial_r) = Ker(\partial_{r-1})$. It is straightforward to see such syzygies are dependent upon the free resolution chosen. As such, to impose a degree of uniqueness we consider the stable syzygy. Here, we consider stability to be an equivalence relation given by $M \sim N$ if and only if $M \oplus \mathbf{Z}[G]^m \cong N \oplus \mathbf{Z}[G]^n$ for some $m,\, n \geq 0$. The r^{th} stable syzygy is then said to be the stable class $\Omega_r(\mathbf{Z}) = [J_r]$.

A natural question to ask is how syzygies interact upon tensoring with another syzygy. By an iterative argument (see Section 2.5), $\Omega_r(\mathbf{Z}) \otimes_{\mathbf{Z}} \Omega_s(\mathbf{Z}) = \Omega_{r+s}(\mathbf{Z})$. However, this does not tell us much about what is actually going on. Consequently, one of the primary aims of this thesis will be to explicitly calculate such interactions for the cases when $q = 2,\, 3$.

In order to discuss these interactions we first need an understanding of what these syzygies actually look like. Unfortunately, such descriptions are not always easy to come by and, in certain circumstances, the syzygies continue to grow ever larger making the situation especially difficult. Nevertheless, under favourable conditions it can be shown that certain stable syzygies have a periodic nature. In such cases we say that $\Omega_r(\mathbf{Z}) = \Omega_{n+r}(\mathbf{Z})$ for some $n \geq 2$ which we call the *periodic cohomology* of G. Such periods are necessarily even and are reflected in the free resolutions; that is, if G has cohomological period n then there exists a free resolution of the form

$$0 \to \mathbf{Z} \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathbf{Z} \to 0.$$

Although most cases do not have this periodic nature, there are a number of significantly important groups that do. For instance, cyclic groups have period 2 (in fact they are the only such groups to have this [55]), dihedral groups D_{4n+2} of order $4n+2$ have period 4 (see [20]), and metacyclic groups $G(p,\, q)$ of order pq have period $2q$ (see [54]). It should be noted, however, that D_{4n} does *not* have this periodic nature. For a classification of all finite soluble groups with finite cohomological period, the reader is directed to [58]. For the classification of nonsoluble groups, the reader is directed to [53].

Whereas this periodic nature simplifies the matter greatly, it is still no easy task to actually describe these syzygies. What is of significant use to us is the (somewhat surprising) existence of free resolutions of $\mathbf{Z}$ that are of *diagonal type*. These are resolutions whose free modules $\mathcal{F}_k$ all have rank 2 when $k \geq 1$, and $\mathcal{F}_0$ is free of rank 1. Furthermore, for each $k \geq 2$ the differential ∂_k has diagonal form

$$\partial_k = \begin{pmatrix} \partial_k^+ & 0 \\ 0 & \partial_k^- \end{pmatrix}.$$

Such resolutions were first observed for groups more general than cyclic groups in the thesis of Strouthos [51]. Here, Strouthos constructed a diagonal resolution for the dihedral group D_6 of order 6. This has since been generalized by Johnson in [24].

Ideally, one would like to generalize this still further for metacyclic groups $G = G(p,\, q)$ (see the beginning of this chapter). Set $\Lambda = \mathbf{Z}[G(p,\, q)]$ and consider the free resolution given by

$$0 \to \mathbf{Z} \to \mathcal{F}_{2q-1} \to \mathcal{F}_{2q-2} \to \cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathbf{Z} \to 0$$

in which each $\mathcal{F}_i$ is a finitely generated free Λ-module. As a starting point first observe $\Omega_1(\mathbf{Z}) = [I_G]$. If $\alpha \in \mathbf{Z}[G]$, then by $[\alpha)$ we mean the right ideal of $\mathbf{Z}[G]$ generated by α; that is, $[\alpha) = \{\alpha\lambda \mid \lambda \in \mathbf{Z}[G]\}$. In a natural way, this can be extended to an ideal generated by a finite number of elements, $[\alpha_1, \ldots,\, \alpha_r)$. With this notation, we have the following decomposition of the augmentation ideal I_G of G as a direct sum

$$I_G \cong \bar{I}_C \oplus [y-1). \tag{1.1.1}$$

Here, $\bar{I}_C$ is the Galois module obtained from the action of C_q on the augmentation ideal I_C of C_p. We will in fact see several important modules arise in this way; that is, as ideals of $\mathbf{Z}[C_p]$ that are invariant under θ. To obtain these modules, we first define a Galois structure on a module M over $\mathbf{Z}[C_p]$ to be an additive automorphism $\Theta : M \to M$ such that $\Theta^q = Id_M$ and $\Theta(m \cdot x) = \Theta(m) \cdot \theta(x)$ for all $m \in M$. If M is finitely generated and free as a $\mathbf{Z}$-module, we define a Galois lattice to be the pair (M, Θ). This becomes a (right) module over Λ via the action

$$m \cdot x^r y^s = \Theta^{-s}(m \cdot x^r).$$

For a proof of (1.1.1) the reader is directed to [17] or [25].

Of particular importance in the decomposition (1.1.1) is that both $\bar{I}_C$ and $[y-1)$ are indecomposable. This behaviour turns out to be repeated at the minimal level of each syzygy. This can be used to 'untwist' the augmentation ideal to form two short exact sequences of the form

$$0 \to ? \to \Lambda \to \bar{I}_C \to 0 \quad \text{and} \quad 0 \to ?? \to \Lambda \to [y-1) \to 0.$$

Consequently, we can effectively 'untwist' a free resolution of $\mathbf{Z}$ over $\mathbf{Z}[G(p, q)]$ to form two separate monogenic[1] infinite resolutions. As such, we shall often speak of the so-called y-strand (this could also be thought of as the lower strand) of the resolution. By this, we mean the exact sequence which is induced up from the standard resolution of C_q. To be precise, let $\mathcal{E}$ be the standard resolution of $\mathbf{Z}$ over $\mathbf{Z}[C_q]$

$$\mathcal{E} = (\cdots \overset{\Sigma_y}{\to} \mathbf{Z}[C_q] \overset{y-1}{\to} \mathbf{Z}[C_q] \overset{\Sigma_y}{\to} \mathbf{Z}[C_q] \overset{y-1}{\to} \mathbf{Z}[C_q] \overset{\Sigma_y}{\to} \mathbf{Z}[C_q] \overset{y-1}{\to} \cdots) \tag{1.1.2}$$

where $\Sigma_y = \sum_{i=0}^{q-1} y^i$. If $j : C_q \to G(p, q)$ is the inclusion, then the y-strand of our resolution is the induced resolution $j_*(\mathcal{E})$,

$$\cdots \overset{\Sigma_y}{\to} \Lambda \overset{y-1}{\to} \Lambda \overset{\Sigma_y}{\to} \Lambda \overset{y-1}{\to} \Lambda \overset{\Sigma_y}{\to} \Lambda \overset{y-1}{\to} \Lambda \overset{\Sigma_y}{\to} \Lambda \overset{y-1}{\to} \cdots \tag{1.1.3}$$

By contrast, the x-strand (or, alternatively, the upper strand) causes a significant amount of bother. A first step was outlined by Remez in [41] (see also [25]) using an explicit form of Rosen's theorem [43]. Here, we set $\zeta_p = exp(2\pi i/p)$ and describe Λ as a fibre product

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d] & \mathcal{T}_q(A, \pi) \arrow[d] \\ \mathbf{Z}[C_q] \arrow[r] & \mathbf{F}_p[C_q] \end{tikzcd} \tag{1.1.4}$$

where $A = \mathbf{Z}[\zeta_p]^\theta$ is the subring of $\mathbf{Z}[\zeta_p]$ fixed by θ, and $\pi = (\zeta_p - 1)^q$. We define the following *quasi-triangular* subring of $M_q(A)$ to be

$$\mathcal{T}_q(A, \pi) = \{X = (x_{rs})_{1\le r,\, s\le q} \in M_q(A) \mid x_{rs} \in (\pi) \text{ if } r > s\}.$$

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[1] It may be useful to note that monogenic modules are often called cyclic modules.

Next, denote the i^{th} row of $\mathcal{T}_q(A,\, \pi)$, considered as a right Λ-module, by $R(i)$[2]. We then have the following decomposition of the quasi-triangulars,

$$\mathcal{T}_q(A,\, \pi) \cong R(1) \oplus \cdots \oplus R(q).$$

It is quite clear that each $R(i)$ is monogenic by composing the obvious projections $\Lambda \twoheadrightarrow \mathcal{T}_q(A,\, \pi)$ and $\mathcal{T}_q(A,\, \pi) \twoheadrightarrow R(i)$ to give $p(i) : \Lambda \twoheadrightarrow R(i)$. Next, we define $K(i) = Ker(p(i))$ and note that in [25] Johnson has shown the existence of the following exact sequence of Λ-modules

$$\begin{tikzcd}
 & & & K(q) \arrow[dr] & & & & K(q-1) \arrow[dr] & & & K(2) \arrow[dr] & & & & K(1) \arrow[dr] & & & \\
0 \arrow[r] & R(1) \arrow[r] & \Lambda \arrow[rr] \arrow[ur] & & \Lambda \arrow[rr] \arrow[dr] & & P(q-1) \arrow[rr] \arrow[ur] & & \Lambda \arrow[r] & \cdots \arrow[rr] \arrow[ur] & & \Lambda \arrow[rr] \arrow[dr] & & P(1) \arrow[rr] \arrow[ur] & & \Lambda \arrow[r] & R(1) \arrow[r] & 0 \\
 & & & & & R(q) \arrow[ur] & & & & & & & R(2) \arrow[ur] & & & & &
\end{tikzcd} \tag{1.1.5}$$

in which $P(1), \ldots,\, P(q-1)$ are projective modules of rank 1 over Λ, and such that $\bigoplus_{i=1}^{q-1} P(i) \cong \Lambda^{q-1}$. It is unknown at present whether each $P(i)$ is indeed a free module. The sequencing conjecture can therefore be stated thus:

Sequencing Conjecture: Each $P(i) \cong \Lambda$ for $1 \leq i \leq q-1$.

It should be noted that the above conjecture was first formulated in this form in [25]. It is also clear in that paper that if the sequencing conjecture has an affirmative answer, then we may build a diagonalised free resolution. We will discuss this further at a later point in the thesis. At present, the sequencing conjecture has been confirmed for:

(i) $G(2n+1,\, 2)$ (see [24])

(ii) $G(7,\, 3)$ (see [41])

(iii) $G(5,\, 4)$ (see [33]).

Furthermore, in [25] Johnson has confirmed the sequencing conjecture for the following small values of p and q:

$$G(7,\, 6); \qquad G(11,\, 5), G(11,\, 10); \qquad G(13,\, 3),\, G(13,\, 4),\, G(13,\, 6); \qquad G(17,\, 4);$$

$$G(19,\, 3),\, G(19,\, 6),\, G(19,\, 9).$$

Beyond the obvious algebraic interest, this does have a topological motivation. Indeed, the third syzygy is of particular importance in the context of low-dimensional topology; specifically, the $\mathcal{R}(2) - \mathcal{D}(2)$ problem. We stress that these topological considerations will have no bearing on the rest of this thesis. Nevertheless, it is perhaps useful to bear in mind the wider area in which these constructions relate.

In essence, the $\mathcal{D}(2)$-problem asks what conditions are necessary to impose upon a finite connected cell complex X of geometrical dimension 3 before it can be homotopy

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[2] We will later see that $\bar{I}_C$ is isomorphic to $R(1)$.

equivalent to one of dimension at most 2. Specifically, if $\widetilde{X}$ represents the universal covering of X, and if

$$H_3(\widetilde{X},\, \mathbf{Z}) = H^3(X,\, \mathcal{B}) = 0$$

for all coefficient bundles $\mathcal{B}$, then the $\mathcal{D}(2)$-problem asks if it is true that X is homotopy equivalent to a finite complex of dimension 2. Of note is that the $\mathcal{D}(2)$-problem is parametrized by the fundamental group. Thus, each finitely presented group G has its own $\mathcal{D}(2)$-problem. We then talk of the $\mathcal{D}(2)$-property holding, or failing, for a specific group G.

It has been shown that for a finitely presented group G, the $\mathcal{D}(2)$-problem is equivalent to an older problem known as the realization problem, or $\mathcal{R}(2)$-problem. This was shown by Johnson in [20], subject to mild conditions upon G which were later shown to be unnecessary by Mannan [30]. To solve the $\mathcal{R}(2)$-problem, one is tasked with taking exact sequences of the form

$$0 \to J \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathbf{Z} \to 0$$

where each $\mathcal{F}_i$ is finitely generated and free over Λ, and where J belongs to $\Omega_3(\mathbf{Z})$. Such exact sequences are called algebraic 2-complexes. It is then a question of whether such exact sequences serve as algebraic models for geometric 2-complexes.

If an affirmative answer to the $\mathcal{R}(2)$-$\mathcal{D}(2)$-problem is to be found, then a number of things need to be explicitly described. First, we need to understand the stable module $\Omega_3(\mathbf{Z})$. Secondly, let $\mathbf{Alg}_2(\mathbf{Z}[G])$ denote the set of homotopy equivalence classes of algebraic 2-complexes over $Id_{\mathbf{Z}[G]}$. We then need to understand $\mathbf{Alg}_2(\mathbf{Z}[G])$ and the fibres $\pi_2 : \mathbf{Alg}_2(\mathbf{Z}[G]) \to \Omega_3(\mathbf{Z})$. Finally, for each algebraic 2-complex A_* over $\mathbf{Z}[G]$, we would need to construct a finite presentation of G such that A_* has a geometric realization. With the above in mind, there is a clear benefit of having a diagonal resolution, and an understanding of how the syzygies interact. This is perhaps further demonstrated by considering the difficulties encountered when we do not have a diagonal resolution. An example of this can be found in [19].

Now, when Johnson provided an affirmative answer to the sequencing conjecture for $G = D_{4n+2}$, he used the finite cohomological period of such dihedral groups. As $G = D_{4n}$ does not have a finite cohomological period, Johnson's argument does not extend to these groups. Nevertheless, some progress has been made in this direction. In [35] O'Shea has shown that $\mathbf{Z}[D_{4n}]$ having the torsion free cancellation property is a sufficient condition for it to have the $\mathcal{D}(2)$-property. Calculations of Swan [57] and Endo and Miyata [14] therefore show an affirmative answer exists for $\mathbf{Z}[D_{4p}]$ when p is prime such that $2 \leq p \leq 31$, or when $p = 47,\, 179,\, 19379$.

Moving beyond dihedral groups, Remez [41] has shown the $\mathcal{D}(2)$ property holds for the group $G(7,\, 3)$. Some progress was also made toward the general case $G(p,\, q)$. By demonstrating the straightness (see Section 2.7) of $\Omega_3(\mathbf{Z})$, Remez thereby reduced the realization problem to one of discussing fullness of the Swan map. We then have just a single homotopy type for a group of the form $G(p,\, q)$. However, such a discussion is by no means straightforward and at present there is no known way to show this.

Related to the discussion of syzygies, we come to the second major construction of concern to this thesis: stably free modules. We say a module S is stably free if it is stably isomorphic to the zero module, i.e. $S \sim 0$. Such modules can be seen as relatively well-behaved projective modules. As such, it is a natural question to ask what these modules actually look like, at least so far as to whether or not they are free. This question has its genesis in the now famous Serre's conjecture of 1955 [45], [28] (although as it happens, Serre suggested the problem as an open one, making no claim that he believed or disbelieved in a positive solution). Serre's problem was directed towards finitely generated projective modules over a polynomial ring in n variables over a field k. Topologically, the question can be posed as asking if every vector bundle over the affine n-space $\mathbf{A}_k^n$ is a trivial bundle, reflecting the idea that an affine n-space should behave like a 'contractible' space in topology.

The question spurred on a wave of research, notably by Bass [1], Quillen [40], Suslin [52] and Seshadri [46]. Of note is a theorem of Hilbert-Serre [44] in which it was shown that the projective modules in question are necessarily stably free. Despite such research, it would require twenty years before a full and affirmative solution was finally presented by Quillen and Suslin (see [40] and [52]). In an interesting turn of history, the solution by Quillen and Suslin was discovered independently in the same month of the same year (January 1976) and using different means.

Our interest in stably free modules can therefore be seen as a continuation of these discussions. Nevertheless, the topological motivation should not be overlooked, for at present there is no known fundamental group G that satisfies the $\mathcal{D}(2)$ property and which admits non-trivial stably free modules over its integral group ring.

The discussion, then, becomes one of which rings have only trivial stably free modules, i.e. free. In this, we are aided by a property known as the *Eichler condition*. Groups have this property when their real Wedderburn decomposition admits no simple Hamiltonian $\mathbf{H}$ factors. By a theorem of Swan and Jacobinski (see [11]) we know $\mathbf{Z}[G]$ admits no nontrivial stably free module if G is a finite group satisfying the Eichler condition. However, this condition is not a necessary condition, as shown by the calculations of Swan in [56]. Here, Swan showed that the generalized quaternions $Q(4n)$ of order $4n$ admit no nontrivial stably free modules over their integral group ring if and only if $n \leq 5$, yet $Q(4n)$ is easily shown to fail the Eichler condition. As it happens, whenever $n > 5$, $\mathbf{Z}[Q(4n)]$ has at least one nontrivial stably free module.

When we make the transition to infinite groups, the Swan-Jacobinski theorem can no longer be called upon. As such, progress is now of a far more delicate nature. Nevertheless, progress has been made by Johnson and others. As a starting point, we note that Johnson has shown in [21] that $k[G \times F_n]$ admits no nontrivial stably free module whenever G is some finite group and k a field. Returning to $\mathbf{Z}$, Johnson [22] has generalized a result of Kamali [26] to show $\mathbf{Z}[Q(8m) \times F]$ has infinitely many isomorphically distinct nontrivial stably free modules, where F is some group which maps surjectively onto F_n. Similarly, O'Shea [35] has shown this result holds for groups of the form $Q(12m) \times F$.

Beyond quaternion groups, Johnson [22] has shown that both $\mathbf{Z}[C_p \times F_n]$ and $\mathbf{Z}[D_{2p} \times F_n]$ admit no nontrivial stably free modules. Johnson has also shown the same applies for $\mathbf{Z}[C_m \times C_\infty]$ for $m \geq 2$. In fact, *whenever* G is a finitely generated abelian group, $\mathbf{Z}[G]$ admits no nontrivial stably free modules (see [23]). By contrast, O'Shea [35] has shown this is not the case for $\mathbf{Z}[C_m \times F_n]$ whenever m is divisible by p^2 for some prime p and $n \geq 2$. Indeed, there are infinitely many isomorphically distinct stably free modules of rank 1. We are similarly met with infinitely many isomorphically distinct nontrivial stably free modules for $\mathbf{Z}[(C_2 \times C_2) \times F_n]$ (see [22]).

1.2 Structure of thesis

As already stated, the overall structure of this thesis will consist of two parts. Part I will be concerned with finite metacyclic groups, and will consist of five chapters including this one. Part II will concern metacyclic groups of infinite type and contains two chapters.

In Chapter 2, we introduce the necessary results to discuss the syzygies of D_{2p} and $G(p, 3)$. We shall, for the most part, omit the proofs and provide references wherever suitable. Chapter 3 will be concerned with the syzygies of $\mathbf{Z}[C_n]$ for $n \geq 2$. Whereas the results of this chapter are certainly known, they do not appear readily in the literature, and the direct calculations appear to be entirely absent. The results of this chapter shall be of use in Chapters 4 and 5.

In Chapter 4 we consider the syzygies of $\mathbf{Z}[D_{2p}]$ and explicitly calculate the various interactions under the tensor product. As we shall see, the majority of our calculations depend only upon n where $p = 2n+1$. The requirement of p being prime is necessary only to ensure the indecomposable nature of our modules, and to prevent our syzygies from becoming 'too big'. The key point in this chapter is that the syzygies decompose into indecomposable modules representing the x-strand and the y-strand. To provide us with a succinct notation, we shall often write $\Omega_r^x(\mathbf{Z})$ for the x-strand of the rth stable syzygy, i.e. those modules arising from the x-strand (or upper strand) of the resolution. So, for example, $K \oplus Y$ is a minimal representative element of $\Omega_0(\mathbf{Z})$, in which K is a minimal module representing $\Omega_0^x(\mathbf{Z})$, and Y is a minimal module representing the y-strand. In fact, as we will see, $Y = [\Sigma_y)$ in this case. Moreover, with K as above, we will show:

Theorem A. *For* $\Lambda = \mathbf{Z}[D_{2p}]$ *(p odd prime), the module K acts as the identity within the stable class; that is,* $K \otimes X \cong X \oplus \Lambda^a$ *for some* $a \geq 0$ *and where X is a representative element of the x-strand of* $\Omega_r(\mathbf{Z})$, $0 \leq r \leq 3$.

For the purposes of this section alone, we shall write Ω_r for a representative module of the x-strand of $\Omega_r(\mathbf{Z})$ $(0 \leq r \leq 3)$ in which $\Omega_0 = K$ with this notation. The author stresses that this is purely in the interests of clarity, and shall be abandoned once we begin to build a more accurate picture of these stable syzygies. We will then explicitly show the following relations:

Theorem B. *For* $\Lambda = \mathbf{Z}[D_{2p}]$, $p = 2n+1$, *the following relations hold when tensoring over* $\mathbf{Z}$*:*

- $\Omega_1 \otimes \Omega_1 \cong \Omega_2 \oplus \Lambda^{n-1}$;
- $\Omega_1 \otimes \Omega_2 \cong \Omega_3 \oplus \Lambda^{n}$;
- $\Omega_1 \otimes \Omega_3 \cong \Omega_0 \oplus \Lambda^{n-1}$.

In particular, the last isomorphism uses the fact that $\Omega_0(\mathbf{Z}) = \Omega_4(\mathbf{Z})$. Using Theorems A and B, along with the fact that K is self dual (see [24]), we therefore have:

Theorem C. *The x-strand of the syzygies* $\Omega_r(\mathbf{Z})$ *of* $\mathbf{Z}[D_{2p}]$ *forms a cyclic group within the stable class of order* 4*, generated by* Ω_1 *with identity* Ω_0*.*

In Chapter 5, we now consider the syzygies of $\mathbf{Z}[G(p, 3)]$, mirroring the techniques of Chapter 4. As before, we denote the representative element of the x-strand of $\Omega_0(\mathbf{Z})$ by K. We then show:

Theorem D. *For $\Lambda = \mathbf{Z}[G(p,\,3)]$, the module K acts as the identity within the stable class; that is, $K \otimes X \cong X \oplus \Lambda^a$ for some $a \geq 0$ and where X is a representative element of the x-strand of $\Omega_r(\mathbf{Z})$, $0 \leq r \leq 5$.*

Theorem E. *With K as in Theorem D, $K \sim K^*$; that is, K is stably self-dual.*

We denote the representative modules of the x-strand of $\Omega_r(\mathbf{Z})$ by Ω_r $(0 \leq r \leq 5)$ where $\Omega_0 = K$. By setting $d = (p-1)/3$ we will explicitly show:

Theorem F. *For $\Lambda = \mathbf{Z}[G(p,\,3)]$, the following relations hold when tensoring over $\mathbf{Z}$:*

(1): $\Omega_1 \otimes \Omega_1 \cong \Omega_2 \oplus \Lambda^{d-1}$;

(2): $\Omega_1 \otimes \Omega_2 \cong \Omega_3 \oplus \Lambda^{2d}$;

(3): $\Omega_1 \otimes \Omega_3 \cong \Omega_4 \oplus \Lambda^{d-1}$;

(4): $\Omega_1 \otimes \Omega_4 \cong \Omega_5 \oplus \Lambda^{2d}$;

(5): $\Omega_1 \otimes \Omega_5 \cong \Omega_0 \oplus \Lambda^{d-1}$.

Again, the last isomorphism uses the fact that $\Omega_0(\mathbf{Z}) = \Omega_6(\mathbf{Z})$. By putting this all together we therefore have the following result:

Theorem G. *The x-strand of the syzygies $\Omega_r(\mathbf{Z})$ of $\mathbf{Z}[G(p,\,3)]$ forms a cyclic group within the stable class of order 6, generated by Ω_1 with identity Ω_0.*

In the process of showing Theorem F, we shall in fact provide an affirmative answer for the sequencing conjecture when $q = 3$. This shall be proven in two parts.

Theorem H. *There exist the following three basic sequences:*

(i)

$$\begin{tikzcd} & & & K(3) \arrow[dr] & & & \\ 0 \arrow[r] & R(1) \arrow[r] & \Lambda \arrow[rr] \arrow[ur] & & \Lambda \arrow[r] & R(3) \arrow[r] & 0; \end{tikzcd} \tag{$\mathcal{B}(1)$}$$

(ii)

$$\begin{tikzcd} & & & K(1) \arrow[dr] & & & \\ 0 \arrow[r] & R(2) \arrow[r] & \Lambda \arrow[rr] \arrow[ur] & & \Lambda \arrow[r] & R(1) \arrow[r] & 0; \end{tikzcd} \tag{$\mathcal{B}(2)$}$$

(iii)

$$\begin{tikzcd} & & & K(2) \arrow[dr] & & & \\ 0 \arrow[r] & R(3) \arrow[r] & \Lambda \arrow[rr] \arrow[ur] & & \Lambda \arrow[r] & R(2) \arrow[r] & 0. \end{tikzcd} \tag{$\mathcal{B}(3)$}$$

It is then a straight forward splicing argument to prove the sequencing conjecture for $q = 3$; that is:

Theorem I. *The sequencing conjecture is true for $q = 3$ and we have the following exact sequence*

$$\begin{tikzcd}
 & & & K(3) \arrow[dr] & & & & K(2) \arrow[dr] & & & & K(1) \arrow[dr] & & & \\
0 \arrow[r] & R(1) \arrow[r] & \Lambda \arrow[rr] \arrow[ur] & & \Lambda \arrow[rr] \arrow[dr] & & \Lambda \arrow[rr] \arrow[ur] & & \Lambda \arrow[rr] \arrow[dr] & & \Lambda \arrow[rr] \arrow[ur] & & \Lambda \arrow[r] & R(1) \arrow[r] & 0. \\
 & & & & & R(3) \arrow[ur] & & & & R(2) \arrow[ur] & & & & &
\end{tikzcd}$$

We conclude Part 1 by generalising the results of Theorems D and E to a general prime q such that $q|p-1$; that is, we show:

Theorem J. *For $\Lambda = \mathbf{Z}[G(p, q)]$, there is a module K which represents the x-strand of $\Omega_0(\mathbf{Z})$. In particular, this acts as the identity within the stable class. In other words, we have $K \otimes X \cong X \oplus \Lambda^a$ for some $a \geq 0$ and where X is a representative element of the x-strand of $\Omega_r(\mathbf{Z})$, $0 \leq r \leq 2q-1$. Furthermore, $K \sim K^*$.*

Chapter 6 signals the start of Part II. As with Chapter 2, this consists of the necessary results for Chapter 7. The overall strategy will be to consider fibre squares, thereby allowing us to construct projective modules over the ring in question by lifting them from two of the corners. In particular, we show:

Theorem K. *There are no nontrivial stably free modules over $\mathbf{Z}[G(p, q) \times F_n]$, where $n \geq 1$.*

Chapter 2

Preamble

Let Λ be an arbitrary ring and denote the units by $U(\Lambda)$. By a (right) Λ-module we shall mean an abelian group M such that:

- $m \cdot 1 = m$ for all $m \in M$;
- $m(\lambda_1\lambda_2) = (m\lambda_1)\lambda_2$ for all $m \in M$, λ_1, $\lambda_2 \in \Lambda$;
- $(m_1 + m_2)\lambda = m_1\lambda + m_2\lambda$ for all m_1, $m_2 \in M$, $\lambda \in \Lambda$;
- $m(\lambda_1 + \lambda_2) = m\lambda_1 + m\lambda_2$ for all $m \in M$, λ_1, $\lambda_2 \in \Lambda$.

A module is said to be free if it has a basis and, in this case, can be thought of as behaving similar to a vector space. When a module fails to have a basis, it is often useful to consider the extent to which it fails to have one. We say that a Λ-module P is projective if there is another Λ-module Q such that $P \oplus Q \cong F$, where F is some Λ-module which is free of unspecified rank. A projective module can therefore be seen to be a generalisation of a free module.

Throughout this thesis we denote the ring of $n \times n$ matrices over Λ by $M_n(\Lambda)$, and the set of $m \times n$ matrices over Λ by $M_{m,n}(\Lambda)$. Of particular interest is the group of invertible $n \times n$ matrices over Λ, which we denote by $GL_n(\Lambda)$ and call the *general linear group*. This will play a larger role in Part II of this thesis.

Recall that if Λ is a commutative ring, and G a (finite) group[1], then the group algebra $\Lambda[G]$ consists of all formal Λ-valued functions with finite support defined on G. This naturally forms a Λ-module under pointwise addition and scalar multiplication. If $g \in G$ then we can make particular note of the element of $\Lambda[G]$ defined by

$$\hat{g}(x) = \begin{cases} 1, & \text{if } x = g; \\ 0, & \text{if } x \neq g. \end{cases}$$

We then have a basis for $\Lambda[G]$ over Λ given by $\{\hat{g}\}_{g \in G}$ thereby allowing us to represent elements of $\Lambda[G]$ as finite linear combinations of elements of G with coefficients in Λ; that is,

$$\alpha = \sum_{g \in G} \lambda_g \hat{g}.$$

---

[1] The finiteness condition here purely reflects the fact that Part I of this thesis concerns only finite metacyclic groups. Nevertheless, many of the results that follow will apply to infinite groups. The details are left to the reader

Trivially, we note $\Lambda[G]$ is free over Λ. Moreover, $\Lambda[G]$ has the structure of a Λ-algebra, where the product is given by

$$(\alpha \cdot \beta)(g) = \sum_{h \in G} \alpha(gh^{-1})\beta(h).$$

The multiplicative identity is clearly seen to be $\hat{1}$ and hence there is the inclusion $\Lambda \hookrightarrow \Lambda[G]$ where $\lambda \mapsto \lambda \cdot \hat{1}$. When there is no confusion, we adopt the notational shorthand of dropping the hat above $\hat{g}$.

For such group rings we define a $\Lambda[G]$-lattice to be a $\Lambda[G]$-module whose underlying Λ-module is finitely generated and projective. As this thesis will be primarily concerned with $\mathbf{Z}[G]$-modules, we can think of $\mathbf{Z}[G]$-lattices as $\mathbf{Z}[G]$-modules whose underlying abelian group is finitely generated and free. Note that any ideal in $\mathbf{Z}[G]$ is a $\mathbf{Z}[G]$-lattice.

Now, we define a group representation by a homomorphism $\rho : G \to GL_n(\Lambda)$. Whenever $\Lambda = \mathbf{F}$ is a field, there are a number of results which aid the computations and more theoretical aspects of the theory. Unfortunately, in the realm of rings, the theory does contain a certain degree of pathology. For instance, Maschke's theorem fails the transition to general rings, and $\mathbf{Z}$ is no longer projective as a $\mathbf{Z}[G]$ module unless $|G| = 1$. Nevertheless, as we shall see in Section 2.2, the integral representation theory of lattices does have some nice behaviour which we will be able to exploit.

Despite the added complications of integral representation theory, there are sufficient 'nice' qualities for us to utilise. The remainder of this section, while still aimed at finite groups, is also applicable to more general groups. For our purposes, we will never need anything more troubling than a countable group. With this, we find that our rings are not 'too big' when $\Lambda = \mathbf{Z}$. To reflect this we say that a ring is *weakly coherent* when any submodule of a countably generated module is necessarily countably generated. A straightforward argument shows that any countable ring is weakly coherent. Clearly $\mathbf{Z}[G]$ belongs to this class of rings (provided G is countable).

More useful still is the property of *weak finiteness*. We say a ring Λ is weakly finite if, whenever $\varphi : \Lambda^n \to \Lambda^n$ is a surjective Λ-homomorphism, then φ is bijective. In [6], Cohn attributes the result that $\mathbf{F}[G]$ is weakly finite for any field $\mathbf{F}$ of characteristic zero to an unpublished result of Kaplansky. This was later proven by Montgomery [32] and Passman [37]. A theorem of Cohn [9] then shows that any subring of a weakly finite ring is weakly finite. Hence, the property is shown for $\mathbf{Z}[G]$ also. Furthermore, by results of Cohn (see [6], [7], [8]), this implies that $\mathbf{Z}[G]$ also satisfies the *invariant basis number property* (IBN); that is, for positive integers m, n:

$$\mathbf{Z}[G]^m \cong \mathbf{Z}[G]^n \quad \implies \quad m = n.$$

Henceforth, for the remainder of this chapter we shall relabel $\Lambda = \mathbf{Z}[G]$. It should be noted, however, that in future chapters it will be beneficial to use Λ to denote the integral group ring for specific groups; for example $G = D_{4n+2}$ in Chapter 4. For now, we shall stick with a general finite group G unless explicitly stated otherwise. As a final comment on the conventions throughout this thesis, we denote the category of $\mathbf{Z}[G]$-lattices by $\mathcal{F}(\mathbf{Z}[G])$ or simply $\mathcal{F}(\Lambda)$. It is a full subcategory of Λ-modules and throughout this thesis, we shall always stay within this category unless otherwise stated.

2.1 Dual modules

Throughout, we work with right modules unless otherwise stated. A potential drawback of this approach, however, is that of dual modules. Recall, we have set $\Lambda = \mathbf{Z}[G]$. We define the Λ-dual of a Λ-module M to be

$$M^* = Hom_\Lambda(M, \Lambda).$$

In general, for a right Λ-module M, the dual module M^* is naturally a left module via the action

$$\begin{array}{rcl} \bullet : \Lambda \times Hom_\Lambda(M, \Lambda) & \to & Hom_\Lambda(M, \Lambda) \\ (\lambda \bullet f)(x) & = & \lambda f(x). \end{array}$$

To make this into a right module, we require our rings have an additional structure; namely, a natural involution.

Fortunately for us, group rings offer us just such an involution, thereby allowing us to convert any dual module into a right module via the canonical involution. Let $\tau : \Lambda \to \Lambda$ be the aforementioned involution where

$$\tau\left(\sum_{g\in G} a_g g\right) = \left(\overline{\sum_{g\in G} a_g g}\right) = \left(\sum_{g\in G} a_g g^{-1}\right).$$

Given the canonical involution above, convert the left action $\bullet$ into a right action $*$ by

$$(f * \lambda) = \bar{\lambda} \bullet f,$$

where $\bar{g} = g^{-1}$.

When working within $\mathcal{F}(\Lambda)$ there is the following relationship between a module and its double dual. The reader is directed to [20] for a more thorough exposition.

Proposition 2.1.1. *For all $M \in \mathcal{F}(\Lambda)$, $\nu : M \to M^{**}$ is a natural isomorphism of Λ-modules.*

It should be stressed, however, that in general M is not self-dual; that is, it is *not* true in general that $M \cong M^*$. We shall explore three cases of particular note with regards to duality and double duality in Section 2.2 when we introduce representations. Until then, another nice property is that duality is preserved by the direct sum construction; that is:

Proposition 2.1.2. *Let M, N be Λ-modules; then*

$$(M \oplus N)^* \cong M^* \oplus N^*.$$

A natural property to discuss is the transition between modules over $\mathbf{Z}[H]$ and over $\mathbf{Z}[G]$ where H is some (possibly trivial) subgroup of G. First, recall the $\mathbf{Z}$-dual of M is the lattice

$$M^\star = Hom_{\mathbf{Z}}(M, \mathbf{Z})$$

on which G acts by $(fg)(m) = f(mg^{-1})$. Consider now the following notions of restriction and extension of scalars. Let $i : H \subset G$ be the natural inclusion map of the subgroup H into G. We now induce two maps on the category of finitely generated lattices:

$$i^* : \mathcal{F}(\mathbf{Z}[G]) \to \mathcal{F}(\mathbf{Z}[H]),$$

and

$$i_* : \mathcal{F}(\mathbf{Z}[H]) \to \mathcal{F}(\mathbf{Z}[G]),$$

where i^* is given by restricting scalars to $\mathbf{Z}[H]$, and i_* is given by extending scalars; that is, $i_*(M) = M \otimes_{\mathbf{Z}[H]} \mathbf{Z}[G]$. In the interests of maintaining a concise notation, we shall often write the above tensor product $- \otimes_{\mathbf{Z}[H]} -$ over $\mathbf{Z}[H]$ simply as $- \otimes_H -$.

Proposition 2.1.3 (E-S Stage 1). *If $i : \{1\} \hookrightarrow G$, then for any $L \in \mathcal{F}(\Lambda)$,*

$$Hom_{\mathbf{Z}[G]}(\mathbf{Z}[G],\, L) \cong_{\mathbf{Z}} Hom_{\mathbf{Z}}(\mathbf{Z},\, i^*(L)).$$

Proof. Let $l \in L$ and consider the well defined Λ-homomorphism,

$$\hat{l} : \mathbf{Z}[G] \to L$$

given by $\hat{l}(\lambda) = l\lambda$. Define the mapping,

$$\natural : Hom_{\mathbf{Z}}(\mathbf{Z},\, i^*(L)) \to Hom_{\mathbf{Z}[G]}(\mathbf{Z}[G],\, L),$$

where $\natural(f) = \widehat{f(1)}$. Evidently, $\natural$ is a $\mathbf{Z}$-homomorphism.

It therefore remains to show bijectivity. First suppose $\natural(f) = \natural(g)$, i.e. that $\widehat{f(1)} = \widehat{g(1)}$. Consider an element $\lambda = \sum_{h \in G} \lambda_h h \in \mathbf{Z}[G]$ so that

$$\sum_{h \in G} f(1)\lambda_h h = \sum_{h \in G} g(1)\lambda_h h.$$

It follows that $f(1)\lambda_h = g(1)\lambda_h$ for all $\lambda_h \in \mathbf{Z}$. As $i^*(L)$ is torsion free we therefore conclude $f(1) = g(1)$ and hence $f = g$.

Now, take some $\bar{f} \in Hom_{\mathbf{Z}[G]}(\mathbf{Z}[G],\, L)$. This is clearly determined by where it sends 1. Consequently, this corresponds to some $f \in Hom_{\mathbf{Z}}(\mathbf{Z},\, i^*(L))$; that is, there exists an f such that $\natural(f) = \bar{f}$. ☐

Proposition 2.1.4 (E-S Stage 2). *If $i : \{1\} \hookrightarrow G$, then for any $L \in \mathcal{F}(\Lambda)$,*

$$Hom_{\mathbf{Z}[G]}(L,\, \mathbf{Z}[G]) \cong_{\mathbf{Z}} Hom_{\mathbf{Z}}(i^*(L),\, \mathbf{Z}).$$

Proof. Let $\epsilon : \mathbf{Z}[G] \to \mathbf{Z}$ denote the augmentation homomorphism where $\epsilon(g) = 1$. We next induce the following $\mathbf{Z}$-module homomorphism,

$$\epsilon_* : Hom_{\mathbf{Z}[G]}(L,\, \mathbf{Z}[G]) \to Hom_{\mathbf{Z}}(i^*(L),\, \mathbf{Z}).$$

To show that ϵ_* is an isomorphism, we first consider the following special case:

(I): ϵ_* is an isomorphism when $L = \mathbf{Z}[G]$.

Evidently, $Hom_{\mathbf{Z}[G]}(\mathbf{Z}[G],\, \mathbf{Z}[G]) \cong \mathbf{Z}[G]$ which is the free $\mathbf{Z}$-module with basis $\{g\}_{g \in G}$. Since $\mathbf{Z}[G]$ is a lattice we have $i^*(\mathbf{Z}[G]) \cong \mathbf{Z}^{|G|}$. Hence, $Hom_{\mathbf{Z}}(i^*(\mathbf{Z}[G]),\, \mathbf{Z}) \cong \mathbf{Z}^{|G|}$ is also a free **Z**-module with basis $\{g^*\}_{g\in G}$ where

$$g^*(h) = \begin{cases} 1, & h = g; \\ 0, & h \neq g. \end{cases}$$

Clearly, $\epsilon_*(g) = g^*$, and thus ϵ_* is an isomorphism of **Z**-modules.

Now, as ϵ_* is additive, (I) clearly generalises to:

(II): ϵ_* is an isomorphism when $L = \mathbf{Z}[G]^n$ for $n \geq 1$.

Given this, we can now prove that ϵ_* is an isomorphism for any $\mathbf{Z}[G]$-lattice L.

(III): ϵ_* is injective for any $\mathbf{Z}[G]$-lattice L.

As L is finitely generated, we take a surjective $\mathbf{Z}[G]$-homomorphism,

$$\pi : \mathbf{Z}[G]^m \to L$$

and extend this to form an exact sequence thus

$$\mathbf{Z}[G]^m \overset{\pi}{\to} L \to 0 \to 0.$$

We then construct the following commutative diagram with exact rows as follows,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] \arrow[d, "\cong"] & Hom_{\mathbf{Z}[G]}(L, \mathbf{Z}[G]) \arrow[r] \arrow[d, "\epsilon_*"'] & Hom_{\mathbf{Z}[G]}(\mathbf{Z}[G]^m, \mathbf{Z}[G]) \arrow[d, "\epsilon_*"', "\cong"] \\
0 \arrow[r] & 0 \arrow[r] & Hom_{\mathbf{Z}}(i^*(L), \mathbf{Z}) \arrow[r] & Hom_{\mathbf{Z}}(i^*(\mathbf{Z}[G]^m), \mathbf{Z})
\end{tikzcd}$$

The result now follows from the ‘injective Four Lemma’.

(IV): ϵ_* is surjective for any $\mathbf{Z}[G]$-lattice L.

Take the exact sequence

$$0 \to K \overset{j}{\to} \mathbf{Z}[G]^m \overset{\pi}{\to} L \to 0$$

where $K = Ker(\pi)$ and j is the natural injection. It is well-known that i^* is an exact functor (see, for example [22]). So, we have

$$0 \to i^*(K) \overset{i^*(j)}{\to} i^*(\mathbf{Z}[G]^m) \overset{i^*(\pi)}{\to} i^*(L) \to 0$$

is also an exact sequence. Since L, $\mathbf{Z}[G]^m$ are lattices, it follows that their restrictions are simply copies of **Z**. Consequently, the above exact sequence splits as $\mathbf{Z}^{m\cdot|G|} \cong i^*(K) \oplus \mathbf{Z}^l$, where $l = rk_{\mathbf{Z}}(L)$. Hence, $i^*(K)$ is projective as a **Z**-module. Furthermore, it is clear that $i^*(K)$ is also finitely generated. In other words, K is a $\mathbf{Z}[G]$-lattice.

Next, construct the following commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"] & Hom_{\mathbf{Z}[G]}(L,\, \mathbf{Z}[G]) \arrow[r, "\pi^*"] \arrow[d, "\epsilon_*^L"'] & Hom_{\mathbf{Z}[G]}(\mathbf{Z}[G]^m,\, \mathbf{Z}[G]) \arrow[r, "j^*"] \arrow[d, "\epsilon_*"', "\cong"] & Hom_{\mathbf{Z}[G]}(K,\, \mathbf{Z}[G]) \arrow[d, "\epsilon_*^k"'] \\
0 \arrow[r] & Hom_{\mathbf{Z}}(i^*(L),\, \mathbf{Z}) \arrow[r, "\pi^*"] & Hom_{\mathbf{Z}}(i^*(\mathbf{Z}[G]^m),\, \mathbf{Z}) \arrow[r, "j^*"] & Hom_{\mathbf{Z}}(i^*(K),\, \mathbf{Z})
\end{tikzcd}$$

As ϵ_*^K is injective by (III), it follows that ϵ_*^L is surjective by the 'surjective Four Lemma'. ☐

In fact, the above isomorphisms are merely special cases of what one may tentatively call the 'Eckmann-Shapiro' relations.

Proposition 2.1.5 (Eckmann-Shapiro). *Let $i : H \hookrightarrow G$ be the natural inclusion map of a subgroup H into G. If M is a $\mathbf{Z}[H]$-lattice, and N is a $\mathbf{Z}[G]$-lattice, then there exist the following isomorphisms:*

- $Hom_{\mathbf{Z}[G]}(i_*(M),\, N) \cong Hom_{\mathbf{Z}[H]}(M,\, i^*(N))$*;*
- $Hom_{\mathbf{Z}[G]}(N,\, i_*(M)) \cong Hom_{\mathbf{Z}[H]}(i^*(N),\, M)$*.*

Strictly speaking, the original statement by Eckmann and Shapiro concerns the corresponding isomorphisms in cohomology in which the above is simply the case $n = 0$:

$$Ext^n_{\mathbf{Z}[G]}(i_*(M),\, N) \cong Ext^n_{\mathbf{Z}[H]}(M,\, i^*(N)); \tag{2.1.6}$$

$$Ext^n_{\mathbf{Z}[G]}(N,\, i_*(M)) \cong Ext^n_{\mathbf{Z}[H]}(i^*(N),\, M). \tag{2.1.7}$$

Nevertheless, we shall refer to the isomorphisms of Proposition 2.1.5 as the Eckmann-Shapiro relations throughout. In the interests of succinctness, we shall refrain from providing a proof for Proposition 2.1.5. It suffices to say that the first isomorphism[2] is the well-known result that the extension of scalars functor is a left adjoint of the restriction of scalars. The second isomorphism is not true in general, and the reader is directed to [22] (see appendix B) for a proof.

A related result is sometimes referred to as the 'projection formula for Frobenius reciprocity' in the literature, see [5]. As we have engulfed the more classical Frobenius Reciprocity under the umbrella of the Eckmann-Shapiro relations, we shall henceforth refer to the following simply as 'Frobenius Reciprocity'.

Proposition 2.1.8 (Frobenius Reciprocity). *Let $i : H \subset G$ be the inclusion map of the subgroup H into a finite group G. If M is a $\mathbf{Z}[H]$-module, and N is a $\mathbf{Z}[G]$-module, then there exists an isomorphism*

$$\varphi : i_*(M) \otimes_{\mathbf{Z}} N \xrightarrow{\simeq} i_*(M \otimes_{\mathbf{Z}} i^*(N))$$

Corollary 2.1.9. *Let M be a Λ-lattice of rank $rk_{\mathbf{Z}}(M) = m$; then $(\Lambda)^r \otimes M \cong (\Lambda)^{rm}$.*

---

[2] This is sometimes referred to as Frobenius Reciprocity in the literature.

Proof. Take the trivial subgroup $\{1\}$ and the natural inclusion $i : \{1\} \subset G$. Then $i_*(\mathbf{Z}) = \Lambda$ and it follows from Frobenius reciprocity that,

$$\begin{aligned} (\Lambda)^r \otimes_{\mathbf{Z}} M = & \quad i_*(\mathbf{Z}^r) \otimes M \\ \cong & \quad i_*(\mathbf{Z}^r \otimes i^*(M)) \\ = & \quad i_*(\mathbf{Z}^r \otimes \mathbf{Z}^m) \\ = & \quad i_*(\mathbf{Z}^{rm}) \\ = & \quad (\Lambda)^{rm}. \end{aligned}$$

□

Finally, we consider a result concerning the extension of scalars of the trivial module $\mathbf{Z}$ and self-duality. If $i : H \hookrightarrow G$ is the natural inclusion of a subgroup H into the finite group G, then we induce the extension of scalars functor as before. We therefore have the following result which will be of use in Chapter 4.

Proposition 2.1.10. *With i_* as defined above, the extension of the trivial $\mathbf{Z}[H]$-module is self-dual; that is, $i_*(\mathbf{Z})^* \cong i_*(\mathbf{Z})$.*

Proof. If we think of $\mathbf{Z}$ as the trivial $\mathbf{Z}[H]$-module, then $i_*(\mathbf{Z}) = \mathbf{Z} \otimes_H \Lambda$ and we clearly have the following action of $g \in G$:

$$\begin{aligned} (\alpha \otimes g_i) \cdot g = & \quad \alpha \otimes g_i \cdot g \\ = & \quad \alpha \otimes h \cdot g_j \\ = & \quad \alpha \cdot h \otimes g_j \\ = & \quad \alpha \otimes g_j \end{aligned}$$

where $h \in H$ and $g_j \in G \backslash H$. Hence we may form a set $\{g_1, \ldots, g_k\}$ of coset representatives of $G \backslash H$ such that $\{1 \otimes g_1, \ldots, 1 \otimes g_k\}$ forms a $\mathbf{Z}$-basis for $i_*(\mathbf{Z})$.

Now, let $g \in G$ and observe that $g_j \cdot g \in H g_{\sigma_g(j)}$ where $g_{\sigma_g(j)} \in \{g_1, \ldots, g_k\}$. We intend to show the σ_g is in fact a permutation; that is, we have $(1 \otimes g_j)g = 1 \otimes g_{\sigma_g(j)}$ for all $1 \leq j \leq k$.

It is clearly sufficient to show injectivity of σ_g. So, suppose $g_{j_1}g, g_{j_2}g \in Hg_i$ for some $1 \leq i \leq k$. Thus, for some $h, h' \in H$, $hg_{j_1}g = h'g_{j_2}g$ from which it follows that $hg_{j_1} = h'g_{j_2}$. In other words, we are in the same coset and so $g_{j_1} = g_{j_2}$, as required.

Thus far we have shown σ_g is a permutation on $\{1, \ldots, k\}$ and therefore we have a permutation matrix $\rho_{i_*(\mathbf{Z})}(g^{-1})$ in which

$$(\rho_{i_*(\mathbf{Z})}(g^{-1}))_{ij} = \begin{cases} 1, & j = \sigma_g(i); \\ 0, & j \neq \sigma_g(i). \end{cases}$$

Finally, since $gg^{-1} = 1$ it is straightforward to see that $(\sigma_g)^{-1} = \sigma_{g^{-1}}$. Consequently,

$$(\rho_{i_*(\mathbf{Z})}(g))_{ij} = \begin{cases} 1, & j = \sigma_{g^{-1}}(i); \\ 0, & j \neq \sigma_{g^{-1}}(i) \end{cases} = (\rho_{i_*(\mathbf{Z})}(g^{-1}))_{ji}.$$

Thus, $\rho_{i_*(\mathbf{Z})}(g^{-1}) = \rho_{i_*(\mathbf{Z})}(g)^T$, as required. □

2.2 Integral representation theory

Let G be a group. An n-dimensional $\mathbf{Z}$-representation is a pair (M, ρ) where M is a free $\mathbf{Z}$-module of rank n, and $\rho : G \to GL_n(\mathbf{Z})$ is a group homomorphism. If (N, σ) is another $\mathbf{Z}$-representation, then $\Psi : (M, \rho) \to (N, \sigma)$ is a $(G, \mathbf{Z})$-morphism when $\Psi : M \to N$ is a $\mathbf{Z}$-linear map such that for all $g \in G$, $m \in M$ we have $\Psi(\rho(g)(m)) = \sigma(g)(\Psi(m))$.

Now, for $g \in G$ denote by $\hat{g}$ the element of $\mathbf{Z}[G]$ where

$$\hat{g}(x) = \begin{cases} 1, & \text{if } x = g; \\ 0, & \text{if } x \neq g. \end{cases}$$

The set $\{\hat{g}\}_{g \in G}$ forms a $\mathbf{Z}$-basis for $\mathbf{Z}[G]$. We can associate with (M, ρ) a right $\mathbf{Z}[G]$-module $M(\rho)$ whose underlying abelian group is M, and on which $\mathbf{Z}[G]$ acts by

$$m \bullet \left(\sum_{g \in G} \lambda_g \hat{g} \right) = \sum_{g \in G} \lambda_g \rho(g^{-1})(m).$$

Conversely, if M is a finite dimensional right $\mathbf{Z}[G]$-module, we associate with M a finite dimensional $\mathbf{Z}$-representation $\rho_M : G \to GL_n(\mathbf{Z})$ where

$$\rho_M(g)(m) = m \cdot \hat{g}^{-1}.$$

This correspondence between $\mathbf{Z}$-representations of G and right modules over $\mathbf{Z}[G]$ is clearly $1-1$. Such a viewpoint was initiated by Noether in the 1920s and allows us to consider, among other things, properties of modules by working with the associated representations. We say two n-dimensional representations ρ, σ are *equivalent* if there exists some $X \in GL_n(\mathbf{Z})$ such that $\rho(g)X = X\sigma(g)$ for all $g \in G$. In particular, this is true if and only if the associated $\mathbf{Z}[G]$-modules are isomorphic.

Of particular interest will be the following relationship between dual modules and their representations. Specifically, we may think of the $\mathbf{Z}[G]$-dual of a $\mathbf{Z}[G]$-lattice M as the lattice M^* where G acts by

$$\rho_{M^*}(g) = \rho_M(g^{-1})^T.$$

There are now three special cases of representations to note:

- The regular representation

This is simply the matrix description of the free module of rank 1; that is we consider $\mathbf{Z}[G]$ as a module over itself. If $|G| = n$, this gives rise, via the above correspondence, to the regular representation $\rho_{reg} : G \to GL_n(\mathbf{Z})$; that is,

$$\rho_{reg}(g)(m) = g \cdot m.$$

Each $\rho_{reg}(g)$ is a permutation matrix and so satisfies the orthogonality condition $\rho_{reg}(g)^{-1} = \rho_{reg}(g)^T$. It therefore follows that $\rho_{reg} \equiv \rho_{reg}^*$. Moreover, we may legitimately identify $\mathbf{Z}[G]$ with $\mathbf{Z}[G]^*$ by confusing the canonical basis $\{g\}_{g \in G}$ with its dual basis $\{g^*\}_{g \in G}$. This result generalises easily to give:

Proposition 2.2.1. *If $M \in \mathcal{F}(\mathbf{Z}[G])$, then*

$$M \text{ is free } \Leftrightarrow M^* \text{ is free.}$$

We may extend Proposition 2.2.1 to encompass projectives in a straightforward manner:

Proposition 2.2.2. *If $M \in \mathcal{F}(\mathbf{Z}[G])$, then*

$$M \text{ is projective } \Leftrightarrow M^* \text{ is projective.}$$

- The trivial representation

We consider $\mathbf{Z}$ to be a $\mathbf{Z}[G]$-module in which each group element acts trivially, i.e. each $g \in G$ acts as the identity. In matrix terms, the trivial representation is therefore given by the trivial homomorphism $\tau : G \to GL_1(\mathbf{Z})$. Since $\tau(g) = \tau^*(g) = Id$ for all $g \in G$, it is clear that $\mathbf{Z} \cong_{\mathbf{Z}[G]} \mathbf{Z}^*$.

- The augmentation ideal

The previous two cases were both self dual, and perhaps this was overly deceptive. For, as already stated, it is not a property that is shared by $\mathbf{Z}[G]$-modules in general. Consider the *augmentation map* $\epsilon : \mathbf{Z}[G] \to \mathbf{Z}$, where $\epsilon(g) = 1$. The kernel of ϵ is of significant interest to us, called the *augmentation ideal.* We usually denote the augmentation ideal by $I(G)$, I_G, or simply I when there is no confusion as to which group we are working with. Clearly, $I(G) \in \mathcal{F}(\mathbf{Z}[G])$. Moreover, we have the following well known result:

Proposition 2.2.3. *The augmentation ideal $I(G)$ has a $\mathbf{Z}$-basis given by*

$$\{g - 1_G \mid g \in G \text{ such that } g \neq 1_G\}.$$

Although in general $I(G)$ is not self dual, there is a specific case worth noting where $I(G) \cong I(G)^*$. Suppose $G = C_n =< x \mid x^n = 1 >$ is the cyclic group of order n. Then we write $I(C_n)$ as

$$I(C_n) = span_{\mathbf{Z}}\{x - 1, x^2 - 1, \dots, x^{n-1} - 1\}.$$

By a series of elementary basis changes, we confuse the basis of $I(C_n)$ with it's dual basis; that is:

Proposition 2.2.4. *Let C_n denote the cyclic group of order n; then*

$$I(C_n) \cong_{\mathbf{Z}[C_n]} I(C_n)^*.$$

As it happens, this is the only case for which $I(G)$ is self-dual. This fact follows since self duality of $I(G)$ is equivalent to G having cohomological period 2. As shown by Swan [55] (Lemma 5.2), this is only true when G is cyclic.

It will become necessary throughout this thesis to understand how lattices interact under the tensor product. For this reason, we recall a few basic facts. When M, N are Λ-lattices of ranks m and n, respectively, with corresponding representations ρ_M

and ρ_N, then the tensor product $M \otimes_{\mathbf{Z}} N$ is a lattice of rank mn where the G-action is given by,

$$(v \otimes w) \cdot g = vg \otimes wg.$$

Recall that if $X = (x_{ij})$ and $Y = (y_{ij})$ are two matrices, then $X \otimes Y = (Xy_{ij})$. Consequently, $M \otimes_{\mathbf{Z}} N$ is determined by a representation,

$$\begin{aligned} \rho_{M \otimes N} : \ G &\longrightarrow \ GL_{mn}(\mathbf{Z}) \\ g &\mapsto \ \rho_M(g) \otimes \rho_N(g). \end{aligned}$$

As we shall be tensoring predominantly over $\mathbf{Z}$, we drop the subscript in $- \otimes_{\mathbf{Z}} -$, reserving subscripts only for tensoring over some other ring.

With these considerations in mind, we now move on to considering the representations of lattices in more detail. First, we note the following standard results of tensor products:

Proposition 2.2.5. *If M, N are $\mathbf{Z}[G]$-modules, then*

$$Hom_{\mathbf{Z}}(M,\, N) \cong M^* \otimes N.$$

Proposition 2.2.6. *Let M, N be two Λ-lattices. Then the dual of their tensor product is isomorphic to the tensor product of their duals; that is,*

$$(M \otimes N)^* \cong M^* \otimes N^*.$$

Moreover, $Hom_{\mathbf{Z}}(M,\, N)$ is a lattice on which G acts by $(fg)(v) = (f(vg^{-1}))g$. In particular $M^* \otimes M$ is the matrix ring $M_m(\mathbf{Z})$ on which G acts by conjugation,

$$Ag = \rho_M(g^{-1}) A \rho_M(g).$$

Finally, we introduce a result that will be of use throughout Part I of this thesis.

Proposition 2.2.7. *Let $\{E_\psi\}_{\psi \in \Psi}$ be a $\mathbf{Z}$-basis for the free abelian group A and let $B \subset A$ be an additive subgroup such that $rk_{\mathbf{Z}}(B) \leq m$. Suppose also that there exists a subset $\Phi \subset \Psi$ such that $|\Phi| = m$ and $E_\phi \in B$ for each $\phi \in \Phi$; then*

i) $rk_{\mathbf{Z}}(B) = m$;

ii) $\{E_\phi\}_{\phi \in \Phi}$ is a $\mathbf{Z}$-basis for B;

iii) A/B is torsion free.

2.3 Cyclic Algebras

Let R be a commutative ring and let $\theta : R \to R$ be a ring automorphism of finite order dividing $n \geq 2$, i.e. we have $\theta^n = Id$. We define the fixed ring,

$$R^\theta = \{r \in R \mid \theta(r) = r\}.$$

If $\alpha \in R^\theta$ then we define the *cyclic algebra* $\mathcal{C}_n(R,\, \theta;\, \alpha)$ to be the two-sided R-module

$$\mathcal{C}_n(R,\, \theta;\, \alpha) = R \cdot 1 \oplus R \cdot y \oplus \cdots \oplus R \cdot y^{n-1}$$

which is free of rank n over R with basis $\{1, y, \ldots, y^{n-1}\}$. Moreover, we have multiplication determined by

$$y^n = \alpha,\ y \cdot r = \theta(r) \cdot y,\ \text{where } r \in R.$$

Evidently, $\mathcal{C}_n(R, \theta;\, \alpha)$ is an algebra over the fixed point ring R^θ. For ease of notation, whenever $\alpha = 1$, we shall simply denote this by $\mathcal{C}_n(R,\, \theta)$.

The cyclic algebra construction allows us to construct certain group rings from other, simpler, group rings. For example, let $R = \mathbf{Z}[C_3]$ denote the integral group ring of the cyclic group of order 3. We may then construct $\mathbf{Z}[D_6]$, the integral group ring of the dihedral group of order 6. Let $\theta \in Aut(C_3)$ be nontrivial of order 2, i.e. $\theta^2 = Id$. Take 1 to be the element of the fixed ring. It is straightforward to now show

$$\mathbf{Z}[D_6] = \mathcal{C}_2(\mathbf{Z}[C_3],\, \theta;\, 1).$$

Now define a *pointed n-ring* to be a triple $(R,\, \theta;\, \alpha)$ such that $\theta : R \to R$ satisfies $\theta^n = Id$ and $\alpha \in R^\theta$. We in fact have a category of pointed n-rings with morphism

$$f : (R, \theta;\, \alpha) \to (S, \psi;\, \beta)$$

such that $f : R \to S$ is a ring homomorphism where $f(\alpha) = \beta$, and $f \circ \theta = \psi \circ f$. Formally, the cyclic algebra construction is functorial on pointed n-rings. However, it should be noted that we do also want to allow the case where $\theta^n = Id$ but $ord(\theta) \neq n$. For consider the above example of $\mathbf{Z}[C_3]$ in which $\theta^2 = Id$. As θ fixes $1 + x + x^2$, then θ induces a ring automorphism on the quotient $I(C_3)^* = \mathbf{Z}[C_3]/(1 + x + x^2)$. Likewise, the augmentation ideal $I(C_3)$ is stable under θ and so θ induces the identity automorphism on the quotient $\mathbf{Z} = \mathbf{Z}[C_3]/I(C_3)$. Thus, we can apply the cyclic algebraic construction to both $I(C_3)^*$ and $\mathbf{Z}$. This is important in the context of fibre squares (see Section 6.2), for which we have the following crucial result:

Proposition 2.3.1. *The cyclic algebra construction $\mathcal{C}_n$ preserves fibre squares of pointed n-rings.*

This result will be of significant use in Part II of this thesis.

2.4 The Ext^1 functor

By $\mathbf{Ext}^1_\Lambda$ we mean the collection of short exact sequences of Λ-modules and Λ-homomorphisms of the form,

$$\mathcal{E} = (0 \to E_+ \xrightarrow{i} E_0 \xrightarrow{p} E_- \to 0). \tag{2.4.1}$$

This becomes a category upon introducing the following commutative diagrams as morphisms

$$\begin{tikzcd}
\mathcal{E}= (0 \arrow[r] \arrow[d, "\varphi"] & E_+ \arrow[r] \arrow[d, "\varphi_+"'] & E_0 \arrow[r] \arrow[d, "\varphi_0"'] & E_- \arrow[r] \arrow[d, "\varphi_-"'] & 0) \\
\mathcal{F}= (0 \arrow[r] & F_+ \arrow[r] & F_0 \arrow[r] & F_- \arrow[r] & 0).
\end{tikzcd}$$

By $\mathbf{Ext}^1_\Lambda(A,\, B)$ we mean the full subcategory of $\mathbf{Ext}^1_\Lambda$ in which $E_+ = B$ and $E_- = A$. When there is no confusion as to the choice of ring, we omit the suffix and write $\mathbf{Ext}^1$ and $\mathbf{Ext}^1(A,\, B)$.

Suppose $\mathcal{E},\, \mathcal{F} \in \mathbf{Ext}^1(A,\, B)$, then we define a *congruence* to be a morphism $\varphi : \mathcal{E} \to \mathcal{F}$ that induces the identity at both ends; that is, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}= (0 \arrow[r] \arrow[d, "\varphi"] & B \arrow[r] \arrow[d, "Id"'] & E_0 \arrow[r] \arrow[d, "\varphi_0"'] & A \arrow[r] \arrow[d, "Id"'] & 0) \\
\mathcal{F}= (0 \arrow[r] & B \arrow[r] & F_0 \arrow[r] & A \arrow[r] & 0).
\end{tikzcd}$$

If such a congruence exists, we write '$\mathcal{E} \equiv \mathcal{F}$'. It is then a straightforward consequence of the Five Lemma that congruence is an equivalence relation on $\mathbf{Ext}^1(A,\, B)$. We denote the collection of such equivalence classes in $\mathbf{Ext}^1(A,\, B)$ under '$\equiv$' by $Ext^1(A,\, B)$.

Observe that $\mathbf{Ext}^1(A,\, B)$ is equivalent to a small category, so that $Ext^1(A,\, B)$ is in fact a set. It is then a well known result that the 'Baer sum' induces the structure of an abelian group on $Ext^1(A,\, B)$. To describe this operation, we first recall some natural constructions on $\mathbf{Ext}^1(A,\, B)$:

- **Pushout**: Let $A,\, B_1,\, B_2$ be Λ-modules; if $f : B_1 \to B_2$ is a Λ-homomorphism and $\mathcal{E} = (0 \to B_1 \xrightarrow{i} E_0 \xrightarrow{\eta} A \to 0) \in \mathbf{Ext}^1(A,\, B_1)$ we put

  $$f_*(\mathcal{E}) = (0 \to B_2 \xrightarrow{j} \varinjlim(f,\, i) \xrightarrow{\epsilon} A \to 0),$$

  where $\varinjlim(f,\, i) = (B_2 \oplus E_0)/Im(f \times -i)$ denotes the colimit and j is the injection $j : B_2 \to \varinjlim(f,\, i)$, $j(x) = [x,\, 0]$. The correspondence $\mathcal{E} \mapsto f_*(\mathcal{E})$ determines the covariant 'pushout' functor $f_* : \mathbf{Ext}^1(A,\, B_1) \to \mathbf{Ext}^1(A,\, B_2)$. Furthermore, there is a natural transformation $\nu_f : Id \to f_*$ obtained as follows:

  $$\begin{tikzcd}
  \mathcal{E} \;= (0 \arrow[r] \arrow[d, "\nu_f"] & B_1 \arrow[r, "i"] \arrow[d, "f"'] & E_0 \arrow[r] \arrow[d, "\nu"'] & A \arrow[r] \arrow[d, "Id"'] & 0) \\
  f_*(\mathcal{E})= (0 \arrow[r] & B_2 \arrow[r] & \varinjlim(f,\, i) \arrow[r] & A \arrow[r] & 0)
  \end{tikzcd}$$

  where $\nu : E_0 \to \varinjlim(f,\, i)$ is the mapping $\nu(x) = [0,\, x]$.

- **Pullback**: Let $A_1,\, A_2,\, B$ be Λ-modules; if $f : A_1 \to A_2$ is a Λ-homomorphism and $\mathcal{E} = (0 \to B \to E_0 \xrightarrow{\eta} A_2 \to 0) \in \mathbf{Ext}^1(A_2,\, B)$ we put

  $$f^*(\mathcal{E}) = (0 \to B \to \varprojlim(\eta,\, f) \xrightarrow{\epsilon} A_1 \to 0),$$

  where $\varprojlim(\eta,\, f) = (E_0 \times_{\eta, f} A_1) = \{(x,\, y) \;:\; \eta(x) = f(y)\}$ denotes the fibre product and $\epsilon : \varprojlim(\eta,\, f) \to A_1$ is the projection $\epsilon(x,\, y) = y$. The correspondence $\mathcal{E} \mapsto f^*(\mathcal{E})$ determines the contravariant 'pullback' functor $f_* : \mathbf{Ext}^1(A_2, B) \to \mathbf{Ext}^1(A_1, B)$. Furthermore, there is a natural transformation $\mu_f : f^* \to Id$ defined by:

$$\begin{tikzcd}
f^*(\mathcal{E}) = (0 \arrow[d, "\mu_f"] \arrow[r, phantom] & 0 \arrow[r] & B \arrow[r] \arrow[d, "Id"'] & \varprojlim(\eta, f) \arrow[r] \arrow[d, "\mu_0"'] & A_1 \arrow[r] \arrow[d, "f"'] & 0) \\
\mathcal{E} = (0 \arrow[r, phantom] & 0 \arrow[r] & B \arrow[r] & E_0 \arrow[r, "\eta"] & A_2 \arrow[r] & 0)
\end{tikzcd}$$

where $\mu_0 : \varprojlim(\eta, f) \to E_0$ is the projection $\mu_0(x, y) = x$.

- **Direct product**: Let A_1, A_2, B_1, B_2 be Λ-modules, and for $r = 1, 2$, let

$$\mathcal{E}_r = (0 \to B_r \to E(r)_0 \to A_r \to 0) \in \mathbf{Ext}^1(A_r, B_r).$$

Then $\mathcal{E}_1 \times \mathcal{E}_2 = (0 \to B_1 \times B_2 \to E(1)_0 \times E(2)_0 \to A_1 \times A_2 \to 0)$ is exact, and we get a functorial pairing

$$\times : \mathbf{Ext}^1(A_1, B_1) \times \mathbf{Ext}^1(A_2, B_2) \to \mathbf{Ext}^1(A_1 \oplus A_2, B_1 \oplus B_2).$$

Now, we define the *external sum*,

$$\oplus : \mathbf{Ext}^1(A, B_1) \times \mathbf{Ext}^1(A, B_2) \to \mathbf{Ext}^1(A, B_1 \oplus B_2)$$

by $\mathcal{E}_1 \oplus \mathcal{E}_2 = \Delta^*(\mathcal{E}_1 \times \mathcal{E}_2)$ where $\Delta : A \to A \times A$ is the diagonal. Next, the addition map $+ : B \times B \to B$ can also be regarded as a Λ-homomorphism,

$$\alpha : B \oplus B \to B, \quad \alpha(b_1, b_2) = b_1 + b_2.$$

By combining the external sum with the pushout, we obtain the 'Baer sum'. Explicitly, let $\mathcal{E}_1, \mathcal{E}_2 \in \mathbf{Ext}^1(A, B)$; then the Baer sum $\mathcal{E}_1 + \mathcal{E}_2$ is given by,

$$\mathcal{E}_1 + \mathcal{E}_2 = \alpha_*(\mathcal{E}_1 \oplus \mathcal{E}_2) = \alpha_*\Delta^*(\mathcal{E}_1 \times \mathcal{E}_2).$$

This gives a functorial pairing

$$+ : \mathbf{Ext}^1(A, B) \times \mathbf{Ext}^1(A, B) \to \mathbf{Ext}^1(A, B).$$

It is a straightforward observation that congruence in $\mathbf{Ext}^1$ is compatible with the Baer sum.

The reader is now directed to Chapter 4 of [22] for proofs of the following:

- $(\mathcal{E}_1 + \mathcal{E}_2) + \mathcal{E}_3 \equiv \mathcal{E}_1 + (\mathcal{E}_2 + \mathcal{E}_3)$;
- $\mathcal{E}_1 + \mathcal{E}_2 \equiv \mathcal{E}_2 + \mathcal{E}_1$;
- By $\mathcal{T}$, we denote the trivial extension

$$\mathcal{T} = (0 \to B \xrightarrow{i_B} B \oplus A \xrightarrow{\pi_A} A \to 0),$$

where $i_B(b) = (b,\, 0)$ and $\pi_A(b,\, a) = a$. Whenever there exists a congruence $\mathcal{E} \equiv \mathcal{T}$ we say that $\mathcal{E}$ *splits*. We then have

$$\mathcal{E} + \mathcal{T} \equiv \mathcal{E} \equiv \mathcal{T} + \mathcal{E}.$$

- If $\mathcal{E} \in \mathbf{Ext}^1(A,\, B)$ is defined as that of (2.4.1) with $E_+ = B$ and $E_- = A$, we denote by $-\mathcal{E}$ the extension

  $$-\mathcal{E} = (0 \to B \xrightarrow{i} E_0 \xrightarrow{-p} A \to 0).$$

  We then have,

  $$\mathcal{E} + (-\mathcal{E}) \equiv \mathcal{T} \equiv (-\mathcal{E}) + \mathcal{E}.$$

By the above, $Ext^1(A,\, B)$ is an abelian group with respect to the Baer sum.

We conclude by introducing the derived module category, and discussing a 'destabilization theorem' that will allow us to 'cancel' excess free modules in our exact sequences. For any two Λ-modules M, N, we say that a homomorphism $f : M \to N$ *factors through a projective module* (written $f \approx 0$) when there exists a projective module P, and a pair of homomorphisms $\eta : M \to P$ and $\xi : P \to N$ such that $f = \xi \circ \eta$. Evidently, this is equivalent to f factoring through a free module. We now define

$$\langle M,\, N\rangle = \{f : M \to N \mid f \approx 0\}.$$

It may be shown that $\langle M,\, N\rangle$ is an additive subgroup of $Hom_\Lambda(M,\, N)$ in which $f \approx g$ if and only if $f - g \approx 0$. We now obtain the *derived module category* $\mathcal{D}er = \mathcal{D}er(\Lambda)$, whose objects are right Λ-modules, and in which, for any two objects M, N, the set of morphisms $Hom_{\mathcal{D}er}(M,\, N)$ is give by

$$Hom_{\mathcal{D}er}(M,\, N) = Hom_\Lambda(M,\, N)/\langle M,\, N\rangle.$$

In particular, $Hom_{\mathcal{D}er}(M,\, N)$ has the natural structure of an abelian group since $\langle M,\, N\rangle$ is a subgroup of $Hom_\Lambda(M,\, N)$. As a final comment, we highlight the important result that two objects M, N are isomorphic in the derived module category $M \cong_{\mathcal{D}er} N$ if and only if $M \oplus P \cong_\Lambda N \oplus Q$, for some projective modules P, Q. For a more detailed exposition, the reader is directed to Chapter 5 of [22].

Next, we say that a module M is coprojective when $Ext^1(M,\, Q) = 0$ for any projective module Q; that is, every such short exact sequence splits. Since the dual of any short exact sequence of Λ-lattices is another short exact sequence, and since the dual of a projective module is projective, it follows that any Λ-lattice is coprojective. We therefore have the following theorem [22] (p. 97):

Proposition 2.4.2 (Johnson's destabilization theorem). *Consider the following exact sequence of* Λ*-modules* $0 \to J \oplus Q_0 \xrightarrow{j} Q_1 \to M \to 0$ *in which* Q_0, Q_1 *are projective; if* M *is coprojective, then* $Q_1/j(Q_0)$ *is projective.*

2.5 Free resolutions and syzygies

As before, we let $\Lambda = \mathbf{Z}[G]$ denote the integral group ring for some finite group G. It is well known that any module M may be written as the quotient of some free module; that is, if M is a Λ-module, then we write

$$M = F_0/K_0.$$

As we are only interested in finitely generated modules, we may take F_0 to be a module with a basis indexed by the generators of M. There is then a surjective map $\varphi : F_0 \to M$ which sends basis elements to generators. It therefore follows that $M \cong F_0/Ker(\varphi)$, as required.

Now, write $K_0 = Ker(\varphi)$, and observe K_0 is a submodule of a finitely generated module, and hence finitely generated (because $\mathbf{Z}[G]$ is Noetherian). It follows that K_0 is itself a quotient $K_0 = F_1/K_1$ of some free module F_1. We may continue this process to yield a sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

which is necessarily exact. Furthermore, each F_r is finitely generated by construction. We call this a *free resolution* of M. In our case, let

$$\cdots \overset{\partial_{n+2}}{\to} F_{n+1} \overset{\partial_{n+1}}{\to} F_n \overset{\partial_n}{\to} \cdots \overset{\partial_2}{\to} F_1 \overset{\partial_1}{\to} F_0 \overset{\partial_0}{\to} \mathbf{Z} \to 0 \qquad (\mathcal{F})$$

be a resolution of the trivial module $\mathbf{Z}$ over Λ such that each F_r is a finitely generated free module. We define the *syzygy modules* $(J_r)_{1\leq r}$ of $\mathcal{F}$ to be the intermediate modules,

$$J_r = Im(\partial_r) = Ker(\partial_{r-1}).$$

This definition allows us to break $\mathcal{F}$ up into a collection of short exact sequences,

$$0 \to J_1 \xrightarrow{i_1} F_0 \xrightarrow{\partial_0} \mathbf{Z} \to 0 \text{ and } 0 \to J_n \xrightarrow{i_n} F_{n-1} \xrightarrow{p_{n-1}} J_{n-1} \to 0.$$

It is quite evident that these depend upon the free resolution chosen. For consider the second short exact sequence above. This may be transformed by the addition of a free module F,

$$0 \to J_n \oplus F \xrightarrow{i_n \oplus Id} F_{n-1} \oplus F \xrightarrow{p'_{n-1}} J_{n-1} \to 0$$

where p'_{n-1} is the obvious composition of p_{n-1} with the projection $F_{n-1} \oplus F \to F_{n-1}$. Clearly, $J_n \oplus F$ is another syzygy related via the free module F. In this sense, there may be many syzygies which are distinct as Λ-modules. To impose a sense of uniqueness we consider the stable class of syzygies. The stability relation between Λ-modules M, M' is understood to be the isomorphism $M \oplus \Lambda^a \cong M' \oplus \Lambda^b$ for some integers a, $b \geq 0$. We then say that M and M' are *stably equivalent* and denote this $M \sim M'$. It is straightforward to show:

Proposition 2.5.1. *The relation '$\sim$' is an equivalence on isomorphism classes of Λ-modules.*

We denote the set of isomorphism classes of modules N such that $N \sim M$ by $[M]$ and call this the *stable module* of M. Given this, the stable syzygy $\Omega_r(\mathbf{Z})$ is therefore defined to be the stable class $[J_r]$ of any such J_r.

Proposition 2.5.2. *Suppose the following are two free resolutions of the Λ-module M:*

$$\cdots F_n \xrightarrow{\delta_n} F_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\delta_0} M \to 0$$

$$\cdots F'_n \xrightarrow{\delta'_n} F'_{n-1} \xrightarrow{\delta'_{n-1}} \cdots \xrightarrow{\delta'_2} F'_1 \xrightarrow{\delta'_1} F'_0 \xrightarrow{\delta'_0} M \to 0$$

If $J_r = Ker(\delta_{r-1})$ *and* $J'_r = Ker(\delta'_{r-1})$ *as before, then* $J_r \sim J'_r$ *for each* r*.*

To prove this, we use the following well known result:

Proposition 2.5.3 (Schanuel's Lemma)**.** *Suppose we have two short exact sequences of* Λ*-modules*

$$0 \to K \to F \to M \to 0$$

and

$$0 \to K' \to F' \to M \to 0$$

in which F*,* F' *are free (or, indeed, projective); then,*

$$K \oplus F' \cong K' \oplus F$$

and hence $K \sim K'$*.*

Proof of Proposition 2.5.2. Proceed by induction. First, when $n = 1$ we have the short exact sequences

$$0 \to J_1 \to F_0 \to M \to 0$$

and

$$0 \to J'_1 \to F'_0 \to M \to 0.$$

By Schanuel's Lemma, $J_1 \oplus F'_0 \cong J'_1 \oplus F_0$, i.e. $J_1 \sim J'_1$.

Now suppose the statement holds for $n = r - 1$, i.e. for some a, $b \geq 0$ we have:

$$J_{r-1} \oplus \Lambda^a \cong J'_{r-1} \oplus \Lambda^b. \tag{2.5.4}$$

Consider when $n = r$; we have

$$0 \to J_r \to F_{r-1} \to J_{r-1} \to 0$$

and

$$0 \to J'_r \to F'_{r-1} \to J'_{r-1} \to 0.$$

By using (2.5.4), and modifying the above exact sequences, we obtain:

$$0 \to J_r \to F_{r-1} \oplus \Lambda^a \to J_{r-1} \oplus \Lambda^a \to 0$$

and

$$0 \to J'_r \to F'_{r-1} \oplus \Lambda^b \to J_{r-1} \oplus \Lambda^a \to 0.$$

By once more using Schanuel's Lemma, we obtain the desired isomorphism

$$J_r \oplus F'_{r-1} \oplus \Lambda^b \cong J'_r \oplus F_{r-1} \oplus \Lambda^a,$$

i.e. $J_r \sim J'_r$. □

Proposition 2.5.2 can therefore be reinterpreted as:

Proposition 2.5.5. *The stable syzygy* $\Omega_r(\mathbf{Z})$ *is independent of the choice of free resolution* $\mathcal{F}$*.*

Proposition 2.5.6. *Let $\Omega_r(\mathbf{Z})$ and $\Omega_s(\mathbf{Z})$ be stable syzygies of the trivial Λ-module $\mathbf{Z}$; then*

$$\Omega_r(\mathbf{Z}) \otimes \Omega_s(\mathbf{Z}) = \Omega_{r+s}(\mathbf{Z}).$$

Proof. To prove this we first demonstrate:

$$\Omega_r(\mathbf{Z}) \otimes \Omega_1(\mathbf{Z}) = \Omega_{r+1}(\mathbf{Z}). \tag{2.5.7}$$

As the tensor product is associative, we may iterate this process to reach the desired conclusion. As before, we proceed by induction. For ease of notation, for each r write Ω_r for a representative element of $\Omega_r(\mathbf{Z})$. Now, consider the case for $r = 1$,

$$0 \to \Omega_1 \to \Lambda^a \to \mathbf{Z} \to 0$$

for some $a \geq 1$. Now apply $- \otimes \Omega_1$,

$$0 \to \Omega_1 \otimes \Omega_1 \to \Lambda^a \otimes \Omega_1 \to \mathbf{Z} \otimes \Omega_1 \to 0$$

and note that we may write $\Lambda^a \otimes \Omega_1 \cong (\Lambda \otimes \Omega)^a$. We therefore have

$$0 \to \Omega_1 \otimes \Omega_1 \to (\Lambda \otimes \Omega_1)^a \to \Omega_1 \to 0.$$

By Corollary 2.1.9 this can be rewritten as

$$0 \to \Omega_1 \otimes \Omega_1 \to \Lambda^b \to \Omega_1 \to 0$$

for some $b \geq 1$. Thus, $\Omega_1 \otimes \Omega_1 = \Omega_2$, as required.

Now suppose we have shown the result for $r - 1$; that is,

$$\Omega_{r-1} \otimes \Omega_1 = \Omega_r. \tag{2.5.8}$$

Consider the exact sequence,

$$0 \to \Omega_r \to \Lambda^\alpha \to \Omega_{r-1} \to 0$$

for some $\alpha \geq 1$, and apply $- \otimes \Omega_1$ as before

$$0 \to \Omega_r \otimes \Omega_1 \to \Lambda^\alpha \otimes \Omega_1 \to \Omega_{r-1} \otimes \Omega_1 \to 0.$$

By (2.5.8), $\Omega_{r-1} \otimes \Omega_1 = \Omega_r$, and by once again identifying $\Lambda^\alpha \otimes \Omega_1 \cong (\Lambda \otimes \Omega_1)^\alpha \cong \Lambda^\beta$ for some $\beta \geq 1$, we have

$$0 \to \Omega_r \otimes \Omega_1 \to \Lambda^\beta \to \Omega_r \to 0$$

and hence $\Omega_r \otimes \Omega_1 = \Omega_{r+1}$. The result now follows by our earlier remark. □

Evidently, Proposition 2.5.6 is telling us something about what is going on when these syzygies interact. However it isn't telling us much. As such, it will be our aim in Chapters 3, 4 and 5 to discuss how these syzygies interact outside of the stable class. In particular, for the metacyclic groups in question, we will demonstrate the existence of a cyclic group formed within the stable class. The order of this group turns out to be related to what we call the periodic cohomology of the group G, which we now define.

First, for a finite group G we may say, *prima facie*, that there are two possibilities within the stable category: either

(i) the stable syzygies $(\Omega_r(\mathbf{Z}))_{r\in\mathbf{Z}}$ are isomorphically distinct; or

(ii) $\Omega_r(\mathbf{Z}) \cong \Omega_s(\mathbf{Z})$ for some r, $s \in \mathbf{Z}$ where $r \neq s$.

As it happens, most finite groups belong to type (i), but there are a number of important examples belonging to type (ii). Two such examples are the cyclic groups C_m of order m, and the metacyclic groups $G(p, q)$ of order pq.

The categorisation of finite groups within these two types depends upon the Sylow subgroup structure. Specifically, when for each odd prime p the Sylow p-subgroup is cyclic and the Sylow 2-subgroup is either cyclic or generalized quaternion, then G belongs to type (ii). For a classification of all finite soluble groups satisfying these conditions, the reader is directed to [58], and for finite non-soluble groups to [53].

Now, we say that $n > 0$ is a (free) *cohomological period* of G if $\Omega_{r+n}(\mathbf{Z}) \cong \Omega_r(\mathbf{Z})$ for all $r \in \mathbf{Z}$. We now have the following useful result (see [20], Chapter 7):

Proposition 2.5.9. *Whenever G is a finite group of (free) cohomological period $n \in \mathbf{N}$, then the following conditions are equivalent:*

$\mathcal{F}_1(n)$: $\Omega_{r+n}(\mathbf{Z}) \cong \Omega_r(\mathbf{Z})$ *for all* $r \in \mathbf{Z}$*;*

$\mathcal{F}_2(n)$: $\Omega_{r+n}(\mathbf{Z}) \cong \Omega_r(\mathbf{Z})$ *for at least one* $r \in \mathbf{Z}\backslash\{0\}$*;*

$\mathcal{F}_3(n)$: *There exists an exact sequence in* $\mathcal{F}(\Lambda)$ *of the form*

$$0 \to \mathbf{Z} \to F_{n-1} \to \cdots \to F_0 \to \mathbf{Z} \to 0$$

where each F_i is finitely generated and free over Λ.

When the above free modules are merely projective, we say G has cohomological period n. The two are related as the cohomological period divides the (free) cohomological period. So, if the cohomological period of G is n, then its free period is δn where $\delta \geq 1$. In particular, δ divides the order of the projective class group $\widetilde{K}_0(\mathbf{Z}[G])$.

Next, note that the cohomological period is necessarily even. Consequently, the smallest possible non-trivial period is $n = 2$. This is realised in the case of cyclic groups. If we describe C_n as, $C_n = \langle x \mid x^n = 1\rangle$ then there is a free resolution of period 2 given by:

$$0 \to \mathbf{Z} \xrightarrow{\epsilon^*} \Lambda \xrightarrow{x-1} \Lambda \xrightarrow{\epsilon} \mathbf{Z} \to 0$$

where ϵ is the augmentation map, and ϵ^* is its dual. The converse was also shown to hold true by Swan (see [55], p. 205); that is, if $n = 2$ is a cohomological period of G, then G is necessarily cyclic. We return to this in Chapter 3.

The next case is when $n = 4$. Unlike the previous case, this does not have a single type of group associated with this cohomological period. For example, both D_{4n+2} and $Q(4n)$ for $n \geq 2$ have (free) cohomological period 4 but are clearly not isomorphic. In Chapter 4, we shall focus our attention on the former. It should be noted, however, that the order $4n + 2$ is necessary, for the dihedral groups of order $4n$ do not have a finite free cohomological period. For a general metacyclic group $G(p, q)$ we note this has cohomological period $2q$.

2.6 Stably free modules

We say that a Λ-module S is stably free when it is stably equivalent to the zero module; that is, when $S \oplus \Lambda^a \cong F$, where $a \geq 1$ and F is a free module of unspecified rank. Trivially, any free module is stably free and an obvious question is when (if ever) the converse is true. When any stably free Λ-module is necessarily trivial (i.e. free), we shall say that Λ has *stably free cancellation* (SFC).

By a theorem of Gabel (see [15]) any stably free module that is not finitely generated is necessarily trivial. Consequently, the question is now posed as to whether every finitely generated stably free module is trivial. In general this is not the case. For instance, in [56] Swan considered the integral group rings of the generalized quaternions $Q(4n)$, $n \geq 2$,

$$Q(4n) =< x,\, y \,|\, x^n = y^2,\, xyx = y > .$$

Specifically, Swan showed that $\mathbf{Z}[Q(4n)]$ admits no nontrivial stably free modules if and only if $n \leq 5$, and at least one nontrivial stably free module exists whenever $n \geq 6$.

Nevertheless, there are still cases of importance where there are no nontrivial stably frees. In the realm of finite groups, it is a sufficient condition for our group to satisfy the Eichler condition. To understand this first recall that, by Wedderburn's Theorem, we have the following decomposition of the real group ring,

$$\mathbf{R}[G] \cong \prod_{i=1}^{m} M_{d_i}(\mathcal{D}_i)$$

where $\mathcal{D}_i \cong \mathbf{C}$, $\mathbf{R}$ or $\mathbf{H}$, the ring of Hamiltonian quaternions. We say that G satisfies the *Eichler condition* when $\mathbf{H}$ is *not* a factor of $\mathbf{R}[G]$. We then have the following form of a result due to Swan and Jacobinski (see [11], p.324):

Theorem 2.6.1 (Swan-Jacobinski)**.** *If G satisfies the Eichler condition, then $\mathbf{Z}[G]$ has SFC.*

It should be noted, however, that this is not a necessary condition. Indeed, consider the following real Wedderburn decomposition of the generalized quaternions,

$$\mathbf{R}[Q(4n)] \cong \begin{cases} \mathbf{R}^{(4)} \times \mathbf{M}_2(\mathbf{R})^{(n-2)/2} \times \mathbf{H}^{(n/2)}, & \text{n even;} \\ \mathbf{R}^{(2)} \times M_2(\mathbf{R})^{(n-1)/2} \times \mathbf{C} \times \mathbf{H}^{(n-1)/2}, & \text{n odd.} \end{cases}$$

Clearly, $Q(4n)$ fails to be Eichler, and yet we have already discussed the stably free modules when $n \leq 5$ - they are trivial.

One of our main goals in this thesis will be to understand the syzygies of certain metacyclic groups. As such, we note the following:

Proposition 2.6.2. *D_{2n} satisfies the Eichler condition.*

Proof. Make the identification $\mathbf{Q}[C_n] = \mathbf{Q}[x]/(x^n - 1)$, where C_n is the cyclic group of order n. As an initial observation,

$$\mathbf{Q}[x]/(x^n - 1) \cong \mathbf{Q} \times \mathbf{Q}[x]/(x^{n-1} + \cdots + x + 1).$$

To go one step further, we fully factorise $x^n - 1$ into a product of cyclotomic polynomials

$$x^n - 1 = \prod_{d|n} c_d(x)$$

and define $\mathbf{Q}(d) = \mathbf{Q}[x]/(c_d(x))$. As is well known, each $c_d(x)$ is an irreducible polynomial over $\mathbf{Q}$. Hence $\mathbf{Q}(d)$ is a field. As such, we have the following *rational* Wedderburn decomposition of C_n,

$$\mathbf{Q}[C_n] \cong \bigoplus_{d|n} \mathbf{Q}(d). \tag{2.6.3}$$

In particular, we observe C_n is necessarily Eichler when considering the group ring over $\mathbf{R}$.

We now use the cyclic algebraic construction of Section 2.3 to find the Wedderburn decomposition of D_{2n}. Observe that for any $n \geq 3$,

$$\mathbf{Q}[D_{2n}] \cong \mathcal{C}_2(\mathbf{Q}[C_n],\, \theta,\, 1),$$

where $\theta : \mathbf{Q}[C_n] \to \mathbf{Q}[C_n]$ is the involution on group elements given by $\theta(g) = g^{-1}$. Under the isomorphism of (2.6.3) this induces an involution $\gamma_d : \mathbf{Q}(d) \to \mathbf{Q}(d)$, which is the identity for $d = 1,\, 2$, and complex conjugation otherwise. In the latter instance, we take a primitive dth-root of unity ζ_d, and write $\mu_d = \zeta_d + \bar{\zeta}_d$. Then the fixed field of $\mathbf{Q}(d)$ under complex conjugation is $\mathbf{Q}(\mu_d)$. It is then straightforward to see

$$\mathcal{C}_2(\mathbf{Q}(d),\, \gamma_d) \cong \begin{cases} \mathbf{Q} \times \mathbf{Q}, & d = 1,\, 2; \\ M_2(\mathbf{Q}(\mu_d)), & d \geq 3. \end{cases}$$

Thus, for any $n \geq 3$, we have the following *rational* Wedderburn decomposition of D_{2n},

$$\mathbf{Q}[D_{2n}] \cong \prod_{d|n} \mathcal{C}_2(\mathbf{Q}(d),\, \gamma_d) \cong \begin{cases} \mathbf{Q} \times \mathbf{Q} \times \prod_{d|n,\, d\geq 3} M_2(\mathbf{Q}(\mu_d)), & n = 2m+1 \\ \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \times \mathbf{Q} \times \prod_{d|n,\, d\geq 3} M_2(\mathbf{Q}(\mu_d)), & n = 2m. \end{cases} \tag{2.6.4}$$

The result now follows as evidently no factor in the real decomposition of D_{2n} can be represented by $\mathbf{H}$. □

Corollary 2.6.5. *$\mathbf{Z}[D_{2n}]$ has SFC.*

When considering more general metacyclic groups, we note the following rational Wedderburn decomposition of $G(p,\, q) = C_p \rtimes C_q$,

$$\mathbf{Q}[G(p,\, q)] \cong \mathbf{Q}[C_q] \times M_q(K_0)$$

where $K_0 = \{x \in \mathbf{Q}(\zeta_p) \mid \theta(x) = x\}$ is the fixed field (and centre) of $\mathcal{C}_q(\mathbf{Q}(\zeta_p),\, \theta)$ and $\zeta_p = exp(2\pi i/p)$. In the interests of succinctness, we omit a proof. The reader is directed to [41] (see Chapter 4). Once again, it is quite clear that no such factor can be represented by $\mathbf{H}$ when considering the Wedderburn decomposition over $\mathbf{R}$. As such, we have:

Proposition 2.6.6. *The metacyclic group $G(p, q)$ of order pq satisfies the Eichler condition. In particular, $\mathbf{Z}[G(p, q)]$ has SFC.*

2.7 The tree structure of stable syzygies

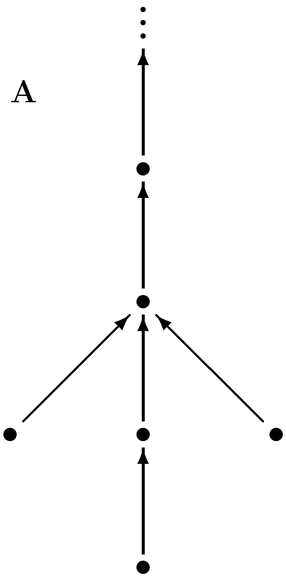
Consider once more the stable module $[M]$ of a finitely generated Λ-module M. We may view this graphically by adopting a strategy first introduced by Dyer and Sieradski [13]. Here, the stable module can be expressed as a 'tree with roots' where each vertex is the isomorphism class of a module $N \in [M]$, and where we draw an arrow $N_1 \to N_2$ when $N_2 \cong N_1 \oplus \Lambda$. Each module N has a unique arrow exiting the vertex given by $N \to N \oplus \Lambda$. Consequently, the only way a nontrivial loop in $[M]$ can occur is if $N \cong N \oplus \Lambda^a$ for some $a > 0$, a possibility precluded by supposing our ring Λ has what we call the *surjective rank property* (see [22], p. 7). Here, we suppose that for any surjective Λ-homomorphism $\varphi : \Lambda^N \to \Lambda^n$ we necessarily have $n \leq N$. Note that this is sometimes referred to in the literature as requiring that Λ has *unbounded generating number*. While trivial for a finite group, we note that this property is also present in the infinite groups of Part II of this thesis. We note the following chain of implications (see [7]):

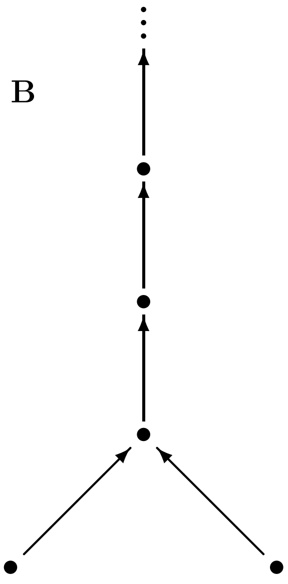
$$WF \ \Rightarrow \ SR \ \Rightarrow \ IBN,$$

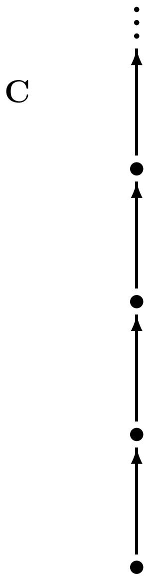
where the above are as explained at the beginning of this chapter. In particular, as integral group rings are weakly finite, they therefore have the SR property, as required.

These finiteness conditions are also of use in the context of a well-defined rank for stably free modules. Suppose S is some finitely generated stably free Λ-module, i.e. $S \oplus \Lambda^a \cong \Lambda^b$ for some integers $a, b \geq 1$. For any weakly finite ring we have a well-defined positive rank for stably free modules. We write this as $rk(S) = b - a$.

Given the above we now portray a stable module $[M]$ as a 'tree with roots'. The key observation here is that this tree does not extend infinitely downwards. In other words, we can define a minimal module M_0 to be a module that does not contain a summand isomorphic to Λ. Graphically, these minimal modules are the roots of our tree. Whenever G is finite, there are three types of tree structure for $[M]$:







Here, **A** can represent the stable class of [0] when $\Lambda = \mathbf{Z}[Q(24)]$, where $Q(24)$ denotes the quaternion group of order 24. The second tree, denoted by **B**, can represent the stable class $\Omega_3(\mathbf{Z})$ over $\mathbf{Z}[Q(32)]$, the integral group ring of the quaternion group of order 32. Finally, **C** can represent the stable class $\Omega_3(\mathbf{Z})$ over $\mathbf{Z}[D_{4n+2}]$, the integral group ring of the dihedral groups of order $4n+2$.

In particular, those stable modules whose tree structure is that of **C** share a common property. We say that a Λ-lattice M has the *cancellation property* when, for any Λ-lattice N such that $rk_{\mathbf{Z}}(M) \leq rk_{\mathbf{Z}}(N)$, we have

$$N \oplus \Lambda^a \cong M \oplus \Lambda^b \;\Rightarrow\; N \cong M \oplus \Lambda^{b-a}.$$

Consequently, we say that the stable module $[M]$ has the *strong cancellation property* if every $N \in [M]$ has the cancellation property. We then have the following relationship with the above tree structures (see [20], Chapter 3):

Proposition 2.7.1. *Let M be a finitely generated Λ-lattice; then the stable module $[M]$ has the strong cancellation property if and only if $[M]$ is straight.*

We have already seen a special case of the cancellation property; namely, SFC. In relation to this, we have discussed the Eichler condition and its role in the Swan-Jacobinski Theorem. However, it should also be noted that this theorem says decidedly more than we previously stated. For a Λ-lattice M, we write $M_{\mathbf{R}} = M \otimes \mathbf{R}$. Such a lattice is said to be Eichler if the endomorphism ring $End_{\Lambda_{\mathbf{R}}}(M_{\mathbf{R}})$ has no simple Hamiltonian factor, **H**. In particular, when Λ satisfies the Eichler condition, any Λ-lattice M is Eichler. We then have the following form of Swan-Jacobinski:

Theorem 2.7.2 (Swan-Jacobinski)**.** *Let M be an Eichler lattice over Λ such that $M \cong M_0 \oplus \Lambda$ for some module M_0; then M has the cancellation property.*

Now, we say that a module M satisfies the *weak cancellation property* whenever $M \oplus \Lambda$ satisfies the cancellation property. In light of this, we may view the above form of Swan-Jacobinski as saying any Λ-lattice has the weak cancellation property if Λ is Eichler. The reader is directed to Chapter 3 of [20] for more details.

Next, we define a *fork* to mean a tree structure with a finite number of 'roots' at the minimal level, and no branching above level 1. Thus, a fork structure looks like either **B** or (trivially) **C**. In particular, if M has the weak cancellation property, then it necessarily has the structure of a fork. Relating this to the syzygies from earlier, we note that any odd stable syzygy necessarily has a fork structure [20]; that is:

Proposition 2.7.3. *For each $n \geq 0$, $\Omega_{2n+1}(\mathbf{Z})$ is a fork.*

This result is of particular interest in the context of the $\mathcal{R}(2)-\mathcal{D}(2)$ problem. Less important in this context are the even syzygies. Nevertheless, Johnson demonstrated in [20] that there is one other possibility for such syzygies; namely, what we call a *crow's foot*. This is a tree structure in which there is just one module at the minimal level and a finite number of at least one at level 1. There is no branching at level 2 or above. Such trees look like either **A** or (trivially) **C**.

When considering what our syzygies look like, we have the following useful result [20]:

Proposition 2.7.4. *Let G be a finite group such that $\mathbf{Z}[G]$ has SFC; then the augmentation ideal $I(G)$ is the unique minimal representative of $\Omega_1(\mathbf{Z})$. In particular, $\Omega_1(\mathbf{Z})$ is straight.*

Furthermore, duality $M \mapsto M^*$ induces a $1-1$ correspondence $\Omega_1(\mathbf{Z}) \leftrightarrow \Omega_{-1}(\mathbf{Z})$. In other words, we have the following corollary:

Corollary 2.7.5. *Let G be a finite group such that $\mathbf{Z}[G]$ has SFC; then $(I(G))^*$ is the unique minimal representative of $\Omega_{-1}(\mathbf{Z})$, i.e. $\Omega_{-1}(\mathbf{Z})$ is straight.*

The situation for even syzygies is decidedly more complicated. Nevertheless, we do have the following result (see [20], Proposition 29.5, p. 122):

Proposition 2.7.6. *Let M_0 be a minimal representative of $\Omega_{2n}(\mathbf{Z})$, where $\mathbf{Z}$ is the trivial Λ-module. If Λ satisfies the Eichler condition, there are two possibilities:*

(i) if $rk_{\mathbf{Z}}(M_0) > 1$, then $\Omega_{2n}(\mathbf{Z})$ is a fork; or

(ii) if $M_0 \cong \mathbf{Z}$, then $\Omega_{2n}(\mathbf{Z})$ is straight.

2.8 Indecomposable modules

As we shall see in Chapter 4, we have the unusual result that the stable class of an indecomposable module (in our case $\mathbf{Z}$) decomposes nontrivially. As pointed out by Johnson in [24], this paradoxical nature of stable modules seems to have been first discussed in a paper by Gruenberg and Roggenkamp [17] (although they attribute the original observation to E.C. Dade). As we are primarily concerned with metacyclic groups, we find that we can still say something about the cancellation of the component parts. Whereas we shall leave the specifics to the relevant chapters, we note the role of indecomposable modules in the cancellation of some of the constituent parts. For now, it will be sufficient to state the classification of indecomposable modules over $\mathbf{Z}[G(p, q)]$ due to Pu [39]. Of particular use will be the quasi-triangular subring $\mathcal{T}_q(A, \pi)$ of $M_q(A)$, as defined in the previous chapter (see also Example 6.2.12). This decomposes as a direct sum of right ideals

$$\mathcal{T}_q(A, \pi) \cong R(1) \oplus \cdots \oplus R(q)$$

in which $R(i)$ is the i^{th} row of $\mathcal{T}_q(A, \pi)$.

Next, recall the reduced projective class group $\widetilde{K}_0(\Lambda)$ which is constructed as follows. Let $\mathcal{P}(\Lambda)$ denote the set of isomorphism classes of finitely generated projective Λ-modules. This becomes an abelian monoid under direct sum, $[P] + [Q] = [P \oplus Q]$. The *Grothendieck group* $K_0(\Lambda)$ is then the universal abelian group obtained from $\mathcal{P}(\Lambda)$. We now define the *reduced Grothendieck group* (or reduced projective class group) to be the quotient $\widetilde{K}_0(\Lambda) = K_0(\Lambda)/[\Lambda]$, in which $[\Lambda]$ is the subgroup generated by the class of Λ. In the context of free resolutions, we may ignore the indecomposable modules not arising from the identity element in $\widetilde{K}_0(\Lambda)$ (see [16]). We now have the following special case of Pu:

Proposition 2.8.1. *Let p, q be prime numbers such that $q|p-1$. There are a total of $2 + q + 2^{q-1} + 2^q$ distinct non-isomorphic genera*[3] *of indecomposable modules for $\Lambda = \mathbf{Z}[C_p \rtimes C_q]$.*

---

[3] Here, we adopt the usual convention of saying two Λ-lattices M, N belong to the same genus if they are isomorphic when localised at (p) for any prime p, i.e. $M_{(p)} \cong N_{(p)}$.

In the manner of [41], we list the indecomposable modules as follows:

I. There are three indecomposable modules over $\mathbf{Z}[C_q]$ that become modules over Λ via the quotient map $G(p,\,q) \to C_q$:

   (i) The trivial module (rank 1);

   (ii) The augmentation ideal, $I_Q = Ker(\mathbf{Z}[C_q] \to \mathbf{Z})$ (rank $q-1$);

   (iii) The group ring itself $\mathbf{Z}[C_q]$ (rank q).

II. There are q distinct indecomposable modules over $\mathcal{C}_q(\mathbf{Z}[\zeta_p],\,\theta) \cong \mathcal{T}_q(A,\,\pi)$ of rank $p-1$:

   (iv) $R(i) \cong (\zeta_p - 1)^e\mathbf{Z}[\zeta_p]$, where $0 \le e \le q-1$ and $R(0) = R(q)$.

   These are distinct Λ-modules via the twisting relation $y\zeta_p^r = \zeta_p^{\theta_*(r)}y$.

We may think of the above as the 'basic' indecomposable modules. The remaining genera of indecomposable modules then arise in the form of non-split extensions

$$0 \to X \to ? \to Y \to 0$$

where X is the direct sum of possible combinations (without repeat) of $R(i)$, and $Y = \mathbf{Z}$, I_Q or $\mathbf{Z}[C_q]$. The proof of this can be found in [10], [39] and, in the form of a specific case, in [41].

III. There is one extension when $Y = \mathbf{Z}$:

   (v) $0 \to R(1) \to \bar{\Lambda}_0 \to \mathbf{Z} \to 0$ (rank p).

IV. There are 2^{q-1} indecomposable non-split extensions when $Y = I_Q$. As $Ext^1_\Lambda(I_Q,\,R(1)) = 0$, such extensions cannot contain $R(1)$. Consequently, there are a total of $q-1$ indecomposable modules in X. Let k_1 denote the number of distinct type II modules combined in X that is contained in an extension with I_Q. There exist:

   (vi) $\sum_{k_1=1}^{q-1}\binom{q-1}{k_1}$ extensions of the form $0 \to X \to V_c \to I_Q \to 0$ where $1 \le c \le 2^{q-1}-1$. In particular, we note $rk_{\mathbf{Z}}(V_c) = (p-1)k_1 + (q-1)$.

V. There are $2^q - 1$ indecomposable non-split extensions for $Y = \mathbf{Z}[C_q]$. Here there are no split extensions, and so there are a total of q indecomposable modules in X. Let k_2 denote the number of distinct type II modules combined in X that is contained in an extension with $\mathbf{Z}[C_q]$. Then there exist:

   (vii) $\sum_{k_2=1}^{q}\binom{q}{k_2}$ extensions of the form $0 \to X \to Y_d \to \mathbf{Z}[C_q] \to 0$ where $1 \le d \le 2^q - 1$. In particular, $rk_{\mathbf{Z}}(Y_d) = (p-1)k_2 + q$.

Any other indecomposable modules belong to the non-trivial elements of $\widetilde{K}_0(\Lambda)$ and are therefore of no consequence in the context of free resolutions. As we shall see, the importance of the above list is that it tells us that there are a limited number of $\mathbf{Z}$-ranks that an indecomposable module can be in any given group ring $\mathbf{Z}[G(p,\,q)]$. This fact will allow us to deduce some properties of the tree structures of syzygies.

As a final comment, consider the stable class $[R(i)]$ of $R(i)$ (considered as a $\mathbf{Z}[G(p,\, q)]$-module). We have the following result:

$$\text{For each } i \in \{1, \dots,\, q\} \text{ the stable class } [R(i)] \text{ is straight.} \tag{2.8.2}$$

A proof of this can (essentially) be found in Chapter 6 of [41] using [39]. However, the proof of this is far from clear and so we provide an alternative proof that was shown to the author by Prof. F. E. A. Johnson. The downside of this approach is that it will rely upon several ideas and results introduced in Part II of this thesis. As these ideas will not be directly required elsewhere in Part I, we will postpone a proof of this until Section 6.8.

Chapter 3

The syzygies of $\mathbf{Z}[C_n]$

As we observed in the previous chapter, the smallest possible non-trivial cohomological period ($k = 2$) is realized in the case of cyclic groups. If we describe the cyclic group of order n as $C_n = \langle x \mid x^n = 1\rangle$ then there is a free resolution of period 2 given by:

$$0 \to \mathbf{Z} \xrightarrow{\epsilon^*} \Lambda \xrightarrow{x-1} \Lambda \xrightarrow{\epsilon} \mathbf{Z} \to 0$$

where ϵ is the augmentation map, and ϵ^* is its dual. Throughout this chapter we denote the integral group ring of C_n by $\Lambda = \mathbf{Z}[C_n]$ and the augmentation ideal as $I = ker(\epsilon)$. We can now read off the syzygies from the above free resolution:

$$\Omega_r(\mathbf{Z}) = \begin{cases} \mathbf{Z}, & r \equiv 0 (mod\, 2); \\ I, & r \equiv 1 (mod\, 2). \end{cases}$$

In Proposition 2.5.6, we saw $\Omega_1(\mathbf{Z}) \otimes \Omega_1(\mathbf{Z}) = \Omega_2(\mathbf{Z})$ allowing us to form a cyclic group of order 2, generated by I. In this chapter we will explicitly examine $I \otimes I$ to better understand what is happening. In particular, this will give us a description that will be of fundamental use in Chapters 4 and 5. It is useful to bear in mind that, as we are tensoring over $\mathbf{Z}$, tensoring with $\mathbf{Z}$ acts like the identity. We therefore have the following table:

$\otimes$	$\mathbf{Z}$	I
$\mathbf{Z}$	$\mathbf{Z}$	I
I	I	??

3.1 The isomorphism $I \otimes I \cong \mathbf{Z} \oplus$ free

The ultimate aim will be to show

$$I \otimes I \cong \mathbf{Z} \oplus \Lambda^{n-2}. \tag{3.1.1}$$

However, we stress that it is the precise description of this isomorphism that will be the main result of this chapter. Now, to justify the $n-2$, first recall that I can be written as

$$I = span_{\mathbf{Z}}\{x - 1, x^2 - 1, \dots, x^{n-1} - 1\},$$

and in particular $rk_{\mathbf{Z}}(I) = n - 1$. Since $rk_{\mathbf{Z}}(\Lambda) = n$, we deduce that we necessarily have $n-2$ copies of Λ. Proving this result will take up the remainder of this chapter.

To start, consider the standard exact sequence,

$$0 \to I \xrightarrow{i} \Lambda \xrightarrow{\epsilon} \mathbf{Z} \to 0$$

and dualise[1],

$$0 \to \mathbf{Z} \xrightarrow{\epsilon^*} \Lambda \xrightarrow{i^*} I^* \to 0$$

where $\epsilon^*(1) = \Sigma = \sum_{r=0}^{n-1} x^r$ is central. Therefore, $Im(\epsilon^*)$ is the two-sided ideal of Λ, generated by Σ. Consequently, $I^* \cong \Lambda/(\Sigma)$ is naturally a ring. Moreover, when $n = p$, prime, we can think of I^* as the cyclotomic ring $\mathbf{Z}[\zeta_p]$. Next, we put $\nu_r = i^*(x^r)$ where $\nu_0 = 1$, and observe that we can write $\nu_r = (\nu_1)^r = \nu^r$. If we think of I^* as a Λ-module, then I^* has a $\mathbf{Z}$-basis $\{1, \nu, \nu^2, \dots, \nu^{n-2}\}$, where $\nu^{n-1} = -1 - \nu - \cdots - \nu^{n-2}$ and the action of x is to multiply by ν.

It may also be useful to recall Proposition 2.2.4 in which we observed $I \cong I^*$. Nevertheless, with the next two chapters in mind, it will be beneficial to distinguish between them.

Now, if $n = 2$ then (3.1.1) is immediate. We therefore let $n \geq 3$ and define the following for $1 \leq r \leq n-2$:

$$V(r) = span_{\mathbf{Z}}\{\nu^{r+k} \otimes \nu^k \mid 0 \leq k \leq n-1\} \subset I^* \otimes I^*.$$

Proposition 3.1.2. *For each $1 \leq r \leq n-2$, we have $V(r) \cong \Lambda$.*

Proof. We will prove the map $f_r : \Lambda \to V(r)$ which sends $x^k \mapsto \nu^{r+k} \otimes \nu^k$ is an isomorphism. This map is clearly surjective by the definition of $V(r)$. To prove that it is injective, we need to show that the defining set of $V(r)$ is linearly independent.

To start, observe that $\nu^n = 1$ and consider,

$$\lambda_1(\nu^r \otimes 1) + \lambda_2(\nu^{r+1} \otimes \nu) + \cdots + \lambda_{n-1}(\nu^{r+n-2} \otimes \nu^{n-2}) + \lambda_n(\nu^{r-1} \otimes \nu^{n-1}) = 0.$$

To show linear independence, we will utilise the fact that $\{\nu^i \otimes \nu^j \mid 0 \leq i, j \leq n-2\}$ is a $\mathbf{Z}$-basis of $I^* \otimes I^*$, i.e. we need to rewrite every term of the form $- \otimes \nu^{n-1}$ or $\nu^{n-1} \otimes -$. To do so, we use $\nu^{n-1} = -1 - \nu - \cdots - \nu^{n-2}$, and rewrite the above sum as

$$(\lambda_1\nu^r - \lambda_n\nu^{r-1}) \otimes 1 + (\lambda_2\nu^{r+1} - \lambda_n\nu^{r-1}) \otimes \nu + \quad \cdots + (\lambda_{n-2}\nu^{r+n-3} - \lambda_n\nu^{r-1}) \otimes \nu^{n-3} + \\ +(\lambda_{n-1}\nu^{r+n-2} - \lambda_n\nu^{r-1}) \otimes \nu^{n-2} = 0.$$

If this sum equals zero, then it follows from the linear independence of $\{\nu^i \otimes \nu^j \mid 0 \leq i, j \leq n-2\}$ that each term $- \otimes \nu^j = 0$. Start with those terms of the form $- \otimes 1$, and observe $\lambda_1\nu^r - \lambda_n\nu^{r-1} = 0$ if and only if $\lambda_1 = \lambda_n = 0$. The above can now be rewritten a final time as

$$\lambda_2\nu^{r+1} \otimes \nu + \cdots + \lambda_{n-2}\nu^{r+n-3} \otimes \nu^{n-3} + \lambda_{n-1}\nu^{r+n-2} \otimes \nu^{n-2} = 0. \qquad (3.1.3)$$

It is now clear that $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda_n = 0$. If we are to be strictly formal, then we should note that ν^{n-1} can appear on the left hand side of the a term of the form $- \otimes \nu^j$ for some $1 \leq j \leq n-2$. However, even in this case, this will not

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[1] The reader is reminded i^* can either refer to the dualisation of i, or to the restriction of scalars induced from i. While not ideal, it should be clear from the context which we mean. For this chapter, we will always mean the dualisation of i.

change that fact that each $\lambda_i = 0$. For suppose $\lambda_i \nu^{r+i-1} \otimes \nu^{i-1} = \lambda_i \nu^{n-1} \otimes \nu^{i-1}$. For the sum of (3.1.3) to be zero, each term of the form $- \otimes \nu^j = 0$. In particular,

$$\lambda_i(-1 - \nu - \cdots - \nu^{n-2}) \otimes \nu^{i-1} = 0.$$

However, by again utilising the linear independence of $\{\nu^i \otimes \nu^j \mid 0 \leq i,\, j \leq n-2\}$ it then follows that each $-\lambda_i \nu^k \otimes \nu^{i-1} = 0$ and so $\lambda_i = 0$.

As explained above, we can now define the map $f_r : \Lambda \to V(r)$ by $x^k \mapsto \nu^{r+k} \otimes \nu^k$, for $0 \leq k \leq n-1$. This is clearly a Λ-homomorphism that maps basis elements to basis elements. □

Proposition 3.1.4. *For any* $1 \leq r \leq n-2$,

$$V(r) \cap (V(1) + \cdots + V(r-1) + V(r+1) + \cdots + V(n-2)) = \{0\}.$$

Proof. Suppose $v_r \in V(r) \cap (V(1) + \cdots + V(r-1) + V(r+1) + \cdots + V(n-2))$. We can write this as

$$\sum_{i=1}^{n} \mu_{r,\,i} \nu^{r+i-1} \otimes \nu^{i-1} = \sum_{j=1,\, j \neq r}^{n-2} \left(\sum_{k=1}^{n} \eta_{j,\,k} \nu^{j+k-1} \otimes \nu^{k-1} \right).$$

This can be rewritten as

$$\sum_{i=1}^{n-2} \left(\sum_{j=1}^{n} \lambda_{i,\,j} \nu^{i+j-1} \otimes \nu^{j-1} \right) = 0$$

where $\lambda_{i,\,j} = \eta_{i,\,j}$ if $i \neq r$ and $\lambda_{r,\,j} = -\mu_{r,\,j}$. By replacing ν^{n-1} with $\sum_{l=0}^{n-2} \nu^l$, we have

$$\sum_{i=1}^{n-2} \left(\sum_{j=1}^{n-1} \lambda_{i,\,j} \nu^{i+j-1} - \lambda_{i,\,n} \nu^{i+n-1} \otimes \nu^{j-1} \right) = 0 \tag{3.1.5}$$

We first show $\lambda_{i,\,n} = 0$ for each $i \in \{1, \ldots, n-2\}$. Begin by setting $j = 1$. Write

$$\sum_{i=1}^{n-2} (\lambda_{i,\,1} \nu^i - \lambda_{i,\,n} \nu^{i+n-1}) \otimes 1 = 0$$

and observe $n \leq i + n - 1 \leq 2n - 3$, i.e. $\nu^{i+n-1} = \nu^{i-1}$ varies between 1 and ν^{n-3}. As ν^i varies between ν and ν^{n-2} we end up with

$$(-\lambda_{1,\,n} + (\text{other terms not including } 1)) \otimes 1 = 0.$$

Thus, $\lambda_{1,\,n} = 0$.

Next, let $\mathcal{T}(k)$ be the statement,

$$\mathcal{T}(k) : \quad \lambda_{k,\,n} = 0.$$

Note that we have already shown $\mathcal{T}(1)$ is true, and suppose $\mathcal{T}(k)$ is true for some $k \in \{1, \ldots, n-3\}$. If we set $j = k+1$, we have

$$\sum_{i=1}^{n-2} (\lambda_{i,\,k+1}\nu^{i+k} - \lambda_{i,\,n}\nu^{i-1}) \otimes \nu^k = 0.$$

First observe $i + k \equiv n - 1\ (mod\, n)$ if and only if $i \equiv n - (k+1)\ (mod\, n)$. We therefore have the term $\lambda_{n-k-1,\,k+1}\nu^{n-1}$. Before we replace this term, we make two observations. First, $i+k \not\equiv k-1\,(mod n)$ for any $i \in \{1, \ldots, n-2\}$. Secondly, observe $i - 1 \equiv k - 1\ (mod\, n)$ if and only if $i = k$. Thus, the term $\lambda_{k,\,n}\nu^{k-1}$ vanishes by the inductive assumption. So, when we replace $\lambda_{n-k-1,\,k+1}\nu^{n-1}$ by $-\lambda_{n-k-1,\,k+1}\sum_{l=0}^{n-2}\nu^l$ we end up with

$$(-\lambda_{n-k-1,\,k+1}\nu^{k-1} + (\text{other terms not including } \nu^{k-1})) \otimes \nu^k = 0.$$

By the linear independence of $\{\nu^i \otimes \nu^j \,|\, 0 \leq i,\, j \leq n-2\}$ we conclude $\lambda_{n-k-1,\,k+1} = 0$.

When $i = k+1$ we have the term $\lambda_{k+1,\,k+1}\nu^{2k+1} - \lambda_{k+1,\,n}\nu^k$. As $\nu^{i+k} \not\equiv \nu^k\,(mod n)$ for any $i \in \{1, \ldots, n-2\}$, we have

$$(-(\lambda_{n-k-1,\,k+1} + \lambda_{k+1,\,n})\nu^k + (\text{other terms not including } \nu^k)) \otimes \nu^k = 0$$

and so

$$(-\lambda_{k+1,\,n}\nu^k + (\text{other terms not including } \nu^k)) \otimes \nu^k = 0.$$

Thus, $\lambda_{k+1,n} = 0$, as required. We have therefore shown $\mathcal{T}(k) \Rightarrow \mathcal{T}(k+1)$ and so $\mathcal{T}(k)$ is true for each $k \in \{1, \ldots, n-2\}$. We can therefore rewrite (3.1.5) as

$$\sum_{i=1}^{n-2}\left(\sum_{j=1}^{n-1} \lambda_{i,\,j}\nu^{i+j-1} \otimes \nu^{j-1}\right) = 0. \tag{3.1.6}$$

By the linear independence of $\{\nu^i \otimes \nu^j \mid 0 \leq i,\, j \leq n-2\}$ we conclude that for each $j \in \{1, \ldots, n-1\}$ we have $\sum_{i=1}^{n-2}\lambda_{i,j}\nu^{i+j-1} \otimes \nu^{j-1} = 0$. So we need only worry about the left hand side of each $- \otimes \nu^{j-1}$. If $i + j - 1 \not\equiv n - 1\ (mod\ n)$ for any $i \in \{1, \ldots, n-2\}$, then we have a sum of linearly independent terms, and therefore conclude each $\lambda_{i,j} = 0$ for $1 \leq i \leq n-2$.

Alternatively, suppose $i+j-1 \equiv n-1\,(mod n)$ for $i \equiv n-j\,(mod n)$. As i varies over $n-2$ terms, we note there will be an 'extra' two terms when we replace ν^{n-1} by $-\sum_{l=0}^{n-2}\nu^l$. For notational simplicity, say these two terms are ν^a and ν^b. We then get

$$-\lambda_{n-j,j}\nu^a \otimes \nu^{j-1} - \lambda_{n-j,j}\nu^b \otimes \nu^{j-1} + (\text{other terms not including } \nu^a \text{ or } \nu^b) \otimes \nu^{j-1} = 0.$$

As these terms are all linearly independent, it follows that $\lambda_{n-j,j} = 0$. We can therefore rewrite $\sum_{i=1}^{n-2}\lambda_{i,j}\nu^{i+j-1} \otimes \nu^{j-1}$ as

$$\sum_{i=1}^{n-j-1} \lambda_{i,j}\nu^{i+j-1} \otimes \nu^{j-1} + \sum_{i=n-j+1}^{n-2} \lambda_{i,j}\nu^{i+j-1} \otimes \nu^{j-1} = 0.$$

These terms are all linearly independent and so $\lambda_{i,j} = 0$ for each $i \in \{1, \ldots, n-2\}$. $\square$

We therefore set $V = V(1) \oplus \cdots \oplus V(n-2)$ and observe that $rk_{\mathbf{Z}}(V) = n(n-2)$. Thus, $rk_{\mathbf{Z}}((I^* \otimes I^*)/V) = 1$. So, by considering the underlying abelian group of $(I^* \otimes I^*)/V$, we apply the fundamental theorem of finitely generated abelian groups to see that this is isomorphic to

$$\mathbf{Z} \oplus (\text{finite abelian}).$$

Proposition 3.1.7. *$(I^* \otimes I^*)/V$ is torsion free.*

Proof. Begin by expressing the basis elements of $I^* \otimes I^*$ as:

$$\begin{array}{ccccc}
1 \otimes 1, & 1 \otimes \nu, & \cdots & 1 \otimes \nu^{n-3}, & 1 \otimes \nu^{n-2}, \\
\nu \otimes 1, & \nu \otimes \nu, & \cdots & \nu \otimes \nu^{n-3}, & \nu \otimes \nu^{n-2}, \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\nu^{n-3} \otimes 1, & \nu^{n-3} \otimes \nu, & \cdots & \nu^{n-3} \otimes \nu^{n-3}, & \nu^{n-3} \otimes \nu^{n-2}, \\
\nu^{n-2} \otimes 1, & \nu^{n-2} \otimes \nu, & \cdots & \nu^{n-2} \otimes \nu^{n-3}, & \nu^{n-2} \otimes \nu^{n-2}.
\end{array}$$

Observe that $1 \otimes \nu^{n-1} = -1 \otimes (1+\nu+\cdots+\nu^{n-2})$ and $\nu^{n-1} \otimes \nu = -(1+\nu+\cdots+\nu^{n-2}) \otimes \nu$. Thus, in the above basis of $I^* \otimes I^*$, we can replace $1 \otimes 1$ and $1 \otimes \nu$ by $1 \otimes \nu^{n-1}$ and $\nu^{n-1} \otimes \nu$, respectively. This can be repeated for each 'row' (except for the last 'row'), where we replace $\nu^i \otimes \nu^i$ and $\nu^i \otimes \nu^{i+1}$ by $\nu^i \otimes \nu^{n-1}$ and $\nu^{n-1} \otimes \nu^{i+1}$, respectively $(0 \leq i \leq n-3)$. Thus, by performing elementary basis transformations, the above basis for $I^* \otimes I^*$ can be replaced by the basis

$$\{\nu^{r+k} \otimes \nu^k \mid 1 \leq r \leq n-2,\ 0 \leq k \leq n-1\} \cup \{\nu^{n-2} \otimes \nu^{n-2}\}.$$

It therefore follows from Proposition 2.2.7 that $(I^* \otimes I^*)/V$ is the rank 1 lattice generated by $\natural(\nu^{n-2} \otimes \nu^{n-2})$, where $\natural : I^* \otimes I^* \to (I^* \otimes I^*)/V$ is the natural surjection. Furthermore, as $\mathbf{Z}$ is the only rank 1 lattice over Λ, we observe $(I^* \otimes I^*)/V \cong \mathbf{Z}$. ☐

Continuing on from the above proof, we can perform further basis transformations (see Proposition 4.5.11), and replace $\nu^{n-2} \otimes \nu^{n-2}$ by T, where

$$\begin{array}{rclccccccc}
T & = & 1 \otimes 1 & + & 1 \otimes \nu & + & 1 \otimes \nu^2 & + & \cdots & + \ 1 \otimes \nu^{n-2} \\
 & & & + & \nu \otimes \nu & + & \nu \otimes \nu^2 & + & \cdots & + \ \nu \otimes \nu^{n-2} \\
 & & & & & + & \nu^2 \otimes \nu^2 & + & \cdots & + \ \nu^2 \otimes \nu^{n-2} \\
 & & & & & & & & \ddots & \vdots \\
 & & & & & & & & & + \ \nu^{n-2} \otimes \nu^{n-2}.
\end{array}$$

So $(I^* \otimes I^*)/V$ is generated by $\natural(T)$. As will become apparent in later chapters, this provides us with a more suitable description of the rank 1 lattice isomorphic to $\mathbf{Z}$. Alternatively, we note

$$\begin{array}{rclcccccccc}
Tx & = & \nu \otimes \nu & + & \nu \otimes \nu^2 & + & \nu \otimes \nu^3 & + & \cdots & + \ \nu \otimes \nu^{n-2} & + \ \nu \otimes \nu^{n-1} \\
 & & & + & \nu^2 \otimes \nu^2 & + & \nu^2 \otimes \nu^3 & + & \cdots & + \ \nu^2 \otimes \nu^{n-2} & + \ \nu^2 \otimes \nu^{n-1} \\
 & & & & & + & \nu^3 \otimes \nu^3 & + & \cdots & + \ \nu^3 \otimes \nu^{n-2} & + \ \nu^3 \otimes \nu^{n-1} \\
 & & & & & & & & \ddots & & \vdots \\
 & & & & & & & & & + \ \nu^{n-2} \otimes \nu^{n-2} & + \ \nu^{n-2} \otimes \nu^{n-1} \\
 & & & & & & & & & & + \ \nu^{n-1} \otimes \nu^{n-1}.
\end{array}$$

Evidently, $Tx = T - 1 \otimes [1 + \cdots + \nu^{n-2}] + [\nu + \cdots \nu^{n-1}] \otimes \nu^{n-1}$. Consider the last 'column' of Tx. This may be rewritten as

$$(\nu + \nu^2 + \cdots + \nu^{n-1}) \otimes \nu^{n-1} = -1 \otimes \nu^{n-1} = 1 \otimes 1 + 1 \otimes \nu + \cdots + 1 \otimes \nu^{n-2}.$$

Substituting back into Tx therefore shows $Tx = T$. Furthermore, it is clear that $T \in I^* \otimes I^*$ but $T \notin V$. For notational convenience, we shall adopt the slight abuse of notation by writing T for the monogenic module of rank 1. As the need arises, we shall often switch between using T to denote the element defined above and the monogenic module of rank 1. It should always be clear from the context which we mean. Now, observe $T \cap V = \{0\}$. Consequently, $I^* \otimes I^* \cong_{\mathbf{Z}} T \oplus V$ is an isomorphism over $\mathbf{Z}$. Hence $(I^* \otimes I^*)/V \cong_{\mathbf{Z}} T$ is an isomorphism over $\mathbf{Z}$. As $(I^* \otimes I^*)/V$ is isomorphic to $\mathbf{Z}$, we know x acts trivially on both $(I^* \otimes I^*)/V$ and T. Thus, we have shown $(I^* \otimes I^*)/V \cong T$ is an isomorphism over Λ.

In particular, the above arguments have shown the existence of the following short exact sequence,

$$0 \to V \to I^* \otimes I^* \to T \to 0. \tag{3.1.8}$$

Recall the dual of a short exact sequence of Λ-lattices is another short exact sequence. By the self-duality of $V \cong \Lambda^{n-2}$ and $T \cong \mathbf{Z}$, we end up with the exact sequence

$$0 \to \mathbf{Z} \to I \otimes I \to \Lambda^{n-2} \to 0$$

which splits. We therefore arrive at the desired isomorphism:

Theorem 3.1.9. *$I \otimes I \cong T \oplus V \cong \mathbf{Z} \oplus \Lambda^{n-2}$.*

Chapter 4

The syzygies of $\mathbf{Z}[D_{4n+2}]$

Throughout this chapter we set $G = D_{2p}$, the dihedral group of order $2p$, p prime. We will find it useful to write $p = 2n + 1$. We then have the following description for G,

$$D_{4n+2} =< x,\, y \mid x^{2n+1} = y^2 = 1,\, yx = x^{2n}y > .$$

As we shall see, many of our calculations will not overtly require $2n + 1$ to be prime. Rather, the necessity of this is to ensure our syzygies do not get ‘too big’, and to ensure our modules of interest are indecomposable (by using results such as those of Pu’s paper outlined in Section 2.8). This does raise the question of whether the results of this chapter may be extended to non-prime integers p. However, these considerations will not be discussed in this thesis and throughout this chapter p will always be a prime number that can be expressed as $p = 2n + 1$.

Hereafter, we denote the integral group ring of G by $\Lambda = \mathbf{Z}[D_{2p}]$, and the integral group ring of C_p by $\Lambda_0 = \mathbf{Z}[C_p]$. Associated to Λ_0 is the canonical injection $i : \Lambda_0 \hookrightarrow \Lambda$. Similarly, we have the canonical injection $j : \mathbf{Z}[C_2] \hookrightarrow \Lambda$. Recall that by $[\alpha)$ we mean the right ideal generated by α; that is $[\alpha) = \{\alpha\lambda \mid \lambda \in \Lambda\}$. In particular, any ideal in Λ is a Λ-lattice, i.e. a Λ-module whose underlying abelian group is finitely generated and free. When considering the stable class of the right ideal $[\alpha)$, we will write this simply as $[\alpha]$.

As already noted in Section 2.5, Λ has cohomological period four. Consequently, the trivial module $\mathbf{Z}$ has a projective resolution of period 4 over Λ; that is, there exists an exact sequence of Λ modules of the form

$$0 \to \mathbf{Z} \to P_3 \to P_2 \to P_1 \to P_0 \to \mathbf{Z} \to 0$$

in which each P_i is a finitely generated projective Λ-module. We are interested in the case where each P_i is finitely generated free.

By a *diagonalised free resolution over* Λ we mean an exact sequence of Λ-modules of the form

$$\cdots \to F_m \overset{\partial_m}{\to} F_{m-1} \overset{\partial_{m-1}}{\to} \cdots \to F_1 \overset{\partial_1}{\to} F_0 \to \mathbf{Z} \to 0 \tag{4.0.1}$$

in which $F_0 \cong \Lambda$ and for each $i \geq 1$, F_i is a free Λ-module of rank 2, i.e. $F_i \cong \Lambda^2$ for $i \geq 1$. Moreover, for each $i \geq 2$ the differential ∂_i has the diagonal form

$$\partial_i = \begin{pmatrix} \partial_i^+ & 0 \\ 0 & \partial_i^- \end{pmatrix} .$$

As a starting point to constructing such resolutions, we first note that the augmentation ideal I_G of G decomposes as the direct sum of two indecomposable modules. First, let M be a Λ_0-lattice. We transform M into a module over Λ via a Galois action. To do so first define a *Galois structure* on M to be an additive automorphism $\Theta : M \to M$ such that $\Theta^2 = Id_M$ and $\Theta(m \cdot x) = \Theta(m) \cdot \theta(x)$ for all $m \in M$ where θ is our chosen automorphism of C_p (in this case $\theta(x) = x^{-1}$). A *Galois lattice* shall then mean a pair (M, Θ) where M is a lattice over Λ_0 and Θ is a Galois structure on M. We then make a Galois lattice (M, Θ) into a (right) Λ-lattice via the action,

$$m \cdot x^a = mx^a;$$

$$m \cdot y = \Theta^{-1}(m), \quad (\Theta(m) = my^{-1} = my).$$

If $J \subset \Lambda_0$ is ideal such that $\theta(J) = J$, then we put $\bar{J} = (J, \Theta_J)$ where Θ_J is simply the restriction of θ to J. With this notation, we have the following decomposition of the augmentation ideal I_G,

$$I_G \cong \bar{I}_C \oplus [y - 1). \tag{4.0.2}$$

For a proof of this the reader is directed to [17], [24] or [25]. In the interest of clarity, we provide a proof in the case of $G = D_{2p}$ below (Proposition 4.1.13).

Consider now the y-strand (or lower strand) of our desired diagonal resolution (as explained in Section 1.1). The standard resolution of the trivial $\mathbf{Z}[C_2]$-module was seen in the previous chapter to have the form,

$$0 \to \mathbf{Z} \overset{\epsilon^*}{\to} \mathbf{Z}[C_2] \overset{y-1}{\to} \mathbf{Z}[C_2] \overset{\epsilon}{\to} \mathbf{Z} \to 0.$$

Here, ϵ is the usual augmentation map, and ϵ^* its dual. In a natural way we can then induce the following diagonal resolution of period two

$$\cdots \overset{\Sigma_y}{\to} \Lambda \overset{y-1}{\to} \Lambda \overset{\Sigma_y}{\to} \Lambda \overset{y-1}{\to} \Lambda \overset{\Sigma_y}{\to} \Lambda \overset{y-1}{\to} \Lambda \overset{\Sigma_y}{\to} \Lambda \overset{y-1}{\to} \cdots \tag{4.0.3}$$

where $\epsilon^*(1) = \Sigma_y = y + 1$.

The x-strand (or upper strand), however, is decidedly more complex and a lot more work is required. First, observe that the cyclic algebra $\mathcal{C}_2(I_C^*, \theta)$ is another description of the induced module $i_*(I_C^*)$. However, we have already observed that $\mathcal{C}_2(I_C^*, \theta) \cong \mathcal{T}_2(A, \pi)$ by an explicit form of Rosen's Theorem (see also [24]). We therefore have

$$i_*(I_C) \cong R(1) \oplus R(2).$$

It is quite clear that each $R(i)$ is monogenic by composing the obvious projections $\Lambda \twoheadrightarrow i_*(I_C)$ and $i_*(I_C) \twoheadrightarrow R(i)$ to give $p(i) : \Lambda \twoheadrightarrow R(i)$ for $i = 1, 2$. We define $K(i) = Ker(p(i))$ and, using the calculations of Johnson in [24], there is an exact sequence:

$$\begin{tikzcd}
 & & & K(2) \arrow[dr] & & & & K(1) \arrow[dr] & & & \\
0 \arrow[r] & R(1) \arrow[r] & \Lambda \arrow[ur] \arrow[rr] & & \Lambda \arrow[dr] \arrow[rr] & & \Lambda \arrow[ur] \arrow[rr] & & \Lambda \arrow[r] & R(1) \arrow[r] & 0 \\
 & & & & & R(2) \arrow[ur] & & & & &
\end{tikzcd} \tag{4.0.4}$$

In fact, the above exact sequence (4.0.4) can be suitably modified to form a periodic exact sequence that extends infinitely in both directions and repeats with period four

$$\cdots \to \Lambda \xrightarrow{\partial_0^+} \Lambda \xrightarrow{\partial_3^+} \Lambda \xrightarrow{\partial_2^+} \Lambda \xrightarrow{\partial_1^+} \Lambda \xrightarrow{\partial_0^+} \cdots$$

in which we give the ∂_i^+ below. By 'entwining' this with the sequence of (4.0.3), we therefore yield another exact sequence, again repeating with period four and again extending infinitely in both directions:

$$\cdots \longrightarrow \Lambda \oplus \Lambda \xrightarrow{\begin{pmatrix} \partial_0^+ & 0 \\ 0 & y+1 \end{pmatrix}} \Lambda \oplus \Lambda \xrightarrow{\begin{pmatrix} \partial_3^+ & 0 \\ 0 & y-1 \end{pmatrix}} \Lambda \oplus \Lambda \xrightarrow{\begin{pmatrix} \partial_2^+ & 0 \\ 0 & y+1 \end{pmatrix}} \Lambda \oplus \Lambda \xrightarrow{\begin{pmatrix} \partial_1^+ & 0 \\ 0 & y-1 \end{pmatrix}} \Lambda \oplus \Lambda \xrightarrow{\begin{pmatrix} \partial_0^+ & 0 \\ 0 & y+1 \end{pmatrix}} \Lambda \oplus \Lambda \longrightarrow \cdots$$

This may now be suitably truncated to form the following diagonal resolution of period four;

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\epsilon^*} \Lambda \xrightarrow{\begin{pmatrix} \partial_3^+ \\ y-1 \end{pmatrix}} \Lambda \oplus \Lambda \xrightarrow{\begin{pmatrix} \partial_2^+ & 0 \\ 0 & y+1 \end{pmatrix}} \Lambda \oplus \Lambda \xrightarrow{(\partial_1^+,\ y-1)} \Lambda \xrightarrow{\epsilon} \mathbf{Z} \longrightarrow 0. \tag{4.0.5}$$

The above was explicitly constructed by utilizing the following descriptions for $R(1)$, $R(2)$, $K(1)$ and $K(2)$:

$$K(2) = \quad K \quad \cong [\Sigma_x,\, y-1) \cong [(1-y)\theta + \Sigma_x y), \tag{4.0.6}$$

$$R(1) = \quad P \quad \cong [(x^n-1)(y-1)), \tag{4.0.7}$$

$$K(1) = \quad L \quad \cong [\Sigma_x,\, y+1) \cong [(1+y)\theta - \Sigma_x y), \tag{4.0.8}$$

$$R(2) = R \cong [(x^n - x^{n+1})(y+1)) = [(y-1)(x^{n+1}-x^n) = [(y-1)(x-1)), \tag{4.0.9}$$

where $\Sigma_x = 1 + x + \cdots + x^{2n}$ and $\theta = 1 + x + \cdots + x^{n-1}$. In the notation of (4.0.5), we have:

$$\partial_0^+ = (1-y)\theta + \Sigma_x \cdot y;$$

$$\partial_1^+ = (x^n-1)(y-1);$$

$$\partial_2^+ = (1+y)\theta - \Sigma_x \cdot y;$$

$$\partial_3^+ = (x^n - x^{n+1})(y+1).$$

For the remainder of this chapter we adopt Johnson's notation [24] of K, L, P and R as above.

An obvious benefit of the above diagonal resolution is that one can now simply read off the syzygies as follows:

$$\Omega_r(\mathbf{Z}) \sim \begin{cases} [K] \oplus [y+1], & r \equiv 0 \bmod 4; \\ [P] \oplus [y-1], & r \equiv 1 \bmod 4; \\ [L] \oplus [y+1], & r \equiv 2 \bmod 4; \\ [R] \oplus [y-1], & r \equiv 4 \bmod 4. \end{cases}$$

In this sense, P is a representative element of part of the first syzygy. We aim to explicitly show there is a relationship between P and a representative element of part of the second syzygy, namely L. Likewise, we show there is a relationship between P and R, representing part of the third syzygy. Finally we show there is a relationship between P and K, representing part of the fourth syzygy (= zeroth syzygy). In particular, we shall show that there is a group structure of order four within the stable class, generated by P with identity K. We therefore restate Theorems A and B in this form for clarity:

Theorem A: With K defined as above, $K \otimes ? \cong ? \oplus \Lambda^r$, for some $r \geq 0$ and where $? = K$, P, L or R.

Theorem B: With, K, P, L, R as defined above, we have the following relations:

1. $P \otimes P \cong L \oplus \Lambda^{n-1}$;
2. $P \otimes L \cong R \oplus \Lambda^n$;
3. $P \otimes R \cong K \oplus \Lambda^{n-1}$.

It may be instructive to briefly outline the structure of this chapter. In Sections 4.1 and 4.3, we outline the necessary results of [24] regarding the modules P, R, K, L. In addition, Section 4.3 will also discuss the indecomposable nature of K, L. Section 4.2 will be concerned with the tree structures of $[P]$, $[R]$, highlighting the fact that they are necessarily straight. Once this preliminary work is finished, we prove Theorem A in Section 4.4. We then dedicate one section for each of the required isomorphisms in Theorem B. To conclude the chapter we use the results proven to provide an alternative proof to that of Johnson, providing an affirmative answer to the sequencing conjecture discussed in Section 1.1. It should be noted, however, that this cannot be used to construct an explicit description of the diagonal resolution. Nevertheless, it does provide a method to construct such resolutions for a given q when an explicit description appears out of reach. This will be of use in Chapter 5.

4.1 The modules P and R

In keeping with the notation of [24] we set,

$$\pi = (x^n - 1)(y - 1) \tag{4.1.1}$$

$$\rho = (y - 1)(x^{n+1} - x^n) = (x^n - x^{n+1})(y + 1) \tag{4.1.2}$$

$$\tilde{\rho} = (y - 1)(x - 1). \tag{4.1.3}$$

Clearly $\tilde{\rho} = \rho \cdot x^{n+1}$ and $\rho = \tilde{\rho} \cdot x^n$ so that $[\rho) = [\tilde{\rho})$. We then define

$$P = [\pi), \quad R = [\rho) = [\tilde{\rho}). \tag{4.1.4}$$

The author stresses the importance of the expressions denoted by π, ρ and $\tilde{\rho}$. In addition, we write $\Sigma_x = 1 + x + \cdots + x^{2n}$, which we observe is central in Λ. In the interest of clarity, we now outline the results of [24] that will be of use to us throughout this chapter. First, we note the following $\mathbf{Z}$-basis for R.

Proposition 4.1.5. *The* Λ*-module* R *has* $\mathbf{Z}$*-rank* $2n$ *and* $\mathbf{Z}$*-basis*

$$\{(y - 1)(x^r - 1) \mid 1 \leq r \leq 2n\}.$$

Proof. First, consider the modules $[x-1)$ and $[y-1)$ and, in particular, the module given by their intersection, $[x-1)\cap[y-1)$. We show this latter module is, in fact, R. First, recall the augmentation ideal I_G of Λ has $\mathbf{Z}$-basis $\{g-1 \mid g \in D_{4n+2}\backslash\{1\}\}$ which, by performing elementary basis transformations, may be written

$$\{x^r-1,\, y-1,\, (y-1)(x^s-1) \mid 1 \le r,\, s \le 2n\}.$$

From the identities, $x^r-1=(x-1)\sum_{s=0}^{r-1}x^s$ and $yx^r-1=(x^r-1)+(y-1)x^r$ it follows

$$I_G=[x-1)+[y-1).$$

However, this sum is certainly not direct and we have an exact sequence

$$0 \to [x-1)\cap[y-1) \to [x-1)\oplus[y-1) \to I_G \to 0. \tag{4.1.6}$$

Claim 1. $[x-1)$ *has* $\mathbf{Z}$*-basis* $\{x^r-1,\,(y-1)(x^s-1) \mid 1 \le r,\, s \le 2n\}$*. In particular,* $rk_{\mathbf{Z}}([x-1))=4n$*.*

To prove Claim 1, denote the augmentation ideal of $\mathbf{Z}[C_{2n+1}]$ by I_C. Next, regard $[x-1)$ as the induced module $[x-1)=I_C\otimes_{\mathbf{Z}[C_{2n+1}]}\Lambda$. As Λ is a free module of rank 2 over $\mathbf{Z}[C_{2n+1}]$, it follows that

$$rk_{\mathbf{Z}}([x-1))=2rk_{\mathbf{Z}}(I_C)=4n. \tag{4.1.7}$$

Now, it is clear that $x^r-1\in[x-1)$ for $1\le r\le 2n$, and we observe

$$(y-1)(x^r-1)=(x^{2n+1-r}-1)y-(x^r-1)=(x-1)\left(\sum_{s=0}^{2n-r}x^s\right)y-(x^r-1)\in[x-1).$$

Consequently, by using (4.1.7) and applying Proposition 2.2.7, the claim now follows.

Claim 2. $[y-1)$ *has* $\mathbf{Z}$*-basis* $\{y-1,\,(y-1)(x^r-1) \mid 1 \le r \le 2n\}$*. In particular,* $rk_{\mathbf{Z}}([y-1))=2n+1$*.*

Claim 2 follows similarly to that of Claim 1. Simply regard $[y-1)$ as the induced module $[y-1)=I(C_2)\otimes_{\mathbf{Z}[C_2]}\Lambda$ and proceed as above.

Using Claims 1 and 2 alongside (4.1.6), we calculate $rk_{\mathbf{Z}}([x-1)\cap[y-1))=2n$. However, it is also apparent that $(y-1)(x^r-1)\in[x-1)\cap[y-1)$ for each $1\le r\le 2n$. Thus, by applying Proposition 2.2.7, we find

$$\{(y-1)(x^r-1) \mid 1\le r\le 2n\} \text{ is a } \mathbf{Z}\text{-basis for } [x-1)\cap[y-1). \tag{4.1.8}$$

Finally, we turn to $R=[\tilde{\rho})$. It is clear that $\tilde{\rho}\in[y-1)$ so $R\subset[y-1)$. Furthermore, we can write $\rho=(x^n-x^{n+1})(y+1)=(x-1)(-x^n)(y+1)$ so that $R\subset[x-1)$, i.e. $R\subset[x-1)\cap[y-1)$. Conversely, it is a trivial observation that $(y-1)(x-1)\in R$. Moreover,

$$(y-1)(x^r-1)=(y-1)(x-1)\cdot(1+x+\cdots+x^{r-1})$$

so that $(y-1)(x^r-1)\in R$ for $1\le r\le 2n$. Hence, $[x-1)\cap[y-1)\subset R$ and therefore $R=[x-1)\cap[y-1)$. The result now follows from (4.1.8). ☐

As an addendum to the above proof observe that the method used to prove Claims 1 and 2 also applies to the ideal $[y+1)$. By regarding this as the induced module $[y+1) = \mathbf{Z} \otimes_{\mathbf{Z}[C_2]} \Lambda$ we can once more use Proposition 2.2.7 to show:

$$[y+1) \text{ has } \mathbf{Z}\text{-basis } \{y+1,\ (y+1)(x^r-1) \mid 1 \le r \le 2n\}. \tag{4.1.9}$$

Observe that both $[y-1)$ and $[y+1)$ are self-dual. For instance, the self-duality of $[y+1)$ follows immediately from Proposition 2.1.10. A similar argument applies for $[y-1)$. It must be stressed, however, that $[y-1)$ and $[y+1)$ are *not* isomorphic, as Λ-modules.

Proposition 4.1.10. *The ideal* $[x-1)$ *decomposes as a direct sum*

$$[x-1) = P \oplus R.$$

Proof. Define $Q = [x-1)/R$ and consider the canonical short exact sequence

$$0 \to R \to [x-1) \xrightarrow{\natural} Q \to 0. \tag{$\dagger$}$$

It is sufficient to show $\dagger$ splits over Λ and, in turn, it suffices to show the natural map $\natural : [x-1) \to Q$ restricts to an isomorphism $\natural : Q \overset{\simeq}{\to} P$.

To this end recall that in the proof of Proposition 4.1.5 it was shown that $[x-1)$ has a $\mathbf{Z}$-basis $\{(x^r-1),\ (y-1)(x^s-1) \mid 1 \le r,\ s \le 2n\}$. Moreover, by Proposition 4.1.5 itself it is known that R has $\mathbf{Z}$-basis $\{(y-1)(x^r-1) \mid 1 \le r \le 2n\}$. It therefore follows that

$$Q \text{ is torsion free with } \mathbf{Z}\text{-basis } \{\natural(x^r-1) \mid 1 \le r \le 2n\}. \tag{4.1.11}$$

Recall $\pi = (x^n-1)(y-1)$ and define $\tilde{\pi} = \pi x^{n+1}$ so that $\pi = \tilde{\pi} x^n$ and $P = [\pi) = [\tilde{\pi})$. It is straightforward to show

$$\tilde{\pi} = (x-1) + (y-1)(x-1) - (y-1)(x^{n+1}-1)$$

and hence $\natural(\tilde{\pi}) = \natural(x-1)$. In particular, $\natural(\tilde{\pi} \cdot x^r) = \natural((x-1)x^r)$. Next, observe that

$$(x^r-1) = (x-1)\left(\sum_{s=0}^{r-1} x^s\right)$$

so that

$$\natural(x^r-1) = \natural(\tilde{\pi} \cdot [\Sigma_{s=0}^{r-1} x^s]).$$

It therefore follows that $\natural : P \to Q$ is surjective and $rk_{\mathbf{Z}}(P) \ge 2n$. However, as $\pi y = -\pi$ it follows that $P = span_{\mathbf{Z}}\{\pi \cdot x^r \mid 0 \le r \le 2n\}$. Moreover, $\pi \Sigma_x = 0$ since Σ_x is central, and so

$$P = span_{\mathbf{Z}}\{\pi \cdot x^r \mid 1 \le r \le 2n\}.$$

Consequently, $rk_{\mathbf{Z}}(P) \le 2n$ and thus, $rk_{\mathbf{Z}}(P) = 2n = rk_{\mathbf{Z}}(Q)$. As $\natural : P \to Q$ is surjective, we therefore conclude $\natural$ is an isomorphism, as required. □

In the course of the proof, we showed:

Proposition 4.1.12. *The* Λ*-module* P *has* $\mathbf{Z}$*-basis* $\{\pi \cdot x^r \mid 1 \le r \le 2n\}$.

Proposition 4.1.13. $I_G = P \oplus [y-1)$.

Proof. Recall $I_G = [x-1) + [y-1)$ which we can rewrite as $I_G = P + R + [y-1)$. However, $R = [x-1) \cap [y-1)$ so that $I_G = P + [y-1)$. Now, $P \subset [x-1)$ so that $P \cap [y-1) \subset P \cap [x-1) \cap [y-1) \subset P \cap R$. Since $P \cap R = \{0\}$ (from Proposition 4.1.12), it follows that $P \cap [y-1) = \{0\}$. Thus, the sum is direct, as required. □

When characterising P, R, the following will be of particular use. We consider the following three properties for a Λ-lattice M:

$\mathcal{M}(-)$: there exists $\hat{\varphi}_- \in M$ such that $\{\varphi_- \cdot x^r \,|\, 1 \le r \le 2n\}$ is a $\mathbf{Z}$-basis for M and for which $\hat{\varphi}_- \cdot y = -\hat{\varphi}_-$;

$\mathcal{M}(+)$: there exists $\hat{\varphi}_+ \in M$ such that $\{\varphi_+ \cdot x^r \,|\, 1 \le r \le 2n\}$ is a $\mathbf{Z}$-basis for M and for which $\hat{\varphi}_+ \cdot y = \hat{\varphi}_+$;

$\mathcal{M}(\Sigma)$: the identity $m \cdot \Sigma_x = 0$ holds for each $m \in M$.

Proposition 4.1.14. *For a Λ-lattice M, we have $M \cong P$ if and only if M satisfies $\mathcal{M}(-)$ and $\mathcal{M}(\Sigma)$. Likewise, $M \cong R$ if and only if M satisfies $\mathcal{M}(+)$ and $\mathcal{M}(\Sigma)$.*

Using this we can now show two alternative descriptions for P and R. Recall that in Chapter 3 we discussed the augmentation ideal I_C, and its dual I_C^* of $\mathbf{Z}[C_{2n+1}]$. In particular, I_C^* has $\mathbf{Z}$-basis, $\{\nu^r \,|\, 0 \le r \le 2n-1\}$ where $1 + \nu + \cdots + \nu^{2n} = 0$. Now, the action of C_p on I_C^* may be extended in one of two ways to an action of the dihedral group:

- Either: $\nu^r \cdot y = \nu^{-r} = \nu^{2n+1-r}$ for $0 \le r \le 2n-1$;
- or: $\nu^r \cdot y = -\nu^{-r} = -\nu^{2n+1-r}$ for $0 \le r \le 2n-1$.

Under the former, we denote $(I_C^*)_+$, and under the latter we denote $(I_C^*)_-$. Later, particularly in the next chapter, we denote the introduction of the y-action by placing a bar over the Λ_0-module in question.

Proposition 4.1.15. $P \cong (I_C^*)_-$.

Proof. We use the recognition criteria of Proposition 4.1.14. Take $\nu^0 \in (I_C^*)_-$ and note that $\nu^0 \cdot x^r = \nu^r$. Moreover, $\nu^0 \cdot y = -\nu^0$ by our choice of Galois action. Thus, $\mathcal{M}(-)$ is satisfied.

It remains to show $\mathcal{M}(\Sigma)$ is satisfied. Let $\alpha \in (I_C^*)_-$ be written as $\alpha = \Sigma_r a_r \nu^r$. Since $1 + \nu + \nu^2 + \cdots + \nu^{2n} = 0$,

$$\begin{aligned} \nu^r \Sigma_x = & \quad \nu^r(1 + x + x^2 + \cdots x^{2n}) \\ = & \quad \nu^r + \nu^{r+1} + \nu^{r+2} + \cdots + \nu^{r+2n} \\ = & \quad \nu^r(1 + \nu + \nu^2 + \cdots + \nu^{2n}) \quad = \quad 0 \end{aligned}$$

Thus, $M(\Sigma)$ is satisfied and $P \cong (I_C^*)_-$, as required. □

By a similar argument, we have:

Proposition 4.1.16. $R \cong (I_C^*)_+$.

Evidently, P and R are *not* isomorphic as Λ-modules, nor even stably isomorphic. Nevertheless, by considering the representations of P and R, we have:

$$P^* \cong R \text{ and } R^* \cong P. \tag{4.1.17}$$

Finally, return to the quasi-triangular matrices $\mathcal{T}_2(A, \pi) \cong R(1) \oplus R(2)$, where $A = \mathbf{Z}[\zeta_p]^\theta$ and $\pi = (\zeta_p - 1)^2$. It is useful to describe both $R(1)$ and $R(2)$ as Galois modules. First, observe $R(2)$ satisfies the conditions $\mathcal{M}(+)$ and $\mathcal{M}(\Sigma)$ where $\hat{\varphi}_+ = (0, 1) \in R(2)$. By Proposition 4.1.14, we therefore conclude $R(2) \cong R \cong (I_C^*)_+$. In [25], it is shown that:

Proposition 4.1.18. *$R(2)^* \cong R(1)$.*

By Proposition 4.1.18 and (4.1.17) it therefore follows that $R(1) \cong P \cong (I_C^*)_-$.

4.2 The tree structures of the odd syzygies

Using the result of Proposition 2.8.1 we have a complete list of the genera of indecomposable modules over Λ. This can be used to deduce that the tree structures of $[P]$ and $[R]$ are straight. For convenience, we rewrite the indecomposable modules of Section 2.8 for the specific case of $q = 2$. We have a total of 10 indecomposable genera.

I. There are three indecomposable modules over $\mathbf{Z}[C_2]$ that become modules over Λ via the quotient map $D_{2p} \to C_2$:

   (i) The trivial module (rank 1);

   (ii) The augmentation ideal, $I(C_2) = Ker(\mathbf{Z}[C_2] \to \mathbf{Z})$ (rank 1). This may also be thought of as $\mathbf{Z}_-$, the rank 1 module in which x acts trivially and y acts as multiplication by -1;

   (iii) The group ring itself $\mathbf{Z}[C_2]$ (rank 2).

II. There are two distinct indecomposable modules over $\mathcal{T}_2(A, \pi)$ of rank $p - 1$:

   (iv) $R = \overline{\mathbf{Z}[\zeta_p]} \cong (I_C^*)_+$ (rank $p - 1$);

   (v) $P = \overline{(\zeta_p - 1)\mathbf{Z}[\zeta_p]} \cong (I_C^*)_-$ (rank $p - 1$).

III. There is one extension when $Y = \mathbf{Z}$:

   (vi) $0 \to P \to \bar{\Lambda}_0 \to \mathbf{Z} \to 0$ (rank p).

IV. There is one indecomposable non-split extension when $Y = I(C_2)$:

   (vii) $0 \to R \to V_1 \to \mathbf{Z}_- \to 0$ where $rk_{\mathbf{Z}}(V_1) = p$.

V. There are three indecomposable non-split extensions for $Y = \mathbf{Z}[C_2]$:

   (viii) $0 \to R \to \Lambda/R \to \mathbf{Z}[C_2] \to 0$ where $rk_{\mathbf{Z}}(\Lambda/R) = p+1$. We note $\Lambda/R \sim K$;

   (ix) $0 \to P \to \Lambda/P \to \mathbf{Z}[C_2] \to 0$ where $rk_{\mathbf{Z}}(\Lambda/P) = p+1$. We note $\Lambda/P \sim L$;

   (x) $0 \to R \oplus P \to \Lambda \to \mathbf{Z}[C_2] \to 0$.

Proposition 4.2.1. *Let* $M \in \Omega_1(\mathbf{Z})$ *be a minimal representative; then* M *decomposes as* $M \cong M_1 \oplus M_2$*, where* M_1*,* M_2 *are non-trivial indecomposable modules.*

Proof. Any minimal representative of $\Omega_1(\mathbf{Z})$ occurs in the following exact sequence,

$$0 \to M \to \Lambda \to \mathbf{Z} \to 0.$$

By Theorem 7.7 of [22] we can therefore write

$$End_{\mathcal{D}er}(M) \cong End_{\mathcal{D}er}(\mathbf{Z}) \cong \mathbf{Z}/|D_{2p}| \cong \mathbf{Z}/2 \times \mathbf{Z}/p$$

where $\mathcal{D}er$ once again denotes the derived module category in the sense of [20] or [22] (recall the definition given at the end of Section 2.4).

Now, suppose we have the following decomposition of M into indecomposable modules $M \cong M_1 \oplus \cdots \oplus M_r$, where $r > 2$. Apply $End_{\mathcal{D}er}(-)$ and observe $End_{\mathcal{D}er}(M_i)$ is trivial for all $i \geq 3$ (after reordering). By Proposition 5.5 of [22], it therefore follows that M_i is projective for $i \geq 3$. In [54], Swan has shown for any finite group Γ, projective modules over $\mathbf{Z}[\Gamma]$ must have the same $\mathbf{Z}$-rank as a free module. It therefore follows that $rk_{\mathbf{Z}}(M) > 2p$, which is a contradiction as we know $rk_{\mathbf{Z}}(M) = 2p - 1$.

The only other possibility is that M is itself indecomposable. However, by the above list we have no indecomposable module of $\mathbf{Z}$-rank $2p - 1$. Thus, M must decompose as $M \cong M_1 \oplus M_2$ where M_1, M_2 are non-trivial and indecomposable. ☐

Recall (2.8.2), in which we discussed the straightness of each $[R(i)]$. In the case of dihedral groups D_{2p}, this can be rewritten as:

Proposition 4.2.2. *The tree structure of* $[P]$ *is straight.*

and

Proposition 4.2.3. *The tree structure of* $[R]$ *is straight.*

A proof of these two results can be found in Section 6.8.

4.3 The modules K and L

We define the modules K and L to be

$$K = [\Sigma_x,\, y - 1) \text{ and } L = [\Sigma_x,\, y + 1). \tag{4.3.1}$$

As with Section 4.1, we outline the results of [24] necessary to what follows.

Proposition 4.3.2. *The* Λ*-module* K *has* $\mathbf{Z}$*-basis* $\{(y-1)x^s \mid 0 \leq s \leq 2n\} \cup \{\Sigma_x\}$ *and therefore has* $rk_{\mathbf{Z}}(K) = 2n + 2$*. In particular,* Λ/K *is torsion free.*

Proof. Define $K_0 = \{(1 - y)a(x) \mid a(x) \in \mathbf{Z}[C_{2n+1}]\} \subset K$ and observe

$$(y - 1)x^s \cdot y = -(y - 1)x^{2n+1-s}.$$

It therefore follows that K_0 is a Λ-submodule of K. Moreover, $\Sigma_x y = \Sigma_x(y - 1) + \Sigma_x$ from which it follows that K is spanned over $\mathbf{Z}$ by $\{(y-1)x^s \mid 0 \leq s \leq 2n\} \cup \{\Sigma_x\}$. In particular, by starting from the canonical basis for Λ and proceeding by elementary basis transformations, it follows that

$$\{(y-1)x^r \mid 0 \leq r \leq 2n\} \cup \{\Sigma_x\} \cup \{x^s \mid 1 \leq s \leq 2n\}$$

is a $\mathbf{Z}$-basis for Λ. By Proposition 2.2.7, it therefore follows that

$$\{(y-1)x^s \mid 0 \leq s \leq 2n\} \cup \{\Sigma_x\}$$

is a $\mathbf{Z}$-basis for K and Λ/K is torsion free. □

Proposition 4.3.3. *K is monogenic, generated by $(1-y)\theta + \Sigma_x y$, where $\theta = 1 + x + \cdots + x^{n-1}$.*

Proof. Evidently, $(1-y)\theta + \Sigma_x y \in [\Sigma_x,\, y-1)$. However, the identity

$$[(1-y)\theta + \Sigma_x y] \cdot x^{n+1}(1-y) = (y-1)$$

demonstrates that $y-1 \in [(1-y)\theta + \Sigma_x y)$. Thus, $(y-1)\theta \in [(1-y)\theta + \Sigma_x y)$ and therefore $\Sigma_x y \in [(1-y)\theta + \Sigma_x y)$. Consequently, $\Sigma_x = (\Sigma_x y) \cdot y \in [(1-y)\theta + \Sigma_x y)$. □

Proposition 4.3.4. *$\Lambda/K \cong R$.*

Proof. Once again, perform elementary basis transformations to the basis of Λ so that we have the following $\mathbf{Z}$-basis for Λ, $\{x^i \,|\, 1 \leq i \leq 2n\} \cup \{(y-1)x^j \,|\, 0 \leq j \leq 2n\} \cup \{\Sigma_x\}$. Thus, Λ/K has $\mathbf{Z}$-basis $\{\natural(x^i) \mid 1 \leq i \leq 2n\}$ where $\natural : \Lambda \to \Lambda/K$. In particular, $rk_{\mathbf{Z}}(\Lambda/K) = 2n$. Moreover, since $\Sigma_x = 0$ in Λ/K we note $m\Sigma_x = 0$ for all $m \in \Lambda/K$. It remains (by the criteria of Proposition 4.1.14) to show $\natural(1)y = \natural(1)$. However, in Λ we clearly have

$$1 \cdot y = (y-1) + 1$$

and so y acts trivially on $\natural(1)$ as required. The result follows from Proposition 4.1.14. □

Consequently, the module K arises in an exact sequence of the form

$$0 \to K \to \Lambda \to R \to 0. \tag{4.3.5}$$

Using a similar argument to that of Proposition 4.2.1, we can also show $\Omega_0(\mathbf{Z})$ decomposes precisely into two non-trivial, indecomposable components. A detailed exposition of this unusual behaviour may be found in Section 8 of [24]. We express our result in the form provided in that paper:

Proposition 4.3.6. *The stable module $\Omega_0(\mathbf{Z})$ decomposes as*

$$\Omega_0(\mathbf{Z}) = [\mathbf{Z}] = [K] \oplus [y+1].$$

Corollary 4.3.7. *The module K is indecomposable.*

It should be noted, however, that we cannot claim $[K]$ is straight. From Pu's list, we only know any $K \sim K'$ occurs in an exact sequence $0 \to R \to K' \to \mathbf{Z}[C_2] \to 0$.

By defining $L_0 = \{(y+1)a(x) \mid a(x) \in \mathbf{Z}[C_{2n+1}] \subset L\}$ we adopt similar reasoning to the above to show:

Proposition 4.3.8. *The* Λ*-module* L *has* $\mathbf{Z}$*-basis* $\{(y+1)x^s \mid 0 \leq s \leq 2n\} \cup \{\Sigma_x\}$ *and therefore* $rk_{\mathbf{Z}}(L) = 2n+2$*. In particular,* Λ/L *is torsion free.*

Moreover, as $[(1+y)\theta - \Sigma_x y] \cdot x^{n+1}(y+1) = -(y+1)$ it may be similarly shown L is monogenic.

Proposition 4.3.9. L *is monogenic, generated by* $(1+y)\theta - \Sigma_x y$*.*

As with K we have a corresponding exact sequence for L as follows:

$$0 \to L \to \Lambda \to P \to 0. \tag{4.3.10}$$

Proposition 4.3.11. *The module* L *is indecomposable.*

Finally, Johnson has shown that both K and L are self-dual; that is:

$$K^* \cong K; \tag{4.3.12}$$

$$L^* \cong L. \tag{4.3.13}$$

4.4 $K \otimes ? \sim ?$

With the preliminary results of Sections 4.1 - 4.3 in mind, we can now show Theorem A; that is, we show

$$K \otimes ? \cong ? \oplus \Lambda^r \tag{4.4.1}$$

for some $r \geq 0$ and where $? = K,\ P,\ L,\ R$. Recall $K = [y-1,\ \Sigma_x)$ is self-dual and define $K_0 = span_{\mathbf{Z}}\{(y-1),\ (y-1)x, \ldots,\ (y-1)x^{2n}\}$ so that K/K_0 is represented by the class of Σ_x. Observe:

- $\Sigma_x \cdot x = \Sigma_x$; and
- $\Sigma_x \cdot y = y\Sigma_x = (y-1)\Sigma_x + \Sigma_x = \Sigma_x$ in K/K_0.

Thus, x and y act trivially on K/K_0 and it is therefore isomorphic to $\mathbf{Z}$. In particular, we have an exact sequence of the form,

$$0 \to K_0 \to K \to \mathbf{Z} \to 0.$$

Proposition 4.4.2. *If* $j : \mathbf{Z}[C_2] \hookrightarrow \Lambda$*, and* $I_2 = Ker(\epsilon : \mathbf{Z}[C_2] \to \mathbf{Z})$*, then* $j_*(I_2) \cong [y-1)$*.*

Proof. It is straightforward to show $\{(y-1)x^i \mid 0 \leq i \leq 2n\}$ is a $\mathbf{Z}$-basis for $[y-1)$. Next, observe $j_*(I_2) = I_2 \otimes_{\mathbf{Z}[C_2]} \Lambda$ has $\mathbf{Z}$-basis

$$\{(y-1) \otimes_{C_2} x^s \mid 0 \leq s \leq 2n\}.$$

We then define the map $\varphi : j_*(I_2) \to [y-1)$ by

$$\varphi((y-1) \otimes x^s) = (y-1)x^s.$$

It is straightforward to check φ is a Λ-homomorphism between basis elements. ☐

Using the above, we are now in the position to show K acts as the identity within the stable class of our cyclic group of rank 4. Observe that tensoring with any of the P, R, K or L therefore yields the exact sequence

$$0 \to K_0 \otimes ? \to K \otimes ? \to ? \to 0.$$

Proposition 4.4.3. *$j^*(P) \cong j^*(R) \cong \mathbf{Z}[C_2]^n$.*

Proof. Consider the exact sequence $0 \to I_C \to \Lambda_0 \to \mathbf{Z} \to 0$, and apply the exact functor $i_*(-)$ to yield

$$0 \to i_*(I_C) \to \Lambda \to \mathbf{Z}[C_2] \to 0.$$

Next, observe that the induced module $i_*(I_C^*)$ is simply another description of $\mathcal{C}_2(I_C^*, \theta)$. So, using the isomorphism $\mathcal{C}_2(I_C^*, \theta) \cong \mathcal{T}_2(A, \pi)$, and the fact that $i_*(I_C^*) \cong i_*(I_C)$, we have $i_*(I_C) \cong \mathcal{T}_2(A, \pi)$. Since $\mathcal{T}_2(A, \pi) \cong P \oplus R$ (see Proposition 4.1.10), we have the following exact sequence,

$$0 \to R \oplus P \to \Lambda \to \mathbf{Z}[C_2] \to 0.$$

Now apply the exact functor $j^*(-)$,

$$0 \to j^*(R \oplus P) \to \mathbf{Z}[C_2]^{2n+1} \to \mathbf{Z}[C_2] \to 0.$$

This sequence clearly splits and we observe $j^*(R \oplus P)$ is stably free of rank $2n$. As C_2 satisfies the Eichler condition, $\mathbf{Z}[C_2]$ has SFC by Swan-Jacobinski, hence both $j^*(R)$ and $j^*(P)$ are projective $\mathbf{Z}[C_2]$-modules of equal $\mathbf{Z}$-rank.

Now, $\widetilde{K_0}(\mathbf{Z}[C_2]) = 0$ (see, for example, [42]) and so any projective module is necessarily stably free. Using Swan-Jacobinski once more, we conclude $j^*(P)$ and $j^*(R)$ are free, each of rank n. □

Proposition 4.4.4. *$K \otimes R(i) \cong R(i) \oplus \Lambda^n$, for $1 \leq i \leq 2$ where $R(1) \cong P$ and $R(2) \cong R$.*

Proof. Consider the following exact sequence,

$$0 \to K_0 \otimes R(i) \to K \otimes R(i) \to R(i) \to 0.$$

Note that $I_2 \cong \mathbf{Z}_-$, the rank one module where x acts trivially and y acts as multiplication by -1. By two applications of Frobenius Reciprocity (Proposition 2.1.8), and Proposition 4.4.3, we have the following isomorphism:

$$\begin{aligned} j_*(I_2) \otimes R(i) &\cong j_*(I_2 \otimes j^*(R(i))) \\ &\cong j_*(I_2 \otimes \mathbf{Z}[C_2]^n) \\ &\cong j_*(\mathbf{Z}[C_2]^n) \\ &\cong \Lambda^n. \end{aligned}$$

Replacing this in the above exact sequence we therefore get

$$0 \to \Lambda^n \to K \otimes R(i) \to R(i) \to 0$$

which splits, yielding $K \otimes R(i) \cong R(i) \oplus \Lambda^n$, as required. □

Proposition 4.4.5. $j^*(K) \cong \mathbf{Z}[C_2]^{n+1}$.

Proof. Start with the following exact sequence,

$$0 \to K \to \Lambda \to R \to 0$$

and apply $j^*(-)$,

$$0 \to j^*(K) \to \mathbf{Z}[C_2]^{2n+1} \to j^*(R) \to 0.$$

By Proposition 4.4.3, we know $j^*(R) \cong \mathbf{Z}[C_2]^n$ and so the above exact sequence splits, yielding

$$j^*(K) \oplus \mathbf{Z}[C_2]^n \cong \mathbf{Z}[C_n]^{2n+1},$$

i.e. $j^*(K)$ is stably free of rank $n+1$. As $\mathbb{Z}[C_2]$ has SFC, $j^*(K) \cong \mathbf{Z}[C_2]^{n+1}$, as required. □

Since $j^*(P) \cong j^*(R)$ we have the following dual statement:

Proposition 4.4.6. $j^*(L) \cong \mathbf{Z}[C_2]^{n+1}$.

Proposition 4.4.7. $K \otimes K_i \cong K_i \oplus \Lambda^{n+1}$*, for* $1 \leq i \leq 2$ *where* $K_1 = L$ *and* $K_2 = K$[1].

Proof. Consider the exact sequence,

$$0 \to K_0 \otimes K_i \to K \otimes K_i \to K_i \to 0.$$

Using Frobenius Reciprocity and Propositions 4.4.5 and 4.4.6, we have

$$\begin{aligned} j_*(I_2) \otimes K_i \cong &\quad j_*(I_2 \otimes j^*(K_i)) \\ \cong &\quad j_*(I_2 \otimes \mathbf{Z}[C_2]^{n+1}) \\ \cong &\quad j_*(\mathbf{Z}[C_2]^{n+1}) \\ \cong &\quad \Lambda^{n+1}. \end{aligned}$$

For each $1 \leq i \leq 2$, the above exact sequence now splits, yielding the desired isomorphism $K \otimes K_i \cong K_i \oplus \Lambda^{n+1}$. □

Evidently, Theorem A follows directly from Propositions 4.4.4 and 4.4.7.

4.5 $P \otimes P \sim L$

In this section, we show

$$P \otimes P \cong L \oplus \Lambda^{n-1}. \tag{4.5.1}$$

Recall Proposition 4.1.12 in which it was shown that P has $\mathbf{Z}$-basis

$$\{\pi x^r \mid 1 \leq r \leq 2n\}.$$

Consequently, $P \otimes P$ has $\mathbf{Z}$-basis $\{\pi x^i \otimes \pi x^j \mid 1 \leq i,\, j \leq 2n\}$ with $rk_{\mathbf{Z}}(P \otimes P) = 4n^2$. Furthermore, from Proposition 4.3.8 we know L has $\mathbf{Z}$-basis

$$\{(y+1),\, (y+1)x, \ldots,\, (y+1)x^{2n},\, \Sigma_x\},$$

---

[1] It is instructive to observe that $K_i \sim K(i)$. Regarding the main result of Theorem A, however, this distinction matters little since they are both isomorphic over $\mathbf{Z}[C_2]$.

where $\Sigma_x = 1 + x + \cdots + x^{2n}$. By counting **Z**-ranks, it is clear that for the required isomorphism to hold, $n-1$ copies of Λ are required.

In Chapter 3, we constructed the isomorphism $I_C^* \otimes I_C^* \cong_{\Lambda_0} T \oplus V$, where

$$V = V(1) \oplus V(2) \oplus \cdots \oplus V(2n-1)$$

and $V(r) = span_{\mathbf{Z}}\{\nu^{r+k} \otimes \nu^k \mid 0 \leq k \leq 2n\}$. Moreover, in Section 4.1 we introduced the following action of y,

$$\nu^r \cdot y = -\nu^{2n+1-r}$$

As before, $\{\nu^r \mid 0 \leq r \leq 2n-1\}$ is a **Z**-basis for I_C^*, and under this action $(I_C^*)_- \cong P$. We start by constructing the free part. In particular, we show that for $r \geq 2$,

$$V_r = V(r) + V(2n+1-r) \text{ is a } \Lambda\text{-module, and } V_r \cong \Lambda.$$

By Chapter 3 this is clearly true when restricted to modules over Λ_0. To extend this to an isomorphism over Λ, we consider representations.

Proposition 4.5.2. *Define* Ψ *to be the* $(2n+1) \times (2n+1)$ *matrix where*

$$\Psi_{ij} = \begin{cases} 1, & i = 1,\, j = 2n+1; \\ 1, & j = i-1,\, 2 \leq i \leq 2n+1; \\ 0, & o/w. \end{cases}$$

Then $\rho_{V_r}(x^{-1}) = \begin{pmatrix} \Psi & 0 \\ 0 & \Psi \end{pmatrix}$.

Proof. Label

$$\begin{aligned} e_i = &\quad \nu^{r+i-1} \otimes \nu^{i-1},\ 1 \leq i \leq 2n+1 \\ e_{(2n+1)+i} = &\quad \nu^{2n-r+i} \otimes \nu^{i-1},\ 1 \leq i \leq 2n+1. \end{aligned}$$

Then $e_{2n+1} \cdot x = \nu^r \otimes 1 = e_1$ and for $2 \leq i \leq 2n+1$, $e_i \cdot x = \nu^{r+i} \otimes \nu^i = e_{i+1}$. Likewise, we have $e_{4n+2} \cdot x = \nu^{2n+1-r} \otimes 1 = e_{2n+2}$. In general, for $2 \leq i \leq 2n+1$, x acts on $e_{(2n+1)+i}$ by $e_{(2n+1)+i} \cdot x = \nu^{2n+1-r+i} \otimes \nu^i = e_{(2n+1)+i+1}$. The result now follows. ☐

Proposition 4.5.3. *Define the matrix* Φ *by*

$$\Phi_{ij} = \begin{cases} 1, & i = j = 1; \\ 1, & j = 2n+3-i,\, 2 \leq i \leq 2n+1; \\ 0, & o/w. \end{cases}$$

Then $\rho_{V_r}(y) = \begin{pmatrix} 0 & \Phi \\ \Phi & 0 \end{pmatrix}$.

Proof. With the e_i, $e_{(2n+1)+i}$ as defined above, first observe $e_1 \cdot y = \nu^{2n+1-r} \otimes 1 = e_{2n+2}$. Now consider y acting on a general basis element of $V(r)$ for $2 \leq i \leq 2n+1$,

$$e_i \cdot y = (\nu^{r+i-1} \otimes \nu^{i-1})y = (\nu^{2n+2-r-i} \otimes \nu^{2n+2-i}) = (\nu^{2n-r+(2-i)} \otimes \nu^{1-i}) = e_{2n+1+(2n+3-i)}.$$

Thus, for the first $2n+1$ columns, we get zeroes in the first $2n+1$ rows, and then Φ making up the latter $2n+1$ rows.

Now let y act on the basis elements of $V(2n+1-r)$. As before,we have $e_{2n+2} \cdot y = \nu^r \otimes 1 = e_1$. For a general basis element $2 \leq i \leq 2n+1$,

$$e_{2n+1+i} \cdot y = (\nu^{2n-r+i} \otimes \nu^{i-1})y = \nu^{r+(2n+2-i)} \otimes \nu^{2n+2-i} = e_{2n+3-i}.$$

Thus, for the last $2n+1$ columns we get Φ making up the first $2n+1$ rows, and zeroes thereafter. ☐

Proposition 4.5.4. *With* Ψ*, as defined above* $\rho_{reg}(x^{-1}) = \begin{pmatrix} \Psi & 0 \\ 0 & \Psi \end{pmatrix}$.

Proof. Set

$$\begin{aligned} f_i = & \quad x^{i-1}, \quad 1 \leq i \leq 2n+1 \\ f_{2n+1+i} = & \quad yx^{i-1}, \quad 1 \leq i \leq 2n+1. \end{aligned}$$

Clearly, $f_{2n+1} \cdot x = 1 = f_1$ and $f_i \cdot x = x^i = f_{i+1}$ for $1 \leq i \leq 2n$. Similarly, $f_{4n+2} \cdot x = y = f_{2n+2}$ and $f_{2n+1+i} \cdot x = yx^i = f_{2n+2+i}$. The result now follows. ☐

Proposition 4.5.5. *With* Φ *as defined above,* $\rho_{reg}(y) = \begin{pmatrix} 0 & \Phi \\ \Phi & 0 \end{pmatrix}$.

Proof. Let f_i, f_{2n+1+i} be as above. First observe $f_1 \cdot y = y = f_{2n+2}$. For a more general element where $2 \leq i \leq 2n+1$,

$$f_i \cdot y = x^{i-1}y = yx^{2n+2-i} = f_{(2n+1)+2n+3-i}.$$

Similarly $f_{2n+2} \cdot y = 1 = f_1$ For a more general element,

$$f_{2n+1+i} \cdot y = yx^{i-1}y = x^{2n+2-i} = f_{2n+3-i}.$$

The result now follows. ☐

Proposition 4.5.6. *For* $r \geq 2$, $V_r = V(r) + V(2n+1-r) \cong \Lambda$.

Proof. Immediate since $\rho_{V_r}(g) = \rho_{reg}(g)$ for all $g \in D_{4n+2}$, by Propositions 4.5.2 - 4.5.5. ☐

We are therefore left with $T + V(1)$ which, we hope, can be used to give us L. Consider the following map $\psi : I_C^* \otimes I_C^* \to \mathbf{Z}$ defined by,

$$\nu^r \otimes \nu^s \longmapsto \begin{cases} 1, & \text{if } r = s+1; \\ -1, & \text{if } s = r+1; \\ 0, & \text{if } |r-s| \neq 1 \end{cases}$$

where $\mathbf{Z}$ is taken to mean the trivial Λ_0-module and $0 \leq r,\ s \leq 2n-1$.

Proposition 4.5.7. *The map* ψ*, as defined above, is a* Λ_0*-homomorphism.*

Proof. It is straightforward to see ψ is well defined and a $\mathbf{Z}$-homomorphism. By applying the x-action, note that $\psi(\nu^r x \otimes \nu^s x) = \psi(\nu^{r+1} \otimes \nu^{s+1}) = \psi(\nu^r \otimes \nu^s)x$. We therefore conclude that ψ is a Λ_0-homomorphism. ☐

Proposition 4.5.8. *The map* $\psi : (I_C)_- \otimes (I_C)_- \to \mathbf{Z}_-$ *induced from* ψ *above, is a* Λ*-homomorphism.*

Proof. Consider how ψ behaves on $\nu^r \otimes \nu^s$. By Proposition 4.5.7, we need only consider the y-action. There are three cases to consider:

a) $r = s+1$;

b) $s = r+1$;

c) $|r-s| \neq 1$.

For *a*) we have the following:

$$\nu^{s+1}y \otimes \nu^s y = \begin{cases} -\sum_{i=0}^{2n-1} \nu^i \otimes 1, & s=0; \\ -\nu^{2n-1} \otimes \sum_{i=0}^{2n-1} \nu^i, & s=1; \\ \nu^{2n-s} \otimes \nu^{2n+1-s}, & 2 \leq s \leq 2n-1. \end{cases}$$

As such,

$$\begin{aligned} \psi\left(-\sum_{i=0}^{2n-1} \nu^i \otimes 1\right) &= \quad \psi(-1 \otimes 1 - \nu \otimes 1 - \cdots - \nu^{2n-1} \otimes 1) &= -1, \\ \psi\left(-\nu^{2n-1} \otimes \sum_{i=0}^{2n-1} \nu^i\right) &= \psi(-\nu^{2n-1} \otimes 1 - \cdots - \nu^{2n-1} \otimes \nu^{2n-2} - \nu^{2n-1} \otimes \nu^{2n-1}) &= -1, \end{aligned}$$

and finally $\psi\left(\nu^{2n-s} \otimes \nu^{2n+1-s}\right) = -1$. It is worth observing that when $s = 2n-1$, we have

$$\nu^{s+1} \otimes \nu^s = (\nu^{2n} \otimes \nu^{2n-1}) = -1 \otimes \nu^{2n-1} - \cdots - \nu^{2n-2} \otimes \nu^{2n-1} - \nu^{2n-1} \otimes \nu^{2n-1},$$

which is sent to 1 by ψ, as required. In any case, y acts by -1 when $r = s+1$. Likewise, the same is true for $s = r+1$.

Finally, when $|r-s| \neq 1$ there are the following possibilities:

$$\nu^r y \otimes \nu^s y = \begin{cases} 1 \otimes 1, & r = s = 0; \\ 1 \otimes \nu^{2n+1-s}, & r = 0,\ 2 \leq s \leq 2n-1; \\ \nu^{2n} \otimes \nu^{2n}, & r = s = 1; \\ \nu^{2n} \otimes \nu^{2n+1-s}, & r = 1,\ 3 \leq s \leq 2n-1; \\ \nu^{2n+1-r} \otimes 1, & 2 \leq r \leq 2n-1,\ s = 0; \\ \nu^{2n+1-r} \otimes \nu^{2n}, & 3 \leq r \leq 2n-1,\ s = 1; \\ \nu^{2n+1-r} \otimes \nu^{2n+1-s}, & 2 \leq r = s \leq 2n-1. \end{cases}$$

The only cases that do not obviously go to zero are those involving ν^{2n}. First, consider $\nu^r \otimes \nu^s$ where r is fixed such that $1 \leq r \leq 2n-2$, and s varies between $0 \leq s \leq 2n-1$. It is clear that both $r-s=1$ and $s-r=1$ occur as s varies, hence they cancel each other out and the effect is that $\nu^r \otimes \nu^s$ is sent to zero. As such, it is clear that both $\nu^{2n} \otimes \nu^{2n+1-s}$ and $\nu^{2n+1-r} \otimes \nu^{2n}$ are sent to zero, as $3 \leq r,\ s \leq 2n-1$.

For the case $\nu^{2n} \otimes \nu^{2n}$, we sum over all $\nu^r \otimes \nu^s$ where both r, s vary as above. By the above remark, we are left with elements of the form $1 \otimes \nu^s$ and $\nu^{2n-1} \otimes \nu^s$ where $0 \leq s \leq 2n-1$. The only elements sent to something other than zero in the former is of the form $1 \otimes \nu$, and in the latter $\nu^{2n-1} \otimes \nu^{2n-2}$. These are clearly sent to -1 and 1, respectively, and therefore cancel. It follows that ψ will map each of the above elements to zero when $|r-s| \neq 1$. Consequently, y acts by -1 and $\psi : (I_C^*)_- \otimes (I_C^*)_- \to \mathbf{Z}_-$ is a Λ-homomorphism. □

As an aside, it is worth mentioning that we may similarly use the other Galois action defined in Section 4.1. This will yield another Λ-homomorphism. In particular, by combining these two Galois actions, we end up with the following four Λ-homomorphisms,

- $\psi_1 : I(C_{2n+1})^*_+ \otimes I(C_{2n+1})^*_+ \longrightarrow \mathbf{Z}_-$;
- $\psi = \psi_2 : I(C_{2n+1})^*_- \otimes I(C_{2n+1})^*_- \longrightarrow \mathbf{Z}_-$;
- $\psi_3 : I(C_{2n+1})^*_+ \otimes I(C_{2n+1})^*_- \longrightarrow \mathbf{Z}$;
- $\psi_4 : I(C_{2n+1})^*_- \otimes I(C_{2n+1})^*_+ \longrightarrow \mathbf{Z}$.

Nevertheless, only ψ (or ψ_2 in the above list) is of interest to us as only $(I_C^*)_- \cong P$. Now, let $V' = V(2) + \cdots + V(2n-1)$ and observe that V' is free and $V' \subset Ker(\psi)$. Let

$$\natural : [(I_C^*)_- \otimes (I_C^*)_-] \to [(I_C^*)_- \otimes (I_C^*)_-]/V'$$

be the natural map and restrict ψ to $[(I_C^*)_- \otimes (I_C^*)_-]/V'$. We then have the following exact sequence,

$$0 \to Ker(\psi) \to [(I_C^*)_- \otimes (I_C^*)_-]/V' \overset{\psi}{\to} \mathbf{Z}_- \to 0. \tag{4.5.9}$$

To emphasise which part of $(I_C^*)_- \otimes (I_C^*)_-$ remains when we quotient by V', we shall often write this as $\natural(T + V(1))$. Observe that the sum $T + V(1)$ will not be direct over Λ. Our goal now will be to prove:

Proposition 4.5.10. *$Ker(\psi) = span_{\mathbf{Z}}\{\natural(\nu^i \otimes \nu^i) \mid 0 \leq i \leq 2n\}$.*

To do so, we need to prove a number of preliminary results. To make the notation somewhat more readable, we will write p instead of $2n+1$.

Proposition 4.5.11. *For each $j \in \{0, 1, \ldots, p-1\}$,*

$$\natural(\nu^j \otimes \nu^j) = \natural(T) + \sum_{i=j+1}^{j+p-2} \natural(\nu^{i+1} \otimes \nu^i).$$

Proof. We split the proof into four cases; $j = 0$, $1 \leq j \leq p-3$, $j = p-2$ and $j = p-1$. Start with the case $j = 0$ in which we successively subtract the 'columns' from T. We therefore have:

$$\begin{aligned}
1 \otimes 1 = \quad & T + \nu^{p-1} \otimes \nu^{p-2} - (1 + \nu + \cdots + \nu^{p-3}) \otimes \nu^{p-3} - \cdots \\
& \cdots - (1 + \nu + \cdots + \nu^i) \otimes \nu^i - \cdots - (1 + \nu) \otimes \nu \\
= \quad & T + \nu^{p-1} \otimes \nu^{p-2} + (\nu^{p-1} + \nu^{p-2}) \otimes \nu^{p-3} + \cdots \\
& \cdots + (\nu^{p-1} + \cdots + \nu^{i+1}) \otimes \nu^i + \cdots + (\nu^{p-1} + \cdots + \nu^2) \otimes \nu \\
= \quad & T + \textstyle\sum_{i=1}^{p-2} (\nu^{p-1} + \cdots + \nu^{i+1}) \otimes \nu^i.
\end{aligned}$$

However, $\nu^{i+k} \otimes \nu^i \in V(k)$ for some $1 \leq k \leq p-2$. We therefore conclude

$$\natural(1 \otimes 1) = \natural(T) + \sum_{i=1}^{p-2} \natural(\nu^{i+1} \otimes \nu^i). \tag{4.5.12}$$

Next, we look at the region $1 \leq j \leq p-3$. For this, we alter our approach slightly. For $i < j$ we systematically subtract 'rows' from T (where T is written as in Chapter 3). Once we reach j, we then subtract the 'columns'. The result will be the removal of all terms from T except $\nu^j \otimes \nu^j$. To keep track of this we use the variable i when subtracting rows, and k when subtracting columns. Thus, we have:

$$\begin{aligned}
\nu^j \otimes \nu^j = \quad & T + 1 \otimes \nu^{p-1} - \nu \otimes (\nu + \cdots + \nu^{p-2}) - \nu^2 \otimes (\nu^2 + \cdots + \nu^{p-2}) - \cdots \\
& \cdots - \nu^i \otimes (\nu^i + \cdots + \nu^{p-2}) - \cdots - \nu^{j-1} \otimes (\nu^{j-1} + \cdots + \nu^{p-2}) \\
& -(\nu^j + \nu^{j+1}) \otimes \nu^{j+1} - (\nu^j + \nu^{j+1} + \nu^{j+2}) \otimes \nu^{j+2} - \cdots \\
& \cdots - (\nu^j + \cdots + \nu^{j+k}) \otimes \nu^{j+k} - \cdots - (\nu^j + \cdots + \nu^{p-2}) \otimes \nu^{p-2} \\
= \quad & T + 1 \otimes \nu^{p-1} - \textstyle\sum_{i=1}^{j-1} \nu^i \otimes (\nu^i + \cdots + \nu^{p-2}) - \sum_{k=1}^{p-2-j} (\nu^j + \cdots + \nu^{j+k}) \otimes \nu^{j+k} \\
= \quad & T + 1 \otimes \nu^{p-1} + \textstyle\sum_{i=1}^{j-1} \nu^i \otimes (\nu^{p-1} + 1 + \nu + \cdots + \nu^{i-1}) \\
& + \textstyle\sum_{k=1}^{p-2-j} (\nu^{j+k+1} + \cdots + \nu^{p-1} + 1 + \nu + \cdots + \nu^{j-1}) \otimes \nu^{j+k} \\
= \quad & T + \textstyle\sum_{i=0}^{j-1} (\nu^i \otimes \nu^{p-1}) + \sum_{i=1}^{j-1} \nu^i \otimes (\nu^{i-1} + \cdots \nu + 1) \\
& + \textstyle\sum_{k=1}^{p-2-j} (\nu^{j+k+1} + \cdots + \nu^{p-1}) \otimes \nu^{j+k} + \sum_{k=1}^{p-2-j} (\nu^{j-1} + \cdots + \nu + 1) \otimes \nu^{j+k}.
\end{aligned}$$

Now, we observe $\nu^i \otimes \nu^{p-1} \in V(i+1)$ and $1 \leq i+1 \leq j \leq p-3$. As such, we have $\sum_{i=0}^{j-1} \natural(\nu^i \otimes \nu^{p-1}) = \natural(1 \otimes \nu^{p-1})$. A similar observation shows

$$\sum_{i=1}^{j-1} \natural(\nu^i \otimes (\nu^{i-1} + \cdots \nu + 1)) = \sum_{i=1}^{j-1} \natural(\nu^i \otimes \nu^{i-1}).$$

Note $1 \leq p-2-j \leq p-3$ and $2 \leq j+k \leq p-2$. Consequently, each of the following terms belong to some $V(l)$ for $1 \leq l \leq p-3$ and so

$$\sum_{k=1}^{p-2-j} \natural((\nu^{j+k+1} + \cdots + \nu^{p-1}) \otimes \nu^{j+k}) = \sum_{k=1}^{p-2-j} \natural(\nu^{j+k+1} \otimes \nu^{j+k}).$$

Finally, for $1 \leq k \leq p-2-j \leq p-3$, $\nu^i \otimes \nu^{j+k} \in V(l)$ where $2 \leq l \leq p-2$, and so

$$\sum_{k=1}^{p-2-j} \natural((\nu^{j-1} + \cdots + \nu + 1) \otimes \nu^{j+k}) = 0.$$

To conclude, for $1 \leq j \leq p-3$ we have

$$\natural(\nu^j \otimes \nu^j) = \natural(T) + \natural(1 \otimes \nu^{p-1}) + \sum_{i=1}^{j-1} \natural(\nu^i \otimes \nu^{i-1}) + \sum_{k=1}^{p-2-j} \natural(\nu^{j+k+1} \otimes \nu^{j+k}). \tag{4.5.13}$$

We now take $j = p-2$ in which we successively subtract 'rows'. We have:

$$\begin{aligned}\nu^{p-2}\otimes\nu^{p-2} = {} & T + 1\otimes\nu^{p-1} - \nu\otimes(\nu+\cdots+\nu^{p-2}) - \cdots \\ & \cdots - \nu^{i}\otimes(\nu^{i}+\cdots+\nu^{p-2}) - \cdots - \nu^{p-3}\otimes(\nu^{p-3}+\nu^{p-2}) \\ = {} & T + 1\otimes\nu^{p-1} - \textstyle\sum_{i=1}^{p-3}\nu^{i}\otimes(\nu^{i}+\cdots+\nu^{p-2}) \\ = {} & T + 1\otimes\nu^{p-1} + \textstyle\sum_{i=1}^{p-3}\nu^{i}\otimes(\nu^{i-1}+\cdots+1) + \sum_{i=1}^{p-3}\nu^{i}\otimes\nu^{p-1}.\end{aligned}$$

It is straightforward to show $\sum_{i=1}^{p-3}\natural(\nu^{i}\otimes\nu^{p-1}) = 0$, and

$$\sum_{i=1}^{p-3}\natural(\nu^{i}\otimes(\nu^{i-1}+\cdots+1)) = \sum_{i=1}^{p-3}\natural(\nu^{i}\otimes\nu^{i-1}) = \sum_{i=0}^{p-4}\natural(\nu^{i+1}\otimes\nu^{i}).$$

Then

$$\natural(\nu^{p-2}\otimes\nu^{p-2}) = \natural(T) + \natural(1\otimes\nu^{p-1}) + \sum_{i=0}^{p-4}\natural(\nu^{i+1}\otimes\nu^{i}) \tag{4.5.14}$$

We conclude the proof with the case $j = p-1$. For this, we add in the required terms, each of which come from one of the $V(r)$. In other words, we have

$$\begin{aligned}\nu^{p-1}\otimes\nu^{p-1} = {} & T + (\nu+\cdots+\nu^{p-2})\otimes 1 + \cdots + (\nu^{i+1}+\cdots+\nu^{p-2})\otimes\nu^{i} + \cdots\nu^{p-2}\otimes\nu^{p-3} \\ = {} & T + \textstyle\sum_{i=0}^{p-3}(\nu^{i+1}+\cdots+\nu^{p-2})\otimes\nu^{i}.\end{aligned}$$

It is now clear that

$$\natural(\nu^{p-1}\otimes\nu^{p-1}) = \natural(T) + \sum_{i=0}^{p-3}\natural(\nu^{i+1}\otimes\nu^{i}). \tag{4.5.15}$$

□

Corollary 4.5.16.

- $\natural(\nu^2\otimes\nu^2) - \natural(1\otimes 1) = \natural(1\otimes\nu^{p-1}) + \natural(\nu\otimes 1) - \natural(\nu^2\otimes\nu) - \natural(\nu^3\otimes\nu^2)$;
- *For* $3\le j\le p-4$, $\natural(\nu^{j+1}\otimes\nu^{j+1}) - \natural(\nu^{j}\otimes\nu^{j}) = \natural(\nu^{j}\otimes\nu^{j-1}) - \natural(\nu^{j+2}\otimes\nu^{j+1})$;
- $\natural(\nu^{p-1}\otimes\nu^{p-1}) - \natural(\nu^{p-2}\otimes\nu^{p-2}) = \natural(\nu^{p-2}\otimes\nu^{p-3}) - \natural(1\otimes\nu^{p-1})$.

Proof. Using Proposition 4.5.11, we have $\natural(1\otimes 1) = \natural(T) + \sum_{i=1}^{p-2}\natural(\nu^{i+1}\otimes\nu^{i})$ and

$$\natural(\nu^2\otimes\nu^2) = \natural(T) + \sum_{k=1}^{p-4}\natural(\nu^{k+3}\otimes\nu^{k+2}) + \natural(1\otimes\nu^{p-1}) + \natural(\nu\otimes 1).$$

As $3\le k+2\le p-2$ it follows

$$\natural(\nu^2\otimes\nu^2) - \natural(1\otimes 1) = \natural(1\otimes\nu^{p-1}) + \natural(\nu\otimes 1) - \natural(\nu^2\otimes\nu) - \natural(\nu^3\otimes\nu^2)$$

Likewise, we can show $\natural(\nu^{p-1}\otimes\nu^{p-1}) - \natural(\nu^{p-2}\otimes\nu^{p-2}) = \natural(\nu^{p-2}\otimes\nu^{p-3}) - \natural(1\otimes\nu^{p-1})$.

Next, let $3\le j\le p-4$, then

$$\begin{aligned}\natural(\nu^{j+1}\otimes\nu^{j+1}) - \natural(\nu^{j}\otimes\nu^{j}) = {} & \textstyle\sum_{k=1}^{p-3-j}\natural(\nu^{j+k+2}\otimes\nu^{j+k+1}) + \sum_{i=1}^{j}\natural(\nu^{i}\otimes\nu^{i-1}) \\ & -\textstyle\sum_{k=1}^{p-2-j}\natural(\nu^{j+k+1}\otimes\nu^{j+k}) - \nu_{i=1}^{j-1}\natural(\nu^{i}\otimes\nu^{i-1}).\end{aligned}$$

First note $\sum_{i=1}^{j}\natural(\nu^i\otimes\nu^{i-1}) - \sum_{i=1}^{j-1}\natural(\nu^i\otimes\nu^{i-1}) = \nu^j\otimes\nu^{j-1}$. Next, we can rewrite

$$\sum_{k=1}^{p-3-j}\natural(\nu^{j+k+2}\otimes\nu^{j+k+1}) = \sum_{k=2}^{p-2-j}\natural(\nu^{j+k+1}\otimes\nu^{j+k})$$

so that $\sum_{k=2}^{p-2-j}\natural(\nu^{j+k+1}\otimes\nu^{j+k}) - \sum_{k=1}^{p-2-j}\natural(\nu^{j+k+1}\otimes\nu^{j+k}) = -\natural(\nu^{j+2}\otimes\nu^{j+1})$. Thus,

$$\natural(\nu^{j+1}\otimes\nu^{j+1}) - \natural(\nu^j\otimes\nu^j) = \natural(\nu^j\otimes\nu^{j-1}) - \natural(\nu^{j+2}\otimes\nu^{j+1}).$$

□

Proof of Proposition 4.5.10. Clearly $\Omega = span_{\mathbf{Z}}\{\natural(\nu^i\otimes\nu^i)\,|\,0\leq i\leq p-1\}\subset Ker(\psi)$. For the converse, suppose $v\in Ker(\psi)$ and write $v=\sum_{r=0}^{p-1}\alpha_r\natural(\nu^{r+1}\otimes\nu^r)+\alpha_T\natural(T)$, where α_r, $\alpha_T\in\mathbf{Z}$. By the results of Proposition 4.5.11, we can rewrite $\natural(T)$ as $\natural(T)=\natural(1\otimes 1)-\sum_{i=1}^{p-2}\natural(\nu^{i+1}\otimes\nu^i)$. We can therefore rewrite v as

$$v=\sum_{r=0}^{p-1}\beta_r\natural(\nu^{r+1}\otimes\nu^r)+\alpha_T\natural(1\otimes 1),$$

where $\beta_0=\alpha_0$, $\beta_r=\alpha_r-\alpha_T$ for $1\leq r\leq p-2$ and $\beta_{p-1}=a_{p-1}$. Thus, $v\in\Omega$ if and only if $v'=\sum_{r=0}^{p-1}\beta_r\natural(\nu^{r+1}\otimes\nu^r)\in\Omega$.

Next, since $\psi(\natural(\nu^{r+1}\otimes\nu^r))=1$ and $\psi(v')=0$, it follows that $\sum_{r=0}^{p-1}\beta_r=0$. In particular, we can write:

$$\begin{array}{rc}
v' = & \sum_{r=0}^{p-1}\beta_r\natural(\nu^{r+1}\otimes\nu^r)-\sum_{r=0}^{p-1}\beta_r\natural(\nu\otimes 1)\\
= & \sum_{r=0}^{p-1}\beta_r\natural(\nu\otimes 1)x^r-\sum_{r=0}^{p-1}\beta_r\natural(\nu\otimes 1)\\
= & \natural(\nu\otimes 1)(\sum_{r=0}^{p-1}\beta_r x^r-\sum_{r=0}^{p-1}\beta_r)\\
= & \natural(\nu\otimes 1)(\sum_{r=0}^{p-1}\beta_r(x^r-1))\\
= & \natural(\nu\otimes 1)(\sum_{r=0}^{p-1}\beta_r(x-1)(x^{r-1}+\cdots+1))\\
= & \natural(\nu\otimes 1)(x-1)(\sum_{r=0}^{p-1}\beta_r(x^{r-1}+\cdots+1))\\
= & [\natural(\nu^2\otimes\nu)-\natural(\nu\otimes 1)](\sum_{r=0}^{p-1}\beta_r(x^{r-1}+\cdots+1)).
\end{array}$$

It is therefore sufficient to show $\natural(\nu^2\otimes\nu)-\natural(\nu\otimes 1)\in\Omega$. To do so, first note that for $3\leq j\leq p-4$, we can rewrite $\natural(\nu^{j+3}\otimes\nu^{j+3})-\natural(\nu^{j+2}\otimes\nu^{j+2})+\natural(\nu^{j+1}\otimes\nu^{j+1})-\natural(\nu^j\otimes\nu^j)$ as

$$\natural(\nu^{j+2}\otimes\nu^{j+1})-\natural(\nu^{j+4}\otimes\nu^{j+3})+\natural(\nu^j\otimes\nu^{j-1})-\natural(\nu^{j+2}\otimes\nu^{j+1}) = \natural(\nu^j\otimes\nu^{j-1})-\natural(\nu^{j+4}\otimes\nu^{j+3})$$

It follows that $\sum_{j=3}^{p-1}(-1)^j\natural(\nu^j\otimes\nu^j)=\natural(\nu^3\otimes\nu^2)-\natural(1\otimes\nu^{p-1})$ and so

$$\begin{array}{rcl}
\sum_{j=2}^{p-1}(-1)^j\natural(\nu^j\otimes\nu^j)-\natural(1\otimes 1) = & \natural(\nu^3\otimes\nu^2)-\natural(1\otimes\nu^{p-1})+\natural(1\otimes\nu^{p-1})+ \\
& +\natural(\nu\otimes 1)-\natural(\nu^2\otimes\nu)-\natural(\nu^3\otimes\nu^2) \\
= & \natural(\nu\otimes 1)-\natural(\nu^2\otimes\nu).
\end{array}$$

□

With $\natural$ as above, we use Proposition 4.5.6 (and Proposition 2.2.7) to construct the following split short exact sequence of Λ-lattices,

$$0 \to \Lambda^{n-1} \to (I_C^*)_- \otimes (I_C^*)_- \to \natural(T + V(1)) \to 0. \tag{4.5.17}$$

Proposition 4.5.18. *L, $\natural(T + V(1)) \in Ext^1(\mathbf{Z}_-, L_0)$.*

Proof. A similar argument to that of K, shows L occurs in an exact sequence

$$0 \to L_0 \to L \to \mathbf{Z}_- \to 0.$$

Thus, by (4.5.9) and Proposition 4.5.10 it remains to show

$$\Omega = span_{\mathbf{Z}}\{\natural(\nu^i \otimes \nu^i) \mid 0 \leq i \leq 2n\} \cong L_0.$$

To do so, denote the bases of Ω and L_0 by $\{e_i \mid 1 \leq i \leq 2n+1\}$, $\{f_j \mid 1 \leq j \leq 2n+1\}$, respectively, where $e_i = \natural(\nu^{i-1} \otimes \nu^{i-1})$ and $f_j = (y+1)x^{i-1}$. Clearly, we have an isomorphism as abelian groups. The result now follows as both sets of basis elements are easily shown to be equivariant under the actions of x and y. ☐

Corollary 4.5.19. *$L \cong \natural(T + V(1))$.*

Proof. It is sufficient to show L and $\natural(T+V(1))$ belong to the same class of $Ext^1(\mathbf{Z}_-, L_0)$. First, recall (4.1.9) in which $j_*(\mathbf{Z}) = [y+1)$. It is straightforward to see $j_*(\mathbf{Z}) \cong L_0$. Now, using Eckmann-Shapiro we get $Ext^1(\mathbf{Z}_-, j_*(\mathbf{Z})) \cong Ext^1(\mathbf{Z}_-, \mathbf{Z}) \cong \mathbf{Z}/2$. Since L is indecomposable, it clearly does not belong in the trivial class.

For $\natural(T + V(1))$, we first observe this is free as a $\mathbf{Z}[C_2]$-module. Start with the exact sequence

$$0 \to \Lambda^{n-1} \to (I_C^*)_- \otimes (I_C^*)_- \to \natural(T + V(1)) \to 0$$

and apply the exact functor $j^*(-)$ so that we have

$$0 \to \mathbf{Z}[C_2]^{(2n+1)(n-1)} \to j^*((I_C^*)_- \otimes (I_C^*)_-) \to j^*(\natural(T + V(1))) \to 0.$$

As $(I_C^*)_- \cong P$, and since $j^*(P)$ is free of rank n (Proposition 4.4.3), $j^*(\natural(T+V(1)))$ is stably free of rank $n+1$. By the Swan-Jacobinski Theorem, $j^*(\natural(T+V(1)))$ is therefore free of rank $n+1$, i.e. $j^*(\natural(T + V(1))) \cong \mathbf{Z}[C_2]^{n+1}$. Next, $j^*(L_0) \cong \mathbf{Z} \oplus \mathbf{Z}[C_2]$ (see below). So if we suppose $\natural(T + V(1))$ is in the trivial class of $Ext^1(\mathbf{Z}_-, L_0)$, then the exact sequence containing $\natural(T+V(1))$ splits. In particular, so too does the restriction of this exact sequence to $\mathbf{Z}[C_2]$. In other words, the following is a split short exact sequence:

$$0 \to \mathbf{Z} \oplus \mathbf{Z}[C_2]^n \xrightarrow{\iota} \mathbf{Z}[C_2]^{n+1} \to \mathbf{Z}_- \to 0.$$

This exact sequence can be altered so that

$$0 \to \mathbf{Z} \to \mathbf{Z}[C_2]^{n+1}/\iota(\mathbf{Z}[C_2]^n) \to \mathbf{Z}_- \to 0 \tag{4.5.20}$$

is also exact. By Johnson's 'destabilization theorem' (Proposition 2.4.2) $\mathbf{Z}[C_2]^{n+1}/\iota(\mathbf{Z}[C_2]^n)$ is projective. We can therefore construct the following split short exact sequence

$$0 \to \mathbf{Z}[C_2]^n \xrightarrow{\iota} \mathbf{Z}[C_2]^{n+1} \to \mathbf{Z}[C_2]^{n+1}/\iota(\mathbf{Z}[C_2]^n) \to 0.$$

So $\mathbf{Z}[C_2]^{n+1}/\iota(\mathbf{Z}[C_2]^n)$ is stably free, and hence free, of rank 1. Replacing this in (4.5.20), we have

$$0 \to \mathbf{Z} \to \mathbf{Z}[C_2] \to \mathbf{Z}_- \to 0.$$

However, this clearly does not split and so $\natural(T+V(1))$ cannot belong in the trivial class of $Ext^1(\mathbf{Z}_-, L_0)$. Hence, both L and $\natural(T+V(1))$ belong to the non-trivial class. It therefore follows that they are isomorphic, as required. □

Using Corollary 4.5.19 we can replace $\natural(T+V(1))$ with L in (4.5.17). We therefore have the following split the short exact sequence

$$0 \to \Lambda^{n-1} \to (I_C^*)_- \otimes (I_C^*)_- \to L \to 0.$$

By recalling $(I_C^*)_- \cong P$, we have therefore shown:

Proposition 4.5.21. *$P \otimes P \cong L \oplus \Lambda^{n-1}$.*

Furthermore, note the following dual statement:

Proposition 4.5.22. *$R \otimes R \cong L \oplus \Lambda^{n-1}$.*

Proof. By Proposition 4.5.21, we have $P \otimes P \cong L \oplus \Lambda^{n-1}$. Moreover, we know $P^* \cong R$ and thus, by applying Proposition 2.2.6,

$$R \otimes R \cong (P \otimes P)^* \cong (L \oplus \Lambda^{n-1})^* \cong L \oplus \Lambda^{n-1}$$

where the last isomorphism follows from L and Λ^{n-1} being self dual. □

To conclude, we relate this to the result of Section 2.5, $\Omega_1(\mathbf{Z}) \otimes \Omega_1(\mathbf{Z}) = \Omega_2(\mathbf{Z})$. Thus far we have shown the minimal level of $\Omega_1(\mathbf{Z}) = P \oplus [y-1)$. As such,

$$(P \oplus [y-1)) \otimes (P \oplus [y-1)) \cong (P \otimes P) \oplus (P \otimes [y-1)) \oplus ([y-1) \otimes P) \oplus ([y-1) \otimes [y-1)). \tag{4.5.23}$$

By Proposition 4.5.21, $P \otimes P \cong L \oplus \Lambda^{n-1}$. Moreover, recall $K_0 = [y-1) \cong j_*(I_2)$. Therefore, by Frobenius Reciprocity and Corollary 2.1.9,

$$\begin{aligned} P \otimes [y-1) \cong [y-1) \otimes P \cong & \quad j_*(I_2) \otimes P \\ \cong & \quad j_*(I_2 \otimes j^*(P)) \\ \cong & \quad j_*(I_2 \otimes \mathbf{Z}[C_2]^n) \quad \cong j_*(\mathbf{Z}[C_2]^n) \cong \Lambda^n \end{aligned}$$

where we have used the fact that P is free when considered as a $\mathbf{Z}[C_2]$-module. Finally, we come to $[y-1) \otimes [y-1)$.

Proposition 4.5.24. *$j^*([y-1)) \cong I_2 \oplus \mathbf{Z}[C_2]^n$.*

Proof. Recall $j^*(K) \cong \mathbf{Z}[C_2]^{n+1}$ by Proposition 4.4.5 and consider the exact sequence $0 \to K_0 \to K \to \mathbf{Z} \to 0$. Now apply the restriction of scalars functor $j^*(-)$ to yield

$$0 \to j^*(K_0) \to \mathbf{Z}[C_2]^{n+1} \to \mathbf{Z} \to 0.$$

We also have the exact sequence $0 \to I(C_2) \to \mathbf{Z}[C_2] \to \mathbf{Z} \to 0$. By Schanuel's Lemma we therefore have $j^*(K_0) \oplus \mathbf{Z}[C_2] \cong I(C_2) \oplus \mathbf{Z}[C_2]^{n+1}$. The result now follows since $[I_2]$ is straight. □

Using the fact that $j_*(\mathbf{Z}) \cong [y+1) = L_0$, a similar proof now shows:

Proposition 4.5.25. $j^*([y+1)) \cong \mathbf{Z} \oplus \mathbf{Z}[C_2]^n$

Proposition 4.5.26. $[y-1) \otimes [y-1) \cong [y+1) \oplus \Lambda^n$.

Proof. Using Frobenius Reciprocity and Proposition 4.5.24, we have

$$\begin{aligned}[y-1) \otimes [y-1) \cong& \quad j_*(I_2 \otimes j^*([y-1))) \\ \cong& \quad j_*(I_2 \otimes (I_2 \oplus \mathbf{Z}[C_2]^n)) \\ \cong& \quad j_*(I_2 \otimes I_2) \oplus j_*(\mathbf{Z}[C_2]^n).\end{aligned}$$

Now, recall that in Chapter 3 we showed $I_2 \otimes I_2 \cong \mathbf{Z}$. Thus,

$$[y-1) \otimes [y-1) \cong j_*(\mathbf{Z}) \oplus j_*(\mathbf{Z}[C_2]^n) \cong [y+1) \oplus \Lambda^n.$$

□

Combining the above results, (4.5.23) becomes

$$(P \oplus [y-1)) \otimes (P \oplus [y-1)) \cong L \oplus [y+1) \oplus \Lambda^{4n-1}$$

as required.

4.6 $L \otimes P \sim R$

The aim of this section will be to construct the following isomorphism:

$$L \otimes P \cong R \oplus \Lambda^n. \tag{4.6.1}$$

First, recall:

- P has $\mathbf{Z}$-basis $\{\pi x, \pi x^2, \ldots, \pi x^{2n}\}$, $\pi = (x^n-1)(y-1)$;
- L has $\mathbf{Z}$-basis $\{(y+1), (y+1)x, \ldots, (y+1)x^{2n}, \Sigma_x\}$, $\Sigma_x = 1 + x + \cdots + x^{2n}$;
- R has $\mathbf{Z}$-basis $\{(y-1)(x-1), (y-1)(x^2-1), \ldots, (y-1)(x^{2n}-1)\}$.

By counting $\mathbf{Z}$-ranks of $P \otimes L \cong R \oplus \Lambda^a$, we see that $a = n$, thereby explaining the number of copies of Λ in (4.6.1).

In Section 4.3 we defined L_0 to be the Λ-submodule of L with $\mathbf{Z}$-basis

$$\{(y+1), (y+1)x, \ldots, (y+1)x^{2n}\}.$$

Recall that the underlying abelian group of L/L_0 is free abelian of rank 1, generated by the image of Σ_x upon which x acts trivially and y acts by -1. To reflect this, write $L/L_0 \cong \mathbf{Z}_-$ and construct the short exact sequence

$$0 \to L_0 \to L \to \mathbf{Z}_- \to 0.$$

Tensoring with P yields the following short exact sequence,

$$0 \to L_0 \otimes P \to L \otimes P \to \mathbf{Z}_- \otimes P \to 0. \tag{4.6.2}$$

To achieve the goal of this section, it is necessary to investigate $L_0 \otimes P$ and $\mathbf{Z}_- \otimes P$, which we do by considering their respective representations. Trivially, we have:

Proposition 4.6.3. *$\rho_{\mathbf{Z}_-}(x^{-1}) = 1$, and $\rho_{\mathbf{Z}_-}(y) = -1$.*

Corollary 4.6.4. *For any module M,*

$$\rho_{\mathbf{Z}_-\otimes M}(x^{-1}) = \rho_M(x^{-1}) \text{ and } \rho_{\mathbf{Z}_-\otimes M}(y) = -\rho_M(y).$$

Proposition 4.6.5. *The representation of the x-action on $\mathbf{Z}_- \otimes P$ is given by*

$$(\rho_{\mathbf{Z}_-\otimes P}(x^{-1}))_{ij} = \begin{cases} 1, & j = i-1,\ 2 \le i \le 2n; \\ -1, & j = 2n,\ 1 \le i \le 2n; \\ 0, & o/w. \end{cases}$$

Proof. Write $e_i = \pi x^i$, where $1 \le i \le 2n$. Then $e_i \cdot x = e_{i+1}$ for $1 \le i \le 2n-1$, and $e_{2n} \cdot x = \pi$. Since Σ_x is central we note $\pi\Sigma_x = 0$. It follows that $\pi = -\pi x - \pi x^2 - \cdots - \pi x^{2n} = -\sum_{i=1}^{2n} e_i$. As x acts trivially on $\mathbf{Z}_-$, it therefore follows that

$$\rho_{\mathbf{Z}_-\otimes P}(x^{-1}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

as required. ☐

In what follows, it will be convenient to use the following form for $\rho_{\mathbf{Z}_-\otimes P}(x^{-1})$:

Proposition 4.6.6.

$$\rho_{\mathbf{Z}_-\otimes P}(x^{-1}) = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$$

where P_1, P_2, P_3, P_4 are each $n \times n$ blocks such that

$$(P_1)_{ij} = \begin{cases} 1, & j = i-1,\ 2 \le i \le n; \\ 0, & o/w; \end{cases} \qquad (P_2)_{ij} = \begin{cases} -1, & j = n,\ 1 \le i \le n; \\ 0, & o/w; \end{cases}$$

$$(P_3)_{ij} = \begin{cases} 1, & i = 1,\ j = n; \\ 0, & o/w; \end{cases} \qquad (P_4)_{ij} = \begin{cases} 1, & j = i-1,\ 2 \le i \le n; \\ -1, & j = n,\ 1 \le i \le n; \\ 0, & o/w. \end{cases}$$

Proof. Both P_1, P_4 follow in a straightforward manner by simply restricting i, j to the intervals $1 \le i, j \le n$ and $n+1 \le i, j \le 2n$, respectively. For P_2, P_3 it is worth saying a bit more. For the former, we restrict ourselves to the interval $1 \le i \le n$ and $n+1 \le j \le 2n$. By Proposition 4.6.5, we know we have a non-zero entry in this $n \times n$ block only when $j = 2n$ and $1 \le i \le n$, or when $j = i-1$ and $2 \le i \le n$. In the first instance, this gives the -1 as required. In the latter, we note that when i is at its largest (i.e. $i = n$), then $j = n-1$, which is not in the interval we are considering. As such, P_2 only has non-zero entries when $j = 2n$ and $1 \le i \le n$.

Similarly for P_3, we now restrict ourselves to the intervals $n+1 \le i \le 2n$ and $1 \le j \le n$. As before, Proposition 4.6.5 tells us any non-zero entry occurs when

$j = i - 1$ and $n + 1 \leq i \leq 2n$. Now, when i is at its smallest (i.e. $i = n + 1$) then $j = n$ and is within our block of interest. Beyond that however j must be in the interval $n + 1 \leq j \leq 2n$ and therefore of no interest to us for P_3. ☐

Proposition 4.6.7. *The representation of the* y*-action of* $\mathbf{Z}_- \otimes P$ *is given by,*

$$(\rho_{\mathbf{Z}_- \otimes P}(y))_{ij} = \begin{cases} 1, & j = 2n + 1 - i,\ 1 \leq i \leq 2n; \\ 0, & o/w. \end{cases}$$

Proof. Consider the y-action on a general basis element of P,

$$e_i y = \pi x^i y = (x^n - 1)(y - 1)yx^{2n+1-i} = -\pi x^{2n+1-i} = -e_{2n+1-i}$$

By Corollary 4.6.4 we therefore have the desired representation,

$$\rho_P(y) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & -1 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix} = -\rho_{\mathbf{Z}_- \otimes P}(y). \tag{4.6.8}$$

☐

Now repeat the process for the x, y-actions on R.

Proposition 4.6.9. *The representation of the* x*-action on* R *is given by,*

$$(\rho_R(x^{-1}))_{ij} = \begin{cases} 1, & j = i - 1,\ 2 \leq i \leq 2n; \\ -1, & i = 1,\ 1 \leq j \leq 2n; \\ 0, & o/w. \end{cases}$$

Proof. Set $f_i = (y - 1)(x^i - 1)$, where $1 \leq i \leq 2n$. Then,

$$f_i \cdot x = (y - 1)(x^i - 1)x = (y - 1)(x^{i+1} - 1) - (y - 1)(x - 1) = f_{i+1} - f_1$$

for $1 \leq i \leq 2n - 1$, and $f_{2n} \cdot x = -f_1$. The result now follows; that is,

$$\rho_R(x^{-1}) = \begin{pmatrix} -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

☐

A similar proof to that of Proposition 4.6.6 shows:

Proposition 4.6.10.

$$\rho_R(x^{-1}) = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}$$

where R_1, R_2, R_3, R_4 *are each* $n \times n$ *blocks such that*

$$(R_1)_{ij} = \begin{cases} 1, & j = i-1,\, 2 \le i \le n; \\ -1, & i = 1,\, 1 \le j \le n; \\ 0, & o/w; \end{cases} \qquad (R_2)_{ij} = \begin{cases} -1, & i = 1,\, 1 \le j \le n; \\ 0, & o/w; \end{cases}$$

$$(R_3)_{ij} = \begin{cases} 1, & i = 1,\, j = n; \\ 0, & o/w; \end{cases} \qquad (R_4)_{ij} = \begin{cases} 1, & j = i-1,\, 2 \le i \le n; \\ 0, & o/w. \end{cases}$$

Proposition 4.6.11. *The representation of the* y*-action on* R *is given by,*

$$\rho_R(y) = \begin{cases} -1, & j = 2n+1-i; \\ 0, & o/w. \end{cases}$$

In particular, $\rho_R(y) = -\rho_{\mathbf{Z}_- \otimes P}(y)$.

Proof. As before, apply y to a general f_i to yield,

$$f_i y = (y-1)(x^i - 1)y = (y-1)y(x^{2n+1-i} - 1) = -f_{2n+1-i}.$$

The result now follows. □

To show the equivalence of $\mathbf{Z}_- \otimes P$ and R, we define the following $(2n) \times (2n)$ matrix X,

$$X = \begin{pmatrix} 0 & -\alpha^T \\ \alpha & 0 \end{pmatrix} \tag{4.6.12}$$

where

$$\alpha_{ij} = \begin{cases} 1, & j \ge i; \\ 0, & j < i. \end{cases}$$

Proposition 4.6.13. *With* X *as defined in* (4.6.12), $\rho_{\mathbf{Z}_- \otimes P}(x^{-1})X = X\rho_R(x^{-1})$.

Proof. Using Propositions 4.6.6 and 4.6.10,

$$\rho_{\mathbf{Z}_- \otimes P}(x^{-1})X = \begin{pmatrix} P_2\alpha & -P_1\alpha^T \\ P_4\alpha & -P_3\alpha^T \end{pmatrix}, \text{ and } X\rho_R(x^{-1}) = \begin{pmatrix} -\alpha^T R_3 & -\alpha^T R_4 \\ \alpha R_1 & \alpha R_2 \end{pmatrix}.$$

We therefore have four calculations to check. First we show $P_2\alpha = -\alpha^T R_3$. Now, $(P_2\alpha)_{ik} = \sum_{j=1}^{n} (P_2)_{ij}\alpha_{jk} \neq 0$ only if $k \ge j$ and $j = n$, $1 \le i \le n$. In other words, when $k = n$ and $1 \le i \le n$ then $(P_2\alpha)_{i,n} = -1$. For any other entry we get 0.

Now, $(\alpha^T R_3)_{ik} = \sum_{j=1}^{n} \alpha_{ji}(R_3)_{jk} \neq 0$ when $i \ge j$ and $j = 1$, $k = n$. So, when $k = n$ and $i \ge 1$ $(\alpha^T R_3)_{i,n} = 1$ and zero otherwise. Thus, $P_2\alpha = -\alpha^T R_3$ as required.

Next, we show $P_1\alpha^T = \alpha^T R_4$ by first observing $(P_1\alpha^T)_{ik} = \sum_{j=1}^{n} (P_1)_{ij}\alpha_{kj} \neq 0$ when $j \ge k$, $j = i-1$ and $2 \le i \le n$. Putting this together, we see that for $i > k$ where $2 \le i \le n$ and $1 \le k \le n-1$ we have $(P_1\alpha^T)_{ik} = 1$, and zero otherwise.

Likewise, $(\alpha^T R_4)_{ik} = \sum_{j=1}^{n} \alpha_{ji}(R_4)_{jk} \neq 0$ when $i \ge j$, $k = j-1$ and $2 \le j \le n$. So when $i > k$, $1 \le k \le n-1$ and $2 \le i \le n$ we have $(\alpha^T R_4)_{ik} = 1$, and zero otherwise. In other words, we have the desired equality.

For $P_4\alpha = \alpha R_1$ note $(P_4\alpha)_{ik} = \sum_{j=1}^{n} (P_4)_{ij}\alpha_{jk} \neq 0$ when either:

- $k \geq j$, $j = i - 1$ and $2 \leq i \leq n$, in which case we have $+1$; or when
- $k \geq j$, $j = n$ and $1 \leq i \leq n$ when we have -1.

Putting the above two cases together we find that when $k \geq i-1$ and $2 \leq i \leq n$, then $(P_4\alpha)_{ik} = 1$. However if $k = n$, then $(P_4\alpha)_{i,n} \neq 0$ only when $i = 1$. This is due to a cancellation occurring from the $+1$ and -1 whenever $i \geq 2$. Putting this together, we have

$$(P_4\alpha)_{ik} = \begin{cases} 1, & i-1 \leq k \leq n-1,\ 2 \leq i \leq n; \\ -1, & i = 1,\ k = n; \\ 0, & o/w. \end{cases}$$

Adopting a similar approach shows $(\alpha R_1)_{ik} = \sum_{j=1}^{n} \alpha_{ij}(R_1)_{jk} \neq 0$ when either:

- $j \geq i$, $k = j - 1$ and $2 \leq j \leq n$, in which case we have $+1$; or when
- $j \geq i$, $j = 1$ and $1 \leq k \leq n$ when we have -1.

By putting this together we once more get a cancellation between the $+1$, -1 when $i = 1$ and $k < n$, and so for $i = 1$, $k = n$ we get a value of -1. When $2 \leq i \leq n$ and $i - 1 \leq k \leq n - 1$, we get a value of 1. Thus, we conclude $P_4\alpha = \alpha R_1$.

Finally, we show $-P_3\alpha^T = \alpha R_2$. First, $(-P_3\alpha^T)_{ik} = \sum_{j=1}^{n}(-P_3)_{ij}\alpha_{kj} \neq 0$ only if $j \geq k$, $i = 1$ and $j = n$. In other words, we get a value of -1 when $i = 1$, and zero otherwise. Likewise, $(\alpha R_2)_{ik} = \sum_{j=1}^{n} \alpha_{ij}(R_2)_{jk} \neq 0$ only when $j \geq i$, $j = 1$ and $1 \leq k \leq n$, i.e. when $i = 1$. In this case we get -1, and zero otherwise. ☐

Proposition 4.6.14. *With X as defined in (4.6.12), $\rho_{\mathbf{Z}_-\otimes P}(y)X = X\rho_R(y)$.*

Proof. Now, it is quite clear that $\rho_{\mathbf{Z}_-\otimes P}(y)$ may be written in the form

$$\rho_{\mathbf{Z}_-\otimes P}(y) = \begin{pmatrix} 0 & P_0 \\ P_0 & 0 \end{pmatrix}$$

where P_0 is an $n \times n$ block such that

$$(P_0)_{ij} = \begin{cases} 1,\ j = n + 1 - i,\ 1 \leq i \leq n; \\ 0,\ o/w. \end{cases}$$

It is then also clear that

$$\rho_R(y) = \begin{pmatrix} 0 & -P_0 \\ -P_0 & 0 \end{pmatrix}.$$

As such, it is quite clear that

$$\rho_{\mathbf{Z}_-\otimes P}(y)X = \begin{pmatrix} P_0\alpha & 0 \\ 0 & -P_0\alpha^T \end{pmatrix}, \text{ and } X\rho_R(y) = \begin{pmatrix} \alpha^T P_0 & 0 \\ 0 & -\alpha P_0 \end{pmatrix}.$$

It remains to show $\alpha P_0 = P_0\alpha^T$ and $\alpha^T P_0 = P_0\alpha$. Observe $(\alpha P_0)_{ik} = \sum_{j=1}^{n} \alpha_{ij}(P_0)_{jk}$ which is non-zero when $j \geq i$ and $k = n+1-j$; that is, when $k \leq n+1-i$. In this case we get a value of $+1$, and zero otherwise. Likewise, $(P_0\alpha^T)_{ik} = \sum_{j=1}^{n}(P_0)_{ij}\alpha_{kj} \neq 0$ when $j = n + 1 - i$ and $j \geq k$, i.e. when $n + 1 - i \geq k$. Thus, $\alpha P_0 = P_0\alpha^T$. A similar proof shows $\alpha^T P_0 = P_0\alpha$. ☐

Proposition 4.6.15. $\mathbf{Z}_{-} \otimes P \cong R$.

Proof. By Propositions 4.6.13 and 4.6.14, it remains to show X is invertible over $\mathbf{Z}$. Define the $n \times n$ matrix β by

$$\beta_{ij} = \begin{cases} 1, & i = j; \\ -1, & j = i + 1; \\ 0, & o/w. \end{cases}$$

Observe $(\alpha\beta)_{ik} = \sum_{j=1}^{n} \alpha_{ij}\beta_{jk} = \sum_{j \geq i}^{n} \beta_{jk}$. Suppose now that $i = k$, then $(\alpha\beta)_{ii} = 1$ since $\beta_{ji} = 0$ for any $j \neq i$ in the range $i \leq j \leq n$. If $i \neq k$ then there are two cases to consider. If $k < i$ then $(\alpha\beta)_{ik} = 0$ since $\beta_{jk} = 0$ for any $k < i \leq j \leq n$. If $k > i$, then $(\alpha\beta)_{ik} = \beta_{k-1,k} + \beta_{kk} = -1 + 1 = 0$. Thus, $\alpha\beta = I_n$. A similar argument now shows $\beta\alpha = I_n$, i.e. $\alpha^{-1} = \beta$. Finally, we define the matrix

$$Y = \begin{pmatrix} 0 & \beta \\ -\beta^T & 0 \end{pmatrix},$$

and a straightforward calculation shows $XY = I_{2n} = YX$. In other words, X is invertible over $\mathbf{Z}$, as required. □

Next, we show $L_0 \otimes P$ is free. In particular, by counting $\mathbf{Z}$-ranks, it is necessarily free of rank n.

Proposition 4.6.16. $L_0 \otimes P \cong \Lambda^n$.

Proof. Consider $j_*(\mathbf{Z}) \cong L_0$. By Proposition 4.4.3, $j^*(P) \cong \mathbf{Z}[C_2]^n$. So, by using Frobenius reciprocity we have:

$$\begin{aligned} L_0 \otimes P \cong & \quad j_*(\mathbf{Z}) \otimes P \\ \cong & \quad j_*(\mathbf{Z} \otimes \mathbf{Z}[C_2]^n) \\ \cong & \quad j_*(\mathbf{Z}[C_2]^n) \\ \cong & \quad \Lambda^n \end{aligned}$$

□

From Propositions 4.6.15 and 4.6.16 we can now rewrite (4.6.2) as,

$$0 \to \Lambda^n \to L \otimes P \to R \to 0.$$

Dualising yields a split short exact sequence and so $(L \otimes P)^* \cong P \oplus \Lambda^n$. By dualising once again Proposition 2.1.1 yields the desired conclusion; that is,

Proposition 4.6.17. $L \otimes P \cong R \oplus \Lambda^n$.

As with the end of Section 4.5 we explicitly calculate the tensor product at the minimal level of $\Omega_2(\mathbf{Z}) \otimes \Omega_1(\mathbf{Z})$. We have,

$$(L \oplus [y+1)) \otimes (P \oplus [y-1)) \cong (L \otimes P) \oplus (L \otimes [y-1)) \oplus ([y+1) \otimes P) \oplus ([y+1) \otimes [y-1)).$$

By Proposition 4.6.17 we know $L \otimes P \cong R \oplus \Lambda^n$. Using Frobenius Reciprocity,

$$\begin{aligned} [y+1) \otimes P \cong & \; j_*(\mathbf{Z} \otimes j^*(P)) \\ \cong & \; j_*(\mathbf{Z}[C_2]^n) \\ \cong & \; \Lambda^n. \end{aligned}$$

Recall $j^*(L) \cong \mathbf{Z}[C_2]^{n+1}$ from Proposition 4.4.6. Using this and Frobenius reciprocity,

$$\begin{aligned} L \otimes [y-1) \cong & \; j_*(\mathbf{Z}_- \otimes j^*(L)) \\ \cong & \; j_*(\mathbf{Z}[C_2]^{n+1}) \\ \cong & \; \Lambda^{n+1}. \end{aligned}$$

Finally, we come to $[y+1) \otimes [y-1)$ and proceed much as before. Applying Frobenius Reciprocity and Proposition 4.5.24, we get

$$\begin{aligned} [y+1) \otimes [y-1) \cong & \; j_*(\mathbf{Z} \otimes (I_2 \oplus \mathbf{Z}[C_2]^n)) \\ \cong & \; j_*(I_2 \oplus \mathbf{Z}[C_2]^n) \\ \cong & \; j_*(I_2) \oplus j_*(\mathbf{Z}[C_2]^n) \\ \cong & \; [y-1) \oplus \Lambda^n. \end{aligned}$$

Putting this all together, we finally get the desired isomorphism:

$$(L \oplus [y+1)) \otimes (P \oplus [y-1)) \cong R \oplus [y-1) \oplus \Lambda^{4n+1}.$$

4.7 $R \otimes P \sim K$

To conclude the proof of Theorem B, we construct the isomorphism,

$$R \otimes P \cong K \oplus \Lambda^{n-1}. \tag{4.7.1}$$

As before, the number of copies of Λ is justified by counting $\mathbf{Z}$-ranks of $R \otimes P$. It is clear that we must add on $n-1$ copies of Λ to K.

Proposition 4.7.2. *$R \otimes P \cong (\mathbf{Z}_- \otimes L) \oplus \Lambda^{n-1}$.*

Proof. In the previous section we showed $R \cong \mathbf{Z}_- \otimes P$. Using Proposition 4.5.1, we have

$$\begin{aligned} R \otimes P \cong & \; (\mathbf{Z}_- \otimes P) \otimes P \\ \cong & \; \mathbf{Z}_- \otimes (L \oplus \Lambda^{n-1}) \\ \cong & \; (\mathbf{Z}_- \otimes L) \oplus (\mathbf{Z}_- \otimes \Lambda^{n-1}) \;\cong\; (\mathbf{Z}_- \otimes L) \oplus \Lambda^{n-1}. \end{aligned}$$

□

It remains to show $\mathbf{Z}_- \otimes L \cong K$. To this end, recall L has a $\mathbf{Z}$-basis,

$$\{(y+1),\, (y+1)x, \ldots,\, (y+1)x^{2n},\, \Sigma_x\}.$$

and K has $\mathbf{Z}$-basis,

$$\{(y-1),\,(y-1)x,\ldots,\,(y-1)x^{2n},\,\Sigma_x\}.$$

Proposition 4.7.3. *Define the* $(2n+1)\times(2n+1)$ *matrix* Ψ *as before; that is,*

$$\Psi_{ij}=\begin{cases}1, & i=1,\, j=2n+1;\\ 1, & j=i-1,\, 2\le i\le 2n+1;\\ 0, & o/w.\end{cases}$$

Then $\rho_{\mathbf{Z}_-\otimes L}(x^{-1})=\begin{pmatrix}\Psi & 0_{(2n+1)\times 1}\\ 0_{1\times(2n+1)} & 1\end{pmatrix}$.

Proof. Set $E_i=(y+1)x^{i-1}$ for $1\le i\le 2n+1$, and $E_{2n+2}=\Sigma_x$. Clearly, $E_i\cdot x=E_{i+1}$ for $1\le i\le 2n$ and $E_{2n+1}\cdot x=E_1$. Finally, $E_{2n+2}\cdot x=E_{2n+2}$. The result now follows since x acts trivially on $\mathbf{Z}_-$, i.e $\rho_{\mathbf{Z}_-}(x^{-1})=1$. □

Proposition 4.7.4. *Define the* $(2n+1)\times(2n+1)$ *matrix* Φ *as before,*

$$\Phi_{ij}=\begin{cases}1, & i=j=1;\\ 1, & j=2n+3-i,\, 2\le i\le 2n+1;\\ 0, & o/w.\end{cases}$$

Then $\rho_{\mathbf{Z}_-\otimes L}(y)=\begin{pmatrix}-\Phi & -1_{(2n+1)\times 1}\\ 0_{1\times(2n+1)} & 1\end{pmatrix}$.

Proof. With E_i as above, consider how y acts on the basis elements of L. First observe,

$$E_1+E_2+\cdots+E_{2n+1}=\Sigma_x y+E_{2n+2}.$$

Now, $E_1y=E_1$ and for a general E_i where $2\le i\le 2n+1$,

$$\begin{aligned}E_iy=&\ (y+1)x^{i-1}y=(y+1)yx^{2n+2-i} &&=E_{2n+3-i}\\ E_{2n+2}y=&\ y+xy+\cdots+x^{2n}y &&=E_1+E_2+\cdots+E_{2n+1}-E_{2n+2}.\end{aligned}$$

Finally, apply Corollary 4.6.4 so that

$$\rho_L(y)=\begin{pmatrix}1&0&0&\cdots&0&0&1\\ 0&0&0&\cdots&0&1&1\\ 0&0&0&\cdots&1&0&1\\ \vdots&\vdots&\vdots&&\vdots&\vdots&\vdots\\ 0&0&1&\cdots&0&0&1\\ 0&1&0&\cdots&0&0&1\\ 0&0&0&\cdots&0&0&-1\end{pmatrix}=-\rho_{\mathbf{Z}_-\otimes L}(y) \qquad (4.7.5)$$

□

Next, we do the same for K.

Proposition 4.7.6. *With* Ψ *as above,* $\rho_K(x^{-1})=\begin{pmatrix}\Psi & 0_{(2n+1)\times 1}\\ 0_{1\times(2n+1)} & 1\end{pmatrix}$.

Proof. Set $F_i = (y-1)x^{i-1}$ for $1 \leq i \leq 2n+1$, and $F_{2n+2} = \Sigma_x$. As with Proposition 4.7.3, $F_i \cdot x = F_{i+1}$ for $1 \leq i \leq 2n$, $F_{2n+1} \cdot x = F_1$ and $F_{2n+2} \cdot x = F_{2n+2}$. ☐

Proposition 4.7.7. *With* Φ *as above,* $\rho_K(y) = \begin{pmatrix} -\Phi & 1_{(2n+1)\times 1} \\ 0_{1\times(2n+1)} & 1 \end{pmatrix}$.

Proof. First observe

$$F_1 + F_2 + \cdots + F_{2n+1} = \Sigma_x y - F_{2n+1}.$$

Now, $F_1 \cdot y = -F_1$, and y acts on the other F_i as follows,

$$F_i y = (y-1)x^{i-1}y = (y-1)yx^{2n+2-i} = -F_{2n+3-i}.$$

Finally

$$F_{2n+2} \cdot y = y + xy + \cdots + x^{2n}y = F_1 + \cdots + F_{2n+2}.$$

The representation of the y-action for K is therefore

$$\rho_K(y) = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & \cdots & 0 & 0 & 1 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}. \tag{4.7.8}$$

☐

Proposition 4.7.9. $\rho_{\mathbf{Z}_-\otimes L}(g)$ *and* $\rho_K(g)$ *are equivalent for all* $g \in D_{4n+2}$.

Proof. Define

$$X_{ij} = \begin{cases} 1, & i = j,\ 1 \leq i, j \leq 2n+1; \\ -1, & i = j = 2n+2; \\ 0, & o/w \end{cases}$$

i.e. $X = \begin{pmatrix} I_{2n+1} & 0_{(2n+1)\times 1} \\ 0_{1\times(2n+1)} & -1 \end{pmatrix}$. We claim $X\rho_k(g) = \rho_{\mathbf{Z}_-\otimes L}(g)X$ for all $g \in D_{4n+2}$.

First, it is straightforward to see the sizes of each block 'match up' and,

$$\rho_{\mathbf{Z}_-\otimes L}(x^{-1})X = \begin{pmatrix} \Psi & 0_{(2n+1)\times 1} \\ 0_{1\times(2n+1)} & -1 \end{pmatrix} = X\rho_K(x^{-1}).$$

Similarly,

$$\rho_{\mathbf{Z}_-\otimes L}(y)X = \begin{pmatrix} -\Phi & 1_{(2n+1)\times 1} \\ 0_{1\times(2n+1)} & -1 \end{pmatrix} = X\rho_K(y).$$

Since X is clearly invertible, this completes the proof. ☐

Corollary 4.7.10. $\mathbf{Z}_- \otimes L \cong K$.

Proposition 4.7.2 and Corollary 4.7.10 therefore imply:

Proposition 4.7.11. $R \otimes P \cong K \oplus \Lambda^{n-1}$.

Evidently, by combining the main results of Sections 4.5 - 4.7, we have proven Theorem B. Finally, we explicitly demonstrate $\Omega_3(\mathbf{Z}) \otimes \Omega_1(\mathbf{Z}) = \Omega_4(\mathbf{Z})$. We proceed as before and observe

$$(R \oplus [y-1)) \otimes (P \oplus [y-1)) \cong (R \otimes P) \oplus (R \otimes [y-1)) \oplus ([y-1) \otimes P) \oplus ([y-1) \otimes [y-1)).$$

From Proposition 4.7.11, as well as Sections 4.5 and 4.6 we have $R \otimes P \cong K \oplus \Lambda^{n-1}$, $[y-1) \otimes [y-1) \cong [y+1) \oplus \Lambda^n$ and $[y-1) \otimes P \cong \Lambda^n$. As $P^* \cong R$ and $[y-1)$ is self-dual, it also follows that $[y-1) \otimes R \cong \Lambda^n$. Putting this all together, we finally get

$$(R \oplus [y-1)) \otimes (P \oplus [y-1)) \cong K \oplus [y+1) \oplus \Lambda^{4n-1}$$

as required.

4.8 A diagonal resolution for $\mathbf{Z}[D_{4n+2}]$

Using the results of Sections 4.4 - 4.7, we may now construct a free resolution providing an affirmative answer to the sequencing conjecture discussed in the overview of Section 1. The author stresses that, unlike [24], this sequence is not explicit. Nevertheless, this does suggest a method which allows the construction of such resolutions in the case of p, q prime where $q|p-1$ (provided q is given). This shall be particularly useful in the next chapter where we consider the case when $q = 3$. The benefit is that for $q = 3$ no explicit diagonal resolution has been found. However, it should be noted that beyond $q = 3$, the calculations quickly become unmanageable by hand and so a general treatment of $G(p, q)$ is likely to depend upon a different method.

Proposition 4.8.1. *There exists an exact sequence,*

$$\begin{tikzcd}
 & & & K \arrow[dr] & & & & L \arrow[dr] & & & \\
0 \arrow[r] & P \arrow[r] & \Lambda \arrow[ur] \arrow[rr] & & \Lambda \arrow[rr] \arrow[dr] & & \Lambda \arrow[rr] \arrow[ur] & & \Lambda \arrow[r] & P \arrow[r] & 0 \\
 & & & & & R \arrow[ur] & & & & &
\end{tikzcd} \tag{4.8.2}$$

Proof. Recall that we have the exact sequence,

$$0 \to L \to \Lambda \to P \to 0. \tag{4.8.3}$$

By applying the exact functor $- \otimes P$ we get,

$$0 \to L \otimes P \to \Lambda^{2n} \to P \otimes P \to 0$$

which, by Propositions 4.5.21 and 4.6.17, becomes

$$0 \to R \oplus \Lambda^n \xrightarrow{j} \Lambda^{2n} \to L \oplus \Lambda^{n-1} \to 0.$$

Evidently, this can be transformed into the exact sequence,

$$0 \to R \to \Lambda^{2n}/j(\Lambda^n) \to L \oplus \Lambda^{n-1} \to 0.$$

Observe that we have the following exact sequence,

$$0 \to \Lambda^n \xrightarrow{j} \Lambda^{2n} \to \Lambda^{2n}/j(\Lambda^n) \to 0.$$

By Johnson's 'destabilization theorem' (Proposition 2.4.2), $\Lambda^{2n}/j(\Lambda^n)$ is projective and hence the above exact sequence splits. It follows that $\Lambda^{2n}/j(\Lambda^n)$ is stably free of rank n, and hence free by Swan-Jacobinski. Thus, we have the exact sequence,

$$0 \to R \to \Lambda^n \to L \oplus \Lambda^{n-1} \to 0.$$

By dualising and once again applying Proposition 2.4.2 we 'remove' the Λ^{n-1} from the right hand module. Dualising a final time and our exact sequence thereby becomes,

$$0 \to R \to \Lambda \to L \to 0 \tag{4.8.4}$$

to which we once more apply the functor $- \otimes P$. This yields,

$$0 \to K \oplus \Lambda^{n-1} \to \Lambda^{2n} \to R \oplus \Lambda^n \to 0$$

by using Propositions 4.6.17 and 4.7.11. Once again, we use two applications of Proposition 2.4.2 to 'remove' the 'extra' free modules, thereby producing the following exact sequence,

$$0 \to K \to \Lambda \to R \to 0. \tag{4.8.5}$$

Applying the functor $- \otimes P$ one final time yields the exact sequence,

$$0 \to P \oplus \Lambda^n \to \Lambda^{2n} \to K \oplus \Lambda^{n-1} \to 0.$$

Here we have used Propositions 4.4.4 and 4.7.11. As before, we use two applications of Proposition 2.4.2 to remove the 'extra' free module. This yields,

$$0 \to P \to \Lambda \to K \to 0. \tag{4.8.6}$$

It is now straightforward to splice the sequences (4.8.3) - (4.8.6) to construct the desired sequence. ☐

Recall the resolution of the y-strand given in (4.0.3). By combining this with the resolution of (4.8.1) yields the desired resolution:

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\epsilon^*} \Lambda \xrightarrow{\begin{pmatrix} \partial_3^+ \\ y-1 \end{pmatrix}} \Lambda \oplus \Lambda \xrightarrow{\begin{pmatrix} \partial_2^+ & 0 \\ 0 & y+1 \end{pmatrix}} \Lambda \oplus \Lambda \xrightarrow{(\partial_1^+, \ \ y-1)} \Lambda \xrightarrow{\epsilon} \mathbf{Z} \longrightarrow 0. \tag{4.8.7}$$

Chapter 5

The syzygies of $G(p, 3)$

For this chapter let $G = G(p, 3) = C_p \rtimes C_3$ where p is a prime such that $3|p-1$. In this way C_3 acts on C_p via the natural embedding $C_3 \hookrightarrow Aut(C_p)$ and we write

$$C_p \rtimes C_3 =< x, y \mid x^p = y^3, yx = \theta(x)y >$$

where $\theta \in Aut(C_p)$. Throughout this chapter, we denote the integral group ring of G by $\Lambda = \mathbf{Z}[G(p, 3)]$, and write $d = (p-1)/3$.

The aim of this chapter shall be to mirror the techniques of Chapter 4. In particular, we construct a resolution of the form:

$$\begin{tikzcd}
 & & & & & K \arrow[dr] & & & & L \arrow[dr] & & & & L^* \arrow[dr] & & & & & \\
0 \arrow[rr] & & \bar{I}_C \arrow[rr] & & \Lambda \arrow[rr] \arrow[ur] & & \Lambda \arrow[rr] \arrow[dr] & & \Lambda \arrow[rr] \arrow[ur] & & \Lambda \arrow[rr] \arrow[dr] & & \Lambda \arrow[rr] \arrow[ur] & & \Lambda \arrow[rr] & & \bar{I}_C \arrow[rr] & & 0 \\
 & & & & & & & \bar{I}_C^* \arrow[ur] & & & & \overline{(x-1)I_C} \arrow[ur] & & & & & & &
\end{tikzcd}$$

Here, we have once again used the bar notation (such as for $\bar{I}_C$) to represent the Galois module obtained from the action of C_3 (for example, the action of C_3 on I_C). We shall define this formally for $q = 3$ in the next section. To construct the above resolution we prove Theorems D, E, F and G of Section 1.2; that is,

Theorem D: $K \otimes ? \cong ? \oplus \Lambda^r$ for some $r \geq 0$.

Theorem E: K is stably self-dual.

Theorem F: With K, L, L^* as above, we have:

1. $\bar{I}_C \otimes \bar{I}_C \cong L^* \oplus \Lambda^{d-1}$;
2. $\bar{I}_C \otimes L^* \cong \overline{(x-1)I_C} \oplus \Lambda^{2d}$;
3. $\bar{I}_C \otimes \overline{(x-1)I_C} \cong L \oplus \Lambda^{d-1}$;
4. $\bar{I}_C \otimes L \cong \bar{I}_C^* \oplus \Lambda^{2d}$;
5. $\bar{I}_C \otimes \bar{I}_C^* \cong K \oplus \Lambda^{d-1}$.

Whereas the construction of the exact sequence above provides an affirmative answer to the sequencing conjecture, this suffers from the same shortcomings of the previous chapter. Namely, this exact sequence is not explicitly calculated. The missing ingredient is actually to find the specific polynomials that generate the above (monogenic) modules. Nevertheless, this still represents significant progress in this area.

5.1 The modules $R(1)$, $R(2)$ and $R(3)$

We have the following subring of $M_3(A)$ consisting of quasi-triangular matrices

$$\mathcal{T}_3(A,\, \pi) = \{X \in M_3(A) \mid x_{ij} \in (\pi) \text{ if } i > j\}$$

where $A = \mathbf{Z}[\zeta_p]^\theta$ is the subring of $\mathbf{Z}[\zeta_p]$ fixed by θ, and $\pi = (\zeta_p - 1)^3$. As we shall see in Part II of this thesis (see example 6.2.12), $\mathcal{T}_3(A,\, \pi)$ occurs in the following fibre square:

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d] & \mathcal{T}_3(A,\, \pi) \arrow[d] \\ \mathbf{Z}[C_3] \arrow[r] & \mathbf{F}_p[C_3] \end{tikzcd}$$

In particular, if $R(i)$ is the i^{th} row of $\mathcal{T}_3(A,\, \pi)$ considered as a right Λ-module, then we have the following decomposition of $\mathcal{T}_3(A,\, \pi)$,

$$\mathcal{T}_3(A,\, \pi) \cong R(1) \oplus R(2) \oplus R(3).$$

To see this explicitly, we can write

$$R(1) = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;,\; a_i \in A \right\}$$

$$R(2) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \pi a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{pmatrix} \;,\; a_i \in A \right\}$$

$$R(3) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \pi a_1 & \pi a_2 & a_3 \end{pmatrix} \;,\; a_i \in A \right\}.$$

Our initial goal here is to provide a more suitable description for the modules $R(i)$. Recall once more that $Aut(C_p) \cong C_{p-1}$. Since p is chosen such that $3|p-1$ we define $d = (p-1)/3$. We now choose some $\theta \in Aut(C_p)$ such that $ord(\theta) = 3$. Provided we are consistent, the actual choice of θ matters little. Thus, we choose once and for all a θ such that $\theta(x) = x^\alpha$ where $\alpha^3 \equiv 1\ (mod\, p)$, and can therefore legitimately describe $G(p,\, 3)$ as,

$$G = G(p,\, 3) = C_p \rtimes C_3 =< x,\, y \mid x^p = y^3,\, yx = x^\alpha y > .$$

Henceforth, denote $\Lambda = \mathbf{Z}[G(p,\, 3)]$. There are natural inclusions $i : \mathbf{Z}[C_p] \hookrightarrow \Lambda$ and $j : \mathbf{Z}[C_3] \hookrightarrow \Lambda$. As we frequently restrict scalars to those over $\mathbf{Z}[C_p]$ we shall, for the sake of ease, denote $\Lambda_0 = \mathbf{Z}[C_p]$. Additionally, denote the augmentation ideal of Λ_0 by I_C, and that of $\mathbf{Z}[C_3]$ by I_3. Throughout, we denote the trivial module over any group ring by $\mathbf{Z}$.

Once again, we transform a Λ_0-lattice M into a module over Λ via a Galois action. To do so first define a *Galois structure* on M to be an additive automorphism $\Theta : M \to M$ such that $\Theta(m \cdot x) = \Theta(m) \cdot \theta(x)$ for all $m \in M$ and such that $\Theta^3 = Id_M$. A *Galois lattice* shall then mean a pair (M, Θ) where M is a lattice over Λ_0 and Θ is a Galois structure on M. We then make a Galois lattice (M, Θ) into a (right) Λ-lattice via the action,

$$m \cdot x^a = mx^a;$$

$$m \cdot y^b = \Theta^{-b}(m), \quad (\Theta^b(m) = my^{-b}).$$

For us, the significant examples of Galois lattices arise from the ideals of Λ_0 which are invariant under θ. If $J \subset \Lambda_0$ is such an ideal then we put $\bar{J} = (J, \Theta_J)$ where Θ_J is simply the restriction of θ to J. As such, we obtain the following four modules of interest

(i) $\bar{\Lambda}_0$;

(ii) $\bar{I_C}$;

(iii) $\overline{(x-1)I_C}$;

(iv) $\bar{I}_C^*$.

It is straightforward to show:

$$\bar{\Lambda}_0 \cong [\Sigma_y) = span_{\mathbf{Z}}\{\Sigma_y x^i \mid 0 \leq i \leq p-1\}. \tag{5.1.1}$$

Of decidedly more interest, however, is the question of what the modules (ii)-(iv) represent. First recall our definition of basis elements ν^i for $1 \leq i \leq p-1$, as defined in Chapter 4. For (iv) we note y acts on ν^i by $y \cdot \nu^i = \nu^{\theta_*(i)}$. As we are primarily interested in right actions, we make the usual alteration via the standard involution to give $\nu^i \cdot y = \nu^{\theta_*^{-1}(i)} = \nu^{\theta_*^2(i)}$. As it is often unclear when a module is isomorphic to $\bar{I}_C^*$, we note the following criteria (see [25]) that characterise the module $\bar{I}_C^*$ amongst Λ-modules.

Proposition 5.1.2. *Let M be a Λ-lattice and suppose that the following three conditions are satisfied:*

(I) there exists $\mu \in M$ such that $\mu \cdot y = \mu$ and $M = span_{\mathbf{Z}}\{\mu \cdot x^r \mid 0 \leq r \leq p-1\}$;

(II) $rk_{\mathbf{Z}}(M) = p-1$;

(III) $m \cdot \Sigma_x = 0$ for each $m \in M$.

Then $M \cong_\Lambda \bar{I}_C^*$ and $\{\mu \cdot x^r \mid 0 \leq r \leq p-2\}$ is a $\mathbf{Z}$-basis for M.

Returning to the discussion of the modules $R(i)$, we first note that these are pairwise isomorphically distinct (see [41] or [25]); that is:

$$R(i) \cong_\Lambda R(j) \text{ if and only if } i = j. \tag{5.1.3}$$

Moreover, we note the following duality relation:

$$R(i)^* \cong R(4-i). \tag{5.1.4}$$

Now that we have an understanding of how the $R(i)$ interact under duality, we relate this to the modules induced from I_C and I_C^*. It is straightforward to show $R(3)$ satisfies conditions (I)-(III) of Proposition 5.1.2 where $\mu = (0, 0, 1) \in R(3)$ (see [41], Chapter 4.3 or [25]) and so we therefore have:

$$R(3) \cong \bar{I}_C^*. \tag{5.1.5}$$

By (5.1.4), we necessarily have:

$$R(1) \cong \bar{I}_C. \tag{5.1.6}$$

We are now left with $R(2)$ and the reader is once more directed to Chapter 4.3 of [41] to see that, as expected, we have:

$$R(2) \cong \overline{(x-1)I_C}. \tag{5.1.7}$$

It is therefore a straightforward consequence that $\bar{I}_C$, $\overline{(x-1)I_C}$, $\bar{I}_C^*$ are all isomorphically distinct. We conclude by observing the following fact [17], [25]:

$$\text{The augmentation ideal } I_G \text{ of } G(p, 3) \text{ decomposes as } I_G \cong \bar{I}_C \oplus [y-1]. \tag{5.1.8}$$

5.2 Indecomposable modules and tree structures

For clarity, we restate Proposition 5.2.1 in the case where $q = 3$.

Proposition 5.2.1. *There are a total of 17 distinct non-isomorphic genera of indecomposable modules for $\Lambda = \mathbf{Z}[G(p, 3)]$.*

In the manner of [41], we list the indecomposable modules as follows:

I. There are three indecomposable modules over $\mathbf{Z}[C_3]$ that become modules over Λ via the quotient map $G(p, 3) \to C_3$:
   - (i) The trivial module (rank 1);
   - (ii) The augmentation ideal, $I_3 = Ker(\mathbf{Z}[C_3] \to \mathbf{Z})$ (rank 2);
   - (iii) The group ring itself $\mathbf{Z}[C_3]$ (rank 3).

II. There are three distinct indecomposable modules over $\mathcal{T}_3(A, \pi)$:
   - (iv) $\bar{I}_C^* = \overline{\mathbf{Z}[\zeta_p]} \cong R(3)$, where $\zeta_p = exp(2\pi i/p)$ (rank $p-1$);
   - (v) $\bar{I}_C = \overline{(\zeta_p - 1)\mathbf{Z}[\zeta_p]} \cong R(1)$ (rank $p-1$);
   - (vi) $\overline{(x-1)I_C} = \overline{(\zeta_p - 1)^2\mathbf{Z}[\zeta_p]} \cong R(2)$ (rank $p-1$).

   These are distinct Λ-modules via the twisting relation $y\zeta_p^r = \zeta_p^{\theta_*(r)}y$.

The above can be thought of as the 'basic' indecomposable modules. The remaining genera of indecomposable modules then arise in the form of non-split extensions

$$0 \to X \to ? \to Y \to 0$$

where X is the direct sum of possible combinations of $\bar{I}_C^*$, $\bar{I}_C$, $\overline{(x-1)I_C}$ (without repetition), and $Y = \mathbf{Z}$, I_3 or $\mathbf{Z}[C_3]$. The proof of this can be found in [10], [39] and, in the form of a specific case, in [41].

III. There is one extension when $Y = \mathbf{Z}$:

(vii) $0 \to \bar{I}_C \to \bar{\Lambda}_0 \to \mathbf{Z} \to 0$ (rank p).

IV. There are three indecomposable non-split extensions when $Y = I_3$:

(viii) $0 \to \bar{I}_C \to V_1 \to I_3 \to 0$ ($rk_{\mathbf{Z}}(V_1) = p + 1$);

(ix) $0 \to \bar{I}_C^* \to V_2 \to I_3 \to 0$ ($rk_{\mathbf{Z}}(V_2) = p + 1$);

(x) $0 \to \bar{I}_C \oplus \bar{I}_C^* \to [y-1) \to I_3 \to 0$ (rank $2p$).

V. There are three indecomposable non-split extensions for $Y = \mathbf{Z}[C_3]$ and $rk_{\mathbf{Z}}(X) = p - 1$:

(xi) $0 \to \bar{I}_C \to W_1 \to \mathbf{Z}[C_3] \to 0$, ($rk_{\mathbf{Z}}(W_1) = p + 2$);

(xii) $0 \to \overline{(x-1)I_C} \to W_2 \to \mathbf{Z}[C_3] \to 0$, ($rk_{\mathbf{Z}}(W_2) = p + 2$);

(xiii) $0 \to \bar{I}_C^* \to W_3 \to \mathbf{Z}[C_3] \to 0$, ($rk_{\mathbf{Z}}(W_3) = p + 2$).

VI. Set $Q(i) = \Lambda/R(i)$. There are four indecomposable non-split extensions for $Y = \mathbf{Z}[C_3]$ where X is the direct sum of $\bar{I}_C$, $\overline{(x-1)I_C}$ or $\bar{I}_C^*$:

(xiv) $0 \to \bar{I}_C \oplus \overline{(x-1)I_C} \to Q(3) \to \mathbf{Z}[C_3] \to 0$, ($rk_{\mathbf{Z}}(Q(3)) = 2p + 1$);

(xv) $0 \to \bar{I}_C \oplus \bar{I}_C^* \to Q(2) \to \mathbf{Z}[C_3] \to 0$, ($rk_{\mathbf{Z}}(Q(2)) = 2p + 1$);

(xvi) $0 \to \overline{(x-1)I_C} \oplus \bar{I}_C^* \to Q(1) \to \mathbf{Z}[C_3] \to 0$, ($rk_{\mathbf{Z}}(Q(1)) = 2p + 1$);

(xvii) $0 \to \bar{I}_C \oplus \overline{(x-1)I_C} \oplus \bar{I}_C^* \to \Lambda \to \mathbf{Z}[C_3] \to 0$, (rank $3p$).

Any other indecomposable module belongs to the non-trivial elements of $\widetilde{K}_0(\Lambda)$ and are therefore of no consequence in the context of free resolutions. Using the above we can now discuss the tree structures of the syzygies $\Omega_r(\mathbf{Z})$ in some detail. First note the following result of Remez (see [41], p.79):

Proposition 5.2.2. *For any metacyclic group* $\Lambda = \mathbf{Z}[G(p, 3)]$, *set* $Q(i) = \Lambda/R(i)$. *We describe the syzygies at the minimal level of its free resolution by*

$$\Omega_r(\mathbf{Z}) = \begin{cases} R(i+1) \oplus [y-1), & \textit{when } r = 2i + 1; \\ Q(i+1) \oplus [\Sigma_y), & \textit{when } r = 2i. \end{cases}$$

Recall that the free period of $G(p, 3)$ is 6. Thus, by using Proposition 2.7.4 and Corollary 2.7.5, we have:

Proposition 5.2.3. *Both* $\Omega_1(\mathbf{Z})$ *and* $\Omega_5(\mathbf{Z})$ *are straight.*

For the third syzygy we note the following result of Remez (see [41], p.83):

Proposition 5.2.4. *For $\Lambda = \mathbf{Z}[G(p,\, 3)]$, $\Omega_3(\mathbf{Z})$ is straight.*

As already observed, the situation for even syzygies is decidedly more complex. Nevertheless, we do have the following consequence of Proposition 2.7.6:

Proposition 5.2.5. *$\Omega_0(\mathbf{Z}) = \Omega_6(\mathbf{Z})$ is straight.*

As we are interested in the decompositions of these syzygies (and in particular the so-called x-strand) we note the straightness of some of the component parts. A similar argument to that of Proposition 4.2.1 shows:

Proposition 5.2.6. *Let $M \in \Omega_1(\mathbf{Z})$ be a minimal representative; then M decomposes as $M \cong M_1 \oplus M_2$, where M_1, M_2 are non-trivial indecomposable modules.*

In the case of $G(p,\, 3)$, (2.8.2) takes the form:

Proposition 5.2.7. *The tree structure of $[R(1)]$ is straight.*

Proposition 5.2.8. *The tree structure of $[R(2)]$ is straight.*

Proposition 5.2.9. *The tree structure of $[R(3)]$ is straight.*

As with the dihedral case, we have the following result:

Proposition 5.2.10. *The stable module $\Omega_0(\mathbf{Z})$ decomposes as two non-trivial indecomposable modules*

$$\Omega_0(\mathbf{Z}) = [\mathbf{Z}] = [Q(1)] \oplus [y^2 + y + 1].$$

However, unlike the case for the odd syzygies we do not know if there is a suitable and unique module of rank $2p + 1$. Rather, we only know such modules exist in an exact sequence of the form

$$0 \to \overline{(x-1)I_C} \oplus \bar{I}_C^* \to ? \to \mathbf{Z}[C_3] \to 0.$$

As such, we cannot conclude that the tree structure of $[Q(1)]$ is straight. We note that similar results exist for the other two even syzygies, i.e. both decompose as a direct sum of two non-trivial indecomposable modules. Once again, this does not imply the corresponding tree structures of the component parts are straight.

5.3 The sequencing conjecture

Recall the sequencing conjecture discussed in Chapter 1. In [25], the existence of the following exact sequence was shown,

$$\begin{tikzcd}
 & & & & & K(3) \arrow[dr] & & & & K(2) \arrow[dr] & & & & K(1) \arrow[dr] & & & & & \\
0 \arrow[rr] & & R(1) \arrow[rr] & & \Lambda \arrow[ur] \arrow[rr] & & \Lambda \arrow[rr] \arrow[dr] & & P(2) \arrow[ur] \arrow[rr] & & \Lambda \arrow[rr] \arrow[dr] & & P(1) \arrow[ur] \arrow[rr] & & \Lambda \arrow[rr] & & R(1) \arrow[rr] & & 0 \\
 & & & & & & & R(3) \arrow[ur] & & & & R(2) \arrow[ur] & & & & & & &
\end{tikzcd}$$

In the above $K(i) = Ker(p_i : \Lambda \to R(i))$, and $P(1)$, $P(2)$ are projective modules such that their direct sum is isomorphic to the free module Λ^2 of rank 2. The sequencing conjecture in this case asserts that both $P(i)$ are in fact free. One of the results of this chapter is to provide an affirmative answer to this conjecture. To do so we first prove the existence of the so-called 'basic sequences':

Theorem H: There exist the following three basic sequences:

(i)

$$\begin{tikzcd} & & & K(3) \arrow[dr] & & & \\ 0 \arrow[r] & R(1) \arrow[r] & \Lambda \arrow[ur] \arrow[rr] & & \Lambda \arrow[r] & R(3) \arrow[r] & 0; \end{tikzcd}$$

$(\mathcal{B}(1))$

(ii)

$$\begin{tikzcd} & & & K(1) \arrow[dr] & & & \\ 0 \arrow[r] & R(2) \arrow[r] & \Lambda \arrow[ur] \arrow[rr] & & \Lambda \arrow[r] & R(1) \arrow[r] & 0; \end{tikzcd}$$

$(\mathcal{B}(2))$

(iii)

$$\begin{tikzcd} & & & K(2) \arrow[dr] & & & \\ 0 \arrow[r] & R(3) \arrow[r] & \Lambda \arrow[ur] \arrow[rr] & & \Lambda \arrow[r] & R(2) \arrow[r] & 0. \end{tikzcd}$$

$(\mathcal{B}(3))$

It is then straightforward to splice the three basic sequences together to provide an affirmative answer to the sequencing conjecture for $q = 3$. The author notes that part (i) of Theorem H has already been shown by Johnson (see [25]) and, as will be seen, we shall only require part (ii) of Theorem H to prove Theorem I. Nevertheless, in the process of showing the existence of part (iii) we can say quite a bit more about what is happening in these syzygies.

5.4 The module K

Define the module $K = [y - 1,\, \Sigma_x)$.

Proposition 5.4.1. *K has $\mathbf{Z}$-basis* $\mathcal{E} = \{(y^i - 1)x^j \,|\, 1 \le i \le 2,\ 0 \le j \le p-1\} \cup \{\Sigma_x\}$.

Proof. Start by defining

$$K_0 = \{(y-1)a(x) + (y^2-1)b(x) \mid a(x),\, b(x) \in \Lambda_0\} \subset K.$$

It is straightforward to see

$$(y-1)x^i y = (y^2 - y)x^{\theta_*^2(i)} = (y^2-1)x^{\theta_*^2(i)} - (y-1)x^{\theta_*^2(i)}$$

and

$$(y^2-1)x^i y = -(y-1)x^{\theta_*^2(i)}.$$

It follows that K_0 is a Λ-submodule of K. Finally, when considering Σ_x we immediately observe that for $i = 1,\,2$,

$$\Sigma_x y^i = \sum_{j=0}^{p-1}(y^i - 1)x^j + \Sigma_x.$$

Thus, K is spanned over $\mathbf{Z}$ by $\mathcal{E}$. Finally, upon performing elementary basis transformations to the standard $\mathbf{Z}$-basis of Λ, it is evident that

$$\mathcal{E} \cup \{x^i \mid 1 \leq i \leq p-1\}$$

is a $\mathbf{Z}$-basis for Λ. It therefore follows that $\mathcal{E}$ is a $\mathbf{Z}$-basis for K, as required. ☐

Proposition 5.4.2. $\Lambda/K \cong \bar{I}_C^*$.

Proof. Perform elementary basis transformations so that we have the following $\mathbf{Z}$-basis for Λ,

$$\{x^i \mid 1 \leq i \leq p-1\} \cup \{(y^j - 1)x^k \mid 1 \leq j \leq 2,\, 0 \leq k \leq p-1\} \cup \{\Sigma_x\}.$$

Thus, Λ/K has $\mathbf{Z}$-basis $\{\natural(x^i) \mid 1 \leq i \leq p-1\}$ where $\natural : \Lambda \to \Lambda/K$. In particular, we note $rk_{\mathbf{Z}}(\Lambda/K) = p-1$. Moreover, since $\Sigma_x = 0$ in Λ/K we may write

$$\natural(1) = \natural(1)\Sigma_x - \natural(1)x - \cdots - \natural(1)x^{p-1}$$

so that

$$\Lambda/K = span_{\mathbf{Z}}\{\natural(1)x^r \mid 0 \leq r \leq p-1\}.$$

Since $m\Sigma_x = 0$ for all $m \in \Lambda/K$, it remains (by the criteria of Proposition 5.1.2) to show $\natural(1)y = \natural(1)$. However, in Λ we clearly have

$$1 \cdot y = (y-1) + 1$$

and so y acts trivially on $\natural(1)$ as required. The result follows from Proposition 5.1.2. ☐

Recall, we have a surjection $p_i : \Lambda \to R(i)$ by composing the obvious projections $\Lambda \to \mathcal{T}_3(A,\,\pi)$ and $\mathcal{T}_3(A,\,\pi) \to R(i)$. We have previously defined $K(i) = Ker(p_i)$.

Corollary 5.4.3. $K \sim K(3)$.

Proof. By Proposition 5.4.2, the module K arises in an exact sequence of the form

$$0 \to K \to \Lambda \to \bar{I}_C^* \to 0$$

and $\bar{I}_C^* \cong R(3)$ by Proposition 5.1.5. As we also have the exact sequence

$$0 \to K(3) \to \Lambda \to R(3) \to 0$$

then the result follows from Schanuel's Lemma. ☐

Next, we have an exact sequence of the form,

$$0 \to K_0 \to K \to \mathbf{Z} \to 0$$

since y acts trivially on K/K_0. We are now in the position to show K acts as the identity within the stable class when tensoring over $\mathbf{Z}$. Observe that tensoring with any of the $R(i)$ or $K(j)$ therefore yields the exact sequence

$$0 \to K_0 \otimes ? \to K \otimes ? \to ? \to 0.$$

First, we need two preliminary results:

Proposition 5.4.4. *For* $j : \mathbf{Z}[C_3] \hookrightarrow \Lambda$ *the canonical injection,* $j_*(I_3) \cong [y-1)$.

Proof. It is straightforward to show

$$\{(y^i - 1)x^j \mid 1 \leq i \leq 2,\, 0 \leq j \leq p-1\}$$

is a $\mathbf{Z}$-basis for $[y-1)$. Now observe $j_*(I_3) = I_3 \otimes_{\mathbf{Z}[C_3]} \Lambda$ has $\mathbf{Z}$-basis

$$\{(y-1) \otimes_{C_3} x^s,\, (y^2-1) \otimes_{C_3} x^t \mid 0 \leq s,\, t \leq p-1\}.$$

If we define the map $\varphi : j_*(I_3) \to [y-1)$ by

$$\varphi((y^i - 1) \otimes_{C_3} x^s) = (y^i - 1)x^s$$

then it is straightforward to check φ is a Λ-homomorphism between basis elements, i.e. a Λ-isomorphism. ☐

Proposition 5.4.5. *With* j *as above,* $j^*(R(i)) \cong \mathbf{Z}[C_3]^d$ *for each* $1 \leq i \leq 3$.

Proof. We proceed much as in the case $q = 2$. Start with the exact sequence,

$$0 \to R(1) \oplus R(2) \oplus R(3) \to \Lambda \to \mathbf{Z}[C_3] \to 0.$$

Upon applying the exact functor $j^*(-)$ we have the split exact sequence,

$$0 \to j^*(R(1) \oplus R(2) \oplus R(3)) \to \mathbf{Z}[C_3]^p \to \mathbf{Z}[C_3] \to 0.$$

It follows that each $R(i)$ is projective as a $\mathbf{Z}[C_3]$-module. Now, using a result of Rim (see [42]), we know $\widetilde{K_0}(\mathbf{Z}[C_3]) = 0$ and so any projective module is stably free. Using Swan-Jacobinski, we therefore conclude $j^*(R(i)) \cong \mathbf{Z}[C_3]^d$. ☐

Proposition 5.4.6. $K \otimes R(i) \cong R(i) \oplus \Lambda^{2d}$, *where* $1 \leq i \leq 3$ *and* $d = (p-1)/3$ *as before.*

Proof. Consider the following exact sequence,

$$0 \to K_0 \otimes R(i) \to K \otimes R(i) \to R(i) \to 0$$

and recall $K_0 = [y-1)$. By two applications of Frobenius Reciprocity, and Propositions 5.4.4 and 5.4.5, we therefore have the following isomorphism:

$$\begin{aligned} j_*(I_3) \otimes R(i) &\cong j_*(I_3 \otimes j^*(R(i))) \\ &\cong j_*(I_3 \otimes \mathbf{Z}[C_3]^d) \\ &\cong j_*(\mathbf{Z}[C_3]^{2d}) \\ &\cong \Lambda^{2d}. \end{aligned}$$

Replacing $K_0 \otimes R(i)$ with Λ^{2d} in the above exact sequence we therefore get

$$0 \to \Lambda^{2d} \to K \otimes R(i) \to R(i) \to 0$$

which splits, yielding $K \otimes R(i) \cong R(i) \oplus \Lambda^{2d}$, as required. ☐

Proposition 5.4.7. *$j^*(K(i)) \cong \mathbf{Z}[C_3]^{p-d}$ for each $1 \leq i \leq 3$.*

Proof. Start with the following exact sequence,

$$0 \to K(i) \to \Lambda \to R(i) \to 0$$

and apply $j^*(-)$,

$$0 \to j^*(K(i)) \to \mathbf{Z}[C_3]^p \to j^*(R(i)) \to 0.$$

By Proposition 5.4.5, $j^*(R(i)) \cong \mathbf{Z}[C_3]^d$ and so the above exact sequence splits, yielding

$$j^*(K(i)) \oplus \mathbf{Z}[C_3]^d \cong \mathbf{Z}[C_3]^p,$$

i.e. $j^*(K(i))$ is stably free of rank $p-d$. As $\mathbf{Z}[C_3]$ satisfies the Eichler condition, it has SFC and so $j^*(K(i)) \cong \mathbf{Z}[C_3]^{p-d}$. ☐

Proposition 5.4.8. *$K \otimes K(i) \cong K(i) \oplus \Lambda^{2(p-d)}$ for $1 \leq i \leq 3$.*

Proof. Once again, start with the following exact sequence,

$$0 \to K_0 \otimes K(i) \to K \otimes K(i) \to K(i) \to 0.$$

It therefore remains to consider $K_0 \otimes K(i)$. We have

$$\begin{aligned} j_*(I_3) \otimes K(i) &\cong j_*(I_3 \otimes j^*(K(i))) \\ &\cong j_*(I_3 \otimes \mathbf{Z}[C_3]^{(p-d)}) \\ &\cong j_*(\mathbf{Z}[C_3]^{2(p-d)}) \\ &\cong \Lambda^{2(p-d)}. \end{aligned}$$

The above exact sequence now splits, yielding

$$K \otimes K(i) \cong K(i) \oplus \Lambda^{2(p-d)}$$

for each $1 \leq i \leq 3$. ☐

Theorem D therefore follows immediately from Propositions 5.4.6 and 5.4.8. Now, recall that duality $M \mapsto M^*$ induces a one-to-one correspondence $\Omega_r(\mathbf{Z}) \leftrightarrow \Omega_{-r}(\mathbf{Z})$. It therefore follows that $Q(1)^* \sim Q(1)$, $Q(2)^* \sim Q(3)$ and $Q(3)^* \sim Q(2)$. In particular, these exist in exact sequences of the form

$$0 \to Q(i)^* \to \Lambda \to R(4-i) \to 0$$

By Schanuel, we therefore observe $Q(1) \sim K(3) \sim K$, $Q(2) \sim K(1)$ and $Q(3) \sim K(2)$. In particular, we notice

$$K \sim K^* \tag{5.4.9}$$

thereby proving Theorem E. Thus, K acts as the identity within the stable class for the x-strand $\Omega_r^x(\mathbf{Z})$ of the syzygy modules $\Omega_r(\mathbf{Z})$.

5.5 Theorem F(1)

Recall $K(1) = Ker(p_1 : \Lambda \to \bar{I}_C)$. We would like to show,

$$\bar{I}_C^* \otimes \bar{I}_C^* \cong K(1)^* \oplus \Lambda^{d-1}. \tag{5.5.1}$$

Since [**Z**] is straight, it is a straightforward application of Schanuel's Lemma to show this is true over Λ_0. To extend this isomorphism to one over Λ we directly show

$$\bar{I}_C^* \otimes \bar{I}_C^* \cong L \oplus \Lambda^{d-1},$$

for some (yet to be defined) module $L \sim K(1)^*$. It is then clear that (5.5.1) follows from the stable equivalence since $d \geq 2$. An obvious next question is then whether $K(1) \cong L^*$. At present it is unclear whether this is true, however a related problem which may provide an affirmative answer (if indeed one exists) is whether or not $[K(1)^*]$ is straight. Throughout this chapter the straightness (or possible lack thereof) of the even syzygies will provide complications. Nevertheless, for the most part we will be able to circumvent these issues when they arise.

To show the result for L we first construct the free part of $\bar{I}_C^* \otimes \bar{I}_C^*$. Recall that we defined α such that $\theta(x) = x^\alpha$ and recall $\nu^i \cdot y = \nu^{\alpha^2 i}$ and $\nu^i \cdot y^{-1} = \nu^{\alpha i}$. Our goal is to first show:

$$V(r) + V(\alpha r) + V(\alpha^2 r) \cong \Lambda \text{ for } 1 \leq r \leq p-2 \tag{5.5.2}$$

and where $V(r) = span_{\mathbf{Z}}\{\nu^{r+k} \otimes \nu^k \mid 0 \leq k \leq p-1\}$, as before. Note that we are in fact taking $\alpha r\,(mod p)$ and $\alpha^2 r\,(mod p)$ when defining $V(\alpha r)$ and $V(\alpha^2 r)$, respectively. Evidently, (5.5.2) holds as an isomorphism of Λ_0-modules.

Proposition 5.5.3. *Set* $V_r = V(r) + V(\alpha r) + V(\alpha^2 r)$. *The representation of* V_r *with respect to the* x*-action on the defining basis of* V_r *is given by*

$$\rho_{V_r}(x^{-1}) = \begin{pmatrix} \Psi & 0 & 0 \\ 0 & \Psi & 0 \\ 0 & 0 & \Psi \end{pmatrix},$$

where

$$(\Psi)_{ij} = \begin{cases} 1, & i = 1,\, j = p; \\ 1, & j = i-1,\, 2 \leq i \leq p; \\ 0, & o/w. \end{cases}$$

Proof. Denote the basis elements of V_r by:

$$e_i = \begin{cases} \nu^{i+r-1} \otimes \nu^{i-1}, & 1 \leq i \leq p-r; \\ \nu^{i+r-1-p} \otimes \nu^{i-1}, & p-r+1 \leq i \leq p; \end{cases}$$

$$e_{i+p} = \begin{cases} \nu^{i+\alpha r-1} \otimes \nu^{i-1}, & 1 \leq i \leq p-\alpha r; \\ \nu^{i+\alpha r-1-p} \otimes \nu^{i-1}, & p-\alpha r+1 \leq i \leq p; \end{cases}$$

$$e_{i+2p} = \begin{cases} \nu^{i+\alpha^2 r-1} \otimes \nu^{i-1}, & 1 \leq i \leq p-\alpha^2 r; \\ \nu^{i+\alpha^2 r-1-p} \otimes \nu^{i-1}, & p-\alpha^2 r+1 \leq i \leq p. \end{cases}$$

Thus, $V(r)$, $V(\alpha r)$, $V(\alpha^2 r)$ have **Z**-bases $\{e_i \mid 1 \leq i \leq p\}$, $\{e_{i+p} \mid 1 \leq i \leq p\}$ and $\{e_{i+2p} \mid 1 \leq i \leq p\}$, respectively. Consider now the x-action on these basis elements. It is quite clear that $e_i \cdot x = e_{i+1}$ for $1 \leq i \leq p-1$, and $e_p \cdot x = e_1$. Similarly, $e_{i+p} \cdot x = e_{p+i+1}$ for $1 \leq i \leq p-1$, and $e_{2p} \cdot x = e_{p+1}$. Finally, $e_{i+2p} \cdot x = e_{2p+i+1}$ for $1 \leq i \leq p-1$, and $e_{3p} \cdot x = e_{2p+1}$. The result now follows. ☐

Proposition 5.5.4. *With V_r as before, the integral representation of V_r with respect to the action of y^2 is given by*

$$\rho_{V_r}(y) = \begin{pmatrix} 0 & 0 & \Phi^T \\ \Phi^T & 0 & 0 \\ 0 & \Phi^T & 0 \end{pmatrix}$$

where

$$\Phi^T_{ij} = \begin{cases} 1, & i = (j-1)\alpha + 1 \, (mod\, p); \\ 0, & o/w. \end{cases}$$

Proof. With the e_i, e_{i+p}, e_{i+2p} as before, observe $e_i y^{-1} = \nu^{\alpha r + \alpha(i-1)} \otimes \nu^{\alpha(i-1)}$ where we have $\alpha r + \alpha(i-1) - \alpha(i-1) = \alpha r$ and $0 \leq \alpha(i-1) \leq p-1$. Thus, $e_i y^{-1} = e_{j+p}$ for some $1 \leq j \leq p$. Similarly, $e_{j+p} y^{-1} = e_{k+2p}$ and $e_{k+2p} y^{-1} = e_i$ for some $1 \leq i,\, k \leq p$. Note that the last equality follows since $\alpha^3 \equiv 1 \, (mod\, p)$. Thus,

$$\rho_{V_r}(y) \sim \begin{pmatrix} 0 & 0 & \Phi_3 \\ \Phi_1 & 0 & 0 \\ 0 & \Phi_2 & 0 \end{pmatrix}$$

for some $p \times p$ blocks Φ_1, Φ_2, Φ_3.

Clearly, $e_1 y^{-1} = e_{1+p}$. In general we have $e_i y^{-1} = \nu^{\alpha r + \alpha(i-1)} \otimes \nu^{\alpha(i-1)} = e_{\alpha(i-1)+1+p}$, and so

$$(\Phi_1)_{ij} = \begin{cases} 1, & i = (j-1)\alpha + 1 \, (mod\, p); \\ 0, & o/w. \end{cases}$$

Similar arguments now apply to $e_{j+p} y^{-1}$ and $e_{k+2p} y^{-1}$ so that

$$\rho_{V_r}(y) = \begin{pmatrix} 0 & 0 & \Phi^T \\ \Phi^T & 0 & 0 \\ 0 & \Phi^T & 0 \end{pmatrix}$$

where $\Phi^T = \Phi_1 = \Phi_2 = \Phi_3$, as required. ☐

Proposition 5.5.5. *The regular representation of the x-action is given by*

$$\rho_{reg}(x^{-1}) = \begin{pmatrix} \Psi & 0 & 0 \\ 0 & \Psi & 0 \\ 0 & 0 & \Psi \end{pmatrix}.$$

Proof. Set $f_i = x^{i-1}$, $f_{j+p} = yx^{j-1}$, $f_{k+2p} = y^2 x^{k-1}$ where $1 \leq i,\, j,\, k \leq p$. Then $f_i \cdot x = f_{i+1}$ for $1 \leq i \leq p-1$, and $f_p \cdot x = f_1$. Similarly, $f_{j+p} \cdot x = f_{j+p+1}$ for $1 \leq i \leq p-1$, and $f_{2p} \cdot x = f_{p+1}$. Finally, $f_{k+2p} \cdot x = f_{k+2p+1}$ for $1 \leq i \leq p-1$, and $f_{3p} \cdot x = f_{2p+1}$. The result now follows. ☐

Proposition 5.5.6. *The regular representation of the* y^{-1}*-action is given by*

$$\rho_{reg}(y) = \begin{pmatrix} 0 & \Phi^T & 0 \\ 0 & 0 & \Phi^T \\ \Phi^T & 0 & 0 \end{pmatrix}$$

Proof. It is clear that $f_i y^{-1} = f_{k+2p}$ for some $1 \le k \le p$. Similarly, we see that $f_{j+p} y^{-1} = f_i$ and $f_{k+2p} y^{-1} = f_{j+p}$ for some $1 \le i,\, j,\, k \le p$. Thus,

$$\rho_{reg}(y) \sim \begin{pmatrix} 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_3 \\ \Theta_1 & 0 & 0 \end{pmatrix}$$

where Θ_1, Θ_2, Θ_3 are $p \times p$ blocks. Now, $f_i y^{-1} = x^{i-1} y^2 = y^2 x^{\alpha(i-1)} = f_{2p+\alpha(i-1)+1}$ and so,

$$(\Theta_1)_{ij} = \begin{cases} 1, & i = \alpha(j-1)+1; \\ 0, & o/w. \end{cases}$$

In other words, $\Theta_1 = \Phi^T$ where Φ^T is as defined above. Similarly, we consider the action of y^2 on f_{j+p} and f_{k+2p} to show $\Theta_2 = \Theta_3 = \Phi^T$. Replacing Θ_1, Θ_2, Θ_3 above with Φ^T concludes the proof. □

Proposition 5.5.7. $V(r) + V(\alpha r) + V(\alpha^2 r) \cong \Lambda$ *for* $1 \le r \le p-2$.

Proof. All that remains is to show there exists an invertible $(3p) \times (3p)$ matrix X such that $\rho_{reg}(g)X = X\rho_{V_r}(g)$ for all $g \in G(p,\, 3)$. Set

$$X = \begin{pmatrix} 0 & 0 & I_p \\ 0 & I_p & 0 \\ I_p & 0 & 0 \end{pmatrix}$$

and observe

$$\rho_{reg}(x^{-1})X = \begin{pmatrix} 0 & 0 & \Psi \\ 0 & \Psi & 0 \\ \Psi & 0 & 0 \end{pmatrix} = X\rho_{V_r}(x^{-1})).$$

and

$$\rho_{reg}(y)X = \begin{pmatrix} 0 & \Phi^T & 0 \\ \Phi^T & 0 & 0 \\ 0 & 0 & \Phi^T \end{pmatrix} = X\rho_{V_r}(y).$$

Finally, X is clearly invertible as a series of row permutations transform X into I_{3p}. □

Observe that, as $3|p-1$, then we get $d-1$ copies of Λ with two of the $V(r)$'s left over. These two $V(r)$'s along with T will, we hope, represent a module stably isomorphic to $K(1)^*$. To see which two $V(r)$'s are left over (call them $V(s)$, $V(t)$) we first observe that applying y to an element of $V(s)$ (say) will give an element in $V(t)$. Applying y to an element of $V(t)$ will then give an element in what we may think of as '$V(p-1)$'. Applying y once more gives us an element of $V(s)$. With this in mind, write V' for the sum of the $V(i)$ *without* $V(s)$ and $V(t)$, i.e. $V' \cong \Lambda^{d-1}$ and

$V = V' + V(s) + V(t)$. Now let

$$\eta : \bar{I}_C^* \otimes \bar{I}_C^* \to (\bar{I}_C^* \otimes \bar{I}_C^*)/V'$$

be the natural map. As with the dihedral case, we write the above quotient as $\eta(T + V(s) + V(t))$ to emphasise which elements will span the module we hope is stably isomorphic to $K(1)^*$. In particular, we have:

Proposition 5.5.8. $\bar{I}_C^* \otimes \bar{I}_C^* \cong \eta(T + V(\alpha+1) + V(p-\alpha)) \oplus \Lambda^{d-1}$.

Proof. Start by showing $\alpha^2 + \alpha + 1 = 0\ (mod\, p)$. Since $\theta(x) = x^\alpha$ and $\theta^3 = Id$, it follows $p|\alpha^3 - 1$. In particular, $p|\alpha - 1$ or $p|\alpha^2 + \alpha + 1$. As $p \not| \alpha - 1$, the claim is shown. It therefore follows that the only possible choices for s and t are $\alpha + 1$ and $p - \alpha$. In other words, $T + V(\alpha+1) + V(p-\alpha)$ is left over when we have introduced the y-action to $T \oplus V$ (thought of as a Λ_0-module).

Now, $(\bar{I}_C^* \otimes \bar{I}_C^*)/V'$ is torsion free (Proposition 2.2.7). Furthermore, it is clear that $T \cdot g \notin V'$ for any $g \in G(3,\,p)$. Likewise, if e_i is an element of the given $\mathbf{Z}$-basis of $V(\alpha+1)$ or $V(p-\alpha)$, then $e_i \cdot g \notin V'$ for any $g \in G(3,\,p)$. In other words, $(\bar{I}_C^* \otimes \bar{I}_C^*)/V'$ is a Λ-lattice. The result now follows as we have the following split short exact sequence, $0 \to V' \to \bar{I}_C^* \otimes \bar{I}_C^* \to \eta(T + V(\alpha+1) + V(p-\alpha)) \to 0$ since $V' \cong \Lambda^{d-1}$. ☐

Proposition 5.5.9. *Set* $L = \eta(T + V(\alpha+1) + V(p-\alpha))$. *We then have* $K(1) \sim L^*$.

Proof. Recall that $I_G \cong \bar{I}_C \oplus [y-1)$. We can think of $[y-1)$ as $j_*(I_3)$ and note $[y-1)$ is self dual. Consider the following exact sequence,

$$0 \to \bar{I}_C \oplus [y-1) \to \Lambda \to \mathbf{Z} \to 0$$

and apply $- \otimes \bar{I}_C$. Using Frobenius Reciprocity on the middle term yields the exact sequence,

$$0 \to (\bar{I}_C \otimes \bar{I}_C) \oplus ([y-1) \otimes \bar{I}_C) \to \Lambda^{3d} \to \bar{I}_C \to 0.$$

In the proof of Proposition 5.4.6 we have shown $\bar{I}_C \otimes [y-1) \cong \Lambda^{2d}$. The above exact sequence therefore becomes

$$0 \to (\bar{I}_C \otimes \bar{I}_C) \oplus \Lambda^{2d} \xrightarrow{\iota} \Lambda^{3d} \to \bar{I}_C \to 0$$

which becomes

$$0 \to \bar{I}_C \otimes \bar{I}_C \to \Lambda^{3d}/\iota(\Lambda^{2d}) \to \bar{I}_C \to 0.$$

By Johnson's 'destabilization theorem' (Proposition 2.4.2), $\Lambda^{3d}/\iota(\Lambda^{2d})$ is projective. We can therefore construct the the following split exact sequence

$$0 \to \Lambda^{2d} \xrightarrow{\iota} \Lambda^{3d} \to \Lambda^{3d}/\iota(\Lambda^{2d}) \to 0.$$

As the sequence splits, $\Lambda^{3d}/\iota(\Lambda^{2d})$ is stably free of rank d. However, by Swan-Jacobinski, Λ has SFC and so there are no nontrivial stably free modules. It therefore follows that we have the exact sequence

$$0 \to \bar{I}_C \otimes \bar{I}_C \to \Lambda^d \to \bar{I}_C \to 0. \qquad (5.5.10)$$

Now, by Proposition 5.5.8 we have $\bar{I}_C^* \otimes \bar{I}_C^* \cong L \oplus \Lambda^{d-1}$. Since $\bar{I}_C^*$ is the dual of $\bar{I}_C$, we have $\bar{I}_C \otimes \bar{I}_C \cong L^* \oplus \Lambda^{d-1}$. Thus, we substitute this back into (5.5.10) to yield the following

$$0 \to L^* \oplus \Lambda^{d-1} \to \Lambda^d \to \bar{I}_C \to 0$$

and by once more using Proposition 2.4.2 we get

$$0 \to L^* \to \Lambda \to \bar{I}_C \to 0.$$

Since $\bar{I}_C \cong R(1)$, the result now follows from Schanuel's Lemma, i.e. $L^* \sim K(1)$. ☐

To summarise, we have successfully made the transition from the x-strand of the first syzygy, to that of the second syzygy. Since $(\bar{I}_C)^* \cong \bar{I}_C^*$, this may be shown as:

Proposition 5.5.11. *With, L, L^* as above, we have shown:*

- $\bar{I}_C \otimes \bar{I}_C \cong L^* \oplus \Lambda^{d-1}$;
- $\bar{I}_C^* \otimes \bar{I}_C^* \cong L \oplus \Lambda^{d-1}$.

In the next section we continue in this vein to make the transition into the third syzygy. A biproduct of this is an affirmative answer to the sequencing conjecture for $q = 3$.

5.6 Theorem F(2) and the sequencing conjecture

As a consequence of Section 5.5, we have the following exact sequence

$$0 \to L^* \to \Lambda \to \bar{I}_C \to 0. \tag{I}$$

In particular, our work thus far allows us to identify representative elements for the x-strand of $\Omega_r(\mathbf{Z})$ when $r = 0,\,1,\,2$ and $5\,(mod 6)$. Now, in [41] (see pp. 78-79) Remez has shown $\Omega_3^x(\mathbf{Z}) = [\overline{(x-1)I_C}]$ and $\Omega_4^x(\mathbf{Z}) = [Q(3)]$, where $Q(3) = \Lambda/R(3)$. As such, we have the following identifications for the x-strand of $\Omega_r^x(\mathbf{Z})$,

$$\Omega_r^x(\mathbf{Z}) \sim \begin{cases} [K], & r \equiv 0 \ \ (mod\, 6); \\ [\bar{I}_C], & r \equiv 1 \ \ (mod\, 6); \\ [L^*], & r \equiv 2 \ \ (mod\, 6); \\ [\overline{(x-1)I_C}], & r \equiv 3 \ \ (mod\, 6); \\ [Q(3)], & r \equiv 4 \ \ (mod\, 6); \\ [\bar{I}_C^*], & r \equiv 5 \ \ (mod\, 6). \end{cases}$$

Although $\overline{(x-1)I_C}$ is useful, $Q(3)$ is not quite as nice as one would like. In particular, it is unclear whether $K(2) \cong Q(3)$. Nevertheless, we can say something about $[Q(3)]$.

Proposition 5.6.1. $K(2) \sim Q(3) \sim L$.

Proof. Because of the exact sequence (I) above, it is clear that $L \sim Q(3)$ by using the dual form of Schanuel's Lemma. So, we can take $\Omega_4^x(\mathbf{Z}) = [L]$. Moreover, as we noted at the end of Section 5.4, $Q(3) \sim K(2)$. It therefore follows that $L \sim K(2)$, as required. ☐

Proposition 5.6.2. $\bar{I}_C \otimes L^* \cong \overline{(x-1)I_C} \oplus \Lambda^{2d}$.

Proof. Return to (I) and apply $- \otimes \bar{I}_C$,

$$0 \to L^* \otimes \bar{I}_C \to \Lambda^{p-1} \to \bar{I}_C \otimes \bar{I}_C \to 0$$

which, by the result of Proposition 5.5.8, becomes (after dualising)

$$0 \to L \oplus \Lambda^{d-1} \to \Lambda^{3d} \to L \otimes \bar{I}_C^* \to 0,$$

where we have once more used $3d = p-1$. To remove the free module on the left we use Johnson's 'destabilization theorem' (Proposition 2.4.2). Upon dualising again we have the exact sequence

$$0 \to L^* \otimes \bar{I}_C \to \Lambda^{2d+1} \to L^* \to 0. \tag{II}$$

Splicing together (I) and (II) yields the following 'weak basic sequence',

$$\begin{tikzcd}
 & & & L^* \arrow[dr] & & & \\
0 \arrow[r] & L^* \otimes \bar{I}_C \arrow[r] & \Lambda^{2d+1} \arrow[ur] \arrow[rr] & & \Lambda \arrow[r] & \bar{I}_C \arrow[r] & 0.
\end{tikzcd}$$

If $L^* \otimes \bar{I}_C \cong \overline{(x-1)I_C} \oplus \Lambda^{2d}$, then the above becomes a basic sequence in the sense of Theorem H. A simple check of $\mathbf{Z}$-ranks demonstrates the above is an isomorphism of abelian groups. It now remains to extend this isomorphism to one over Λ.

Return to the exact sequence defining $K(2)$ and replace $R(2)$ with $\overline{(x-1)I_C}$,

$$0 \to K(2) \xrightarrow{\iota} \Lambda \xrightarrow{p} \overline{(x-1)I_C} \to 0. \tag{†}$$

As shown above, $K(2) \sim L$ and so $K(2) \oplus \Lambda \cong L \oplus \Lambda$. We can therefore alter (†) appropriately so that

$$0 \to K(2) \oplus \Lambda \xrightarrow{\iota \oplus Id} \Lambda^2 \xrightarrow{p'} \overline{(x-1)I_C} \to 0$$

is also exact. Note that p' is simply the composition of p with the projection $\Lambda^2 \to \Lambda$. We then have

$$0 \to L \oplus \Lambda \to \Lambda^2 \to \overline{(x-1)I_C} \to 0$$

and by using Proposition 2.4.2 and dualising we get the exact sequence

$$0 \to \overline{(x-1)I_C} \to \Lambda \to L^* \to 0.$$

Now, compare this with (II) and apply Schanuel's Lemma; we get

$$(L \otimes \bar{I}_C^*) \oplus \Lambda \cong \overline{(x-1)I_C} \oplus \Lambda^{2d+1}.$$

Recall Proposition 5.2.8 in which $[\overline{(x-1)I_C}]$ is seen to be straight. Consequently, we have cancellation of free modules and can therefore write

$$L \otimes \bar{I}_C^* \cong \overline{(x-1)I_C} \oplus \Lambda^{2d}.$$

The result now follows from the self-duality of $\overline{(x-1)I_C}$. ☐

This clearly proves Theorem F(2). Additionally, we can now prove parts (ii) and (iii) of Theorem H. As part (i) was shown by Johnson [25], this concludes the proof of Theorem H.

Proof of Theorem H. For part (ii), recall we constructed the 'weak basic sequence' in the previous proof. If we apply the result of Proposition 5.6.2 we obtain the exact sequence,

$$\begin{tikzcd}
& & & L^* \arrow[dr] & & & \\
0 \arrow[r] & \overline{(x-1)I_C} \oplus \Lambda^{2d} \arrow[r] & \Lambda^{2d+1} \arrow[ur] \arrow[rr] & & \Lambda \arrow[r] & \bar{I}_C \arrow[r] & 0.
\end{tikzcd}$$

By a final use of Proposition 2.4.2, we therefore build the desired basic sequence

$$\begin{tikzcd}
& & & L^* \arrow[dr] & & & \\
0 \arrow[r] & \overline{(x-1)I_C} \arrow[r] & \Lambda \arrow[ur] \arrow[rr] & & \Lambda \arrow[r] & \bar{I}_C \arrow[r] & 0.
\end{tikzcd}$$

Finally, a straightforward dualisation argument demonstrates part (iii). ☐

Theorem I now follows from Theorem H by splicing together the basic sequences $\mathcal{B}(1) - \mathcal{B}(3)$. However, it may be beneficial to observe that we do not need Theorem H in its entirety to prove Theorem I. As such, we now provide an alternative proof to Theorem I.

Proof of Theorem I. Recall (1.1.5) in which Johnson has built the following sequence

$$\begin{tikzcd}
& & & K(3) \arrow[dr] & & & & K(2) \arrow[dr] & & & & K(1) \arrow[dr] & & & \\
0 \arrow[r] & R(1) \arrow[r] & \Lambda \arrow[ur] \arrow[rr] & & \Lambda \arrow[dr] \arrow[rr] & & P(2) \arrow[ur] \arrow[rr] & & \Lambda \arrow[dr] \arrow[rr] & & P(1) \arrow[ur] \arrow[rr] & & \Lambda \arrow[r] & R(1) \arrow[r] & 0 \\
& & & & & R(3) \arrow[ur] & & & & R(2) \arrow[ur] & & & & &
\end{tikzcd}$$

where $P(1) \oplus P(2) \cong \Lambda^2$. In our notation,

- $K(1) \sim L^*$, $K(2) \sim L$ and $K(3) \sim K$;
- $R(1) \cong \bar{I}_C$, $R(2) \cong \overline{(x-1)I_C}$ and $R(3) \cong \bar{I}_C^*$.

Now, we have shown that in fact we can construct a sequence such that $P(1) \cong \Lambda$ and so $P(2)$ is necessarily stably free of rank 1. Since Λ satisfies the Eichler condition, it follows that Λ has SFC by the Swan-Jacobinski Theorem. Consequently, $P(2) \cong \Lambda$. Using this along with what we have learned of the modules $R(i)$ and $K(i)$, we therefore have the following sequence

$$\begin{tikzcd}
 & & & K \arrow[dr] & & & & L \arrow[dr] & & & & L^* \arrow[dr] & & & \\
0 \arrow[r] & \bar{I}_C \arrow[r] & \Lambda \arrow[ur] \arrow[rr] & & \Lambda \arrow[dr] \arrow[rr] & & \Lambda \arrow[ur] \arrow[rr] & & \Lambda \arrow[dr] \arrow[rr] & & \Lambda \arrow[ur] \arrow[rr] & & \Lambda \arrow[r] & \bar{I}_C \arrow[r] & 0 \\
 & & & & & \bar{I}_C^* \arrow[ur] & & & & \overline{(x-1)I_C} \arrow[ur] & & & & &
\end{tikzcd}$$

thereby providing an affirmative answer to the sequencing conjecture for $q = 3$. ☐

At this point, it is worthwhile stating the nontrivial fact that each $K(i)$ is necessarily monogenic. While it is still unknown whether each $[K(i)]$ is straight, for our purposes it is entirely sufficient that each $[K(i)]$ has the structure of a fork.

5.7 Theorem F(5)

The purpose of this section is to prove Theorem $F(5)$. With the Λ-modules $\bar{I}_C^*$, $\bar{I}_C = (x-1)\bar{I}_C^*$ and K as before, we aim to show

$$\bar{I}_C \otimes \bar{I}_C^* \cong K \oplus \Lambda^{d-1} \tag{5.7.1}$$

where $d = (p-1)/3$. As an initial observation note $\{(\nu-1)\nu^i \otimes \nu^j \mid 0 \le i,\, j \le p-2\}$ forms a $\mathbf{Z}$-basis for $\bar{I}_C \otimes \bar{I}_C^*$ where ν^i is as defined in Chapter 3, and the x, y-actions as explained in Section 5.1. A simple calculation of $\mathbf{Z}$-ranks justifies the number of copies of Λ, and so we turn our attention to directly showing the isomorphism. This can be shown in the following three stages:

Stage 1: Find a useful description for $\bar{I}_C \otimes \bar{I}_C^*$, and find a part of this which is isomorphic to K;

Stage 2: Of what is left, demonstrate this is free;

Stage 3: Deduce that $\bar{I}_C \otimes \bar{I}_C^* \cong K \oplus \Lambda^{d-1}$.

Stage 1: Define the following modules for $1 \le r \le p-2$:

$$U(r) = span_{\mathbf{Z}}\{(\nu-1)\nu^{r+k} \otimes \nu^k \mid 0 \le k \le p-1\} \subset I_C \otimes I_C^*. \tag{5.7.2}$$

Using essentially the same arguments as Chapter 3, we can show:

Fact 1: For each $r \in \{1, \ldots, p-2\}$, $\{(\nu-1)\nu^{r+k} \otimes \nu^k \mid 0 \le k \le p-1\}$ is a $\mathbf{Z}$-basis for $U(r)$. Further, $U(r) \cong_{\Lambda_0} \Lambda_0$ for each r.

Fact 2: $U(r) \cap [U(1) + \cdots + U(r-1) + U(r+1) + \cdots + U(p-2)] = \{0\}$.

Fact 3: Set $U = U(1) \oplus \cdots \oplus U(p-2)$. We have a rank 1 lattice generated by

$$\begin{array}{rcllllll}
S & = & (\nu-1)\otimes 1 & + \ (\nu-1)\otimes\nu & + \ (\nu-1)\otimes\nu^2 & + \ \cdots & + \ (\nu-1)\otimes\nu^{p-2} \\
 & & & + \ (\nu-1)\nu\otimes\nu & + \ (\nu-1)\nu\otimes\nu^2 & + \ \cdots & + \ (\nu-1)\nu\otimes\nu^{p-2} \\
 & & & & + \ (\nu-1)\nu^2\otimes\nu^2 & + \ \cdots & + \ (\nu-1)\nu^2\otimes\nu^{p-2} \\
 & & & & & \ddots & \vdots \\
 & & & & & & + \ (\nu-1)\nu^{p-2}\otimes\nu^{p-2}.
\end{array}$$

In particular, using the fact that $(\nu^i - 1) = (\nu-1)\nu^{i-1} + \cdots + (\nu-1)\nu + (\nu-1)$, we may rewrite S as

$$S = (\nu-1)\otimes 1 + (\nu^2-1)\otimes\nu + \cdots + (\nu^i-1)\otimes\nu^{i-1} + \cdots + (\nu^{p-1}-1)\otimes\nu^{p-2}.$$

Evidently $S \notin U$, but we also notice that $Sx = S$ since

$$\begin{aligned} Sx = {}& S - [(\nu-1)\otimes 1 + \cdots + (\nu-1)\otimes\nu^{p-2}] + [(\nu-1)\nu\otimes\nu^{p-1} + \cdots + (\nu-1)\nu^{p-1}\otimes\nu^{p-1}] \\ = {}& S + (\nu-1)\otimes\nu^{p-1} + (\nu-1)(\nu+\cdots+\nu^{p-1})\otimes\nu^{p-1} \\ = {}& S + (\nu-1)\otimes\nu^{p-1} - (\nu-1)\otimes\nu^{p-1} \\ = {}& S \end{aligned}$$

As with Chapter 3, a near identical argument shows $(I_C \otimes I_C^*)/U$ is torsion free of rank 1. It therefore follows $I_C \otimes I_C^* \cong \mathbf{Z} \oplus \Lambda_0^{p-2}$, in which the module generated by S is isomorphic to $\mathbf{Z}$, i.e. $I_C \otimes I_C^* \cong S \oplus U$.

Upon introducing the y-action, we now need to find the part which represents K, and that which represents Λ^{d-1}. To do this, we first make a number of alterations to the $\mathbf{Z}$-basis of $S \oplus U$. First, we consider the elements of U of the form $-\otimes\nu^i$. The left hand side of these looks like $(\nu-1)\nu^r$ where $i+1 \le r \le i+p-2$. In particular, by summing the first r of these elements, we get

$$\begin{aligned} (\nu-1)(\nu^{i+r} + \cdots + \nu^{i+2} + \nu^{i+1}) = {}& (\nu-1)(\nu^{r-1} + \cdots + \nu + 1)\nu^{i+1} \\ = {}& (\nu^r - 1)\nu^{i+1}. \end{aligned}$$

By elementary basis change, we can therefore replace $U(r)$ by

$$U(r)' = span_{\mathbf{Z}}\{(\nu^r-1)\nu^{i+1}\otimes\nu^i \mid 0 \le i \le p-1\}.$$

This can clearly be done for each $1 \le r \le p-2$ in which $U(1)' = U(1)$. It is straightforward to show $U(r)' \cong_{\Lambda_0} \Lambda_0$, and that

$$U(i)' \cap (U(1)' + \cdots + U(i-1)' + U(i)' + \cdots + U(p-2)') = \{0\}.$$

If we write $U' = U(1)' \oplus \cdots \oplus U(p-2)'$, then we can replace $S \oplus U$ by $S \oplus U'$, as Λ_0-modules.

By introducing the y-action, we proceed to show $S + U(\alpha-1)' + U(\alpha^2-1)' \cong K$. To do so, we begin by setting $U_\alpha := U(\alpha-1)' + U(\alpha^2-1)'$. By considering representations, we show $U_\alpha \cong K_0$.

Proposition 5.7.3. $\rho_{U_\alpha}(x^{-1}) = \begin{pmatrix} \Psi & 0 \\ 0 & \Psi \end{pmatrix}$ *where Ψ is the $p \times p$ block given by*

$$\Psi_{ij} = \begin{cases} 1, & i = 1,\ j = p; \\ 1, & j = i-1,\ 2 \le i \le p; \\ 0, & o/w. \end{cases}$$

Proof. Label $e_i = (\nu^{\alpha-1}-1)\nu^i \otimes \nu^{i-1}$ for $1 \le i \le p$, and $e_{p+i} = (\nu^{\alpha^2-1}-1)\nu^i \otimes \nu^{i-1}$, also for $1 \le i \le p$. It is straightforward to see $e_i \cdot x = e_{i+1}$ for $1 \le i \le p-1$ and

$e_p \cdot x = e_1$. Likewise, $e_{p+i} \cdot x = e_{p+i+1}$ for $1 \leq i \leq p-1$ and $e_{2p} \cdot x = e_{p+1}$. The result now follows. □

Proposition 5.7.4. *$\rho_{U_\alpha}(y) = \begin{pmatrix} -\Phi^T & -\Phi^T \\ \Phi^T & 0 \end{pmatrix}$ where Φ^T is the $p \times p$ block given by*

$$\Phi^T_{ij} = \begin{cases} 1, & i = (j-1)\alpha + 1 \, (mod\, p); \\ 0, & o/w. \end{cases}$$

Proof. With e_i, e_{p+i} as defined above, consider the action of y^2. Start with e_1, then

$$\begin{aligned} [(\nu^{\alpha-1} - 1)\nu \otimes 1]y^2 &= (\nu^{\alpha^2-\alpha} - 1)\nu^\alpha \otimes 1 \\ &= (\nu^{\alpha^2} - \nu^\alpha) \otimes 1 \\ &= (\nu^{\alpha^2} - \nu) \otimes 1 - (\nu^\alpha - \nu) \otimes 1 \\ &= (\nu^{\alpha^2-1} - 1)\nu \otimes 1 - (\nu^{\alpha-1} - 1)\nu \otimes 1 \\ &= e_{p+1} - e_1. \end{aligned}$$

Likewise, for a general $2 \leq i \leq p$ we have:

$$\begin{aligned} [(\nu^{\alpha-1} - 1)\nu^i \otimes \nu^{i-1}]y^2 &= (\nu^{\alpha^2-\alpha} - 1)\nu^{\alpha i} \otimes \nu^{\alpha(i-1)} \\ &= (\nu^{\alpha^2} - \nu + \nu - \nu^\alpha)\nu^{\alpha(i-1)} \otimes \nu^{\alpha(i-1)} \\ &= (\nu^{\alpha^2-1} - 1)\nu^{\alpha(i-1)+1} \otimes \nu^{\alpha(i-1)} - (\nu^{\alpha-1} - 1)\nu^{\alpha(i-1)+1} \otimes \nu^{\alpha(i-1)} \\ &= e_{p+\alpha(i-1)+1} - e_{\alpha(i-1)+1}. \end{aligned}$$

Similarly, when we consider the elements e_{p+i} then, using $\alpha^3 = 1$ $(mod\, p)$, we note:

$$\begin{aligned} [(\nu^{\alpha^2-1} - 1)\nu \otimes 1]y^2 &= (\nu^{1-\alpha} - 1)\nu^\alpha \otimes 1 \\ &= (\nu - \nu^\alpha) \otimes 1 \\ &= -(\nu^{\alpha-1} - 1)\nu \otimes 1 \\ &= -e_1 \end{aligned}$$

and for a general $2 \leq i \leq p$:

$$\begin{aligned} [(\nu^{\alpha^2-1} - 1)\nu^i \otimes \nu^{i-1}]y^2 &= (\nu^{1-\alpha} - 1)\nu^{\alpha i} \otimes \nu^{\alpha(i-1)} \\ &= (\nu - \nu^\alpha)\nu^{\alpha(i-1)} \otimes \nu^{\alpha(i-1)} \\ &= -e_{\alpha(i-1)+1}. \end{aligned}$$

□

Proposition 5.7.5. *$S \cdot y^2 = S + \sum_{i=0}^{p-1}(\nu^{\alpha-1} - 1)\nu^{i+1} \otimes \nu^i$.*

Proof. Write S in the shorter form of $S = \sum_{i=0}^{p-2}(\nu^{i+1} - 1) \otimes \nu^i$, and consider those elements of the form $- \otimes \nu^i$. When we apply y^2 then we note $\nu^i \cdot y^2 = \nu^j \cdot y^2$ if and only if $i = j$ $(mod\, p)$. Furthermore we observe $\alpha i \neq p - \alpha$ $(mod\, p)$, i.e. $- \otimes \nu^{p-\alpha}$ will not appear when we make y^2 act on S. To see why, suppose $\alpha i = p - \alpha$, then $\alpha(i+1) = 0$ $(mod\, p)$ and hence $p|\alpha$ or $p|i+1$, which clearly cannot happen.

Now consider what happens when we apply y^2 to an arbitrary element $-\otimes\nu^i$. We get:

$$\begin{aligned}[(\nu^{i+1}-1)\otimes\nu^i]\cdot y^2 &= (\nu^{\alpha(i+1)}-1)\otimes\nu^{\alpha i}\\ &= (\nu^{\alpha(i+1)}-\nu^{\alpha i+1}+\nu^{\alpha i+1}-1)\otimes\nu^{\alpha i}\\ &= (\nu^{\alpha-1}-1)\nu^{\alpha i+1}\otimes\nu^{\alpha i}+(\nu^{\alpha i+1}-1)\otimes\nu^{\alpha i}\\ &= (\text{element of } U(\alpha-1)')+(\text{part of } S).\end{aligned}$$

Thus, we can write

$$S\cdot y^2 = S-(\nu^{p-\alpha+1}-1)\otimes\nu^{p-\alpha}+\sum_{i\neq p-\alpha}(\nu^{\alpha-1}-1)\nu^{i+1}\otimes\nu^i.$$

However, we also note that $(\nu^{p-\alpha+1}-1)\otimes\nu^{p-\alpha}=-(\nu^{\alpha-1}-1)\nu^{p-\alpha+1}\otimes\nu^{p-\alpha}\in U(\alpha-1)'$. The result now follows. □

Using Proposition 5.7.3, and the fact that $Sx=S$, we have:

Proposition 5.7.6. *Set $K'=S+U_\alpha$; then*

$$\rho_{K'}(x^{-1})=\begin{pmatrix}\Psi & 0 & 0_{p\times1}\\ 0 & \Psi & 0_{p\times1}\\ 0_{1\times p} & 0_{1\times p} & 1_{1\times1}\end{pmatrix}.$$

Likewise, using Propositions 5.7.4 and 5.7.5 we have:

Proposition 5.7.7.

$$\rho_{K'}(y)=\begin{pmatrix}-\Phi^T & -\Phi^T & 1_{p\times1}\\ \Phi^T & 0 & 0_{p\times1}\\ 0_{1\times p} & 0_{1\times p} & 1_{1\times1}\end{pmatrix}$$

When considering the basis elements of K_0, then we have:

Proposition 5.7.8. $\rho_{K_0}(x^{-1})=\begin{pmatrix}\Psi & 0\\ 0 & \Psi\end{pmatrix}$ *where Ψ is the $p\times p$ block given above.*

Proof. Denote the $\mathbf{Z}$-basis of K_0 as follows

$$f_i=(y-1)x^{i-1},\quad 1\le i\le p;$$

$$f_{p+j}=(y^2-1)x^{j-1},\quad 1\le j\le p.$$

Consider how x acts on the basis elements of K_0. First, $f_i\cdot x=f_{i+1}$ for $1\le i\le p-1$, and $f_p\cdot x=(y-1)=f_1$. Similarly, $f_{p+j}\cdot x=f_{p+j+1}$ for $1\le j\le p-1$, and $f_{2p}\cdot x=(y^2-1)=f_{p+1}$. The result now follows. □

Corollary 5.7.9.

$$\rho_K(x^{-1})=\begin{pmatrix}\Psi & 0 & 0_{p\times1}\\ 0 & \Psi & 0_{p\times1}\\ 0_{1\times p} & 0_{1\times p} & 1_{1\times1}\end{pmatrix}.$$

Proposition 5.7.10. $\rho_{K_0}(y)=\begin{pmatrix}0 & \Phi^T\\ -\Phi^T & -\Phi^T\end{pmatrix}$ *where Φ^T is the $p\times p$ block given above.*

Proof. With the f_i and f_{p+j} as above, consider the right action of y^2 on the basis elements of K_0 we note $xy^2 = y^2x^\alpha$, where $\theta(x) = x^\alpha$ as before. Consider now the action of y^2 on f_i. Evidently, $f_1 \cdot y^2 = (1 - y^2) = -f_{p+1}$. For $2 \leq i \leq p$, we have

$$f_i \cdot y^2 = -(y^2-1)x^{\alpha(i-1)} = -f_{p+\alpha(i-1)+1}.$$

Likewise, $f_{p+1} \cdot y^2 = (y - y^2) = (y-1) - (y^2-1) = f_1 - f_{p+1}$ and, for $2 \leq i \leq p$, we have

$$f_{p+i} \cdot y^2 = (y-1)x^{\alpha(i-1)} - (y^2-1)x^{\alpha(i-1)} = f_{\alpha(i-1)+1} - f_{p+\alpha(i-1)+1}.$$

It therefore follows that $\rho_{K_0}(y) = \begin{pmatrix} 0 & \Phi^T \\ -\Phi^T & -\Phi^T \end{pmatrix}$. □

By recalling $\Sigma_x y^2 = \Sigma_x(y^2-1) + \Sigma_x$, we have:

Corollary 5.7.11.

$$\rho_K(y) = \begin{pmatrix} 0 & \Phi^T & 0_{p\times 1} \\ -\Phi^T & -\Phi^T & 1_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 1_{1\times 1} \end{pmatrix}.$$

Proposition 5.7.12. *$S + U_\alpha \cong K$.*

Proof. Set

$$X = \begin{pmatrix} 0 & I_p & 0_{p\times 1} \\ I_p & 0 & 0_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 1_{1\times 1} \end{pmatrix},$$

which is clearly invertible over the integers. It is a straightforward observation that

$$\begin{aligned} \begin{pmatrix} \Psi & 0 & 0_{p\times 1} \\ 0 & \Psi & 0_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 1_{1\times 1} \end{pmatrix} \begin{pmatrix} 0 & I_p & 0_{p\times 1} \\ I_p & 0 & 0_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 1_{1\times 1} \end{pmatrix} &= \begin{pmatrix} 0 & \Psi & 0_{p\times 1} \\ \Psi & 0 & 0_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 1_{1\times 1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & I_p & 0_{p\times 1} \\ I_p & 0 & 0_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 1_{1\times 1} \end{pmatrix} \begin{pmatrix} \Psi & 0 & 0_{p\times 1} \\ 0 & \Psi & 0_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 1_{1\times 1} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} -\Phi^T & -\Phi^T & 1_{p\times 1} \\ \Phi^T & 0 & 0_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 1_{1\times 1} \end{pmatrix} \begin{pmatrix} 0 & I_p & 0_{p\times 1} \\ I_p & 0 & 0_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 1_{1\times 1} \end{pmatrix} &= \begin{pmatrix} -\Phi^T & -\Phi^T & 1_{p\times 1} \\ 0 & \Phi^T & 0_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 1_{1\times 1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & I_p & 0_{p\times 1} \\ I_p & 0 & 0_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 1_{1\times 1} \end{pmatrix} \begin{pmatrix} 0 & \Phi^T & 0_{p\times 1} \\ -\Phi^T & -\Phi^T & 1_{p\times 1} \\ 0_{1\times p} & 0_{1\times p} & 1_{1\times 1} \end{pmatrix}. \end{aligned}$$

In other words, $\rho_{K'}(g)X = X\rho_K(g)$ for all $g \in G(p,\,3)$. □

Next, we show that $(S+U')/(S+U_\alpha)$ is free. Let $\natural$ be the natural surjection $\natural : S+U' \to (S+U')/(S+U_\alpha)$ and set $U'_r = U(r)' + U(\alpha(r+1)-1)' + U(\alpha^2(r+1)-1)'$ for $r \neq \alpha - 1,\ \alpha^2 - 1$. We claim $\natural(U'_r) \cong \Lambda$ for each $r \neq \alpha - 1,\ \alpha^2 - 1$ in which $\natural(U'_r)$ represents the image of U'_r under $\natural$. As before we consider representations. For the x-action it is clear that:

Proposition 5.7.13. *Set* $U_r = \natural(U_r')$. *With* Ψ *as above, we have*

$$\rho_{U_r}(x^{-1}) = \begin{pmatrix} \Psi & 0 & 0 \\ 0 & \Psi & 0 \\ 0 & 0 & \Psi \end{pmatrix}.$$

Proposition 5.7.14.

$$\rho_{U_r}(y) = \begin{pmatrix} 0 & 0 & \Phi^T \\ \Phi^T & 0 & 0 \\ 0 & \Phi^T & 0 \end{pmatrix}$$

where Φ^T *is as above.*

Proof. Start by labelling

$$f_i = (\nu^r - 1)\nu^i \otimes \nu^{i-1},\ f_{p+j} = (\nu^{\alpha(r+1)-1} - 1)\nu^j \otimes \nu^{j-1},\ f_{2p+k} = (\nu^{\alpha^2(r+1)-1} - 1)\nu^k \otimes \nu^{k-1}$$

for $1 \le i,\, j,\, k \le p$. It is clear that $\natural(U(r)' + U(\alpha(r+1)-1)' + U(\alpha^2(r+1)-1)')$ has $\mathbf{Z}$-basis $\{\natural(f_i),\, \natural(f_{p+j}),\, \natural(f_{2p+k}) \mid 1 \le i,\, j,\, k \le p\}$.

First, for f_1 we observe

$$\begin{aligned} f_1 \cdot y^2 = & \quad (\nu^{\alpha r} - 1)\nu^{\alpha} \otimes 1 \\ = & \quad (\nu^{\alpha(r+1)} - \nu^{\alpha}) \otimes 1 \\ = & \quad (\nu^{\alpha(r+1)-1} - 1)\nu \otimes 1 - (\nu^{\alpha-1} - 1)\nu \otimes 1. \end{aligned}$$

It follows that $\natural(f_1) \cdot y^2 = \natural(f_{p+1})$. Likewise, for a general $2 \le i \le p$ we have:

$$\begin{aligned} f_i \cdot y^2 = & \quad (\nu^{\alpha r} - 1)\nu^{\alpha i} \otimes \nu^{\alpha(i-1)} \\ = & \quad (\nu^{\alpha(r+1)} - \nu^{\alpha})\nu^{\alpha(i-1)} \otimes \nu^{\alpha(i-1)} \\ = & \quad (\nu^{\alpha(r+1)-1} - 1)\nu^{\alpha(i-1)+1} \otimes \nu^{\alpha(i-1)} - (\nu^{\alpha-1} - 1)\nu^{\alpha(i-1)+1} \otimes \nu^{\alpha(i-1)}. \end{aligned}$$

Thus, $\natural(f_i)y^2 = \natural(f_{p+\alpha(i-1)+1})$.

By repeating the above argument we can see that:

$$\begin{aligned} f_{p+i} \cdot y^2 = & \quad (\nu^{\alpha^2(r+1)-\alpha} - 1)\nu^{\alpha i} \otimes \nu^{\alpha(i-1)} \\ = & \quad (\nu^{\alpha^2(r+1)} - \nu^{\alpha})\nu^{\alpha(i-1)} \otimes \nu^{\alpha(i-1)} \\ = & \quad (\nu^{\alpha^2(r+1)-1} - 1)\nu^{\alpha(i-1)+1} \otimes \nu^{\alpha(i-1)} - (\nu^{\alpha-1} - 1)\nu^{\alpha(i-1)+1} \otimes \nu^{\alpha(i-1)} \end{aligned}$$

and

$$\begin{aligned} f_{2p+i} \cdot y^2 = & \quad (\nu^{r+1-\alpha} - 1)\nu^{\alpha i} \otimes \nu^{\alpha(i-1)} \\ = & \quad (\nu^{r+1} - \nu^{\alpha})\nu^{\alpha(i-1)} \otimes \nu^{\alpha(i-1)} \\ = & \quad (\nu^r - 1)\nu^{\alpha(i-1)+1} \otimes \nu^{\alpha(i-1)} - (\nu^{\alpha-1} - 1)\nu^{\alpha(i-1)+1} \otimes \nu^{\alpha(i-1)}. \end{aligned}$$

It therefore follows that $\natural(f_{p+i})y^2 = \natural(f_{2p+\alpha(i-1)+1})$ and $\natural(f_{2p+i})y^2 = \natural(f_{\alpha(i-1)+1})$. The result is now shown. ☐

As has been shown previously, we have the following representations for Λ:

Proposition 5.7.15.

$$\rho_{reg}(x^{-1}) = \begin{pmatrix} \Psi & 0 & 0 \\ 0 & \Psi & 0 \\ 0 & 0 & \Psi \end{pmatrix}$$

where Ψ *is as above.*

Proposition 5.7.16.

$$\rho_{reg}(y) = \begin{pmatrix} 0 & \Phi^T & 0 \\ 0 & 0 & \Phi^T \\ \Phi^T & 0 & 0 \end{pmatrix}$$

where Φ^T *is as above.*

Proposition 5.7.17. $\natural(U(r)' + U(\alpha(r+1)-1)' + U(\alpha^2(r+1)-1)') \cong \Lambda$ *for* $r \neq \alpha - 1,\, \alpha^2 - 1$.

Proof. Let $X = \begin{pmatrix} 0 & 0 & I_p \\ 0 & I_p & 0 \\ I_p & 0 & 0 \end{pmatrix}$ which is clearly seen to be invertible over $\mathbf{Z}$. It is now straightforward to show

$$\begin{pmatrix} \Psi & 0 & 0 \\ 0 & \Psi & 0 \\ 0 & 0 & \Psi \end{pmatrix} \begin{pmatrix} 0 & 0 & I_p \\ 0 & I_p & 0 \\ I_p & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \Psi \\ 0 & \Psi & 0 \\ \Psi & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & I_p \\ 0 & I_p & 0 \\ I_p & 0 & 0 \end{pmatrix} \begin{pmatrix} \Psi & 0 & 0 \\ 0 & \Psi & 0 \\ 0 & 0 & \Psi \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & \Phi^T \\ \Phi^T & 0 & 0 \\ 0 & \Phi^T & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & I_p \\ 0 & I_p & 0 \\ I_p & 0 & 0 \end{pmatrix} = \begin{pmatrix} \Phi^T & 0 & 0 \\ 0 & 0 & \Phi^T \\ 0 & \Phi^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & I_p \\ 0 & I_p & 0 \\ I_p & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \Phi^T & 0 \\ 0 & 0 & \Phi^T \\ \Phi^T & 0 & 0 \end{pmatrix}.$$

Then $\rho_{U_r}(g)X = X\rho_{reg}(g)$ for all $g \in G(p,\,3)$. □

Corollary 5.7.18. $(S + U')/(S + U_\alpha) \cong \Lambda^{d-1}$.

Proof. As we are excluding $r = \alpha - 1$ or $r = \alpha^2 - 1$ when defining U_r, then it is clear that neither $\alpha(r+1)-1$ nor $\alpha^2(r+1)-1$ can be equivalent to $\alpha - 1$ or $\alpha^2 - 1$ $(mod p)$. It therefore remains to show we do not 'double-count' in any way. Thus, suppose $\alpha(r+1) - 1 \equiv s\,(mod p)$ for some $s \not\equiv r$. This is equivalent to $r \equiv \alpha^2(s+1) - 1\,(mod p)$ and $\alpha^2(r+1) - 1 \equiv \alpha(s+1) - 1\,(mod\,p)$. If $\alpha(r+1) - 1 \equiv \alpha(s+1) - 1\,(mod\,p)$ then this is equivalent to $r \equiv s\ (mod\ p)$. Finally, if $\alpha(r+1) - 1 \equiv \alpha^2(s+1) - 1\ (mod\ p)$ then this is equivalent to $r \equiv \alpha(s+1) - 1\ (mod\ p)$ and $\alpha^2(r+1) - 1 \equiv s\ (mod\ p)$. Consequently, we can be assured that each 'cycle' of three is distinct from any other. Thus, the only possibility left is that $(S + U')/(S + U_\alpha)$ is free of rank $d - 1$. □

We therefore have the following exact sequence

$$0 \to S + U_\alpha \to S + U' \to \Lambda^{d-1} \to 0.$$

As this clearly splits, we have $S + U' \cong_\Lambda (S + U_\alpha) \oplus \Lambda^{d-1}$. Since $\bar{I}_C \otimes \bar{I}_C^* \cong S + U'$ (thought of as a Λ-module), we apply Proposition 5.7.12, to conclude:

Proposition 5.7.19. $\bar{I}_C \otimes \bar{I}_C^* \cong K \oplus \Lambda^{d-1}$.

5.8 Theorem F(4)

The aim of this chapter is to show

$$\bar{I}_C \otimes L \cong \bar{I}_C^* \oplus \Lambda^{2d} \tag{5.8.1}$$

where L is as defined in Section 5.5. As ever, the number of copies of Λ is justified by a simple calculation of **Z**-ranks.

Proposition 5.8.2. *There exists the following exact sequence of* Λ*-modules,*

$$0 \to K \to \Lambda^{2d+1} \to \bar{I}_C \otimes L \to 0.$$

Proof. By the results of the sequencing conjecture for $q = 3$ (see Section 5.6) we have the following exact sequence

$$0 \to \bar{I}_C^* \to \Lambda \to L \to 0.$$

Apply the exact functor $\bar{I}_C \otimes -$ to yield the exact sequence,

$$0 \to \bar{I}_C \otimes \bar{I}_C^* \to \Lambda^{3d} \to \bar{I}_C \otimes L \to 0.$$

By Theorem F(5) (see Section 5.7), $\bar{I}_C \otimes \bar{I}_C^* \cong K \oplus \Lambda^{d-1}$ and so the above exact sequence becomes

$$0 \to K \oplus \Lambda^{d-1} \to \Lambda^{3d} \to \bar{I}_C \otimes L \to 0.$$

As usual, use Johnson's 'destabilization theorem' (Proposition 2.4.2) to form the required exact sequence,

$$0 \to K \to \Lambda^{2d+1} \to \bar{I}_C \otimes L \to 0.$$

□

Proposition 5.8.3. $\bar{I}_C \otimes L \cong \bar{I}_C^* \oplus \Lambda^{2d}$.

Proof. Recall that we have the exact sequence

$$0 \to K \to \Lambda \to \bar{I}_C^* \to 0.$$

Using this and the exact sequence of Proposition 5.8.2, we apply the dual form of Schanuel's Lemma to obtain the following isomorphism

$$(\bar{I}_C \otimes L) \oplus \Lambda \cong \bar{I}_C{}^* \oplus \Lambda^{2d+1}.$$

Next, we apply Proposition 5.2.9 (i.e. $[\bar{I}_C^*]$ is straight) to 'cancel' the free module on the left; that is, we have the desired isomorphism

$$\bar{I}_C \otimes L \cong \bar{I}_C^* \oplus \Lambda^{2d}.$$

□

Evidently, Proposition 5.8.3 proves Theorem F(4).

5.9 Theorem F(3)

Finally, we show F(3) of Theorem F; that is, we show

$$\bar{I}_C \otimes \overline{(x-1)I_C} \cong L \oplus \Lambda^{d-1} \tag{5.9.1}$$

where $d = (p-1)/3$, as usual. First, observe the following **Z**-basis for $\overline{(x-1)I_C} \otimes \bar{I}_C^*$,

$$\{(\nu-1)^2\nu^i \otimes \nu^j \mid 0 \leq i,\, j \leq p-2\}$$

where ν^i are as defined in Chapter 3, and the x, y-actions are as explained in Section 5.1. It will in fact be easier to work with the above **Z**-basis and then dualise. We now mirror the techniques of Section 5.5 and proceed in three stages:

Stage 1: Show $(x-1)I_C \otimes I_C^* \cong_{\Lambda_0} R \oplus W'$ for some Λ_0-modules R, W';

Stage 2: Upon applying the y-action, find the 'free part';

Stage 3: Of what remains, demonstrate that this is stably isomorphic to L^*. Finally, we dualise to reach the desired conclusion.

Stage 1: Define the following modules for $1 \leq r \leq p-2$:

$$W(r) = span_{\mathbf{Z}}\{(\nu-1)^2\nu^{r+k} \otimes \nu^k \mid 0 \leq k \leq p-1\} \subset (x-1)I_C \otimes I_C^*. \tag{5.9.2}$$

As with Chapter 3 and Section 5.7, we can show:

Fact 1: For each $1 \leq r \leq p-2$, $W(r) \cong_{\Lambda_0} \Lambda_0$.

Fact 2: $W(r) \cap [W(1) + \cdots + W(r-1) + W(r+1) + \cdots + W(p-2)] = \{0\}$.

Fact 3: Set $W = W(1) \oplus \cdots \oplus W(p-2)$. As with Chapter 3, a near identical argument shows $((x-1)I_C \otimes I_C^*)/W$ is torsion free of rank 1. It follows $(x-1)I_C \otimes I_C^* \cong R \oplus W$, where R is the rank 1 lattice generated by

$$\begin{array}{rclllll} R & = & (\nu-1)^2 \otimes 1 & + \; (\nu-1)^2 \otimes \nu & + \cdots & + \; (\nu-1)^2 \otimes \nu^{p-2} \\ & & & + \; (\nu-1)^2\nu \otimes \nu & + \cdots & + \; (\nu-1)^2\nu \otimes \nu^{p-2} \\ & & & & \ddots & \quad \vdots \\ & & & & & + \; (\nu-1)^2\nu^{p-2} \otimes \nu^{p-2}. \end{array}$$

Evidently $R \notin W$, but we also notice that $Rx = R$. In particular, the module generated by R is isomorphic to **Z**. However, when introducing the y-action, we find that the above description turns out to be less than helpful. Consequently, for $1 \leq r \leq p-2$ we define the following modules:

$$W(r)' = span_{\mathbf{Z}}\{(\nu^r-1)^2\nu^{r+k} \otimes \nu^k \mid 0 \leq k \leq p-1\}. \tag{5.9.3}$$

Proposition 5.9.4. *For each* $1 \leq r \leq p-2$, $W(r)' \cong_{\Lambda_0} \Lambda_0$.

Proof. As before, it is sufficient to show the above set is linearly independent. Consider first the case $r = 1$. We start with the relation relation,

$$\lambda_1(\nu-1)^2\nu \otimes 1 + \cdots + \lambda_p(\nu-1)^2 \otimes \nu^{p-1} = 0 \tag{5.9.5}$$

for some $\lambda_i \in \mathbf{Z}$. Using $\nu^{p-1} = -1 - \nu - \cdots - \nu^{p-2}$, this becomes

$$(\nu-1)^2(\lambda_1\nu - \lambda_p) \otimes 1 + \cdots + (\nu-1)^2(\lambda_{p-1}\nu^{p-1} - \lambda_p) \otimes \nu^{p-2} = 0. \tag{5.9.6}$$

It follows that each $\lambda_i = 0$ and so $W(1)' \cong_{\Lambda_0} \Lambda_0$.

Now, consider the case for a general $r \geq 2$ and write

$$\lambda_1(\nu^r - 1)^2\nu^r \otimes 1 + \cdots + \lambda_p(\nu^r - 1)^2\nu^{r-1} \otimes \nu^{p-1} = 0 \tag{5.9.7}$$

where once again each $\lambda_i \in \mathbf{Z}$. As before, by rewriting ν^{p-1} we transform this into

$$(\nu^r-1)^2(\lambda_1\nu^r - \lambda_p\nu^{r-1}) \otimes 1 + \cdots + (\nu^r-1)^2(\lambda_{p-1}\nu^{r-2} - \lambda_p\nu^{r-1}) \otimes \nu^{p-2} = 0. \tag{5.9.8}$$

Clearly, the only way this can be zero, is if each $(\nu^r - 1)^2(\lambda_i\nu^{r-1+i} - \lambda_p\nu^{r-1}) = 0$. Now, to show this implies $\lambda_i = 0$ we will have to do a significant amount of work. As such, it may be useful for the reader to bear in mind a 'road map' of the following proof. We shall focus our attention on the elements of the form $- \otimes 1$. Our intention will be to show $\lambda_p = 0$ so that we are left with a number of terms of the form $\lambda_1\mu_i(\nu-1)^2\nu^i \otimes 1$ where $0 \leq i \leq p-2$, and μ_i are integers that are not all zero. As each of these terms are independent from one another, it follows that $\lambda_1\mu_i = 0$ and hence $\lambda_1 = 0$. We can then repeat this idea to show $\lambda_i = 0$ for all i. That said, a significant amount of work will be needed to show $\lambda_p = 0$. We shall show this by first considering how r varies. By dividing r into a number of regions, we will end up with an element of the form $\lambda_p\mu_i(\nu-1)^2\nu^i \otimes 1$ that does not involve λ_1.

With the above in mind, consider the elements of the form $(\nu^r - 1)^2(-) \otimes 1$. We start by observing $(\nu^r - 1)^2\nu^r = (\nu - 1)^2(\nu^{r-1} + \cdots + \nu + 1)^2\nu^r$. This motivates us to split the computation into three ranges for r:

1. $2 \leq r \leq d$;
2. $d + 1 \leq r \leq 2d$;
3. $2d + 1 \leq r \leq p - 2$.

- If $2 \leq r \leq d$, then $1 \leq r - 1 \leq d - 1$.

When we expand $(\nu^{r-1} + \cdots + r + 1)^2$ the largest term is of degree $2(r-1) < 2d < p-1$, and the smallest is of degree 0. In other words, ν^{p-1} will not appear at any point. Now, the left hand side of $-\otimes 1$ has the form $(\nu^r - 1)^2(\lambda_1\nu^r - \lambda_p\nu^{r-1})$. Multiplying the above by ν^r then the largest term has degree $2(r-1)+r \leq 2(d-1)+d = 3d-2 = p-3 < p-1$, and so once more we will have no cancellation arising from ν^{p-1}. The smallest term will have degree r.

Next, multiply by ν^{r-1}. The largest term will therefore be of degree $p - 4$, and smallest term of degree $r - 1$. Hence, for $- \otimes 1$ the left hand side will be of the form

$$(\nu-1)^2(\lambda_1\nu^{3r-2} + (\text{terms involving } \lambda_1 \text{ and } \lambda_p) - \lambda_p\nu^{r-1}).$$

Therefore, for $(\nu^r - 1)^2(\lambda_1\nu^r - \lambda_p\nu^{r-1}) \otimes 1 = 0$ we have $\lambda_1(\nu-1)^2\nu^{3r-2} \otimes 1 = 0$ and $\lambda_p(\nu-1)^2\nu^{r-1} \otimes 1 = 0$. Hence, $\lambda_1 = \lambda_p = 0$.

- Now let r vary between $d + 1 \leq r \leq 2d$.

Unlike the previous range, we will now have to deal with cancellation in the form of ν^{p-1}. First, we use the following well known expansion of $(\nu^{r-1}+\cdots+\nu+1)^2$; that is,

$$(\nu^{r-1}+\cdots+\nu+1)^2 = \nu^{2r-2}+2\nu^{2r-3}+\cdots+(r-1)\nu^{r}+r\nu^{r-1}+(r-1)\nu^{r-2}+\cdots+2\nu+1$$

and so we have

$$(\nu^{r-1}+\cdots+\nu+1)^2\nu^r = \nu^{3r-2}+\cdots+(r-1)\nu^{2r}+r\nu^{2r-1}+(r-1)\nu^{2r-2}+\cdots+\nu^{r}.$$

Our first observation is that ν^{2r-1} (i.e. the term with the largest coefficient) cannot be ν^{p-1}. To see this, suppose $2r-1 \equiv p-1\,(mod p)$. This is equivalent to $2r \equiv 0\,(mod p)$, i.e. $p|r$. However, this is not possible and it therefore follows that the coefficient of ν^{p-1} (if it appears) is less than r. In particular, this coefficient will pair up with some other ν^i, i.e. we get a cancellation. This cancellation will be key to showing $\lambda_1=\lambda_p=0$.

Before proceeding further, return once more to the expansion of $(\nu^{r-1}+\cdots+1)^2$ (i.e. before we multiply by ν^r). By writing $r=d+i$ for $1\le i\le d$, it is straightforward to see that $\nu^{2r-j}=\nu^{p-1}$ $(2\le j\le 2r)$ if and only if $j\equiv 2i-d\,(mod p)$. This motivates us to further divide our region into another two sub-regions: $d+1\le r\le d+(d/2)$ and $d+(d/2)+1\le r\le 2d$.

- Let r vary over $d+1\le r\le d+(d/2)$.

  Here, ν^{p-1} will not appear until we multiply by ν^r. Once multiplied, we want to know when $\nu^{3r-j}=\nu^{p-1}$ where $2\le j\le 2r$. This happens precisely when $3r-j-p+1=mp$ for some $m\in\mathbf{Z}$. We rewrite this as $3(d+i)-j-3d=mp$, or as $j=3i\,(mod p)$. In fact, since $3\le 3i\le (3d)/2=(p-1)/2$, the $mod p$ turns out to be superfluous.

  It is quite clear that the coefficient in this case is $j-1$. By symmetry, it follows that $j-1$ is a coefficient for some ν^α where $r\le\alpha\le 2r-2$. In particular, $j-1$ is also the coefficient of $\nu^{r+(3i-2)}=\nu^{d+4i-2}$. We conclude, then, that once we cancel the ν^{p-1}, we will certainly not have ν^{d+4i-2}. In other words, we will not have $\lambda_1(\nu-1)^2\nu^{d+4i-2}\otimes 1$.

  We now do the same, but for ν^{r-1}. We intend to show that ν^{d+4i-2} will appear once we have cancelled the ν^{p-1}. We will then be left with λ_p (or some integer multiple of λ_p) on its own. This will imply that $\lambda_p=0$, and hence $\lambda_1=0$. So, we have

  $$\begin{aligned}(\nu^{r-1}+\cdots+1)^2\nu^{r-1} &= \nu^{3r-3}+2\nu^{3r-4}+\cdots\\ &\quad\cdots+(r-1)\nu^{2r-1}+r\nu^{2r-2}+\ (r-1)\nu^{2r-3}+\cdots+2\nu^{r}+\nu^{r-1}.\end{aligned}$$

  By a similar argument to the above, write ν^{3r-k} for $3\le k\le 2r+1$ and observe $\nu^{3r-k}=\nu^{p-1}$ precisely when $k=3i$. The coefficient in this case is $3i-2$. By symmetry, this will also be the coefficient of $\nu^{r-1+3i-3}=\nu^{d+4i-4}$ and, furthermore, $d+4i-4\not\equiv d+4i-2\ (mod\, p)$. Indeed, it is a straightforward observation that the coefficient of ν^{d+4i-2} in this expansion is $3i$. Therefore, once we cancel the ν^{p-1} we will be left with, among other terms, $2\lambda_p(\nu-1)^2\nu^{d+4i-2}\otimes 1$. It follows then, that $2\lambda_p=0$ and hence $\lambda_p=0$.

This leaves us with $\lambda_1(\nu-1)^2(\nu^{r-1}+\cdots+1)^2\nu^r\otimes 1$. As we have already observed, ν^{p-1} will appear. However, this will not cancel all the terms, and those remaining will be of the form $\lambda_1\mu_\alpha(\nu-1)^2\nu^\alpha\otimes 1$, where at least one integer $\mu_\alpha\neq 0$ and $0\leq\alpha\leq p-2$. It therefore follows that $\lambda_1\mu_\alpha=0$ for all α, and so $\lambda_1=0$.

– Let r vary over $d+(d/2)+1\leq r\leq 2d$.

Unlike the previous range, ν^{p-1} will appear when we expand $(\nu^{r-1}+\cdots+1)^2$ (before multiplying by ν^r). In particular, this occurs when $j=2i-d$ and with coefficient $2i-d-1$. We observe $2i-d-1\neq r=d+i$ and so, by symmetry, ν^{2i-d-2} also has coefficient $2i-d-1$. So, when we cancel the terms arising from ν^{p-1} we will eliminate ν^{2i-d-2}. The terms of degree $2i-d-1\leq\alpha\leq p-2$ will remain and have a positive coefficient which we shall discuss shortly. For $i=(d/2)+1$, $\nu^{2(r-1)}=\nu^{p-1}$ and so we have no other terms to worry about. For $d+(d/2)+2\leq r\leq 2d$ we also have terms of degree $2r-(2i-d)+\beta$ where $1\leq\beta\leq 2i-d-2$ and of degree γ where $0\leq\gamma\leq 2i-d-3$. We claim that these latter terms will all cancel with the $(2i-d-1)\nu^{p-1}$.

To see this, observe that the $\nu^{2r-(2i-d)+\beta}=\nu^{\beta-1}$, and that $0\leq\beta-1\leq 2i-d-3$. In other words, the terms 'before' $(2i-d-1)\nu^{p-1}$ pair up with the terms 'after' $(2i-d-1)\nu^{2i-d-2}$. Furthermore, the coefficient of $\nu^{\beta-1}$ is $(2i-d-1)-\beta$, and that of ν^γ is $\gamma+1$. Thus, when $\nu^\gamma=\nu^{\beta-1}$, their coefficients add up to $2i-d-1$ and are therefore eliminated by $(2i-d-1)\nu^{p-1}$.

Finally, we have the terms of degree $2i-d-1\leq\alpha\leq p-2$. Clearly, ν^{r-1} has coefficient r, and so when we cancel this with the term arising from $(2i-d-1)\nu^{p-1}$ we have the coefficient of ν^{r-1} as $r-2i+d+1=2d-i+1$. For $\nu^{r-1+\alpha_1}$ where $1\leq\alpha_1\leq 2d-i$, the coefficient is seen to be $r-\alpha_1$. When we cancel with the terms arising from $(2i-d-1)\nu^{p-1}$ the coefficient of $\nu^{r-1+\alpha_1}$ is seen to be $(2d-i+1)-\alpha_1$. Similarly, the coefficient of $\nu^{r-1-\alpha_2}$ $(1\leq\alpha_2\leq 2d-i)$ is $r-\alpha_2$. When we cancel with the terms arising from $(2i-d-1)\nu^{p-1}$, the coefficient of $\nu^{r-1-\alpha_2}$ is seen to be $(2d-i+1)-\alpha_2$. Putting this all together, $(\nu^{r-1}+\cdots+1)^2$ becomes

$$(\nu^{r-1}+\cdots+1)^2=\nu^{p-2}+2\nu^{p-3}+\cdots+(2d-i+1)\nu^{r-1}+\cdots+2\nu^{2i-d}+\nu^{2i-d-1}.$$

The rest follows in a similar manner to the range $d+1\leq r\leq d+(d/2)$. First, we multiply by ν^r and get

$$(\nu^{r-1}+\cdots+1)^2\nu^r=\nu^{r-2}+2\nu^{r-3}+\cdots+(2d-i+1)\nu^{2r-1}+\cdots+2\nu^{r+2i-d}+\nu^{r+2i-d-1}.$$

A straightforward calculation shows $\nu^{p-1}=\nu^{2r-j}$ if and only if $j=2i-d$. In this case, the coefficient is clearly seen to be $2d-i+1-(j-1)=3(d-i)+2$. Since the coefficient of ν^{r-k} is $k-1$, it follows that the coefficient of ν^{p-1} is paired with the term $\nu^{r-3(d-i+1)}=\nu^{4i-2d-3}$. Thus, once we cancel the ν^{p-1}, we can be sure that $\lambda_1(\nu-1)^2\nu^{4i-2d-3}\otimes 1$ will not appear.

We now multiply $(\nu^{r-1}+\cdots+1)^2$ by ν^{r-1} to yield

$$(\nu^{r-1}+\cdots+1)^2\nu^{r-1}=\nu^{r-3}+2\nu^{r-4}+\cdots+(2d-i+1)\nu^{2r-2}+\cdots+2\nu^{r+2i-d-1}+\nu^{r+2i-d-2}.$$

Our first observation, is that if $r = d + (d/2) + 1$ then $\nu^{2r-2} = \nu^{p-1}$. As $2d-i+1 = (3d)/2$ is the largest coefficient, it will not eliminate any of the other coefficients. In particular, $\nu^{4i-2d-3} = \nu$ will appear. Once we have rewritten ν^{p-1}, ν will have coefficient -2. As such, $-2\lambda_p(\nu-1)^2\nu \otimes 1$ will appear and hence $\lambda_p = 0$. It follows that $\lambda_1 = 0$.

Next, suppose $d + (d/2) + 2 \leq r \leq 2d$. Once again, it is clear that $\nu^{r-k} \neq \nu^{p-1}$ where $3 \leq k \leq p - r + 1$. As such, $\nu^{p-1} = \nu^{2r-j}$ for some $2 \leq j \leq 2d - i + 2$. Clearly, $j = 2i-d$, and the coefficient will be $2d-i+1-(2i-d-2) = 3(d-i+1)$. By symmetry, $\nu^{2r-2+(2i-d-2)} = \nu^{d+4(i-1)}$ will have coefficient $3(d-i+1)$ also. Consequently, this will be eliminated by $3(d-i+1)\nu^{p-1}$. In particular, we note $\nu^{d+4(i-1)} \neq \nu^{4i-2d-3}$. Indeed, we observe $\nu^{4i-2d-3} = \nu^{r-3(d-i+1)}$ and hence has coefficient $3(d-i+1)-2 = 3(d-i)+1$. It therefore follows that, upon cancelling the terms arising from $(3(d-i+1))\nu^{p-1}$ we end up with, among other terms, $-2\lambda_p(\nu-1)^2\nu^{4i-2d-3} \otimes 1$. It therefore follows that $-2\lambda_p = 0$ and hence $\lambda_p = 0$, as required. It follows that $\lambda_1 = 0$.

- Finally, let r vary between $2d+1 \leq r \leq p-2$.

As with the previous range, we can show (through essentially identical reasoning) that

$$(\nu^{r-1} + \cdots + 1)^2 = \nu^{p-2} + 2\nu^{p-3} + \cdots + (2d-i+1)\nu^{r-1} + \cdots + 2\nu^{2i-d} + \nu^{2i-d-1}.$$

However, unlike the previous range, we will no longer have ν^{p-1} when we multiply by ν^r or ν^{r-1}. To see this, first consider

$$(\nu^{r-1} + \cdots + 1)^2\nu^r = \nu^{r-2} + 2\nu^{r-3} + \cdots + (2d-i+1)\nu^{2r-1} + \cdots + 2\nu^{r+2i-d} + \nu^{r+2i-d-1}.$$

By considering the largest term, we notice that $2d-1 \leq r-2 \leq p-4 = 3(d-1)$. For the 'smallest' term we use $r = d+i$ to first write $r + 2i - d - 1 = 3i - 1$. Since $d+1 \leq i \leq 2d-1$, we observe $p + 1 = 3d + 2 \leq 3i - 1 \leq 6d - 4 = 2(p-3)$. Hence, ν^{3i-1} varies between ν and ν^{p-6}. In any case, it is clear that ν^{p-1} will not appear in the expansion of $(\nu^{r-1} + \cdots + 1)^2\nu^r$. A similar argument shows ν^{p-1} will not appear when multiplying $(\nu^{r-1} + \cdots + 1)^2$ by ν^{r-1}.

It follows that the largest term when multiplying by ν^r is ν^{r-2}, and the largest term when multiplying by ν^{r-1} is ν^{r-3}. Similarly, the smallest term when multiplying by ν^r is ν^{3i-1}, and when multiplying by ν^{r-1} is ν^{3i-2}. As such, for $- \otimes 1$, the left hand side will be of the form

$$(\nu-1)^2(\lambda_1\nu^{r-2} + (\text{terms involving } \lambda_1 \text{ and } \lambda_p) - \lambda_p\nu^{3i-2}).$$

A straightforward argument now shows $\lambda_1 = \lambda_p = 0$.

In each case we note $\lambda_1 = \lambda_p = 0$. Returning to (5.9.7) we rewrite this as

$$\lambda_2(\nu^r - 1)^2\nu^{r+1} \otimes \nu + \cdots + \lambda_{p-1}(\nu^r - 1)^2\nu^{r-2} \otimes \nu^{p-2} = 0. \tag{5.9.9}$$

By once more expanding $(\nu^r - 1)^2 = (\nu-1)(\nu^{r-1} + \cdots + \nu + 1)^2$, then (5.9.9) becomes a sum of terms, each of the form $\lambda_i(\nu-1)^2\nu^j \otimes \nu^k$ for some $0 \leq j,\, k \leq p-2$. Since these terms are linearly independent, we therefore conclude that each $\lambda_i = 0$. It therefore follows that $W(r)' \cong_{\Lambda_0} \Lambda_0$, as required. ☐

As with Theorems F(1) and F(5), a similar proof to that of Proposition 3.1.4 (and using similar reasoning to that of Proposition 5.9.4) shows

$$W(r)' \cap (W(1)' + \cdots + W(r-1)' + W(r+1)' + \cdots + W(p-2)') = \{0\}.$$

So we define $W' = W(1)' \oplus \cdots \oplus W(p-2)' \subset (x-1)I_C \otimes I_C^*$. In particular, it is straightforward to show $W \cong_{\Lambda_0} W'$. Thus, we can therefore conclude that

$$(x-1)I_C \otimes I_C^* \cong_{\Lambda_0} R \oplus W'.$$

Stage 2: Upon extending the action of C_p to an action of $G(3,\,p)$, we show $W_r = W(r)' + W(\alpha r)' + W(\alpha^2 r)' \cong_\Lambda \Lambda$ for $1 \leq r \leq p-2$. By repeating the arguments of Propositions 5.5.3 - 5.5.4 and (with only minor notational differences to account for the $(\nu^r - 1)^2$), we therefore have the following results:

Proposition 5.9.10. *Set* $W_r = W(r)' + W(\alpha r)' + W(\alpha^2 r)'$. *Then*

$$\rho_{W_r}(x^{-1}) = \begin{pmatrix} \Psi & 0 & 0 \\ 0 & \Psi & 0 \\ 0 & 0 & \Psi \end{pmatrix},$$

where

$$(\Psi)_{ij} = \begin{cases} 1, & i = 1,\, j = p; \\ 1, & j = i-1,\, 2 \leq i \leq p; \\ 0, & o/w. \end{cases}$$

Proposition 5.9.11. *The integral representation of* W_r *with respect to the* y*-action is given by*

$$\rho_{W_r}(y) = \begin{pmatrix} 0 & 0 & \Phi^T \\ \Phi^T & 0 & 0 \\ 0 & \Phi^T & 0 \end{pmatrix}$$

where

$$\Phi^T = \begin{cases} 1, & i = (j-1)\alpha + 1 \,(mod\, p); \\ 0, & o/w. \end{cases}$$

As with Sections 5.5 and 5.7, a straightforward calculation shows $\rho_{reg}(x^{-1}) = \rho_{W_r}(x^{-1})$ and

$$\rho_{reg}(y) = \begin{pmatrix} 0 & \Phi^T & 0 \\ 0 & 0 & \Phi^T \\ \Phi^T & 0 & 0 \end{pmatrix}$$

where Φ^T is as defined above. By once more taking the invertible matrix

$$X = \begin{pmatrix} 0 & 0 & I_p \\ 0 & I_p & 0 \\ I_p & 0 & 0 \end{pmatrix}$$

it may be shown $\rho_{reg}(g)X = X\rho_{W_r}(g)$ for all $g \in G(p,\,3)$. In other words:

Proposition 5.9.12. $W(r)' + W(\alpha r)' + W(\alpha^2 r)' \cong \Lambda$ *for* $1 \leq r \leq p-2$.

As $3|p-1$ we get $d-1$ copies of Λ, each arising as W_r, for some r. Left over is $R+W(\alpha+1)'+W(p-\alpha)'$ (by the same argument as Proposition 5.5.8). Let $\widetilde{W}$ be the sum of the $W(i)'$ *without* $W(\alpha+1)'$ and $W(p-\alpha)'$, i.e $W'=\widetilde{W}+W(\alpha+1)'+W(p-\alpha)'$ and $\widetilde{W}\cong\Lambda^{d-1}$. Now let κ be the natural surjection

$$\kappa:\overline{(x-1)I_C}\otimes\bar{I}_C^*\to(\overline{(x-1)I_C}\otimes\bar{I}_C^*)/\widetilde{W}.$$

If we set $Y=\kappa(R+W(\alpha+1)'+W(p-\alpha)')$, the image of $R+W(\alpha+1)'+W(p-\alpha)'$ under κ, then:

Proposition 5.9.13. $\overline{(x-1)I_C}\otimes\bar{I}_C^*\cong Y\oplus\Lambda^{d-1}$.

Stage 3: Finally, we demonstrate that what remains is stably isomorphic to L^*. The result of Theorem F(3) is then an immediate consequence of dualisation.

Proposition 5.9.14. $Y\sim L^*$

Proof. As we saw in Section 5.6, we have the short exact sequence,

$$0\to\overline{(x-1)I_C}\to\Lambda\to L^*\to 0.$$

Applying the exact functor $-\otimes\bar{I}_C^*$ this becomes,

$$0\to\overline{(x-1)I_C}\otimes\bar{I}_C^*\to\Lambda^{3d}\to L^*\otimes\bar{I}_C^*\to 0.$$

By the dual form of Proposition 5.8.3 we have $L^*\otimes\bar{I}_C^*\cong\bar{I}_C\oplus\Lambda^{2d}$, and by Proposition 5.9.13 we have $\overline{(x-1)I_C}\otimes\bar{I}_C^*\cong Y\oplus\Lambda^{d-1}$. Replacing this in the above exact sequence yields

$$0\to Y\oplus\Lambda^{d-1}\to\Lambda^{3d}\to\bar{I}_C\oplus\Lambda^{2d}\to 0.$$

As we have done many times throughout this chapter, we use Proposition 2.4.2 to 'cancel' the free modules. We are left with the following short exact sequence,

$$0\to Y\to\Lambda\to\bar{I}_C\to 0.$$

Finally, recall the short exact sequence (also seen in Section 5.6),

$$0\to L^*\to\Lambda\to\bar{I}_C\to 0.$$

The result is now a consequence of Schanuel's Lemma; that is $Y\sim L^*$. ☐

Corollary 5.9.15. $\overline{(x-1)I_C}\otimes\bar{I}_C^*\cong L^*\oplus\Lambda^{d-1}$.

Evidently, (5.9.1) and hence Theorem F(3) follows from the dual statement of Corollary 5.9.15. Combining this result with those of Sections 5.5-5.8 concludes the proof of Theorem F.

We conclude this section with a brief discussion of a possible next step in the research of syzygy modules over $G(p,\,3)$. First, and most obvious, is to better understand the difficult descriptions of L, L^*. However, as we saw with K, which has a much nicer description than either L or L^*, the going is still tough. Indeed, this stems from the fact that at present it is unknown whether there is any branching at the minimal level of the respective tree diagrams of the even syzygies. It is for this reason that the author believes any future research into this area for $G(p,\,3)$ should begin by an attempt to tackle this question.

5.10 The module K: a general case

We conclude this chapter with a generalisation of our methods in Section 5.4. In this section, let p be prime, and q any prime such that $q|p-1$, and in which x, y are generators of C_p, C_q, respectively. As before, we write $d = (p-1)/q$ and choose $\theta \in Aut(C_p)$ such that $ord(\theta) = q$. We can then construct the semi-direct product $G(p, q) = C_p \rtimes_h C_q$, where $h : C_q \to Aut(C_p)$ is the homomorphism given by $h(y)(x) = \theta(x) = x^\alpha$ for some $2 \leq \alpha \leq p-1$. In particular, we can write this as

$$G(p, q) = \langle x, y \mid x^p = y^q = 1, yxy^{-1} = x^\alpha \rangle.$$

Once more, we set $\Lambda = \mathbf{Z}[G(p, q)]$ and observe the following fibre square decomposition for Λ (see [25], [41] or Example 6.2.12):

$$\begin{tikzcd}
\Lambda \arrow[r] \arrow[d] & \mathcal{T}_q(A, \pi) \arrow[d] \\
\mathbf{Z}[C_q] \arrow[r] & \mathbf{F}_p[C_q]
\end{tikzcd}$$

where $A = \mathbf{Z}[\zeta_p]^\theta$ is the subring of $\mathbf{Z}[\zeta_p]$, fixed by θ. It is of note that θ acts on $\mathbf{Z}[\zeta_p]$ via the isomorphism $Gal(\mathbf{Q}(\zeta_p)/\mathbf{Q}) \cong C_{p-1}$.

Now, if $R(i)$ is the i^{th} row of $\mathcal{T}_q(A, \pi)$, then $\mathcal{T}_q(A, \pi)$ decomposes as a direct sum of right Λ-modules thus

$$\mathcal{T}_q(A, \pi) \cong R(1) \oplus \cdots \oplus R(q).$$

In particular, the $R(i)$ are pairwise isomorphically distinct and we have the following duality relation

$$R(i)^* \cong R(q+1-i). \tag{5.10.1}$$

By composing the projections $\Lambda \twoheadrightarrow \mathcal{T}_q(A, \pi)$ and $\mathcal{T}_q(A, \pi) \twoheadrightarrow R(i)$ it follows that each $R(i)$ is monogenic. Hence, for each $i \in \{1, \ldots, q\}$ there is an exact sequence

$$\mathcal{X}(i) = (0 \to K(i) \to \Lambda \to R(i) \to 0).$$

Moreover, in [25] it was shown for $i \in \{1, \ldots, q-1\}$ that there exists an exact sequence

$$\mathcal{W}(i) = (0 \to R(i+1) \to P(i) \to K(i) \to 0),$$

in which each $P(i)$ is projective of rank 1 over Λ, and $\bigoplus_{i=1}^{q-1} P(i) \cong \Lambda^{q-1}$. For $i = q$ there is an exact sequence,

$$\mathcal{W}(q) = (0 \to R(1) \to \Lambda \to K(q) \to 0).$$

Finally, we relate this to the stable syzygies $\Omega_i(\mathbf{Z})$ for $0 \leq r \leq 2q-1$. In [41] it was shown that the minimal level of $\Omega_r(\mathbf{Z})$ can be described as

$$\Omega_r(\mathbf{Z}) = \begin{cases} R(i+1) \oplus [y-1), & r = 2i+1; \\ Q(i+1) \oplus [\Sigma_y), & r = 2i. \end{cases}$$

Above, $\Sigma_y = y^{q-1} + \cdots + y + 1$ and $Q(i) = \Lambda/R(i)$.

Proposition 5.10.2. *$Q(1) \sim K(q)$ and $Q(i+1) \sim K(i)$ for $1 \le i \le q-1$.*

Proof. Begin with $Q(1)$, which is defined by the exact sequence

$$0 \to R(1) \to \Lambda \to Q(1) \to 0.$$

By dualising, and using (5.10.1), we have

$$0 \to Q(1)^* \to \Lambda \to R(q) \to 0.$$

By Schanuel's Lemma, it therefore follows that $Q(1)^* \oplus \Lambda \cong K(q) \oplus \Lambda$. However, by the duality relation $\Omega_r(\mathbf{Z})^* \cong \Omega_{2q-r}(\mathbf{Z})$, a straightforward argument now shows $Q(1)^* \sim Q(1)$. Thus, $Q(1) \sim K(q)$, as required. In particular, since even syzygies have a fork structure (Proposition 2.7.6), $Q(1) \oplus \Lambda \cong K(q) \oplus \Lambda$.

For a general i, start with the exact sequence

$$0 \to K(i) \to \Lambda \to R(i) \to 0.$$

Dualising and using (5.10.1) gives the exact sequence

$$0 \to R(q+1-i) \to \Lambda \to K(i)^* \to 0$$

and so $K(i)^* \sim Q(q+1-i)$ by Schanuel. However, $Q(q+1-i)$ is a minimal representative of the x-strand of $\Omega_{2(q-i)}(\mathbf{Z})$. Since $\Omega_{2(q-i)}(\mathbf{Z})^* = \Omega_{2i}(\mathbf{Z})$, it follows that $Q(q+1-i)^* \sim Q(i+1)$. Thus, $K(i) \sim Q(i+1)$. $\square$

Corollary 5.10.3. *$K(q)^* \sim K(q)$ and $K(i)^* \sim K(q-i)$ for $1 \le i \le q-1$.*

Now, we define the module $K = [y^{q-1}-1, \ldots, y^2-1, y-1, \Sigma_x)$. Both the statement and result of Proposition 5.1.2 extends to a general prime q. As such, the proofs of Propositions 5.4.1 and 5.4.2 may be easily modified to show:

Proposition 5.10.4. *K has $\mathbf{Z}$-basis*

$$\mathcal{E} = \{(y^i-1)x^j \mid 1 \le i \le q-1,\, 0 \le j \le p-1\} \cup \{\Sigma_x\}.$$

Proposition 5.10.5. *$\Lambda/K \cong \bar{I}_C^*$.*

Corollary 5.10.6. *$K \sim K(q)$.*

Recalling that $K(q)^* \sim K(q)$, we now have:

Corollary 5.10.7. *$K^* \sim K$.*

Finally, as with the case $q = 3$ we have the following exact sequence,

$$0 \to K_0 \to K \to \mathbf{Z} \to 0$$

as x, y act trivially on K/K_0. It to this exact sequence that we shall turn to when proving $K \otimes X \cong X \oplus \Lambda^a$, where $a \ge 1$ and $X = R(i)$ or $K(i)$ for some $1 \le i \le q$. However, unlike Section 5.4, it is not true in general that $\widetilde{K}_0(\mathbf{Z}[C_q]) = 0$. We therefore adopt a different strategy. First, observe the following result concerning $R(1)$ (see [25], Corollary 1.9):

Proposition 5.10.8. $R(1) \otimes [y-1) \cong \Lambda^{d(q-1)}$.

Corollary 5.10.9. $K(1) \otimes [y-1) \cong \Lambda^{(q-1)(p-d)}$.

Proof. Consider first the exact sequence

$$0 \to K(1) \to \Lambda \to R(1) \to 0.$$

As $[y-1)$ is free as an additive group, then $-\otimes_{\mathbf{Z}}[y-1)$ is an exact functor on underlying abelian groups. In particular,

$$0 \to K(1) \otimes [y-1) \to \Lambda^{(q-1)p} \to \Lambda^{d(q-1)} \to 0$$

is also exact. Note we have used Frobenius Reciprocity and $rk_{\mathbf{Z}}([y-1)) = (q-1)p$ in the above exact sequence. Moreover, the exact sequence clearly splits and so $K(1) \otimes [y-1)$ is stably free of rank $(q-1)(p-d)$. However, stably free modules over Λ are free and so the result follows. □

As both $[y-1)$ and $\Lambda^{d(q-1)}$ are self-dual then:

Corollary 5.10.10. $R(q) \otimes [y-1) \cong \Lambda^{d(q-1)}$.

A similar proof to that of Corollary 5.10.9 now shows:

Corollary 5.10.11. $K(q) \otimes [y-1) \cong \Lambda^{(q-1)(p-d)}$.

For each $1 \leq i \leq q$ let $\mathcal{T}(i)$ be the statement:

$\mathcal{T}(i): \quad R(i)\otimes[y-1) \cong \Lambda^{d(q-1)}$, $K(i)\otimes[y-1) \cong \Lambda^{(q-1)(p-d)}$ and $P(i)\otimes[y-1) \cong \Lambda^{(q-1)p}$.

First, we have the following:

Proposition 5.10.12. $\mathcal{T}(i) \implies \mathcal{T}(i+1)$ *for* $1 \leq i \leq q-1$.

Proof. Suppose $\mathcal{T}(i)$ is true. Using the fact that $[y-1)$ and Λ are self-dual, we have $R(q+1-i) \otimes [y-1) \cong \Lambda^{d(q-1)}$. Furthermore, $K(i)^* \oplus \Lambda \cong K(q-i) \oplus \Lambda$ by Corollary 5.10.3. As such,

$$(K(q-i) \otimes [y-1)) \oplus \Lambda^{(q-1)p} \cong (K(i) \otimes [y-1))^* \oplus \Lambda^{(q-1)p}$$

and so $K(q-i) \otimes [y-1)$ is stably free (and hence free by Swan-Jacobinski) of rank $(q-1)(p-d)$.

Now, for $K(q-i)$ we have the exact sequence

$$0 \to K(q-i) \to \Lambda \to R(q-i) \to 0.$$

By once more tensoring with $[y-1)$ this becomes

$$0 \to \Lambda^{(q-1)(p-d)} \to \Lambda^{(q-1)p} \to R(q-i) \otimes [y-1) \to 0$$

which clearly splits. As such, $R(q-i) \otimes [y-1)$ is stably free (hence free) of rank $d(q-1)$. Dualising therefore yields $R(i+1) \otimes [y-1) \cong \Lambda^{d(q-1)}$, as required. Moreover, utilising the exact sequence

$$0 \to K(i+1) \to \Lambda \to R(i+1) \to 0$$

a similar argument now shows $K(i+1) \otimes [y-1) \cong \Lambda^{(q-1)(p-d)}$.

It therefore remains to show $P(i+1) \otimes [y-1) \cong \Lambda^{(q-1)p}$. Begin by dualising $K(i+1)$. A similar argument as above shows $K(q-i-1) \otimes [y-1) \cong \Lambda^{(q-1)(p-d)}$. Using the exact sequence $\mathcal{X}(q-i-1)$ we again apply similar reasoning to above to show $R(q-i-1) \otimes [y-1) \cong \Lambda^{d(q-1)}$. Dualising therefore yields $R(i+2) \otimes [y-1) \cong \Lambda^{d(q-1)}$. Finally, consider the exact sequence $\mathcal{W}(i+1)$,

$$0 \to R(i+2) \to P(i+1) \to K(i+1) \to 0$$

and apply the exact functor $- \otimes [y-1)$. We therefore get the split exact sequence

$$0 \to R(i+2) \otimes [y-1) \to P(i+1) \otimes [y-1) \to K(i+1) \otimes [y-1) \to 0.$$

We therefore conclude $P(i+1) \otimes [y-1) \cong \Lambda^{(q-1)p}$, as required. ☐

Proposition 5.10.13. *$\mathcal{T}(i)$ is true for all $i \in \{1, \dots, q\}$.*

Proof. First observe that $\mathcal{T}(q)$ is true by Corollaries 5.10.10 and 5.10.11. Due to Proposition 5.10.12, it suffices to show $\mathcal{T}(1)$. However, by Proposition 5.10.8 and Corollary 5.10.9, it only remains to show $P(1) \otimes [y-1) \cong \Lambda^{(q-1)p}$.

To do so, observe $K(q-1) \otimes [y-1) \cong \Lambda^{(q-1)(p-d)}$. Using similar arguments to those of Proposition 5.10.12 we therefore deduce $R(q-1) \otimes [y-1) \cong \Lambda^{d(q-1)}$ and hence so is its dual; that is, $R(2) \otimes [y-1) \cong \Lambda^{d(q-1)}$. We therefore apply the exact functor $- \otimes [y-1)$ to the exact sequence $\mathcal{W}(2)$ to yield the split exact sequence,

$$0 \to R(2) \otimes [y-1) \to P(1) \otimes [y-1) \to K(1) \otimes [y-1) \to 0.$$

The result now follows. ☐

Proof of Theorem J. By Corollary 5.10.7, we know K is stably self-dual. It therefore remains to show $K \otimes R(i) \cong \Lambda^a$ and $K \otimes K(i) \cong \Lambda^b$ for some $a, b \geq 1$. To do so, return to the exact sequence

$$0 \to K_0 \to K \to \mathbf{Z} \to 0.$$

and recall $K_0 = [y-1)$. We now apply the exact functor $- \otimes R(i)$ so that

$$0 \to K_0 \otimes R(i) \to K \otimes R(i) \to R(i) \to 0$$

is exact. By Proposition 5.10.13, this sequence splits and so $K \otimes R(i) \cong R(i) \oplus \Lambda^{d(q-1)}$.

Next, we apply the exact functor $- \otimes K(i)$ so that

$$0 \to K_0 \otimes K(i) \to K \otimes K(i) \to K(i) \to 0$$

is exact. Once more, we use Proposition 5.10.13 to deduce that the above sequence splits. We therefore conclude $K \otimes K(i) \cong K(i) \oplus \Lambda^{(q-1)(p-d)}$. ☐

Part II

Metacyclic groups of infinite type

Chapter 6

Preliminaries

Recall that in Chapter 2 we defined a finitely generated module S to be stably free if it is stably equivalent to a free module. Such modules will be the main discussion of this latter part of the thesis. In particular, we discuss a class of rings that satisfy the SFC condition discussed in Section 2.6. This class is of the form $\mathbf{Z}[G(p,\, q) \times F_n]$, where F_n is the free group of rank $n \geq 1$. This may be seen as an attempt to generalise the results of Swan from $\mathbf{Z}[G]$ to $A[G]$, where $A = \mathbf{Z}[F_n]$ and continues the work of Johnson (see [20], [21], [22]) discussed in Chapter 1.

It is therefore the aim of this chapter to introduce the necessary results that will allow us to discuss stably free modules in detail. Roughly, the strategy will be to formalise a technique originating with Bass and Milnor whereby we break up the ring in question into smaller chunks that we know satisfy SFC. If we can then 'glue' these pieces together in such a way that any stably free module is lifted from the decomposed rings, then we can show that there are no nontrivial stably free modules. The next chapter will then utilise this material to consider the stably free modules over $\mathbf{Z}[G(p,\, q) \times F_n]$, demonstrating that they are necessarily trivial.

6.1 The general and restricted linear groups

Let Λ be some ring with unit group $U(\Lambda)$. For $n \geq 2$, the matrix ring $M_n(\Lambda)$ has the canonical Λ-basis $\{\epsilon(i,\, j) \mid 1 \leq i,\, j \leq n\}$, where we write $(\epsilon(i,\, j))_{r,\, s} = \delta_{ir}\delta_{js}$. Let $\lambda \in \Lambda$ and $\delta \in U(\Lambda)$, and recall the elementary row and column operations expressed using the elementary invertible matrices

$$\begin{aligned} E(i,\, j; \lambda) = &\quad I_n + \lambda\epsilon(i,\, j) \quad (i \neq j); \\ D(i,\, \delta) = &\quad I_n + (\delta - 1)\epsilon(i,\, i). \end{aligned}$$

We denote the set of invertible $n \times n$ matrices over Λ by $GL_n(\Lambda)$ and, whenever $n \geq 2$, we denote the subgroup generated by the matrices $E(i,\, j; \lambda)$ by $E_n(\Lambda)$. Likewise, we have the subgroup $D_n(\Lambda)$ of $GL_n(\Lambda)$ defined by $D_n(\Lambda) = \{D(1,\, \delta) \mid \delta \in U(\Lambda)\}$. In particular, we may confuse $D_n(\Lambda)$ with $U(\Lambda)$ in a natural way.

As is well known (see [20], [28]), $D_n(\Lambda)$ normalises $E_n(\Lambda)$. Consequently we define the *restricted linear group* $GE_n(\Lambda)$ to be the subgroup of $GL_n(\Lambda)$ given as the internal product

$$GE_n(\Lambda) = D_n(\Lambda) \cdot E_n(\Lambda).$$

In general $GE_n(\Lambda)$ is a proper subgroup of $GL_n(\Lambda)$. However, there are significant instances in which $GL_n(\Lambda) = GE_n(\Lambda)$. When this is the case we say that Λ is *weakly Euclidean.* Clearly, this condition implies any invertible $n \times n$ matrix may be reduced to a diagonal matrix $D(1, \delta)$ by performing a series of elementary row and column operations $E(i, j; \lambda)$. In other words, each $X \in GL_n(\Lambda)$ has a Smith Normal Form.

A useful generalisation of this is that of weakly m-Euclidean. When $m \leq n$ we embed $GL_m(\Lambda)$ into $GL_{m+1}(\Lambda)$ by identifying $X \in GL_m(\Lambda)$ with its 'suspension',

$$\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} \in GL_{m+1}(\Lambda).$$

This has the obvious generalisation of embedding of $GL_m(\Lambda)$ into $GL_n(\Lambda)$, and that of $GL(-) = \lim_{\rightarrow} GL_n(-)$ and $E(-) = \lim_{\rightarrow} E_n(-)$. Furthermore, $E_n(\Lambda)$ is normalised by $GL_m(\Lambda)$ and so we have the subgroup $GL_m(\Lambda) \cdot E_n(\Lambda)$ of $GL_n(\Lambda)$. We then say that Λ is *weakly m-Euclidean* when $GL_n(\Lambda) = GL_m(\Lambda) \cdot E_n(\Lambda)$ for $m \leq n$.

To conclude, observe that E_n and GE_n are functorial under ring homomorphism. If we let $\varphi : A \to B$ be a surjective ring homomorphism then it is well known that the induced map $\varphi_* : E_n(A) \to E_n(B)$ is also surjective. Unfortunately, this property does not hold in general for $D_n(-)$, $GE_n(-)$ or $GL_n(-)$. Clearly, if such a property holds for $D_n(-)$ then it also holds for $GE_n(-)$. We say that the ring homomorphism φ *has the lifting property for units* when the induced map on units $\varphi_* : U(A) \to U(B)$ is surjective. If this is the case then the induced maps on $D_n(-)$ and hence $GE_n(-)$ are clearly surjective also.

6.2 Fibre square decompositions for finitely generated modules

The techniques introduced in the next two sections have their genesis in the work of Milnor [31] and were further developed by Swan in [56] to understand stably free modules over group rings. The aim is to analyse the structure of projective modules (and in particular stably free modules) via the decomposition of rings into fibre squares.

We start by considering a corner of rings and ring homomorphisms. Suppose we have a trio of rings $\mathcal{A} = (A_+, A_- A_0)$, along with a pair of ring homomorphisms $\varphi_+ : A_+ \to A_0$ and $\varphi_- : A_- \to A_0$. This gives rise to what we call a *corner*, denoted by $\mathcal{A}$:

$$\mathcal{A} = \left\{ \begin{tikzcd} & A_- \arrow[d, "\varphi_-"] \\ A_+ \arrow[r, "\varphi_+"] & A_0 \end{tikzcd} \right.$$

We 'complete the square' by taking a 'twisted product' of A_+ and A_- over A_0. We call this the *fibre product* and define it by,

$$A_+ \times_{A_0} A_- = \{(a_+, a_-) \in A_+ \times A_- \mid \varphi_+(a_+) = \varphi_-(a_-)\}$$

where addition and multiplication are defined component wise. Set $A = A_+ \times_{A_0} A_-$ and construct the following canonical fibre square, denoted by $\widehat{\mathcal{A}}$:

$$\widehat{\mathcal{A}} = \left\{ \begin{tikzcd} A \arrow[r, "\pi_-"] \arrow[d, "\pi_+"] & A_- \arrow[d, "\varphi_-"] \\ A_+ \arrow[r, "\varphi_+"] & A_0 \end{tikzcd} \right. \tag{6.2.1}$$

where π_+ and π_- are the projections from A onto A_+ and A_-, respectively. In general, any such square is called a *fibre square* if A is mapped isomorphically onto to the fibre product given above by $\pi_+ \times \pi_-$. It is straightforward to verify that this is equivalent to requiring the following sequence is exact,

$$0 \to A \xrightarrow{\begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix}} A_+ \oplus A_- \xrightarrow{(\varphi_+, -\varphi_-)} A_0. \tag{6.2.2}$$

Consequently, we note the following:

Proposition 6.2.3. *Let $\mathcal{F}$: Rings $\to$ Rings be a left exact functor; that is, a functor which is left exact when considered as a functor on the underlying additive groups. If $\widehat{\mathcal{A}}$ is a fibre square as in (6.2.1), then so is $\mathcal{F}(\widehat{\mathcal{A}})$:*

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(\pi_-)"] \arrow[d, "\mathcal{F}(\pi_+)"] & \mathcal{F}(A_-) \arrow[d, "\mathcal{F}(\varphi_-)"] \\ \mathcal{F}(A_+) \arrow[r, "\mathcal{F}(\varphi_+)"] & \mathcal{F}(A_0) \end{tikzcd} \tag{6.2.4}$$

Corollary 6.2.5. *Let $\widehat{\mathcal{A}}$ be a fibre square as in (6.2.1). Then*

$$\begin{tikzcd} A \otimes \mathbf{Z}[G] \arrow[r, "\pi_- \otimes Id"] \arrow[d, "\pi_+ \otimes Id"] & A_- \otimes \mathbf{Z}[G] \arrow[d, "\varphi_- \otimes Id"] \\ A_+ \otimes \mathbf{Z}[G] \arrow[r, "\varphi_+ \otimes Id"] & A_0 \otimes \mathbf{Z}[G] \end{tikzcd}$$

is also a fibre square.

Proof. Any group algebra $\mathbf{Z}[G]$ is free as an additive group; thus $- \otimes_{\mathbf{Z}} \mathbf{Z}[G]$ is a functor *Rings* $\to$ *Rings*, which is exact as a functor on the underlying additive groups. We may therefore apply Proposition 6.2.3. ☐

Proposition 6.2.6. *Suppose*

$$\begin{tikzcd} (A,\, \theta;\, \alpha) \arrow[r, "\pi_-"] \arrow[d, "\pi_+"] & (A_-,\, \theta_-;\, \alpha_-) \arrow[d, "\varphi_-"] \\ (A_+,\, \theta_+;\, \alpha_+) \arrow[r, "\varphi_+"] & (A_0,\, \theta_0;\, \alpha_0) \end{tikzcd}$$

is a fibre square of pointed n*-rings in which* $A = A_+ \times_{A_0} A_-$. *Then*

$$\begin{tikzcd}
\mathcal{C}_n(A,\, \theta;\, \alpha) \arrow[r, "\mathcal{C}_n(\pi_-)"] \arrow[d, "\mathcal{C}_n(\pi_+)"] & \mathcal{C}_n(A_-,\, \theta_-;\, \alpha_-) \arrow[d, "\mathcal{C}_n(\varphi_-)"] \\
\mathcal{C}_n(A_+,\, \theta_+;\, \alpha_+) \arrow[r, "\mathcal{C}_n(\varphi_+)"] & \mathcal{C}_n(A_0,\, \theta_0;\, \alpha_0)
\end{tikzcd}$$

is also a fibre square, where $\mathcal{C}_n(-)$ *is the cyclic algebraic construction discussed in Section 2.3.*

Using the above we may construct fibre squares for more complicated group rings starting from simpler rings that we, hopefully, better understand. As would be expected, this starting point is most useful when it is from a group ring of a cyclic group.

Proposition 6.2.7. *Let* I *be an ideal in a ring* Λ, *and suppose* $f : \Lambda \to \Gamma$ *is a ring homomorphism such that* $f|_I : I \to f(I)$ *is bijective. Then,*

$$\begin{tikzcd}
\Lambda \arrow[r, "\natural"] \arrow[d, "f"] & \Lambda/I \arrow[d, "f_*"] \\
\Gamma \arrow[r, "\natural"] & \Gamma/f(I)
\end{tikzcd} \tag{6.2.8}$$

is a fibre square.

Proof. It is sufficient to show the following sequence is exact:

$$0 \to \Lambda \xrightarrow{\begin{pmatrix} f \\ \natural \end{pmatrix}} \Gamma \oplus (\Lambda/I) \xrightarrow{\begin{pmatrix} \natural, & -f_* \end{pmatrix}} \Gamma/f(I).$$

For exactness at Λ, suppose $\lambda \in Ker \begin{pmatrix} f \\ \natural \end{pmatrix}$. Then $\natural(\lambda) = 0$ and hence $\lambda \in I$. As f is bijective on I, and since $f(\lambda) = 0$, we conclude $\lambda = 0$. Therefore, the sequence is exact at Λ.

Next we consider exactness at $\Gamma \oplus (\Lambda/I)$. First, suppose $(\gamma, \mu)^T \in Im \begin{pmatrix} f \\ \natural \end{pmatrix}$, then there is some $\lambda \in \Lambda$ such that $f(\lambda) = \gamma$ and $\natural(\lambda) = \mu$. Now,

$$\begin{pmatrix} \natural, & -f_* \end{pmatrix} \begin{pmatrix} f(\lambda) \\ \natural(\lambda) \end{pmatrix} = \natural(f(\lambda)) - f_*(\natural(\lambda)) = 0$$

and therefore $Im \begin{pmatrix} f \\ \natural \end{pmatrix} \subset Ker \begin{pmatrix} \natural, & -f_* \end{pmatrix}$. For the reverse inclusion, we let $\begin{pmatrix} \gamma \\ \natural(\lambda) \end{pmatrix} \in Ker \begin{pmatrix} \natural, & -f_* \end{pmatrix}$; that is, $\begin{pmatrix} \natural, & -f_* \end{pmatrix} \begin{pmatrix} \gamma \\ \natural(\lambda) \end{pmatrix} = 0$. Since $\natural(\gamma) - f_*(\natural(\lambda)) = 0$, by commutativity we have $\natural(\gamma) - \natural(f(\lambda)) = 0$, i.e. $\gamma - f(\lambda) \in f(I)$. We can therefore choose some (unique) $\delta \in I$ such that $f(\delta) = f(\lambda) - \gamma$. Finally, then,

$$\begin{pmatrix} f \\ \natural \end{pmatrix} (\lambda - \delta) = \begin{pmatrix} f(\lambda - \delta) \\ \natural(\lambda - \delta) \end{pmatrix} = \begin{pmatrix} \gamma \\ \natural(\lambda) \end{pmatrix}$$

and $Ker \left(\natural, \quad -f_*\right) \subset Im \begin{pmatrix} f \\ \natural \end{pmatrix}$ thus showing exactness at $\Gamma \oplus (\Lambda/I)$. □

Corollary 6.2.9. *Let G be a finite group and let H be a normal subgroup of G. Then the following is a fibre square:*

$$\begin{tikzcd}
\mathbf{Z}[G] \arrow[r, "\natural"] \arrow[d, "f"'] & \mathbf{Z}[G]/(\Sigma_H) \arrow[d] \\
\mathbf{Z}[G/H] \arrow[r] & (\mathbf{Z}/|H|)[G/H]
\end{tikzcd}$$

where $\Sigma_H = \sum_{h \in H} h$.

Proof. Let $f : \mathbf{Z}[G] \to \mathbf{Z}[G/H]$ be given by,

$$f\left(\sum_{g \in G} a_g g\right) = \sum_{g \in G} a_g \natural(g)$$

and let $I = (\Sigma_H) \cdot \mathbf{Z}[G]$. Hence $f(I) = |H| \cdot \mathbf{Z}[G/H]$. To apply Proposition 6.2.7, it remains to show $f|_I : I \to f(I)$ is bijective.

Clearly, $f|_I$ is surjective and so it suffices to show injectivity. To this end we suppose

$$f\left(\Sigma_H \sum_{g \in G} a_g g\right) = |H| \sum_{g \in G} a_g \natural(g) = 0.$$

It follows that $\sum_{g \in G} a_g g \in Ker(f)$. Since $Ker(f) = Im(h_1 - 1, \ldots, h_m - 1)$, where $h_1, \ldots, h_m$ generate H, we may write $\sum_{g \in G} a_g g = (h_1 - 1)\lambda_1 + \cdots + (h_m - 1)\lambda_m$ for some $\lambda_1, \ldots, \lambda_m \in \mathbf{Z}[G]$. However, from this we clearly observe that $\Sigma_H \cdot \sum_{g \in G} a_g g = 0$, and thus conclude f is bijective on I. The result now follows. □

Example 6.2.10 (Cyclic group of order p)**.** *Let p be a prime number and consider the cyclic group of order p, C_p. By Lagrange, we have only two subgroups, the trivial subgroup and the whole group. If we apply Corollary 6.2.9 with $H = C_p$ then we have the following fibre Square*

$$\begin{tikzcd}
\mathbf{Z}[C_p] \arrow[r] \arrow[d, "\varepsilon"] & \mathbf{Z}[C_p]/(\Sigma) \arrow[d] \\
\mathbf{Z} \arrow[r] & \mathbf{F}_p
\end{tikzcd}$$

where $\Sigma = \sum_{g \in C_p} g$ and $\varepsilon : \mathbf{Z}[G] \to \mathbf{Z}$ is the augmentation map. The kernel of this map is the augmentation ideal and in the case of cyclic groups only we have an isomorphism $\mathbf{Z}[C_p]/(\Sigma) \cong I_C^* \cong I_C$, the augmentation ideal.

Example 6.2.11 (Dihedral group of order $2p$)**.** *Start with the fibre square decomposition of* $\mathbf{Z}[C_p]$*:*

$$\begin{tikzcd}
\mathbf{Z}[C_p] \arrow[r] \arrow[d, "\varepsilon"] & \mathbf{Z}[C_p]/(\Sigma) \arrow[d] \\
\mathbf{Z} \arrow[r] & \mathbf{F}_p
\end{tikzcd}$$

Take $\theta(x) = x^{-1}$*, the canonical involution for* $x \in \mathbf{Z}[C_p]$*. Clearly* $\theta^2 = Id$ *and* $1 \in (\mathbf{Z}[C_p])^{\theta}$*. We can therefore apply the cyclic algebraic construction* $\mathcal{C}_2(-)$ *to the above fibre square and identify* $\mathcal{C}_2(\mathbf{Z}[C_p], \theta; 1) = \mathbf{Z}[D_{2p}]$*,* $\mathcal{C}_2(\mathbf{Z}, \theta) = \mathbf{Z}[C_2]$ *and* $\mathcal{C}_2(\mathbf{F}_p, \theta) = \mathbf{F}_p[C_2]$*. We therefore have the following fibre square decomposition for the dihedral group of order* $2p$*,* D_{2p}*:*

$$\begin{tikzcd}
\mathbf{Z}[D_{2p}] \arrow[r] \arrow[d, "\varepsilon"] & \mathcal{C}_2(\mathbf{Z}[C_p]/(\Sigma), \theta) \arrow[d] \\
\mathbf{Z}[C_2] \arrow[r] & \mathbf{F}_p[C_2]
\end{tikzcd}$$

Example 6.2.12 (Metacyclic group of infinite type)**.** *We proceed in two stages. First we build a fibre square model for the metacyclic group* $G(p, q)$ *of order* pq*. Next, we apply the exact functor* $- \otimes \mathbf{Z}[F_n]$*, where* F_n *is the free group of rank* $n \geq 1$*. As before, consider the canonical fibre square model for* $\mathbf{Z}[C_p]$ *(p prime), and identify* $\mathbf{Z}[C_p]$ *with* $\mathbf{Z}[x]/(x^p - 1)$*. Observe that* $x^p - 1 = (x-1)(x^{p-1} + \cdots + x + 1)$ *and* $\mathbf{Z}[x]/(x^{p-1} + \cdots + x + 1) \cong \mathbf{Z}[\zeta_p]$ *where* $\zeta_p = exp(2\pi i/p)$*. We therefore have the following fibre square,*

$$\begin{tikzcd}
\mathbf{Z}[C_p] \arrow[r] \arrow[d] & \mathbf{Z}[\zeta_p] \arrow[d] \\
\mathbf{Z} \arrow[r] & \mathbf{F}_p
\end{tikzcd} \tag{$\dagger$}$$

in line with the conclusion of Example 6.2.10. Now, let $\alpha : C_p \to C_p$ *be a generator of* $Aut(C_p)$*. In particular,* $ord(\alpha) = p-1$ *as* $Aut(C_p) \cong C_{p-1} \cong Gal(\mathbf{Q}(\zeta_p) : \mathbf{Q})$*. Choose a prime* q *such that* $q|p-1$ *and put* θ *such that* $\theta = \alpha^{(p-1)/q}$*. Clearly,* $ord(\theta) = q$*.*

Now, θ *induces a ring automorphism of order* q *on* $\mathbf{Z}[C_p]$*. In particular,* θ *fixes* Σ_x *and so* θ *induces a ring automorphism on* $I_C^* = \mathbf{Z}[C_p]/(\Sigma_x)$*, the integral duel of the augmentation ideal. Likewise,* I_C *is stable under* θ *and so* θ *induces the identity automorphisms on* $\mathbf{Z}$ *and* $\mathbf{F}_p = \mathbf{Z}/p$*. As such, we may apply* $\mathcal{C}_q(-, \theta; 1)$ *to* $(\dagger)$ *to obtain another fibre square:*

$$\begin{tikzcd}
\mathcal{C}_q(\mathbf{Z}[C_p], \theta) \arrow[r] \arrow[d] & \mathcal{C}_q(\mathbf{Z}[\zeta_p], \theta) \arrow[d] \\
\mathcal{C}_q(\mathbf{Z}, \theta) \arrow[r] & \mathcal{C}_q(\mathbf{F}_p, \theta)
\end{tikzcd} \tag{$\ddagger$}$$

We identify $\mathcal{C}_q(\mathbf{Z}[C_p], \theta)$ *with* $\mathbf{Z}[C_p \rtimes C_q]$*. Moreover, as* θ *acts trivially on* $\mathbf{Z}$ *and* $\mathbf{F}_p$*, we make the identifications* $\mathcal{C}_q(\mathbf{Z}, \theta) = \mathbf{Z}[C_q]$ *and* $\mathcal{C}_q(\mathbf{F}_p, \theta) = \mathbf{F}_p[C_q]$*. Thus,* $(\ddagger)$ *may be rewritten as:*

$$\begin{tikzcd}
\mathbf{Z}[C_p \rtimes C_q] \arrow[r] \arrow[d] & \mathcal{C}_q(\mathbf{Z}[\zeta_p],\, \theta) \arrow[d] \\
\mathbf{Z}[C_q] \arrow[r] & \mathbf{F}_p[C_q]
\end{tikzcd} \tag{$\ddagger'$}$$

Evidently, the biggest obstacle to utilising the above fibre square lies in the top right corner. The key to resolving the issue is a decomposition of $\mathcal{C}_q(\mathbf{Z}[\zeta_p], \theta)$ *into ideals due to Rosen [43], although a more accessible treatment is given in the thesis of Remez [41]. We give the statement here in a format most of use to us.*

Proposition 6.2.13 (Rosen)**.** *Let A be the fixed ring of $\mathbf{Z}[\zeta_p]$ under θ and consider the ring of quasi-triangular matrices,*

$$\mathcal{T}_q(A, \pi) = \{X \in M_q(A) \mid x_{rs} \in (\pi) \text{ if } r > s\},$$

where $\pi = (\zeta_p - 1)^q$. Then there is an isomorphism,

$$\mathcal{C}_q(\mathbf{Z}[\zeta_p],\, \theta) \cong \mathcal{T}_q(A, \pi).$$

The proof of this is omitted in favour of succinctness. Nevertheless, with this result we now apply the functor $- \otimes_{\mathbf{Z}} \mathbf{Z}[F_n]$ to ($\ddagger'$) thus obtaining the following fibre square:

$$\begin{tikzcd}
\mathbf{Z}[(C_p \rtimes C_q) \times F_n] \arrow[r] \arrow[d] & \mathcal{T}_q(A,\, \pi)[F_n] \arrow[d] \\
\mathbf{Z}[C_q \times F_n] \arrow[r] & \mathbf{F}_p[C_q \times F_n]
\end{tikzcd} \tag{$\heartsuit$}$$

It is to ♡ that we will be interested in for the final chapter of this thesis.

6.3 Projective modules over fibre squares

Consider the canonical fibre square $\widehat{\mathcal{A}}$, as given in (6.2.1). When Milnor first discussed the construction of projective modules over fibre squares (see [31]), there were two conditions imposed upon the square:

- Hypothesis 1: Suppose $A = A_1 \times_\varphi A_2$ is the fibre product. In particular, there is one, and only one, element $a \in A$ such that $\pi_+(a) = a_+$ and $\pi_-(a) = a_-$;
- Hypothesis 2: At least one of the two homomorphisms φ_+, φ_- is surjective.

This section can be seen as an attempt to discuss and extend these two conditions to suit our purposes. This will allow us to naturally progress to the question of whether A can have nontrivial stably frees.

Suppose M is a module over A, then this determines a triple

$$(M_+,\, M_-\,; \alpha(M))$$

where $M_+ = M \otimes_{\pi_+} A_+$ and $M_- = M \otimes_{\pi_-} A_-$ are modules over A_+, A_-, respectively. Moreover, $\alpha(M)$ is the canonical A_0-isomorphism such that the following commutes:

$$\begin{tikzcd}(M \otimes_{\pi_+} A_+) \otimes_{\varphi_+} A_0 \arrow[r, "\alpha(M)"] \arrow[d, "\nu_+"] & (M \otimes_{\pi_-} A_-) \otimes_{\varphi_-} A_0 \arrow[d, "\nu_-"] \\ M \otimes_{\varphi_+\pi_+} A_0 \arrow[r, "Id"] & M \otimes_{\varphi_-\pi_-} A_0\end{tikzcd} \tag{6.3.1}$$

where $\nu_\sigma : (M \otimes_{\pi_\sigma} A_\sigma) \otimes A_0 \to M \otimes_{\varphi_\sigma \pi_\sigma} A_0$ is the canonical isomorphism for $\sigma = +, -$.

Now suppose the converse; i.e. that we have a triple $(M_+, M_-; \alpha)$, where M_+, M_- are modules over A_+, A_- respectively such that $\alpha : M_- \otimes_{\varphi_-} A_0 \xrightarrow{\simeq} M_+ \otimes_{\varphi_+} A_0$ is a specific A_0-isomorphism. We then obtain an A-module $\langle M_+, M_-; \alpha\rangle$ given by

$$\langle M_+, M_-; \alpha\rangle = \{(m_+, m_-) \in M_+ \times M_- \mid \alpha(m_- \otimes 1) = m_+ \otimes 1\}$$

with the A-action given by

$$(m_+, m_-) \cdot \lambda = (m_+ \cdot \pi_+(\lambda), m_- \cdot \pi_-(\lambda)).$$

In particular, observe that

$$\begin{aligned} \alpha(m_- \cdot \pi_-(\lambda) \otimes 1) &= \alpha(m_- \otimes \varphi_-\pi_-(\lambda)) \\ &= m_+ \otimes \varphi_+\pi_+(\lambda) \\ &= m_+\pi_+(\lambda) \otimes 1 \end{aligned}$$

and so $(m_+, m_-) \cdot \lambda \in \langle M_+, M_-; \alpha\rangle$. This action then forms a right A-module.

Now, the key point to note is that every finitely generated projective A-module arises in this way. See [22] for a proof of this.

Proposition 6.3.2. *Let P be a finitely generated projective module over A; then*

$$P \cong \langle P_+, P_-; \alpha(P)\rangle.$$

An obvious special case to observe is when we are looking at a free module of rank n over A. By the above, this may be written

$$\langle A_+^n, A_-^n : \alpha(A^n)\rangle.$$

In particular, we note that there is a canonical isomorphism $\natural : A_+ \otimes A_0 \xrightarrow{\simeq} A_- \otimes A_0$ which can be confused with $Id : A_0 \to A_0$. Thus, for each $n \geq 1$, we say the *globally free module* of rank n over A has the form

$$F_n(A) = \langle A_+^n, A_-^n; Id_n\rangle.$$

A finitely generated module P is said to be *globally projective* if there is another module Q such that $P \oplus Q \cong F_n(A)$.

An A-module M is said to be *locally projective* (resp. *locally free*) if M_σ is projective (resp. free) for $\sigma = +, -$. Quite clearly, any globally projective module is locally projective, but the converse need not be true. In fact, for this converse to hold we need to be careful as to how we 'patch' these modules together. We therefore consider an added condition:

- $\mathcal{P}$*atch*: For each $n \geq 1$ and each $\alpha \in GL_n(A_0)$ there exists a $k \geq 1$ and $\beta \in GL_k(A_0)$ such that $\alpha \oplus \beta = [h_+][h_-]$ for some $h_+ \in GL_{n+k}(A_+)$ and $h_- \in GL_{n+k}(A_-)$, and where $[h_\sigma] = h_\sigma \otimes Id_{A_0}$.

The following is essentially due to Bass and Milnor, although the present formulation is due to Johnson [22].

Proposition 6.3.3. *Consider a fibre square of the form (6.2.1) which satisfies* $\mathcal{P}$*atch, then a finitely generated* A*-module* P *is globally projective if and only if it is locally projective.*

The next issue becomes one of actually understanding when a fibre square satisfies $\mathcal{P}$atch. There are two cases of note which guarantees we have this patching property. We say that a fibre square $\widehat{\mathcal{A}}$ of the form (6.2.1) is $\bar{E}$*-trivial* when the double coset

$$\overline{E}(\widehat{\mathcal{A}}) = E(A_+)\backslash E(A_0)/E(A_-) = \{*\}.$$

In other words, any $[X] \in E(A_0)$ can be written $[X] = [\varphi_+(X_+)][\varphi_-(X_-)]$ for some $X_+ \in E_n(A_+)$ and $X_- \in E_m(A_-)$ (for some m, n). We have the following relationship between $\mathcal{P}$atch, and $\bar{E}$-triviality. A proof can be found in Chapter 3 of [22].

Proposition 6.3.4. *Let* $\widehat{\mathcal{A}}$ *be a fibre square of the form 6.2.1. If* $\widehat{\mathcal{A}}$ *is* $\bar{E}$*-trivial, then* $\widehat{\mathcal{A}}$ *satisfies* $\mathcal{P}$*atch.*

There is an obvious generalisation of $\bar{E}$-triviality. For $n \geq 2$ we say that $\widehat{\mathcal{A}}$ is $\bar{E}_n$-trivial if

$$\overline{E}_k(\widehat{\mathcal{A}}) = E_k(A_+)\backslash E_k(A_0)/E_k(A_-) = \{*\}$$

whenever $k \geq n$. Trivially, if $\widehat{\mathcal{A}}$ is $\bar{E}_n$-trivial and $n \leq N$, then $\mathcal{A}$ is $\bar{E}_N$-trivial. Evidently, this means that $\bar{E}_2$-triviality is the strongest of these conditions. A straightforward stabilisation argument shows:

Proposition 6.3.5. *If* $\widehat{\mathcal{A}}$ *is* $\bar{E}_n$*-trivial for some* n*, then* $\widehat{\mathcal{A}}$ *is* $\bar{E}$*-trivial.*

The second patching condition of note is a much simpler condition to check, and was the original patching condition considered by Milnor. We say that a fibre square $\widehat{\mathcal{A}}$ of the form (6.2.1) is Milnor when:

- Milnor: At least one of φ_+, φ_- is surjective.

Proposition 6.3.6. *Suppose* $\widehat{\mathcal{A}}$ *is a fibre square satisfying Milnor's condition; then* $\widehat{\mathcal{A}}$ *is* $\bar{E}_2$*-trivial.*

Proof. Without loss of generality suppose $\varphi_+ : A_+ \to A_0$ is surjective; then for each $k \geq 2$ the induced homomorphism

$$\varphi_+ : E_k(A_+) \to E_k(A_0)$$

is surjective. Thus, $E_k(A_+)\backslash E_k(A_0)$ consists of a single point. It follows trivially that $E_k(A_+)\backslash E_k(A_0)/E_k(A_-)$ consists of a single point. □

Corollary 6.3.7. *If* $\widehat{\mathcal{A}}$ *is a fibre square satisfying Milnor's condition, then* $\widehat{\mathcal{A}}$ *satisfies* $\mathcal{P}$*atch.*

Consequently, whenever $\widehat{\mathcal{A}}$ is a Milnor square, a locally free module over A is necessarily projective. The issue is now to decide when it is stably free. As we will see in the next section, this is a far more delicate matter.

6.4 Recognition criteria for stably free cancellation

As before, we work over fibre squares of the form (6.2.1). The aim will now be to understand when the trivial stably free modules (i.e. free modules) survive the transition from A_+, A_- to A. Start by imposing the condition of $\bar{E}$-triviality on our fibre square. Denote the set of isomorphism classes of finitely generated locally free modules of rank n over A by $\mathcal{LF}_n(\widehat{\mathcal{A}})$. Likewise, denote the set of isomorphism classes of stably free modules of rank n over A by $\mathcal{SF}_n(A)$. We then have the following (see pp. 52-55 of [22]):

Proposition 6.4.1. *Let the fibre square (6.2.1) be $\bar{E}$-trivial such that the corner ring A_0 is weakly m-Euclidean. Let S_+, S_- be stably free modules of rank $n \geq m$ over A_+, A_- respectively such that*

$$S_+ \otimes A_0 \cong S_- \otimes A_0 \cong A_0^n;$$

then there exists a stably free module S over A such that $\pi_+(S) \cong S_+$ and $\pi_-(S) \cong S_-$. In particular, if A_0 has SFC, then

$$\pi_+ \times \pi_- : \mathcal{SF}_n(A) \to \mathcal{SF}_n(A_+) \times \mathcal{SF}_n(A_-)$$

is surjective.

This tells us when nontrivial stably free modules over A_+, A_- survive to be nontrivial over A. If we now take the stably free modules over A_+, A_- to be trivial, we would like to know when the stably free modules over A are trivial. In [22] (p. 41), it was shown:

Proposition 6.4.2. *There is a bijection,*

$$\nu_n : \overline{GL_n}(\widehat{\mathcal{A}}) = GL_n(A_+)\backslash GL_n(A_0)/GL_n(A_-) \xrightarrow{\simeq} \mathcal{LF}_n(\widehat{\mathcal{A}}).$$

We now define two stabilisation operators. First, $s_{n,k} : \overline{GL_n}(\widehat{\mathcal{A}}) \to \overline{GL_{n+k}}(\widehat{\mathcal{A}})$ is defined where $s_{n,k}([\alpha]) = [\alpha \oplus I_k]$, for some $k \geq 1$. Second, we define the stabilisation operator on $\mathcal{LF}_n(\widehat{\mathcal{A}})$ induced from the correspondence $P \mapsto P \oplus (A_+^k, A_-^k; I_k)$ by,

$$\sigma_{n,k} : \mathcal{LF}_n(\widehat{\mathcal{A}}) \to \mathcal{LF}_{n+k}(\widehat{\mathcal{A}}).$$

We then have the following commutative diagram

$$\begin{tikzcd}
\overline{GL_n}(\widehat{\mathcal{A}}) \arrow[r, "s_{n,k}"] \arrow[d, "\nu_n"] & \overline{GL_{n+k}}(\widehat{\mathcal{A}}) \arrow[d, "\nu_{n+k}"] \\
\mathcal{LF}_n(\widehat{\mathcal{A}}) \arrow[r, "\sigma_{n,k}"] & \mathcal{LF}_{n+k}(\widehat{\mathcal{A}})
\end{tikzcd} \tag{6.4.3}$$

where ν_n, ν_{n+k} are bijections due to Proposition 6.4.2.

Now, write $*$ for the class of A^{n+k} in $\mathcal{LF}_{n+k}(\widehat{\mathcal{A}})$ and observe that for a locally free A-module S of rank n we have:

$$S \text{ is stably free if and only if } \sigma_{n,k}([S]) = * \text{ for some } k \geq 0. \tag{6.4.4}$$

Alternatively, we can think of this as requiring $s_{n,k}([\alpha]) = [I_{n+k}]$. This observation motivates us to define the set $\mathcal{Z}_n(\mathcal{A}) = \{\eta \in \overline{GL_n}(\widehat{\mathcal{A}}) \mid s_{n,k}(\eta) = * \text{ for some } k \geq 1\}$, and put

$$\mathcal{Z}(\widehat{\mathcal{A}}) = \coprod_{n\geq 1} \mathcal{Z}_n(\widehat{\mathcal{A}}).$$

We call $\mathcal{Z}(\widehat{\mathcal{A}})$ the *singular set*. From (6.4.4) it follows that

$$\nu_n : \mathcal{Z}_n(\widehat{\mathcal{A}}) \xrightarrow{\simeq} \mathcal{SF}_n(A) \cap \mathcal{LF}_n(\widehat{\mathcal{A}}) \text{ is bijective.} \tag{6.4.5}$$

We say that $\widehat{\mathcal{A}}$ is *locally n-free* when $\mathcal{SF}_n(A_+) = \mathcal{SF}_n(A_-) = \{*\}$, and *of locally free type* when it is locally n-free for all n. Suppose $\widehat{\mathcal{A}}$ is locally n-free, then every stably free A-module of rank n is locally free. Thus, $\mathcal{SF}_n(A) = \mathcal{SF}_n(A) \cap \mathcal{LF}_n(\widehat{\mathcal{A}})$ and (6.4.5) becomes

$$\nu_n : \mathcal{Z}_n(\mathcal{A}) \xrightarrow{\simeq} \mathcal{SF}_n(A) \text{ is a bijection when } \mathcal{A} \text{ is locally } n\text{-free.} \tag{6.4.6}$$

It follows that to understand when a stably free module over A is necessarily trivial, it is enough to consider when $\mathcal{Z}_n(\widehat{\mathcal{A}})$ is trivial. An obvious condition to guarantee this is for $s_{n,k}$ to be injective for all $k \geq 1$; that is:

Proposition 6.4.7. *If $\widehat{\mathcal{A}}$ is locally n-free and each $s_{n,k} : \overline{GL_n}(\widehat{\mathcal{A}}) \to \overline{GL_{n+k}}(\widehat{\mathcal{A}})$ is injective for $k \geq 1$, then $\mathcal{SF}_n(A) = \{*\}$.*

Corollary 6.4.8. *Suppose that, for each $k \geq 0$, $\widehat{\mathcal{A}}$ is locally $(n+k)$-free, and that each $s_{n,k}$ is bijective; then A has no nontrivial stably free modules of rank $\geq n$.*

Proof. For $1 \leq k < m$, note that $s_{n,m} = s_{n+k,m-k} \circ s_{n,k}$. Therefore, if $s_{n,k}$ and $s_{n,m}$ are both bijective, then so too is $s_{n+k,m-k}$. So if $\mathcal{A}$ is locally $(n+k)$-free, it follows that $\mathcal{SF}_{n+k}(A) = \{*\}$, as required. ☐

This observation will be sufficient for the majority of our purposes in Chapter 7. Nevertheless, we still have use for the following 'recognition criteria' for the component corners. We say $\widehat{\mathcal{A}}$ is *pointlike in dimension one* when

$$\overline{GL_1}(\widehat{\mathcal{A}}) = U(A_+)\backslash U(A_0)/U(A_-)$$

is a singleton; that is, we can 'lift' the units in such a way that any unit $u_0 \in U(A_0)$ can be written $u_0 = \varphi_-(u_-)\varphi_+(u_+)$, where $u_\sigma \in U(A_\sigma)$ $(\sigma = \pm)$. We therefore have the following result [22]:

Proposition 6.4.9 (Recognition Criterion)**.** *Let $\widehat{\mathcal{A}}$ be of locally free type that is pointlike in dimension one and satisfies Milnor's patching condition. If A_0 is weakly Euclidean, then A has SFC.*

6.5 Stably free cancellation

The results of the previous section mean that we can reduce the situation of stably free modules over A to those over the component corner rings. To understand these it may become necessary to decompose these into yet simpler rings. This process may be continued until we are left with understanding stably free modules over the 'simplest' constituent rings. The aim of this section, and the next two, will be to understand these base rings and the properties they possess.

For completeness, we include a stronger property than SFC. We say that a ring Λ is *projective free* when any finitely generated projective Λ-module is free. Evidently, a stably free module S is projective but the converse need not be true. For consider the 2×2 matrices over the complex numbers, $R = M_2(\mathbf{C})$. By Wedderburn this is semisimple and we may consider the simple modules,

$$P = \begin{bmatrix} \mathbf{C} & 0 \\ \mathbf{C} & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & \mathbf{C} \\ 0 & \mathbf{C} \end{bmatrix}$$

thereby decomposing $M_2(\mathbf{C})$ as $M_2(\mathbf{C}) \cong P \oplus Q$. Now, if P is stably free, then $P \oplus R^m \cong R^n$, and so $2 + 4m = 4n$, i.e. $1 + 2m = 2n$.

Thus, we see that projective freeness is a stronger property than SFC. It should also be noted that Gabel's theorem regarding infinitely generated stably free modules does not work in the wider realm of projective modules. As such, it is necessary to include the 'finitely generated' hypothesis in statements concerning projective modules. However these considerations are not especially pertinent to us and the above discussion is more than ample for our purposes. We conclude by simply noting a theorem of Kaplansky [27] which tells us that projective modules cannot be 'too big'; specifically, any projective module is a direct sum of countably generated modules.

With this in mind, consider now which rings have SFC (or indeed, projective freeness). Quite clearly, any field is projective free since any vector space has a basis. More generally, it is well known that if Λ is a PID, then every submodule of a free Λ-mod is itself free (for instance, see Lang [29]). Consequently, every projective module over a PID is itself free and therefore:

Proposition 6.5.1. *If Λ is a PID, then Λ is projective free, and hence has SFC.*

For the noncommutative analogue of a PID, often referred to in the literature as a *free ideal ring*, the reader is directed to [7], [8] for more details in relation to projective freeness. In the next section, we will discuss a further generalisation, namely that of Dedekind domains. We will see that these too have SFC and will be of particular use in our later discussions.

Before that, we discuss some more general properties that will be directly applicable to our concerns. First, observe the following result for abelian groups. For a proof, the reader is directed to [23].

Proposition 6.5.2. *Let G be a finitely generated abelian group; then the group ring $\mathbf{Z}[G]$ has SFC.*

As we shall often 'build' more complex rings from, hopefully simpler, rings we provide a brief exposition of some standard constructions which preserve SFC. The reader is directed to [22] for proofs of these. Recall that an ideal $\mathbf{m}$ of Λ is *radical* when $\mathbf{m} \subset rad(\Lambda)$, the *Jacobson radical* of Λ. Moreover, if $\Lambda/rad(\Lambda)$ is a division algebra then we say that Λ is *local*. From Nakayama's Lemma and a result of Bourbaki [4], we have the following:

Proposition 6.5.3. *Let* $\mathbf{m}$ *be a two-sided radical ideal in a ring* Λ *and let* M *be a finitely presented flat* Λ*-module; then,*

$$M \otimes_\Lambda (\Lambda/\mathbf{m}) \textit{ is free over } \Lambda/\mathbf{m} \implies M \textit{ is free over } \Lambda.$$

Clearly, any finitely generated projective module P satisfies the hypotheses of Proposition 6.5.3. Further, if S is a stably free module, then the extension $S \otimes_\Lambda (\Lambda/\mathbf{m})$ is stably free over $\Lambda/\mathbf{m}$. We therefore have the following:

Proposition 6.5.4. *Let* $\mathbf{m}$ *be a two-sided radical ideal in a ring* Λ*; then,*

$$\Lambda/\mathbf{m} \textit{ has SFC} \implies \Lambda \textit{ has SFC.}$$

A special case of Proposition 6.5.4 is when $\mathbf{m} = rad(\Lambda)$ and Λ is a local ring. We therefore conclude that if Λ is a local ring, then it necessarily has SFC. In fact, it is a result of Kaplansky's (see [27]) that any projective module over a local ring is necessarily free. Another useful property follows from Morita equivalence.

Proposition 6.5.5. *If* Λ *has SFC, then* $M_n(\Lambda)$ *has SFC.*

Finally, we consider products of rings.

Proposition 6.5.6.

$$\Lambda_1 \times \Lambda_2 \textit{ has SFC if and only if } \Lambda_i \textit{ has SFC for } i = 1, 2.$$

6.6 Dedekind domains and free groups

A Dedekind domain Λ is a commutative integral domain for which every nonzero proper ideal factors into primes. Equivalently, any pair of ideals $\mathfrak{a} \subset \mathfrak{b}$, there exists an ideal $\mathfrak{c}$ such that $\mathfrak{a} = \mathfrak{b}\mathfrak{c}$. It is straightforward to see that in a Dedekind domain, any nonzero prime ideal is necessarily maximal. A classical example of a Dedekind domain, and one we shall frequently use, is the ring of algebraic integers in a number field. For a proof of this, the reader is directed to [31]. In particular, we have:

Proposition 6.6.1. *Any Dedekind domain* Λ *has SFC.*

This is essentially a consequence of a classification due to Steinitz (c.1911) for finitely generated torsion free modules over Dedekind domains (see [4], [49], [50]). In particular, this result will be useful alongside the theory of free groups.

Consider a ring Λ and a set X. Denote by $\Lambda\langle X\rangle$ the free algebra over Λ on the set X. By letting F_n be the free group on the set $X = \{x_1, \ldots, x_n\}$, we may describe the group algebra $\Lambda[F_n]$ as the localisation $\Lambda\langle X, X^{-1}\rangle$ of $\Lambda\langle X\rangle$ by formally inverting each $x_i \in X$. Moreover, $\Lambda[F_n]$ is isomorphic to the free product,

$$\Lambda[F_n] = \Lambda[x_1, x_1^{-1}] * \cdots * \Lambda[x_n, x_n^{-1}]$$

where the coefficients Λ are identified in the various copies. In particular, we note the special case when $|X| = 1$. This allows us to represent the group algebra $\Lambda[C_\infty]$, where C_∞ is the infinite cyclic group, as the ring $\Lambda[x, x^{-1}]$ of Laurent polynomials in the variable x and with coefficients in Λ.

Following [21], we see that when Λ is a (possibly noncommutative) PID, then both $\Lambda[x]$ and $\Lambda[x,\, x^{-1}]$ are left and right Noetherian domains of global dimension at most 2. It can be shown that,

$$\Lambda[x] \text{ is projective free } \Longleftrightarrow \Lambda[x,\, x^{-1}] \text{ is projective free.} \tag{6.6.2}$$

Using 6.6.2, along with Seshadri's result that $\Lambda[x]$ is projective free [46] (for Λ a commutative PID), we conclude that $\Lambda[x,\, x^{-1}]$ is also projective free. This was extended by Bass [1] to show:

Proposition 6.6.3. *Let* Λ *be a commutative PID and* $X = \{x_1, \dots, x_n\}$*; then* $\Lambda[F_n] = \Lambda\langle X,\, X^{-1}\rangle$ *is projective free.*

However, it should be noted that this does not extend to noncommutative PIDs. For we can consider the integral quaternion algebra $\left(\frac{-1,\,-1}{\mathbf{Z}}\right)$. For any odd prime p, the localisation $\Omega_{(p)}$ is a PID. However, in [22] Johnson has shown $\Omega_{(p)}[C_\infty]$ has infinitely many isomorphically distinct stably free modules of rank 1.

If we now restrict ourselves to stably free modules, then Bass' generalisation of Seshadri's argument shows [2]:

Proposition 6.6.4. *If* Λ *is a Dedekind domain, then* $\Lambda\langle X,\, X^{-1}\rangle$ *has SFC.*

Next, we consider the special case where $\Lambda = \mathcal{D}$ division ring. We may write,

$$\mathcal{D}[F_n] = \mathcal{D}[x_1,\, x_1^{-1}] * \cdots * \mathcal{D}[x_n,\, x_n^{-1}]$$

in which each $\mathcal{D}[x_i,\, x_i^{-1}]$ is a (left and right) PID (see [22], p. 170). By Proposition 6.5.1 $\mathcal{D}[x_i,\, x_i^{-1}]$ is therefore seen to be projective free. A result of Dicks and Sontag [12] now shows:

Proposition 6.6.5. *If* $\mathcal{D}$ *is a division ring, then* $D[F_n]$ *is projective free.*

For completeness, we briefly discuss the corresponding situation for free abelian algebras. Now, the situation is entirely different and decidedly ill-behaved. If we define the set X as before, then for $n \geq 1$, we let C_∞^n denote the free abelian group on the set X. For a ring Λ, the group algebra $\Lambda[C_\infty^n]$ is isomorphic to the ring of Laurent polynomials,

$$\Lambda[C_\infty^n] = \Lambda[x_1,\, x_1^{-1}, \dots,\, x_n,\, x_n^{-1}].$$

If Λ is a Dedekind domain, then $\Lambda[C_\infty^n]$ has SFC (see [28], p.189). Next, observe that F_n and C_∞^n coincide when $n = 1$. As such, when $\mathcal{D}$ is a division ring, $\mathcal{D}[C_\infty]$ has SFC. However, it has been shown in [28] that requiring $\mathcal{D}[C_\infty^n]$ $(n > 1)$ to have SFC is equivalent to requiring that $\mathcal{D}$ is commutative. Indeed, by generalizing the arguments of Dicks-Sontag [12], Ojurangen-Sridharan [34] and Parimala-Sridharan [36], we observe that whenever $\mathcal{D}$ is a noncommutative division ring and $n \geq 2$, then $\mathcal{D}[C_\infty^n]$ possesses nontrivial stably free modules.

Nevertheless, if k is a field such that $char(k) > 0$, then $k[C_\infty^n \times \Phi]$ has SFC for any finite group Φ (see [21]). When dealing with a field of characteristic zero, we need to be more careful. If $n \geq 2$ and $char(k) = 0$, then for $k[C_\infty^n \times \Phi]$ to have SFC, it is necessary that Φ satisfies the generalized Eichler property[1]. However, it should

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[1] As this is of no further use to us in this thesis, the reader is directed to [21] for a definition of this.

be noted that the converse of this statement is not true in general. In spite of this, Johnson points out [21] that as a consequence of Stafford (see [48]) the converse does hold when we consider polynomials, rather than Laurent polynomials.

We conclude this section by considering a property shared by free groups which will be of use in the next section. We begin by considering a group G and two subsets of G, denoted by X and Y. We say that $g \in G$ is represented as a product in X and Y when $g = xy$ for some $x \in X$ and $y \in Y$. Further, we say that g is *uniquely represented* as a product in X and Y when, in addition, we require that if $g = x'y'$ with $x' \in X$ and $y' \in Y$, then $x = x'$ and $y = y'$. Now, we say that G satisfies the *two unique products* condition (abbreviated to $\mathcal{TUP}$) when, given finite subsets X, Y of G with $|X| \geq 2$ and $|Y| \geq 2$, at least two elements of G are uniquely represented as products in X, Y.

It is a straightforward observation that no nontrivial finite group can satisfy $\mathcal{TUP}$. In particular, we note that every $\mathcal{TUP}$ group is torsion free. Moreover, from the work of Higman [18], we note that free groups have the $\mathcal{TUP}$ property.

Now, we say that a group ring $\Lambda[G]$ over Λ has *only trivial units* when every $\lambda \in U(\Lambda[G])$ has the form $\lambda = ug$ for $u \in U(\Lambda)$ and $g \in G$. We then have the following result due to Passman [38]:

Proposition 6.6.6. *Let G be a $\mathcal{TUP}$ group; then for any (possibly noncommutative) integral domain Λ, $\Lambda[G]$ has only trivial units.*

6.7 Weakly Euclidean

We conclude the required legwork by discussing what sort of ring is weakly Euclidean. Recall that a ring Λ is said to be weakly Euclidean when

$$GL_n(\Lambda) = GE_n(\Lambda) = D_n(\Lambda) \cdot E_n(\Lambda)$$

for all $n \geq 2$. A ring homomorphism $\pi : A \to B$ is said to have the *lifting property for units* when the induced map on units $\pi_* : U(A) \to U(B)$ is a surjection. Further, we say that π has the *strong lifting property for units* when, in addition, we have the following:

$$a \in U(A) \iff \pi(a) \in U(B).$$

With this, we now have the following Recognition Criterion, see [22].

Proposition 6.7.1. *Let $\pi : A \to B$ be a surjective ring homomorphism with the strong lifting property for units. If B is weakly Euclidean, then so too is A.*

Now, it is quite clear that any division ring, being a PID is weakly Euclidean. More generally, from [47], we see that any commutative Euclidean domain is weakly Euclidean. If we now consider a (possibly noncommutative) local ring Λ, then we have the canonical homomorphism $\pi : \Lambda \to \Lambda/rad(\Lambda)$. It is straightforward to see that π has the strong lifting property for units and since $\Lambda/rad(\Lambda)$ is a division ring, we have:

Proposition 6.7.2. *If Λ is a (possibly noncommutative) local ring, then Λ is weakly Euclidean.*

Similar to our results of Section 6.5, we observe the following two results [22]:

Proposition 6.7.3. *If $\Lambda_1, \dots, \Lambda_n$ are weakly Euclidean rings, then so too is $\Lambda_1 \times \cdots \times \Lambda_n$.*

Proposition 6.7.4. *Let Λ be a weakly Euclidean ring. Then $M_n(\Lambda)$ is also weakly Euclidean.*

Moreover, by considering the free groups of the previous section, we note the following theorem of Cohn [7]:

Proposition 6.7.5. *For any division ring D, the group ring $D[F_n]$ is weakly Euclidean.*

The following is proved in [22]:

Proposition 6.7.6. *Let $\pi : A \to B$ be a surjective ring homomorphism with the strong lifting property for units and suppose that the ideal $Ker(\pi)$ is nilpotent; if G is a group such that $B[G]$ has only trivial units, then the induced homomorphism $\pi_* : A[G] \to B[G]$ has the strong lifting property for units.*

As already noted, the free group F_n satisfies the $\mathcal{TUP}$ condition and therefore for a division ring D the group ring $D[F_n]$ has only trivial units. If we now consider a local ring Λ, then we may set $D = \Lambda/rad(\Lambda)$ where $rad(\Lambda)$ is nilpotent. This clearly satisfies the hypotheses of Proposition 6.7.6 and so we conclude that the induced homomorphism $\Lambda[F_n] \to D[F_n]$ has the strong lifting property for units. By Propositions 6.7.1 and 6.7.5 we conclude:

Proposition 6.7.7. *If Λ is a local ring for which $rad(\Lambda)$ is nilpotent, then the group ring $\Lambda[F_n]$ is weakly Euclidean.*

6.8 The cancellation properties of the modules $R(i)$

We conclude this chapter by tying up a loose end from Part I of this thesis. For this section, write $\Lambda = \mathbf{Z}[G(p, q)]$ for the integral group ring of the metacyclic group given in Example 6.2.12, and recall $\mathcal{T}_q(A, \pi)$ decomposes as a direct sum of right ideals

$$\mathcal{T}_q(A, \pi) \cong R(1) \oplus \cdots R(q),$$

where $R(i)$ is the i^{th} row of $\mathcal{T}_q(A, \pi)$. By considering $R(i)$ as a Λ-module, we write $[R(i)]$ for the stable class of $R(i)$. In particular, we note that $R(i)$ is a minimal element of $[R(i)]$. At various points throughout Chapters 4 and 5 we used the fact that $[R(i)]$ is straight for each $i \in \{1, \dots, q\}$. The following proof was shown to the author by Prof. F. E. A. Johnson, and has the benefit being much clearer than that of [41], albeit at the cost of a bit more work.

Let R be a ring, and denote by $\mathcal{P}(R)$, the set of isomorphism classes of finitely generated projective R-modules. As previously noted, this is a commutative monoid under direct sum

$$\begin{array}{ccc} \mathcal{P}(R) \times \mathcal{P}(R) & \to & \mathcal{P}(R) \\ (P\ ,\ Q) & \mapsto & P \oplus Q. \end{array}$$

For $q \geq 2$, define $\mathfrak{R}_q(R) = \{(r_1, \ldots, r_q) \mid r_i \in R\}$, which is a R-$M_q(R)$ bimodule. If Q is a right R-module, we define

$$\widetilde{Q} = Q \otimes_R \mathfrak{R}_q(R)$$

so that $\widetilde{Q}$ is a right $M_q(R)$-module. We then have the following form of Morita's Theorem:

For any $q \geq 2$ the correspondence $Q \mapsto \widetilde{Q}$ gives an isomorphism of monoids $\mathcal{P}(R) \xrightarrow{\simeq} \mathcal{P}(M_q(R))$. (6.8.1)

If we specialise R to the case of a commutative PID, then we can describe $\mathcal{P}(R)$:

If R is a commutative principal ideal domain, then $\mathcal{P}(R) \cong \mathbf{N}$. (6.8.2)

Furthermore, if we write $R^q = \underbrace{R \times \cdots \times R}_{q}$, then:

If R is a commutative principal ideal domain, then $\mathcal{P}(R^q) \cong \mathbf{N}^q$. (6.8.3)

In the isomorphism of (6.8.3), the generators of $\mathcal{P}(R^q)$ correspond to the primitive idempotents of R^q.

Next, for a commutative principal ideal domain A such that $(\pi) \triangleleft A$ is a maximal ideal, we have the quasi-triangular subring of $M_q(A)$,

$$\mathcal{T}_q(A, \pi) = \{(x_{rs} \in M_q(A) \mid x_{rs} \in (\pi) \text{ if } r > s\}.$$

We then have the following fibre square decomposition of $\mathcal{T}_q(A, \pi)$ (see Proposition 7.2.1):

$$\begin{tikzcd} \mathcal{T}_q(A, \pi) \arrow[r] \arrow[d] & M_q(A) \arrow[d, "\nu"] \\ \mathcal{T}_q(A/\pi) \arrow[r, "j"] & M_q(A/\pi). \end{tikzcd} \tag{6.8.4}$$

Note that $\mathcal{T}_q(A, \pi)$ decomposes as a direct sum of right ideals

$$\mathcal{T}_q(A, \pi) = R(1) \oplus \cdots \oplus R(q),$$

where $R(i)$ is the i^{th} row of $\mathcal{T}_q(A, \pi)$. Evidently, each $R(i)$ is projective over $\mathcal{T}_q(A, \pi)$. In the special case where $\pi = \{0\}$ we write $\mathcal{T}_q(A)$ which decomposes as

$$\mathcal{T}_q(A) = \mathcal{R}(1) \oplus \cdots \oplus \mathcal{R}(q).$$

For R a commutative principal ideal domain, observe that there is an obvious surjective ring homomorphism $\delta : \mathcal{T}_q(R) \to R^q$ given by $\delta(X) = (X_{11}, \ldots, X_{qq})$. The kernel of this homomorphism is the nilpotent ideal of strictly upper triangular matrices and is contained in the Jacobson radical of $\mathcal{T}_q(R)$ (see the proof of Proposition

7.2.2). It then follows from (6.8.3) and Nakayama's Lemma that:

$$\text{If } R \text{ is a commutative PID, then } \mathcal{P}(\mathcal{T}_q(R)) \cong \mathbf{N}^q \text{ with generators } \mathcal{R}(1), \dots, \mathcal{R}(q). \tag{6.8.5}$$

Proposition 6.8.6. *Let A be a commutative principal ideal domain, and let $\pi \in A$ generate a maximal ideal in A. Then $\mathcal{P}(\mathcal{T}_q(A, \pi)) \cong \mathbf{N}^q$ with generators $R(1), \dots, R(q)$.*

Proof. Return once more to the fibre square given in (6.8.4). If P is a finitely generated projective module over $\mathcal{T}_q(A, \pi)$, then by Milnor's classification P can be described as a triple $P \cong (P_+, P_-; \alpha)$ in which $P_+ \in \mathcal{P}(\mathcal{T}_q(A/\pi))$, $P_- \in \mathcal{P}(M_q(A))$ and $\alpha : j_*(P_+) \xrightarrow{\simeq} \nu_*(P_-)$ is an isomorphism over $M_q(A/\pi)$.

Now, let $\mathbf{c} = (c_1, \dots, c_q) \in \mathbf{N}^q$ and write

$$P_+(\mathbf{c}) = \bigoplus_{i=1}^{q} \mathcal{R}(i)^{c_i} \text{ and } P_-(\mathbf{c}) = \widetilde{A}^{|\mathbf{c}|}$$

where $|\mathbf{c}| = \sum_{i=1}^{q} c_i$. In particular, we observe $j_*(P_+(\mathbf{c})) \cong \nu_*(P_-(\mathbf{c}))$. In this case, we refer to $\mathbf{c}$ as the *type* of P. It now follows from (6.8.1) and (6.8.2) that $\mathcal{P}(M_q(A)) \cong \mathbf{N}$. Likewise, since A/π is a field, it follows from (6.8.5) that $\mathcal{P}(\mathcal{T}_q(A/\pi)) \cong \mathbf{N}^q$. Thus, by Milnor's classification, any $P \in \mathcal{P}(\mathcal{T}_q(A, \pi))$ is of type $\mathbf{c}$ for some $\mathbf{c} \in \mathbf{N}^q$. In particular,

$$\bigoplus_{i=1}^{q} R(i)^{c_i} \text{ is of type } \mathbf{c}. \tag{6.8.7}$$

It therefore suffices to show that, up to isomorphism, $\bigoplus_{i=1}^{q} R(i)^{c_i}$ is the unique projective module over $\mathcal{T}_q(A, \pi)$ of type $\mathbf{c}$.

To show this, first note that by Milnor's classification, we may write

$$\bigoplus_{i=1}^{q} R(i)^{c_i} = (P_+(\mathbf{c}), P_-(\mathbf{c}); \gamma)$$

where we regard $\gamma : j_*(P_+(\mathbf{c})) \xrightarrow{\simeq} \nu_*(P_-(\mathbf{c}))$ as a 'basepoint isomorphism'. Next, suppose $(P_+(\mathbf{c}), P_-(\mathbf{c}); \alpha)$ is also of type $\mathbf{c}$. Relative to γ, α can be described as an element of $GL_q((A/\pi)^{|\mathbf{c}|}) \cong GL_{|\mathbf{c}|q}(A/\pi)$. However, as A/π is a field, then it is weakly Euclidean. As such, α can be expressed as a product $\alpha = \Delta \cdot E$, where Δ is a diagonal matrix with entries $\Delta_{ii} \in U(A/\pi)$ and $E = E_1 \cdots E_N$ is a product of transvections with $det(E_r) = 1$ for each $r \in \{1, \dots, N\}$.

Now, each Δ_{ii} is located in the automorphism group of the corresponding $\mathcal{R}(i)$ so that $\Delta \in Im(Aut(P_+(\mathbf{c})) \to Gl_{|\mathbf{c}|q}(A))$. Furthermore, since $A \to A/\pi$ is surjective, then so too is $E_{|\mathbf{c}|q}(A) \to E_{|\mathbf{c}|q}(A/\pi)$. Hence, $E \in Im(GL_{|\mathbf{c}|q}(A) \to GL_{|\mathbf{c}|q}(A/\pi))$. Therefore, by Milnor $(P_+(\mathbf{c}), P_-(\mathbf{c}); \alpha) \cong (P_+(\mathbf{c}), P_-(\mathbf{c}); \gamma)$, i.e $P \cong \bigoplus_{i=1}^{q} R(i)^{c_i}$. So $\bigoplus_{i=1}^{q} R(i)^{c_i}$ is the unique projective module of type $\mathbf{c}$, as required. □

Let P be a finitely generated projective module over $\mathcal{T}_q(A, \pi)$. As already seen, P can be described as a triple $(P_+, P_-; \alpha)$ where $P_+ \in \mathcal{P}(\mathcal{T}_q(A/\pi))$, $P_- \in \mathcal{P}(M_q(A))$ and $\alpha : j_*(P_+) \xrightarrow{\simeq} \nu_*(P_-)$ is an isomorphism over $M_q(A/\pi)$. We say that P is of type (s, Q) when

$$P_+ \cong_{\mathcal{T}_q(A/\pi)} \mathcal{R}(s) \text{ and } P_- \cong_{M_q(A)} \widetilde{Q}$$

where Q is a projective module over A. Of primary concern to us will be modules of type $(s,\, A)$, where we consider A as an ideal in its own right.

Proposition 6.8.8. *Let A be a Dedekind domain, $\pi \in A$ a prime and $1 \leq s \leq q$; then, up to isomorphism, $R(s)$ is the unique projective module over $\mathcal{T}_q(A,\, \pi)$ of type $(s,\, A)$.*

Proof. As $R(s)$ is of type $(s,\, A)$, we write $R(s) = (\mathcal{R}(s),\, \widetilde{A};\, \beta)$, where we regard the isomorphism $\beta : j_*(\mathcal{R}(s)) \xrightarrow{\simeq} \nu_*(\widetilde{A})$ describing $R(s)$ as a 'basepoint isomorphism'. Next, let $P = (\mathcal{R}(s),\, \widetilde{A};\, \alpha)$ be another module of type $(s,\, A)$. Relative to β, α can be described as a element of $GL_q(A/\pi)$. However, A/π is a field, and so we can write α as a product $\alpha = \Delta \cdot E$. Here, Δ is a diagonal matrix with entries $\Delta_{ii} \in U(A/\pi)$ and $E = E_1 \cdots E_N$ is a product of transvections with determinant $det(E_r) = 1$. Evidently, $\Delta \in Im(U(\mathcal{T}_q(A/\pi)) \to GL_q(A/\pi))$, and since $A \to A/\pi$ is a surjection, then $E \in Im(GL_q(A) \to GL_q(A/\pi))$. From Milnor's 'isomorphism criterion', it follows that $(\mathcal{R}(s),\, \widetilde{A};\, \alpha) \cong (\mathcal{R}(s),\, \widetilde{A}; \beta)$, i.e. $P \cong R(s)$. ☐

Corollary 6.8.9. *Let A be a Dedekind domain, and $\pi \in A$ be prime; then each stable class $[R(s)]$ is straight over $\mathcal{T}_q(A,\, \pi)$.*

Proof. Let $\mathbf{E}$ be the field of fractions of A. As $\mathcal{T}_q(A,\, \pi)$ is an order in the simple $\mathbf{E}$-algebra $M_q(\mathbf{E})$ it suffices, by Swan-Jacobinski, to show

$$X \oplus \mathcal{T}_q(A,\, \pi) \cong_{\mathcal{T}_q(A,\, \pi)} R(s) \oplus \mathcal{T}_q(A,\, \pi) \;\Rightarrow\; X \cong_{\mathcal{T}_q(A,\, \pi)} R(s).$$

First, observe that X is necessarily a finitely generated projective module over $\mathcal{T}_q(A,\, \pi)$. Furthermore, both $\mathcal{P}(\mathcal{T}_q(A/\pi))$ and $\mathcal{P}(M_q(A)) \cong \mathcal{P}(A)$ are cancellation monoids, isomorphic to $\mathbf{N}^q$ and $\mathbf{N} \oplus \widetilde{K_0}(A)$, respectively. Thus X has the same local type as $R(s)$ and so $X \cong_{\mathcal{T}_q(A,\, \pi)} R(s)$ by Proposition 6.8.8. ☐

Now, let p, q be primes such that $q|p-1$ and let $\Lambda = \mathbf{Z}[G(p,\, q)]$. We now restrict A to the fixed ring $A = \mathbf{Z}[\zeta_p]^\theta$, and $\pi = (\zeta_p - 1)^q$, as with Example 6.2.12. We then have the following fibre square decomposition of Λ:

$$\begin{tikzcd} \mathbf{Z}[C_p \rtimes C_q] \arrow[r] \arrow[d] & \mathcal{T}_q(A,\, \pi) \arrow[d] \\ \mathbf{Z}[C_q] \arrow[r] & \mathbf{F}_p[C_q]. \end{tikzcd} \tag{$\ddagger'$}$$

Evidently, $\mathcal{T}_q(A,\, \pi)$, and hence each $R(s)$, acquires the structure of a Λ-module by coinduction from the homomorphism $\Lambda \twoheadrightarrow \mathcal{T}_q(A,\, \pi)$. We can now finally prove (2.8.2); that is:

Proposition 6.8.10. *Each stable class $[R(s)]$ is straight over Λ.*

Proof. For any Λ-module M, we put $M_{\mathbf{Q}} = M \otimes_{\mathbf{Z}} \mathbf{Q}$ so that $M_{\mathbf{Q}}$ is a module over the semisimple ring $\Lambda_{\mathbf{Q}} = \mathbf{Q}[G(p,\, q)]$. By Swan-Jacobinski, it is sufficient to prove,

$$X \oplus \Lambda \cong_\Lambda R(s) \oplus \Lambda \;\Rightarrow\; X \cong_\Lambda R(s),$$

for some Λ-module X. As $X_{\mathbf{Q}} \oplus \Lambda_{\mathbf{Q}} \cong_{\Lambda_{\mathbf{Q}}} R(s)_{\mathbf{Q}} \oplus \Lambda_{\mathbf{Q}}$, then by Wedderburn's Theorem we see that $X_{\mathbf{Q}} \cong_{\Lambda_{\mathbf{Q}}} R(s)_{\mathbf{Q}}$. It now follows easily that

$$Hom_{\Lambda}(X, \mathbf{Z}[C_q]) = 0.$$

Next, let $h : X \oplus \Lambda \xrightarrow{\simeq} R(s) \oplus \Lambda$ be a Λ-isomorphism and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & X \oplus \mathcal{T}_q(A, \pi) \arrow[r, "i"] & X \oplus \Lambda \arrow[r, "p"] \arrow[d, "h"'] & \mathbf{Z}[C_q] \arrow[r] & 0 \\
0 \arrow[r] & R(s) \oplus \mathcal{T}_q(A, \pi) \arrow[r, "i'"] & R(s) \oplus \Lambda \arrow[r, "p'"] & \mathbf{Z}[C_q] \arrow[r] & 0.
\end{tikzcd}$$

Since $Hom_{\Lambda}(X, \mathbf{Z}[C_q]) = 0$, we can complete the above to a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \oplus \mathcal{T}_q(A, \pi) \arrow[r, "i"] \arrow[d, "h_-"] & X \oplus \Lambda \arrow[r, "p"] \arrow[d, "h"] & \mathbf{Z}[C_q] \arrow[r] \arrow[d, "h_+"] & 0 \\
0 \arrow[r] & R(s) \oplus \mathcal{T}_q(A, \pi) \arrow[r, "i'"] & R(s) \oplus \Lambda \arrow[r, "p'"] & \mathbf{Z}[C_q] \arrow[r] & 0.
\end{tikzcd}$$

in which h_+ is necessarily surjective. Furthermore, as $\mathbf{Z}[C_q]$ is a free abelian group of finite rank, it follows that h_+ is a Λ-isomorphism. Hence, so too is h_-.

Now, as the Λ-module structure on $R(s) \oplus \mathcal{T}_q(A, \pi)$ is coinduced from $\mathcal{T}_q(A, \pi)$, it follows that the Λ-module X is coinduced from a $\mathcal{T}_q(A, \pi)$-module which we denote by $\widehat{X}$. Then h_- defines an isomorphism of $\mathcal{T}_q(A, \pi)$-modules,

$$h_- : \widehat{X} \oplus \mathcal{T}_q(A, \pi) \to R(s) \oplus \mathcal{T}_q(A, \pi).$$

By Corollary 6.8.9, $\widehat{X} \cong_{\mathcal{T}_q(A, \pi)} R(s)$. As the Λ-module structures on X and $R(s)$ are coinduced from $\mathcal{T}_q(A, \pi)$ then $X \cong_{\Lambda} R(s)$. ☐

Chapter 7

Stably free modules over $\mathbf{Z}[G(p,\, q) \times F_n]$

With the preliminary work of Chapter 6 done, we now discuss the stably free modules over $\mathbf{Z}[G(p,\, q) \times F_n]$ where p, q are primes such that $q|p-1$. In particular, it is shown that $\mathbf{Z}[G(p,\, q) \times F_n]$ admits no nontrivial stably free module. Recall that we constructed a fibre model for $\mathbf{Z}[G(p,\, q) \times F_n]$ in Section 6.2

$$\begin{tikzcd}
\mathbf{Z}[(C_p \rtimes C_q) \times F_n] \arrow[r] \arrow[d] & \mathcal{T}_q(A,\, \pi)[F_n] \arrow[d] \\
\mathbf{Z}[C_q \times F_n] \arrow[r] & \mathbf{F}_p[C_q \times F_n]
\end{tikzcd} \qquad (\heartsuit)$$

which clearly satisfies Milnor Patching. Our focus now becomes that of understanding each of the corners in turn in the hope that $\heartsuit$ is locally free. Of use to us throughout this chapter will be the following result:

Proposition 7.0.1. $\mathbf{F}_p[C_q] \cong \underbrace{\mathbf{F}_p \times \cdots \mathbf{F}_p}_{q}.$

Proof. First, $\mathbf{F}_p[C_q]$ is isomorphic to the ring $\mathbf{F}_p[x]/(x^q - 1)$. If we write $f(x) = x^q - 1$, then $f'(x) = qx^{q-1}$ and we observe $gcd(f,\, f') = 1$ as $gcd(p,\, q) = 1$. As such, f has no repeated roots and the factorisation of f in $\mathbf{F}_p[x]$ is

$$f(x) = \prod_i f_i(x)$$

in which each $f_i(x)$ is a distinct irreducible factor. By the Chinese Remainder Theorem, we therefore have an isomorphism of rings

$$\mathbf{F}_p[C_q] \cong \bigoplus_i \mathbf{F}_p[x]/(f_i(x)).$$

The result now follows since $x^q - 1$ splits completely in $\mathbf{F}_p[x]$ (it has q distinct roots). This is a direct consequence of the fact that $U(\mathbf{F}_p)$ is a cyclic group of order $p-1$. Since $q|p-1$, there are precisely q elements a whose order divides q. As each of these satisfy $a^q = 1$, there are q roots of $x^q - 1$. $\square$

Corollary 7.0.2. $\mathbf{F}_p[C_q \times F_n] \cong \underbrace{\mathbf{F}_p[F_n] \times \cdots \mathbf{F}_p[F_n]}_{q}$.

Corollary 7.0.3. $\mathbf{F}_p[C_q \times F_n]$ *is weakly Euclidean; that is,*

$$GL_n(\mathbf{F}_p[C_q \times F_n]) = U(\mathbf{F}_p[C_q \times F_n])E_n(\mathbf{F}_p[C_q \times F_n])$$

for each $n \geq 2$.

Proof. By Corollary 7.0.2, we may write $\mathbf{F}_p[C_q \times F_n]$ as q copies of $\mathbf{F}_p[F_n]$. By Proposition 6.7.5, each of these copies is weakly Euclidean. The result now follows from Proposition 6.7.3. ☐

Proposition 7.0.4. *Let*

$$\begin{tikzcd} R \arrow[r, "\pi_-"] \arrow[d, "\pi_+"'] & R_- \arrow[d, "\varphi_-"'] \\ R_+ \arrow[r, "\varphi_+"] & R_0 \end{tikzcd}$$

be a fibre square, and let G *be a group. If* $R_+[G]$, $R_-[G]$ *have only trivial units, then so also does* $R[G]$.

Proof. Suppose $\alpha \in U(R[G])$ and write $\alpha = \sum_{g \in G} a_g g$. Observe $\pi_+(\alpha) \in U(R_+[G])$. Since $R_+[G]$ has only trivial units, there exists $h \in G$ such that $\pi_+(\alpha) = \pi_+(a_h)h$. Similarly, $\pi_-(\alpha) \in U(R_-[G])$ and the same argument shows there exists some $k \in G$ such that $\pi_-(\alpha) = \pi_-(a_k)k$.

Now, $\varphi_+\pi_+(\alpha) \in U(R_0[G])$ so that $\varphi_+\pi_+(\alpha) \neq 0$. Hence $\varphi_+\pi_+(a_h) \neq 0$. Likewise $\varphi_-\pi_-(\alpha) \neq 0$ and so $\varphi_-\pi_-(a_k) \neq 0$. Since $\varphi_+\pi_+(\alpha) = \varphi_-\pi_-(\alpha)$, we can rewrite this as $\varphi_+\pi_+(a_h)h = \varphi_-\pi_-(a_k)k$. As $\{g \mid g \in G\}$ is a basis for $R_0[G]$, and as both $\varphi_+\pi_+(a_h) \neq 0$, $\varphi_-\pi_-(a_k) \neq 0$, it follows that $h = k$.

Next, put $\alpha' = a_h h$ so that $\pi_+(\alpha') = \pi_+(\alpha)$ and $\pi_-(\alpha') = \pi_-(\alpha)$. Using the fact that $\pi_+ \times \pi_- : R[G] \to R_+[G] \times R_-[G]$ is injective, it follows that $\alpha = \alpha'$. Hence $supp(\alpha) = \{h\}$, i.e. $|supp(\alpha)| = 1$. We conclude α is a trivial unit, as required. ☐

Corollary 7.0.5. $U(\mathbf{Z}[C_q \times F_n]) \cong U(\mathbf{Z}[C_q]) \times F_n$.

Proof. First, decompose $\mathbf{Z}[C_q]$ into the following fibre square model,

$$\begin{tikzcd} \mathbf{Z}[C_q] \arrow[r, "\pi_-"] \arrow[d, "\pi_+"'] & \mathbf{Z}[\zeta_q] \arrow[d] \\ \mathbf{Z} \arrow[r] & \mathbf{F}_q \end{tikzcd}$$

where $\zeta_q = exp(2\pi i/q)$. Since $\mathbf{Z}$, $\mathbf{Z}[\zeta_q]$ are integral domains, $\mathbf{Z}[F_n]$, $\mathbf{Z}[\zeta_q][F_n]$ have only trivial units (Proposition 6.6.6). The result follows from Proposition 7.0.4. ☐

It should be noted, however, that the argument of Proposition 7.0.4 fails if R_0 is allowed to be the zero ring. In this instance $A \cong A_+ \times A_-$ and we can always find nontrivial units for any nontrivial group G. For if g, $h \in G$ such that $g \neq h$, we can simply choose $\alpha = (1, 0)g + (0, 1)h$ and $\beta = (1, 0)g^{-1} + (0, 1)h^{-1}$. It is quite clear that $\alpha\beta = \beta\alpha = (1, 1)$ but neither is trivial.

Proposition 7.0.6. $U(\mathbf{F}_p[C_q \times F_n]) \cong U(\mathbf{F}_p[C_q]) \times F_n^q$.

Proof. By Proposition 6.6.6 $U(\mathbf{F}_p[F_n]) \cong U(\mathbf{F}_p) \times F_n$. The result now follows from Corollary 7.0.2. □

Corollary 7.0.7. *$U(\mathbf{F}_p[C_q \times F_n])/U(\mathcal{T}_q(A, \pi)[F_n])$ is finite; in fact*

$$|U(\mathbf{F}_p[C_q \times F_n])/U(\mathcal{T}_q(A, \pi)[F_n])| \leq |U(\mathbf{F}_p[C_q])/U(\mathcal{T}_q(A, \pi))|.$$

Proof. Write $\mathcal{T}_q = \mathcal{T}_q(A, \pi)$ and observe $U(\mathcal{T}_q[F_n])$ contains a copy of F_n^q, namely the diagonal matrices

$$\Delta(\gamma_1, \dots, \gamma_q) = \begin{pmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_q \end{pmatrix}$$

where $\gamma_i \in F_n$. Combining this with the obvious inclusion $U(\mathcal{T}_q) \subset U(\mathcal{T}_q[F_n])$ gives an injection $U(\mathcal{T}_q) \times F_n^q \hookrightarrow U(\mathcal{T}_q[F_n])$. Hence, we have a surjection

$$U(\mathbf{F}_p[C_q \times F_n])/U(\mathcal{T}_q) \times F_n^q \twoheadrightarrow U(\mathbf{F}_p[C_q \times F_n])/U(\mathcal{T}_q[F_n]). \tag{7.0.8}$$

It follows from Proposition 7.0.6 that we have bijections

$$U(\mathbf{F}_p[C_q \times F_n])/U(\mathcal{T}_q) \times F_n^q \leftrightarrow U(\mathbf{F}_p[C_q]) \times F_n^q/U(\mathcal{T}_q) \times F_n^q \leftrightarrow U(\mathbf{F}_p[C_q])/U(\mathcal{T}_q). \tag{7.0.9}$$

Combining this with the surjection of (7.0.8) we conclude that we have a surjection $U(\mathbf{F}_p[C_q])/U(\mathcal{T}_q) \twoheadrightarrow U(\mathbf{F}_p[C_q \times F_n])/U(\mathcal{T}_q[F_n])$. The result now follows as $U(\mathbf{F}_p[C_q])/U(\mathcal{T}_q)$ is finite. □

7.1 $\mathbf{Z}[C_q \times F_n]$ has SFC

We now demonstrate that the stably free modules over $\mathbf{Z}[C_q \times F_n]$ are trivial. The results of this section can be found in Chapter 10 of [22]. Nevertheless, we include the details here for completeness. Start by decomposing $\mathbf{Z}[C_q]$ in the usual way:

$$\begin{tikzcd} \mathbf{Z}[C_q] \arrow[r] \arrow[d] & \mathbf{Z}[\zeta_q] \arrow[d] \\ \mathbf{Z} \arrow[r] & \mathbf{F}_q \end{tikzcd}$$

On applying the functor $- \otimes_{\mathbf{Z}} \mathbf{Z}[F_n]$ and Corollary 6.2.5 we obtain another Milnor square:

$$\begin{tikzcd} \mathbf{Z}[C_q \times F_n] \arrow[r] \arrow[d] & (\mathbf{Z}[\zeta_q])[F_n] \arrow[d] \\ \mathbf{Z}[F_n] \arrow[r] & \mathbf{F}_q[F_n] \end{tikzcd} \tag{♣}$$

Our intention will now be to apply Proposition 6.4.9. First, $\mathbf{Z}$ is a Dedekind domain and so $\mathbf{Z}[F_n]$ has SFC by Proposition 6.6.4. Similarly, if we regard $\mathbf{Z}[\zeta_q]$ as the ring of integers in the cyclotomic field $\mathbf{Q}(\zeta_q)$ then it is seen to be a Dedekind domain also, and so $\mathbf{Z}[\zeta_q][F_n]$ has SFC by the same reasoning. Consequently, $\mathbf{Z}[C_q \times F_n]$ is locally free.

Now, $\mathbf{F}_q$ is a finite field and so it follows that $\mathbf{F}_q[F_n]$ is weakly Euclidean (by Proposition 6.7.5). It remains to lift the units. By Proposition 6.6.6, $\mathbf{F}_q[F_n]$ has only trivial units, i.e. any unit $u \in U(\mathbf{F}_q[F_n])$ can be written $u = \lambda g$ for $\lambda \in U(\mathbf{F}_q)$ and $g \in F_n$. As has been seen, this latter property is one shared both by $\mathbf{Z}[F_n]$ and $\mathbf{Z}[\zeta_q^*][F_n]$.

Proposition 7.1.1. *There exists a surjective map on units,*

$$\varphi : U(\mathbf{Z}[\zeta_q][F_n]) \twoheadrightarrow U(\mathbf{F}_q[F_n]).$$

Proof. Note that $\mathbf{Z}[\zeta_q]$ is a $\mathbf{Z}$-lattice of rank $q-1$ in $\mathbf{Q}(\zeta_q)$, and $(\zeta_q - 1)\mathbf{Z}[\zeta_q]$ has index q in $\mathbf{Z}[\zeta_q]$. Moreover, $(\zeta_q - 1)^{q-1} = qu$ for some unit $u \in U(\mathbf{Z}[\zeta_q])$. If $2 \leq k \leq q-1$, then ζ_q^k is also a primitive qth root of unity so that $(\zeta_q^k - 1)^{q-1} = qw$, for some $w \in U(\mathbf{Z}[\zeta_q])$. As such, we have:

$$\frac{(\zeta_q^k - 1)}{(\zeta_q - 1)} = 1 + \zeta_q + \cdots + \zeta_q^{k-1} \in U(\mathbf{Z}[\zeta_q])$$

Further, as ζ_q has order q, and $|U(\mathbf{F}_q)| = q-1$, it follows that the canonical homomorphism $\mathbf{Z}[\zeta_q] \to \mathbf{F}_q$ sends $\zeta_q \mapsto 1$. In particular, $1 + \zeta_q + \cdots \zeta_q^{k-1} \mapsto k \in U(\mathbf{F}_q)$, i.e. $\mathbf{Z}[\zeta_q] \to \mathbf{F}_q$ induces a surjection on the units $U(\mathbf{Z}[\zeta_q]) \twoheadrightarrow U(\mathbf{F}_q)$. It follows that the induced map on units $U(\mathbf{Z}[\zeta_q][F_n]) \twoheadrightarrow U(\mathbf{F}_q[F_n])$ is surjective. ☐

Corollary 7.1.2. $\mathbf{Z}[C_q \times F_n]$ *has SFC.*

Proof. By Proposition 7.1.1, ♣ is pointlike in dimension one. As $\mathbf{F}_q[F_n]$ is weakly Euclidean, and $\mathbf{Z}[F_n]$, $(\mathbf{Z}[\zeta_q])[F_n]$ have SFC, the hypotheses of Proposition 6.4.9 are satisfied. The result follows. ☐

It is possible to somewhat generalize the result of this section to encompass a cyclic group of order m, where m is not necessarily prime. It has been shown by Bass and Murthy [3] that $\mathbf{Z}[C_m \times C_\infty]$ has SFC for $m \geq 2$, although a more direct proof can be found in [22]. However, if one tries to fully generalize this section, that is to consider $C_m \times F_n$, then the arguments breaks down. To this end, O'Shea [35] has shown that $\mathbf{Z}[C_m \times F_n]$ has infinitely many non-isomorphic stably free modules of rank 1 provided that $n \geq 2$ and $m \equiv 0 \ (mod\ p^2)$ for some prime p.

7.2 Top right corner of ♡ has SFC

Next we consider the top right corner, i.e. $\mathcal{T}_q(A,\, \pi)[F_n]$. Consider the following decomposition,

$$\begin{tikzcd}
\mathcal{T}_q(A,\, \pi) \arrow[r, "j"] \arrow[d, "\varphi"'] & M_q(A) \arrow[d, "\psi"'] \\
\mathcal{T}_q(A/\pi) \arrow[r, "i"] & M_q(A/\pi)
\end{tikzcd} \qquad (*)$$

where i and j are injections into the respective matrix rings.

Proposition 7.2.1. *The commutative square $*$ is a fibre square.*

Proof. Consider $X \in \mathcal{T}_q(A/\pi)$ and $Y \in M_q(A)$ such that $i(X) = \psi(Y)$. As φ is a surjection, there is an $\bar{X} \in \mathcal{T}_q(A,\, \pi)$ such that $\varphi(\bar{X}) = X$ and which we may inject into $M_q(A)$. By the commutativity of $*$ it is clear that $\psi(j(\bar{X})) = i(X)$, i.e. $(X,\, j(\bar{X})) \in \mathcal{T}_q(A/\pi) \times_{i,\psi} M_q(A)$. In particular, $j(\bar{X}) - Y \in Ker(\psi)$ and so there exists some $Z \in \mathcal{T}_q(A,\, \pi)$ such that $j(Z) = j(\bar{X}) - Y$, i.e. $Y = j(\bar{X} - Z)$. Moreover, $\varphi(\bar{X} - Z) = \varphi(\bar{X})$.

Furthermore, if we suppose there exist $\bar{X},\, \bar{X}' \in \mathcal{T}_q(A,\, \pi)$ such that $(\varphi \times j)(\bar{X}) = (\varphi \times j)(\bar{X}')$, then it follows that $j(\bar{X}) = j(\bar{X}')$. As j is an injection, it is necessary that $\bar{X} = \bar{X}'$ and $*$ is therefore a fibre square. ☐

Tensoring the above commutative square with $- \otimes_{\mathbf{Z}} \mathbf{Z}[F_n]$ now yields another fibre square:

$$\begin{tikzcd}
\mathcal{T}_q(A,\, \pi)[F_n] \arrow[r, "j"] \arrow[d, "\varphi"'] & M_q(A[F_n]) \arrow[d, "\psi"'] \\
\mathcal{T}_q((A/\pi)[F_n]) \arrow[r, "i"] & M_q((A/\pi)[F_n])
\end{tikzcd} \qquad (\diamondsuit)$$

Observe $A = \mathbf{Z}[\zeta_p]^\theta$ is the ring of algebraic integers in the fixed field $\mathbf{Q}(\zeta_p)^\theta$, and therefore a Dedekind domain. It follows that $A[F_n]$ has SFC by Proposition 6.6.4, and therefore so does $M_q(A[F_n])$ by Proposition 6.5.5.

Now, as π is the unique prime in A over p, and as A is a Dedekind domain, it follows that A/π is the finite field $\mathbf{F}_p$. Consequently, we observe that $(A/\pi)[F_n]$ is weakly Euclidean (by Proposition 6.7.5), and so too is $M_q((A/\pi)[F_n])$.

Proposition 7.2.2. *$\mathcal{T}_q((A/\pi)[F_n])$ has SFC.*

Proof. Consider the obvious surjection,

$$\psi : \mathcal{T}_q((A/\pi)[F_n]) \twoheadrightarrow \underbrace{(A/\pi)[F_n] \times \cdots \times (A/\pi)[F_n]}_{q}$$

in which $\psi(X) = (X_{11}, \ldots,\, X_{qq})$. In particular, we observe that each $(A/\pi)[F_n]$ has SFC and hence so does any finite product of these. To deduce that $\mathcal{T}_q((A/\pi)[F_n])$ has SFC we seek to apply Bourbaki-Nakayama (Proposition 6.5.4) and need show $Ker(\psi)$ is radical. If $X \in Ker(\psi)$, then X is strictly upper triangular, i.e.

$$X \sim \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}.$$

Evidently, for any $T_1,\, T_2 \in \mathcal{T}_q((A/\pi)[F_n])$, we have $(T_1 X T_2)^m = 0$ for some m. Thus, $I_q + T_1 X T_2$ is a unit and so $X \in rad(\mathcal{T}_q((A/\pi)[F_n]))$. Hence $Ker(\psi)$ is radical and we apply Bourbaki-Nakayama to show $\mathcal{T}_q((A/\pi)[F_n])$ has SFC, as claimed. ☐

Corollary 7.2.3. *$\mathcal{T}_q(A,\, \pi)[F_n]$ has SFC.*

Proof. By Propositions 7.2.1 and 7.2.2, as well as the discussion in between, it suffices to show $\diamondsuit$ is pointlike in dimension one. To this end we consider $X \in GL_q((A/\pi)[F_n])$.

As $(A/\pi)[F_n]$ is weakly Euclidean, we write $X = \Delta E$ for $E \in E_q((A/\pi)[F_n])$, and

$$\Delta = \begin{pmatrix} \delta & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

where $\delta \in U((A/\pi)[F_n])$. Clearly, $\Delta \in \mathcal{T}_q((A/\pi)[F_n])$ and since $\natural$ is surjective, it follows that there exists an $\widetilde{E} \in E_q(A[F_n])$ such that $\natural(\widetilde{E}) = E$. As such, $X = i(\Delta)\natural(\widetilde{E})$, i.e. $\diamondsuit$ is pointlike in dimension one. We therefore conclude by the Recognition Criterion (Proposition 6.4.9) that $\mathcal{T}_q(A,\, \pi)[F_n]$ has SFC. ☐

7.3 $\mathbf{Z}[G(p,\, q) \times F_n]$ has SFC

It follows from Sections 7.1 - 7.2 that $\heartsuit$ is of locally free type. As such, for all $k \geq 1$ we have $\mathcal{Z}_k(\heartsuit) \cong \mathcal{SF}_k(\mathbf{Z}[G(p,\, q) \times F_n])$. Our aim will now be to show $\mathcal{Z}_k(\heartsuit)$ is trivial for each $k \geq 1$. In the interest of a succinct notation, we relabel $\mathcal{A} = \mathbf{Z}[C_q]$, $\mathcal{B} = \mathbf{F}_p[C_q]$ and $\mathcal{T}_q = \mathcal{T}_q(A,\, \pi)$.

Start by observing $\mathbf{Z}[G(p,\, q)]$ is a retract of $\mathbf{Z}[G(p,\, q) \times F_n]$, i.e. there are ring homomorphisms $i : \mathbf{Z}[G(p,\, q)] \to \mathbf{Z}[G(p,\, q) \times F_n]$ and $r : \mathbf{Z}[G(p,\, q) \times F_n] \to \mathbf{Z}[G(p,\, q)]$ such that $r \circ i = 1$. We therefore have the following collection of mappings,

$$\begin{tikzcd}
\mathcal{LF}_1(\ddagger') \arrow[r, "\sigma_{1,\,1}"] \arrow[d, "i_1"'] & \mathcal{LF}_2(\ddagger') \arrow[r, "\sigma_{2,\,1}"] \arrow[d, "i_2"'] & \mathcal{LF}_3(\ddagger') \arrow[r, "\sigma_{3,\,1}"] \arrow[d, "i_3"'] & \mathcal{LF}_4(\ddagger') \arrow[r, "\sigma_{4,\,1}"] \arrow[d, "i_4"'] & \cdots \\
\mathcal{LF}_1(\heartsuit) \arrow[r, "\sigma_{1,\,1}"] \arrow[d, "r_1"'] & \mathcal{LF}_2(\heartsuit) \arrow[r, "\sigma_{2,\,1}"] \arrow[d, "r_2"'] & \mathcal{LF}_3(\heartsuit) \arrow[r, "\sigma_{3,\,1}"] \arrow[d, "r_3"'] & \mathcal{LF}_4(\heartsuit) \arrow[r, "\sigma_{4,\,1}"] \arrow[d, "r_4"'] & \cdots \\
\mathcal{LF}_1(\ddagger') \arrow[r, "\sigma_{1,\,1}"] & \mathcal{LF}_2(\ddagger') \arrow[r, "\sigma_{2,\,1}"] & \mathcal{LF}_3(\ddagger') \arrow[r, "\sigma_{3,\,1}"] & \mathcal{LF}_4(\ddagger') \arrow[r, "\sigma_{4,\,1}"] & \cdots
\end{tikzcd}$$

where $\ddagger'$ is the fibre square defined in Example 6.2.12, and i_k, r_k are the maps induced from i, r, respectively. Since $r \circ i = 1$, it follows that each r_k is surjective. Moreover, by Swan-Jacobinski $\mathcal{LF}_k(\ddagger') \equiv \mathcal{LF}_{k+1}(\ddagger')$ for each $k \geq 1$. In particular,

$$|\mathcal{LF}_1(\ddagger')| = |U(\mathbf{Z}[C_q])\backslash U(\mathbf{F}_p[C_q])/U(\mathcal{T}_q)| = N \tag{7.3.1}$$

for some finite N. We now compare this to the size of $\mathcal{LF}_k(\heartsuit)$ for which, we recall, there is a bijection $\nu_n : \overline{GL_k}(\heartsuit) \xrightarrow{\simeq} \mathcal{LF}_k(\heartsuit)$.

Proposition 7.3.2. *For each $k \geq 1$, $\mathcal{LF}_k(\heartsuit)$ is finite. In particular, $\mathcal{LF}_k(\heartsuit) \equiv \mathcal{LF}_k(\ddagger')$.*

Proof. From Corollary 7.0.3, $\mathcal{B}[F_n] = \mathbf{F}_p[C_q \times F_n]$ is weakly Euclidean, i.e.

$$GL_k(\mathcal{B}[F_n]) = GE_k(\mathcal{B}[F_n]) = U(\mathcal{B}[F_n])E_k(\mathcal{B}[F_n]). \tag{7.3.3}$$

Now, it is clear that $U(\mathcal{T}_q[F_n])E_k(\mathcal{T}_q[F_n]) \subset GL_k(\mathcal{T}_q[F_n])$ and so there is a natural surjection

$$GL_k(\mathcal{B}[F_n])/U(\mathcal{T}_q[F_n])E_k(\mathcal{T}_q[F_n]) \twoheadrightarrow GL_k(\mathcal{B}[F_n])/GL_k(\mathcal{T}_q[F_n]).$$

Next, the surjection $\natural : \mathcal{T}_q[F_n] \to \mathcal{B}[F_n]$ induces a surjection

$$\natural_* : E_k(\mathcal{T}_q[F_n]) \to E_k(\mathcal{B}[F_n]).$$

Combining this with (7.3.3) we therefore have the surjection

$$U(\mathcal{B}[F_n])/U(\mathcal{T}_q[F_n]) \twoheadrightarrow GL_k(\mathcal{B}[F_n])/GL_k(\mathcal{T}_q[F_n]). \tag{7.3.4}$$

Now, recall that $U(\mathcal{B})/U(\mathcal{T}_q)$ is finite. Furthermore, from Corollary 7.0.7 there is a surjection $U(\mathcal{B})/U(\mathcal{T}_q) \twoheadrightarrow U(\mathcal{B}[F_n])/U(\mathcal{T}_q[F_n])$. As such, we combine this with (7.3.4) to deduce

$$|GL_k(\mathcal{B}[F_n])/GL_k(\mathcal{T}_q[F_n])| \leq |U(\mathcal{B})/U(\mathcal{T}_q)|$$

and hence

$$|\overline{GL_k}(\heartsuit)| = |GL_k(\mathcal{A}[F_n])\backslash GL_k(\mathcal{B}[F_n])/GL_k(\mathcal{T}_q[F_n])| \leq |U(\mathcal{A})\backslash U(\mathcal{B})/U(\mathcal{T}_q)| = |\mathcal{LF}_1(\ddagger')|.$$

Finally, we know $r_k : \mathcal{LF}_k(\heartsuit) \to \mathcal{LF}_k(\ddagger')$ is surjective. However,

$$|\mathcal{LF}_k(\heartsuit)| = |\overline{GL_k}(\heartsuit)| \leq |\mathcal{LF}_1(\ddagger')| = |\mathcal{LF}_k(\ddagger')| = N$$

from which it follows that r_k is an isomorphism, as required. □

Corollary 7.3.5. *For each* $k \geq 1$*,* $\mathcal{LF}_k(\heartsuit) \equiv \mathcal{LF}_{k+1}(\heartsuit)$*.*

Theorem K: For p, q prime numbers such that $q|p-1$, the integral group ring $\mathbf{Z}[(C_p \rtimes C_q) \times F_n]$ has no non-trivial stably free modules; that is, it has SFC.

Proof. By Corollary 7.3.5 we have $\mathcal{LF}_k(\heartsuit) \equiv \mathcal{LF}_{k+1}(\heartsuit)$ for each $k \geq 1$. By (6.4.3) $s_{n,k}$ is therefore bijective for each $k \geq 1$. By Corollary 6.4.8, $\mathcal{Z}_k(\heartsuit) = \{*\}$ for each $k \geq 1$. □

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