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FOR A LEARNABLE MATHEMATICS IN THE DIGITAL CULTURE[1]

ABSTRACT. I begin with some general remarks concerning the co-evolution of representational forms and mathematical meanings. I then discuss the changed roles of mathematics and novel representations that emerge from the ubiquity of computational models, and briefly consider the implications for learning mathematics. I contend that a central component of knowledge required in modern societies involves the development of a meta-epistemological stance – i.e. developing a sense of mechanism for the models that underpin social and professional discourses. I illustrate this point in relation to recent research in which I am investigating the mathematical epistemology of engineering practice. Finally, I map out one implication for the design of future mathematical learning environments with reference to some data from the "Playground Project".

KEY WORDS: Representations, mathematical epistemology, culture, computers, meta-epistemological stance, situated abstraction

INTRODUCTION

I would like to begin by thanking my hosts for inviting me to present this opening lecture. I have for some time looked at CIEAEM from afar, as a place where both cognitive and social questions of mathematics education are elaborated, and I am glad, finally, to have first-hand experience of its particular attractions: as an aside, I believe that synthesising cognitive, epistemological and socio-cultural analyses is a pressing need of current research in our field. This belief will form a backdrop for much of what I have to say, but I will not be able to do justice to the assertion here.

It is a particular pleasure to be present at a meeting of CIEAEM, as fifty or so years ago, one of its founding members, Caleb Gattegno, was a professor of mathematics and teacher training at my own institution. I was recently interested to learn that one of his colleagues remembers that he was "always experimenting with the latest gadgetry from coloured blocks to electrical devices". (Dixon, 1986, p.73).

Apart from expressing the affinity that I feel with Gattegno's obsession – I confess a similar affliction – I believe there is a deeper point here, that concerns educational matters in general, and mathematics education in particular. What would Gattegno have made of the modern computer, which can deliver in a 2 kg laptop more computational power than a roomful of machinery of his time, while connected wirelessly to 3 billion web pages? It goes without saying that he would probably have fallen in love with this new gadgetry, at least as much as I have, and that it would not be too difficult for him to see how it might be exploited in the service of mathematics education. But there are two issues which I think would have been interesting to explore with him and which, in his absence, I will explore with you.

The first concerns an essential difference between the computer and all other gadgets with which Gattegno might have played. I do not want to argue at all that "the computer" (an empty phrase) has this or that inescapable impact on learning. Far from it. But I do want to emphasise that it can, and that its potential for such transformative change stems in no small part from its ability to be reconstructed by teachers and learners themselves. This is what Seymour Papert referred to as the "protean" quality of the computer: like Proteus, it can be changed (even change itself) into any number of forms.

The second issue concerns the difficulty of exploiting this protean quality. Gattegno's (Cuisenaire) rods are easy to translate into a computer version – I would be fairly sure that a rudimentary search on the web would reveal more than one version. But what would such a version add? From a didactical point of view, this is an important question. A deeper question concerns the ways in which the knowledge which is modelled by the rods may be transformed by the change of medium from wood to pixel? What, in any case, does it mean to talk of knowledge "modelled" by a technology: Are we justified in assuming that the knowledge survives intact across technological transitions? Such epistemological questions return us to didactical ones: how might the novel epistemologies, and the manipulable and dynamic representational form of a model lead to transformative learning from both a cognitive and social perspective?

These questions take us to a range of issues that cannot be explored in any depth within this lecture, but they emphasise the complexity of understanding our societies' cultural encounter with computational power. They can, perhaps, be seen as generalisations of the recognition of representational form as a key component of mathematical expression, and as a contribution to our collective quest to understand better how students make sense of mathematical ideas. Kaput & Shaffer (in press) have recently drawn our attention to the socio-genetic basis for representational systems for mathematics, and illustrated how representations and notational systems have evolved over time and shaped our ways of reading and writing the world – including the mathematical world. They have also illustrated how mathematical notations developed over millennia, how they consistently served only an intellectual élite, and how their evolution with static media institutionalised the relationship between knowledge and its privileged representations. (See also Kaput, Noss & Hoyles, in press).

I would like to offer a first illustration of this point by reference to a non-mathematical example. Consider one of the most famous paintings in the world: Vermeer's *View of Delft* (a monochrome version is illustrated in Figure 1). It is, as everyone knows, a view of the city – "just" a view. I say "just" because it represents, as faithfully as possible, what the city actually looked like. Of course it would be more accurate to say "what Vermeer actually saw", but this is not the place for a philosophical debate on the mediation of the observer in the description of reality. It is just a view, and a faithful one at that.

Figure 1: Vermeer's View of Delft

What do we see? Water, clouds, shadows, buildings (some still standing today), people walking, perhaps just standing. It seems almost an impertinent question: we see what we see. But the question seems facile because we are looking back in time. In his book, "The World on Paper", David Olson (1994) shows how Dutch art of the seventeenth century came to challenge the hitherto hegemonic role of text as a central way of understanding the world and demonstrates how, by exchanging narrative depth for surface description, Vermeer's school changed the meanings associated with the medium of paint. As one of Vermeer's contemporaries commented, he came to "see clouds as clouds and not as symbols of the heavens!".

Neither has text always been a medium for representing the world: It was partly the advent of printing that had allowed text to be seen as representation, the novel technology providing a common representational format for its production and interpretation. This technical and social change gave voice to the descriptive power of text, which in turn loosened its dependence on authorship; meaning gained autonomy from the author. Now Vermeer came to challenge textual description as a way of representing how things are, a change which startled his contemporaries and into whose interpretation his contemporaries had to be acculturated[2]. How hard it is for us, looking at this beautiful painting, to understand what a significant change it represented; it *is* difficult to appreciate the extent to which we have learned what appears to us to be natural. It is, moreover, a salutary reminder that representational systems play a crucial role in making sense and that they enable individuals and communities to communicate and construct meanings in ways which would be (literally) unthinkable without them.

The ways that meanings and representations become intertwined, and more fundamentally, how transparent the representational medium becomes over time, is illustrated by the following example (see Tufte, 1983 p. 28).

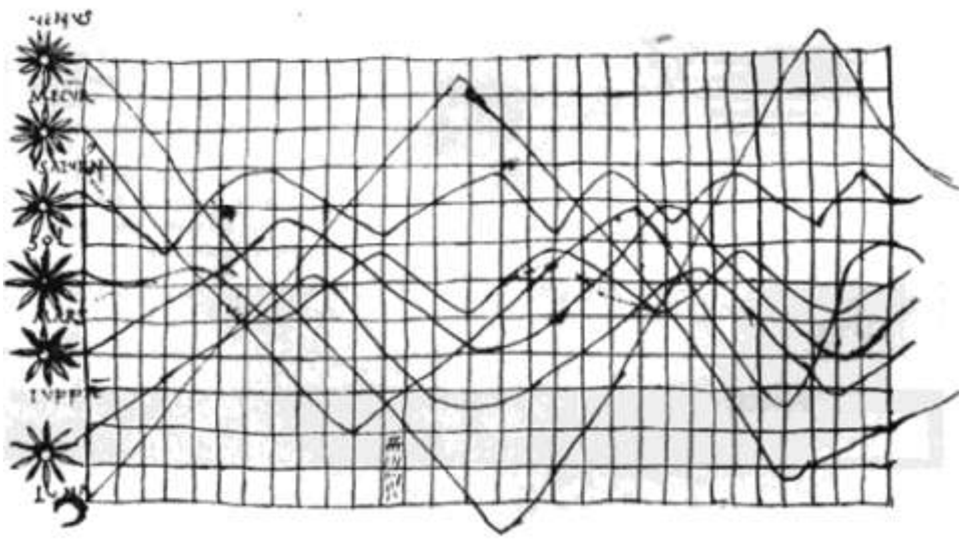

Figure 2: A tenth century time series. Tufte, 1983: p. 28

Figure 2 shows a tenth century time series graph showing the inclination of planetary orbits. According to Tufte, this is the first example of a time series graph which still exists, and it is one thousand years old. It is not uninteresting that it exists in an educational text for monastery schools, suggesting perhaps that the explicit desire to communicate is an important factor shaping the ecology of representations. But what is fascinating is that the next extant time series was 800 years later. Not only did it take nearly a millennium for this representational form to become accepted, but also, once accepted, it became *the* representational form of choice for communicating certain kinds of information: to the extent that between 1974 and 1980, 75% of all published graphics are – according to Tufte – time series graphs.

I have no way of checking whether Tufte is correct in the details, but we can assume that he is right in essentials. The key implication for mathematical meaning is that once a representational form has entered our culture, we are hard-pressed to consider it as other than the essential mathematical notion itself. (A good test for assessing the invisibility of a representation is to make an unusual but apparently trivial change: for example, re-express the equation $y= mx + c$ so that m and c are the variables, and x and y are the constants. We will see below, that "arbitrary" changes of this kind are not so trivial when computational technologies are involved).

As a final comment on the power of representation, I want now to prefigure an issue which will constitute one of the key themes of this lecture; namely, the ways in which mathematical knowledge enters into professional discourses. Here is a quote from a practicing engineer … (I will describe this work in more detail below, together with an appropriate attribution).

"…engineering knowledge encapsulates a lot of past history, getting to the point where you generalise all behaviour of all beams in one equation that you could almost teach to a ten year old, is an astonishing condensation of millions of man *[sic]* years of effort. That 10 year old today can do things that Newton couldn't do".

This is an astute observation, and one which has been made by others (see, for example, diSessa's discussion of the ways in which Galileo's elaboration of the physics of motion is transformed by algebra, which was unavailable to Galileo, almost to the point of triviality (diSessa, 2000). It applies equally at a cultural and social level, as much as at an individual one. The engineer who made this statement played a key role in the building of the new roof in the British Museum in London (see Figure 3) a structure with more than 3000 tiles, no two of which are the same, consisting of some 3 kilometres of steel beams claimed accurate to within 3mm. The design – let alone the construction – of such an edifice would, of course, have been impossible without a computer. But the key point is not only that the computer enabled a particular design: it is that the very design was shaped by the computer, the computer gave its designers a new way to think, not just a tool for calculating the details of what they had already thought. The novelty of the computer's contribution lies at the intersection of cognitive and social, assisting individual engineers and architects to turn their imagined structure into reality, while simultaneously affording the means for the broader community to imagine what had hitherto been unimaginable.

This structure is an example of the way in which computational representations are reshaping cultures, and mathematical epistemologies – not simply changing the ways things are calculated. This representational transformation is at the heart of modern "post-industrial" societies but it is, in my opinion, insufficiently theorized in our field. I would like now to turn to the questions it raises.

Figure 3: The roof of the Great Court at the British Museum, London.

MATHEMATICS IN THE KNOWLEDGE ECONOMIES

Modern societies in the developed world – the so-called knowledge economies – are mathematised to an unprecedented degree. The systems which control our social and professional lives are essentially mathematical, algorithms are ubiquitous. In such a world, the coordination of personal and mathematical models is increasingly necessary: that is, in order for the individual to make sense of her community and her world, it is increasingly necessary for her to think about the relations between the elements of the models that underpin it, and these are mathematical relations. That does not mean, of course, that each individual has to be a mathematician: but it does mean that without some way of accessing the mathematical bases of the models which drive the systems, at best a partial and at worst a misleading view of those systems, and their impact on the lives of individuals and communities, can be formed. Elsewhere (see Noss, 1997) I have discussed the economic and social implications of this situation, but I do not want to dwell on them here.

I do, instead, want to think about the implications for learning mathematics. Consider just one simple example, which I have schematically laid out in Figure 4. If the time series graph is the most favoured graphic of our time, then surely the table of numerical data must rank a close second. The generation of such tables with a spreadsheet is so commonplace that spreadsheet proficiency may have some justifiable claim to having become part of a new component of literacy, a medium of expression of relationships between – not just numbers – but between whole columns of numbers.

(Insert Figure 4 about here)

Figure 4: Some representational forms for the function f.

It is not unreasonable, in this situation, to suggest that the new and shared cultural form may well change the balance of what it is 'natural' to express and how. For example, the traditionally favoured representation of the function (one of the functions) which is 'represented' by the table in the form f(x) = 3x + 4 is different from the recursive definition with which the spreadsheet was constructed

$$f(0) = 4$$

$$f(N) = f(N-1) + 3$$

This recursive definition is usually seen as more complex in pedagogical terms, and in the UK at least, does not figure in the compulsory curriculum at all. This is despite the undoubted fact that many, perhaps most, learners of mathematics tend to see – at least until they are encouraged to do otherwise – the relationships between the numbers in a vertical manner (add 3), rather than the familiar closed form representation (see Cuoco, 1995, for the implications of employing computational media to support the thinking of functions).

The graphical representations which can be constructed at the touch of a button include not only the traditional one (shown bottom left in Figure 4) but some bizarre ones – no less accessible to the unsuspecting spreadsheet user: the bottom right one is thought-provoking but, as far as I can see, somewhat useless! Whatever else graphical representations such as this illustrate, they suggest that the ability to critique novel representations is at least as important as the interpretation of conventional ones.

(Insert figures 5a, 5b and 5c about here)

While I am about it, it is worth representing the graph of the equation y=3x+4 in the graphing calculator program which comes free with my computer (Avitzur *et al.* 2000). Or z=3x+4: Note here that this time, an "arbitrary" change of the name of the variable y has much more than arbitrary consequences! Or z=3xy+4. This last is not, of course the same function, and neither is it two-dimensional (see Figure 5). In a dynamic medium, watching the surface spin around and exploring what kind of a thing it is, one is encouraged to think the unthinkable: are three dimensions necessarily harder to understand than two? Should we just continue to take for granted that an orderly procession from two dimensions to (for some) three is somehow "natural"? What is natural anyway, now that it is more natural to visualise on a computer screen than any other way? Surely there is a case for believing that we have at least as rich a set of intuitions derived from running our hands over surfaces, than those derived from the thought experiment of travelling along a one-dimensional line (set of points) representing a function[3]? There is surprisingly little research that challenges existing epistemological and didactical assumptions in this way (for one exception, see Papert, 1996). Nevertheless, there are at least some clear directions in which research might proceed, and I will discuss them in the concluding section.

The changes in representational forms which computational technologies make available challenge us to rethink the kinds of didactical sequences and hierarchies of knowledge that are appropriate in learning environments. Similarly, on a social level, there are equal challenges to confront. I will start by explaining what I mean when I say that the individual needs to understand something of the models which underpin social and professional practices.

Let me take a recent example. One of the striking aspects of the discussion which surrounded the US presidential election, was the extent to which media reports – at least in the UK – based their assessment of who "really" won on statistical data. The news media were awash with information concerning the numbers of votes, the percentages of the votes in each county, information regarding poll data, exit polls, and so on.

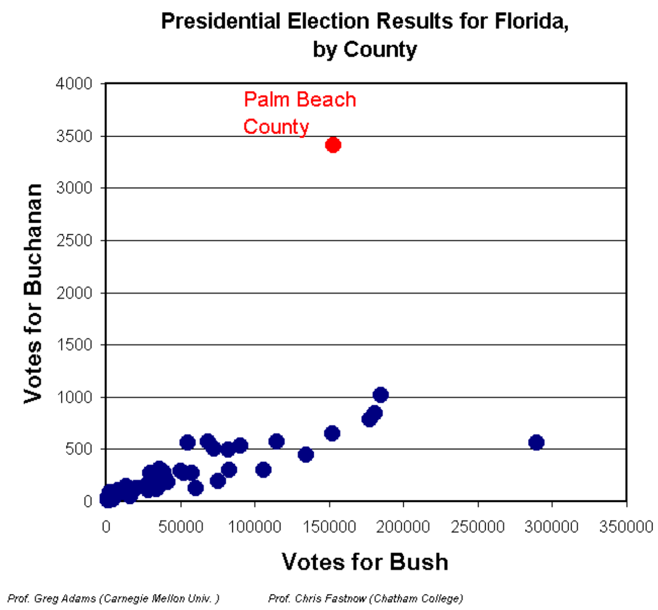

Figure 6: Presidential Results for Florida by County: Bush versus Buchanan. (Adams, G. and Fastnow, C).

Among all this, I found some interesting examples of something more than just information: Figure 6 illustrates one of a number of plots I found on the web, produced by two statisticians (one from Carnegie Mellon and one from Chatham College). The example shown is a plot of Bush votes versus Buchanan votes, by county in the state of Florida. As you can see, with the exception of one outlier – a rather important one given that it corresponds to the vote in the disputed territory of Palm Beach – there is a high correlation between Bush and Buchanan votes. The model is based on the following hypothesis: the more conservative the county (and therefore the more Bush was likely to score compared to Gore) the more likely that the Buchanan vote – though much smaller – would be correspondingly higher, given that Buchanan was an extreme conservative. And so, as Figure 6 shows, it turns out. This graph, however, does not only convey information. Inscribed within the conventional and omnipresent scatter graph representational form, it provides us with knowledge (not, alas, certainty): about the unlikeliness of the result in Palm Beach, about the uniqueness of that result, and about the truth of the underlying model concerning Bush and Buchanan votes.

This example takes me to a second key theme of my lecture which exists at the intersection of the social and individual realms: that the ability to understand this kind of transformation of information into knowledge – of critiquing representations and developing a feel for models – will be a crucial facet of what it means to be a functioning person in the twenty-first century. The difficulty is that the election example is one in which, like so many others, the mathematics is not immediately evident; mathematical work has to be done in order to throw light on the variables and relationships involved. Of course, the US election was about much more than just a mathematical analysis, but as we have seen, mathematical modelling certainly helps to make sense of it. In fact, the invisibility of mathematics has been a developing characteristic of the later part of the old century: as mathematics plays a more and more significant role in running systems of all kinds, it becomes simultaneously less and less visible. A recent report from the Society for Industrial and Applied Mathematics, in describing the role that mathematics plays in work, puts it thus:

> Mathematics is alive and well, but living under different names…. Mathematics is often invisible outside the technical work group because its role in a successful project is not highlighted or publicised, especially to higher management…
>
> Mathematics in Industry (1998)

The workplace serves as a suitable exemplar for more general concerns about the roles that mathematics plays in individuals' and communities' lives, and how it seems to disappear into activities while becoming increasingly ubiquitous below the surface. It is therefore to the workplace that I now turn.

INTO THE WORKPLACE

In a series of studies beginning in the mid nineteen-nineties, Celia Hoyles, Stefano Pozzi and myself have been studying the mathematical components of professional expertise, in an attempt to throw light on a number of topics related to the ways in which mathematical meanings are constructed, how we might characterise the nature of mathematical knowledge, and how we may find some mechanism to replace "transfer" as a primary metaphor for the ways that mathematical knowledge is used in general. Specifically, we have asked how the discourse of work is shaped by mathematics, and reciprocally, how mathematical knowledge is shaped and applied by the discourse of work. These studies are reported in Noss and Hoyles (1996a); Pozzi, Noss and Hoyles (1998); Noss, Pozzi and Hoyles, (1999); Noss, Hoyles and Pozzi (2000); Hoyles, Noss and Pozzi (2001); Noss, Hoyles and Pozzi (in press).

Our studies have involved investment bank employees, nurses and commercial pilots. Most recently, Phillip Kent and myself have undertaken a study with structural engineers, and it is this study from which I tore out of context the comment from the British Museum engineer concerning Newton. I am going to focus on this most recent study, but before I do so, I want to summarise what we found in our earlier studies, particularly with the nurses.

First, there is a disjunction between visible mathematics and what happens in practice. That is, while routine practice seems at first sight to involve little more than the "visible" mathematics of school, largely consisting of simple arithmetic, in practice the strategies employed owe little to taught procedures. In fact, the numerical strategies employed – for example in drug administration – are intimately tied into the artifacts of the practice, the particular drug, its package labelling, or the kind of units in which it is administered. In our nursing study, for example, the nurses had a strong sense of the relationships involved in drug administration, but they almost never used school taught algorithms, and they were sometimes surprised that the correct knowledge they mobilised could be expressed in general terms.

The second striking finding was that numerical routines were overlaid by implicit models, in which the professionals had to elaborate – in a more or less articulated way – which quantities determined the behaviour of the system (e.g., how drug concentration varies over time in the body) and the qualitative and quantitative relationships between them. This became evident in "breakdown" incidents we witnessed, situations in which the normal routines of the practice were somehow ruptured by a non-standard set of circumstances. In such an event, the elements of the situation – including the mathematical elements – were laid bare by the participants, and the artefacts and discourse of the setting became a territory on which we, as observers, were able to get a clearer view.

Finally, we found that where numerical manipulation is involved, numbers are a part of quantitative relationships but do not enter as "pure"'entities. That is, while it seems at first sight that numerical relationships enter professional practice as numbers and operations *per se*, the reality is that these are often seen as one property of an artefact, along with others which are at least as salient for the professional involved.

To summarise, the findings of these earlier studies left us trying not only to elaborate how mathematics gets used in practice, but also to characterise just what kinds of mathematical knowledge professionals have. In order to look more closely at this question, we turned to structural engineering, a profession in which mathematics appears to play a much greater role, and in which those involved have a relatively high level of mathematical preparation.

Our studies have taken us into a large London-based firm of engineers, and into a variety of specialisms. We have interviewed about 20 engineers, followed email trails, attended some project meetings, and reviewed a considerable number of documents, including mathematical textbooks for engineers. Mathematical knowledge is distributed in the firm – there are, for example, a few "analysts" whose specialism is to employ mathematical analysis (sometimes instantiated in computer programs), computer programmers as well as engineers whose engagement with mathematics is – according to them – only superficial. We will meet one or two of these in a moment. Before we do, I should state that a particularly fruitful line of enquiry that Kent and I are pursuing, is to look more closely at the interfaces between these different sub-communities of engineers, and to try to understand what kind of epistemological and cognitive transformations of mathematical objects and relationships occur across them. This will be the subject of further study, and I will not pay much attention to it here.

Instead, as my focus here is on the social rather than the individual, I will look at a broader aspect of this distributed knowledge, not as it applies to an individual engineer in a particular moment, but how it applies in a particular setting, in terms of the broad community of engineers. I will start by introducing you to an engineer, who told us:

"We only <u>use</u> 5% of the mathematics that was in our college courses"

and another who asserted:

"Once you've left university you don't use that maths, 'squared' or 'cubed' is the most complex thing you do. For the vast majority of the engineers here, an awful lot of the mathematics they were taught, I won't say learnt, doesn't surface again. There are a few specialists, less than 2% of the engineers in this company, who spend their lives doing the mathematics which we struggled through at university".

We viewed these assertions with a critical eye, as our earlier work with other professionals, together with a very wide range of other studies by workers in the field, reports that most professionals in most occupations flatly deny that mathematics plays any part in their activities[4]. This is, we believe, related to the transitions across interfaces between and within practices; as the mathematical knowledge of individuals and communities travels across these boundaries, the balance between what is routine and novel, what is pragmatic and theoretical changes and shifts, tending to fade into transparency parts of mathematical practice. This point is related to Artigue's (in press) distinction between the pragmatic and epistemological values of *techniques*.

In our earlier studies, we characterised breakdowns as local, at least in a temporal sense; that is, they tended to occur as a short-term rupture in the routine of the practice, a "hiccup" that was usually remedied (or worked around) in a short time, involving either the person to whom the breakdown had originally happened, or her immediate colleagues. In engineering as I have explained, the system is more distributed – in the sense that there is a wide division of mathematical labour[5]– and we will take the opportunity to look more closely at a breakdown within the broader system.

Figure 7: The Millennium Bridge wobbled alarmingly when it was opened, due to the large number of pedestrians crossing it.

Figure 7 depicts the Millennium Bridge in the centre of London, a beautiful new footbridge across the river Thames that was due to open in the Millennium year. Unfortunately, on opening day, a huge number of pedestrians began to walk across the bridge, and it started to sway alarmingly; two days later it was closed. As part of our project, we have interviewed some of the engineers responsible for its design, and for applying the remedy to its wobble. But, thanks to the openness of the engineering company involved, much of the information is available on their website (see www.arup.com/MillenniumBridge/).

Since the bridge began its wobble, there has arisen a mythology concerning what went wrong and why. Most pervasively, the design engineers involved stand accused of being "unable" to calculate how the bridge would behave. At its most ill-informed, this takes on a striking implication for mathematical education, as in this extract from the Times Educational Supplement, in a recent review of a Royal Society report on geometry by the Joint Mathematical Council, UK:

> … many teenagers are struggling to understand shapes, even after gaining A-levels, because of a lack of specialist maths teachers. As a consequence, students taking up science and engineering courses at university are floundering, we are told. This is serious stuff. A thorough grounding in geometry is essential if Britain is to maintain its cutting-edge in areas such as genetics, drug design and architecture. We may have led the world with Dolly the sheep and the human genome project, but the fiasco over the wobbly millennium bridge over the Thames suggests we should not be complacent.
>
> *Times Educational Supplement: 10/08/01*

I do not for one moment want to offer consolation for the disappearance of geometry in the curriculum in the UK. On the contrary, my sympathies are entirely with the writers of the report, and the sentiments expressed in the TES article: the lack of geometry in the UK curriculum is perhaps one of the most lamentable aspects of its current sorry state[6] But this was emphatically not the problem with the Millennium Bridge.

In fact, the problem lay not in the mathematical calculations, but in the modelling of the bridge's behaviour. Engineering is dominated by codes of practice, precise codifications of engineering design knowledge, which specify in detail all the main elements of a structure, the tolerances involved, rules for calculation and so on. They typically contain formulae, into which the engineer is expected to insert the relevant values, based on the specifics of the project. Bridge design, not surprisingly, is the subject of similarly detailed codes, and footbridges form a well-documented subset of these. In all these, and here lies the source of the difficulty, not a single one tackles the question of lateral vibration. All footbridge design is predicated on the following assumption: that the design of a footbridge must take careful account of vertical vibrations but that horizontal or lateral vibrations will tend to cancel each other out and can safely be ignored.

As it happened, this assumption proved false in the case of the new bridge, on which many thousands tried to cross at once. The small lateral forces exerted by people as they walked set up a corresponding lateral vibration in the bridge. And once that vibration started, instead of cancelling each other out, people started to compensate for their unsteadiness by applying countervailing lateral forces in synchrony with the bridge. This in turn made the vibration of the bridge more pronounced. The difficulty was not that the mathematicians had failed to calculate: it was that the building codes on which the design was based failed to take any account of what turned out to be the crucial variable[7]. As a senior engineer working on the bridge remarked, "Conventional wisdom is so strong that walking input is vertical and 2 hertz that it blinds you to the fact that there's also a 1-hertz horizontal force." And the result was that none of the extensive computational modelling which preceded the building of the bridge took any account of lateral vibrations.

I would like to draw attention to one particular aspect of this story. The mathematical labour within the community of engineers is, like that within the single company we studied, highly distributed; there is mathematical knowledge locked within the codes and there is mathematical knowledge encapsulated within the computational models. There is, too, mathematical knowledge employed by the engineers who put the bridge together, in the sense that they had the task of making sense of the codes, applying them in the special case of the bridge, and interpreting what they meant. Each of these mathematical practices has its own epistemology, and each differs substantially, and in different ways, from mathematical knowledge as it is taught in university engineering or mathematics departments. Thus the division of mathematical labour within the community, and the transformation of mathematical knowledge across the boundaries of these divisions, obscures the models that underpin the practice, no less than the examples I spoke of at the outset. Models are ubiquitous, and they are universally obscured.

I want to conclude this part of my talk by stressing what is and what is not important about this example in relation to learning and teaching mathematics. What is less important is just how many individuals need constantly to access the precise details of the models that underpin social and professional existence. There are not many, certainly not a majority, although I am convinced there are more than is evident at first sight. On the other hand, models are genuinely pervasive; everybody – I mean everybody – needs to know what a model *is* even if they cannot build one; all individuals are required to interpret the idea of a model even if they cannot calculate the implications of a particular example; anybody who uses a spreadsheet needs to have some feel for how the numbers get there, why the macros work, and what activates the buttons. (This last example is motivated by an episode regarding a bank employee's cavalier use of a spreadsheet, reported in Noss and Hoyles, 1996a).

What kind of knowledge is this? The example of the Millennium Bridge, though only illustrative, provides a clue. It is knowing *that* things work in programmed ways rather than (necessarily) *how.* It is knowing that there are assumptions instantiated in the choice of variables, and that there are relationships between them. It is representational knowledge about connections between variables, rather than calculation knowledge about their detailed interrelationships. It concerns the interpretation of models which others have built, and sharing and critiquing of them, together with the different representational forms in which they may be expressed. And finally, it involves knowing something about how knowledge is communicated to others who interact with other parts of the same system, or other, linked systems. I will call this knowledge about knowledge, a meta-epistemological stance.

There is much more that might be said about the cognitive and epistemological aspects of these kinds of situated understanding, but I will resist dealing with them here for reasons of time and space (see Noss, in preparation, for a discussion of these and related questions). Instead, I want now to consider some of the implications of all this for the design of learning systems.

DEVELOPING A META-EPISTEMOLOGICAL STANCE: BUILDING A SENSE OF MECHANISM

A key implication for the development of a meta-epistemological stance, is the need to design mathematical learning environments that make mechanisms manipulable and visible. I will illustrate this approach by describing something of a project based in London which has engaged us for nearly three years. In the *Playground Project*, Celia Hoyles and myself are directing a project that involves a group of researchers based in several European countries to develop a system with which young children, aged less than 8 years old, can play, share, construct and rebuild computer games[8]

Our goal is to put children in the role of game designers and game producers, rather than merely consumers of games produced and designed by adults.

We have chosen the domain of computer games for two main reasons: The first is cultural, the second mathematical. From a cultural point of view, videogames and their associated cultural artefacts (such as animated film, and interactive video) are the most pervasive feature of children's culture in the late twentieth and early twenty-first century. Like them or loathe them, they speak to millions of children and represent a vehicle for huge waves of popular culture (at the time of writing, the Pokémon craze has come and gone; another is surely on its way).

Tapping into children's culture is a necessary (but certainly not sufficient) element of trying to develop the kind of meta-epistemological stance we require. The challenge is considerable, for one effect of the digital revolution is precisely that few things invite inspection: one cannot know how a digital watch works by opening it, what makes the washing machine start and stop, how a speedometer works – all these mechanisms, which once may have offered at least *some* children a chance to investigate how things work, are closed; no user serviceable parts, no learnable mechanisms.

As I said, cultural resonance is not sufficient. The second reason for our choice of video games is that they represent an arena for exploration of interesting mathematical and scientific phenomena. Videogames represent a closed formal system of rules. When this touches that, make this happen. If the speed is greater than x, set y to something. Whenever the joystick button is pressed, make this object change colour. Games in general represent most children's first brush with what it means to operate within a formal system (if you pass go, collect 200 pounds); videogames are a mathematical instantiation of a formal system. We aim to go one step further, in affording children an opportunity to explore the world of formal systems, build and rebuild them for themselves, in accessing and bringing to life the mathematical instantiation hidden beneath a typical videogame.

In formally expressing what she wants to happen, a child can – we hope – become engaged in expressing mathematically interesting phenomena. Until now, this expression has meant interacting with strings of text in the form of computer programming languages. Without doubt, this has provided a considerable opportunity for exploration of and with quasi-algebraic systems; children's engagement with Logo has been well-documented and there is a substantial corpus of encouraging work in this respect (for one view of this, see, for example, diSessa, Hoyles and Noss, 1995; Noss and Hoyles, 1996b).

The problem is that many of the children with whom we were working could not read or write with any sophistication. So we based our design and implementation of a system on a new programming system, called *ToonTalk*, a Turing equivalent programming language whose source code is animated[9]. What this means is that the language is a real language – not just a simplified toy with strict upper limits of what can be expressed. And it means that the source code actually *is* what you see on the screen: animated robots who are "trained" to do tasks (programmed); arithmetic operations are performed by a mouse (an animated cartoon mouse, not the thing we hold in our hands!); even cutting and pasting is done by cartoon characters such as an animated vacuum cleaner.

It is extremely difficult to give a flavour of *ToonTalk:* one needs to see the animated system in order to gain a sense of its power and its affordances. Actually, this prefigures a subtle difficulty with the system, in that "reading" a program actually involves "observing" a robot carrying out its tasks. The insertion of a temporal dimension into the process of reading and editing programs is a fundamental re-representation of what a program actually is, what kind of knowledge a program represents – as well as a substantial difference in what it means to write a program. There is no guarantee that new representational forms are better than old ones; only that we need to consider them.

Two further difficulties emerged as soon as we started working with the *ToonTalk* system. First, like all its predecessors, including Logo, it emerged that while it was very straightforward to build simple programs, the construction of more interesting and complex ones was considerably more difficult. Our solution was to build a new layer of "behaviours", transparent and functional game elements whose construction and effect was more or less self evident, and which could be combined to build more complex functionalities. These program elements or behaviours functioned like components, except that they were completely open for inspection and reconstruction by the child.

Second, we found that *ToonTalk* was not tuned for the task we had in mind. It was a relatively new system that had not been built with the demanding role of video-game construction in mind. This presented us with both a challenge and an opportunity. A challenge, because so many functionalities we required just had to be built from scratch. An opportunity, because it gave us the chance to help to build a programming system that offered what we needed – a learnable new set of representations for programming.

I cannot go into details here: for some background and results of the project, see Noss, Hoyles, Gurtner, Adamson and Lowe (in press); Hoyles, Noss, and Adamson (in press); Kaput, Hoyles, and Noss, (in press). Instead, I will try to illustrate our approach, and how it might answer at least some of the requirements for developing a meta-epistemological stance. In so doing, I will outline the system in a little more detail, while recognising that it is very difficult to appreciate in a static medium (streamed video will shortly be available at www.ioe.ac.uk/playground).

An illustrative episode.

Mitchell is an eight year-old boy in an inner-city school. He has been helping the researchers to design and debug the Playground system for about a year. He has participated in several experimental sessions, and is a member of his school's computer club, in which all Playground sessions take place. He is playing a game where he controls a character called 'dusty', the creature with two legs on the right side of Figure 8. Dusty shoots out flowers every time the force joystick trigger is pressed (it jumps in the hand as the button is depressed), and the player collects points by hitting an animated target moving vertically up the left-hand edge of the screen. Whenever it is hit by a flower, the target changes shape into another animated character. Mitchell finds it very easy to play and achieve high scores since he can move his character as close to the target as he pleases. He is having fun, but it is hard to know what he might be learning — tempting even to dismiss this as just another arcade game.

Figure 8:. A screenshot of the game. Dusty is shooting flowers from right to left. A 'target' is moving up the left-hand edge of the screen. The target changes shape as it is hit by a flower.

Mitchell decides that the game would be more fun if it was competitive. So, at his request, the researcher adds another player character, this one controlled by the mouse. Although she talks through what she is doing, the programming is done by her, with Mitchell watching as she copies the original Dusty and replaces its joystick behaviours with mouse behaviours. Making these modifications is facilitated by the modularity of the behaviours or components – we call them "animagadgets'", as they are represented by animated pictures – and is essentially achieved by replacing one component, "I move with joystick" with another "I move with mouse". Any object has its program on its back – there is a direct connection between a thing and what the thing does, accessible merely by "flipping" it over. Although Mitchell can read these descriptions, he can, if he wishes, listen to the behaviours *say* what they do (a choice of voices is available) and there is, in addition, an animated representation of what the behaviour does (visible in static form in the top right-hand corner of Figure 9).

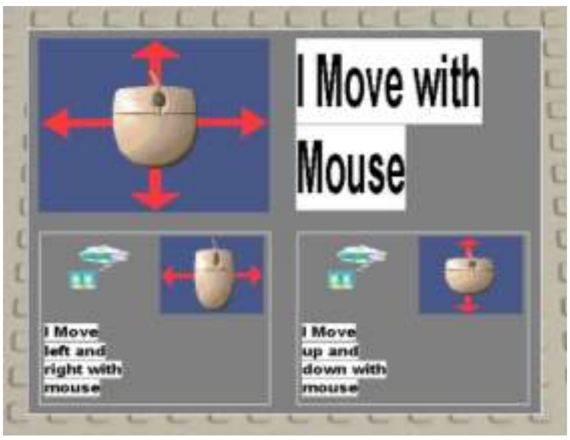

Figure 9: The ‘I move with mouse’ gadget, and the two sub-gadgets moving the mouse horizontally and vertically.

One last aspect of the "I move with mouse" behaviour, is that it actually consists of two sub-gadgets, "I move left and right with mouse", and "I move up and down with mouse" (the same applied, of course, to the joystick gadgets). Both of these sub-gadgets are visible at the bottom of Figure 9. As it turned out, the way that we had designed the gadget had unexpected consequences.

Mitchell plays the two-player game with the researcher. Suddenly, he tells everyone in the room to close their eyes – "no peeping!". He deftly removes the "I move left and right with mouse" component of the "I move with mouse" behaviour from the back of his opponent’s character – effectively disabling her mouse by restricting it only to vertical movements, while he has two-dimensional control and can get as close as he likes to the target! (see Figure 10).

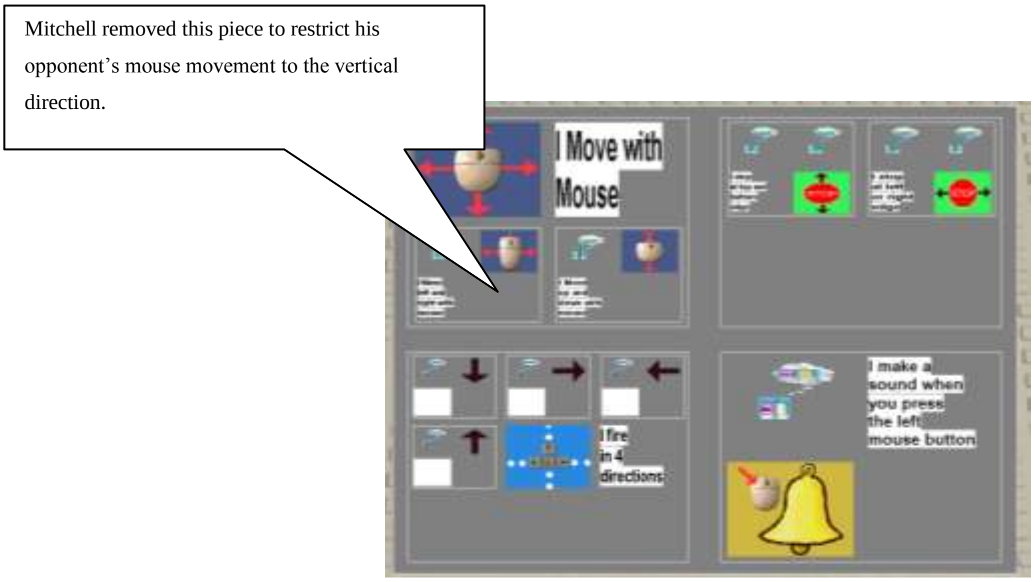

Figure 10: The reverse of the opponent’s player character. Mitchell removed the horizontal component of the "I move with mouse" gadget.

When we are all allowed to open our eyes, Mitchell triumphantly demands a rematch which, unsurprisingly – now that his opponent's motion is restricted to vertical movements – he wins conclusively, much to his satisfaction.

What can we learn from this episode? First, it is important to acknowledge the level of engagement that Mitchell displayed. We began with a desire to tap into children's culture, and we did so not only in the name of "motivation", but with a desire to engage children in personally meaningful projects, projects which they cared about enough to immerse themselves in the quasi-formal world in which we put them.

Second, I would like to raise a methodological issue. The kind of event characterised by Mitchell's trick was relatively rare. In terms of school curricula, teacher take-up, assessment procedures and so on, designing a system which fails to produce regular, measurable, and straightforward effects of use may not be much use. I suggest that this is a short-sighted view; indeed, there is a case to be made that in exploring the potential of digital technologies, and moreover, in studying the interplay between design and learning outcomes, we should precisely – as diSessa puts it – "design for rare events" (diSessa *et al,* 1995). Designing software that elicits the same response from all who use it may be attractive but it more or less guarantees that little will change. In the mathematical realm, we have more than our share of tried and tested routines that are supposed to elicit just this kind of homogenous response – although they mostly fail to do so. The design of systems that encourage heterogeneous responses carries risks, but it increases the possibility of real change.

We cannot know how Mitchell came to the realisation that the two-dimensional movement of the mouse was built from vertical and horizontal components; how he hit on the idea of the trick; or what, exactly, he learned while he was watching the researcher redesign the game (an interesting question, as constructivist orthodoxy – at least as widely misconceived – might suggest that he would not learn much "just" by watching). In mathematical language (which, of course, he did not use), we might say that he had realised that two-dimensional motion could be decomposed into the vector sum of two simpler motions; and even more, that removing one of these, would still result in a movable system, but one which was constrained only to move in a single dimension.

What was Mitchell doing when he removed that behaviour and how did our design facilitate that? He could see a direct link between a goal (disabling an opponent's player or even further, winning!) and a programming action. In other words his knowledge about the game, his goals and intentions (and the possibilities for disrupting these for another player) were built into the environment by designing programming code as manipulable, decomposable objects. It is reasonably certain that the modularity of the code, and the easily-visible description of its function (a description which does not, of course, play any functional role in its execution) must have been important factors.

I would like to draw attention to one final point concerning the way in which Mitchell expressed his knowledge. He didn't speak it, at least not until we quizzed him. He expressed it in action, manipulating the screen objects with his mouse. By definition, therefore, he was constrained in his expression by the specificities of the tools he had at his disposal, or more properly, by his relationship in activity to the tools. He did not express a mathematical abstraction in a recognisable form, but he expressed an abstraction nonetheless, one that was situated within the tools and activity structures of the setting. This issue is essentially tangential to the theme of this paper, and I will not explore it further (for more on situated abstraction, see, for example, Noss, Hoyles and Pozzi, in press); but I would like to flag the point as Hoyles and myself are beginning to think that this kind of situated abstraction is characteristic of many, though not all, work settings.

CONCLUDING REMARKS

In this lecture, I have tried to elaborate some of the mathematical demands that characterise the social and cultural life of the twenty-first century, and to map out one implication for the design of future mathematical learning environments. There is not a great deal of research that challenges existing epistemological and didactical assumptions of curricula, even though almost all mathematical curricula were designed in a pre-computational era. Discussion of these questions is, however, an emerging theme within the literature (see, for example, Papert, 1995). A key component of this and future work has been the forging of organic links between cognitive and sociocultural approaches, and the critical examination of privileged representation systems in favour of alternative, more accessible ones.

At the heart of digital technology is an irony for education. While computational power is increasingly used to render mathematical knowledge less visible and less apparent, it is digital technology that allows the best hope for the design of mathematical learning environments that convey a sense of a meta-epistemological stance. Moreover, it is computational technologies that make it feasible to design new representational systems and to introduce dynamic forms into hitherto static inscriptions, as well as offering new windows onto the kinds of thinking that could develop. The study of these new forms, and their potential for epistemological transformation, remains a major priority for research.

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NOTES

1. This paper is adapted from the plenary talk delivered at the 53rd International Conference of the "Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques", Verbania, Italy, July 21-27, 2001.

2. Of course in the nineteenth and twentieth centuries, the relationship between meaning and depiction changed again fundamentally.

3. This observation should be read in the context of a recent Royal Society (2001) report of the Joint Mathematical Council UK on *Teaching and Learning Geometry 11-19,* which recommends a greater emphasis on work in 3-dimensions.

4. I recognise, of course, that many occupations involve virtually no mathematical activity, and that even when it might, computational technology is often used to remove it from the individual's focus (the usual example is checkout clerks in supermarkets). But there are many who *do* use mathematics in various ways and these are our focus here.

5. Kent and myself are preparing a paper which will focus on this aspect.

6. Furthermore, it is interesting that when we find engineers willing to admit that mathematics is important for their tasks, they identify geometry as the most important mathematical topic.

7. There are many incidental aspects. One is that the behaviour of the bridge had been well documented in a somewhat obscure Japanese journal some years before, but had not been brought to the attention of the designers. Similarly, since the Millennium Bridge's wobble emerged, it has become evident that there are several – perhaps many – bridges that have displayed similar behaviour, and many more that would do so if they were subjected to the same volume of pedestrian traffic.

8. The Playground Project is funded by the European Union, Grant No. 29329. The other partners are in Cambridge, Porto, Bratislava and Stockholm. See http://www.ioe.ac.uk/playground/. I acknowledge the collaboration of all the partners, and in particular our London-based colleagues Ross Adamson, Miki Grahame and Sarah Lowe.

9. See www.toontalk.com for information about ToonTalk. ToonTalk's creator is Ken Kahn, a consultant to the Playground Project.

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