The Invention of Dimension

Steven T. Bramwell

University College London, London Centre for Nanotechnology and Department of Physics and Astronomy, 17-19 Gordon Street, London WC1H 0AJ, UK.

The term "dimensions" of a quantity may conjure up a variety of images, not all of them in sharp focus: perhaps a tool for checking equations or changing scale, a mysterious quality of nature, or a magical means of discovery. Of all the ideas of measurement theory, dimensions are perhaps the most valuable, yet equally, the most elusive. What explains their complex character ?

The modern concept of dimension starts with Maxwell (1863), who synthesised earlier formulations by Fourier, Weber and Gauss [1]. In doing so he added a nuance that we acknowledge today whenever we refer to the dimensions of (say) g (= 9.81 m/s^2) as "distance over time squared" rather than just the dimensional exponents (1, -2). By referring to the "dimensions of a quantity", Maxwell seemed to imply that real things have natural dimensions. In the same spirit he designated units of mass, length and time as "fundamental units".

The consequence of Maxwell's choice was both inspiration and confusion. In the hands of virtuosos like Lord Rayleigh and Osborne Reynolds, dimensional analysis quickly became a powerful tool for discovery – identification of the "Reynolds number" that describes the complexity of fluid flow, is a classic achievement. More generally, the method identifies relationships that are consistent with the laws of physics, and involve only quantities that are relevant to a problem. Identifying these – even if only by inspired guesswork - can give huge savings of time in experiment and clear theoretical guidance [2,3].

However, following Maxwell, was a sense that new fundamental laws could be discovered by dimensional analysis. The product of magnetic and electric constants (inverse square rooted) shared dimensions with speed and indeed turned out to be the speed of light. Bohr's atom was motivated by the fact that Planck's constant shares dimensions with angular momentum [1]. But Bridgman (1922) insisted that dimensions are a matter of convention: human choices, like units [2]. They can't be used to discover new fundamental laws, only relations that derive from existing laws. In 1954 Maxwell's term "fundamental units" was replaced with "base units" [1], just to lay down the law.

This "operational" conclusion about dimensions may seem, even today, just a little too bleak for physicists, who are keen to get to the truth about nature, not just to measure it. However, even if dimensions were demystified a century ago, physics was far from finished with them. The late twentieth century saw the creation of a firm theoretical basis for dimensional analysis as well as the elucidation of a new concept: that of "anomalous dimension".

The big breakthrough was the invention of renormalisation group methods, which now pervade many areas of theoretical physics, from particle physics to condensed matter and fluid turbulence. Starting with a microscopic model one can "integrate out" shorter length scales and determine how the influence of various coupling parameters evolves as one "zooms out" from the microscopic to the macroscopic scale. Such methods can justify the choices of which couplings and variables to choose in dimensional analysis: they are the ones that remain relevant at large scales. In this way the basic idea of dimensional analysis is strongly justified.

But renormalisation group further calculates numbers. A good example is its ability to calculate "anomalous dimensions". This property of some physical systems echoes the concepts of fractal geometry: for example the length of an idealised fractal coastline is proportional to the length of the measuring rod raised to some power, the anomalous dimension. In fact, dimensional analysis shows[3] that anomalous dimensions are necessarily the exponents of dimensionless clusters. In reality, these can only be generated by the physics of the problem.

For example, at a critical point in condensed matter (such as the gas-liquid or ferromagnetic critical point), correlations typically decay as distance to a power $-(d-2+\eta)$, where d is the spatial dimensionality and η is the anomalous dimension. But here the "distance" is in fact the ratio of physical distance l to a microscopic distance a (typically atomic size) – both length scales, and all between, remain relevant. One way [4] of introducing such a dimensionless ratio of dissimilar length scales is via a logarithmic integral, $\int_a^l (1/r)\,dr = \log(l/a)$. This commonly occurs in two-dimensional (2D) systems, where the integral is related to the fundamental solution of the Laplace equation. This enables many 2D systems – magnets, superfluids, crystals - to show anomalous dimensions or criticality, over a broad temperature range.

It is clear that dimensions have a life beyond the SI brochure. Over the years their stock has risen, fallen and risen again, but some of their mystery and magic has always endured. This is surely because our changing concept of dimensions reflects the evolution of physics itself – a subject that will always be concerned with the problems of how to scale, how to distinguish between relevant and irrelevant factors and how to use mathematics to get to the truth about nature.

References

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