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Systems of cubic forms in many variables

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Abstract. We consider a system of R cubic forms in n variables, with integer coefficients, which define a smooth complete intersection in projective space. Provided $n \geq 25R$, we prove an asymptotic formula for the number of integer points in an expanding box at which these forms simultaneously vanish. In particular, we obtain the Hasse principle for systems of cubic forms in $25R$ variables, previous work having required that $n \gg R^2$. One conjectures that $n \geq 6R + 1$ should be sufficient. We reduce the problem to an upper bound for the number of solutions to a certain auxiliary inequality. To prove this bound we adapt a method of Davenport.

1. Introduction

1.1. Main result. Let $c_1, \dots, c_R$ be homogeneous cubic forms in n variables $x_1, \dots, x_n$ with integer coefficients. We treat the simultaneous Diophantine equations

$$c_1(\vec{x}) = 0, \quad \dots, \quad c_R(\vec{x}) = 0$$

and the corresponding projective variety in $\mathbb{P}^{n-1}_{\mathbb{Q}}$, which we call $V(c_1, \dots, c_R)$. We assume throughout that the c_i generate the ideal of $V(c_1, \dots, c_R)$, and are linearly independent. The cubic case of a classic result of Birch gives us:

Theorem 1.1 (Birch [2]). *Let $\mathcal{B}$ be a box in $\mathbb{R}^n$, contained in the box $[-1, 1]^R$, and having sides of length at most 1 which are parallel to the coordinate axes. For each $P \geq 1$, write*

$$N_{c_1,\dots,c_R}(P) = \#\{\vec{x} \in \mathbb{Z}^n : \vec{x}/P \in \mathcal{B},\ c_1(\vec{x}) = 0, \dots, c_R(\vec{x}) = 0\}.$$

If the variety $V(c_1, \dots, c_R)$ is a smooth complete intersection, and the inequality

$$n \geq 8R^2 + 9R \tag{1.1}$$

holds, then for all $P \geq 1$, some $\mathfrak{I} \geq 0$ depending only on the c_i and $\mathcal{B}$, and some $\mathfrak{S} \geq 0$ depending only on the c_i, we have

$$N_{c_1,\dots,c_R}(P) = \mathfrak{I}\mathfrak{S}P^{n-3R} + O(P^{n-3R-\delta}), \tag{1.2}$$

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where the implicit constant depends only on the forms c_i, and the positive real number δ depends only on R. If the variety $V(c_1, \dots, c_R)$ has a smooth point over $\mathbb{Q}_p$ for each prime p, and a smooth real point whose homogeneous co-ordinates lie in $\mathcal{B}$, then $\mathfrak{S}$ and $\mathfrak{I}$ are positive.

In particular, the latter theorem follows from [2, Theorem 1], on inserting the bound $\dim V^* \leq R - 1$ for the dimension of the variety V^* occurring in that result. This bound follows from [4, Lemma 3.1] whenever $V(c_1, \dots, c_R)$ is a smooth complete intersection. See [18, Lemma 1.1] for details.

In Section 1.3 we prove:

Theorem 1.2. *In Theorem* 1.1 *we may replace* (1.1) *with the condition*

$$n \geq 25R. \tag{1.3}$$

This sharpens (1.1) as soon as $R \geq 3$. For example when $R = 3$ and $V(c_1, c_2, c_3)$ is a smooth complete intersection, Theorem 1.2 applies when $n \geq 75$, whereas Birch's theorem requires $n \geq 99$.

The "square-root cancellation" heuristic suggests that in place of (1.1) the condition $n \geq 6R + 1$ should suffice, see for example [3, discussion around formula (1.5)]. By handling systems of forms in $O(R)$ variables we come within a constant factor of this conjecture.

Our strategy is an extension of our previous work [18]. In forthcoming papers we further generalise this approach to treat systems of R forms with degree $d \geq 2$, with rational or real coefficients.

1.2. Related work. We begin with the case when the forms $c_i(\vec{x})$ are diagonal.

In the case of a single diagonal form c, Baker [1] proves that $V(c)$ has a rational point whenever $n \geq 7$.

Brüdern and Wooley [7, 8, 11] treat diagonal systems in $n \geq 6R + 1$ variables, the best value of n possible with the classical circle method. In particular, they prove the Hasse principle for $V(c_1, \dots, c_R)$ whenever the c_i are diagonal, $V(c_1, \dots, c_R)$ is smooth and $n \geq 6R + 1$. They also prove an asymptotic formula of the type (1.2) whenever $n \geq 6R + 3$ holds, or when $R = 2$ and $n \geq 14$ holds [5, 6, 9]. In the case $R = 2$ they prove a Hasse principle for certain pairs of diagonal cubics in as few as eleven variables [10].

Returning to the case of general (not necessarily diagonal) forms, we consider the case $R = 1$. Let c be a cubic form. Hooley [17] proves that if $n = 8$, the variety $V(F)$ is smooth, and the box $\mathcal{B}$ is sufficiently small and centred at a point at which the Hessian determinant of F does not vanish, then a smoothly weighted version of the asymptotic formula (1.2) holds. In this work he assumes a Riemann hypothesis for a certain modified Hasse–Weil L-function. When $n = 9$, he proves the same result unconditionally [16], with the weaker error term $O(P^{n-3}(\log P)^{-\delta})$ in place of the $O(P^{n-3-\delta})$ from (1.2). Heath-Brown [15] proves that if $n \geq 14$, then $V(c)$ always has a rational point, regardless of whether it is singular.

In the case $R = 2$, Dietmann and Wooley [14] have shown that $V(c_1, c_2)$ always has a rational point when $n \geq 827$, whether or not it is smooth.

In the general case $R \geq 1$, Schmidt [19] shows that $V(c_1, \dots, c_R)$ always has a rational point if $n \geq (10R)^5$. Recent work of Dietmann [13] improves this condition to

$$n \geq 400{,}000R^4.$$

1.3. Reduction to an auxiliary inequality. To prove Theorem 1.2 we will use Theorem 1.3 from the author's previous work [18]. This will reduce the problem to proving an upper bound for the number of solutions to the following auxiliary inequality.

Definition 1.3. For any $k \in \mathbb{N}$ and $\vec{t} \in \mathbb{R}^k$, we write $\|\vec{t}\|_\infty = \max_i |t_i|$ for the supremum norm. When $c(\vec{x})$ is a cubic form in n variables with real coefficients, we define a symmetric matrix

$$H_c(\vec{x}) = \frac{1}{\|c\|_\infty}\left(\frac{\partial^2 c(\vec{x})}{\partial x_i \partial x_j}\right)_{1\le i,j\le n}, \tag{1.4}$$

where

$$\|c\|_\infty = \frac{1}{6}\max_{i,j,k\in\{1,\dots,n\}}\left|\frac{\partial^3 c(\vec{x})}{\partial x_i x_j x_k}\right|.$$

Thus $H_c(\vec{x})$ is the Hessian of the cubic form $c(\vec{x})/\|c\|_\infty$, which has been normalised so that 1 is the absolute value of its largest coefficient. For each $B \ge 1$ we put $N_c^{\mathrm{aux}}(B)$ for the number of pairs $(\vec{x}, \vec{y})$ of integer vectors with

$$\|\vec{x}\|_\infty, \|\vec{y}\|_\infty \le B, \quad \|H_c(\vec{x})\vec{y}\|_\infty < B.$$

We show that this definition of the counting function $N_c^{\mathrm{aux}}(B)$ agrees with the one given in [18, Definition 1.1]. There we consider a degree d polynomial f and a system of multilinear forms $\vec{m}^{(f)}(\vec{x}^{\,(1)}, \dots, \vec{x}^{\,(d-1)})$, and when $d = 3$ and $f(\vec{x}) = c(\vec{x})$, we see that

$$\vec{m}^{(f)}(\vec{x}^{\,(1)}, \vec{x}^{\,(2)}) = \|c\|_\infty H_c(\vec{x}^{\,(1)})\vec{x}^{\,(2)}.$$

One can then check that the definitions agree. The case $d = 3$ of [18, Theorem 1.3] therefore states that:

Theorem 1.4. *Let the counting function $N_{c_1,\dots,c_R}(P)$ be as in Theorem* 1.1. *Suppose that for some $C_0 \ge 1$ and $\mathcal{C} > 3R$, we have*

$$N^{\mathrm{aux}}_{\vec{\beta}\cdot\vec{c}}(B) \le C_0 B^{2n-8\mathcal{C}} \tag{1.5}$$

for all $\vec{\beta} \in \mathbb{R}^R$ and $B \ge 1$, where we write $\vec{\beta}\cdot\vec{c}$ for $\beta_1 c_1 + \cdots + \beta_R c_R$. Then we have

$$N_{c_1,\dots,c_R}(P) = \mathfrak{S}\mathfrak{S}P^{n-3R} + O(P^{n-3R-\delta})$$

for all $P \ge 1$, where the implicit constant depends at most on C_0, $\mathcal{C}$ and the c_i, and the positive constant δ depends at most on $\mathcal{C}$ and R. Here the constants $\mathfrak{S}$ and $\mathfrak{S}$ are as in Theorem 1.1.

We give the following bound for the counting function $N_c^{\mathrm{aux}}(B)$. The proof occupies the bulk of this paper and is completed in Section 6.

Proposition 1.5. *We call a set $\mathcal{K}$ of real cubic forms in n variables a* closed cone *if*

(i) *for all $c \in \mathcal{K}$ and $\lambda \ge 0$ we have $\lambda c \in \mathcal{K}$,*

(ii) *$\mathcal{K}$ is closed in the real linear space of cubic forms in n variables.*

Let $\mathcal{K}$ be a closed cone as above, and let $N_c^{\mathrm{aux}}(B)$ be as in Definition 1.3. *If we set*

$$\sigma_{\mathcal{K}} = 1 + \max_{c \in \mathcal{K} \setminus \{0\}} \dim \operatorname{Sing} V(c), \tag{1.6}$$

so that $\sigma_{\mathcal{K}} \in \{0, \dots, n-1\}$, then for all $\epsilon > 0$, $c \in \mathcal{K}$ and $B \geq 1$ we have

$$N_c^{\mathrm{aux}}(B) \ll_{\mathcal{K},\epsilon} B^{n+\sigma_{\mathcal{K}}+\epsilon}. \tag{1.7}$$

Note that without the normalising factor $1/\|c\|_\infty$ in (1.4) this result would be false, since we would then have $N_c^{\mathrm{aux}}(B) \gg_n B^{2n}$ whenever $\|c\|_\infty \leq \frac{1}{B}$. We will outline the proof of the proposition after deducing Theorem 1.2.

Proof of Theorem 1.2. Suppose that (1.3) holds. We claim that for all $B \geq 1$, $\epsilon > 0$ and $\vec{\beta} \in \mathbb{R}^R$ we have

$$N_{\vec{\beta}\cdot\vec{c}}^{\mathrm{aux}}(B) \ll_{c_1,\dots,c_R,\epsilon} B^{n+R-1+\epsilon}, \tag{1.8}$$

where $\vec{\beta} \cdot \vec{c}$ is as in Theorem 1.4. If we set $\mathcal{C} = (n - R + \frac{1}{2})/8$ and let C_0 be sufficiently large in terms of the forms c_i, we can then apply Theorem 1.4. For (1.8) implies (1.5) on setting $\epsilon = \frac{1}{2}$ in (1.8). Moreover, we have $\mathcal{C} > 3R$, by (1.3). So the hypotheses of Theorem 1.4 are satisfied, and Theorem 1.2 follows.

Setting $\mathcal{K} = \{\vec{\beta} \cdot \vec{c} : \vec{\beta} \in \mathbb{R}^R\}$ in Proposition 1.5, we see that (1.8) follows from (1.7) unless $\sigma_{\mathcal{K}} > R - 1$ holds. Suppose for a contradiction that we have $\sigma_{\mathcal{K}} > R - 1$.

By the definition in (1.6) there must be $\vec{\beta} \in \mathbb{R}^R \setminus \{\vec{0}\}$ with

$$\dim \operatorname{Sing} V(\vec{\beta} \cdot \vec{c}) \geq R - 1. \tag{1.9}$$

We may assume that $V(c_1, \dots, c_R) = V(c_1, \dots, c_{R-1}, \vec{\beta} \cdot \vec{c})$ holds, after permuting the c_i if necessary. We have

$$V(c_1, \dots, c_{R-1}) \cap \operatorname{Sing} V(\vec{\beta} \cdot \vec{c}) \subset \operatorname{Sing} V(c_1, \dots, c_R)$$

since $V(c_1, \dots, c_R)$ is a complete intersection, and so

$$\begin{aligned} \dim \operatorname{Sing} V(c_1, \dots, c_R) &\geq \dim V(c_1, \dots, c_{R-1}) + \dim \operatorname{Sing} V(\vec{\beta} \cdot \vec{c}) - (n-1) \\ &= \dim \operatorname{Sing} V(\vec{\beta} \cdot \vec{c}) - (R-1). \end{aligned}$$

Thus (1.9) implies that $\dim \operatorname{Sing} V(c_1, \dots, c_R) \geq 0$, which contradicts the assumption in the theorem that $\operatorname{Sing} V(c_1, \dots, c_R) = \emptyset$. □

1.4. Outline of remaining steps. To prove Proposition 1.5 we adapt the argument used to prove Lemma 3 in Davenport [12], and subsequently a somewhat more general result in [19, Section 5]. These authors consider the counting function defined by

$$N_c^{\text{aux-eq}}(B) = \#\{(\vec{x}, \vec{y}) \in (\mathbb{Z}^n)^2 : \|\vec{x}\|_\infty, \|\vec{y}\|_\infty \leq B,\ H_c(\vec{x})\vec{y} = \vec{0}\}$$

for a cubic form c with integer coefficients. Davenport proves that either $N_c^{\text{aux-eq}}(B)$ is small, or there is a large rational linear space on which c vanishes. In order to briefly sketch his line of reasoning, we define some additional notation.

Definition 1.6. Define

$$\|H_c(\vec{x})\|_\infty = \max_{i,j} |H_c(\vec{x})_{ij}|.$$

Let $\lambda_{c,1}(\vec{x}), \dots, \lambda_{c,n}(\vec{x})$ be the eigenvalues of the real symmetric matrix $H_c(\vec{x})$, listed with multiplicity and in order of decreasing absolute value. Observe that

$$|\lambda_{c,1}(\vec{x})| \leq n\|H_c(\vec{x})\|_\infty \leq n^2\|\vec{x}\|_\infty. \tag{1.10}$$

For each $i \in \{1, \dots, n\}$ let $\vec{D}^{(c,i)}(\vec{x})$ be the vector of all $i \times i$ minors of $H_c(\vec{x})$, arranged in some order. This is a vector of degree i homogeneous forms in the variables $\vec{x}$, with real coefficients. Let $J_{\vec{D}^{(c,i)}}(\vec{x})$ be the Jacobian matrix $(\partial D_j^{(c,i)}(\vec{x})/\partial x_k)_{jk}$.

Davenport's argument runs as follows.

(1) Let $\sigma \in \{0, \dots, n-1\}$. Suppose that we have $N_c^{\text{aux-eq}}(B) \gg B^{n+\sigma}$ for some sufficiently large implicit constant. The contribution to this count from any one vector $\vec{x}$ is at most $O(B^{n-\operatorname{rank} H_c(\vec{x})})$. So there must be an integer b in the set $\{0, \dots, n-1\}$ such that at least $\gg B^{\sigma+b}$ integer points $\vec{x}$ satisfy both $\operatorname{rank} H_c(\vec{x}) = b$ and $\|\vec{x}\|_\infty \leq B$.

(2) If σ and b are as in (1), then one can deduce that there is an integer point $\vec{x}^{\,(0)}$ satisfying the condition $\operatorname{rank} H_c(\vec{x}^{\,(0)}) = b$ such that the tangent space to the affine variety $\vec{D}^{(c,b+1)}(\vec{x}) = \vec{0}$ at the point $\vec{x}^{\,(0)}$ has dimension $\sigma + b + 1$ or more. Equivalently,

$$\operatorname{rank} H_c(\vec{x}^{\,(0)}) = b \quad \text{and} \quad \operatorname{rank} J_{\vec{D}^{(c,b+1)}}(\vec{x}^{\,(0)}) \leq n - \sigma - b - 1$$

both hold. This follows from [12, Lemma 2].

(3) If c has integral coefficients and there exists a vector $\vec{x}^{\,(0)}$ as in (2), then it follows that there exist linear subspaces X, Y of $\mathbb{Q}^n$, with dimensions $\sigma + b + 1$ and $n - b$, respectively, such that for all $\vec{X} \in X$ and $\vec{Y}, \vec{Y}' \in Y$ the equality $\vec{Y}^T H_c(\vec{X})\vec{Y}' = 0$ holds. See [19, Lemma 4] or [12, proof of Lemma 3].

(4) We conclude that if $N_c^{\text{aux-eq}}(B) \gg B^{n+\sigma}$, then there are spaces X, Y as in (3). In that case the space Z defined by $Z = X \cap Y$ is a rational linear space, with dimension at least $\sigma + 1$, such that for all $\vec{Z} \in Z$ the equality $c(\vec{Z}) = 0$ holds.

Our setting differs in three ways from that of Schmidt and Davenport. First, we consider the inequality $\|H_c(\vec{x})\vec{y}\|_\infty \leq B$ rather than the equation $H_c(\vec{x})\vec{y} = \vec{0}$. Second, for us the cubic form $c(\vec{x})$ may have real coefficients. And third, rather than concluding that $c(\vec{x})$ has a rational linear space of zeroes, we seek to show that the variety $V(c)$ is very singular.

1.5. Structure of this paper. In Section 2 and Sections 4–6 we will modify each of the four steps (1)–(4) above to accommodate the three changes described at the end of Section 1.4. In the remaining section, Section 3, we prove some technical lemmas relating the minors and eigenvalues of real matrices.

1.6. Notation. Throughout, we let c, $\|\vec{t}\|_\infty$, $\|c\|_\infty$, $H_c(\vec{x})$ and $N_c^{\text{aux}}(B)$ be as in Definition 1.3, and we let $\|H_c(\vec{x})\|_\infty$, $\lambda_{c,i}(\vec{x})$, $\vec{D}^{(c,i)}(\vec{x})$ and $J_{\vec{D}^{(c,i)}}(\vec{x})$ be as in Definition 1.6. We do not require algebraic varieties to be irreducible, and we adopt the convention that $\dim \emptyset = -1$. We use Vinogradov's $\ll$ notation and big-O notation in the usual way.

2. The eigenvalues of the Hessian matrix $H_c(\vec{x})$

In this section we show that if the counting function $N_c^{\text{aux}}(B)$ from Definition 1.3 is large, there are many integer points $\vec{x}$ for which the eigenvalues of $H_c(\vec{x})$ lie in some fixed dyadic ranges. Namely, we will show that there are many integer points in a set $K_k(E_1, \dots, E_{k+1})$ defined as follows. This corresponds to step (1) from Section 1.4.

Definition 2.1. Suppose that $k \in \{0, \dots, n\}$ and that $E_1, \dots, E_{k+1} \in \mathbb{R}$ such that the inequalities $E_1 \geq \dots \geq E_{k+1} \geq 1$ hold. Then we define $K_k(E_1, \dots, E_{k+1})$ to be the set of all vectors $\vec{x}$ in $\mathbb{R}^n$ satisfying the following conditions: the inequality $\|\vec{x}\|_\infty \leq B$ holds, and we have

$$\frac{1}{2}E_i < |\lambda_{c,i}(\vec{x})| \leq E_i$$

whenever $1 \leq i \leq k$ holds, and we have

$$|\lambda_{c,i}(\vec{x})| \leq E_{k+1}$$

whenever $k+1 \leq i \leq n$ holds.

Lemma 2.2. *Let H be a real symmetric $n \times n$ matrix and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the matrix H, listed with multiplicity and in order of decreasing absolute value. Let $C \geq 1$ and $B \geq 1$, and suppose that $|\lambda_1| \leq CB$ holds. Set*

$$N_H(B) = \#\{\vec{y} \in \mathbb{Z}^n : \|\vec{y}\|_\infty \leq B,\ \|H\vec{y}\|_\infty \leq B\}.$$

Then we have

$$N_H(B) \ll_{C,n} \min_{1 \leq i \leq n} \frac{B^n}{1 + |\lambda_1 \cdots \lambda_i|}.$$

Proof. The integral vectors $\vec{y}$ counted by $N_H(B)$ are all contained in the box $\|\vec{y}\|_\infty \leq B$ and in the ellipsoid $\{\vec{t} \in \mathbb{R}^n : \vec{t}^{\,T} H^T H \vec{t} \leq nB^2\}$, which has principal radii $|\lambda_i|^{-1}\sqrt{n}B$. Hence

$$N_H(B) \ll_n \prod_{i=1}^{n} \min\{1 + |\lambda_i|^{-1}\sqrt{n}B, B\}$$

and as $|\lambda_i| \leq CB$ holds, this is

$$\leq \prod_{i=1}^{n} \min\{2C|\lambda_i|^{-1}\sqrt{n}B, B\}.$$

It follows that

$$N_H(B) \ll_{C,n} B^n \prod_{i=1}^{n} \min\{|\lambda_i|^{-1}, 1\}.$$

Since the inequalities $|\lambda_1| \geq \dots \geq |\lambda_n|$ hold, we deduce that

$$\begin{aligned} N_H(B) &\ll_{C,n} B^n \min\left\{1, \frac{1}{|\lambda_1|}, \frac{1}{|\lambda_1\lambda_2|}, \dots, \frac{1}{|\lambda_1 \cdots \lambda_n|}\right\} \\ &\ll \min_{1 \leq i \leq n} \frac{B^n}{1 + |\lambda_1 \cdots \lambda_i|}, \end{aligned}$$

as claimed in the lemma. □

Corollary 2.3. *Let $N_c^{\mathrm{aux}}(B)$ be as in Definition* 1.3*, let $\lambda_{c,i}(\vec{x})$ and $\vec{D}^{(c,i)}(\vec{x})$ be as in Definition* 1.6*, and let $K_k(E_1, \dots, E_{k+1})$ be as in Definition* 2.1*. For any $B \geq 1$, one of the following alternatives holds. Either*

$$\frac{N_c^{\mathrm{aux}}(B)}{B^n(\log B)^n} \ll_n \#\{\mathbb{Z}^n \cap K_0(1)\}, \tag{2.1}$$

or there is $k \in \{1, \dots, n-1\}$ and there are $e_1, \dots, e_k \in \mathbb{N}$ satisfying $\log B \gg_n e_1 \geq \cdots \geq e_k$ and

$$\frac{2^{e_1+\cdots+e_k} N_c^{\mathrm{aux}}(B)}{B^n(\log B)^n} \ll_n \#\{\mathbb{Z}^n \cap K_k(2^{e_1}, \dots, 2^{e_k}, 1)\}, \tag{2.2}$$

or there are $e_1, \dots, e_n \in \mathbb{N}$ satisfying $\log B \gg_n e_1 \geq \cdots \geq e_n$ and

$$\frac{2^{e_1+\cdots+e_n} N_c^{\mathrm{aux}}(B)}{B^n(\log B)^n} \ll_n \#\{\mathbb{Z}^n \cap K_{n-1}(2^{e_1}, \dots, 2^{e_n})\}. \tag{2.3}$$

Proof. Note that in the case that $k = n$, there are no values of i satisfying $k+1 \leq i \leq n$, so the last condition in the definition of $K_k(E_1, \dots, E_{k+1})$ is vacuously true and can be omitted. In particular, if $k = n$ then (2.3) follows from (2.2), because

$$K_n(2^{e_1}, \dots, 2^{e_n}, 1) \subset K_{n-1}(2^{e_1}, \dots, 2^{e_n}).$$

So it is enough to prove that either (2.1) holds or there exist integers k and $e_1, \dots, e_k$ satisfying the inequalities $1 \leq k \leq n$ and $\log B \gg_n e_1 \geq \cdots \geq e_n$ such that (2.2) holds.

The set $K_0(1)$, together with the sets $K_k(2^{e_1}, \dots, 2^{e_k}, 1)$, partition the box $\|\vec{x}\|_\infty \leq B$ into disjoint pieces. So, if we let

$$N_{H_c(\vec{x})}(B) = \#\{\vec{y} \in \mathbb{Z}^n : \|\vec{y}\|_\infty \leq B,\ \|H_c(\vec{x})\vec{y}\|_\infty \leq B\},$$

then we have

$$N_c^{\mathrm{aux}}(B) = \sum_{\substack{\vec{x} \in \mathbb{Z}^n \\ \vec{x} \in K_0(1)}} N_{H_c(\vec{x})}(B) + \sum_{\substack{1 \leq k \leq n \\ e_1 \geq \cdots \geq e_k \geq 1 \\ e_1 \ll_n \log B}} \sum_{\substack{\vec{x} \in \mathbb{Z}^n \\ \vec{x} \in K_k(2^{e_1}, \dots, 2^{e_k}, 1)}} N_{H_c(\vec{x})}(B). \tag{2.4}$$

The total number of terms on the right-hand side of (2.4) is $O_n((\log B)^n)$ at most, so it follows that either

$$\sum_{\substack{\vec{x} \in \mathbb{Z}^n \\ \vec{x} \in K_0(1)}} N_{H_c(\vec{x})}(B) \gg_n \frac{N_c^{\mathrm{aux}}(B)}{(\log B)^n} \tag{2.5}$$

holds, or else there are $1 \leq k \leq n$ and $e_1 \geq \cdots \geq e_k \geq 1$ such that

$$\sum_{\substack{\vec{x} \in \mathbb{Z}^n \\ \vec{x} \in K_k(2^{e_1}, \dots, 2^{e_k}, 1)}} N_{H_c(\vec{x})}(B) \gg_n \frac{N_c^{\mathrm{aux}}(B)}{(\log B)^n}. \tag{2.6}$$

If (2.5) holds, then the trivial bound $N_{H_c(\vec{x})}(B) \ll_n B^n$ implies (2.1). Suppose instead that (2.6) holds.

By (1.10), for each real vector $\vec{x}$ the bound

$$|\lambda_{c,1}(\vec{x})| \ll_n B$$

holds. So we may apply Lemma 2.2 with the choice $H = H_c(\vec{x})$ and some C depending on n only. This shows that

$$N_{H_c(\vec{x})}(B) \ll_n \frac{B^n}{2^{e_1+\cdots+e_k}}.$$

Substituting this into (2.6), we see that (2.2) holds, as claimed. □

3. Intermission: Eigenvalues and minors

Here we collect some results about the eigenvalues and minors of real matrices which will be needed in Section 4. We need the following relatively straightforward fact; we include a proof for the reader's convenience.

Lemma 3.1. *For each $k, \ell \in \mathbb{N}$, let*

$$T_{k,\ell} = \{\vec{a} \in \mathbb{N}^k : 1 \le a_1 < \cdots < a_k \le \ell\}.$$

This set has $\binom{\ell}{k}$ members. For each $k, \ell, m \in \mathbb{N}$ such that $k \le \min\{\ell, m\}$, and each $\ell \times m$ real matrix L, define an $\binom{\ell}{k} \times \binom{m}{k}$ real matrix $L^{[k]}$ by

$$L^{[k]} = (L^{[k]}_{\vec{a}\vec{b}})_{\vec{a}\in T_{k,\ell},\vec{b}\in T_{k,m}}, \quad L^{[k]}_{\vec{a}\vec{b}} = \det((L_{a_i b_j})_{1\le i,j\le k}),$$

so that the $L^{[k]}_{\vec{a}\vec{b}}$ are the $k \times k$ minors of L. For all $\ell \times m$ matrices L and all $m \times n$ matrices M we have

$$(LM)^{[k]} = L^{[k]}M^{[k]}$$

for all $k \le \min\{\ell, m, n\}$. That is, we have

$$(LM)^{[k]}_{\vec{a}\vec{b}} = \sum_{\vec{w}\in T_{k,m}} L^{[k]}_{\vec{a}\vec{w}} M^{[k]}_{\vec{w}\vec{b}}. \tag{3.1}$$

Proof. Let $\vec{e}^{\,(1)}, \ldots, \vec{e}^{\,(m)}$ be the standard basis of $\mathbb{R}^m$. Fix $L, \vec{a}, \vec{b}$; then each side of (3.1) is an alternating multilinear form in those k columns of M whose indices appear in the vector $\vec{b}$. This is some k-tuple of m-vectors.

Given the value of an alternating multilinear form at the k-tuple $\vec{e}^{\,(z_1)}, \ldots, \vec{e}^{\,(z_k)}$ for each $\vec{z} \in T_{k,m}$, one can extend by linearity and the alternating property to find its value at any k-tuple of m-vectors. In other words, it suffices to check (3.1) when, for some $\vec{z} \in T_{k,m}$, the $k \times k$ submatrix $(M_{z_i b_j})_{1\le i,j\le k}$ is the identity and all other entries of M are zero. In this case both sides of (3.1) are equal to $L^{[k]}_{\vec{z}\vec{b}}$. □

Our main result of this section is the following technical lemma.

Lemma 3.2. *Let M be a real $m \times n$ matrix. Recall that $M^T M$ is positive semidefinite and symmetric. Let the eigenvalues of $M^T M$ be $\Lambda_1^2, \ldots, \Lambda_n^2$ in decreasing order, where the Λ_i are nonnegative and in decreasing order. That is, the Λ_i are the* singular values *of M.*

In particular, if M is a symmetric matrix, then the Λ_i are exactly the absolute values of the eigenvalues of M, by diagonalisation.

Given a natural number k with $k \leq \min(m,n)$, let $\vec{D}^{(k)}$ be the vector of $k \times k$ minors of M, arranged in some order. Then we have:

(i) *The maximum norm $\|\vec{D}^{(k)}\|_\infty$ satisfies*

$$\|\vec{D}^{(k)}\|_\infty \asymp_{m,n} \Lambda_1 \cdots \Lambda_k. \tag{3.2}$$

(ii) *There is a k-dimensional linear space $V \subset \mathbb{R}^n$ such that for all $\vec{v} \in V$,*

$$\|M\vec{v}\|_\infty \gg_{m,n} \|\vec{v}\|_\infty \Lambda_k. \tag{3.3}$$

We may take V to be a span of k standard basis vectors $\vec{e}^{\,(i)}$ in $\mathbb{R}^n$.

(iii) *For any $C \geq 1$, either there is an $(n-k+1)$-dimensional linear subspace X of $\mathbb{R}^n$ such that*

$$\|M\vec{X}\|_\infty \leq C^{-1}\|\vec{X}\|_\infty \quad \textit{for all } \vec{X} \in X, \tag{3.4}$$

or there is a k-dimensional linear subspace V of $\mathbb{R}^n$, spanned by standard basis vectors of $\mathbb{R}^n$, such that

$$\|M\vec{v}\|_\infty \gg_{m,n} C^{-1}\|\vec{v}\|_\infty \quad \textit{for all } \vec{v} \in V.$$

Proof. Part (i). First we prove the result on the assumption that M^TM is diagonal. Let the sets $T_{k,\ell}$ and the matrices $L^{[k]}$ be as in Lemma 3.1. Since M^TM is diagonal with diagonal entries Λ_i^2, we have

$$\sum_{\vec{a} \in T_{k,n}} \Lambda_{a_1}^2 \cdots \Lambda_{a_k}^2 = \sum_{\vec{a} \in T_{k,n}} (M^TM)^{[k]}_{\vec{a}\vec{a}} = \sum_{\substack{\vec{a} \in T_{k,n} \\ \vec{w} \in T_{k,m}}} (M^{[k]}_{\vec{w}\vec{a}})^2, \tag{3.5}$$

by (3.1). The left-hand side of (3.5) is $\asymp_n \Lambda_1^2 \cdots \Lambda_k^2$, and the right-hand side is $\asymp_{m,n} \|\vec{D}^{(k)}\|_\infty^2$, so this proves (3.2).

Let O be an $n \times n$ orthogonal matrix such that O^TM^TMO is diagonal. Let $\vec{C}^{(k)}$ be the vector of $k \times k$ minors of MO. We claim that the norms $\|\vec{C}^{(k)}\|_\infty$ and $\|\vec{D}^{(k)}\|_\infty$ are of comparable size.

Lemma 3.1 shows that

$$(MO)^{[k]} = M^{[k]}O^{[k]},$$

and since we have

$$(O^T)^{[k]}O^{[k]} = I^{[k]} \quad \text{and} \quad (O^T)^{[k]}_{\vec{a}\vec{b}} = O^{[k]}_{\vec{b}\vec{a}},$$

it follows that $O^{[k]}$ is orthogonal. Hence the maximum norm of the entries satisfies

$$\|\vec{C}^{(k)}\|_\infty = \|(MO)^{[k]}\|_\infty \asymp_{m,n} \|M^{[k]}\|_\infty = \|\vec{D}^{(k)}\|_\infty.$$

So in proving (3.2) we may assume that M^TM is diagonal. The result follows.

Part (ii). By permuting the rows and columns of M, we may assume that

$$\|\vec{D}^{(k)}\|_\infty = |\det(M_{ij})_{1 \leq i,j \leq k}|. \tag{3.6}$$

Let $\vec{v}$ be in the span of the first k basis vectors. If (3.3) holds for all such $\vec{v}$, then we have proved the lemma. Since $v_i = 0$ for $i > k$, one finds that

$$\left(\begin{array}{ccc|c} M_{11} & \cdots & M_{1n} & \\ \vdots & \ddots & \vdots & \\ M_{k1} & \cdots & M_{kn} & \\ \hline 0_{(n-k)\times k} & & & I_{n-k} \end{array}\right)\left(\vec{v} \;\middle|\; \begin{array}{c} 0_{1\times(n-1)} \\ \hline I_{n-1} \end{array}\right) = \left(\begin{array}{ccc|c} (M\vec{v})_1 & M_{12} & \cdots & M_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ (M\vec{v})_k & M_{k2} & \cdots & M_{kn} \\ \hline 0_{(n-k)\times k} & & & I_{n-k} \end{array}\right),$$

where we have divided each matrix into three blocks, and $0_{p\times q}$ stands for a $p \times q$ block of zeroes. By (3.6) we have $\pm v_1 \cdot \|\vec{D}^{(k)}\|_\infty$ as the determinant of the left-hand side. Expanding the determinant of the right-hand side in the first column, we find that it is equal to

$$\begin{aligned} \pm\|\vec{D}^{(k)}\|_\infty v_1 &= \sum_{\ell=1}^{k}(-1)^{\ell+1}(M\vec{v})_\ell \det\Big(\big(M_{ij}\big)_{\substack{i=1,\dots,k;\, i\neq\ell \\ j=2,\dots,k}}\Big) \\ &\ll k\|M\vec{v}\|_\infty \|\vec{D}^{(k-1)}\|_\infty. \end{aligned}$$

Note the $(k-1)\times(k-1)$ determinant in which i runs over $1,\dots,k$ with the value ℓ omitted, and j runs over $2,\dots,k$.

By part (i), this implies that $\Lambda_k v_1 \ll_{m,n} \|M\vec{v}\|_\infty$, so provided that $\|\vec{v}\|_\infty = |v_1|$, then (3.3) holds.

If we apply the same permutation both to the v_i and to the first k rows of M, then both sides of our claim (3.3) and our assumption (3.6) remain the same. By applying such a permutation, we may assume $\|\vec{v}\|_\infty = |v_1|$, and so we have proved (3.3).

Part (iii). Let X be the span of the Λ_i^2-eigenvectors of M^TM, where i runs from k up to n. As the matrix M^TM is symmetric, we have $\vec{X}^TM^TM\vec{X} \ll_n \|\vec{X}\|_\infty^2\Lambda_k^2$ for all $\vec{X} \in X$, and so

$$\|M\vec{X}\|_\infty \ll_{m,n} \|\vec{X}\|_\infty\Lambda_k$$

for all $\vec{X} \in X$. Therefore either this space X satisfies (3.4), or the bound $\Lambda_k \gg_{m,n} C^{-1}$ holds and the existence of the space V follows by part (ii). □

4. Counting points in the sets $K_k(E_1, \dots, E_{k+1})$

In this section we estimate the number of integer points in the sets $K_k(E_1,\dots,E_{k+1})$ from Definition 2.1. We give the following result.

Lemma 4.1. *Let c and $H_c(\vec{x})$ be as in Definition* 1.3*, $\lambda_{c,i}(\vec{x})$, $\vec{D}^{(c,i)}(\vec{x})$, $J_{\vec{D}^{(c,i)}}(\vec{x})$ as in Definition* 1.6*, and $K_k(E_1,\dots,E_{k+1})$ as in Definition* 2.1*. Suppose that $B, C \geq 1$, $\sigma \in \{0,\dots,n-1\}$, $k \in \{0,\dots,n-\sigma-1\}$ and that $CB \geq E_1 \geq \cdots \geq E_{k+1} \geq 1$. Then at least one of the following holds:*

(I)$_k$ *The set $K_k(E_1,\dots,E_{k+1})$ may be covered by a collection of at most*

$$O_{C,n}(B^\sigma(E_1\cdots E_{k+1})E_{k+1}^{-\sigma-k-1})$$

boxes in $\mathbb{R}^n$ of side E_{k+1}. Such a box contains $O_n(E_{k+1}^n)$ integral points, so it follows that

$$\#\{\mathbb{Z}^n \cap K_k(E_1,\dots,E_{k+1})\} \ll_{C,n} B^\sigma(E_1\cdots E_{k+1})E_{k+1}^{n-\sigma-k-1}. \tag{4.1}$$

(II)$_k$ *There exist an integer* $1 \le b \le k$, *a* $(\sigma + b + 1)$*-dimensional linear subspace* X *of* $\mathbb{R}^n$, *and a point* $\vec{x}^{\,(0)} \in K_b(E_1, \dots, E_{b+1})$ *such that*

$$E_{b+1} < C^{-1} E_b$$

and

$$\|J_{\vec{D}^{(c,b+1)}}(\vec{x}^{\,(0)})\vec{X}\|_\infty \le C^{-1}\|\vec{D}^{(c,b)}(\vec{x}^{\,(0)})\|_\infty \|\vec{X}\|_\infty \quad \textit{for all } \vec{X} \in X. \tag{4.2}$$

(III) *There is a* $(\sigma + 1)$*-dimensional linear subspace* X *of* $\mathbb{R}^n$ *such that*

$$\|H_c(\vec{X})\|_\infty \le C^{-1}\|\vec{X}\|_\infty \quad \textit{for all } \vec{X} \in X, \tag{4.3}$$

with $\|H_c(\vec{X})\|_\infty$ *as in Definition* 1.6.

We have subscripted the first two items to emphasize their dependence on k*; note that item* (III) *has no such dependence.*

In Corollary 5.2 below, we will use Lemma 4.1 to bound quantities (2.1) and (2.2) from Corollary 2.3. Before proving the lemma, we give a comparison with step (2) in Section 1.4.

If there are many integer points $\vec{x}$ for which rank $H_c(\vec{x}) = b$ holds, then step (2) gives us a point $\vec{x}^{\,(0)}$ for which the matrix $J_{\vec{D}^{(c,b+1)}}(\vec{x}^{\,(0)})$ has a kernel of dimension $(\sigma + b + 1)$ or more and rank $H_c(\vec{x}^{\,(0)}) = b$ holds.

If there are many integer points $\vec{x}$ for which $\vec{x} \in K_k(E_1, \dots, E_{k+1})$, then (4.1) is false and so either (II)$_k$ or (III) must hold. Of these, case (II)$_k$ gives us a point $\vec{x}^{\,(0)}$ such that $J_{\vec{D}^{(c,b+1)}}(\vec{x}^{\,(0)})$ is small on a $(\sigma + b + 1)$-dimensional space. Moreover, it states that

$$\vec{x}^{\,(0)} \in K_b(E_1, \dots, E_{b+1}) \quad \text{and} \quad E_{b+1} < C^{-1} E_b,$$

so that the $(b + 1)$st eigenvalue of the matrix $H_c(\vec{x}^{\,(0)})$ is about C times smaller than the bth eigenvalue. Thus (II)$_k$ gives us a point $\vec{x}^{\,(0)}$ for which in some sense $J_{\vec{D}^{(c,b+1)}}(\vec{x}^{\,(0)})$ is close to having a kernel of dimension at least $(\sigma + b + 1)$ and $H_c(\vec{x}^{\,(0)})$ is close to having rank b.

The third case (III) is less directly comparable to step (2). We suggest that it could correspond to the case $b = 0$ of step (2).

Proof of Lemma 4.1. The proof is by induction on k. Let c, C, B, and σ be fixed.

The case $k = 0$. Let $k = 0$, let $CB \ge E_1 \ge 1$ and suppose that alternative (III) does not hold. We claim that alternative (I)$_0$ holds, that is, $K_0(E_1)$ is covered by $O_{C,n}(B^\sigma / E_1^\sigma)$ boxes of side E_1.

As (III) is false, applying Lemma 3.2 (iii) to the matrix of the linear map $\vec{x} \mapsto H_c(\vec{x})$ shows that there is an $(n - \sigma)$-dimensional subspace V of $\mathbb{R}^n$ with

$$\|H_c(\vec{v})\|_\infty \gg_n C^{-1}\|\vec{v}\|_\infty \quad \text{for all } \vec{v} \in V. \tag{4.4}$$

For each $\vec{z} \in \mathbb{R}^n$, let $A_0(\vec{z})$ be the box in $\mathbb{R}^n$ defined by

$$A_0(\vec{z}) = \{\vec{z} + \vec{u} + \vec{v} : \vec{u} \in V^\perp,\ \vec{v} \in V,\ \|\vec{u}\|_\infty \le E_1,\ \|\vec{v}\|_\infty \le B\}.$$

Now $K_0(E_1)$ is contained in the box $\|\vec{x}\|_\infty \le B$. It follows that we can cover $K_0(E_1)$ with a collection of $O_{C,n}(B^\sigma / E_1^\sigma)$ boxes of the form $A_0(\vec{z})$, each one of which is centred at a point $\vec{z}$ belonging to $K_0(E_1)$. We will show below that for each $\vec{z} \in K_0(E_1)$, the intersection $A_0(\vec{z}) \cap K_0(E_1)$ is contained in a box of side $O_{C,n}(E_1)$. It follows that $K_0(E_1)$ is covered by $O_{C,n}(B^\sigma / E_1^\sigma)$ boxes of side E_1, as claimed.

It remains to let $\vec{z} \in K_0(E_1)$ and let $\vec{y} \in A_0(\vec{z}) \cap K_0(E_1)$, and to deduce that the bound $\|\vec{y} - \vec{z}\|_\infty \ll_{C,n} E_1$ must hold.

By the definition of $K_0(E_1)$ we have $|\lambda_{c,1}(\vec{y})| \leq E_1$ and $|\lambda_{c,1}(\vec{z})| \leq E_1$, and the bounds $\|H_c(\vec{y})\|_\infty \ll_n E_1$ and $\|H_c(\vec{z})\|_\infty \ll_n E_1$ follow by (1.10). So we have

$$\|H_c(\vec{y} - \vec{z})\|_\infty \ll_n E_1. \tag{4.5}$$

Let $\vec{u} \in V^\perp$ and let $\vec{v} \in V$ such that $\vec{y} = \vec{z} + \vec{u} + \vec{v}$ holds. Since $\vec{y}$ lies in $A_0(\vec{z})$, we have $\|\vec{u}\|_\infty \leq E_1$, and with (4.5) this implies that

$$\|H_c(\vec{v})\|_\infty \ll_n E_1.$$

By (4.4) it follows that $\|\vec{v}\|_\infty \ll_n CE_1$, and hence that $\|\vec{y} - \vec{z}\|_\infty \ll_{C,n} E_1$, as claimed.

The inductive step. Let $k \geq 1$ and $CB \geq E_1 \geq \cdots \geq E_{k+1} \geq 1$. We suppose that $(\mathrm{II})_k$ and (III) are both false, and claim that $(\mathrm{I})_k$ holds.

By induction, at least one of $(\mathrm{I})_{k-1}$, $(\mathrm{II})_{k-1}$, or (III) holds. Note that of these (III) is false by assumption, and $(\mathrm{II})_{k-1}$ is false since it implies $(\mathrm{II})_k$, and so $(\mathrm{I})_{k-1}$ must hold.

Suppose for the time being that

$$E_{k+1} < C^{-1}E_k. \tag{4.6}$$

The contrary case is almost trivial and will be dealt with at the end of the proof. We claim that

$$K_k(E_1, \dots, E_{k+1}) = \bigcup_V K_k^{(C,V)}(E_1, \dots, E_{k+1}), \tag{4.7}$$

where V runs over those $(n - \sigma - k)$-dimensional subspaces of $\mathbb{R}^n$ which are spanned by standard basis vectors, and we define

$$\begin{aligned} &K_k^{(C,V)}(E_1, \dots, E_{k+1}) \\ &\quad = \big\{\vec{x} \in K_k(E_1, \dots, E_{k+1}) : \|J_{\vec{D}^{(c,k+1)}}(\vec{x})\vec{v}\|_\infty \geq C^{-1}\|\vec{D}^{(c,k)}(\vec{x})\|_\infty\|\vec{v}\|_\infty \\ &\qquad\qquad \text{for all } \vec{v} \in V\big\}. \end{aligned} \tag{4.8}$$

We have assumed that $E_{k+1} < C^{-1}E_k$ and that the case $b = k$ of $(\mathrm{II})_k$ is false. So the case $b = k$ of (4.2) must be false for every $\vec{x}^{\,(0)} \in K_k(E_1, \dots, E_{k+1})$ and every $(\sigma + b + 1)$-dimensional subspace X of $\mathbb{R}^n$.

That is, for any $\vec{x}^{\,(0)} \in K_k(E_1, \dots, E_{k+1})$ and any $(\sigma + k + 1)$-dimensional linear subspace X of $\mathbb{R}^n$, we must have

$$\|J_{\vec{D}^{(c,k+1)}}(\vec{x}^{\,(0)})\vec{X}\|_\infty > C^{-1}\|\vec{D}^{(c,k)}(\vec{x}^{\,(0)})\|_\infty\|\vec{X}\|_\infty$$

for some $\vec{X} \in X$. Applying Lemma 3.2 (iii) with the choice $M = J_{\vec{D}^{(c,k+1)}}(\vec{x}^{\,(0)})$ shows that for each $\vec{x}^{\,(0)} \in K_k(E_1, \dots, E_{k+1})$ there is an $(n - \sigma - k)$-dimensional subspace V of $\mathbb{R}^n$, spanned by standard basis vectors, such that

$$\|J_{\vec{D}^{(c,k+1)}}(\vec{x}^{\,(0)})\vec{v}\|_\infty \geq C^{-1}\|\vec{D}^{(c,k)}(\vec{x}^{\,(0)})\|_\infty\|\vec{v}\|_\infty \quad \text{for all } \vec{v} \in V. \tag{4.9}$$

This proves (4.7). So to prove $(\mathrm{I})_k$ it suffices to show that for each $(n - \sigma - k)$-dimensional space V, the set (4.8) is covered by a union of $O_{C,n}(B^\sigma(E_1 \cdots E_{k+1})E_{k+1}^{-\sigma-k-1})$ boxes of side E_{k+1}.

Let $\epsilon > 0$ be a sufficiently small constant depending at most on C and n, and for each $\vec{z} \in \mathbb{R}^n$ set

$$A_k(\vec{z}) = \{\vec{z} + \vec{u} + \vec{v} : \vec{u} \in V^{\perp},\ \vec{v} \in V,\ \|\vec{u}\|_\infty \leq E_{k+1},\ \|\vec{v}\|_\infty \leq \epsilon E_k\}. \tag{4.10}$$

Recall from the start of this inductive step that $(\mathrm{I})_{k-1}$ holds, and so $K_{k-1}(E_1, \dots, E_k)$ is covered by a collection of $O_{C,n}(B^\sigma(E_1 \cdots E_k)E_k^{-\sigma-k})$ boxes of side E_k. We can subdivide each of these boxes into $O_{C,n}(E_k^{\sigma+k}/E_{k+1}^{\sigma+k})$ sub-boxes of the form $A_k(\vec{z})$. Since

$$K_k^{(C,V)}(E_1, \dots, E_{k+1}) \subset K_{k-1}(E_1, \dots, E_k),$$

it follows that the set $K_k^{(C,V)}(E_1, \dots, E_{k+1})$ may be covered by a collection of no more than $O_{C,n}(B^\sigma(E_1 \cdots E_{k+1})E_{k+1}^{-\sigma-k-1})$ boxes of the form $A_k(\vec{z})$, each of which is centred at a point $\vec{z}$ belonging to the set $K_k^{(C,V)}(E_1, \dots, E_{k+1})$. We will show below that for each such box $A_k(\vec{z})$, the intersection $A_k(\vec{z}) \cap K_k^{(C,V)}(E_1, \dots, E_{k+1})$ is covered by a box of side $O_{C,n}(E_{k+1})$. It follows that each set (4.8) is covered by $O_{C,n}(B^\sigma(E_1 \cdots E_{k+1})E_{k+1}^{-\sigma-k-1})$ boxes of side E_{k+1}, and by the comments after (4.9) this proves the lemma.

Let $\vec{z} \in K_k^{(C,V)}(E_1, \dots, E_{k+1})$ and let $\vec{y} \in A_k(\vec{z}) \cap K_k^{(C,V)}(E_1, \dots, E_{k+1})$. The claim is that $\|\vec{y} - \vec{z}\|_\infty \ll_{C,n} E_{k+1}$ holds. Let $\vec{u} \in V^{\perp}$ and let $\vec{v} \in V$ such that $\vec{y} = \vec{z} + \vec{u} + \vec{v}$, and note that since $\vec{y} \in A_k(\vec{z})$, we have

$$\|\vec{u}\|_\infty \leq E_{k+1}, \quad \|\vec{v}\|_\infty \leq \epsilon E_k. \tag{4.11}$$

Now the jth partial derivatives of the $(k+1) \times (k+1)$ minors $\vec{D}^{(c,k+1)}(\vec{x})$ are linear combinations of the minors $\vec{D}^{(c,k+1-j)}(\vec{x})$ with coefficients of size at most $O_n(1)$. So we have

$$\left\| \frac{\partial^j \vec{D}^{(c,k+1)}(\vec{x})}{\partial x_{i_1} \cdots \partial x_{i_j}} \right\|_\infty \ll_n \|\vec{D}^{(c,k+1-j)}(\vec{x})\|_\infty,$$

and Taylor expansion shows that

$$\begin{aligned}
&\vec{D}^{(c,k+1)}(\vec{z} + \vec{u} + \vec{v}) - \vec{D}^{(c,k+1)}(\vec{z}) \\
&\quad = J_{\vec{D}^{(c,k+1)}}(\vec{z}).(\vec{u} + \vec{v}) + O_n\big(\|\vec{u} + \vec{v}\|_\infty^2 \|\vec{D}^{(c,k-1)}(\vec{z})\|_\infty + \cdots \\
&\qquad + \|\vec{u} + \vec{v}\|_\infty^k \|\vec{D}^{(c,1)}(\vec{z})\|_\infty + \|\vec{u} + \vec{v}\|_\infty^{k+1}\big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|J_{\vec{D}^{(c,k+1)}}(\vec{z})\vec{v}\|_\infty \ll_n\ & \|\vec{D}^{(c,k+1)}(\vec{y})\|_\infty + \|\vec{D}^{(c,k+1)}(\vec{z})\|_\infty \\
&+ \|\vec{u}\|_\infty \|\vec{D}^{(c,k)}(\vec{z})\|_\infty + \cdots \\
&+ \|\vec{u}\|_\infty^k \|\vec{D}^{(c,1)}(\vec{z})\|_\infty + \|\vec{u}\|_\infty^{k+1} \\
&+ \|\vec{v}\|_\infty^2 \|\vec{D}^{(c,k-1)}(\vec{z})\|_\infty + \cdots \\
&+ \|\vec{v}\|_\infty^k \|\vec{D}^{(c,1)}(\vec{z})\|_\infty + \|\vec{v}\|_\infty^{k+1}.
\end{aligned} \tag{4.12}$$

Since $\vec{y}, \vec{z} \in K_k(E_1, \dots, E_{k+1})$, Lemma 3.2 (i) gives us the bounds

$$\|\vec{D}^{(c,j)}(\vec{z})\|_\infty \asymp_n \prod_{i=1}^{j} E_i, \quad \|\vec{D}^{(c,k+1)}(\vec{y})\|_\infty \asymp_n \prod_{i=1}^{k+1} E_i, \tag{4.13}$$

and since $\vec{z} \in K_k^{(C,V)}(E_1, \dots, E_{k+1})$, it follows from (4.8) that

$$\|J_{\vec{D}^{(c,k+1)}}(\vec{z})\vec{v}\|_\infty \gg_n C^{-1}\|\vec{v}\|_\infty \prod_{i=1}^{k} E_i. \tag{4.14}$$

Substituting (4.13) and (4.14) into (4.12) yields

$$C^{-1}\|\vec{v}\|_\infty \prod_{i=1}^{k} E_i \ll_n \prod_{i=1}^{k+1} E_i + \|\vec{v}\|_\infty^2 \prod_{i=1}^{k-1} E_i + \cdots + \|\vec{v}\|_\infty^k E_1 + \|\vec{v}\|_\infty^{k+1}$$
$$+ \|\vec{u}\|_\infty \prod_{i=1}^{k} E_i + \cdots + \|\vec{u}\|_\infty^k E_1 + \|\vec{u}\|_\infty^{k+1}.$$

Applying the bounds from (4.11) and the inequalities $E_1 \geq \cdots \geq E_{k+1}$, we deduce that

$$C^{-1}\|\vec{v}\|_\infty \prod_{i=1}^{k} E_i \ll_n \prod_{i=1}^{k+1} E_i + \epsilon\|\vec{v}\|_\infty \prod_{i=1}^{k} E_i.$$

Since ϵ is assumed to be small in terms of C and n, it follows that $\|\vec{v}\|_\infty \ll_n CE_{k+1}$ holds and hence that $\|\vec{y} - \vec{z}\|_\infty \ll_{C,n} E_{k+1}$ holds. By the comments after (4.10), this proves the lemma.

It remains to consider the case when (4.6) is false, that is, when $E_{k+1} \geq C^{-1}E_k$ holds. At the start of the inductive step we supposed that $(\mathrm{I})_{k-1}$ holds, so the set $K_{k-1}(E_1, \dots, E_k)$ may be covered by $O_{C,n}(B^\sigma(E_1 \cdots E_k)E_k^{-\sigma-k})$ boxes of side E_k. We have

$$K_k(E_1, \dots, E_{k+1}) \subset K_{k-1}(E_1, \dots, E_k),$$

and so the set $K_k(E_1, \dots, E_{k+1})$ is also covered by this collection of boxes. Since the estimate $E_{k+1} \geq C^{-1}E_k$ holds, we can divide each of these boxes into $O_{C,n}(1)$ boxes of side E_{k+1}. This proves $(\mathrm{I})_k$. □

5. Small values of a trilinear form

Part (3) of Davenport's argument from Section 1.4 starts from a point $\vec{x}$ for which the matrices $H_c(\vec{x})$ and $J_{\vec{D}^{(c,b+1)}}(\vec{x})$ have prescribed ranks, and finds linear spaces X, Y such that the equation

$$\vec{Y}^T H_c(\vec{X})\vec{Y}' = 0$$

holds for all $\vec{X} \in X$ and $\vec{Y}, \vec{Y}' \in Y$. Our analogue is the following pair of results, which give linear spaces on which the trilinear form $\vec{Y}^T H_c(\vec{t})\vec{Y}'$ is small. One may recover Davenport's result by setting $b = \operatorname{rank} H_c(\vec{x}^{\,(0)})$, $X = \ker J_{\vec{D}^{(c,b+1)}}(\vec{x}^{\,(0)})$ and restricting $\vec{t}$ to lie in the space X

Lemma 5.1. *Let $c(\vec{x})$ be as in Definition* 1.3*, and let $\lambda_{c,i}(\vec{x})$ and $J_{\vec{D}^{(c,i)}}(\vec{x})$ be as in Definition* 1.6*. Suppose that $b \in \{1, \dots, n-1\}$ and that $\vec{x}^{\,(0)} \in \mathbb{R}^n$. Then, provided $\vec{D}^{(c,b)}(\vec{x}^{\,(0)})$ is nonzero, there exists an $(n-b)$-dimensional linear subspace Y of $\mathbb{R}^n$ such that for all $\vec{Y}, \vec{Y}' \in Y$ and all $\vec{t} \in \mathbb{R}^n$ we have*

$$\vec{Y}^T H_c(\vec{t})\vec{Y}' \ll_n \left(\frac{\|J_{\vec{D}^{(c,b+1)}}(\vec{x}^{\,(0)})\vec{t}\|_\infty}{\|\vec{D}^{(c,b)}(\vec{x}^{\,(0)})\|_\infty} + \frac{|\lambda_{c,b+1}(\vec{x}^{\,(0)})| \cdot \|\vec{t}\|_\infty}{|\lambda_{c,b}(\vec{x}^{\,(0)})|} \right) \|\vec{Y}\|_\infty \|\vec{Y}'\|_\infty. \tag{5.1}$$

We prove Lemma 5.1 at the end of this section, after deducing

Corollary 5.2. *Let c, $H_c(\vec{x})$ and $N_c^{\mathrm{aux}}(B)$ be as in Definition* 1.3. *For any $B, C \geq 1$ and any $\sigma \in \{0, \dots, n-1\}$, one of the following alternatives holds: Either*

$$N_c^{\mathrm{aux}}(B) \ll_{C,n} B^{n+\sigma}(\log B)^n, \tag{5.2}$$

or else there exist positive-dimensional linear subspaces X and Y of $\mathbb{R}^n$, satisfying the condition $\dim X + \dim Y = n + \sigma + 1$, *such that*

$$|\vec{Y}^T H_c(\vec{X})\vec{Y}'| \ll_n C^{-1}\|\vec{Y}\|_\infty\|\vec{X}\|_\infty\|\vec{Y}'\|_\infty \quad \textit{for all } \vec{X} \in X,\ \vec{Y}, \vec{Y}' \in Y. \tag{5.3}$$

Proof. Lemma 4.1 shows that for any $k \in \{0, \dots, n-\sigma-1\}$ and any $E_1, \dots, E_{k+1} \in \mathbb{R}$ satisfying

$$CB \geq E_1 \geq \cdots \geq E_{k+1} \geq 1,$$

one of $(\mathrm{I})_k$, $(\mathrm{II})_k$, or (III) must hold. Suppose first that in every case alternative $(\mathrm{I})_k$ holds. By (4.1), we then have

$$\#\{\mathbb{Z}^n \cap K_k(E_1, \dots, E_{k+1})\} \ll_{C,n} B^\sigma(E_1 \cdots E_{k+1})E_{k+1}^{n-\sigma-k-1} \tag{5.4}$$

for every $k \in \{0, \dots, n-\sigma-1\}$ and every $CB \geq E_1 \geq \cdots \geq E_{k+1} \geq 1$. Now Corollary 2.3 shows that either

$$\frac{N_c^{\mathrm{aux}}(B)}{B^n(\log B)^n} \ll_n \#\{\mathbb{Z}^n \cap K_0(1)\}, \tag{5.5}$$

or

$$\frac{2^{e_1+\cdots+e_k}N_c^{\mathrm{aux}}(B)}{B^n(\log B)^n} \ll_n \#\{\mathbb{Z}^n \cap K_k(2^{e_1}, \dots, 2^{e_k}, 1)\}, \tag{5.6}$$

where $k \in \{1, \dots, n-1\}$ and the inequalities $B \gg_n 2^{e_1} \geq \cdots \geq 2^{e_{k+1}} \geq 1$ hold, or

$$\frac{2^{e_1+\cdots+e_n}N_c^{\mathrm{aux}}(B)}{B^n(\log B)^n} \ll_n \#\{\mathbb{Z}^n \cap K_{n-1}(2^{e_1}, \dots, 2^{e_n})\}, \tag{5.7}$$

where the inequalities $B \gg_n 2^{e_1} \geq \cdots \geq 2^{e_n} \geq 1$ hold. We may assume that C is sufficiently large in terms of n, so that $CB \geq 2^{e_1}$ holds in (5.6)–(5.7). Substituting the bound (5.4) into each of (5.5)–(5.7) proves the conclusion (5.2).

Suppose next that alternative (III) holds in Lemma 4.1. In this case we let $Y = \mathbb{R}^n$, and the conclusion (5.3) follows from (4.3).

It remains to treat the case when there exist $k \in \{0, \dots, n-\sigma-1\}$ and $E_1, \dots, E_{k+1} \in \mathbb{R}$ satisfying $CB \geq E_1 \geq \cdots \geq E_{k+1} \geq 1$ such that alternative $(\mathrm{II})_k$ holds in Lemma 4.1. This means that there exist an integer b with $1 \leq b \leq k$, a point $\vec{x}^{\,(0)} \in K_b(E_1, \dots, E_{b+1})$, and a $(\sigma+b+1)$-dimensional linear subspace X of $\mathbb{R}^n$ such that

$$E_{b+1} < C^{-1}E_b \tag{5.8}$$

and

$$\|J_{\vec{D}^{(c,b+1)}}(\vec{x}^{\,(0)})\vec{X}\|_\infty \leq C^{-1}\|\vec{D}^{(c,b)}(\vec{x}^{\,(0)})\|_\infty\|\vec{X}\|_\infty \quad \text{for all } \vec{X} \in X. \tag{5.9}$$

Since $\vec{x}^{(0)} \in K_b(E_1, \dots, E_{b+1})$, the inequalities $\frac{1}{2}E_i < \lambda_{c,i}(\vec{x}^{(0)}) \le E_i$ hold. Therefore (5.8) implies

$$\lambda_{c,b+1}(\vec{x}^{(0)}) < 2C^{-1}\lambda_{c,b}(\vec{x}^{(0)}). \tag{5.10}$$

Note that (5.10) implies that $\lambda_{c,b}(\vec{x}^{(0)}) \neq 0$ so by Lemma 3.2 (i) we have $\vec{D}^{(c,b)}(\vec{x}^{(0)}) \neq 0$. Hence we may apply Lemma 5.1. This gives us an $(n-b)$-dimensional space Y such that for all $\vec{Y}, \vec{Y}' \in Y$ and all $\vec{t} \in \mathbb{R}^n$ the bound (5.1) holds. Taking $\vec{t} = \vec{X}$ in (5.1) and substituting in the bounds (5.9) and (5.10) shows that (5.3) holds. Since $\dim X = \sigma + b + 1$ and $\dim Y = n - b$, we have $\dim X + \dim Y = n + \sigma + 1$, as required. □

Proof of Lemma 5.1. We imitate the proof of Lemma 3 in Davenport [12], which begins by considering the following easy "warm-up" problem. Suppose we were to look for $n - b$ linearly independent vectors $\vec{y}$ at which $H_c(\vec{x}^{(0)})\vec{y}$ vanishes. One approach would be as follows. One can construct matrices $L^{(i)}$, $M^{(i)}$ for $i = 1, \dots, n-b$, with entries in $\{0, \pm 1\}$, such that the vectors

$$\vec{y}^{(i)}(\vec{x}) = L^{(i)}\vec{D}^{(c,b)}(\vec{x}) \tag{5.11}$$

satisfy

$$H_c(\vec{x})\vec{y}^{(i)}(\vec{x}) = M^{(i)}\vec{D}^{(c,b+1)}(\vec{x}). \tag{5.12}$$

In particular, one can take the components of $\vec{y}^{(i)}(\vec{x})$ to be polynomials of the form $\pm D_j^{(b)}(\vec{x})$, and the components of $H_c(\vec{x})\vec{y}^{(i)}(\vec{x})$ to be polynomials of the form $\pm D_j^{(b+1)}(\vec{x})$.

If $H_c(\vec{x}^{(0)})\vec{y} = \vec{0}$ had exactly $n - b$ linearly independent solutions $\vec{y}$, we would have $\vec{D}^{(c,b+1)}(\vec{x}^{(0)}) = \vec{0}$, while $\vec{D}^{(c,b)}(\vec{x}^{(0)})$ would be nonzero. We would then have $n - b$ solutions $\vec{Y}^{(k)}$ defined by

$$\vec{Y}^{(k)} = \frac{\vec{y}^{(k)}(\vec{x}^{(0)})}{\|\vec{D}^{(c,b)}(\vec{x}^{(0)})\|_\infty} \quad (1 \le k \le n-b), \tag{5.13}$$

and if we chose our matrices $L^{(i)}$, $M^{(i)}$ appropriately, these would be linearly independent.

We now return to the proof of the lemma. Assume for the time being that $L^{(i)}$, $M^{(i)}$, $\vec{y}^{(i)}(\vec{x})$ and $\vec{Y}^{(i)}$ satisfying (5.11)–(5.13) are given, and let $\vec{x}^{(0)}$ be as in the lemma. Let $\vec{t} \in \mathbb{R}^n$. Let $\partial_{\vec{t}}$ be the directional derivative along $\vec{t}$ defined by $\sum t_i \frac{\partial}{\partial x_i}$, and apply $\partial_{\vec{t}}$ to both sides of (5.12). This shows that

$$[\partial_{\vec{t}} H_c(\vec{x})]\vec{y}^{(i)}(\vec{x}) + H_c(\vec{x})[\partial_{\vec{t}}\vec{y}^{(i)}(\vec{x})] = M^{(i)}[\partial_{\vec{t}}\vec{D}^{(c,b+1)}(\vec{x})]. \tag{5.14}$$

Now we have

$$\partial_{\vec{t}}\vec{D}^{(c,k)}(\vec{x}) = J_{\vec{D}^{(c,k)}}(\vec{x})\vec{t},$$

and together with (5.11) and (5.14) this shows that

$$H_c(\vec{t})\vec{y}^{(i)}(\vec{x}) = M^{(i)}J_{\vec{D}^{(c,b+1)}}(\vec{x})\vec{t} - H_c(\vec{x})L^{(i)}\partial_{\vec{t}}\vec{D}^{(c,b)}(\vec{x}).$$

Premultiplying by $\vec{y}^{(j)}(\vec{x})^T$ and using (5.12) gives

$$\begin{aligned}\vec{y}^{(j)}(\vec{x})^T H_c(\vec{t})\vec{y}^{(i)}(\vec{x}) = {} & \vec{y}^{(j)}(\vec{x})^T M^{(i)} J_{\vec{D}^{(c,b+1)}}(\vec{x})\vec{t} \\ & - [M^{(i)}\vec{D}^{(c,b+1)}(\vec{x})]^T[L^{(i)}\partial_{\vec{t}}\vec{D}^{(c,b)}(\vec{x})].\end{aligned} \tag{5.15}$$

Now Lemma 3.2 (i) shows that

$$\frac{\|\vec{D}^{(c,b+1)}(\vec{x})\|_\infty}{\|\vec{D}^{(c,b)}(\vec{x})\|_\infty} \ll_n |\lambda_{c,b+1}(\vec{x})|, \quad \frac{\|\partial_{\vec{t}}\vec{D}^{(c,b)}(\vec{x})\|_\infty}{\|\vec{D}^{(c,b)}(\vec{x})\|_\infty} \ll_n \frac{\|\vec{t}\|_\infty}{|\lambda_{c,b}(\vec{x})|}$$

and substituting these bounds into (5.15) gives

$$\vec{Y}^{(j)T} H_c(\vec{t})\vec{Y}^{(i)} \ll_n \frac{\|J_{\vec{D}^{(c,b+1)}}(\vec{x}^{(0)})\vec{t}\|_\infty}{\|\vec{D}^{(c,b)}(\vec{x}^{(0)})\|_\infty} + \frac{|\lambda_{c,b+1}(\vec{x}^{(0)})| \cdot \|\vec{t}\|_\infty}{|\lambda_{c,b}(\vec{x}^{(0)})|}, \tag{5.16}$$

where the $\vec{Y}^{(k)}$ are as in (5.13).

The idea is now to let Y be the span of the $\vec{Y}^{(k)}$ and deduce (5.1) from (5.16). Since we are looking for an $(n-b)$-dimensional space Y, we will need $\vec{Y}^{(1)}, \dots, \vec{Y}^{(n-b)}$ to be linearly independent. In order to prove (5.1) we require the following slightly stronger statement. We claim there are $L^{(i)}$, $M^{(i)}$, $\vec{y}^{(i)}(\vec{x})$ and $\vec{Y}^{(i)}$ satisfying (5.11)–(5.13) such that the linear combination defined by $\vec{Y} = \sum_{i=1}^{n-b} \gamma_i \vec{Y}^{(i)}$ satisfies $\|\vec{\gamma}\|_\infty \ll_n \|\vec{Y}\|_\infty$ for every vector $\vec{\gamma}$ in real $(n-b)$-space. The lemma then follows, with Y being the span of the $\vec{Y}^{(i)}$, on expressing $\vec{Y}, \vec{Y}'$ as linear combinations of the $\vec{Y}^{(i)}$ and applying (5.16).

For the remainder of the proof we will assume for simplicity that the $b \times b$ minor of $H_c(\vec{x}^{(0)})$ with largest absolute value is the minor in the lower right-hand corner, that is, we will assume that

$$\|\vec{D}^{(c,b)}(\vec{x}^{(0)})\|_\infty = \left|\det\Big((H_c(\vec{x}^{(0)})_{k\ell})_{\substack{k=n-b+1,\dots,n\\ \ell=n-b+1,\dots,n}}\Big)\right|. \tag{5.17}$$

In general (5.17) holds after permuting the rows and columns of the matrix $H_c(\vec{x})$ and one can then apply the same permutations throughout the rest of our construction of $\vec{Y}^{(i)}$, every time the matrix $H_c(\vec{x})$ appears.

Define $\vec{y}^{(1)}(\vec{x}), \dots, \vec{y}^{(n-b)}(\vec{x})$ by

$$y_j^{(i)}(\vec{x}) = \begin{cases} (-1)^{n-b} \det\Big((H_c(\vec{x})_{k\ell})_{\substack{k=n-b+1,\dots,n\\ \ell=n-b+1,\dots,n}}\Big) & \text{if } j = i, \\ (-1)^j \det\Big((H_c(\vec{x})_{k\ell})_{\substack{k=n-b+1,\dots,n\\ \ell=i,n-b+1,\dots,n;\ \ell\neq j}}\Big) & \text{if } j > n-b, \\ 0 & \text{otherwise,} \end{cases}$$

where $(\ell = i, n-b+1, \dots, n;\ \ell \neq j)$ means that ℓ first takes the value i and then runs over the numbers $n-b+1, \dots, n$ with j omitted. Now this is of the form (5.11), and one can check that

$$(H_c(\vec{x})\vec{y}^{(i)}(\vec{x}))_j = \begin{cases} (-1)^{n-b} \det\Big((H_c(\vec{x})_{k\ell})_{\substack{k=j,n-b+1,\dots,n\\ \ell=i,n-b+1,\dots,n,}}\Big) & \text{if } j \leq n-b \\ 0 & \text{otherwise,} \end{cases}$$

which is of the form (5.12). Define a matrix Q by

$$Q = \left(\vec{Y}^{(1)} \,\middle|\, \cdots \,\middle|\, \vec{Y}^{(n-b)} \,\middle|\, \vec{e}^{(n-b+1)} \,\middle|\, \cdots \,\middle|\, \vec{e}^{(n)}\right),$$

or equivalently by

$$Q = \left(\frac{\vec{y}^{(1)}(\vec{x}^{(0)})}{\|\vec{D}^{(c,b)}(\vec{x}^{(0)})\|_\infty} \,\middle|\, \cdots \,\middle|\, \frac{\vec{y}^{(n-b)}(\vec{x}^{(0)})}{\|\vec{D}^{(c,b)}(\vec{x}^{(0)})\|_\infty} \,\middle|\, \vec{e}^{(n-b+1)} \,\middle|\, \cdots \,\middle|\, \vec{e}^{(n)}\right),$$

so that the entries Q_{ij} have absolute value at most 1. Then one sees from (5.17) that

$$Q = \begin{pmatrix} I_{n-b} & 0_{b\times b} \\ \widetilde{Q} & I_b \end{pmatrix},$$

where $\widetilde{Q}$ is some $(n-b)\times(n-b)$ matrix. In particular, $\det Q = 1$, and so the entries of Q^{-1} are bounded in terms of n. It follows that if $\vec{Y} = \sum_{i=1}^{n-b} \gamma_i \vec{Y}^{(i)}$, then

$$\gamma_i = (Q^{-1}\vec{Y})_i \ll_n \|\vec{Y}\|_\infty,$$

as claimed. □

6. Constructing singular points on $V(c)$

Corollary 5.2 shows that either $N_c^{\mathrm{aux}}(B)$ is small, or there are spaces X, Y of large dimension on which $\vec{Y}^T H_c(\vec{X})\vec{Y}'$ is small. To prove Proposition 1.5 we show that the second alternative implies that $V(c)$ is singular. This is our analogue of Davenport's step (4), as described in Section 1.4.

Proof of Proposition 1.5. Suppose for a contradiction that the result is false. Then for every $N \in \mathbb{N}$ there is $c_N \in \mathcal{K}$ with

$$N_{c_N}^{\mathrm{aux}}(B) \geq NB^{n+\sigma_{\mathcal{K}}}(\log B)^n.$$

By Corollary 5.2, this implies that there are linear subspaces X_N, Y_N of $\mathbb{R}^n$ such that

$$\dim X_N + \dim Y_N = n + \sigma_{\mathcal{K}} + 1$$

holds and for all $\vec{X} \in X_N$ and $\vec{Y}, \vec{Y}' \in Y_N$, we have

$$|\vec{Y}^T H_{c_N}(\vec{X})\vec{Y}'| \leq N^{-1}\|\vec{Y}\|_\infty \|\vec{X}\|_\infty \|\vec{Y}'\|_\infty.$$

If we multiply c_N by a constant, then the matrix $H_{c_N}(\vec{x})$ does not change. So we may assume that for each N the equality $\|c_N\|_\infty = 1$ holds. After passing to a subsequence, we have $c_N \to c$ as $N \to \infty$, and it follows that there are subspaces X, Y of $\mathbb{R}^n$ such that

$$\dim X + \dim Y = n + \sigma_{\mathcal{K}} + 1$$

and

$$\vec{Y}^T H_c(\vec{X})\vec{Y}' = 0 \quad \text{for all } \vec{X} \in X,\ \vec{Y}, \vec{Y}' \in Y. \tag{6.1}$$

Let $b \in \{0, \dots, n-\sigma-1\}$ such that

$$\dim X = n - b, \quad \dim Y = \sigma_{\mathcal{K}} + b + 1.$$

Let $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ be a basis of $\mathbb{R}^n$ such that $\vec{x}^{(b+1)}, \dots, \vec{x}^{(n)}$ is a basis of X.

Let $[Y]$ be the projective linear space in $\mathbb{P}^{n-1}_{\mathbb{R}}$ associated to Y. Take homogeneous co-ordinates $\vec{y}$ on $[Y]$, so that $\vec{y}$ takes values in Y.

Let W be the projective variety cut out in $[Y]$ by the b equations

$$W : \vec{y}^T H_c(\vec{x}^{(i)})\vec{y} = 0 \quad (i = 1, \dots, b), \tag{6.2}$$

so that

$$\dim W \geq \dim[Y] - b = \sigma_{\mathcal{K}}.$$

We claim that W is contained in the singular locus of the projective hypersurface $V(c)$. It follows that $\dim \operatorname{Sing} V(c) \geq \sigma_{\mathcal{K}}$, which is a contradiction, by (1.6).

Now (6.1) implies that for every $\vec{y} \in Y$ we have

$$\vec{y}^T H_c(\vec{x}^{(i)})\vec{y} = 0 \quad (i = b+1, \dots, n).$$

So if we let $\vec{y} \in Y$ such that (6.2) holds, then we have

$$\vec{y}^T H_c(\vec{x})\vec{y} = 0 \quad \text{for all } \vec{x} \in \mathbb{R}^n.$$

This implies that $\vec{\nabla}_{\vec{y}} c(\vec{y}) = \vec{0}$ holds, by the definition (1.4). It follows that every point of W is contained in $\operatorname{Sing} V(c)$, as claimed. □

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