Spanning forests and $OSP(N|2M)$-invariant σ-models

Sergio Caracciolo

Università di Milano, Dip. di Fisica and INFN, Sezione di Milano
Via Celoria 16, I-20133 Milano, ITALY
`Sergio.Caracciolo@mi.infn.it`

Alan D. Sokal

Department of Physics, New York University
4 Washington Place, New York, NY 10003, USA
and
Department of Mathematics, University College London
Gower Street, London WC1E 6BT, UK
`sokal@nyu.edu`

Andrea Sportiello

LIPN, UMR CNRS 7030, Université Paris-Nord
99, avenue J.-B. Clément, 93430 Villetaneuse, FRANCE
`Andrea.Sportiello@lipn.univ-paris13.fr`

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Dedicated to our friend Tony Guttmann
on the occasion of his ≈70th birthday

Abstract

The present paper is part of our ongoing work on $OSP(N|2M)$ supersymmetric σ-models, their relation with the Potts model at $q = 0$ and spanning forests, and the rigorous analytic continuation of the partition function as an entire function of $N - 2M$, a feature first predicted by Parisi and Sourlas in the 1970's. Here we accomplish two main steps. First, we analyze in detail the role of the Ising variables that arise when the constraint in the $OSP(1|2)$ model is solved, and we point out two situations in which the Ising and forest variables decouple. Second, we establish the analytic continuation for the $OSP(N|2M)$ model in some special cases: when the underlying graph is a forest, and for the Nienhuis action on a cubic graph. We also make progress in understanding the series-parallel graphs.

1 Introduction

Around the time of Tony's 60th birthday, we showed [19] (see also [20, 35]) how the generating function of spanning forests in a graph, which arises as the $q \to 0$ limit of the partition function of the q-state Potts model [39, 52, 53, 59], can be represented as a Grassmann integral involving a quadratic (Gaussian) term together with a special nearest-neighbor four-fermion interaction. More precisely, let G be a finite graph with vertex set V and edge set E, and let $\mathbf{w} = \{w_e\}_{e \in E}$ be a set of edge weights. We introduce at each vertex $i \in V$ a pair of Grassmann variables ψ_i, $\bar{\psi}_i$ obeying the usual rules for Grassmann integration [11, 62]. We then have the following identity:

$$\int \mathcal{D}(\psi, \bar{\psi}) \, \exp\Bigl[t \sum_{i \in V} \bar{\psi}_i \psi_i \;+\; \sum_{e = ij \in E} w_{ij} f_{ij}^{(t)} \Bigr] \;=\; \sum_{F \in \mathcal{F}(G)} \left(\prod_{e \in F} w_e \right) t^{k(F)} \tag{1.1a}$$

$$= \; t^{|V|} \sum_{F \in \mathcal{F}(G)} \left(\prod_{e \in F} \frac{w_e}{t} \right) , \tag{1.1b}$$

where $f_{ij}^{(t)}$ denotes the special combination of Grassmann variables

$$f_{ij}^{(t)} \;\stackrel{\rm def}{=}\; (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j) \;-\; t \, \bar{\psi}_i \psi_i \bar{\psi}_j \psi_j \tag{1.2}$$

(see also [20] for further discussion of the role played by this combination); here the sums on the right-hand side of (1.1) run over spanning forests F in G, and $k(F)$ denotes the number of connected components (i.e., the number of component trees) in the forest F. Furthermore, the fermionic model (1.1) possesses a hidden $OSP(1|2)$ supersymmetry.

In [20] the representation (1.1) was generalized to study spanning hyperforests in a hypergraph; and in [9, 10] explicit results were obtained for spanning hyperforests in the complete hypergraph.

In [19, 20] we also discussed briefly how the fermionic model (1.1) can be mapped, at least in perturbation theory[1], onto either of the following two models:

- An $OSP(1|2)$-invariant σ-model with spins taking values in the unit supersphere in $\mathbb{R}^{1|2}$: that is, a model with a superfield $\mathbf{n}_i = (\sigma_i, \psi_i, \bar{\psi}_i)$ where σ_i is a bosonic (real) variable and $\psi_i, \bar{\psi}_i$ are a pair of Grassmann variables, satisfying the constraint
$$\mathbf{n}_i \cdot \mathbf{n}_i \;\equiv\; \sigma_i^2 + 2t \bar{\psi}_i \psi_i \;=\; 1 \, . \tag{1.3}$$
Here the $OSP(1|2)$ supersymmetry is manifest.

- An $O(N)$-invariant σ-model with spins taking values in the unit sphere in $\mathbb{R}^N$ (also known as an N-vector model), analytically continued to $N = -1$.

[1] The perturbative renormalization flow for spanning forests in two dimensions is discussed in more detail in [17, 19].

In both cases the sign of the coupling in the σ-model is opposite to that in the spanning-forest model: thus, the spanning-forest model with positive weights ($w_e/t > 0$) corresponds to an *anti*ferromagnetic σ-model.

One aim of the present paper is to discuss these latter mappings in more detail, and to illustrate them in some simple examples. In particular we wish to clarify the stipulation "at least in perturbation theory", which arose in [19,20] from the fact that in solving the constraint (1.3) via

$$\sigma_i \;=\; \pm(1-2t\bar{\psi}_i\psi_i)^{1/2} \;=\; \pm(1-t\bar{\psi}_i\psi_i) \tag{1.4}$$

we took only the + sign and neglected the second solution. Here we shall look more carefully at the role of the Ising variables that arise from taking seriously the $\pm$ sign in (1.4).[2] That is, we shall solve the constraint (1.3) as

$$\sigma_i \;=\; \mu_i(1-t\bar{\psi}_i\psi_i) \tag{1.5}$$

where $\mu_i = \pm 1$ is an Ising variable, and study the model of Ising variables μ coupled to fermionic variables $\psi, \bar{\psi}$ that results from this substitution. We shall moreover take as our starting point an $OSP(1|2)$-invariant σ-model with arbitrary $OSP(1|2)$-invariant two-spin Boltzmann weights $W_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j)$. We shall show that an *exact* mapping between the $OSP(1|2)$-invariant σ-model and the fermionic model (1.1) can be obtained in certain cases where the Ising variables decouple from the fermionic variables, or equivalently from the sum over forests. In all other cases, the σ-model maps onto a model (4.13) in which the fermionic and Ising variables are coupled, or equivalently to a model (4.15) in which the forest sum and the Ising variables are coupled.

A second goal of the present paper is to study in more detail the connection between the $OSP(1|2)$-invariant σ-model and the $O(N)$-invariant σ-model analytically continued to $N=-1$. We view this as a special case of a more general connection between an $OSP(N|2M)$-invariant σ-model with spins taking values in the unit supersphere of $\mathbb{R}^{N|2M}$ — that is, a model with a superfield $\mathbf{n}_i = (\boldsymbol{\sigma}_i, \psi_i^{(1)}, \bar{\psi}_i^{(1)}, \ldots, \psi_i^{(M)}, \bar{\psi}_i^{(M)})$ where $\boldsymbol{\sigma}_i \in \mathbb{R}^N$ and $\psi_i^{(\alpha)}, \bar{\psi}_i^{(\alpha)}$ are Grassmann variables ($1 \le \alpha \le M$), satisfying the constraint

$$\mathbf{n}_i \cdot \mathbf{n}_i \;\equiv\; \boldsymbol{\sigma}_i^2 \,+\, 2\sum_{\alpha=1}^{M} \bar{\psi}_i^{(\alpha)}\psi_i^{(\alpha)} \;=\; 1 \tag{1.6}$$

— and an $O(N')$-invariant σ-model analytically continued to $N' = N-2M$. We hope in the future to be able to give (perhaps for Tony's 80th birthday) a mathematically rigorous proof of the following conjecture:

Conjecture 1.1 For $1 \le i \le n$, let $\mathbf{n}_i = (\boldsymbol{\sigma}_i, \psi_i^{(1)}, \bar{\psi}_i^{(1)}, \ldots, \psi_i^{(M)}, \bar{\psi}_i^{(M)})$ be superfields in $\mathbb{R}^{N|2M}$. Consider a generic integral

$$Z_f(N,M) \;=\; (2\pi)^{-nN/2} \int \mathcal{D}\boldsymbol{\sigma}\; \mathcal{D}(\bar{\psi},\psi)\; f(\{\mathbf{n}_i \cdot \mathbf{n}_j\}) \prod_{i=1}^{n} \delta(\mathbf{n}_i \cdot \mathbf{n}_i - 1)\;, \tag{1.7}$$

[2] See also [35, section 2] for some earlier work on this issue.

where f is a smooth (C^∞) function of the matrix of scalar products

$$\mathbf{n}_i \cdot \mathbf{n}_j \;=\; \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \,+\, \sum_{\alpha=1}^{M} (\bar{\psi}_i^{(\alpha)} \psi_j^{(\alpha)} - \psi_i^{(\alpha)} \bar{\psi}_j^{(\alpha)}) \;. \tag{1.8}$$

Then we conjecture that

(a) the integral (1.7) can be given a rigorous nonperturbative definition for arbitrary integers $N \ge 1$ and $M \ge 0$;

(b) the resulting function $Z_f(N, M)$ has a natural analytic continuation to an entire analytic function of the complex variable N (for each integer $M \ge 0$); and

(c) this analytically continued function depends on N and M only via the combination $N - 2M$.

If true, this conjecture would make rigorous, in the context of lattice σ-models (at least on finitely many sites), the intuitive idea that "one pair of fermions equals -2 bosons" [15, 43, 46]. In particular, it would show that the case $N = 1$, $M = 1$ of an $OSP(1|2)$-invariant σ-model is equivalent to the case $N = -1$, $M = 0$ of an $O(N)$-invariant σ-model analytically continued to $N = -1$.

In the present paper we do not purport to prove Conjecture 1.1 in general, but we shall show, by explicit computation, that it holds in some special cases:

(i) When f factorizes in a forest-like fashion: that is, $f = \prod_{\langle ij \rangle \in F} W_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j)$ for some forest F and some smooth functions W_{ij}. See Sections 6 and 7.

(ii) More generally, when f factorizes in a series-parallel fashion: that is, $f = \prod_{\langle ij \rangle \in E(G)} W_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j)$ for some series-parallel graph G and some smooth functions W_{ij}. See Sections 8 and 9.

(iii) When f is a Boltzmann weight of Nienhuis [45] '$O(N)$ loop model' form: that is, $f = \prod_{\langle ij \rangle \in E(G)} (1 + \beta_{ij}\, \mathbf{n}_i \cdot \mathbf{n}_j)$ for some graph G of maximum degree ≤ 3. See Section 5.

More precisely, in cases (i) and (iii) we will fully verify Conjecture 1.1. Case (ii) is similar to case (i), but we face some technical obstructions, and some amount of additional work (not contained in the present paper) will be needed in order to construct the analytic continuation as an entire function of $N - 2M$ and verify its agreement with $Z_f(N, M)$ fro all $N \ge 1$ and $M \ge 0$.

The plan of this paper is as follows: In Section 2 we clarify the meaning of integrals over "superspace", and we prove a lemma concerning the handling of delta functions. In Section 3 we define the models to be considered. In Section 4 we discuss the solution of the constraint in $OSP(1|2)$ and analyze the role of the Ising variables. In Section 5 we discuss the special case of a Nienhuis action in the $OSP(N|2M)$ model. In Section 6 we compute the general one-link integral for the $OSP(N|2M)$

model, and we use this in Section 7 to compute the partition function whenever the underlying graph G is a forest. Similarly, in Section 8 we compute a reduction formula for two links in series, under the hypothesis $N \geq 2$, and we use this in Section 9 to produce a recursion formula for the partition function whenever G is a series-parallel graph. In Section 10 we describe a reduction formula from a generic $(k+1)$-body interaction W, in terms of a k-body interaction W', showing that the map from W to W' is a function of $N - 2M$ only, whenever $N \geq k$. In Section 11 we discuss in more detail the $\mathfrak{osp}(N|2M)$ supersymmetry, and show how it is nonlinearly realized on the unit supersphere when the constraint has been explicitly solved. In Section 12 we discuss some puzzles concerning the critical behavior of the spanning-forest and $OSP(1|2)$ models on regular lattices. Finally, in Appendix A we discuss the analytic continuation of the distribution $(1-x^2)_+^\lambda$, which plays an important role in this paper, and in Appendix B we discuss briefly the supergroup $OSP(N|2M)$.

2 Preliminaries on integration over superspace

2.1 Brief review of Grassmann algebra

Let us begin by clarifying what we mean by integration over a "superspace" $\mathbb{R}^{N|2M}$ parametrized by N bosonic variables $x_1, \ldots, x_N$ and M pairs of fermionic variables $\psi_1, \bar{\psi}_1, \ldots, \psi_M, \bar{\psi}_M$. We assume that the reader is familiar with the basic ideas of Grassmann algebras and Grassmann–Berezin integration, as presented, for instance, in the books [11, 62] or in the pedagogical exposition of [21, Appendix A]. We simply recall the following basic facts:

If R is a commutative ring with identity element, the Grassmann algebra $R[\chi_1, \ldots, \chi_n]_{\rm Grass}$ — which we henceforth abbreviate as $R[\chi]$ — is the noncommutative algebra with coefficients in R and generators $\chi_1, \ldots, \chi_n$ obeying the relations $\chi_i^2 = 0$ and $\chi_i \chi_j + \chi_j \chi_i = 0$ for all i, j.[3] Every element $f \in R[\chi]$ can be uniquely written in the form

$$f \;=\; \sum_{I \subseteq [n]} f_I \, \chi^I \tag{2.1}$$

with coefficients $f_I \in R$, where the sum runs over subsets $I = \{i_1, \ldots, i_p\}$ of $[n] = \{1, \ldots, n\}$ with $i_1 < \ldots < i_p$, and $\chi^I = \chi_{i_1} \cdots \chi_{i_p}$. Here the term $f_\varnothing$ that contains no factors χ_i is termed the *body* of f, and the rest $\sum_{I \neq \varnothing} f_I \, \chi^I$ is termed the *soul* of f. Every even element $f \in R[\chi]_{\rm even}$ commutes with the entire Grassmann algebra; every odd element $f \in R[\chi]_{\rm odd}$ is nilpotent of order 2 (i.e. $f^2 = 0$); and every "pure soul" element $f \in R[\chi]_+$ (i.e. one with $f_\varnothing = 0$) is nilpotent (i.e. $f^\ell = 0$ for some sufficiently large ℓ).[4]

If $\Phi(x) = \sum_{k=0}^\infty c_k x^k$ is a formal power series with coefficients in R or $\mathbb{Z}$, we can apply it to any $f \in R[\chi]_+$ because f is nilpotent and the sum is therefore

[3] If the coefficient ring R contains an element $\frac{1}{2}$ — as it will in all the cases considered here — then $\chi_i^2 = 0$ is of course a consequence of the $i = j$ case of $\chi_i \chi_j + \chi_j \chi_i = 0$.

[4] In [21, Proposition A.9] it is proven that ℓ is at most $\lfloor n/2 \rfloor + 2$.

finite; this defines $\Phi(f)$. More generally, if $\mathbf{x} = (x_1, \ldots, x_m)$ are indeterminates and $\Phi(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{N}^m} c_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$ is a formal power series with coefficients in R or $\mathbb{Z}$ (here $\mathbf{x}^{\mathbf{k}}$ is a shorthand for $\prod_{i=1}^m x_i^{k_i}$), we can apply it to any $f_1, \ldots, f_m \in R[\chi]_+ \cap R[\chi]_{\rm even}$ because all the f_i are nilpotent and $R[\chi]_{\rm even}$ is commutative; this defines $\Phi(f_1, \ldots, f_m)$. In particular, if $R = \mathbb{R}$ and F is an infinitely differentiable real-valued function on $\mathbb{R}^m$ and $\mathbf{x}_0$ is a point in $\mathbb{R}^m$, we can take Φ to be the Taylor expansion of F around the point $\mathbf{x}_0$; this allows us to define, by Taylor expansion, $F(f_1, \ldots, f_m)$ for arbitrary even elements $f_1, \ldots, f_m$ of the Grassmann algebra, by setting $\mathbf{x}_0$ equal to the body of $(f_1, \ldots, f_m)$ and expanding in powers of the soul (which is nilpotent). *We make the convention throughout this paper that $F(f_1, \ldots, f_m)$ is always to be interpreted in this way, whenever F is a function (or distribution) of m variables and $f_1, \ldots, f_m$ are even elements of a Grassmann algebra.* For instance, $F(a + b\chi_1\chi_2)$ (where F is a differentiable function of one variable and $a, b \in \mathbb{R}$) is in our convention simply a shorthand for $F(a) + bF'(a)\chi_1\chi_2$.

From now on we fix the coefficient ring R to be the real numbers $\mathbb{R}$. The Grassmann algebra $\mathbb{R}[\chi_1, \ldots, \chi_n]$ is therefore also a vector space over $\mathbb{R}$.

Now consider a function h from $\mathbb{R}^N$ to $\mathbb{R}[\chi_1, \ldots, \chi_n]$. We can write

$$h(\mathbf{x}) \;=\; \sum_{I \subseteq [n]} h_I(\mathbf{x})\, \chi^I \tag{2.2}$$

where $\mathbf{x} = (x_1, \ldots, x_N)$ and each h_I is a real-valued function. Since by definition

$$\int \mathcal{D}\chi\, \chi^I \;=\; \begin{cases} 1 & \text{if } I = [n] \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

we can define the "integral over superspace $\mathbb{R}^{N|n}$" as

$$\int d\mathbf{x}\, \mathcal{D}\chi\, h(\mathbf{x}) \;=\; \int d\mathbf{x}\, h_{[n]}(\mathbf{x}) \tag{2.4}$$

whenever the function $h_{[n]}$ is integrable. We also introduce formally the "superfield" $\mathbf{n} = (x_1, \ldots, x_N, \chi_1, \ldots, \chi_n)$ and abbreviate $d\mathbf{x}\, \mathcal{D}\chi = d\mathbf{n}$.

In the remainder of this paper, the Grassmann generators $\chi_1, \ldots, \chi_n$ will be the $2M$ fermionic variables $\psi_1, \bar\psi_1, \ldots, \psi_M, \bar\psi_M$. Thus, the integration measure $\mathcal{D}(\psi, \bar\psi)$ denotes

$$\mathcal{D}(\psi, \bar\psi) \;=\; \prod_{i=1}^M d\psi_i\, d\bar\psi_i \;, \tag{2.5}$$

where the terms in the product can be taken in any order because each pair $d\psi_i\, d\bar\psi_i$ is Grassmann-even. However, we shall sometimes use an integration measure $\mathcal{D}(\bar\psi, \psi)$ in which the $\bar\psi$ come first: thus

$$\mathcal{D}(\bar\psi, \psi) \;=\; \prod_{i=1}^M d\bar\psi_i\, d\psi_i \;=\; (-1)^M\, \mathcal{D}(\psi, \bar\psi) \;. \tag{2.6}$$

Roughly speaking, $\mathcal{D}(\psi,\bar{\psi})$ seems to be more natural in Gaussian Grassmann integrals and more generally in forest identities like (1.1), and hence is the convention used in [19–21]; while $\mathcal{D}(\bar{\psi},\psi)$ is more natural for defining $OSP(N|2M)$-invariant integrals like (1.7) and (3.3) because it avoids a sign $(-1)^M$ in (3.10). When comparing formulae using the two conventions we must not forget the sign $(-1)^M$ in (2.6)! [When we consider a model with many lattice sites i, we will have Grassmann generators $\psi_i^{(\alpha)},\bar{\psi}_i^{(\alpha)}$ for $1 \le \alpha \le M$ at each site i, and $\mathcal{D}(\psi,\bar{\psi})$ and $\mathcal{D}(\bar{\psi},\psi)$ will denote products over both i and α; then the sign in (2.6) will be $(-1)^{nM}$ where n is the number of lattice sites.]

The foregoing considerations now give an unambiguous meaning to "integrals over superspace". Consider, for instance, the integral (1.7)/(1.8) where f is a smooth (C^∞) function. Here each $\mathbf{n}_i \cdot \mathbf{n}_j$ is an even element of the Grassmann algebra, with "body" $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$. Therefore, in accordance with the convention adopted earlier in this section, $f(\{\mathbf{n}_i \cdot \mathbf{n}_j\})$ is an element of the Grassmann algebra defined by Taylor expansion around $\{\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\}$: that is, it is a mapping of the form (2.2) from the collection of real variables $\mathbf{x} = \{\boldsymbol{\sigma}_i\}$ to the Grassmann algebra, with coefficient functions h_I that are suitable derivatives of f. Furthermore, by the same convention, each $\delta(\mathbf{n}_i \cdot \mathbf{n}_i - 1)$ is to be interpreted in the same way, by Taylor expansion around $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_i$: that is, it is a distribution defined on the collection of real variables $\boldsymbol{\sigma}_i$ and taking values in the Grassmann algebra, with coefficient distributions h_I that are suitable derivatives of delta functions. Finally, the integral over $\mathbf{x}$ of the product of a smooth function and a distribution is defined in the usual way (that is, derivatives of delta functions act by integration by parts). It follows that the bosonic-fermionic integral $I_f(N,M)$ written in (1.7)/(1.8) is well-defined whenever f is a C^∞ function; in fact, $I_f(N,M)$ equals some specific distribution (a complicated sum of delta functions and their derivatives) acting on f. [If we were to consider instead a model with unbounded fields — that is, without delta functions to restrict the integration over $\{\boldsymbol{\sigma}_i\}$ to a bounded domain — then we would need to assume that f is C^∞ *and* that f and all its derivatives are rapidly decreasing (or at least integrable).]

2.2 A lemma on handling delta functions

In the $OSP(N|2M)$-invariant supersymmetric σ-models to be considered in this paper, we shall have to deal with delta functions of the form $\delta(\mathbf{n}_i \cdot \mathbf{n}_i - 1)$ where $\mathbf{n}_i = (\boldsymbol{\sigma}_i, \psi_i^{(1)}, \bar{\psi}_i^{(1)}, \ldots, \psi_i^{(M)}, \bar{\psi}_i^{(M)})$ is a superfield and the scalar product is defined as

$$\mathbf{n}_i \cdot \mathbf{n}_i \;\equiv\; \boldsymbol{\sigma}_i^2 \,+\, 2\sum_{\alpha=1}^{M} \bar{\psi}_i^{(\alpha)} \psi_i^{(\alpha)} \;. \tag{2.7}$$

In the special case $N = M = 1$, this becomes $\delta(\sigma_i^2 + 2\bar{\psi}_i\psi_i - 1)$. There are two ways of dealing with such delta functions: The first approach, which is the one demanded by the convention that we adopted in the preceding subsection, is to expand $\delta(\sigma_i^2+2\bar{\psi}_i\psi_i-1)$ in Taylor series around the argument's "body" σ_i^2-1, in powers of its "soul" $2\bar{\psi}_i\psi_i$. The second approach, which was already mentioned in the introduction [cf. (1.4)/(1.5)], is to formally solve the constraint $\sigma_i^2 + 2\bar{\psi}_i\psi_i - 1 = 0$ by writing

$\sigma_i = \mu_i(1 - \bar{\psi}_i\psi_i)$ where $\mu_i = \pm 1$. Here we shall prove a general lemma showing that the two approaches give the same result. This lemma will therefore provide the justification for our later use of the second approach, which leads to somewhat simpler computations.

Recall the well-known relation

$$\delta[h(x)] \;=\; \sum_i \frac{1}{|h'(x_i)|}\, \delta(x - x_i) \tag{2.8}$$

where x is a real variable, $h(x)$ is a smooth real-valued function, and the sum runs over the zeros x_i of h (we assume that all such zeros are simple, i.e. $h'(x_i) \neq 0$). This relation can equivalently be written as

$$\int dx\; g(x)\, \delta[h(x)] \;=\; \sum_i \frac{g(x_i)}{|h'(x_i)|} \tag{2.9}$$

for all smooth real-valued functions $g(x)$. We wish to show that (2.8)/(2.9) holds also when the functions $h(x)$ and $g(x)$ take values in the even subalgebra of a Grassmann algebra over $\mathbb{R}$: here the zeros x_i (which will now be even elements of the Grassmann algebra rather than just real numbers) will be defined by "solving the constraint" $h(x) = 0$, using the Taylor-expansion interpretation of $h(x)$ as demanded by the convention adopted in the preceding subsection; and $g(x)$, $\delta[h(x)]$, $g(x_i)$ and $h'(x_i)$ will likewise be defined by Taylor expansion in accordance with the same convention.

Let us first discuss how to define the zeros x_i of $h(x)$. By the convention adopted in the preceding subsection, $h(x)$ is to be interpreted, for an arbitrary even element x of the Grassmann algebra, by Taylor-expanding h around the "body" of x in powers of its "soul". But if we have already defined $h(x_0)$ in this way for some even element x_0 of the Grassmann algebra, we can then define $h(x)$ whenever $x - x_0$ is "pure soul" by Taylor-expanding around x_0; the two results must be the same. (All these Taylor expansions are finite by nilpotence, so it is a simple algebraic fact that they fit together in the expected way.) We will use this fact to define the zeros x_i of $h(x)$ by induction on the dimension of the Grassmann algebra. More precisely, consider a Grassmann algebra over $\mathbb{R}$ with generators $\chi_1, \ldots, \chi_n$. Now write

$$h(x) \;=\; \sum_{i=0}^{n} h_i(x) \tag{2.10}$$

where $h_0(x)$ is real-valued and $h_i(x)$ has a factor χ_i but no factors χ_j for $j > i$ [that is, $h_i(x) = \chi_i\psi_i(x)$ where $\psi_i(x)$ takes values in the odd part of the Grassmann subalgebra generated by $\chi_1, \ldots, \chi_{i-1}$.][5] Observe that this means in particular that $h_i(x)h_i(y) = 0$ for all x, y. Now define the partial sums

$$h^{(m)}(x) \;=\; \sum_{i=0}^{m} h_i(x)\;. \tag{2.11}$$

[5] Of course $h_1(x) = 0$, but we will not need to exploit this fact.

We will inductively define the zeros $x_i^{(m)}$ of $h^{(m)}(x)$, starting from the zeros $x_i^{(0)}$ of $h_0(x)$; the final value $x_i^{(n)}$ will be the desired zero x_i of $h(x)$. So consider some integer m with $1 \le m \le n$, and suppose that the zeros $x_i^{(m-1)}$ of $h^{(m-1)}(x)$ have already been defined. Writing

$$h^{(m)}(x) \;=\; h^{(m-1)}(x) \,+\, h_m(x)\;, \tag{2.12}$$

we Taylor-expand $h^{(m)}(x)$ around $x = x_i^{(m-1)}$, yielding

$$\begin{aligned} h^{(m)}(x) \;&=\; h_m(x_i^{(m-1)}) \,+\, \left[h^{(m-1)\prime}(x_i^{(m-1)}) + h'_m(x_i^{(m-1)})\right] (x - x_i^{(m-1)}) \\ &\qquad + O\big((x - x_i^{(m-1)})^2\big) \end{aligned} \tag{2.13}$$

since $h^{(m-1)}(x_i^{(m-1)}) = 0$ by the inductive hypothesis. Hence $h^{(m)}(x) = 0$ is solved by

$$x_i^{(m)} \;=\; x_i^{(m-1)} \;-\; \frac{h_m(x_i^{(m-1)})}{h^{(m-1)\prime}(x_i^{(m-1)}) + h'_m(x_i^{(m-1)})} \tag{2.14a}$$

$$\;=\; x_i^{(m-1)} \;-\; \frac{h_m(x_i^{(m-1)})}{h^{(m-1)\prime}(x_i^{(m-1)})} \tag{2.14b}$$

where the absence of higher-order terms in (2.14a) follows from $h_m(x_i^{(m-1)})^2 = 0$, while the second equality follows from $h_m(x_i^{(m-1)})\, h'_m(x_i^{(m-1)}) = 0$ which in turn follows from $h_m(x)h_m(y) = 0$. This completes the definition of the zeros x_i of $h(x)$. [6]

The proof of (2.8)/(2.9) will likewise be by induction on the dimension of the Grassmann algebra, using again the decomposition (2.10)/(2.11). Now, the convention adopted in the preceding subsection decrees that $\delta[h^{(m)}(x)]$ is defined by Taylor-expanding δ around the "body" of $h^{(m)}(x)$ — which is the same as the "body" of $h_0(x)$ — in powers of the "soul" of $h^{(m)}(x)$. But, by the same convention, $\delta[h^{(m-1)}(x)]$ is defined by Taylor-expanding δ around the "body" of $h^{(m-1)}(x)$ — which is again the same as the "body" of $h_0(x)$ — in powers of the "soul" of $h^{(m-1)}(x)$. If we have already carried out this latter expansion, then $\delta[h^{(m)}(x)]$ can equivalently be defined by Taylor-expanding δ around the point $h^{(m-1)}(x)$ in powers of $h_m(x)$; the two results must be the same. Moreover, this latter expansion terminates at first order because

[6] We shall get the same zeros if we perform the induction with the generators $\chi_1, \ldots, \chi_n$ in a different order. This is true although not easily verified. A possible proof consists in inspecting the result of *two* induction steps, and verify that the zeros $x_i^{(m)}$, expressed in terms of $x_i^{(m-2)}$, $h^{(m-2)}(x)$ and the monomials of h whose Grassmann variable of smallest index is either χ_m or χ_{m-1}, is symmetric in the indices m and $m-1$. Symmetry under consecutive transpositions, then, implies the purported full permutation symmetry.

$h_m(x)^2 = 0$. We therefore have

$$\delta[h^{(m)}(x)] \;=\; \delta[h^{(m-1)}(x)] \;+\; h_m(x)\,\frac{d}{dh^{(m-1)}}\delta[h^{(m-1)}(x)] \tag{2.15a}$$

$$\;=\; \delta[h^{(m-1)}(x)] \;+\; h_m(x)\,\frac{1}{h^{(m-1)\prime}(x)}\,\frac{d}{dx}\delta[h^{(m-1)}(x)] \tag{2.15b}$$

$$\;=\; \sum_i \frac{1}{|h^{(m-1)\prime}(x_i^{(m-1)})|}\left[\delta(x - x_i^{(m-1)}) \;+\; \frac{h_m(x)}{h^{(m-1)\prime}(x)}\,\delta'(x - x_i^{(m-1)})\right], \tag{2.15c}$$

where in the last step we twice applied the formula (2.8) for $h^{(m-1)}$, which is valid by the induction hypothesis.

Let us now use (2.15) to calculate an integral of the form

$$\int dx\; g(x)\,\delta[h^{(m)}(x)] \tag{2.16}$$

where $g(x)$ is a smooth function of x taking values in the even part of the whole Grassmann algebra. We have

$$\int dx\; g(x)\,\delta[h^{(m)}(x)] \tag{2.17a}$$

$$= \sum_i \frac{1}{|h^{(m-1)\prime}(x_i^{(m-1)})|}\left[g(x_i^{(m-1)}) \;-\; \frac{d}{dx}\,\frac{g(x)\,h_m(x)}{h^{(m-1)\prime}(x)}\bigg|_{x=x_i^{(m-1)}}\right] \tag{2.17b}$$

$$= \sum_i \frac{1}{|h^{(m-1)\prime}(x_i^{(m-1)})|}\left[g\Big(x_i^{(m-1)} - \frac{h_m(x_i^{(m-1)})}{h^{(m-1)\prime}(x_i^{(m-1)})}\Big) \;-\; g(x)\,\frac{d}{dx}\,\frac{h_m(x)}{h^{(m-1)\prime}(x)}\bigg|_{x=x_i^{(m-1)}}\right] \tag{2.17c}$$

$$= \sum_i \frac{1}{|h^{(m-1)\prime}(x_i^{(m-1)})|}\left[g(x_i^{(m)}) \;-\; g(x)\,\frac{d}{dx}\,\frac{h_m(x)}{h^{(m-1)\prime}(x)}\bigg|_{x=x_i^{(m-1)}}\right] \tag{2.17d}$$

$$= \sum_i \frac{g(x_i^{(m)})}{|h^{(m-1)\prime}(x_i^{(m-1)})|}\left[1 \;-\; \frac{d}{dx}\,\frac{h_m(x)}{h^{(m-1)\prime}(x)}\bigg|_{x=x_i^{(m-1)}}\right] \tag{2.17e}$$

where the first equality used (2.15c), the second equality used $h_m(x_i^{(m-1)})^2 = 0$, the third equality used (2.14b), and the fourth equality used (2.14b) and $h_m(x)h_m(y) = 0$.

On the other hand,

$$
\begin{aligned}
h^{(m)\prime}(x_i^{(m)}) &= h^{(m-1)\prime}(x_i^{(m)}) + h_m'(x_i^{(m)}) &\text{(2.18a)}\\
&= \bigl[h^{(m-1)\prime}(x_i^{(m-1)}) + h_m'(x_i^{(m-1)})\bigr] \\
&\qquad - \bigl[h^{(m-1)\prime\prime}(x_i^{(m-1)}) + h_m''(x_i^{(m-1)})\bigr] \frac{h_m(x_i^{(m-1)})}{h^{(m-1)\prime}(x_i^{(m-1)})} &\text{(2.18b)}\\
&= \bigl[h^{(m-1)\prime}(x_i^{(m-1)}) + h_m'(x_i^{(m-1)})\bigr] - h^{(m-1)\prime\prime}(x_i^{(m-1)}) \frac{h_m(x_i^{(m-1)})}{h^{(m-1)\prime}(x_i^{(m-1)})} &\text{(2.18c)}\\
&= h^{(m-1)\prime}(x_i^{(m-1)}) \left[1 + \frac{d}{dx} \frac{h_m(x)}{h^{(m-1)\prime}(x)}\bigg|_{x=x_i^{(m-1)}}\right] &\text{(2.18d)}
\end{aligned}
$$

where the first equality used (2.14b) and $h_m(x_i^{(m-1)})^2 = 0$, and the second equality used $h_m(x_i^{(m-1)})\, h_m''(x_i^{(m-1)}) = 0$ which in turn follows from $h_m(x)h_m(y) = 0$. Taking the reciprocal, we have

$$
\frac{1}{|h^{(m)\prime}(x_i^{(m)})|} = \frac{1}{|h^{(m-1)\prime}(x_i^{(m-1)})|} \left[1 - \frac{d}{dx} \frac{h_m(x)}{h^{(m-1)\prime}(x)}\bigg|_{x=x_i^{(m-1)}}\right] , \tag{2.19}
$$

again as a consequence of $h_m(x)h_m(y) = 0$. It follows that (2.17e) can be rewritten as

$$
\sum_i \frac{g(x_i^{(m)})}{|h^{(m)\prime}(x_i^{(m)})|} , \tag{2.20}
$$

which completes the inductive proof of (2.9).

3 Definition of general $OSP(N|2M)$ σ-model

In this paper we shall consider σ-models with spins taking values in the unit supersphere of $\mathbb{R}^{N|2M}$, with arbitrary $OSP(N|2M)$-invariant two-body interactions. More specifically, let $G = (V, E)$ be a finite graph with vertex set V and edge set E; we write $n = |V|$ for the number of vertices. We then introduce, at each vertex $i \in V$, a superfield $\mathbf{n}_i = (\boldsymbol{\sigma}_i, \psi_i^{(1)}, \bar\psi_i^{(1)}, \ldots, \psi_i^{(M)}, \bar\psi_i^{(M)})$ where $\boldsymbol{\sigma}_i \in \mathbb{R}^N$ is an N-component bosonic variable and $\psi_i^{(\alpha)}, \bar\psi_i^{(\alpha)}$ are Grassmann variables ($1 \le \alpha \le M$). We equip the "superspace" $\mathbb{R}^{N|2M}$ with the scalar product

$$
\mathbf{n}_i \cdot \mathbf{n}_j = \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j + \sum_{\alpha=1}^{M} (\bar\psi_i^{(\alpha)} \psi_j^{(\alpha)} - \psi_i^{(\alpha)} \bar\psi_j^{(\alpha)}) . \tag{3.1}
$$

We then consider the most general $OSP(N|2M)$-invariant two-body Boltzmann weight on the graph G, namely

$$
f(\{\mathbf{n}_i \cdot \mathbf{n}_j\}) = \prod_{\langle ij\rangle \in E} W_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j) \tag{3.2}
$$

where the functions W_{ij} are assumed smooth (C^∞) but are otherwise arbitrary. The partition function is therefore

$$Z \;=\; \int \mathcal{D}\mathbf{n} \prod_{\langle ij\rangle \in E} W_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j) \prod_{i\in V} \delta(\mathbf{n}_i \cdot \mathbf{n}_i - 1) \tag{3.3}$$

where

$$\mathcal{D}\mathbf{n} \;=\; \prod_{i\in V} d\mathbf{n}_i \tag{3.4}$$

and

$$d\mathbf{n}_i \;=\; d\boldsymbol{\sigma}_i \, d\bar{\psi}_i \, d\psi_i \;. \tag{3.5}$$

Note that we have used here the "$\bar{\psi}$-first" convention for the fermionic integration measure. In some instances we find it more convenient to use instead the "ψ-first" convention: we thus define

$$\widehat{Z} \;=\; (-1)^{nM} \, Z \tag{3.6}$$

[cf. (2.6)].

When discussing the analytic continuation in N and M, we find it convenient to introduce an additional factor $(2\pi)^{-nN/2}$:

$$Z' \;=\; (2\pi)^{-nN/2} \, Z \;. \tag{3.7}$$

We are then considering the generic integral (1.7)/(1.8), but restricted to the case in which the integrand f has the two-body form (3.2).

Let us say a bit more about the normalization of the single-site integrals. Consider first the case without fermions, i.e. $M = 0$ and hence $\mathbf{n}_i = \boldsymbol{\sigma}_i$. The (unnormalized) measure of integration with respect to $\boldsymbol{\sigma}_i$ appearing in (3.3) is $d\boldsymbol{\sigma}_i \, \delta(\boldsymbol{\sigma}_i^2 - 1)$, and this measure has total mass

$$\int\limits_{\mathbb{R}^N} d\boldsymbol{\sigma} \, \delta(\boldsymbol{\sigma}^2 - 1) \;=\; \int\limits_0^\infty dr \, r^{N-1} \, \frac{1}{2r} \delta(r-1) \int\limits_{S^{N-1}} d^{N-1}\Omega \;=\; \frac{\mathcal{S}_N}{2} \tag{3.8}$$

where $d^{N-1}\Omega$ is solid angle on the unit sphere in $\mathbb{R}^N$ and

$$\mathcal{S}_N \;=\; \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)} \tag{3.9}$$

is the surface area of the unit sphere in $\mathbb{R}^N$.

Now consider the general case with M pairs of fermions. The (unnormalized) measure of integration with respect to $\mathbf{n}_i$ appearing in (3.3) is $d\mathbf{n}_i \, \delta(\mathbf{n}_i^2 - 1)$, and, as we show in a moment, its total mass is

$$\int\limits_{\mathbb{R}^{N|2M}} d\mathbf{n} \, \delta(\mathbf{n}^2 - 1) \;=\; (2\pi)^M \, \frac{\mathcal{S}_{N-2M}}{2} \;=\; \frac{\mathcal{S}_N}{2} \, 2^M \, \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-2M}{2}\right)} \;=\; \frac{2^M \, \pi^{N/2}}{\Gamma\left(\frac{N-2M}{2}\right)} \;. \tag{3.10}$$

Let us give two derivations of (3.10). The first is based on "solving the constraint" as explained in Section 2.2:

$$\int\limits_{\mathbb{R}^{N|2M}} d\mathbf{n}\, \delta(\mathbf{n}^2 - 1) \;=\; \int\limits_{\mathbb{R}^N} d\boldsymbol{\sigma} \int \mathcal{D}(\bar{\psi}, \psi)\; \delta(\boldsymbol{\sigma}^2 + 2\bar{\psi}\cdot\psi - 1) \tag{3.11a}$$

$$= \mathcal{S}_N \int\limits_0^\infty dr\, r^{N-1} \int \mathcal{D}(\bar{\psi}, \psi)\; \delta(r^2 + 2\bar{\psi}\cdot\psi - 1) \tag{3.11b}$$

$$= \frac{\mathcal{S}_N}{2} \int \mathcal{D}(\bar{\psi}, \psi)\; (1 - 2\bar{\psi}\cdot\psi)^{(N-2)/2} \tag{3.11c}$$

$$= \frac{\mathcal{S}_N}{2}\, 2^M\, \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-2M}{2}\right)}\;, \tag{3.11d}$$

where $\bar{\psi}\cdot\psi$ is a shorthand for $\sum_{\alpha=1}^M \bar{\psi}^{(\alpha)}\psi^{(\alpha)}$, and the final step involves binomial expansion and extraction of the term $\psi^{(1)}\bar{\psi}^{(1)}\cdots\psi^{(M)}\bar{\psi}^{(M)}$. The second derivation is based directly on Taylor expansion of the delta function:

$$\int\limits_{\mathbb{R}^{N|2M}} d\mathbf{n}\, \delta(\mathbf{n}^2 - 1) \;=\; \int\limits_{\mathbb{R}^N} d\boldsymbol{\sigma} \int \mathcal{D}(\bar{\psi}, \psi)\; \delta(\boldsymbol{\sigma}^2 + 2\bar{\psi}\cdot\psi - 1) \tag{3.12a}$$

$$= \mathcal{S}_N \int\limits_0^\infty dr\, r^{N-1} \int \mathcal{D}(\bar{\psi}, \psi)\; \delta(r^2 + 2\bar{\psi}\cdot\psi - 1) \tag{3.12b}$$

$$= \mathcal{S}_N \int\limits_0^\infty dr\, r^{N-1} \int \mathcal{D}(\bar{\psi}, \psi)\; \frac{(2\bar{\psi}\cdot\psi)^M}{M!}\; \delta^{(M)}(r^2 - 1) \tag{3.12c}$$

$$= \mathcal{S}_N\, (-2)^M \int\limits_0^\infty dr\, r^{N-1}\, \delta^{(M)}(r^2 - 1) \tag{3.12d}$$

$$= \mathcal{S}_N\, (-2)^M\, \frac{1}{2} \int\limits_0^\infty dy\, y^{(N-2)/2}\, \delta^{(M)}(y - 1) \tag{3.12e}$$

$$= \frac{\mathcal{S}_N}{2}\, 2^M \int\limits_0^\infty dy\, \delta(y - 1) \left(\frac{d}{dy}\right)^M y^{(N-2)/2} \tag{3.12f}$$

$$= \frac{\mathcal{S}_N}{2}\, 2^M\, \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-2M}{2}\right)}\;, \tag{3.12g}$$

where in the Taylor-expansion step we showed only the term $(2\bar{\psi}\cdot\psi)^M/M!$ that makes a nonzero contribution to the Grassmann integral.

In view of (3.10), we shall define the normalized partition function as

$$Z_{\rm norm} \;=\; \left((2\pi)^M \, \frac{\mathcal{S}_{N-2M}}{2} \right)^{-n} Z \;=\; \left((2\pi)^{-(N-2M)/2} \, \frac{\mathcal{S}_{N-2M}}{2} \right)^{-n} Z' \;; \tag{3.13}$$

it has the property that $Z_{\rm norm} = 1$ when all $W_{ij} \equiv 1$. Thus, when our goal is to obtain formulae that depend on N and M only via the combination $N-2M$, we can use either $Z_{\rm norm}$ or Z', but not Z. However, when our goal is to obtain formulae that depend only on $N-2M$ *and* are entire (rather than just meromorphic) functions of the complex variable $N-2M$, we are obliged to use Z', as $Z_{\rm norm}$ will exhibit poles arising from the poles of the gamma function in $\mathcal{S}_{N-2M}$ [cf. (3.9)].

4 The case of $OSP(1|2)$: Solution of the constraint

In this section we consider the special case $N = M = 1$, in which there is one bosonic component and one pair of fermionic components:

$$\mathbf{n}_i \;=\; (\sigma_i, \psi_i, \bar\psi_i) \tag{4.1}$$

with the scalar product[7]

$$\mathbf{n}_i \cdot \mathbf{n}_j \;=\; \sigma_i \sigma_j + \bar\psi_i \psi_j + \bar\psi_j \psi_i \;. \tag{4.2}$$

We use (in this section only) the "ψ-first" convention for the integration measure, namely

$$d\sigma_i \, d\psi_i \, d\bar\psi_i \; \delta(\mathbf{n}_i^2 - 1) \;. \tag{4.3}$$

As explained in (1.4)/(1.5), the constraint $\mathbf{n}_i \cdot \mathbf{n}_i = 1$ is solved by writing

$$\mathbf{n}_i \;=\; \bigl(\mu_i (1 - 2\bar\psi_i \psi_i)^{1/2}, \bar\psi_i, \psi_i \bigr) \;=\; \bigl(\mu_i (1 - \bar\psi_i \psi_i), \bar\psi_i, \psi_i \bigr) \tag{4.4}$$

where $\mu_i = \pm 1$ is an Ising variable. The integration measure then becomes

$$d\sigma_i \, d\psi_i \, d\bar\psi_i \; \delta(\mathbf{n}_i^2 - 1) \;=\; \sum_{\mu_i = \pm 1} \frac{d\psi_i \, d\bar\psi_i}{2\,(1 - 2\bar\psi_i \psi_i)^{1/2}} \;=\; \frac{1}{2} \sum_{\mu_i = \pm 1} d\psi_i \, d\bar\psi_i \; e^{\bar\psi_i \psi_i} \;, \tag{4.5}$$

while the scalar product becomes

$$\mathbf{n}_i \cdot \mathbf{n}_j = \mu_i \mu_j (1 - \bar\psi_i \psi_i)(1 - \bar\psi_j \psi_j) + (\bar\psi_i \psi_j + \bar\psi_j \psi_i) \;. \tag{4.6}$$

The partition function is therefore

$$\widehat{Z} \;=\; 2^{-|V|} \sum_{\{\mu\}} \int \! \mathcal{D}(\psi, \bar\psi) \prod_{i \in V} \exp[\bar\psi_i \psi_i] \prod_{\langle ij \rangle \in E} W_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j) \tag{4.7}$$

[7] Here we implicitly set $t = 1$ with respect to the formalism described in the Introduction [cf. (1.1) ff.]. If we wanted to restore t, the scalar product would be $\mathbf{n}_i \cdot \mathbf{n}_j = \sigma_i \sigma_j + t(\bar\psi_i \psi_j + \bar\psi_j \psi_i)$.

[cf. (3.6)].

In what follows it turns out to be convenient to perform the change of variables

$$\begin{aligned} \psi_i &= \mu_i \eta_i && (4.8a) \\ \bar{\psi}_i &= \mu_i \bar{\eta}_i && (4.8b) \end{aligned}$$

which can be regarded as a gauge transformation of the fermionic variables. Note that $\bar{\psi}_i \psi_i = \bar{\eta}_i \eta_i$, so that

$$\mathbf{n}_i \;=\; \mu_i (1 - \bar{\eta}_i \eta_i, \bar{\eta}_i, \eta_i) \; . \qquad (4.9)$$

Moreover, the measure (4.5) equals the same thing with $\psi, \bar{\psi}$ replaced by $\eta, \bar{\eta}$. It follows from (4.1)/(4.2)/(4.9) that

$$\mathbf{n}_i \cdot \mathbf{n}_j \;=\; \mu_i \mu_j \, (1 - f_{ij}) \qquad (4.10)$$

where f_{ij} is defined as

$$f_{ij} \;=\; (\bar{\eta}_i - \bar{\eta}_j)(\eta_i - \eta_j) \;-\; \bar{\eta}_i \eta_i \bar{\eta}_j \eta_j \qquad (4.11)$$

[i.e. (1.2) with $t = 1$ and $\psi, \bar{\psi}$ replaced by $\eta, \bar{\eta}$]. As a consequence, the weight

$$\begin{aligned} W_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j) &= W_{ij}\left(\mu_i \mu_j - \mu_i \mu_j f_{ij}\right) && (4.12a) \\ &= W_{ij}(\mu_i \mu_j) \;-\; \mu_i \mu_j f_{ij} W'_{ij}(\mu_i \mu_j) && (4.12b) \\ &= W_{ij}(\mu_i \mu_j) \exp\left[-\mu_i \mu_j f_{ij} \frac{W'_{ij}(\mu_i \mu_j)}{W_{ij}(\mu_i \mu_j)} \right] && (4.12c) \end{aligned}$$

depends on the function $W_{ij}(x)$ only via the four numbers $W_{ij}(\pm 1)$ and $W'_{ij}(\pm 1)$. [Here the transition to (4.12c) assumes that $W_{ij}(\pm 1) \neq 0$.] The expression for the partition function then reads

$$\widehat{Z} \;=\; 2^{-|V|} \sum_{\{\mu\}} \int \mathcal{D}(\bar{\eta}, \eta) \prod_i \exp\left[\bar{\eta}_i \eta_i\right] \prod_{\langle ij \rangle \in E} W_{ij}(\mu_i \mu_j) \exp\left[-\mu_i \mu_j f_{ij} \frac{W'_{ij}(\mu_i \mu_j)}{W_{ij}(\mu_i \mu_j)} \right] , \qquad (4.13)$$

which is a model of fermionic variables coupled to Ising variables.

By using the fundamental identity (1.1) with w_{ij} replaced by

$$w_{ij}(\mu) \;=\; -\mu \frac{W'_{ij}(\mu)}{W_{ij}(\mu)} \; , \qquad (4.14)$$

we can obtain an alternate expression as a forest sum coupled to Ising variables:

$$\begin{aligned} \widehat{Z} &= 2^{-|V|} \sum_{\{\mu\}} \left(\prod_{\langle ij \rangle \in E} W_{ij}(\mu_i \mu_j) \right) \sum_{F \in \mathcal{F}(G)} \prod_{\langle ij \rangle \in F} w_{ij}(\mu_i \mu_j) && (4.15a) \\ &= 2^{-|V|} \sum_{\{\mu\}} \sum_{F \in \mathcal{F}(G)} \left(\prod_{\langle ij \rangle \in E \setminus F} W_{ij}(\mu_i \mu_j) \right) \left(\prod_{\langle ij \rangle \in F} \left(-\mu_i \mu_j W'_{ij}(\mu_i \mu_j) \right) \right) && (4.15b) \end{aligned}$$

where $\mathcal{F}(G)$ denotes the set of (edge sets of) spanning forests in G. [Note that in (4.15b) we do not need to assume that $W_{ij}(\pm 1) \neq 0$.]

We would now like to consider two special cases in which the Ising variables $\{\mu\}$ can be decoupled from the forest sum:

Case I: $\boldsymbol{W_{ij}(1) = W_{ij}(-1)}$ for all $\boldsymbol{\langle ij \rangle \in E}$. In this case we can simplify (4.15a) to

$$\widehat{Z} \;=\; W_+ \, 2^{-|V|} \sum_{\{\mu\}} \sum_{F \in \mathcal{F}(G)} \prod_{\langle ij \rangle \in F} w_{ij}(\mu_i \mu_j) \tag{4.16a}$$

$$\;=\; W_+ \, 2^{-|V|} \sum_{F \in \mathcal{F}(G)} \sum_{\{\mu\}} \prod_{\langle ij \rangle \in F} w_{ij}(\mu_i \mu_j) \tag{4.16b}$$

where

$$W_+ \;=\; \prod_{\langle ij \rangle \in E} W_{ij}(1) \; . \tag{4.17}$$

Now the Ising model on the forest F factorizes over its component trees, and then (because of the $\mu \to -\mu$ global symmetry) over its links, so that

$$\widehat{Z} \;=\; W_+ \sum_{F \in \mathcal{F}(G)} 2^{k(F)-|V|} \prod_{\langle ij \rangle \in F} \big[w_{ij}(1) + w_{ij}(-1) \big] \tag{4.18a}$$

$$\;=\; W_+ \, Z_{\text{forests},\pm} \tag{4.18b}$$

where

$$Z_{\text{forests},\pm} \;=\; \sum_{F \in \mathcal{F}(G)} \prod_{\langle ij \rangle \in F} \frac{w_{ij}(1) + w_{ij}(-1)}{2} \tag{4.19}$$

since $|V| - |F| = k(F)$ for any forest F. Thus, the partition function is the trivial prefactor W_+ times a forest sum.

Case II: $\boldsymbol{w_{ij}(1) = w_{ij}(-1)}$ for all $\boldsymbol{\langle ij \rangle \in E}$. Here we get from (4.15a)

$$\widehat{Z} \;=\; Z_{\text{Ising}} \, Z_{\text{forests},+} \tag{4.20}$$

where

$$Z_{\text{Ising}} \;=\; 2^{-|V|} \sum_{\{\mu\}} \prod_{\langle ij \rangle} W_{ij}(\mu_i \mu_j) \tag{4.21}$$

and

$$Z_{\text{forests},+} \;=\; \sum_{F \in \mathcal{F}(G)} \prod_{\langle ij \rangle \in F} w_{ij}(1) \; . \tag{4.22}$$

So the spin and forest sums again completely decouple, but now (unlike in case I) the spin part is in general a non-trivial Ising model (with $\mu \to -\mu$ global symmetry).

Examples of Boltzmann weights:

1. If $W_{ij}(x)$ is an even function for all $\langle ij \rangle \in E$, then both Case I and Case II apply. Hence both (4.17)–(4.19) and (4.20)–(4.22) hold, with $Z_{\rm Ising} = W_+$ and $Z_{\rm forests,\pm} = Z_{\rm forests,+}$. For instance, with the "standard RP^{N-1}" Boltzmann weight $W_{ij}(x) = e^{(\beta_{ij}/2)x^2}$, we have $W_{ij}(\pm 1) = e^{\beta_{ij}/2}$ and $w_{ij}(\pm 1) = -\beta_{ij}$.

2. For the standard N-vector Boltzmann weight $W_{ij}(x) = e^{\beta_{ij} x}$, we have $W_{ij}(\pm 1) = e^{\pm \beta_{ij}}$ and $w_{ij}(\pm 1) = \mp \beta_{ij}$. So this weight does not belong to Case I or Case II except in the trivial situation $\beta_{ij} = 0$.

3. For the Nienhuis [45] Boltzmann weight $W_{ij}(x) = 1 + \beta_{ij} x$, we have $W_{ij}(\pm 1) = 1 \pm \beta_{ij}$ and $w_{ij}(\pm 1) = \mp \beta_{ij}/(1 \pm \beta_{ij})$. So this weight also does not belong to Case I or Case II except in the trivial situation $\beta_{ij} = 0$.

Please note, in examples 1 and 2, that the sign of the coupling in the σ-model is opposite to that in the spanning-forest model: thus, a ferromagnetic σ-model ($\beta_{ij} > 0$) corresponds to a spanning-forest model with negative weights ($w_{ij} < 0$), while an antiferromagnetic σ-model ($\beta_{ij} < 0$) corresponds to a spanning-forest model with positive weights ($w_{ij} > 0$). The same holds in the Nienhuis example provided that $|\beta_{ij}| < 1$.

Examples of graphs:

1. If G consists of two sites connected by a single link, then from (4.15b)

$$\widehat{Z} \;=\; \frac{W(1) \,+\, W(-1) \,-\, W'(1) \,+\, W'(-1)}{2} \;. \tag{4.23}$$

More generally, if the graph G is a forest, then *every* subset $F \subseteq E$ is a spanning forest, so the sum (4.15b) decouples and factorizes over links, leading to

$$\widehat{Z} \;=\; \prod_{\langle ij \rangle \in E} \widetilde{W}_{ij} \tag{4.24}$$

where

$$\widetilde{W}_{ij} \;=\; \frac{W_{ij}(1) \,+\, W_{ij}(-1) \,-\, W'_{ij}(1) \,+\, W'_{ij}(-1)}{2} \;. \tag{4.25}$$

We remark that for the RP^{N-1} Boltzmann weight $W_{ij}(x) = e^{(\beta_{ij}/2)x^2}$ we have $\widetilde{W}_{ij} = (1 - \beta_{ij}) e^{\beta_{ij}/2}$, while for the standard N-vector Boltzmann weight $W_{ij}(x) = e^{\beta_{ij} x}$ we have

$$\widetilde{W}_{ij} \;=\; \cosh \beta_{ij} \,-\, \beta_{ij} \sinh \beta_{ij} \tag{4.26a}$$

$$\;=\; \frac{1}{2} \Big[(1 - \beta_{ij}) e^{\beta_{ij}} \,+\, (1 + \beta_{ij}) e^{-\beta_{ij}} \Big] \;. \tag{4.26b}$$

2. If G is the n-cycle, then every subset $F \subseteq E$ other than E itself is a spanning

forest. We then have from (4.15b)

$$\widehat{Z} \;=\; \prod_{\langle ij\rangle\in E} \widetilde{W}_{ij} \;-\; 2^{-|V|}\sum_{\{\mu\}}\prod_{\langle ij\rangle\in E} \big(-\mu_i\mu_j W'_{ij}(\mu_i\mu_j)\big) \tag{4.27a}$$

$$\;=\; \prod_{\langle ij\rangle\in E} \widetilde{W}_{ij} \;-\; 2^{-|V|}\sum_{\{\mu\}}\prod_{\langle ij\rangle\in E} -\biggl(\frac{W'_{ij}(1)-W'_{ij}(-1)}{2} \;+\; \frac{W'_{ij}(1)+W'_{ij}(-1)}{2}\mu_i\mu_j\biggr) \tag{4.27b}$$

$$\;=\; \prod_{\langle ij\rangle\in E} \widetilde{W}_{ij} \;-\; (-1)^n \prod_{\langle ij\rangle\in E} \frac{W'_{ij}(1)-W'_{ij}(-1)}{2} \;-\; (-1)^n \prod_{\langle ij\rangle\in E} \frac{W'_{ij}(1)+W'_{ij}(-1)}{2} \;. \tag{4.27c}$$

Now, if one of the final two products in (4.27c) happens to vanish [because for some $\langle ij\rangle \in E$ we have $W'_{ij}(1) = \pm W'_{ij}(-1)$], then $\widehat{Z}$ is indeed proportional to a forest sum

$$Z_{\rm forests} \;=\; \sum_{F\in\mathcal{F}(C_n)} \prod_{\langle ij\rangle\in F} x_{ij} \;=\; \prod_{\langle ij\rangle\in E(C_n)} (1+x_{ij}) \;-\; \prod_{\langle ij\rangle\in E(C_n)} x_{ij} \tag{4.28}$$

for a suitable choice of weights x_{ij}. But in the general situation this is *not* the case, and $\widehat{Z}$ *cannot* merely be interpreted as a forest sum. For instance, for the standard N-vector Boltzmann weight $W_{ij}(x) = e^{\beta_{ij}x}$, we have

$$\widehat{Z} \;=\; \prod_{\langle ij\rangle\in E} (\cosh\beta_{ij} - \beta_{ij}\sinh\beta_{ij}) \;-\; (-1)^n \prod_{\langle ij\rangle\in E} \beta_{ij}\sinh\beta_{ij} \;-\; (-1)^n \prod_{\langle ij\rangle\in E} \beta_{ij}\cosh\beta_{ij} \;, \tag{4.29}$$

which is not simply a forest sum.

Finally, let us check the formula (4.23) for two sites connected by a single link in another way. Consider a pure N-vector model (i.e. with no fermions) and partition function

$$Z \;=\; \widehat{Z} \;=\; \int W(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \prod_{i=1}^{2} \delta(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_i - 1)\, d\boldsymbol{\sigma}_i \;. \tag{4.30}$$

Then a well-known computation of the angular integrals gives, for $N > 1$,

$$Z \;=\; \widehat{Z} \;=\; \frac{\mathcal{S}_{N-1}}{2} \int\limits_{-1}^{1} (1-x^2)^{(N-3)/2}\, W(x)\, dx \;. \tag{4.31}$$

Now, the right-hand side of (4.31) is manifestly an analytic function of N for $\operatorname{Re} N > 1$. But, thanks to the gamma-function prefactor (recall that $\mathcal{S}_N$ is given by (3.9)), it is actually an *entire* analytic function of N: this is proven in Appendix A, where we show how $g_\lambda = (1-x^2)_+^\lambda/\Gamma(\lambda)$ can be defined as a tempered-distribution-valued entire analytic function of λ. Moreover, when λ is a negative integer, g_λ equals an explicit linear combination (A.24) of delta functions at $x = \pm 1$ and their derivatives. In particular, for $\lambda = 0$ and $\lambda = -1$ we have, respectively,

$$g_0 \;=\; \frac{1}{2}(\delta_1 + \delta_{-1}) \tag{4.32}$$

$$g_{-1} \;=\; -\frac{1}{4}(\delta_1 + \delta_{-1} + \delta_1' - \delta_{-1}') \tag{4.33}$$

Therefore, (4.31) analytically continued to $N = 1$ is

$$Z \;=\; \widehat{Z} \;=\; \frac{W(1) + W(-1)}{2}\;, \tag{4.34}$$

which is of course the correct result for the Ising model. Similarly, (4.31) analytically continued to $N = -1$ is

$$Z \;=\; \widehat{Z} \;=\; \frac{W(1) \,+\, W(-1) \,-\, W'(1) \,+\, W'(-1)}{2}\;, \tag{4.35}$$

which agrees with (4.23). [An easy rule of thumb allows to fix overall normalizations with non need of calculations, by using the property, established at the end of Section 3, that $Z_{\rm norm}$ is valued 1 when all $W_{ij} \equiv 1$.]

This example will be reconsidered in detail in Section 6, in the full generality of the $OSP(N|2M)$ model.

5 The case of the Nienhuis action

Nienhuis [45] has adopted a specific choice for the two-body Bolztmann weight, in order to obtain a combinatorial connection between the $O(N)$-invariant σ-model and a gas of self-avoiding loops, when the lattice is at most trivalent. Here we would like to observe that Nienhuis' proof generalizes almost trivially to an $OSP(N|2M)$-invariant supersymmetric σ-model, thereby providing a proof of Conjecture 1.1 in the special case of the Nienhuis action on a graph of maximum degree ≤ 3. After giving this proof, we will look more closely at the case $N = M = 1$.

5.1 $OSP(N|2M)$ model with Nienhuis action

Let $G = (V, E)$ be a graph in which each vertex has degree at most 3, with $|V| = n$; and consider the partition function of an $OSP(N|2M)$-invariant supersymmetric σ-model on G, with the Nienhuis Boltzmann weight

$$W_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j) \;=\; 1 + \beta_{ij}\, \mathbf{n}_i \cdot \mathbf{n}_j \tag{5.1}$$

where $\{\beta_{ij}\}_{\langle ij\rangle \in E}$ are parameters. The partition function is therefore

$$Z \;=\; \int \prod_{\langle ij\rangle \in E} (1 + \beta_{ij}\, \mathbf{n}_i \cdot \mathbf{n}_j) \prod_{i \in V} \delta(\mathbf{n}_i^2 - 1)\, d\mathbf{n}_i\;. \tag{5.2}$$

Now expand the product $\prod_{\langle ij\rangle\in E}(1+\beta_{ij}\,\mathbf{n}_i\cdot\mathbf{n}_j)$, and let us call an edge $\langle ij\rangle$ "occupied" if the factor $\beta_{ij}\mathbf{n}_i\cdot\mathbf{n}_j$ is taken; otherwise call it "vacant". Let $A\subseteq E$ be the set of all occupied edges. We therefore have

$$Z \;=\; \sum_{A\subseteq E}\int\prod_{\langle ij\rangle\in A}\beta_{ij}\,\mathbf{n}_i\cdot\mathbf{n}_j\,\prod_{i\in V}\delta(\mathbf{n}_i^2-1)\,d\mathbf{n}_i\;. \tag{5.3}$$

At each site i we will then need to integrate a monomial in the components of $\mathbf{n}_i$ against the measure $\delta(\mathbf{n}_i^2-1)\,d\mathbf{n}_i$; and this monomial will be of degree at most 3 because by hypothesis at most three edges are incident on any vertex. Let us write the components of the superfield $\mathbf{n}_i$ in the order $\mathbf{n}_i=(\sigma_i^{(1)},\ldots,\sigma_i^{(N)},\psi_i^{(1)},\ldots,\psi_i^{(M)},\bar\psi_i^{(1)},\ldots,\bar\psi_i^{(M)})$; then we can write $\beta_{ij}\,\mathbf{n}_i\cdot\mathbf{n}_j$ in components as

$$\beta_{ij}\,\mathbf{n}_i\cdot\mathbf{n}_j \;=\; \sum_{\alpha,\beta=1}^{N+2M}\beta_{ij}\,n_i^\alpha\,G_{\alpha\beta}\,n_j^\beta \tag{5.4}$$

where G is the matrix

$$G \;=\; \left(\begin{array}{c|c|c} I_N & 0 & 0\\ \hline 0 & 0 & I_M\\ \hline 0 & -I_M & 0\end{array}\right) \;=\; \left(\begin{array}{c|c} I_N & 0\\ \hline 0 & J_{2M}\end{array}\right) \tag{5.5}$$

(note that $G^{-1}=G^{\mathrm{T}}$). We then have the single-site integrals

$$\int 1\;\delta(\mathbf{n}_i^2-1)\,d\mathbf{n}_i \;=\; (2\pi)^M\,\frac{\mathcal{S}_{N-2M}}{2} \tag{5.6a}$$

$$\int n_i^\alpha\;\delta(\mathbf{n}_i^2-1)\,d\mathbf{n}_i \;=\; 0 \tag{5.6b}$$

$$\int n_i^\alpha n_i^\beta\;\delta(\mathbf{n}_i^2-1)\,d\mathbf{n}_i \;=\; (2\pi)^M\,\frac{\mathcal{S}_{N-2M}}{2}\;\times\;\frac{1}{N-2M}\,(G^{\mathrm{T}})_{\alpha\beta} \tag{5.6c}$$

$$\int n_i^\alpha n_i^\beta n_i^\gamma\;\delta(\mathbf{n}_i^2-1)\,d\mathbf{n}_i \;=\; 0 \tag{5.6d}$$

[Note that if we multiply (5.6c) by $G_{\alpha\beta}$ and sum over α,β, we get an additional factor $\mathrm{tr}(G^2)=N-2M$ on the right-hand side, in agreement with (5.6a). Note also that both (5.6a) and (5.6c) are entire analytic functions of $N-2M$, thanks to the gamma function in the denominator of (3.9).] It follows that the integral in (5.3) vanishes unless the graph (V,A) has degree 0 or 2 at every vertex i; in other words, A must be a disjoint union of cycles. For a cycle of length k we have a product of $2k$ matrices — G from each edge according to (5.4), and G^{T} from each vertex according to (5.6c) — linked together as a trace; and $GG^{\mathrm{T}}=I$. Any contribution to the matrix product is either purely bosonic or purely fermionic because the matrix G is block-diagonal with respect to the first N and last $2M$ components.

If the contribution is fermionic, there is a minus sign arising from the reordering $n_1 n_2 n_2 n_3 \cdots n_{k-1} n_k n_k n_1 = -n_1 n_1 n_2 n_2 n_3 \cdots n_{k-1} n_k n_k$ (here we have suppressed the upper indices for brevity). Therefore, the trace gives a factor $N - 2M$. We have in addition a factor $1/(N-2M)$ from each vertex of the cycle [from (5.6c)] and a factor β_{ij} from each edge of the cycle [from (5.4)]. We conclude that [cf. (3.13)]

$$Z_{\rm norm} \;=\; \sum_{\substack{A \subseteq E \\ A\,\text{cycles}}} (N-2M)^{c(A)} \prod_{\langle ij \rangle \in A} \frac{\beta_{ij}}{N-2M} \;, \tag{5.7}$$

where the sum runs over subsets $A \subseteq E$ that are a disjoint union of cycles, and $c(A)$ is the number of cycles in A. This formula is valid for all integers $N \geq 1$ and $M \geq 0$; it reduces to Nienhuis' [45] formula when $M = 0$.

Using (3.9) and (3.13), we can equivalently write (5.7) as

$$Z' \;=\; \left(\frac{2^{-(N-2M+2)/2}}{\Gamma\left(\frac{N-2M+2}{2}\right)} \right)^{n} \sum_{\substack{A \subseteq E \\ A\,\text{cycles}}} (N-2M)^{k(A)} \prod_{\langle ij \rangle \in A} \beta_{ij} \;, \tag{5.8}$$

where $k(A)$ denotes the number of connected components in the graph (V, A) [that is, the number of cycles plus the number of isolated vertices]. Please note that the right-hand side of (5.8) depends on N and M only via the combination $N - 2M$, and moreover defines an entire analytic function of $N - 2M$ (promoted now to a complex variable) that agrees with the partition function (3.7) for integers $N \geq 1$ and $M \geq 0$. We have therefore completely verified Conjecture 1.1 in the special case of the Nienhuis action on a graph of maximum degree ≤ 3.

5.2 The case $N = M = 1$

Let us now specialize to $N = M = 1$: then (5.7) becomes

$$Z_{\rm norm} \;=\; \sum_{\substack{A \subseteq E \\ A\,\text{cycles}}} (-1)^{c(A)} \prod_{\langle ij \rangle \in A} (-\beta_{ij}) \;. \tag{5.9}$$

On the other hand, for $N = M = 1$ we also have the explicit solution from Section 4: here (4.15b) specialized to $W_{ij}(x) = 1 + \beta_{ij} x$ gives

$$\widehat{Z} \;=\; 2^{-|V|} \sum_{\{\mu\}} \sum_{F \in \mathcal{F}(G)} \prod_{\langle ij \rangle \in E \setminus F} (1 + \beta_{ij} \mu_i \mu_j) \prod_{\langle ij \rangle \in F} (-\beta_{ij} \mu_i \mu_j) \;. \tag{5.10}$$

Now expand the product of binomials $1 + \beta_{ij} \mu_i \mu_j$. Let us say that an edge is "black" if it is an edge of the forest F, "red" if it comes from a term $\beta_{ij} \mu_i \mu_j$ of the binomial, and "white" if it comes from a term 1 of the binomial. We use the fact that

$$\sum_{\mu_i} \mu_i^k \;=\; 1 + (-1)^k \tag{5.11}$$

in order to sum over the spin variables. This gives a constraint: the number of red plus black edges incident upon each vertex must be even. And since the graph G has maximum degree 3, this number must be either 0 or 2. So the red and black edges together form a disjoint union of cycles (let us call this set A), while the black edges alone must contain no cycles since they form a forest. Therefore, in each cycle C of A, the edges can be colored arbitrarily red or black except that it is forbidden to color all edges black. The total weight of such a cycle is therefore $\prod(\beta_{ij} - \beta_{ij}) - \prod(-\beta_{ij}) = -\prod(-\beta_{ij})$, so we are left with

$$\widehat{Z} \;=\; \sum_{\substack{A \subseteq E \\ A\,\text{cycles}}} (-1)^{c(A)} \prod_{\langle ij\rangle \in A} (-\beta_{ij}) \,. \tag{5.12}$$

This agrees with (5.9) because

$$\frac{Z_{\rm norm}}{\widehat{Z}} \;=\; \left((-2\pi)^M \frac{\mathcal{S}_{N-2M}}{2} \right)^{-n} \;=\; \left(\frac{(-2)^M\, \pi^{N/2}}{\Gamma\left(\frac{N-2M}{2}\right)} \right)^{-n} \tag{5.13}$$

[cf. (3.6)/(3.13)], which equals 1 when $N = M = 1$.

6 One-link kernel

In this section we study the $OSP(N|2M)$ σ-model on two sites connected by a single link, integrating over one of the two spins while holding the other one fixed. That is, we will study the integral

$$I_{W;N,M}(\mathbf{n}_1) \;=\; (2\pi)^{-N/2} \int W(\mathbf{n}_1 \cdot \mathbf{n}_2)\, \delta(\mathbf{n}_2 \cdot \mathbf{n}_2 - 1)\, d\mathbf{n}_2 \tag{6.1}$$

where W is an arbitrary smooth function and $\mathbf{n}_1$ is an arbitrary superfield (not necessarily of length 1). Our goal is to show the following:

Proposition 6.1 *When $N - 2M > 1$, we have*

$$(2\pi)^{-N/2} \int W(\mathbf{n}_1 \cdot \mathbf{n}_2)\, \delta(\mathbf{n}_2 \cdot \mathbf{n}_2 - 1)\, d\mathbf{n}_2 \;=\; F_{W;N,M}(\mathbf{n}_1 \cdot \mathbf{n}_1) \,, \tag{6.2}$$

where

$$F_{W;N,M}(x) \;=\; (2\pi)^{-(N-2M)/2}\, \frac{\mathcal{S}_{N-2M-1}}{2} \int\limits_{-1}^{1} (1-t^2)^{(N-2M-3)/2}\, W(tx^{1/2})\, dt \tag{6.3}$$

is defined for $x \geq 0$. [Then $F_{W;N,M}(\mathbf{n}_1 \cdot \mathbf{n}_1)$ is defined by our usual Taylor-series prescription: note that $\mathbf{n}_1 \cdot \mathbf{n}_1 = \boldsymbol{\sigma}_1^2 + 2\sum_{\alpha=1}^{M} \bar\psi_1^{(\alpha)} \psi_1^{(\alpha)}$ has a "body" $\boldsymbol{\sigma}_i^2$ that is generically > 0.]

Please note from (6.3) that $F_{W;N,M}$ depends on N and M only via the combination $N-2M$, in agreement with the idea expressed in Conjecture 1.1.

To prove Proposition 6.1 in a simple way we can make use of super-rotation invariance as defined in Appendix B, we shall see this in the following, now, as a warm up, we will proceed as follows: We will generalize (6.2)/(6.3) by letting $\mathbf{n}_1$ (resp. $\mathbf{n}_2$) be a superfield with N bosonic components and M_1 (resp. M_2) fermionic components, with the scalar product defined as

$$\mathbf{n}_i \cdot \mathbf{n}_j \;=\; \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \;+ \sum_{\alpha=1}^{\min(M_1,M_2)} (\bar{\psi}_i^{(\alpha)} \psi_j^{(\alpha)} - \psi_i^{(\alpha)} \bar{\psi}_j^{(\alpha)}) \;. \tag{6.4}$$

We claim that (6.2)/(6.3) holds whenever $0 \le M_1 \le M_2$. We will prove this by an outer induction on M_2 together with an inner induction on M_1. For brevity we write $M_2 = M$.

Base cases $M_1 = 0$ (with $M_2 = M \ge 0$ arbitrary). When $M_1 = 0$, the superfield $\mathbf{n}_1$ has only the bosonic components $\boldsymbol{\sigma}_1$; let us assume that these have squared length $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1 = x > 0$. We now separate the integration over $\boldsymbol{\sigma}_2$ into its component t along the direction of $\boldsymbol{\sigma}_1$ and its components along the $N-1$ directions orthogonal to $\boldsymbol{\sigma}_1$, which we take to be of length r; we then split the latter integral into its radial and angular parts. We have

$$\begin{aligned} &I_{W;N,M}(\mathbf{n}_1) \\ &\quad=\; (2\pi)^{-N/2}\, \mathcal{S}_{N-1} \int\limits_{-\infty}^{\infty} dt\, W(tx^{1/2}) \int\limits_{0}^{\infty} dr\, r^{N-2} \int \mathcal{D}_M(\bar{\psi},\psi)\, \delta(t^2 + r^2 + 2\,\bar{\psi}\cdot\psi - 1) &&\text{(6.5a)} \\ &\quad=\; (2\pi)^{-N/2}\, \frac{\mathcal{S}_{N-1}}{2} \int\limits_{-\infty}^{\infty} dt\, W(tx^{1/2}) \int \mathcal{D}_M(\bar{\psi},\psi)\, (1 - t^2 - 2\,\bar{\psi}\cdot\psi)^{(N-3)/2} &&\text{(6.5b)} \end{aligned}$$

where $\mathcal{S}_{N-1}$ is defined by (3.9), and $\bar{\psi}\cdot\psi$ is a shorthand for $\sum_{\alpha=1}^{M} \bar{\psi}^{(\alpha)}\psi^{(\alpha)}$. Now binomial expansion gives

$$\int \mathcal{D}_M(\bar{\psi},\psi)\, (1 - t^2 - 2\,\bar{\psi}\cdot\psi)^{(N-3)/2} \;=\; 2^M\, \frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N-2M-1}{2}\right)}\, (1-t^2)^{(N-2M-3)/2} \;, \tag{6.6}$$

so using (3.9) we get the desired result.

Inductive step. We continue our proof for the first part of the proposition, *without* making use of the hypothesis on the rotational symmetry of the result.

Suppose now that $1 \le k \le M$, and assume that the proposition is valid when $(M_1, M_2) = (k-1, M)$ and also when $(M_1, M_2) = (k-1, M-1)$; then we will prove it for $(M_1, M_2) = (k, M)$. We have

$$\begin{aligned} W(\mathbf{n}_1 \cdot \mathbf{n}_2) \;&=\; W\!\left[\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \sum_{\alpha=1}^{k-1} (\bar{\psi}_1^{\alpha}\psi_2^{\alpha} + \bar{\psi}_2^{\alpha}\psi_1^{\alpha}) + (\bar{\psi}_1^{k}\psi_2^{k} + \bar{\psi}_2^{k}\psi_1^{k})\right] &&\text{(6.7a)} \\ &=\; W \;+\; (\bar{\psi}_1^{k}\psi_2^{k} + \bar{\psi}_2^{k}\psi_1^{k})\, W' \;+\; \bar{\psi}_1^{k}\psi_2^{k}\bar{\psi}_2^{k}\psi_1^{k}\, W'' &&\text{(6.7b)} \end{aligned}$$

where on the second line W and its derivatives are evaluated at the point $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \sum_{\alpha=1}^{k-1}(\bar{\psi}_1^\alpha \psi_2^\alpha + \bar{\psi}_2^\alpha \psi_1^\alpha)$. The contribution proportional to W' vanishes in the Grassmann integral, because in the other terms of the integrand (i.e. the delta function) ψ_2^k and $\bar{\psi}_2^k$ appear only in the combination $\bar{\psi}_2^k \psi_2^k$. For the contribution proportional to W, we can use the inductive hypothesis for $(M_1, M_2) = (k-1, M)$ because $\bar{\psi}_1^k$ and ψ_1^k do not appear; the result is

$$
\begin{aligned}
I_1 &= F_{W;N,M}\Bigg(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1 \,+\, 2\sum_{\alpha=1}^{k-1} \bar{\psi}_1^\alpha \psi_1^\alpha\Bigg) && \text{(6.8a)} \\
&= (2\pi)^{-(N-2M)/2}\, \frac{\mathcal{S}_{N-2M-1}}{2} \int\limits_{-1}^{1} dt\, (1-t^2)^{(N-2M-3)/2}\, W(ty) && \text{(6.8b)}
\end{aligned}
$$

where

$$
y \;=\; \Bigg(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1 + 2\sum_{\alpha=1}^{k-1} \bar{\psi}_1^\alpha \psi_1^\alpha\Bigg)^{1/2} . \tag{6.9}
$$

Finally, the contribution proportional to W'' is

$$
-\bar{\psi}_1^k \psi_1^k\, (2\pi)^{-N/2} \int W''\Bigg[\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \sum_{\alpha=1}^{k-1}(\bar{\psi}_1^\alpha \psi_2^\alpha + \bar{\psi}_2^\alpha \psi_1^\alpha)\Bigg]\; \bar{\psi}_2^k \psi_2^k\; \delta(\mathbf{n}_2 \cdot \mathbf{n}_2 - 1)\, d\mathbf{n}_2 \tag{6.10}
$$

where of course $d\mathbf{n}_2 = d\boldsymbol{\sigma}_2\, \mathcal{D}(\bar{\psi}_2, \psi_2)$. We can perform the integral over $\bar{\psi}_2^k$ and ψ_2^k, yielding

$$
\int d\bar{\psi}_2^k\, d\psi_2^k\;\; \bar{\psi}_2^k \psi_2^k\;\; \delta(\mathbf{n}_2 \cdot \mathbf{n}_2 - 1) \;=\; -\,\delta(\mathbf{n}_2' \cdot \mathbf{n}_2' - 1) \tag{6.11}
$$

where $\mathbf{n}_2'$ is the superfield $\mathbf{n}_2$ with the components ψ_2^k and $\bar{\psi}_2^k$ removed. Hence (6.10) equals

$$
\bar{\psi}_1^k \psi_1^k\, (2\pi)^{-N/2} \int W''\Bigg[\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \sum_{\alpha=1}^{k-1}(\bar{\psi}_1^\alpha \psi_2^\alpha + \bar{\psi}_2^\alpha \psi_1^\alpha)\Bigg]\; \delta(\mathbf{n}_2' \cdot \mathbf{n}_2' - 1)\, d\mathbf{n}_2' \,. \tag{6.12}
$$

This is an integral involving $M-1$ fermions, to which we can apply the inductive hypothesis for $(M_1, M_2) = (k-1, M-1)$; the result is

$$
\begin{aligned}
I_2 &= \bar{\psi}_1^k \psi_1^k\, F_{W;N,M-1}\Bigg(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1 \,+\, 2\sum_{\alpha=1}^{k-1} \bar{\psi}_1^\alpha \psi_1^\alpha\Bigg) && \text{(6.13a)} \\
&= \bar{\psi}_1^k \psi_1^k\, (2\pi)^{-(N-2M+2)/2}\, \frac{\mathcal{S}_{N-2M+1}}{2} \int\limits_{-1}^{1} dt\, (1-t^2)^{(N-2M-1)/2}\, W''(ty) && \text{(6.13b)}
\end{aligned}
$$

where y is again given by (6.9). Now

$$\int_{-1}^{1} dt\,(1-t^2)^{(N-2M-1)/2}\,W''(ty) \;=\; \frac{1}{y}\int_{-1}^{1} dt\,(1-t^2)^{(N-2M-1)/2}\,\frac{d}{dt}\,W'(ty) \tag{6.14a}$$

$$= -\frac{1}{y}\int_{-1}^{1} dt\,W'(ty)\,\frac{d}{dt}\,(1-t^2)^{(N-2M-1)/2} \tag{6.14b}$$

$$= -\frac{1}{y}\,\frac{d}{dy}\int_{-1}^{1} dt\,W(ty)\,\frac{1}{t}\,\frac{d}{dt}\,(1-t^2)^{(N-2M-1)/2} \tag{6.14c}$$

$$= \frac{N-2M-1}{y}\,\frac{d}{dy}\int_{-1}^{1} dt\,W(ty)\,(1-t^2)^{(N-2M-3)/2}\,, \tag{6.14d}$$

where the boundary terms in the integration by parts vanish whenever $N-2M>1$. By collecting the contributions I_1 and I_2 together, we get [using (3.9)]

$$I_{W;N,M}(\mathbf{n}_1) \;=\; (2\pi)^{-(N-2M)/2}\,\frac{\mathcal{S}_{N-2M-1}}{2}\int_{-1}^{1} dt\,(1-t^2)^{(N-2M-3)/2}\left[1+\bar\psi_1^k\psi_1^k\,\frac{1}{y}\,\frac{d}{dy}\right]W(ty) \tag{6.15a}$$

$$= (2\pi)^{-(N-2M)/2}\,\frac{\mathcal{S}_{N-2M-1}}{2}\int_{-1}^{1} dt\,(1-t^2)^{(N-2M-3)/2}\,W\!\left[t\left(y^2+2\,\bar\psi_1^k\psi_1^k\right)^{1/2}\right] \tag{6.15b}$$

$$= (2\pi)^{-(N-2M)/2}\,\frac{\mathcal{S}_{N-2M-1}}{2}\int_{-1}^{1} dt\,(1-t^2)^{(N-2M-3)/2}\,W(tx^{1/2})\,, \tag{6.15c}$$

which concludes the proof. □

Let us now concentrate on the case in which also the superfield $\mathbf{n}_1$ has unit norm. We obtain a natural extension of Proposition 6.1, which has the seeked range of validity in N and M.

Proposition 6.2 *Under the hypothesis that $\mathbf{n}_1\cdot\mathbf{n}_1=1$, we have that for arbitrary $N\geq 1$ and $M\geq 0$,*

$$(2\pi)^{-N/2}\int W(\mathbf{n}_1\cdot\mathbf{n}_2)\,\delta(\mathbf{n}_2\cdot\mathbf{n}_2-1)\,d\mathbf{n}_2 \;=\; \frac{1}{\sqrt{2\pi}}2^{-\frac{N-2M-1}{2}}\langle g_{\frac{N-2M-1}{2}},W\rangle \tag{6.16}$$

where the distribution g_λ is defined as in Proposition A.2.

Let R be a $OSP(N|2M)$ super-rotation as defined in Appendix B, in the connected component of the group containing the identity (if $N=1$, while if $N\geq 2$ the group is connected). Then we have $R\mathbf{n}_i \cdot R\mathbf{n}_j = \mathbf{n}_i \cdot \mathbf{n}_j$ for $i, j = 1, 2$, and the Berezinian of R is 1. As a result

$$
\begin{aligned}
I_{W;N,M}(\mathbf{n}_1) &= (2\pi)^{-N/2} \int W(\mathbf{n}_1 \cdot \mathbf{n}_2)\, \delta(\mathbf{n}_2 \cdot \mathbf{n}_2 - 1)\, d\mathbf{n}_2 \\
&= (2\pi)^{-N/2} \int W(R\mathbf{n}_1 \cdot R\mathbf{n}_2)\, \delta(R\mathbf{n}_2 \cdot R\mathbf{n}_2 - 1)\, d\mathbf{n}_2 \qquad (6.17) \\
&= (2\pi)^{-N/2} \int W(R\mathbf{n}_1 \cdot \mathbf{n}'_2)\, \delta(\mathbf{n}'_2 \cdot \mathbf{n}'_2 - 1)\, d\mathbf{n}'_2 \;=\; I_{W;N,M}(R\mathbf{n}_1)
\end{aligned}
$$

Thus, instead of evaluating $I_{W;N,M}(\mathbf{n}_1)$ in full generality, we can perform a rotation R at our choice and evaluate $I_{W;N,M}(R\mathbf{n}_1)$. In particular, as discussed in Appendix B.2 (see in particular Proposition B.1), we can choose R as to make the vector $\mathbf{n}'_1 = R\mathbf{n}_1$ purely bosonic. In this case, we are in the situation of the proof of Proposition 6.1, for the base case $M_1 = 0$, i.e. we have determined (6.5b) and (6.6) under the sole requirement that $N \geq 1$ and $M \geq 0$.

The trouble is, we cannot just substitute (6.6) into (6.5b), in full generality, within the realm of functions, because the function $(1-t^2)^{\lambda-1}$ has a non-integrable singularity in $t = -1$ and $t = 1$ whenever $\lambda \leq 0$, i.e., $N - 2M \leq 1$.

To begin with, let us continue the calculation under the hypothesis $N - 2M > 1$. Now, the factors $\mathcal{S}_{N-1}$ in (6.5b) and $\Gamma\left(\frac{N-1}{2}\right) / \Gamma\left(\frac{N-2M-1}{2}\right)$ in (6.6) combine to reproduce the normalization in the definition of the distribution $g_\lambda = \frac{1}{\Gamma(\lambda)}(1 - x^2)_+^{\lambda-1}$, so that we get, as desired

$$
\begin{aligned}
&I_{W;N,M}(\mathbf{n}_1) \\
&\quad = (2\pi)^{-N/2}\, \frac{\mathcal{S}_{N-1}}{2} 2^M \, \frac{\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N-2M-1}{2}\right)} \int_{-1}^{1} dt\, (1 - t^2)^{(N-2M-3)/2}\, W(t) \qquad (6.18a) \\
&\quad = \frac{1}{\sqrt{2\pi}} 2^{-\frac{N-2M-1}{2}} \int_{-1}^{1} dt\, \frac{(1 - t^2)^{(N-2M-3)/2}}{\Gamma\left(\frac{N-2M-1}{2}\right)}\, W(t) \qquad (6.18b) \\
&\quad = \frac{1}{\sqrt{2\pi}} 2^{-\frac{N-2M-1}{2}} \langle g_{\frac{N-2M-1}{2}}, W \rangle \qquad (6.18c)
\end{aligned}
$$

Let us inspect more closely what we have obtained. The formula (6.18c) manifestly depends on N and M only via the combination $N - 2M$. Initially N and M are of course integers satisfying $N \geq 1$ and $M \geq 0$, but the right-hand side of (6.18c) makes sense for any real N and M satisfying $N - 2M > 1$, or indeed for any complex N and M satisfying $\mathrm{Re}(N - 2M) > 1$. Furthermore, (6.18c) can be analytically continued to an entire analytic function of the complex variable $N-2M$: this is discussed in Appendix A. The key point is that the factor $\mathcal{S}_{N-2M-1} \propto 1/\Gamma\left(\frac{N-2M-1}{2}\right)$ in (6.18c) provides precisely the gamma-function denominator that allows this expression to be analytically continued (for a fixed smooth function W_{ij}) as an entire analytic function of $N - 2M$: see Proposition A.2 with $\lambda = (N - 2M - 1)/2$, so one is tempted to imagine that (6.18) has a larger range of validity, as we now shall prove.

In Proposition A.2 a crucial defining property of g_λ is observed, namely equation (A.23), which, by the Bernstein's method (see Appendix A.3), allows to extend the domain of definition of the distribution. So, when $-1 < N - 2M \leq 1$, we can reconsider equation (6.6), and, instead of performing a complete binomial expansion, we can single out the variables $\bar{\psi}^{(M)}$ and $\psi^{(M)}$, and perform a binomial expansion on the remaining Grassmann variables. Calling $\bar{\psi}'$ and ψ' the vectors $\bar{\psi}$ and ψ restricted to the first $M-1$ components, we get

$$\begin{aligned}
&\int \mathcal{D}_M(\bar{\psi}, \psi)\, (1 - t^2 - 2\, \bar{\psi} \cdot \psi)^{(N-3)/2} \\
&\qquad = \int d\bar{\psi}^{(M)} d\psi^{(M)} \int \mathcal{D}_{M-1}(\bar{\psi}', \psi')\, (1 - t^2 - 2\bar{\psi}^{(M)}\psi^{(M)} - 2\, \bar{\psi}' \cdot \psi')^{(N-3)/2} \\
&\qquad = \int d\bar{\psi}^{(M)} d\psi^{(M)}\, 2^{M-1} \frac{\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N-2M+1}{2})} (1 - t^2 - 2\bar{\psi}^{(M)}\psi^{(M)})^{(N-2M-1)/2}
\end{aligned} \tag{6.19}$$

Now, according to the prescription of Section 2, and using (A.17), we have

$$\frac{1}{2} \int d\bar{\psi} d\psi\, (1 - t^2 - 2\bar{\psi}\psi)^\lambda \;=\; \lambda (1 - t^2)^{\lambda - 1} \;=\; \left(\lambda - \frac{t}{2} \frac{d}{dt} \right) (1 - t^2)^\lambda \tag{6.20}$$

At this point, it is legitimate to interpret d/dt as a distributional derivative, and, as a result, we have proven the proposition also in the range $-1 < N - 2M \leq 1$, where the distribution g on the right-hand side of (6.16) is recognized at the very end of the derivation, by mean of property (A.23).

For the range $-3 < N - 2M \leq -1$, we shall just repeat the argument above recursively. Namely, instead of performing a complete binomial expansion on variables $\bar{\psi}'$ and ψ', we can single out the variables $\bar{\psi}^{(M-1)}$ and $\psi^{(M-1)}$, and perform a binomial expansion on the remaining Grassmann variables. We get a nested variant of (6.20), which accordingly requires (A.19) in place of (A.18)

$$\begin{aligned}
&\frac{1}{4} \int d\bar{\psi} d\psi\, d\bar{\eta} d\eta\, (1 - t^2 - 2\bar{\psi}\psi - 2\bar{\eta}\eta)^\lambda \\
&\qquad = \frac{1}{2} \lambda \int d\bar{\psi} d\psi\, (1 - t^2 - 2\bar{\psi}\psi)^{\lambda - 1} \\
&\qquad = \lambda (\lambda - 1)(1 - t^2)^{\lambda - 2} \;=\; \left(\lambda - 1 - \frac{t}{2} \frac{d}{dt} \right) \left(\lambda - \frac{t}{2} \frac{d}{dt} \right) (1 - t^2)^\lambda \,.
\end{aligned} \tag{6.21}$$

The generalisation of this reasoning is straightforward.

We have thus constructed an entire analytic function of $N - 2M$ that agrees with the partition function Z' whenever $N \geq 1$ and $M \geq 0$ are integers. This is the crucial ingredient that, in the following section, allow us to prove Conjecture 1.1 for the special case when f is a forest product.

The expression on the right-hand side of (6.16), when $(N - 2M - 1)/2$ is a negative integer, involves a combination of delta's at ± 1 and its derivatives, described by (A.24). We also know that, as illustrated in Section 2, integrals in superspace shall be interpreted through the Taylor expansion of the integrand function in powers of the soul part of the variables, and yet again, for $N = 1$ and M strictly positive, this leads to a linear combination of the integrand and its derivative evaluated at ± 1. The proposition above has established, in an abstract and indirect way, that these combinations shall coincide in the case of the one-link kernel. Nonetheless, it would be interesting to have a direct combinatorial explanation of this fact, along the lines of the proof of Proposition 6.1, without exploiting the rotational invariance of the kernel. We believe that this verification will require additional combinatorial tools, which will possibly shed light on the pattern behind the full Conjecture 1.1, and we defer it to future work.

7 Forests

We can now compute the partition function (3.7) whenever the graph G is a forest. It suffices to repeatedly use Proposition 6.2 (indeed, at each vertex 1 the delta functions will eventually enforce $\mathbf{n}_1 \cdot \mathbf{n}_1 = 1$), taking at each stage the vertex 2 to be a leaf (i.e. a vertex having only one edge incident on it). If G is a forest, (ij) is such a leaf edge, and $G' = G \setminus (ij)$, one step of the procedure gives

$$Z'_{G,\{W\}} = Z'_{G',\{W\}} \frac{1}{\sqrt{2\pi}} 2^{-\frac{N-2M-1}{2}} \langle g_{\frac{N-2M-1}{2}}, W_{ij} \rangle \tag{7.1}$$

Finally, the integral over the last vertex in each tree is

$$(2\pi)^{-N/2} \int d\mathbf{n}\, \delta(\mathbf{n} \cdot \mathbf{n} - 1) \;=\; (2\pi)^{-(N-2M)/2} \frac{\mathcal{S}_{N-2M}}{2} \tag{7.2}$$

[cf. (3.10)]. It follows that the partition function (3.7) is

$$Z' \;=\; (\pi^{-\frac{1}{2}} 2^{-\frac{N-2M}{2}})^{|V(G)|} \left(\pi^{-\frac{N-2M-1}{2}} \frac{\mathcal{S}_{N-2M}}{2} \right)^{k(G)} \prod_{\langle ij \rangle \in E} \langle g_{\frac{N-2M-1}{2}}, W_{ij} \rangle \tag{7.3}$$

where $k(G)$ is the number of connected components of the graph G (i.e. the number of trees in G, including isolated vertices).

8 Reduction formula for two links in series

In this section we will derive a reduction formula for two links in series in a general $OSP(N|2M)$ model: that is, we will study the integral

$$I_{W_{12},W_{23};N,M}(\mathbf{n}_1, \mathbf{n}_3) \;=\; (2\pi)^{-N/2} \int W_{12}(\mathbf{n}_1 \cdot \mathbf{n}_2)\, W_{23}(\mathbf{n}_2 \cdot \mathbf{n}_3)\, \delta(\mathbf{n}_2 \cdot \mathbf{n}_2 - 1)\, d\mathbf{n}_2 \tag{8.1}$$

where W_{12} and W_{23} are arbitrary smooth functions and $\mathbf{n}_1, \mathbf{n}_3$ are arbitrary superfields (not necessarily of length 1). Our goal is to show the following:

Proposition 8.1 *When $N \geq 2$ and $N - 2M > 1$, we have*

$$(2\pi)^{-N/2} \int W_{12}(\mathbf{n}_1 \cdot \mathbf{n}_2)\, W_{23}(\mathbf{n}_2 \cdot \mathbf{n}_3)\, \delta(\mathbf{n}_2 \cdot \mathbf{n}_2 - 1)\, d\mathbf{n}_2$$
$$= F_{W_{12},W_{23};N,M}(\mathbf{n}_1 \cdot \mathbf{n}_1,\, \mathbf{n}_3 \cdot \mathbf{n}_3,\, \mathbf{n}_1 \cdot \mathbf{n}_3)\ , \tag{8.2}$$

where

$$F_{W_{12},W_{23};N,M}(X_{11}, X_{33}, X_{13}) \;=\; (2\pi)^{-(N-2M)/2}\, \frac{\mathcal{S}_{N-2M-2}}{2} \;\times$$
$$\iint\limits_{s^2+t^2 \leq 1} ds\, dt\, (1 - s^2 - t^2)^{(N-2M-4)/2}\, W_{12}(x_1 s + y_1 t)\, W_{23}(x_3 s + y_3 t) \tag{8.3}$$

and x_1, y_1, x_3, y_3 are shorthand for

$$x_1 \;=\; (X_{11}/2)^{1/2} \left(1 + \frac{X_{13}}{X_{11}^{1/2} X_{33}^{1/2}} \right)^{1/2} \tag{8.4a}$$

$$y_1 \;=\; (X_{11}/2)^{1/2} \left(1 - \frac{X_{13}}{X_{11}^{1/2} X_{33}^{1/2}} \right)^{1/2} \tag{8.4b}$$

$$x_3 \;=\; (X_{33}/2)^{1/2} \left(1 + \frac{X_{13}}{X_{11}^{1/2} X_{33}^{1/2}} \right)^{1/2} \tag{8.4c}$$

$$y_3 \;=\; -\,(X_{33}/2)^{1/2} \left(1 - \frac{X_{13}}{X_{11}^{1/2} X_{33}^{1/2}} \right)^{1/2} \tag{8.4d}$$

Here $F_{W_{12},W_{23};N,M}(X_{11}, X_{33}, X_{13})$ is defined for $X_{11}, X_{33} > 0$ and $-X_{11}^{1/2} X_{33}^{1/2} < X_{13} < X_{11}^{1/2} X_{33}^{1/2}$. [Then $F_{W_{12},W_{23};N,M}(\mathbf{n}_1 \cdot \mathbf{n}_1,\, \mathbf{n}_3 \cdot \mathbf{n}_3,\, \mathbf{n}_1 \cdot \mathbf{n}_3)$ is defined by our usual Taylor-series prescription: note that the "bodies" of $\mathbf{n}_1 \cdot \mathbf{n}_1$, $\mathbf{n}_3 \cdot \mathbf{n}_3$ and $\mathbf{n}_1 \cdot \mathbf{n}_3$ satisfy generically the required inequalities.]

Please note from (8.3) that $F_{W_{12},W_{23};N,M}$ depends on N and M only via the combination $N - 2M$, in agreement with the idea expressed in Conjecture 1.1. Please note also two special cases where the formula simplifies:

1) When W_{23} is a constant function, we can use the $O(2)$ invariance of the integration measure in (8.3) to replace (x_1, y_1) by $(X_{11}^{1/2}, 0)$; then (8.2)/(8.3) reduces to Proposition 6.1.

2) When $X_{11} = X_{33} = 1$ — as will be the case in our applications to $OSP(N|2M)$-

invariant σ-models — then (8.4) simplifies to

$$x_1 = \left(\frac{1+X_{13}}{2}\right)^{1/2} \tag{8.5a}$$

$$y_1 = \left(\frac{1-X_{13}}{2}\right)^{1/2} \tag{8.5b}$$

$$x_3 = \left(\frac{1+X_{13}}{2}\right)^{1/2} \tag{8.5c}$$

$$y_3 = -\left(\frac{1-X_{13}}{2}\right)^{1/2} \tag{8.5d}$$

Our proof of (8.2)–(8.4) will follow the general pattern of our analysis of the one-link integral in Section 6, but will be somewhat more complicated because of the dependence on two superfields rather than one, and hence on three scalar products rather than one.

Analogously to Section 6, therefore, we assume that $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are superfields with the same number N of bosonic components but with M_1, M_2, M_3 pairs of fermions, respectively; the scalar product is defined as in (6.4). Throughout most of the proof, we will assume that $M_3 = 0$. We claim that (8.2)–(8.4) holds whenever $M_3 = 0$ and $0 \le M_1 \le M_2$. We will prove this by an outer induction on M_2 together with an inner induction on M_1. At the last stage we will invoke $OSP(N|2M)$ invariance to remove the restriction to $M_3 = 0$. For brevity we write $M_2 = M$.

Base cases $M_1 = M_3 = 0$ (with $M_2 = M \ge 0$ arbitrary). Here $\mathbf{n}_1$ and $\mathbf{n}_3$ are purely bosonic. By $O(N)$ invariance we can assume without loss of generality that $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_3$ lie in the 12-plane, so that $\boldsymbol{\sigma}_1 = (x_1, y_1, 0, \ldots, 0)$ and $\boldsymbol{\sigma}_3 = (x_3, y_3, 0, \ldots, 0)$; we then write $\boldsymbol{\sigma}_2 = (s, t, \mathbf{z})$ where $|\mathbf{z}| = r$. We have

$$\begin{aligned} I_{W_{12},W_{23};N,M}(\mathbf{n}_1,\mathbf{n}_3) \;&=\; (2\pi)^{-N/2}\, \mathcal{S}_{N-2} \int\limits_{-\infty}^{\infty} ds \int\limits_{-\infty}^{\infty} dt \int\limits_{0}^{\infty} dr\, r^{N-3} \;\times \\ & W_{12}(x_1 s + y_1 t)\, W_{23}(x_3 s + y_3 t) \int \mathcal{D}_M(\bar\psi,\psi)\, \delta(s^2+t^2+r^2+2\bar\psi\cdot\psi-1)\,. \end{aligned} \tag{8.6}$$

Now

$$\int\limits_{0}^{\infty} dr\, r^{N-3}\, \delta(s^2+t^2+r^2+2\bar\psi\cdot\psi-1) \;=\; \frac{1}{2}\,(1-s^2-t^2-2\bar\psi\cdot\psi)^{(N-4)/2} \tag{8.7}$$

and

$$\int \mathcal{D}_M(\bar\psi,\psi)\,(1-u^2-2\,\bar\psi\cdot\psi)^{(N-4)/2} \;=\; 2^M\, \frac{\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-2M-2}{2}\right)}\,(1-u^2)^{(N-2M-4)/2} \tag{8.8}$$

[cf. (6.6)]. So using (3.9) we get

$$
\begin{aligned}
I_{W_{12},W_{23};N,M}(\mathbf{n}_1,\mathbf{n}_3) \;&=\; (2\pi)^{-(N-2M)/2}\,\frac{\mathcal{S}_{N-2M-2}}{2}\;\times \\
&\iint\limits_{s^2+t^2\le 1} dr\,dt\,(1-s^2-t^2)^{(N-2M-4)/2}\,W_{12}(x_1 s+y_1 t)\,W_{23}(x_3 s+y_3 t)\;. \qquad (8.9)
\end{aligned}
$$

Now we have the relations

$$
\begin{aligned}
\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_1 &= X_{11} = x_1^2+y_1^2 && (8.10a)\\
\boldsymbol{\sigma}_3\cdot\boldsymbol{\sigma}_3 &= X_{33} = x_3^2+y_3^2 && (8.10b)\\
\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3 &= X_{13} = x_1x_3+y_1y_3 && (8.10c)
\end{aligned}
$$

And we can also assume, using $O(2)$ invariance, that the bisector between $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_3$ lies along the 1-axis, i.e. that $y_1/x_1 = -y_3/x_3$. Solving these four relations for x_1, y_1, x_3, y_3 in terms of X_{11}, X_{33}, X_{13} gives (8.4).[8]

Remark. Alternative choices of the "fourth relation" are also possible: for instance, we could choose $y_1 = 0$, yielding

$$
\begin{aligned}
x_1 &= X_{11}^{1/2} && (8.11a)\\
y_1 &= 0 && (8.11b)\\
x_3 &= \frac{X_{13}}{X_{11}^{1/2}} && (8.11c)\\
y_3 &= \left(X_{33}-\frac{X_{13}^2}{X_{11}}\right)^{1/2} && (8.11d)
\end{aligned}
$$

— which leads to a formula different from (8.3)/(8.4), but equivalent to it by virtue of the $O(2)$ invariance of the integration measure in (8.3). Similarly, we could choose $y_3 = 0$, yielding

$$
\begin{aligned}
x_1 &= \frac{X_{13}}{X_{33}^{1/2}} && (8.12a)\\
y_1 &= \left(X_{11}-\frac{X_{13}^2}{X_{33}}\right)^{1/2} && (8.12b)\\
x_3 &= X_{33}^{1/2} && (8.12c)\\
y_3 &= 0 && (8.12d)
\end{aligned}
$$

We prefer the choice $y_1/x_1 = -y_3/x_3$ because it yields a formula (8.3)/(8.4) that is manifestly symmetric between $\mathbf{n}_1$ and $\mathbf{n}_3$. But we will find the alternative choice (8.12) useful in simplifying the computations in the inductive step of the proof.

[8] This is one of eight possible solutions; the others are obtained from this one by changing the signs of x_1 and x_3, and/or of y_1 and y_3, and/or of x_1 and y_1 and X_{13}.

Inductive step. Suppose now that $1 \leq k \leq M$, and assume that the proposition is valid when $(M_1, M_2, M_3) = (k-1, M, 0)$ and also when $(M_1, M_2, M_3) = (k-1, M-1, 0)$; then we will prove it for $(M_1, M_2, M_3) = (k, M, 0)$. We have

$$\begin{aligned} W_{12}(\mathbf{n}_1 \cdot \mathbf{n}_2) &= W_{12}\left[\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \sum_{\alpha=1}^{k-1} (\bar{\psi}_1^\alpha \psi_2^\alpha + \bar{\psi}_2^\alpha \psi_1^\alpha) + (\bar{\psi}_1^k \psi_2^k + \bar{\psi}_2^k \psi_1^k)\right] && \text{(8.13a)} \\ &= W_{12} \,+\, (\bar{\psi}_1^k \psi_2^k + \bar{\psi}_2^k \psi_1^k)\, W'_{12} \,+\, \bar{\psi}_1^k \psi_2^k \bar{\psi}_2^k \psi_1^k \, W''_{12} && \text{(8.13b)} \end{aligned}$$

where on the second line W_{12} and its derivatives are evaluated at the point $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \sum_{\alpha=1}^{k-1} (\bar{\psi}_1^\alpha \psi_2^\alpha + \bar{\psi}_2^\alpha \psi_1^\alpha)$. Of course $W_{23}(\mathbf{n}_2 \cdot \mathbf{n}_3) = W_{23}(\boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_3)$ because $\mathbf{n}_3$ is purely bosonic. The contribution proportional to W'_{12} vanishes in the Grassmann integral because in the other terms of the integrand (i.e. the delta function) ψ_2^k and $\bar{\psi}_2^k$ appear only in the combination $\bar{\psi}_2^k \psi_2^k$. For the contribution proportional to W_{12}, we can use the inductive hypothesis for $(M_1, M_2, M_3) = (k-1, M, 0)$ because $\bar{\psi}_1^k$ and ψ_1^k do not appear; the result is

$$\begin{aligned} I_1 &= F_{W_{12}, W_{23}; N, M}\left(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1 \,+\, 2\sum_{\alpha=1}^{k-1} \bar{\psi}_1^\alpha \psi_1^\alpha ,\; \boldsymbol{\sigma}_3 \cdot \boldsymbol{\sigma}_3 ,\; \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_3\right) && \text{(8.14a)} \\ &= (2\pi)^{-(N-2M)/2} \, \frac{\mathcal{S}_{N-2M-2}}{2} \,\times \\ &\qquad \iint\limits_{s^2+t^2 \leq 1} ds\, dt\, (1 - s^2 - t^2)^{(N-2M-4)/2} \, W_{12}(x_1 s + \widetilde{y}_1 t)\, W_{23}(x_3 s) && \text{(8.14b)} \end{aligned}$$

where we choose the version (8.12) and hence have

$$\begin{aligned} x_1 &= \frac{X_{13}}{X_{33}^{1/2}} && \text{(8.15a)} \\ \widetilde{y}_1 &= \left(\widetilde{X}_{11} - \frac{X_{13}^2}{X_{33}}\right)^{1/2} && \text{(8.15b)} \\ x_3 &= X_{33}^{1/2} && \text{(8.15c)} \end{aligned}$$

where

$$\widetilde{X}_{11} \;=\; \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_1 \,+\, 2\sum_{\alpha=1}^{k-1} \bar{\psi}_1^\alpha \psi_1^\alpha \;. \tag{8.16}$$

Finally, the contribution proportional to W''_{12} is

$$-\bar{\psi}_1^k \psi_1^k \,(2\pi)^{-N/2} \int W''_{12}\left[\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \sum_{\alpha=1}^{k-1} (\bar{\psi}_1^\alpha \psi_2^\alpha + \bar{\psi}_2^\alpha \psi_1^\alpha)\right] \bar{\psi}_2^k \psi_2^k W_{23}(\boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_3)\; \delta(\mathbf{n}_2 \cdot \mathbf{n}_2 - 1) d\mathbf{n}_2 \tag{8.17}$$

where of course $d\mathbf{n}_2 = d\boldsymbol{\sigma}_2\, \mathcal{D}(\bar{\psi}_2, \psi_2)$. We can perform the integral over $\bar{\psi}_2^k$ and ψ_2^k, yielding

$$\int d\bar{\psi}_2^k \, d\psi_2^k \;\; \bar{\psi}_2^k \psi_2^k \;\; \delta(\mathbf{n}_2 \cdot \mathbf{n}_2 - 1) \;=\; -\,\delta(\mathbf{n}'_2 \cdot \mathbf{n}'_2 - 1) \tag{8.18}$$

where $\mathbf{n}_2'$ is the superfield $\mathbf{n}_2$ with the components ψ_2^k and $\bar\psi_2^k$ removed. Hence (8.17) equals

$$\bar\psi_1^k\psi_1^k\,(2\pi)^{-N/2}\!\int\! W_{12}''\left[\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+\sum_{\alpha=1}^{k-1}(\bar\psi_1^\alpha\psi_2^\alpha+\bar\psi_2^\alpha\psi_1^\alpha)\right]\,W_{23}(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3)\,\delta(\mathbf{n}_2'\cdot\mathbf{n}_2'-1)\,d\mathbf{n}_2'\,. \tag{8.19}$$

This is an integral involving $M-1$ fermions, to which we can apply the inductive hypothesis for $(M_1,M_2,M_3)=(k-1,M-1,0)$; the result is

$$\begin{aligned}
I_2 \;&=\; \bar\psi_1^k\psi_1^k\,F_{W_{12}'',W_{23};N,M-1}\Big(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_1\;+\;2\sum_{\alpha=1}^{k-1}\bar\psi_1^\alpha\psi_1^\alpha,\;\boldsymbol{\sigma}_3\cdot\boldsymbol{\sigma}_3,\;\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3\Big) && \text{(8.20a)}\\
&=\; \bar\psi_1^k\psi_1^k\,(2\pi)^{-(N-2M+2)/2}\,\frac{\mathcal{S}_{N-2M}}{2}\;\times\\
&\qquad\iint\limits_{s^2+t^2\le 1} ds\,dt\,(1-s^2-t^2)^{(N-2M-2)/2}\,W_{12}''(x_1s+\widetilde{y}_1t)\,W_{23}(x_3s)\,. && \text{(8.20b)}
\end{aligned}$$

Now

$$\begin{aligned}
&\iint\limits_{s^2+t^2\le 1} ds\,dt\,(1-s^2-t^2)^{(N-2M-2)/2}\,W_{12}''(x_1s+\widetilde{y}_1t)\,W_{23}(x_3s)\\
&= \iint\limits_{s^2+t^2\le 1} ds\,dt\,(1-s^2-t^2)^{(N-2M-2)/2}\,\frac{1}{\widetilde{y}_1}\frac{\partial}{\partial t}W_{12}'(x_1s+\widetilde{y}_1t)\,W_{23}(x_3s) && \text{(8.21)}
\end{aligned}$$

and, up to boundary terms in the integration by parts which vanish whenever $N-2M>2$,

$$\begin{aligned}
&= -\frac{1}{\widetilde{y}_1}\iint\limits_{s^2+t^2\le 1} ds\,dt\,W_{12}'(x_1s+\widetilde{y}_1t)\,W_{23}(x_3s)\,\frac{\partial}{\partial t}(1-s^2-t^2)^{(N-2M-2)/2} && \text{(8.22)}\\
&= -\frac{1}{\widetilde{y}_1}\frac{\partial}{\partial\widetilde{y}_1}\iint\limits_{s^2+t^2\le 1} ds\,dt\,W_{12}(x_1s+\widetilde{y}_1t)\,W_{23}(x_3s)\,\frac{1}{t}\frac{\partial}{\partial t}(1-s^2-t^2)^{(N-2M-2)/2} && \text{(8.23)}\\
&= \frac{N-2M-2}{\widetilde{y}_1}\frac{\partial}{\partial\widetilde{y}_1}\iint\limits_{s^2+t^2\le 1} ds\,dt\,W_{12}(x_1s+\widetilde{y}_1t)\,W_{23}(x_3s)\,(1-s^2-t^2)^{(N-2M-4)/2}\,, && \text{(8.24)}
\end{aligned}$$

By collecting the contributions I_1 and I_2 together, we get [using (3.9)]

$$
\begin{aligned}
& I_{W_{12},W_{23};N,M}(\mathbf{n}_1,\mathbf{n}_3) \\
&\quad = (2\pi)^{-(N-2M)/2}\,\frac{\mathcal{S}_{N-2M-2}}{2} \times \\
&\qquad \iint\limits_{s^2+t^2\le 1} ds\,dt\,(1-s^2-t^2)^{(N-2M-4)/2}\left[1+\bar{\psi}_1^k\psi_1^k\,\frac{1}{\widetilde{y}_1}\,\frac{\partial}{\partial\widetilde{y}_1}\right] W_{12}(x_1 s+\widetilde{y}_1 t)\,W_{23}(x_3 s) \qquad (8.25\text{a}) \\
&\quad = (2\pi)^{-(N-2M)/2}\,\frac{\mathcal{S}_{N-2M-2}}{2}\iint\limits_{s^2+t^2\le 1} ds\,dt\,(1-s^2-t^2)^{(N-2M-4)/2}\,W_{12}(x_1 s+y_1 t)\,W_{23}(x_3 s) \qquad (8.25\text{b})
\end{aligned}
$$

where y_1 is given by (8.12b) and satisfies $y_1^2=\widetilde{y}_1^2+2\bar{\psi}_1^k\psi_1^k$. This completes the proof for the case $M_3=0$.

To remove the restriction that $M_3=0$, we invoke $OSP(N|2M)$ invariance to rotate $\mathbf{n}_3$ to be purely bosonic, as described in Appendix B.2.

9 Series-parallel graphs

A graph G is called *series-parallel* if it can be obtained from a forest by a finite (possibly empty) sequence of series and parallel extensions of edges (i.e. replacing an edge by two edges in series or by two edges in parallel).

The results of the preceding sections allow us to compute the partition function (3.7) whenever the graph G is series-parallel, $N\ge 2$ and $N-2M>1$. It suffices to repeatedly perform series and parallel reductions:

- By Proposition 8.1, two edges in series, with Boltmann weights W_{12} and W_{23}, are equivalent to a single edge with a Boltzmann weight $W_{13}^{\rm eff}$ given by (8.2)/(8.3) evaluated at $\mathbf{n}_1\cdot\mathbf{n}_1=\mathbf{n}_3\cdot\mathbf{n}_3=1$.

- Two edges in parallel, with Boltzmann weights $W_{12}^{\rm A}$ and $W_{12}^{\rm B}$, are of course equivalent to a single edge with Boltzmann weight $W_{12}^{\rm A}W_{12}^{\rm B}$.

After finitely many such reductions, we arrive at a forest, for which the partition function is given by (7.3).

The resulting formula for the partition function — which is initially valid for integers $N\ge 2$ and $M\ge 0$ satisfying $N-2M>1$ — is manifestly an analytic function of the complex variable $N-2M$ that is analytic in $\mathrm{Re}(N-2M)>1$. Moreover, we believe that this formula can be analytically continued to an *entire* analytic function of $N-2M$: the argument would be similar to that in Appendix A for the distribution $(1-x^2)_+^\lambda$ on $\mathbb{R}$, but applied instead to the distribution $(1-s^2-t^2)_+^\lambda$ on $\mathbb{R}^2$ that arises in (8.3). More generally, as illustrated in the following in Section 10, we may be led to study the distribution $(1-\sum_{i=1}^d x_i^2)_+^\lambda$ on $\mathbb{R}^d$. This seems viable, as the Bernsetin–Sato pair of this polynomial is easily evinced from the one of the $d = 1$ case.[9] We leave the details for future work.

10 A general reduction formula

In this section we will derive a reduction formula for a generic smooth function of $k+1$ supervectors in $\mathbb{R}^{N|2M}$. This generalises the analysis of Section 6 ($k = 1$), and of Section 8 ($k = 2$, and a function of a factorised form). Ideally, this approach may lead to a viable inductive approach to Conjecture 1.1, however, as we will see in detail in what follows, our present treatment is not complete.

Let $W = W(\{x_i\}_{1\le i\le k}, \{y_{ij}\}_{1\le i<j\le k})$ be a smooth function of its arguments, and let $\mathbf{n}_1, \ldots, \mathbf{n}_k$ be unit supervectors. We want to study the integral

$$I_{W;N,M}(\mathbf{n}_1, \ldots, \mathbf{n}_k) \;=\; (2\pi)^{-N/2} \int W(\{\mathbf{n}_i \cdot \mathbf{n}\}, \{\mathbf{n}_i \cdot \mathbf{n}_j\})\, \delta(\mathbf{n}\cdot\mathbf{n} - 1)\, d\mathbf{n} \tag{10.1}$$

and determine that the map I satisfies the three conditions presented in Conjecture 1.1. However, we can only show the following:

Proposition 10.1 *Let D^n denote the unit disk in $\mathbb{R}^n$. When $N \ge k$ and $M \ge 0$, we have*

$$(2\pi)^{-N/2} \int W(\{\mathbf{n}_i \cdot \mathbf{n}\}, \{\mathbf{n}_i \cdot \mathbf{n}_j\})\, \delta(\mathbf{n}\cdot\mathbf{n} - 1)\, d\mathbf{n} \;=\; F_{W;N,M}(\{\mathbf{n}_i \cdot \mathbf{n}_j\})\;, \tag{10.2}$$

where

$$\begin{aligned} F_{W;N,M}(\{\mathbf{n}_i \cdot \mathbf{n}_j\}) \;&=\; (2\pi)^{-(N-2M)/2}\, \frac{\mathcal{S}_{N-2M-k}}{2} \;\times \\ &\int_{D^k} d\mathbf{t}\, (1 - |\mathbf{t}|^2)^{(N-2M-k-2)/2}\, W(\{\boldsymbol{\sigma}_i \cdot \mathbf{t}\}, \{\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\}) \end{aligned} \tag{10.3}$$

and $\boldsymbol{\sigma}_i$ are any set of vectors valued in $\mathcal{G}^k_{\mathrm{even}}$ satisfying $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \mathbf{n}_i \cdot \mathbf{n}_j$ for all i, j (the set of such vectors is non-empty).

Our proof will follow the general pattern of our analysis of the one-link integral in Section 6, Proposition 6.2. Analogously to Section 6, therefore, we make use of Proposition B.1, now in its full generality, in order to show that

$$F_{W;N,M}(\{\mathbf{n}_i \cdot \mathbf{n}_j\}) = F_{W;N,M}(\{\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\}) \tag{10.4}$$

[9] In particular, equation (A.18) generalises to

$$\left(\lambda - \sum_{i=1}^{d} \frac{x_i}{2}\frac{d}{dx_i}\right)\left(1 - \sum_{i=1}^{d} x_i^2\right)_+^{\lambda} \;=\; \lambda \left(1 - \sum_{i=1}^{d} x_i^2\right)_+^{\lambda-1}.$$

for some vectors $\boldsymbol{\sigma}_i \in \mathcal{G}^k_{\rm even}$ (in fact, we could even choose them so that $(\boldsymbol{\sigma}_i)_j = 0$ for $j \geq i$). For simplicity of notation, we will assume that the $\boldsymbol{\sigma}_i$'s span the subspace of $\mathbb{R}^N$ with components $1, \ldots, k$.

For the variables of integration, we then write $\mathbf{n} = (\mathbf{t}, \mathbf{z}, \psi, \bar{\psi})$ where $\mathbf{t} \in \mathbb{R}^k$, $\mathbf{z} \in \mathbb{R}^{N-k}$, and $|\mathbf{z}| = r$. We have

$$\begin{aligned} I_{W_{12},W_{23};N,M}(\mathbf{n}_1, \mathbf{n}_3) &= (2\pi)^{-N/2}\, \mathcal{S}_{N-k} \int\limits_{\mathbb{R}^k} d\mathbf{t} \int\limits_0^\infty dr\, r^{N-k-1} \times \\ & W(\{\boldsymbol{\sigma}_i \cdot \mathbf{t}\}, \{\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j\}) \int \mathcal{D}_M(\bar{\psi}, \psi)\, \delta(\mathbf{t}^2 + r^2 + 2\bar{\psi}\cdot\psi - 1) \;. \end{aligned} \tag{10.5}$$

Now

$$\int\limits_0^\infty dr\, r^{N-k-1}\, \delta(\mathbf{t}^2 + r^2 + 2\bar{\psi}\cdot\psi - 1) \;=\; \frac{1}{2}\,(1 - \mathbf{t}^2 - 2\bar{\psi}\cdot\psi)^{(N-k-2)/2} \tag{10.6}$$

and

$$\int \mathcal{D}_M(\bar{\psi}, \psi)\,(1 - \mathbf{t}^2 - 2\,\bar{\psi}\cdot\psi)^{(N-k-2)/2} \;=\; 2^M\, \frac{\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-2M-2}{2}\right)}\,(1 - \mathbf{t}^2)^{(N-2M-k-2)/2} \tag{10.7}$$

[cf. (6.6)]. So using (3.9) we can conclude.

11 The $\mathfrak{osp}(N|2M)$ supersymmetry

The goal of this section is to discuss in more detail the $\mathfrak{osp}(N|2M)$ supersymmetry of our $OSP(N|2M)$-invariant σ-models. We begin by writing down the customary linear realization of the supersymmetry algebra $\mathfrak{osp}(N|2M)$ acting on the "superspace" $\mathbb{R}^{N|2M}$ parametrized by the superfield $\mathbf{n} = (\sigma_1, \ldots, \sigma_N,\, \psi_1, \ldots, \psi_M,\, \bar{\psi}_1, \ldots, \bar{\psi}_M)$. We then derive the nonlinear realization of $\mathfrak{osp}(N|2M)$ corresponding to its action on the unit supersphere in $\mathbb{R}^{N|2M}$ when the constraint $\mathbf{n}\cdot\mathbf{n} = 1$ is solved for σ_1 in terms of $\sigma_2, \ldots, \sigma_N$, $\psi_1, \ldots, \psi_M$, $\bar{\psi}_1, \ldots, \bar{\psi}_M$ and an Ising variable μ.

This section is closely modelled on [20, Section 7], with alterations to work on a superspace of arbitrary dimension rather than just $N = M = 1$, and in the second part to insert the Ising variable μ in appropriate places.

At each vertex $i \in V$ we have a superfield $\mathbf{n}_i = (\sigma_i^1, \ldots, \sigma_i^N,\, \psi_i^1, \ldots, \psi_i^M,\, \bar{\psi}_i^1, \ldots, \bar{\psi}_i^M)$, where $\boldsymbol{\sigma}_i = (\sigma_i^1, \ldots, \sigma_i^N) \in \mathbb{R}^n$ and $\psi_i^1, \ldots, \psi_i^M$, $\bar{\psi}_i^1, \ldots, \bar{\psi}_i^M$ are Grassmann variables. The "superspace" $\mathbb{R}^{N|2M}$ is equipped with the scalar product

$$\mathbf{n}_i \cdot \mathbf{n}_j \;=\; \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \;+\; \sum_{\alpha=1}^M (\bar{\psi}_i^\alpha \psi_j^\alpha - \psi_i^\alpha \bar{\psi}_j^\alpha) \;. \tag{11.1}$$

In what follows, we shall for brevity suppress the site indices i. Our global symmetry transformations act simultaneously on all sites, so in restoring the site indices various formulae would have to be prefaced by $\sum_{i \in V}$.

11.1 The Lie superalgebras $\mathfrak{gl}(N|M)$ and $\mathfrak{osp}(N|2M)$

We recall [22,40,41,44,49,51] that a *Lie superalgebra* (over $\mathbb{R}$) is a $\mathbb{Z}_2$-graded algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ (over $\mathbb{R}$) in which the product $\langle \cdot\,, \cdot \rangle$ satisfies a $\mathbb{Z}_2$-graded version of the antisymmetry and Jacobi identities:

$$\langle a, b\rangle \;=\; -(-1)^{\alpha\beta}\langle b, a\rangle \qquad \text{for } a \in \mathcal{A}_\alpha,\ b \in \mathcal{A}_\beta \tag{11.2}$$

and

$$(-1)^{\alpha\gamma}\langle a, \langle b, c\rangle\rangle \;+\; (-1)^{\alpha\beta}\langle b, \langle c, a\rangle\rangle \;+\; (-1)^{\beta\gamma}\langle c, \langle a, b\rangle\rangle \;=\; 0 \qquad \text{for } a \in \mathcal{A}_\alpha,\ b \in \mathcal{A}_\beta,\ c \in \mathcal{A}_\gamma\ . \tag{11.3}$$

In particular, if $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is a $\mathbb{Z}_2$-graded associative algebra (over $\mathbb{R}$) and we define the supercommutator

$$\langle A, B\rangle \;=\; AB - (-1)^{\alpha\beta} BA \qquad \text{for } a \in \mathcal{A}_\alpha,\ b \in \mathcal{A}_\beta\ , \tag{11.4}$$

then $\mathcal{A}$ equipped with $\langle \cdot\,, \cdot \rangle$ is a Lie superalgebra. As a special case, if $V = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space (over $\mathbb{R}$), then the associative algebra $\mathrm{End}(V)$ of endomorphisms of V [that is, linear maps of V into itself] carries a natural $\mathbb{Z}_2$-grading, and $\mathrm{End}(V)$ equipped with the supercommutator $\langle \cdot\,, \cdot \rangle$ is a Lie superalgebra, denoted $\mathfrak{gl}(V) = \mathfrak{gl}(V_0, V_1)$ and called the *general Lie superalgebra of* V. We also write $\mathfrak{gl}(N|M) = \mathfrak{gl}(\mathbb{R}^N, \mathbb{R}^M)$.

Let us fix a basis in $\mathfrak{gl}(N|M)$ by defining some matrices of size $(N+M)\times(N+M)$ as follows. We use uppercase Latin letters running from 1 to $N+M$, and define $s(A) = 0$ if $1 \le A \le N$ (i.e. A is bosonic) and $s(A) = 1$ if $N+1 \le A \le N+M$ (i.e. A is fermionic); we also define $s(A,B) = s(A) + s(B) \bmod 2$. Then let E_{AB} be the matrix with 1 in entry AB and zeroes elsewhere. These basis elements satisfy the supercommutation relations

$$\langle \mathsf{E}_{AB}, \mathsf{E}_{CD}\rangle \;=\; \delta_{BC}\mathsf{E}_{AD} \;-\; (-1)^{s(A,B)\, s(C,D)}\delta_{AD}\mathsf{E}_{CB}\ . \tag{11.5}$$

Alternatively, we can make the subspaces explicit by using lowercase Latin letters for the bosonic indices, which run from 1 to N, and Greek letters for the fermionic indices, which run from 1 to M. We then let E_{ij} be the matrix with 1 in entry ij and zeroes elsewhere; the matrices $\mathsf{E}_{i\alpha}$, $\mathsf{E}_{\alpha i}$ and $\mathsf{E}_{\alpha\beta}$ are defined analogously. Then the matrices of the form E_{ij} and $\mathsf{E}_{\alpha\beta}$ form a basis of the even subspace $\mathfrak{gl}(N|M)_0$, while the matrices $\mathsf{E}_{i\alpha}$ and $\mathsf{E}_{\alpha i}$ form a basis of the odd subspace $\mathfrak{gl}(N|M)_1$. These basis elements satisfy the commutation/anticommutation relations

$$\begin{aligned}
[\mathsf{E}_{ij}, \mathsf{E}_{kl}] &= \delta_{jk}\mathsf{E}_{il} - \delta_{il}\mathsf{E}_{kj} && (11.6a)\\
[\mathsf{E}_{ij}, \mathsf{E}_{k\alpha}] &= \delta_{jk}\mathsf{E}_{i\alpha} && (11.6b)\\
[\mathsf{E}_{ij}, \mathsf{E}_{\alpha k}] &= -\delta_{ik}\mathsf{E}_{\alpha j} && (11.6c)\\
[\mathsf{E}_{ij}, \mathsf{E}_{\alpha\beta}] &= 0 && (11.6d)\\
\{\mathsf{E}_{i\alpha}, \mathsf{E}_{j\beta}\} &= 0 && (11.6e)\\
\{\mathsf{E}_{i\alpha}, \mathsf{E}_{\beta j}\} &= \delta_{\alpha\beta}\mathsf{E}_{ij} + \delta_{ij}\mathsf{E}_{\beta\alpha} && (11.6f)\\
[\mathsf{E}_{i\alpha}, \mathsf{E}_{\beta\gamma}] &= \delta_{\alpha\beta}\mathsf{E}_{i\gamma} && (11.6g)\\
\{\mathsf{E}_{\alpha i}, \mathsf{E}_{\beta j}\} &= 0 && (11.6h)\\
[\mathsf{E}_{\alpha i}, \mathsf{E}_{\beta\gamma}] &= -\delta_{\alpha\gamma}\mathsf{E}_{\beta i} && (11.6i)\\
[\mathsf{E}_{\alpha\beta}, \mathsf{E}_{\gamma\delta}] &= \delta_{\beta\gamma}\mathsf{E}_{\alpha\delta} - \delta_{\alpha\delta}\mathsf{E}_{\gamma\beta} && (11.6j)
\end{aligned}$$

We now define [22,40,41,44,49,51] the Lie superalgebra $\mathfrak{osp}(N|2M)$ as a subalgebra of $\mathfrak{gl}(N|2M)$. As will be seen in the next subsection, $\mathfrak{osp}(N|2M)$ can be regarded as the Lie superalgebra of infinitesimal rotations in the "superspace" $\mathbb{R}^{N|2M}$ that leave invariant the scalar product (11.1). It is therefore generated by three types of transformations:

(i) Infinitesimal rotations in the N bosonic variables. These form a grading-even subalgebra that is isomorphic to $\mathfrak{so}(N)$.

(ii) Infinitesimal "rotations" in the $2M$ fermionic variables. These form a grading-even subalgebra that is isomorphic to $\mathfrak{sp}(2M)$.

(iii) Transformations that mix the bosonic and fermionic variables. These transformations are grading-odd and do not form a subalgebra.

More precisely, the even subspace $\mathfrak{osp}(N|2M)_0$ consists of matrices of the form

$$M_0 = \left(\begin{array}{c|cc} A & \multicolumn{2}{c}{0} \\ \hline 0 & B_0 & B_+ \\ & B_- & -B_0^{\mathrm{T}} \end{array}\right) \tag{11.7}$$

where A is antisymmetric, B_+ and B_- are symmetric, and B_0 is arbitrary; these matrices form a subalgebra isomorphic to $\mathfrak{so}(N) \oplus \mathfrak{sp}(2M)$. The odd subspace $\mathfrak{osp}(N|2M)_1$ consists of matrices of the form

$$M_1 = \left(\begin{array}{c|cc} 0 & C_- & C_+ \\ \hline -C_+^{\mathrm{T}} & \multicolumn{2}{c}{0} \\ C_-^{\mathrm{T}} & & \end{array}\right) \tag{11.8}$$

where C_- and C_+ are arbitrary. With respect to the $\mathbb{Z}$-grading that assigns a "charge" $0, +1, -1$ to the three subspaces of $V = \mathbb{R}^N \oplus (\mathbb{R}^M \oplus \mathbb{R}^M)$, the matrices A and B_0 have charge 0, the matrices $B_\pm$ have charge ± 2, and the matrices $C_\pm$ have charge ± 1.

Let us fix a basis in $\mathfrak{osp}(N|2M)$ by defining some matrices of size $(N+2M) \times (N+2M)$ as follows. We use lowercase Latin letters for the bosonic indices, which run from 1 to N; Greek letters for the first fermionic indices, which run from 1 to M; and barred Greek letters for the second fermionic indices, which also run from 1 to M. We then define the following new matrices in terms of the matrices E used for $\mathfrak{gl}(N|2M)$:

- A_{ij} is the matrix $\mathsf{E}_{ij} - \mathsf{E}_{ji}$.
- $(\mathsf{B}_0)_{\alpha\beta}$ is the matrix $\mathsf{E}_{\alpha\beta} - \mathsf{E}_{\overline{\beta}\overline{\alpha}}$.
- $(\mathsf{B}_+)_{\alpha\beta}$ is the matrix $\mathsf{E}_{\alpha\overline{\beta}} + \mathsf{E}_{\beta\overline{\alpha}}$. Note in particular that $(\mathsf{B}_+)_{\alpha\alpha}$ has a single nonzero entry, with value 2.
- $(\mathsf{B}_-)_{\alpha\beta}$ is analogous but reversing barred and unbarred.
- $(\mathsf{C}_-)_{i\alpha}$ is the matrix $\mathsf{E}_{i\alpha} + \mathsf{E}_{\overline{\alpha}i}$.
- $(\mathsf{C}_+)_{i\alpha}$ is the matrix $\mathsf{E}_{i\overline{\alpha}} - \mathsf{E}_{\alpha i}$.

Then a generic element M of $\mathfrak{osp}(N|2M)$ can be written as

$$\begin{aligned} M \;=\; & \sum_{i<j} a_{ij}\mathsf{A}_{ij} + \sum_{\alpha,\beta}(b_0)_{\alpha\beta}(\mathsf{B}_0)_{\alpha\beta} + \sum_{\alpha\le\beta}(b_+)_{\alpha\beta}(\mathsf{B}_+)_{\alpha\beta} + \sum_{\alpha\le\beta}(b_-)_{\alpha\beta}(\mathsf{B}_-)_{\alpha\beta} \\ & + \sum_{i,\alpha}(c_-)_{i\alpha}(\mathsf{C}_-)_{i\alpha} + \sum_{i,\alpha}(c_+)_{i\alpha}(\mathsf{C}_+)_{i\alpha} \end{aligned} \tag{11.9}$$

for suitable coefficients $a_{ij}, (b_0)_{\alpha\beta}, (b_+)_{\alpha\beta}, (b_-)_{\alpha\beta}, (c_-)_{i\alpha}, (c_+)_{i\alpha}$. These basis elements satisfy a list of commutation/anticommutation relations as follows. First of all, some of the commutators vanish: $[\mathsf{A}_{ij}, (\mathsf{B}_0)_{\alpha\beta}] = [\mathsf{A}_{ij}, (\mathsf{B}_\pm)_{\alpha\beta}] = 0$ because $[X, Y] = 0$ whenever X and Y are in commuting even subalgebras; and $[(\mathsf{B}_\pm)_{\alpha\beta}, (\mathsf{B}_\pm)_{\gamma\delta}] = [(\mathsf{B}_\pm)_{\alpha\beta}, (\mathsf{C}_\pm)_{i\gamma}] = 0$ because in $\mathfrak{osp}(N|2M)$ there are no operators of charge ± 3

or ±4. The nontrivial commutation/anticommutation relations are then

$$\begin{aligned}
[\mathsf{A}_{ij}, \mathsf{A}_{kl}] &= \delta_{jk}\mathsf{A}_{il} + \delta_{il}\mathsf{A}_{jk} + \delta_{ik}\mathsf{A}_{lj} + \delta_{jl}\mathsf{A}_{ki} && (11.10a)\\
[\mathsf{A}_{ij}, (\mathsf{C}_\pm)_{k\alpha}] &= \delta_{jk}(\mathsf{C}_\pm)_{i\alpha} - \delta_{ik}(\mathsf{C}_\pm)_{j\alpha} && (11.10b)\\
[(\mathsf{B}_0)_{\alpha\beta}, (\mathsf{B}_0)_{\gamma\delta}] &= \delta_{\gamma\beta}(\mathsf{B}_0)_{\alpha\delta} - \delta_{\alpha\delta}(\mathsf{B}_0)_{\gamma\beta} && (11.10c)\\
[(\mathsf{B}_0)_{\alpha\beta}, (\mathsf{B}_+)_{\gamma\delta}] &= \delta_{\gamma\beta}(\mathsf{B}_+)_{\alpha\delta} + \delta_{\delta\beta}(\mathsf{B}_+)_{\alpha\gamma} && (11.10d)\\
[(\mathsf{B}_0)_{\alpha\beta}, (\mathsf{B}_-)_{\gamma\delta}] &= -\delta_{\alpha\delta}(\mathsf{B}_-)_{\gamma\beta} - \delta_{\alpha\gamma}(\mathsf{B}_-)_{\delta\beta} && (11.10e)\\
[(\mathsf{B}_0)_{\alpha\beta}, (\mathsf{C}_+)_{i\gamma}] &= \delta_{\beta\gamma}(\mathsf{C}_+)_{i\alpha} && (11.10f)\\
[(\mathsf{B}_0)_{\alpha\beta}, (\mathsf{C}_-)_{i\gamma}] &= -\delta_{\alpha\gamma}(\mathsf{C}_-)_{i\beta} && (11.10g)\\
[(\mathsf{B}_+)_{\alpha\beta}, (\mathsf{B}_-)_{\gamma\delta}] &= \delta_{\beta\gamma}(\mathsf{B}_0)_{\alpha\delta} + \delta_{\alpha\gamma}(\mathsf{B}_0)_{\beta\delta} + \delta_{\beta\delta}(\mathsf{B}_0)_{\alpha\gamma} + \delta_{\alpha\delta}(\mathsf{B}_0)_{\beta\gamma} && (11.10h)\\
[(\mathsf{B}_\pm)_{\alpha\beta}, (\mathsf{C}_\mp)_{i\gamma}] &= -\delta_{\beta\gamma}(\mathsf{C}_\pm)_{i\alpha} - \delta_{\alpha\gamma}(\mathsf{C}_\pm)_{i\beta} && (11.10i)\\
\{(\mathsf{C}_\pm)_{i\alpha}, (\mathsf{C}_\pm)_{j\beta}\} &= \pm\delta_{ij}(\mathsf{B}_\pm)_{\alpha\beta} && (11.10j)\\
\{(\mathsf{C}_+)_{i\alpha}, (\mathsf{C}_-)_{j\beta}\} &= \delta_{\alpha\beta}\mathsf{A}_{ij} - \delta_{ij}(\mathsf{B}_0)_{\alpha\beta} && (11.10k)
\end{aligned}$$

11.2 Linear realization of $\mathfrak{gl}(N|M)$ and $\mathfrak{osp}(N|2M)$ on superspace

The general Lie superalgebra $\mathfrak{gl}(N|M)$ acts in a natural way on the "superspace" $\mathbb{R}^{N|M}$ generated by bosonic variables $\sigma_1, \ldots, \sigma_N$ and fermionic variables $\chi_1, \ldots, \chi_M$. To show this, let us introduce the standard differential operators

$$d_i \;=\; \partial/\partial\sigma_i\,, \quad \partial_\alpha \;=\; \partial/\partial\chi_\alpha\,. \tag{11.11}$$

We can then consider the $\mathbb{Z}_2$-graded associative algebra generated by all the σ, χ and d, ∂ together with their usual commutation/anticommutation relations: this is the super-analogue of the Weyl algebra[10] and is known as the *Weyl superalgebra* $\mathcal{A}_{N|M}$ (also called the *Weyl-Clifford (super)algebra* [22, section 5.1]). When $\mathcal{A}_{N|M}$ is equipped with the supercommutator (11.4), it becomes a Lie superalgebra. Inside the Lie superalgebra $\mathcal{A}_{N|M}$ we can consider the subalgebra $\mathcal{L}_{N|M}$ of "vector fields" $\sum_i P_i d_i + \sum_\alpha Q_\alpha \partial_\alpha$ where P_i, Q_α are polynomials in σ and χ.[11] We then introduce the following elements of $\mathcal{L}_{N|M}$:

$$\begin{aligned}
\mathcal{E}_{ij} &= \sigma_i d_j && (11.12a)\\
\mathcal{E}_{i\alpha} &= \sigma_i \partial_\alpha && (11.12b)\\
\mathcal{E}_{\alpha i} &= \chi_\alpha d_i && (11.12c)\\
\mathcal{E}_{\alpha\beta} &= \chi_\alpha \partial_\beta && (11.12d)
\end{aligned}$$

[10] See [23] for an excellent introduction to the Weyl algebra.

[11] That is, P_i, Q_α belong to the Grassmann algebra $\mathcal{G}_{N|M} = \mathbb{R}[\sigma_1, \ldots, \sigma_N][\chi_1, \ldots, \chi_M]_{\text{Grass}}$, i.e. the Grassmann algebra with generators $\chi_1, \ldots, \chi_M$ and coefficient ring being the ring $\mathbb{R}[\sigma_1, \ldots, \sigma_N]$ of polynomials with real coefficients in $\sigma_1, \ldots, \sigma_N$.

It can be straightforwardly verified that these elements of $\mathcal{L}_{N|M}$ behave under the supercommutator (11.4) exactly as do the corresponding basis elements of $\mathfrak{gl}(N|M)$ in (11.6). Therefore, the map $\mathsf{E}_{ij} \mapsto \mathcal{E}_{ij}$, etc. is a Lie-superalgebra homomorphism of $\mathfrak{gl}(N|M)$ into $\mathcal{L}_{N|M}$; we call it the *standard representation* of $\mathfrak{gl}(N|M)$ on the "superspace" $\mathbb{R}^{N|M}$.

Since $\mathfrak{osp}(N|2M)$ is a subalgebra of $\mathfrak{gl}(N|2M)$, its action on the "superspace" $\mathbb{R}^{N|2M}$ can be defined immediately by restriction of the action of $\mathfrak{gl}(N|2M)$. It is convenient to view $\mathbb{R}^{N|2M}$ as generated by bosonic variables $\sigma_1, \ldots, \sigma_N$ and fermionic variables $\psi_1, \ldots, \psi_M$, $\bar{\psi}_1, \ldots, \bar{\psi}_M$ and equipped with the scalar product (11.1). We then introduce the differential operators

$$d_i \;=\; \partial/\partial\sigma_i\,, \quad \partial_\alpha \;=\; \partial/\partial\psi_\alpha\,, \quad \overline{\partial}_\alpha \;=\; \partial/\partial\bar{\psi}_\alpha \tag{11.13}$$

and consider the corresponding Weyl superalgebra $\mathcal{A}_{N|2M}$. When $\mathcal{A}_{N|2M}$ is equipped with the supercommutator (11.4), it becomes a Lie superalgebra. Inside the Lie superalgebra $\mathcal{A}_{N|2M}$ we can consider the subalgebra $\mathcal{L}_{N|2M}$ of "vector fields" $\sum_i P_i d_i + \sum_\alpha (Q_\alpha \partial_\alpha + R_\alpha \overline{\partial}_\alpha)$ where P_i, Q_α, R_α are polynomials in $\sigma, \psi, \bar{\psi}$. The action (11.12) of $\mathfrak{gl}(N|2M)$ on $\mathbb{R}^{N|2M}$, restricted to $\mathfrak{osp}(N|2M)$, is immediately deduced from the definitions of A, B and C in terms of E (cf. page 39 above) and the isomorphism (11.12):

$$\mathcal{A}_{ij} \;=\; \mathcal{E}_{ij} - \mathcal{E}_{ji} \;=\; \sigma_i d_j - \sigma_j d_i \tag{11.14a}$$

$$(\mathcal{B}_0)_{\alpha\beta} \;=\; \mathcal{E}_{\alpha\beta} - \mathcal{E}_{\overline{\beta}\overline{\alpha}} \;=\; \psi_\alpha \partial_\beta - \bar{\psi}_\beta \overline{\partial}_\alpha \tag{11.14b}$$

$$(\mathcal{B}_+)_{\alpha\beta} \;=\; \mathcal{E}_{\alpha\overline{\beta}} + \mathcal{E}_{\beta\overline{\alpha}} \;=\; \psi_\alpha \overline{\partial}_\beta + \psi_\beta \overline{\partial}_\alpha \tag{11.14c}$$

$$(\mathcal{B}_-)_{\alpha\beta} \;=\; \mathcal{E}_{\overline{\alpha}\beta} + \mathcal{E}_{\overline{\beta}\alpha} \;=\; \bar{\psi}_\alpha \partial_\beta + \bar{\psi}_\beta \partial_\alpha \tag{11.14d}$$

$$(\mathcal{C}_+)_{i\alpha} \;=\; \mathcal{E}_{i\overline{\alpha}} - \mathcal{E}_{\alpha i} \;=\; \sigma_i \overline{\partial}_\alpha - \psi_\alpha d_i \tag{11.14e}$$

$$(\mathcal{C}_-)_{i\alpha} \;=\; \mathcal{E}_{i\alpha} + \mathcal{E}_{\overline{\alpha} i} \;=\; \sigma_i \partial_\alpha + \bar{\psi}_\alpha d_i \tag{11.14f}$$

Since $\mathsf{E}_{ij} \mapsto \mathcal{E}_{ij}$, etc. is a Lie-superalgebra homomorphism of $\mathfrak{gl}(N|2M)$ into $\mathcal{L}_{N|2M}$, it follows immediately that $\mathsf{A}_{ij} \mapsto \mathcal{A}_{ij}$, etc. is a Lie-superalgebra homomorphism of $\mathfrak{osp}(N|2M)$ into $\mathcal{L}_{N|2M}$. That is, the generators (11.14) behave under the supercommutator (11.4) exactly as do the corresponding basis elements of $\mathfrak{osp}(N|2M)$ in (11.10).

Now equip $\mathbb{R}^{N|2M}$ with the "quadratic form"

$$\mathbf{n}\cdot\mathbf{n} \;=\; \sum_{i=1}^{N} \sigma_i^2 + 2\sum_{\alpha=1}^{M} \bar{\psi}_\alpha \psi_\alpha \,. \tag{11.15}$$

It is easily verified that all the $\mathfrak{osp}(N|2M)$ generators (11.14) annihilate $\mathbf{n}\cdot\mathbf{n}$. Indeed, with a bit more work it can be shown that $\mathfrak{osp}(N|2M)$ is precisely the subalgebra of $\mathfrak{gl}(N|2M)$ whose images in the standard representation annihilate $\mathbf{n}\cdot\mathbf{n}$. This justifies our assertion that $\mathfrak{osp}(N|2M)$ can be regarded as the Lie superalgebra of infinitesimal rotations in the "superspace" $\mathbb{R}^{N|2M}$ that leave invariant the scalar product (11.1).

Finally, let us make an observation that will be useful later. The space $\mathcal{L}_{N|2M}$ equipped with the supercommutator is of course a Lie superalgebra, and the image of $\mathfrak{osp}(N|2M)$ under the homomorphism (11.14) is a subalgebra of this Lie superalgebra. But we can also consider either of these two spaces of differential operators with respect to the *ordinary* product, and then consider the (associative) polynomial algebras that they generate; these are subalgebras of the Weyl superalgebra $\mathcal{A}_{N|2M}$. Now, it turns out that whenever N and M are both nonzero, the polynomial algebra generated by $\mathcal{A}$, $\mathcal{B}_0$, $\mathcal{B}_\pm$ and $\mathcal{C}_\pm$ is in fact generated by $\mathcal{C}_\pm$ alone. This is a corollary of (11.10j) and (11.10k), which imply that

$$
\begin{aligned}
(\mathcal{B}_\pm)_{\alpha\beta} &= \pm\big[(\mathcal{C}_\pm)_{i\alpha}(\mathcal{C}_\pm)_{i\beta} + (\mathcal{C}_\pm)_{i\beta}(\mathcal{C}_\pm)_{i\alpha}\big] && \text{for all } \alpha,\beta \text{ and all } i && (11.16a)\\
(\mathcal{B}_0)_{\alpha\beta} &= -\big[(\mathcal{C}_+)_{i\alpha}(\mathcal{C}_-)_{i\beta} + (\mathcal{C}_-)_{i\beta}(\mathcal{C}_+)_{i\alpha}\big] && \text{for all } \alpha,\beta \text{ and all } i && (11.16b)\\
\mathcal{A}_{ij} &= (\mathcal{C}_+)_{j\alpha}(\mathcal{C}_-)_{i\alpha} + (\mathcal{C}_-)_{i\alpha}(\mathcal{C}_+)_{j\alpha} && \text{for all } i\neq j \text{ and all } \alpha && (11.16c)
\end{aligned}
$$

It follows that any polynomial in $\sigma_1,\ldots,\sigma_N,\psi_1,\ldots,\psi_M,\bar\psi_1,\ldots,\bar\psi_M$ that is annihilated by all the $\mathcal{C}_\pm$ is in fact annihilated by all of $\mathfrak{osp}(N|2M)$.

A corollary of this fact was used in [20], to simplify the verification that our expressions f_{ij} of (1.2), as well as their generalizations f_A, are indeed annihilated by the $\mathfrak{osp}(1|2)$ analogues of our generators (11.14). For this reason we expect the general version of this fact to be similarly useful in future work.

11.3 Nonlinear realization of $\mathfrak{osp}(N|2M)$ on unit supersphere

Now let us consider a σ-model in which the superfields $\mathbf{n}$ are constrained to lie on the unit supersphere in $\mathbb{R}^{N|2M}$, i.e. to satisfy the constraint

$$
\mathbf{n}\cdot\mathbf{n} \;\equiv\; \sum_{i=1}^{N}\sigma_i^2 + 2\sum_{\alpha=1}^{M}\bar\psi_\alpha\psi_\alpha \;=\; 1\,. \qquad (11.17)
$$

We solve this constraint for σ_1 in terms of $\boldsymbol{\sigma}' = (\sigma_2,\ldots,\sigma_N)$ and $\psi_1,\ldots,\psi_M$, $\bar\psi_1,\ldots,\bar\psi_M$ by writing

$$
\sigma_1 \;=\; \mu\left(1-\sum_{i=2}^{N}\sigma_i^2 - 2\sum_{\alpha=1}^{M}\bar\psi_a\psi_a\right)^{1/2} \qquad (11.18)
$$

where $\mu=\pm1$ is an Ising variable. [This expression can then be expanded, if we wish, as a polynomial in the $\psi,\bar\psi$ by exploiting nilpotence; but for $M>1$ the resulting formula is not as simple as (1.4)/(1.5).]

Since all the $\mathfrak{osp}(N|2M)$ generators (11.14) annihilate $\mathbf{n}\cdot\mathbf{n}$, it follows that $\mathfrak{osp}(N|2M)$ acts also on the unit supersphere $\mathbf{n}\cdot\mathbf{n}=1$. We would like to compute explicitly this action when the supersphere is parametrized by $\sigma_2,\ldots,\sigma_N$, $\psi_1,\ldots,\psi_M$, $\bar\psi_1,\ldots,\bar\psi_M$ and μ. The generators of this action will be elements of the Lie superalgebra $\mathcal{L}_{N-1|2M}$ whose generators are $\psi_\alpha,\bar\psi_\alpha,\partial_\alpha,\overline{\partial}_\alpha$ ($1\le\alpha\le M$) as before, but now σ_i,d_i only for $2\le i\le N$. Clearly all the $\mathfrak{osp}(N|2M)$ generators that do not involve $i=1$ are simply given by (11.14) as before. We therefore need only figure out how to define

$\mathcal{A}_{1j}$, $(\mathcal{C}_+)_{1\alpha}$ and $(\mathcal{C}_-)_{1\alpha}$ within $\mathcal{L}_{N-1|2M}$ so as to maintain the commutation relations (11.10). It turns out that the simplest guess works: in the definitions (11.14) we replace σ_1 by the composite expression (11.18), and replace d_1 by zero. That is, we define

$$\begin{aligned} \mathcal{A}_{1j} &= \sigma_1(\boldsymbol{\sigma'}, \psi, \bar{\psi})\, d_j && (11.19a)\\ (\mathcal{C}_+)_{1\alpha} &= \sigma_1(\boldsymbol{\sigma'}, \psi, \bar{\psi})\, \overline{\partial}_\alpha && (11.19b)\\ (\mathcal{C}_-)_{1\alpha} &= \sigma_1(\boldsymbol{\sigma'}, \psi, \bar{\psi})\, \partial_\alpha && (11.19c) \end{aligned}$$

where $\sigma_1(\boldsymbol{\sigma'}, \psi, \bar{\psi})$ denotes the right-hand side of (11.18).

In order to prove the validity of the homomorphism defined above, we shall repeat the verification of all the pertinent relations in (11.10) — namely, (11.10a,b,f,g,i,j,k) — with one or more of the Latin indices specialised to 1. [In principle, we shall have a shortcut on the verification of (11.10a,b) coming from the polynomial representation (11.16), but we avoid to exploit this fact here.]

The following relations appear repeatedly in this verification:

$$\begin{aligned} \overline{\partial}_\alpha \sigma_1(\boldsymbol{\sigma'}, \psi, \bar{\psi}) &= -\psi_\alpha\, \sigma_1(\boldsymbol{\sigma'}, \psi, \bar{\psi})^{-1} && (11.20a)\\ \partial_\alpha \sigma_1(\boldsymbol{\sigma'}, \psi, \bar{\psi}) &= \bar{\psi}_\alpha\, \sigma_1(\boldsymbol{\sigma'}, \psi, \bar{\psi})^{-1} && (11.20b)\\ \mathrm{d}_i \sigma_1(\boldsymbol{\sigma'}, \psi, \bar{\psi}) &= -\sigma_i\, \sigma_1(\boldsymbol{\sigma'}, \psi, \bar{\psi})^{-1} && (11.20c) \end{aligned}$$

Note that these relations have no dependence on μ, because μ is a common factor on both sides. The verification of the commutation relations can be performed by straightforward calculus. Let us just make explicit a few examples, in order to highlight the involved cancellations. A first case is the verification of

$$[(\mathcal{B}_0)_{\alpha\beta}, (\mathcal{C}_+)_{1\gamma}] = \delta_{\beta\gamma}(\mathcal{C}_+)_{1\alpha} \tag{11.21}$$

and in fact

$$\begin{aligned} [(\mathcal{B}_0)_{\alpha\beta}, (\mathcal{C}_+)_{1\gamma}] &= (\psi_\alpha \partial_\beta - \bar{\psi}_\beta \overline{\partial}_\alpha)\sigma_1 \overline{\partial}_\gamma - \sigma_1 \overline{\partial}_\gamma (\psi_\alpha \partial_\beta - \bar{\psi}_\beta \overline{\partial}_\alpha)\\ &= (\psi_\alpha \bar{\psi}_\beta + \bar{\psi}_\beta \psi_\alpha)\sigma_1^{-1} \overline{\partial}_\gamma + \delta_{\beta\gamma}\sigma_1 \overline{\partial}_\alpha = \delta_{\beta\gamma}\sigma_1 \overline{\partial}_\alpha = \delta_{\beta\gamma}(\mathcal{C}_+)_{1\alpha} \end{aligned} \tag{11.22}$$

and similarly, for $2 \le i \le N$ we shall have

$$\{(\mathcal{C}_+)_{i\alpha}, (\mathcal{C}_-)_{1\beta}\} = -\delta_{\alpha\beta}\mathcal{A}_{1i} \tag{11.23}$$

and in fact

$$\begin{aligned} \{(\mathcal{C}_+)_{i\alpha}, (\mathcal{C}_-)_{1\beta}\} &= (\sigma_i \overline{\partial}_\alpha - \psi_\alpha d_i)\sigma_1 \partial_\beta + \sigma_1 \partial_\beta (\sigma_i \overline{\partial}_\alpha - \psi_\alpha d_i)\\ &= (-\sigma_i \psi_\alpha + \psi_\alpha \sigma_i)\sigma_1^{-1} \partial_\beta - \delta_{\alpha\beta}\sigma_1 d_i = -\delta_{\alpha\beta}\sigma_1 d_i = -\delta_{\alpha\beta}\mathcal{A}_{1i} \end{aligned} \tag{11.24}$$

The same equation, when both Latin indices are equal to 1, shall give instead

$$\{(\mathcal{C}_+)_{1\alpha}, (\mathcal{C}_-)_{1\beta}\} = -(\mathcal{B}_0)_{\alpha\beta} \tag{11.25}$$

and in fact

$$\begin{aligned}\{(\mathcal{C}_+)_{1\alpha}, (\mathcal{C}_-)_{1\beta}\} &= \sigma_1\overline{\partial}_\alpha\sigma_1\partial_\beta + \sigma_1\partial_\beta\sigma_1\overline{\partial}_\alpha \\ &= -\sigma_1\psi_\alpha\sigma_1^{-1}\partial_\beta + \sigma_1\bar{\psi}_\beta\sigma_1^{-1}\overline{\partial}_\alpha = -\psi_\alpha\partial_\beta + \bar{\psi}_\beta\overline{\partial}_\alpha = -(\mathcal{B}_0)_{\alpha\beta}\end{aligned} \tag{11.26}$$

Finally, for what concerns relation

$$[(\mathcal{C}_+)_{1\alpha}, \mathcal{A}_{1j}] = (\mathcal{C}_+)_{j\alpha} \tag{11.27}$$

we have

$$\begin{aligned}[(\mathcal{C}_+)_{1\alpha}, \mathcal{A}_{1j}] &= \sigma_1\overline{\partial}_\alpha\sigma_1\mathrm{d}_j - \sigma_1\mathrm{d}_j\sigma_1\overline{\partial}_\alpha \\ &= -\sigma_1\psi_\alpha\sigma_1^{-1}\mathrm{d}_j + \sigma_1\sigma_j\sigma_1^{-1}\partial_\alpha = -\psi_\alpha\mathrm{d}_j + \sigma_j\overline{\partial}_\alpha = (\mathcal{C}_+)_{j\alpha}\,.\end{aligned} \tag{11.28}$$

12 Some puzzles concerning the critical behavior

In this paper we have obtained identities between partition functions of different models: notably, the $OSP(1|2)$-invariant σ-model (4.7) is equivalent to the model (4.13) of fermionic variables coupled to Ising variables, and also to the model (4.15) of spanning forests coupled to Ising variables. Moreover, in those cases where the Ising variables decouple (as discussed in Section 4), the σ-model is equivalent to the purely fermionic model (1.1a) and to the spanning-forest model (1.1b).

Of course, it is also well known [39, 52, 53, 59] that the spanning-forest model is equivalent to the $q \to 0$ limit of q-state Potts model in the Fortuin–Kasteleyn random-cluster representation.

Putting these identities together leads to some puzzling questions concerning the critical behavior of these models, which we divide into two parts: those concerning the upper critical dimension (4 or 6), and those concerning the critical behavior in dimension 2.

12.1 Upper critical dimension

The q-state Potts model has a field-theoretic realization that includes a cubic interaction [1, 2, 29, 61] and hence is expected to have upper critical dimension 6. More precisely, one expects [25] that the upper critical dimension is 6 for ferromagnetic Potts (= random-cluster) models whenever $0 \le q < 2$; for the Ising model $q = 2$, the upper critical dimension is of course 4, while for $q > 2$ one expects that in high enough dimension (and in particular in all dimensions ≥ 4) there are only first-order transitions. This scenario is supported by Monte Carlo simulations of the spanning-forest model ($q = 0$) in dimensions 3,4,5 [25] and of percolation ($q = 1$) in dimensions 3 [58, 60], 4 [4, 47] and 5 [47], as well as by high-temperature series expansions for generic q in all dimensions [30]. Moreover, the estimated critical exponents are in reasonable agreement with those predicted by the field-theoretic renormalization group in dimension $6 - \epsilon$ [1, 29].

On the other hand, the field-theoretic realization of the $OSP(1|2)$-invariant σ-model has only quartic interactions $(\mathbf{n} \cdot \mathbf{n})^2$ and hence is expected to have upper critical dimension 4; the same reasoning applies to the $O(N)$-invariant σ-model at $N = -1$. We thus have a paradox: two models are equivalent, but one has upper critical dimension 6 and the other appears to have upper critical dimension 4.

A first possible explanation is the fact that there is a crucial sign reversal between the spanning-forest model and the σ-model, i.e. the spanning-forest model with positive weights maps onto an *antiferromagnetic* σ-model. Of course, for the ordinary interaction $W_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j) = \exp(\beta_{ij}\mathbf{n}_i \cdot \mathbf{n}_j)$ and for a bipartite underlying graph (as, e.g., a hypercubic lattice in arbitrary dimension d), uniform antiferromagnetic couplings $\beta_{ij} = -\beta$ and uniform ferromagnetic couplings $\beta_{ij} = \beta$ are related to one another by a gauge transformation $\mathbf{n}_{(i_1,\ldots,i_d)} \to (-1)^{i_1+\cdots+i_d}\mathbf{n}_{(i_1,\ldots,i_d)}$. However, as pointed out in Section 4, we do not obtain a simple correspondence with spanning forests for such an interaction. The simplest case in which the Ising degrees of freedom decouple from the forests is the case in which we have the local spin-reversal symmetry, e.g. for the *quadratic* versions, either of the Nienhuis model, $W_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j) = 1 + \beta_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j)^2$, or of the exponential weights, $W_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j) = \exp(\beta_{ij}(\mathbf{n}_i \cdot \mathbf{n}_j)^2)$. However, the gauge-transformation argument given above does not hold for either of these models, and it seems that numerical investigations of models in this family, in the antiferromagnetic regime, are absent in the literature. Thus, it is conceivable that models in this universality class have upper critical dimension 6 for the phase transition in the antiferromagnetic sector of the phase diagram.

Another piece of information in this puzzle was found recently by Fei, Giombi, Klebanov and Tarnopolsky [26], who studied a field-theoretic model with one bosonic field σ and M pairs of fermionic fields $\psi, \bar{\psi}$, with the $Sp(2M)$-invariant cubic interaction $g_1\sigma\bar{\psi}\psi + (g_2/6)\sigma^3$. These authors found, in the $6-\epsilon$ expansion computed through order ϵ^3, infrared fixed points with pure imaginary values (g_1^*, g_2^*). They found, moreover, that a special structure emerges for $M = 1$ because $g_2^* = 2g_1^*$: then the interaction is proportional to $\sigma^3 + 3\sigma\bar{\psi}\psi = (\mathbf{n} \cdot \mathbf{n})^{3/2}$ where $\mathbf{n} = (\sigma, \psi, \bar{\psi})$, which implies an enhancement of the symmetry from $Sp(2)$ to $OSP(1|2)$. Finally, they found that both the operator dimensions and the sphere free energy at $M = 1$ match those found [1, 29] for the q-state Potts model at $q = 0$. They concluded that these results provide "strong evidence that the $OSp(1|2)$ symmetric IR fixed point of the cubic theory ... describes the second order transitions in the ferromagnetic $q = 0$ Potts model, which exist in $2 < d < 6$". Nevertheless, it remains unclear why a field theory with interaction $(\mathbf{n} \cdot \mathbf{n})^{3/2}$ and pure imaginary coupling should be expected to be in the same universality class as the $OSP(1|2)$-symmetric field theory with interaction $(\mathbf{n} \cdot \mathbf{n})^2$ and real coupling.

12.2 Dimension 2

On any two-dimensional regular lattice, the $OSP(1|2)$-invariant σ-model — or equivalently, the $O(N)$-invariant σ-model at $N = -1$ — is perturbatively asymptotically free when the coupling T is *negative*: the perturbative renormalization flow for this model is discussed in briefly in [19] for the square lattice, and in more detail in [17] for the triangular lattice. It then follows immediately from the identities discussed in [19, 20] and the present paper that the spanning-forest model with *positive* weight

$w > 0$ should be asymptotically free, i.e. have its ferromagnetic critical point at $w_c = +\infty$ with a computable behavior $\xi \propto e^{aw} w^b$ as $w \uparrow +\infty$ [17, 19]. A transfer-matrix study [39] on the square and triangular lattices gave support to this prediction, though the results were inconclusive because the strip widths were small ($L \leq 10$).[12]

Subtler questions arise in the regime $w < 0$, where the spanning-forest model corresponds to the $q \to 0$ limit of an antiferromagnetic Potts model, with the connection $w = v/q$ where $v = e^J - 1$ is the Potts coupling. Because the relevant Potts model is antiferromagnetic, the behavior is expected to be lattice-dependent. However, on any two-dimensional lattice one expects that there is an antiferromagnetic critical curve $v_{\mathrm{AF}}(q)$ defined in the region $0 \leq q \leq q_*$ (usually $q_* = 4$) and having $v_{\mathrm{AF}}(0) = 0$ with finite nonzero slope $v'_{\mathrm{AF}}(0) = w_{\mathrm{crit,AF}} < 0$. One also expects [39, 50, 55] that there is a Berker–Kadanoff (BK) phase — that is, a massless phase with algebraically decaying correlation functions — in the open interval from $v_{\mathrm{AF}}(q)$ down to another curve $v_-(q)$, with the critical behavior of this BK phase being controlled by an attractive fixed point $v_{\mathrm{BK}}(q)$ that lies between $v_-(q)$ and $v_{\mathrm{AF}}(q)$.

For the square lattice, Baxter's [7] solution yields $v_{\mathrm{AF}}(q) = -2 + \sqrt{4-q}$ and hence $w_{\mathrm{crit,AF}} = -1/4$; this prediction was confirmed by transfer-matrix computations [39]. For the triangular lattice, only numerical estimates of the curve $v_{\mathrm{AF}}(q)$ are known [37, 38, 55]; the slope at $q = 0$ is estimated to be $w_{\mathrm{crit,AF}} = -0.1753 \pm 0.0002$ [39]. On the other hand, the curve $v_-(q)$ on the triangular lattice [8] passes through the point $(q, v) = (0, -3)$, so that by duality we conclude that $w_{\mathrm{crit,AF}} = -1/3$ on the hexagonal lattice. Similar exact values or numerical estimates of $w_{\mathrm{crit,AF}}$ for all the Archimedean lattices, their duals and their medials can be found in [37].

The critical behavior on the square lattice at the antiferromagnetic critical point $w_{\mathrm{crit,AF}} = -1/4$, and more generally along the antiferromagnetic critical curve $v_{\mathrm{AF}}(q) = -2 + \sqrt{4-q}$, was studied by Saleur [50] and more recently by Jacobsen, Saleur and collaborators [16, 33–36] via a detailed analysis of the Bethe-ansatz equations (see also the review [32]). It appears [33, 35, 36] that the behavior on the antiferromagnetic critical curve is governed by a continuum theory with two bosons, one compact and one non-compact, with a total central charge $c = 2 - 6/t$ where $q = 4\cos^2(\pi/t)$; furthermore, this continuum theory is apparently the $SL(2, \mathbb{R})/U(1)$ σ-model [16, 34].[13] Moreover, at $q = 0$ the theory has an $OSP(2|2)$ symmetry that is spontaneously broken down to $OSP(1|2)$, and the continuum limit is a free theory of one boson and one pair of fermions [33, 35, 36], which central charge $c = -1$.

But this fact raises a conundrum, which was already raised in [19]. Jacobsen and Saleur [35, pp. 452–453] put it clearly:

---

[12] As we suggested already 10 years ago [19], it would be interesting to make a Monte Carlo test of the predicted asymptotic freedom, at large correlation lengths, along the lines of [18]. Alas, we never got around to doing this, and to our knowledge no one else has done it either. One more project for Tony's 80th birthday ...

[13] This same σ-model arises in a recent study [56, 57] of the theta point in an $O(n)$ model on the square lattice. This suggests the intriguing idea that there might be a relationship between the square-lattice antiferromagnetic Potts model and the square-lattice $O(n)$ model. We thank Jesper Jacobsen for drawing our attention to these references.

> The fact that the critical point of the supersphere sigma model appears to be the free theory is a bit surprising. One would have expected instead the $OSP(1|2)$ symmetry to be restored in a non-trivial way, and the model maybe to coincide with the critical $O(n = -1)$ model of Nienhuis [45], discussed more thoroughly as an $OSP(1|2)$ model in Ref. [48]. The latter model however has central charge $c = -\frac{3}{5}$, and definitely is very different from the one we obtain.[14] This difference might be related to the difference between the hemi-supersphere and the full supersphere sigma model.

Indeed, as we showed in Section 4, the Ising variables do *not* decouple from the forest variables in the case of the Nienhuis action; and this coupling of variables could well effect the critical behavior. Alternatively,

> It might also be that some of the simplifications in the definition of the solvable $O(n)$ critical model in [45] (such as the definition of the Boltzmann weight) affect the physics in an unexpected way when n is negative, with, as a result, less universality than expected.

It appears that this question is still open.

A Analytic continuation of P^λ

Let $P(x_1, \ldots, x_n)$ be a polynomial in n variables with real coefficients, not identically equal to zero, and let $\Omega \subseteq \mathbb{R}^n$ be an open set such that $P > 0$ on Ω and $P = 0$ on $\partial\Omega$. Then, if λ is any complex number satisfying $\operatorname{Re}\lambda > 0$, the function P^λ is well-defined on Ω and polynomially bounded, and thus defines a tempered distribution P^λ_Ω by the formula

$$\langle P^\lambda_\Omega, \varphi\rangle \;=\; \int\limits_\Omega P(x)^\lambda \, \varphi(x)\, dx \tag{A.1}$$

for any test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, the function $\lambda \mapsto \langle P^\lambda_\Omega, \varphi\rangle$ is analytic on the half-plane $\operatorname{Re}\lambda > 0$, with complex derivative given by

$$\frac{d}{d\lambda}\langle P^\lambda_\Omega, \varphi\rangle \;=\; \int\limits_\Omega [P(x)^\lambda \, \log P(x)]\, \varphi(x)\, dx\ . \tag{A.2}$$

In other words, P^λ_Ω is a tempered-distribution-valued analytic function of λ on the right half-plane. We want to know whether P^λ_Ω can be analytically continued to the whole complex plane as a meromorphic function of λ.

This problem was first posed by Gel'fand [27] at the 1954 International Congress of Mathematicians. It was answered affirmatively in 1969 independently by Bernstein

[14] [Remark by the present authors] The Nienhuis $O(N)$ model for $-2 \le N \le 2$ has central charge $c = 1 - 6/[m(m+1)]$ where $N = 2\cos(\pi/m)$ and $1 \le m \le \infty$ [5, 6, 54]. For $N = -1$ $(m = 3/2)$ this yields $c = -3/5$.

and Gel'fand [13] and Atiyah [3], using deep results from algebraic geometry (Hironaka's resolution of singularities [31]). A few years later, Bernstein [12] produced a much simpler proof based on algebraic considerations. In this appendix we shall solve this problem by explicit computation for the univariate polynomial $P(x) = 1 - x^2$ that plays a central role in the present paper. We shall then conclude by briefly sketching Bernstein's general theory, which we believe will play a crucial role in the proof of our Conjecture 1.1.

A.1 The distribution x_+^λ

Let us begin with a warm-up problem. In dimension $n = 1$, let $P(x) = x$ and $\Omega = (0, \infty)$. Then, for $\operatorname{Re}\lambda > -1$, a tempered distribution x_+^λ is defined by

$$\langle x_+^\lambda, \varphi \rangle \;=\; \int_0^\infty x^\lambda\, \varphi(x)\, dx\;. \tag{A.3}$$

The analytic continuation of x_+^λ to the whole complex λ-plane is discussed in detail in the book of Gel'fand and Shilov [28, pp. 47–49, 55–58]. The key fact is that, for $\operatorname{Re}\lambda > -1$ and any integer $m \ge 0$, we have the identity

$$\begin{aligned}\langle x_+^\lambda, \varphi \rangle \;&=\; \int_0^1 x^\lambda \left[\varphi(x) - \varphi(0) - x\varphi'(0) - \ldots - \frac{x^{m-1}}{(m-1)!}\, \varphi^{(m-1)}(0) \right] dx \\ &\qquad + \int_1^\infty x^\lambda\, \varphi(x)\, dx \;+\; \sum_{k=1}^{m} \frac{\varphi^{(k-1)}(0)}{(k-1)!\,(\lambda+k)}\;. \end{aligned} \tag{A.4}$$

The right-hand side of (A.4) then defines an analytic continuation of x_+^λ to the half-plane $\operatorname{Re}\lambda > -m-1$ ($\lambda \neq -1, -2, \ldots, -m$). It follows that x_+^λ can be continued to the whole complex λ-plane as a meromorphic function with simple poles at $\lambda = -1, -2, \ldots$ having residues

$$\operatorname{Res}_{\lambda=-k}(x_+^\lambda) \;=\; \frac{(-1)^{k-1}}{(k-1)!}\, \delta^{(k-1)}\;. \tag{A.5}$$

The same result can be obtained by an alternate argument. Note first that, for $\operatorname{Re}\lambda > 0$, integration by parts in (A.3) shows that $-\langle x_+^\lambda, \varphi' \rangle = \langle \lambda x_+^{\lambda-1}, \varphi \rangle$, or in other words that the distributional derivative of x_+^λ is $\lambda x_+^{\lambda-1}$. More generally, for $\operatorname{Re}\lambda > m-1$, the mth distributional derivative of x_+^λ is $\lambda(\lambda-1)\cdots(\lambda-m+1)x_+^{\lambda-m}$. This fact can be used to define an analytic continuation of x_+^λ to the half-plane $\operatorname{Re}\lambda > -m-1$ ($\lambda \neq -1, -2, \ldots, -m$) by the formula

$$x_+^\lambda \;=\; \frac{(x_+^{\lambda+m})^{(m)}}{(\lambda+1)(\lambda+2)\cdots(\lambda+m)}\;. \tag{A.6}$$

Taking $\lambda \to -m$ in this formula, we conclude that

$$\operatorname{Res}_{\lambda=-m}(x_+^\lambda) \;=\; \frac{(-1)^{m-1}}{(m-1)!}\, \theta^{(m)} \;=\; \frac{(-1)^{m-1}}{(m-1)!}\, \delta^{(m-1)}\;, \tag{A.7}$$

in agreement with (A.5).

The foregoing results are most conveniently stated if we normalize x_+^λ, dividing by a gamma function in order to remove the poles; it is convenient also to shift the exponent from λ to $\lambda - 1$. We then have:

Proposition A.1 *The distribution*

$$f_\lambda \;\equiv\; \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \;, \tag{A.8}$$

which is initially defined for $\operatorname{Re}\lambda > 0$*, can be analytically continued to an entire function of* λ*. This entire function satisfies*

$$\frac{d}{dx} f_\lambda \;=\; f_{\lambda-1} \;. \tag{A.9}$$

Furthermore, at $\lambda = -k$ *(*k *integer* ≥ 0*) its value is*

$$f_{-k} \;=\; \delta^{(k)} \;. \tag{A.10}$$

A.2 The distribution $(1-x^2)_+^\lambda$

In dimension $n = 1$, let $P(x) = 1 - x^2$ and $\Omega = (-1, 1)$. Then, for $\operatorname{Re}\lambda > -1$, a tempered distribution $(1-x^2)_+^\lambda$ is defined by

$$\langle (1-x^2)_+^\lambda, \varphi\rangle \;=\; \int\limits_{-1}^{1} (1-x^2)^\lambda \, \varphi(x)\, dx \;. \tag{A.11}$$

The analytic continuation of $(1-x^2)_+^\lambda$ to the whole complex λ-plane is discussed briefly in the book of Gel'fand and Shilov [28, pp. 183–185]; we would like to complete that discussion (and correct one error in it). For $\operatorname{Re}\lambda > -1$ we can write

$$\begin{aligned}
\langle (1-x^2)_+^\lambda, \varphi\rangle \;&=\; \int\limits_0^1 (1-x^2)^\lambda \, [\varphi(x) + \varphi(-x)]\, dx \\
&=\; 2^\lambda \int\limits_0^1 y^\lambda \left(1 - \frac{y}{2}\right)^\lambda [\varphi(1-y) + \varphi(-1+y)]\, dy \;.
\end{aligned} \tag{A.12}$$

Now define

$$\psi_{\pm,\lambda}(y) \;=\; \left(1 - \frac{y}{2}\right)^\lambda \varphi\big(\pm(1-y)\big) \tag{A.13}$$

for $0 \le y \le 1$. As in the preceding subsection, we can write

$$\begin{aligned}
\int\limits_0^1 y^\lambda \, \psi_{\pm,\lambda}(y)\, dy \;&=\; \int\limits_0^1 y^\lambda \left[\psi_{\pm,\lambda}(y) - \psi_{\pm,\lambda}(0) - y\psi'_{\pm,\lambda}(0) - \ldots - \frac{y^{m-1}}{(m-1)!}\, \psi_{\pm,\lambda}^{(m-1)}(0)\right] dy \\
&\quad + \sum_{k=1}^{m} \frac{\psi_{\pm,\lambda}^{(k-1)}(0)}{(k-1)!\,(\lambda+k)} \;.
\end{aligned} \tag{A.14}$$

The right-hand side of (A.14), combined with (A.12), defines an analytic continuation of $(1-x^2)_+^\lambda$ to the half-plane $\mathrm{Re}\,\lambda > -m-1$ ($\lambda \neq -1,-2,\ldots,-m$). Therefore, $(1-x^2)_+^\lambda$ can be continued to the whole complex λ-plane as a meromorphic function with simple poles at $\lambda = -1,-2,\ldots$ having residues

$$\mathrm{Res}_{\lambda=-k}\Big(\langle(1-x^2)_+^\lambda, \varphi\rangle\Big) \;=\; \frac{2^{-k}}{(k-1)!}\left[\psi_{+,-k}^{(k-1)}(0) + \psi_{-,-k}^{(k-1)}(0)\right] . \tag{A.15}$$

Straightforward calculations using the binomial formula yield

$$\mathrm{Res}_{\lambda=-k}\Big((1-x^2)_+^\lambda\Big) \;=\; \sum_{j=0}^{k-1} \frac{2^{j+1-2k}}{j!} \binom{2k-2-j}{k-1} \left[\delta_1^{(j)} \;+\; (-1)^j \delta_{-1}^{(j)}\right] \tag{A.16}$$

where δ_a denotes a delta distribution located at a [i.e. $\delta_a(x) = \delta(x-a)$].

The same result can be obtained by an alternate argument. Note first the identity

$$\left(\lambda - \frac{x}{2}\frac{d}{dx}\right)(1-x^2)^\lambda \;=\; \lambda(1-x^2)^{\lambda-1} . \tag{A.17}$$

For $\mathrm{Re}\,\lambda > 0$, integration by parts in (A.11) then shows that

$$\left(\lambda - \frac{x}{2}\frac{d}{dx}\right)(1-x^2)_+^\lambda \;=\; \lambda(1-x^2)_+^{\lambda-1} \tag{A.18}$$

where d/dx is interpreted as a distributional derivative. More generally, for $\mathrm{Re}\,\lambda > m-1$,

$$\left(\lambda - m + 1 - \frac{x}{2}\frac{d}{dx}\right)\cdots\left(\lambda - \frac{x}{2}\frac{d}{dx}\right)(1-x^2)_+^\lambda \;=\; \lambda(\lambda-1)\cdots(\lambda-m+1)(1-x^2)_+^{\lambda-m} . \tag{A.19}$$

This can be used to define an analytic continuation of $(1-x^2)_+^\lambda$ to the half-plane $\mathrm{Re}\,\lambda > -m-1$ ($\lambda \neq -1,-2,\ldots,-m$) by the formula

$$(1-x^2)_+^\lambda \;=\; \frac{1}{(\lambda+1)(\lambda+2)\cdots(\lambda+m)}\left(\lambda+1 - \frac{x}{2}\frac{d}{dx}\right)\cdots\left(\lambda+m - \frac{x}{2}\frac{d}{dx}\right)(1-x^2)_+^{\lambda+m} . \tag{A.20}$$

Taking $\lambda \to -m$ in this formula, we conclude that

$$\begin{aligned}\mathrm{Res}_{\lambda=-m}\big((1-x^2)_+^\lambda\big) \;&=\; \frac{(-1)^{m-1}}{(m-1)!}\left(-m+1 - \frac{x}{2}\frac{d}{dx}\right)\cdots\left(0 - \frac{x}{2}\frac{d}{dx}\right)\chi_{[-1,1]} \\ &=\; \frac{(-1)^{m-1}}{(m-1)!}\left(-m+1 - \frac{x}{2}\frac{d}{dx}\right)\cdots\left(-1 - \frac{x}{2}\frac{d}{dx}\right)\tfrac{1}{2}\big[\delta_1 + \delta_{-1}\big] .\end{aligned} \tag{A.21}$$

Once again, these results are most conveniently stated if we divide by a gamma function in order to remove the poles. We obtain:

Proposition A.2 *The distribution*

$$g_\lambda \;\equiv\; \frac{(1-x^2)_+^{\lambda-1}}{\Gamma(\lambda)}\;, \tag{A.22}$$

which is initially defined for $\operatorname{Re}\lambda > 0$*, can be analytically continued to an entire function of* λ*. This entire function satisfies*

$$\left(\lambda - 1 - \frac{x}{2}\frac{d}{dx}\right) g_\lambda \;=\; g_{\lambda-1}\;. \tag{A.23}$$

Furthermore, at $\lambda = -k$ *(*k *integer* ≥ 0*) its value is*

$$g_{-k} \;=\; (-1)^k \sum_{j=0}^{k} \frac{2^{j-1-2k}\,(2k-j)!}{j!\,(k-j)!} \left[\delta_1^{(j)} \;+\; (-1)^j \delta_{-1}^{(j)}\right]. \tag{A.24}$$

Remark 1. *Formally* we also have the identity

$$-\,\frac{1}{2x}\frac{d}{dx}\,g_\lambda \;=\; g_{\lambda-1}\;. \tag{A.25}$$

The trouble is that $-1/2x$ is singular at $x=0$, so that $(-1/2x)\,d/dx$ is not a well-defined operator on the space of distributions. But we can save the situation as follows: Let $h(x)$ be a polynomially bounded C^∞ function that agrees with $-1/2x$ on an open set $U \subset \mathbb{R}$. Then, if φ is a test function supported in U and $\operatorname{Re}\lambda > 0$, integration by parts in (A.11) shows that

$$\left\langle h(x)\,\frac{d}{dx}\,g_\lambda \;-\; g_{\lambda-1}, \varphi\right\rangle \;=\; 0\;. \tag{A.26}$$

Once we have established (by other means) that g_λ has an analytic continuation to the whole complex plane, it follows immediately that (A.26) must hold for all λ. In other words, for all λ we can assert that the distribution $h(x)(d/dx)g_\lambda - g_{\lambda-1}$ is supported on $\mathbb{R}\setminus U$. On the other hand, it follows easily from (A.22) that when $\lambda = -k$ (k integer ≥ 0), the distribution g_{-k} is supported on $\{-1,1\}$. Therefore, if we take U to be an open neighborhood of $\{-1,1\}$ and $h(x)$ to be a function that agrees there with $-1/2x$, we can conclude that

$$h(x)\,\frac{d}{dx}\,g_{-k} \;=\; g_{-k-1}\;. \tag{A.27}$$

It can be checked explicitly that (A.24) satisfies (A.27). Since

$$g_0 \;=\; \frac{1}{2}\,(\delta_1 + \delta_{-1}) \;=\; \delta(x^2-1) \tag{A.28}$$

it follows that

$$\begin{aligned} g{-}k \;&=\; \left(-\tfrac{1}{2x}\tfrac{d}{dx}\right)^k \tfrac{1}{2}\,(\delta_1+\delta_{-1}) &\text{(A.29a)}\\ &=\; (-1)^k \left(\tfrac{d}{d(x^2)}\right)^k \delta(x^2-1)\,, &\text{(A.29b)}\end{aligned}$$

(where $-1/2x$ is a shorthand for $h(x)$ as above).

Analogous reasoning shows that if P is any real polynomial in one variable, all of whose real roots $x_1, x_2, \ldots$ are simple, one has

$$\left. \frac{P(x)_+^{\lambda-1}}{\Gamma(\lambda)} \right|_{\lambda=-k} \;=\; \sum_n \frac{1}{|P'(x_n)|} \left(\frac{1}{P'(x)} \frac{d}{dx} \right)^k \delta(x - x_n) \tag{A.30}$$

(where $1/P'(x)$ should, strictly speaking, be replaced by a smooth function that agrees with it in a neighborhood of all the roots of P). This formula appears in [28, p. 185].

On the other hand, [28, p. 183] gives a purported alternative formula for g_{-k}, namely (after correcting the minus sign inside the square brackets to a plus sign)

$$\left. \frac{(1-x^2)_+^{\lambda-1}}{\Gamma(\lambda)} \right|_{\lambda=-k} \;=\; \frac{(-1)^k}{2^{k+1} x^k} \, [\delta_1^{(k)} + \delta_{-1}^{(k)}] \; .$$

This formula gives the correct answer for $k = 0, 1$ but is *incorrect* for $k \geq 2$.

A.3 Bernstein's method

Let us now return to the general case of a real polynomial $P(x_1, \ldots, x_n) \not\equiv 0$ and an open set $\Omega \subseteq \mathbb{R}^n$. In 1969, Bernstein and Gel'fand [13] and Atiyah [3] used Hironaka's theorem on the resolution of singularities [31] to reduce the study of P_Ω^λ, locally in the neighborhood of one of its zeros, to the basic example of x_+^λ. The result is the following:

Theorem A.3 (Bernstein and Gel'fand [13], Atiyah [3]) *Let $P(x_1, \ldots, x_n)$ be a polynomial in n variables with real coefficients, not identically equal to zero, and let $\Omega \subseteq \mathbb{R}^n$ be an open set such that $P > 0$ on Ω and $P = 0$ on $\partial\Omega$. Then the distribution P_Ω^λ extends analytically to a meromorphic distribution-valued function of λ in the whole complex plane. Its poles lie on a finite number of arithmetic progressions $\lambda = -r/s, -(r+1)/s, -(r+2)/s, \ldots$ with $r, s > 0$. No pole has order exceeding n.*

In 1972, Bernstein [12] obtained a much simpler proof of this theorem by a method that generalizes the second method shown in the preceding two subsections. The idea is that, given P, we attempt to find a polynomial-coefficient partial differential operator $Q(\lambda, x, \partial/\partial x)$ and a polynomial $b(\lambda)$ satisfying

$$Q(\lambda, x, \partial/\partial x) \, P(x)^\lambda \;=\; b(\lambda) \, P(x)^{\lambda-1} \; . \tag{A.31}$$

We call any pair (Q, b) satisfying (A.31) a *Bernstein–Sato pair* for P. The set of all b for which there exists a Q satisfying (A.31) is easily seen to be an ideal in the polynomial ring $\mathbb{R}[\lambda]$; if nonempty, this ideal is generated by a unique monic polynomial $b_P(\lambda)$, called the *Bernstein–Sato polynomial* (or *b-function*) of P. The remarkable fact is that Bernstein–Sato pairs always exist:

Theorem A.4 (Bernstein's theorem [12]) *For any real polynomial $P(x)$ in n variables, there exist a polynomial-coefficient partial differential operator $Q(\lambda, x, \partial/\partial x)$ and a polynomial $b(\lambda) \not\equiv 0$ satisfying*

$$Q(\lambda, x, \partial/\partial x)\, P(x)^{\lambda} \;=\; b(\lambda)\, P(x)^{\lambda-1} \,. \tag{A.32}$$

The proof of this theorem is purely algebraic, and is based on the study of modules over the ring of differential operators with polynomial coefficients. A pedagogical account of this theory is given in the book of Coutinho [23]; see also [14] and [42, Chapter 8].

It is now easy to construct the desired meromorphic continuation of P_Ω^λ. Note first that, by virtue of the vanishing of P on $\partial\Omega$, (A.32) implies the distributional identity

$$Q(\lambda, x, \partial/\partial x)\, P_\Omega^{\lambda} \;=\; b(\lambda)\, P_\Omega^{\lambda-1} \tag{A.33}$$

for $\operatorname{Re}\lambda$ sufficiently large (how large depends on the degree of Q). Now let

$$b(\lambda) \;=\; b_0 \prod_{i=1}^{r} (\lambda + \lambda_i) \tag{A.34}$$

($b_0 \neq 0$) be the factorization of $b(\lambda)$ over the complex numbers. Then the distribution

$$F_\lambda \;=\; \frac{P_\Omega^{\lambda-1}}{\prod\limits_{i=1}^{r} \Gamma(\lambda + \lambda_i)} \tag{A.35}$$

satisfies

$$Q(\lambda - 1, x, \partial/\partial x)\, F_\lambda \;=\; F_{\lambda-1} \tag{A.36}$$

for $\operatorname{Re}\lambda$ sufficiently large, and this formula can be used to continue F_λ to the entire complex plane. We have therefore proven:

Corollary A.5 (Bernstein [12]) *Suppose that P, Q and b are as in Theorem A.4, with $b(\lambda) = b_0 \prod\limits_{i=1}^{r} (\lambda + \lambda_i)$. Then the distribution*

$$F_\lambda \;=\; \frac{P_\Omega^{\lambda-1}}{\prod\limits_{i=1}^{r} \Gamma(\lambda + \lambda_i)} \,, \tag{A.37}$$

which is initially defined for $\operatorname{Re}\lambda > 0$, can be analytically continued to an entire function of λ, which moreover satisfies

$$Q(\lambda - 1, x, \partial/\partial x)\, F_\lambda \;=\; F_{\lambda-1} \,. \tag{A.38}$$

B The supergroup $OSP(N|2M)_{\mathcal{G}}$

Here we consider the group of supermatrices M, valued in a Grassmann algebra $\mathcal{G}$ over $\mathbb{R}$, which leave invariant the norm of supervectors (more details are given later on). We call this group the *supergroup* $OSP(N|2M)_{\mathcal{G}}$. These are the global transformations, of which the tangent space at the identity has been discussed in Section 11.

Recall that, for a matrix M in $GL(N|M)$, the transposition involves also a sign. The corresponding transpose matrix is denoted as M^{ST}, and is called the *supertranspose* of M.

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \qquad\qquad M^{\mathrm{ST}} = \left(\begin{array}{c|c} A^{\mathrm{T}} & C^{\mathrm{T}} \\ \hline -B^{\mathrm{T}} & D^{\mathrm{T}} \end{array}\right) \tag{B.1}$$

where the entries in A and D belong to $\mathcal{G}_{\mathrm{even}}$ and those in B and C belong to $\mathcal{G}_{\mathrm{odd}}$.

Such a definition is adopted in order to have the fundamental relation

$$(M\mathbf{n})^{\mathrm{T}} = \mathbf{n}^{\mathrm{T}} M^{\mathrm{ST}} \tag{B.2}$$

by appropriately dealing with anticommutation of odd elements in the Grassmann algebra.

It follows that the necessary condition for M to be in $OSP(N|2M)_{\mathcal{G}}$ becomes

$$M^{\mathrm{ST}} G M = G \tag{B.3}$$

where G is defined as in (5.5).

Supermatrices also have a notion of *superdeterminant* (also called *Berezinian* [62]

$$\mathrm{Ber}\, M \stackrel{\mathrm{def}}{=} \det(A - BD^{-1}C)(\det D)^{-1} \,. \tag{B.4}$$

This quantity appears, in particular, as the Jacobian of a linear transformation on a superfield. It is easily verified that matrices satisfying (B.3) have have $(\mathrm{Ber}\, M)^2 = 1$.

B.1 The supergroup $OSP(1|2)_{\mathcal{G}}$

Fix a Grassmann algebra $\mathcal{G}$ over $\mathbb{R}$, with even subspace $\mathcal{G}_{\mathrm{even}}$ and odd subspace $\mathcal{G}_{\mathrm{odd}}$. Consider a column super-vector

$$\mathbf{n} = \begin{pmatrix} \sigma \\ \psi \\ \bar{\psi} \end{pmatrix} \tag{B.5}$$

where $\sigma \in \mathcal{G}_{\mathrm{even}}$ and $\psi, \bar{\psi} \in \mathcal{G}_{\mathrm{odd}}$. Now consider the super-matrix

$$M = \begin{pmatrix} u & \bar{\chi} & \chi \\ \eta & a & b \\ \bar{\eta} & c & d \end{pmatrix} \tag{B.6}$$

where $u, a, b, c, d \in \mathcal{G}_{\text{even}}$ and $\bar{\chi}, \chi, \bar{\eta}, \eta \in \mathcal{G}_{\text{odd}}$. Then

$$\mathbf{n}' \;=\; \begin{pmatrix} \sigma' \\ \psi' \\ \bar{\psi}' \end{pmatrix} \;=\; M\mathbf{n} \;=\; \begin{pmatrix} u\sigma + \bar{\chi}\psi + \chi\bar{\psi} \\ \eta\sigma + a\psi + b\bar{\psi} \\ \bar{\eta}\sigma + c\psi + d\bar{\psi} \end{pmatrix} \tag{B.7}$$

is a super-vector of the same type as $\mathbf{n}$. We say that M belongs to the supergroup $OSP(1|2)_{\mathcal{G}}$ in case the norm of an arbitrary super-vector $\mathbf{n}$ stays unchanged under the action of M, that is

$$\mathbf{n}\cdot\mathbf{n} \;\stackrel{\text{def}}{=}\; \sigma^2 + 2\bar{\psi}\psi \;=\; \sigma'^2 + 2\bar{\psi}'\psi' \;\stackrel{\text{def}}{=}\; \mathbf{n}'\cdot\mathbf{n}' \,. \tag{B.8}$$

This gives the equations

$$\begin{aligned} 1 &= u^2 + 2\bar{\eta}\eta && \text{(B.9a)} \\ 1 &= \bar{\chi}\chi + ad - bc && \text{(B.9b)} \\ 0 &= u\bar{\chi} + a\bar{\eta} - c\eta && \text{(B.9c)} \\ 0 &= u\chi + b\bar{\eta} - d\eta && \text{(B.9d)} \end{aligned}$$

The first equation is solved by

$$u \;=\; \mu\,(1 - \bar{\eta}\eta) \,, \tag{B.10}$$

where $\mu = \pm 1$. Then the last two equations are solved by

$$\begin{aligned} \bar{\chi} &= \mu\,(1+\bar{\eta}\eta)(c\eta - a\bar{\eta}) \;=\; \mu\,(c\eta - a\bar{\eta}) && \text{(B.11a)} \\ \chi &= \mu\,(1+\bar{\eta}\eta)(d\eta - b\bar{\eta}) \;=\; \mu\,(d\eta - b\bar{\eta}) && \text{(B.11b)} \end{aligned}$$

We are then left with

$$ad - bc \;=\; 1 - \bar{\chi}\chi \;=\; 1 - (c\eta - a\bar{\eta})(d\eta - b\bar{\eta}) \;=\; 1 + \bar{\eta}\eta\,(ad - bc) \,, \tag{B.12}$$

which implies

$$ad - bc \;=\; 1 - \bar{\chi}\chi \;=\; 1 + \bar{\eta}\eta \,. \tag{B.13}$$

If we define a', b', c', d' by $a = (1 + \frac{1}{2}\bar{\eta}\eta)a'$ and likewise for b, c, d, we conclude that

$$M \;=\; \begin{pmatrix} \mu\,(1-\bar{\eta}\eta) & \mu\,(c'\eta - a'\bar{\eta}) & \mu\,(d'\eta - b'\bar{\eta}) \\ \eta & a'(1+\frac{1}{2}\bar{\eta}\eta) & b'(1+\frac{1}{2}\bar{\eta}\eta) \\ \bar{\eta} & c'(1+\frac{1}{2}\bar{\eta}\eta) & d'(1+\frac{1}{2}\bar{\eta}\eta) \end{pmatrix} \tag{B.14}$$

where $\mu = \pm 1$ and $a', b', c', d' \in \mathcal{G}_{\text{even}}$ satisfy $a'd' - b'c' = 1$, or, in other words,

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL(2, \mathcal{G}_{\text{even}}) \equiv Sp(2, \mathcal{G}_{\text{even}}) \,. \tag{B.15}$$

Thus the space of parameters of $OSP(1|2)$ is isomorphic to $\mathbb{Z}_2 \times Sp(2, \mathcal{G}_{\text{even}}) \times (\mathcal{G}_{\text{odd}})^2$.

Various special cases of the general expression above can be considered. For example, the case $a' = d' = 1$, $b' = c' = \eta = \bar{\eta} = 0$ and $\mu = -1$ is a valid choice, and from now on we can restrict our attention to the case $\mu = +1$, i.e. the associated coset of the group (which is a connected component) containing the identity, without loss of generality.

The subgroup with $\mu = +1$, $\eta = \bar{\eta} = 0$, and a', b', c', d' in $\mathbb{R}$ (i.e. being "pure body") is isomorphic to the group of symplectic rotations, $Sp(2, \mathbb{R})$. Thus, the subgroup with arbitrary parameters in $Sp(2, \mathcal{G}_{\text{even}})$ is its natural Grassmann-valued generalization.

Finally, an interesting transformation is one in which $\mu = +1$ and $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) = I_2$

$$M = \begin{pmatrix} 1 - \bar{\eta}\eta & -\bar{\eta} & \eta \\ \eta & 1 + \frac{1}{2}\bar{\eta}\eta & 0 \\ \bar{\eta} & 0 & 1 + \frac{1}{2}\bar{\eta}\eta \end{pmatrix} . \tag{B.16}$$

The two subgroups, associated to the parameters in $\mathbb{Z}_2$ and in $Sp(2, \mathcal{G}_{\text{even}})$, and the transformations above, involving the parameters in $(\mathcal{G}_{\text{odd}})^2$, generate the full $OSP(1|2)_{\mathcal{G}}$ group.

B.2 Transformations in the supergroup $OSP(N|2M)_{\mathcal{G}}$

The supergroup $OSP(N|2M)_{\mathcal{G}}$ has an obvious subgroup $O(N)_{\mathcal{G}} \times Sp(2M)_{\mathcal{G}}$, generalising the subgroup $\mathbb{Z}_2 \times Sp(2, \mathcal{G}_{\text{even}})$ depicted above for the case $(N, M) = (1, 2)$.

However, as we have seen above, and in Section 11 at the level of infinitesimal transformations, there exist also interesting transformations that mix the fermionic and bosonic variables. In particular, for every bosonic index $1 \le i \le N$ and every fermionic index $1 \le \alpha \le M$, we have the family of transformations $M \in S_{i,\alpha}$, the set of matrices acting as in (B.16) on the corresponding subspace $\mathbb{R}^{1|2}$ of $\mathbb{R}^{N|2M}$, and as the identity on the other components.

These trasformations allow in particular to rotate an arbitrary superfield $\mathbf{n} = (\sigma_1, \ldots, \sigma_N, \psi_1, \ldots, \psi_M, \bar{\psi}_1, \ldots, \bar{\psi}_M)$ into $\mathbf{n}' = (\sqrt{\mathbf{n} \cdot \mathbf{n}}, 0, 0, \ldots, 0)$.

For example, one can start by performing a $O(N)$ rotation, to transform $\mathbf{n}$ into $\mathbf{n}'' = (\sqrt{\sigma \cdot \sigma}, 0, \ldots, 0, \psi_1, \ldots, \psi_M, \bar{\psi}_1, \ldots, \bar{\psi}_M)$, and then apply iteratively trasformations in $S_{i,\alpha}$, for $\alpha = M, M-1, \ldots, 1$, using the fact that

$$\begin{pmatrix} 1 - \frac{1}{\sigma^2}\bar{\psi}\psi & \frac{1}{\sigma}\bar{\psi} & -\frac{1}{\sigma}\psi \\ -\frac{1}{\sigma}\psi & 1 + \frac{1}{2\sigma^2}\bar{\psi}\psi & 0 \\ -\frac{1}{\sigma}\bar{\psi} & 0 & 1 + \frac{1}{2\sigma^2}\bar{\psi}\psi \end{pmatrix} \begin{pmatrix} \sigma \\ \psi \\ \bar{\psi} \end{pmatrix} = \begin{pmatrix} (\sigma^2 + 2\bar{\psi}\psi)^{1/2} \\ 0 \\ 0 \end{pmatrix} . \tag{B.17}$$

More generally, let $X = \left(\begin{smallmatrix} B \\ F \end{smallmatrix}\right)$ be a $(N + 2M) \times k$ matrix, with the top $N \times k$ block B valued in $\mathcal{G}_{\text{even}}$ (*bosonic*), and the bottom $(2M) \times k$ block F valued in $\mathcal{G}_{\text{odd}}$ (*fermionic*). Then we have

Proposition B.1 *If $k \le N$, there exists a matrix $R \in OSP(N|2M)$ such that $X' =$*

XR has the form

$$X' = \left(\begin{array}{ccc} * & * & * \\ 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \\ & \vdots & \\ \hline 0 & 0 & 0 \\ & \vdots & \\ 0 & 0 & 0 \end{array}\right) \tag{B.18}$$

If $k \geq N$, there exists a matrix $R \in OSP(N|2M)$ such that $X' = XR$ has the form

$$X' = \left(\begin{array}{cccccccc} * & * & \cdots & * & * & * & \cdots \\ 0 & * & \cdots & * & * & * & \cdots \\ 0 & 0 & \cdots & * & * & * & \cdots \\ & & \ddots & & & & \\ 0 & 0 & \cdots & 0 & * & * & \cdots \\ \hline 0 & 0 & \cdots & 0 & 0 & * & \cdots \\ \vdots & & & \vdots & & & \\ 0 & 0 & \cdots & 0 & 0 & * & \cdots \end{array}\right) \tag{B.19}$$

Indeed, by a $O(N)$ rotation the upper 'bosonic' block can be put in the form described in the proposition. Then, by a sequence of suitable transformations as in (B.17), performed on subspaces with indices $(i, \alpha) = \big((1,1), \ldots, (1,M), (2,1), \ldots, (2,M), \cdots, (h,1), \ldots, (h,M)\big)$, where $h = \min(N,k)$, the fermionic components $(1,1)$ and $(\bar{1},1)$, $(2,1)$ and $(\bar{2},1)$, $\ldots$, $(M,1)$ and $(\overline{M},1)$, $(1,2)$ and $(\bar{1},2)$, and so on, in this order, can be set to zero, and this without affecting the components that have already been set to zero (at this aim it is crucial to have set the bosonic part in triangular form).

This proposition is exploited in Sections 6 (for $k = 1$), 8 (for $k = 2$) and 10 (for arbitrary k).

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