Geometrical Diagrams as Representation and Communication: A Functional Analytic Framework

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Abstract:

Although diagrams are considered part and parcel of mathematics, mainstream mathematicians exhibit prejudice against the use of diagrams in public. Adopting a multimodality social semiotics approach, I consider diagrams as a semiotic mode of representation and communication which enable us to construct mathematical meaning. Mathematics is a multimodal discourse, where different modes of representation and communication are used, such as (spoken and written) language, algebraic notations, visual forms and gestures. These different modes have different meaning potentials. I suggest an analytic framework that can be used as a tool to analyse the kinds of meanings afforded by diagrams in mathematical discourse, focusing on geometry.

Starting from characteristics of diagrams identified in the literature, I construct the framework using an iterative methodology tested with data from classrooms in the UK and the Occupied Palestinian territories and from textbooks. The classroom data consist of approximately 350 written mathematical texts in English and Arabic produced by 13- and 14-year-old students as a response to two geometrical problems, accompanied by audio and video records of their verbal and gestural interactions with each other while solving the problems.

I then present the critical aspects of the development journey of the framework followed by a discussion of each of the three (meta)functions: ideational, interpersonal and textual. Each of these functions is illustrated by examples of diagrams from mathematical texts collected from the empirical data, textbooks and the Internet. Because I consider mathematics to be a social and cultural practice, I discuss the issue of culture and language in relation to the meanings of diagrams.

Lastly, I discuss the implications of the study on representation and communication in mathematical discourse, with possible applications for the framework in learning and teaching mathematics.

Declaration and Word Count

I hereby declare that, except where explicit attribution is made, the work presented in this thesis is entirely my own.

Word count (exclusive of list of figures and tables, references of figures and bibliography): **75,221** words.

Jehad Alshwaikh

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Here, where the hills slope before the sunset and the chasm of time
near gardens whose shades have been cast aside
we do what prisoners do
we do what the jobless do
we sow hope

Mahmoud Darwish

(A State of Siege)

For Ward, my son

With love and hope

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Acknowledgments

The PhD journey is an interesting and difficult journey of learning which, fortunately and unfortunately, has an end. The fact that I am writing this acknowledgment means that I am almost there!

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1   Introduction

The study of mathematics went through many historical changes and developments before scholars reached the current dominant view that mathematics is formal, abstract and symbolic. In its early development, research in mathematics education tried to answer questions about mathematics such as what mathematics is and how it should be taught or learned (Kilpatrick, 1992). Being influenced by psychology over many years, research in mathematics education focused mainly on the study of the behaviour of mathematicians, mathematical thinking, transfer, and other aspects. By the end of the 1970s and beginning of the 1980s, research in mathematics education evolved to focus on language and the relationship between language and learning and teaching mathematics. One of the main influential works in this area is the work of the linguist Michael Halliday, his Systemic Functional Linguistics (SFL) and the notion of register. This movement was extended in two directions: research about language as a social semiotic system and research about discourse. Both have been influenced by the notion of communication.

Communication is a social process (Halliday, 1985; Kress, Jewitt, Ogborn, & Tsatsarelis, 2001; Lemke, 1990) in which humans make use of different semiotic resources (modes) available to make meaning. Halliday (1985) argues in his systemic functional linguistics approach that in these human communicational acts, any human act fulfils three essential functions: ideational, interpersonal and textual. Our ideas (states of affairs) about the world are represented and communicated in the ideational function. The interpersonal function is realised by the social relations constructed by participants in the act of communication. The textual meaning is realised as these representations get presented in a coherent way.

The recognition of the importance of communication to mathematics learning and teaching was prompted by the seminal work of Pimm (1987) and the publication of the *Curriculum and Evaluation Standards for School Mathematics*, and later the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989, 2000a). This view was extended to the study of the discourse of classroom and of learning, drawing on different approaches from disciplines such as sociology, sociolinguistic and social semiotics (Barwell, 2008). The work of Morgan

(1996b) and O'Halloran (2005) has opened research in mathematics education to the SFL approach. The dominant view, however, was that language is the 'only' means (mode) to communicate, the mono-mode of communication.

The notion of the monomodality of language was challenged and extended by the work of Kress and his colleagues (e.g. Jewitt & Kress, 2003; Kress & Van Leeuwen, 2001, 2006). Adopting the Hallidayan SFL, they argue that communication is (and always has been) multimodal, where multi-modes such as images, diagrams and gestures are used to convey meaning. Furthermore, they use the term 'multimodality' to describe communication as a multimodal act where multiple modes of communication occur simultaneously, and each of them contributes to the construction of an 'overall' or a 'unified' meaning (Kress et al., 2001; Lemke, 1999; O'Halloran, 2004a). Thus, for a better understanding of the construction of meaning, of the meaning-making process, all modes should be considered.

Mathematics discourse is a form of communication (Pimm, 1987; Sfard, 2008), and, thus, it is multimodal, where different modes of communication take place, such as verbal language, algebraic notations, visual forms and gesture (Morgan, 1996b; O'Halloran, 2005; Radford, Bardini, & Sabena, 2007). These different modes may offer different meanings, or they may convey one set of meanings (Kress & Van Leeuwen, 2006). The verbal language in (mathematical) texts, for instance, despite its power, has limited ability 'to represent spatial relations such as the angles of a triangle (..) or irrational ratios' (Lemke, 1999, p. 174). Thus we need diagrams or algebraic notations to represent these qualities or quantities. In the same manner, gestures help in representing dynamic acts, which both language and visual representations are limited in their ability to represent. It is the deployment of all these (and other) modes which carries the 'unified' meaning (Lemke, 1999).

Morgan's linguistic approach to mathematical texts (Morgan, 1996b) offers descriptive tools to describe and interpret features of the verbal mode in the written/spoken mathematical texts based on Halliday's SFL. Tools for the description of the other modes, such as the diagrammatic and the gestural, 'are less fully developed from the systemic functional perspective' (Morgan, 2006, p. 226). Therefore, what I set out to do in this study is to offer such tools. In other words, I extend Morgan's linguistic framework for analysing mathematical discourse to

include diagrams and gestures. In doing so, I adopt Kress's multimodality approach to complement Morgan's framework.

This study, thus, takes the multimodal nature of mathematics discourse and examines geometric diagrams and the potential mathematical meaning they may offer. To a lesser extent, it also looks (or begins to look) at gestures and their contribution to mathematical meaning. In other words, this study is about communication in general and communication in mathematics in particular. It attempts to construct a framework to describe geometrical diagrams and to analyse their role in constructing mathematical meaning. It offers a 'visual grammar' (Kress & Van Leeuwen, 2006) that will allow us to read geometric diagrams.

To build the framework, a method of visual analysis has been developed, using tools derived from the visual grammar of Kress and Van Leeuwen (2006), together with interpretative techniques derived from Morgan's linguistic approach to mathematical texts (Morgan, 1996b). In other words, informed by the visual grammar, in some instances, I try to derive the potential mathematical meaning from the diagram, and, in other cases, I start from the possible mathematical meaning and see what visual indicators could be used to convey it. The main challenge from the visual grammar point of view was to find visual indicators presented in the diagram. Using the interpretative techniques derived from Morgan's work, on the other hand, raised a number of questions:

- How is mathematical activity represented in the diagram?

- What relationships are constructed in the diagram between the producer of the diagram and the viewer?

- How is the mathematical text organised, and what is the relationship between the visual (the diagram) and the verbal modes?

This framework was developed through an iterative approach in which an early version of the framework was informed by the literature and then was tested through application to the data collected in schools in the UK and in the Occupied Palestinian Territories (OPT). A refined version of the framework emerged, which in turn was developed through the 'same' process. In all, four versions were developed, where each of them lent itself to the development of the next version (the journey of the

development of the framework is described in Chapter 4). The validity and the generalisability of the framework were investigated through different types of data.

In addition to my personal motive to conduct this study in two different languages and cultures, two other motives led me to that decision. First, the theoretical approach I adopt toward mathematics and diagrams is that doing mathematics is a social and cultural practice (Morgan, 1996b; Pimm, 1991a), and, hence, there is a need to understand the cultural context of each group to inform the process of analysis. The second motive is to offer a different context for the generalisability of the framework.

Thus, sources of data were varied in order to achieve the validity and generalisability of the framework and to understand the context of situation and the context of culture. The data collected were textbooks, students' mathematical texts, the Internet, group problem solving and observation. The classroom data consisted of approximately 350 written mathematical texts in English and Arabic produced by 13- and 14-year-old students as a response to two geometrical problems (tasks) and audio-video records of their verbal interactions with each other while solving the problems.

While the main focus of this study is the diagrammatic mode, gestures were present. During the iterative watching of the video records of students' communication about the geometric problems, I noticed their frequent use of gestures. This led me to look at the gestural mode as well. An early version of a framework to read gestures is also offered.

The construction of the diagrammatic framework contributes to a more thorough understanding of the social character of doing mathematics. It suggests a way to look at how mathematical activity (and the picture of mathematics) is presented in diagrams and the role of human beings in doing mathematics. It also attempts to read the social relationship between the author of the diagram and the viewer/reader through the visual marks presented in geometric diagrams. Moreover, the framework might be used to look at the overall arrangement of mathematical texts, including the visual, the gestural and the verbal and the interaction between them.

The two suggested frameworks, together with Morgan's linguistic framework, thus offer analytic tools to look at the multimodal modes of communication and representation of mathematical discourse.

While these two frameworks were confined to school mathematics, some of the suggested features (visual or gestural marks), however, might be used to look at geometric diagrams and gestures beyond that context. All of these aspects will be dealt with through the different chapters of this study.

The order of the thesis:

This study may be seen, on one hand, as another attempt, in addition to the existing research (Morgan, 1996a; O'Halloran, 1999), to extend Halliday's SFL framework in mathematics. On the other hand, it may be see as one of the first attempts to extend the multimodality approach in mathematics. I have tried, therefore, to arrange the study to highlight its multimodal nature (see Figure 1-1).

In Chapters 2 and 3, I engage the literature which presents a background for the study. The aim is to establish the context of the relevant literature in which the study claims its position and significance. The main argument here is that language alone gives only a partial picture of mathematical communication, and that there is a need to include other modes of communication, such as diagrams and gesture. Geometric diagrams, for example, have been a significant feature of mathematical texts (in Greek mathematics, diagram was synonymous to mathematics itself (Netz, 1999)) until mathematicians started to exhibit prejudice against the use of diagrams in mathematical texts as part of a philosophical development in 'Western' culture in the mid-seventeenth century.

Because of the nature of the study, an iterative approach, as a methodology, has been used for the development of the intended framework(s), and it informs the data sources and the data collection. The iterative approach facilitated interaction between the suggested framework and the collected data. This approach includes suggesting a framework, applying it to the collected data to check its applicability, and then culling feedback to be used in developing subsequent versions. A detailed account of this methodology is presented in Chapter 4. Afterward, the thesis may be read in two

parallel but complementary routes; the reading of the diagrammatic mode which is the main focus of the thesis and the reading of the gestural mode.

Before presenting the suggested framework to read diagrams, I present a general description of how that framework has been developed in Chapter 5, in which I describe some of the major steps which led to the developed diagrammatic framework. This is followed by four chapters (6-9) that describe in detail the suggested framework according to the potential mathematical meanings conveyed by the diagrams.

The ideational meaning is delivered in two chapters, 6 & 7. While Chapter 6 focuses on narrative diagrams which are distinguished by the presence of action, Chapter 7 describes conceptual diagrams which are distinguished by the absence of action, presenting mathematical objects. The interpersonal function of diagrams is presented in Chapter 8 in which I mainly look at diagrams as a communicative act in which a social relationship is established between the producer of the diagram and the viewer/reader of it (a teacher, for instance). In contrast to the discussion of the ideational and the interpersonal meanings, in which the focus is diagrams, the discussion of the textual function in Chapter 9 extends the scene to include the whole mathematical text, including other modes of representation such as the verbal (written) mode.

The parallel framework, the gestural, is presented in Chapter 10 at an early stage of development, addressing only the ideational meaning. Having offered two frameworks to read diagrams and gestures, I then attempt to analyse students' communication during their solution to one of the two geometric problems offered in this thesis. This analysis takes into consideration the three modes of communication together – the diagrammatic, the gestural and the verbal (spoken and written).

Finally, the conclusion and the implications of the study are presented in Chapter 11.

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Diagrammatic mode

Gestural mode
(part of ch.10)

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Figure 1-1 : A summary of the thesis

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*Numbers in parentheses refer to the numbers of the corresponding chapters.

2 Communication and language in mathematics

1.   Introduction and plan of the chapter:

The basic argument of this thesis is that doing mathematics is a social practice (Morgan, 1996b; Pimm, 1987). Social practices are practices in which people represent their experiences about the world in order to understand it, and, in doing so, they interact with others or with themselves and ultimately present a coherent account of that interaction (Halliday, 1985). Moreover, these practices 'are established patterns of activity and interaction' (Morgan, 2010, Personal Communication). This means that a social practice entails not only communication among the participants but also representation and the modes of communication and representation they use.

This chapter is about communication and language in mathematics, in which I intend to set the background of this study. I start by presenting the concept of communication in general, focusing on the Hallidayan SFL, which emphasises the use of language as a dominant mode of communication, and on the multimodality social semiotics approach, which extends that view to include other modes of communication such as visual representation and gestures. Then I move to the mathematical discourse and (re)visit the communicative acts in it, focusing on language.

In the development of this chapter, I move toward establishing the need for a multimodal approach with its basic argument that language alone presents only a part of the communicative act in mathematics (or in other discourses) and that in order to more fully understand it, we must take into consideration the other modes of communication such as the diagrammatic and the gestural modes, which will be the focus of the next chapter.

At the end of this chapter, I introduce a shortcut-list to the most salient relevant concepts used in this chapter, which will be used throughout the thesis.

2. Communication and language

First, I want to agree with Kress's (1997, p. xv) opinion that:

> [t]he first and real question for education, and for schools, concerns human dispositions (...) which will be required by young people for productive engagement with the world (...).

That 'productive engagement' requires people to communicate in order to understand their environments and change them.

Communication involves interaction and representation (Halliday, 1985; Kress & Van Leeuwen, 2001). Interaction is about doing something to others or for them or acting upon them, such as doing favours, telling stories, arguing, etc. When people interact with each other, they have to have something to interact 'about': content or meaning, for example, as in what the favour or the story is about or the subject and claims of the argument. They (re)present their experiences or stories or arguments in specific forms which they consider 'as the most apt and plausible in the given context' (Kress & Van Leeuwen, 2006, p. 13). Furthermore, when people communicate, they communicate 'about meaning rather than about information' (Kress, 1988a, p. 4). In other words, to communicate is to make meaning (Kress, 2003).

In that sense, communication is a social and cultural activity (Kress, 1988b; Lemke, 1990; Morgan, 2009; Pimm, 1987) embedded in a form of social engagement in a 'wider social environment'. This engagement involves others (or oneself), an audience or a community; '[c]ommunication is always the creation of community' (Lemke, 1990, p. x). It may also involve rhetoric (Kress et al., 2001). When people communicate, they make use of different semiotic resources (modes) available to make meaning. Modes are resources shaped and offered by a culture for representation and meaning-making (Kress et al., 2001), such as language, images and gestures.

For a long time, language has been viewed as the central mode of communication and representation. Parenthetically, I note that this view is changing – see the discussion about the other modes of representation and communication at the end of this chapter and in Chapter 3. There are many studies about the relationship between the structure of language as a meaning-making system and the social structure (e.g. Fairclough, 2003; Halliday, 1978; Hodge & Kress, 1993). A seminal work is the

Systemic Functional Linguistics (SFL) approach suggested by the linguist Halliday (e.g. 1978; 1985; 2002; 2003) from the social semiotics point of view. He (1985) argues that any text fulfils three essential (meta)functions: ideational, interpersonal and textual. While the ideational function represents our ideas about the world, the interpersonal function represents the social relationships constructed by the participants in the act of communication. The textual function is concerned with the coherence of the text.

Text is a form of social exchange of meanings in a particular context that takes place in an interactive event, i.e. a communicative act using language and other meaning-making systems (Halliday & Hasan, 1985, p. 11). In other words, as Morgan (2006) considers, a text is any coherent unit of meaning that 'may be written or spoken, formal or informal, long or short, produced monologically by a single writer/speaker or dialogically by several in interaction' (p. 225). Thus, a piece of writing could be a text, a record of a meeting might be a text, and an image also could be a text.

Influenced by the SFL approach (among other theoretical approaches), Fairclough's (2003) work, Critical Discourse Analysis (CDA), focuses, on one hand, on the text itself (linguistic analysis level) and, on the other hand, moves beyond that, to the discourse level, meaning an analysis of the relationship between the text and the social context in which that text was produced.

A 'similar' starting point was established by Kress's work. Kress, together with Hodge, focused on the linguistic level (Hodge & Kress, 1993) as the departure point for their analysis, and then moved to the discourse level, to the field of social semiotics, in which they consider the social structures and the meaning-making process 'as the proper standpoint from which to attempt the analysis of [the multiplicity of] meaning systems' (Hodge & Kress, 1988, p. vii). That multiplicity of meaning systems is the focus of Kress's later work (Kress, 1997; Kress et al., 2001; Kress & Van Leeuwen, 2001, 2006) in which he, and others, have developed the notion of *multimodality*.

The main argument of multimodality is that language is no longer the central mode of communication and representation, and, furthermore, there is a need to look at the contribution of other modes, such as images and gestures, in the meaning making process (Kress et al., 2001). The mode in which something is expressed or

represented makes a difference and contributes to the meaning (Kress & Van Leeuwen, 2006). Hence, there is a need to develop distinctive frameworks to 'read' the different modes. These frameworks must be derived from the specific characteristics of the modes themselves (Kress et al., 2001).

In the rest of this chapter, I look at how these notions — communication, language, and multimodality — were adopted in mathematics and mathematics education, focusing on communication and language. The other modes of communication, namely diagrams and gestures, will be the focus of the next chapter.

3. Communication, language and mathematics

Seeing mathematics as a social activity entails the consideration of communication in it. As Pimm (1987, p. xvii) puts it: 'Mathematics is, among other things, a social activity, deeply concerned with communication'. As with any other form of communication, the focus was on language and its role in teaching and learning mathematics.

Research about the relationship between mathematics and language has developed over the last three decades (e.g. Austin & Howson, 1979; Halliday, 1975). The follower of that research and development (especially the work of Morgan in: Morgan, 1996b, 2000; Morgan, Ferrari, Duval, & Høines, 2005) may notice how the view of that relationship has changed from the view that mathematics has its own language, namely a mathematical language of symbols and special technical vocabulary, to the notion of a 'mathematics register' (Halliday, 1975). Later, the research also developed to talk about mathematics as a discourse having distinctive features, including language (e.g. Sfard, 2008). More recently, scholars study mathematics as a multimodal discourse that uses multisemiotic modes, such as language, diagrams and gestures (e.g. Lemke, 2003; Morgan, 2006; Morgan & Alshwaikh, 2009; O'Halloran, 2005; Radford et al., 2007).

Mathematics as a language

The dominant view of mathematics used to be, and may be still, that mathematics has its own specialised language: the mathematical language, which is basically symbols

alongside numbers and other specialist mathematical vocabulary and notations (Morgan, 2000, 2009). Alongside this view, mathematicians and mathematics educators considered verbal language as an 'imperfect, imprecise and ambiguous version of the symbolic systems of mathematics' (Morgan et al., 2005, p. 789). Learning mathematics, according to this view, is the 'acquisition' of that mathematical language and the ability to read and speak it (Aiken, 1971, 1972). This view can be noticed in the titles of some studies or exams such as Mathematical Vocabulary Test (e.g. Olander & Ehmer, 1971) which presume that mathematics is a language that students need to 'acquire' in order to understand mathematical meaning.

Students' mathematics-learning problems or difficulties were attributed to their lack of understanding of the mathematical terms and vocabulary (Austin & Howson, 1979). In their detailed review about language and mathematics education, Austin & Howson (1979) raised many issues and questions. One of the issues they referred to was the 'movement' between mathematical symbolism and natural language, as in 6+2=8 and 8-2=6. While in the first term, the equal sign means 'makes', in the second term it means 'leaves' (p. 177). The relationship between natural language and mathematical symbolism and specialist vocabulary was investigated by Halliday (1975), who introduced the notion of a register.

Mathematics register

Rather than seeing mathematics itself as a language focusing on vocabularies, Halliday (1975), at a 1974 conference held in Kenya for linguists and mathematics educators, introduced the notion of a register, which he defines as:

> a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings. (...) It is the meanings, including the styles of meaning and modes of argument, that constitute a register, rather than the words and structures as such. (p. 65)

Halliday continues in the same page

> We can refer to a 'mathematics register', in the sense of the meanings that belongs to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it used for mathematical purposes.

In order to develop the register of mathematics in a specific language, new words or structures have to be created. In English, Halliday mentioned several examples of words which have been reinterpreted or borrowed from other languages such as: set, point, sum, series, exceed, multiply, right-angled triangle, lowest common multiple, and permutation (Halliday, 1975, pp. 65-66). He also referred to structure, such as 'the sum of the series to n terms' and 'each term is one greater than the term which precedes it' (p. 67).

The notion of a mathematics register has been brought into mathematics education by various studies (Chapman, 2003a; Morgan, 1995; Pimm, 1987). Pimm's (1987) seminal work utilises the metaphor of 'mathematics as language' to look at teaching mathematics in the classroom. This is a metaphor, meaning that Pimm does not consider mathematics as a natural language 'in the sense that French and Arabic are' (Pimm, 1991a, p. 17) but rather uses linguistic terms as an alternative way to look at mathematics. As he states:

> to structure the concept of *mathematics* in terms of that of *language*,
> but with the primary intention of illuminating mathematics teaching and
> learning. (Pimm, 1987, p. xiv, italics in original)

Looking at mathematics in terms of language entails bringing some linguistic features to mathematics, such as metaphor. See Chapter 7 in the present study for Pimm's distinction between extra-mathematical metaphors – e.g., *a diagram is a picture* – and structural metaphors – e.g., *spherical triangles*. Metaphor may be one of the reasons that some find it difficult to learn and teach mathematics. As Halliday (1975, pp. 71-72) suggests:

> it [mathematics] has a great deal of metaphor and even poetry in it, and
> it is precisely here the difficulties often reside.

Ambiguity, among other things (see for example, Aiken, 1972; Chapman, 2003a; Durkin & Shire, 1991b; Sfard & Lavie, 2005), is one of the difficulties which has been investigated. Durkin & Shire (1991a), for instance, list many words commonly used in school mathematics such as: as great as, difference, differentiation, figure, integration, rational, square, etc. Pimm (1987, p. 8) mentions, as one among many examples of difficulties, the response of a nine-year-old to the written question, 'What is the difference between 24 and 9?' The child replied, 'One has two numbers in it and the other has one.' I provide another example, in the field of geometry, from an interview with an 11-year-old exploring how students know to recognise

geometric figure such as rectangle, square and rhombus (Alshwaikh, 2005). In Arabic, the word rhombus is *ma'een* or *mo'ayyan* (مـعـيـن), but this word has other meanings such as assist or help. Here is the excerpt, where JA is the researcher, and S is the student:

JA: How do you recognise the rhombus?

S: It assists (helps) the square and the rectangle.

Polysemy, the study of words with multiple possible meanings, is a branch of linguistics that explored the relationship between vocabulary and mathematics learning (e.g. Forrester & Pike, 1997). Actually the work of Halliday is not the only source for investigating the relationship between language and mathematics education. For example, there are strands that are concerned with metaphor and metonymy (e.g. Pimm, 1991b; Presmeg, 1998), which draw on Jakobson (structural linguistics) or Lakoff (cognitive linguistics) and strands that take post-structuralist or post-modern perspectives, drawing on Derrida and Barthes (e.g. Brown, 1996, 2001). Other theoretical sources used in mathematics education include Peircean semiotics, discursive psychology and work in second language learning.

The ambiguity aspect, moreover, has been investigated in the research about the relationship between mathematics education and language. Barwell (2005, p. 125), for instance, argues that 'ambiguity can be seen as a resource for participants' if the social and the discursive perspective is considered. In other words, according to Barwell, learning mathematics is not just learning mathematical vocabulary, but rather an act that involves mathematical communication and interaction, in which ambiguity plays an important role in articulating mathematical thinking and mathematical discourse.

Mathematics as discourse

'[C]ommunication [is] one of the central concerns of anyone interested in mathematics education' (Pimm, 1987, p. xvii). Indeed, researchers of the relationship between mathematics and language began, in the 1980's to focus on communication, representation and the concept of discourse. Communication and representation have been addressed in the Standards for mathematics teaching and learning suggested by

the National Council of Teachers of Mathematics (National Council of Teachers of Mathematics, 2000a).

Morgan's (2009, p. 4) comment about the title of Pimm's book addresses the way in which language can be seen as communication:

> [the title] pointed to three important characteristics of the language in which I was interested: it is mathematical in some sense; it is for communication, so involves some form of social engagement; and it is situated within a particular context. It is thus not only the form of the language that is significant but also the role that it plays in interactions between individuals and in the broader social context.

Moreover, the concept of a mathematics register and the concept of discourse were 'enormously useful' in providing Morgan (1996b, p. 3) 'with ways of thinking (and writing) about language.' The social engagement within a particular context refers to the notion of conceiving mathematics as a social practice (e.g. Morgan, 2001; Pimm, 1987; Sfard, 2008). This view, according to Barwell (2008), emerges in viewing and studying language as discourse based on sociological perspectives. He reviews the major perspectives that affect research on the role of discourse in mathematics and mathematics education: sociological and socio-cultural, social semiotics and post-structuralism.

In her detailed research review, Schleppegrell (2007) revisits different theoretical approaches which consider the relationship between language and mathematics education such as constructivism, sociocultural perspective and social semiotics. While the first emphasises the role of the individual in constructing mathematical knowledge, the sociocultural perspective stresses the role of the social and cultural context. While Schleppegrell (2007) considers the role of social semiotics in synthesising these two views, she adopts O'Halloran's (2005) approach which considers only three semiotics systems in mathematical discourse (natural language, mathematics symbolism and visual displays), leaving out other modes such as gestures which, as I will show in the next chapter, constitute an evolving area of mathematics education research. Furthermore, Schleppegrell (2007) also points out the increasing interest of mathematics education research in the notion of discourse and communication. Sfard's (2008) commognitive (communicational approach to cognition) approach to mathematics discourse, in which she considers thinking to be communication, is an example of this research interest. However, communication is

always associated with another term, representation. Communication and representation are inseparable (Kress et al., 2001), and what is represented is communicated.

Representation is also an issue that accompanied the development of language not only from the 'social engagement' point of view, but also from the cognitive point of view. 'The concept of representation has been one of the most talked about concepts over the last two decades in mathematics education' (Radford, 2003, p. 40). For example, the *Journal of Mathematical Behavior* had two consecutive Special Issues in 1998 (Volume 17, numbers 1 and 2) edited by Claude Janvier and Gerald Goldin which were devoted to the discussion about representations. I consider this issue in my discussion of diagrams in the next chapter.

Mathematics as a multimodal (multi-semiotic) discourse

Communication, representation and discourse are concepts within the focus of the social semiotics and multimodality perspective that this study adopts (see, at the end of this chapter, the definitions I use for all these concepts). The basic relevant aspect of this discussion is that doing mathematics involves making use of not only language but also other modes of communication such as diagrams and gestures (Morgan & Alshwaikh, 2009; Morgan et al., 2005).

The interaction between mathematics and other disciplines such as sociology and sociolinguistics (recontextualization, in Bernstein's terms as presented by Lerman (2000)) led to different perspectives in the field of research in mathematics education. Conceiving of mathematics as a social practice advanced the research about language and mathematics education further toward the concept of language 'in use' in communication and discourse.

The multimodal social semiotics (Jewitt & Kress, 2003; Kress et al., 2001; Kress & Van Leeuwen, 2001, 2006), furthermore, considers communication to be inevitably multimodal, where different modes of communication take place such as verbal language, algebraic notations, visual forms and gesture (Morgan & Alshwaikh, 2009; O'Halloran, 2005; Radford, Edwards, & Arzarello, 2009). Figure 2-1, taken from McInnes & Murison (1992) as presented in Veel (1999, p. 188), is an example of a mathematical text which is multimodal, using words, image, action and symbols. It is

reasonable to ask about the role visual forms may play or what meaning they offer –
different, complementary or new to mathematical texts.
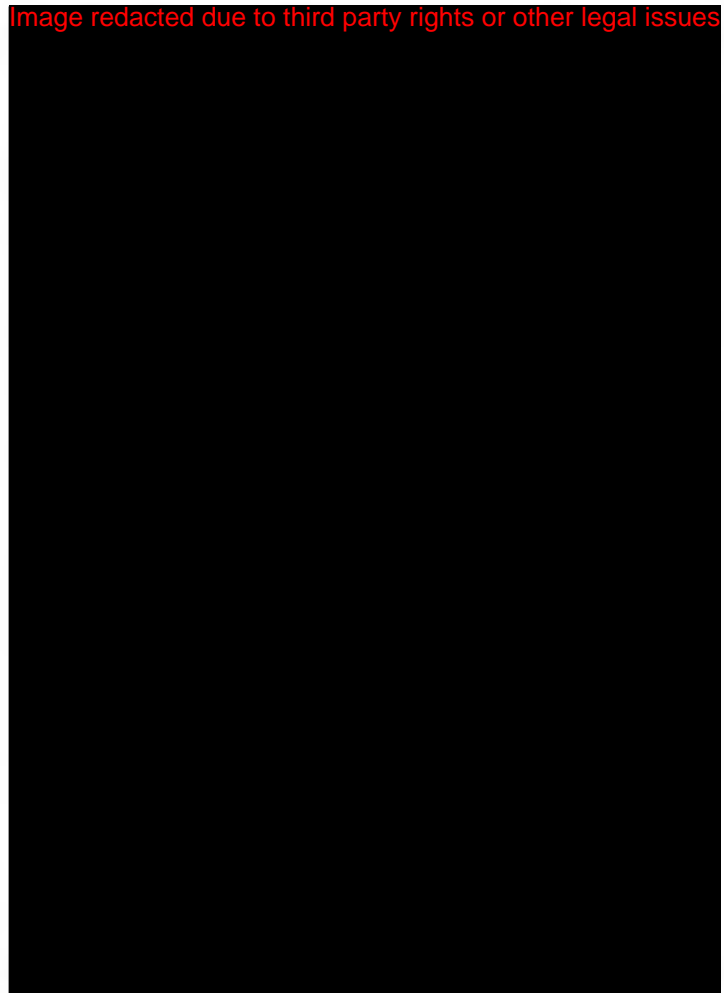
Figure 2-1 : Multimodal mathematical text
(taken from McInnes & Murison (1992) as presented in (Veel, 1999, p. 188))

While mathematical texts deploy different modes of communication and
representation, it may be argued that a generic type of framework provided by Kress
(e.g. Kress & Van Leeuwen, 2006) would be sufficient to read
mathematical/geometrical diagrams. However, mathematical texts, practices and
discourse have distinctive features which are different from other discourses (e.g.
Halliday & Martin, 1993; Sfard, 2008), especially in meaning potential. The
mainstream thinking among mathematicians is that mathematics is abstract, formal
and timeless (Morgan, 2001). There are, however, different views about mathematics
among mathematicians and mathematics education researchers and among each of

these two groups. Sfard (2008) argues that word use, visual mediators and discursive routines are distinctive features of mathematical discourse.

Radford et al. (2007), moreover, offered a detailed analysis to show how gestures can contribute to the way in which students solve mathematical problems and to demonstrate the need to consider modes other than language, which alone does not present that unified meaning.

The role of diagram, or the diagrammatic mode, and of gestures, or the gestural mode, are the focus of the next chapter. Before moving to the next chapter, I summarise the main concepts mentioned in the current chapter, which will recur throughout the current study.

4. Definitions

Social practice: Social practice is an established pattern of activity and interaction in which people represent their experiences about the world in order to understand it, and in doing so, they interact with others or with themselves and ultimately present a coherent account of these experiences.

Communication: 'Communication is about meaning rather than about information.' (Kress, 1988a, p. 4). In other words, to communicate is to make meaning (Kress, 2003). Moreover, communication entails an audience and creates community (Lemke, 1990). '[I]t is impossible to think about communication without thinking about cultural contexts and meanings' (Kress, 1988a, p. 13). Culture provides, or individuals in a specific culture develop, resources and systems (modes) to make meaning.

While representation 'focuses on what the individual wishes to represent about the thing represented', communication 'focuses on how that is done in the environment of making that representation suitable for a specific other, a particular audience'. Hence, communication and representation 'are inseparable – representation is always communicated' (Kress et al., 2001, p. 4).

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Representation: Representation is a motivated sign in which the sign-makers present their interest of the thing represented. Representation, in that sense, has a form or signifier coupled with a carrier of meaning which is signified.

Sign: Sign, in social semiotics, is a semiotic object, a 'product of a social process' (Jewitt, 2003a, p. 46) or 'the carrier of a meaning' (Kress et al., 2001) that consists of (or is materially realised by). a form (signifier) and a meaning (signified). Sign is motivated by the interest of the sign-maker. A geometric diagram is an example of a sign in which drawing is the form that conveys a mathematical meaning.

Text: Text is a form of social exchange of meanings in a particular context that takes place in an interactive event: a communicative act using language and other meaning-making systems (Halliday & Hasan, 1985, p. 11). For instance, a mathematical piece of writing, a record of a meeting and an image (diagram, for example) could be texts.

Diagram: A diagram (in geometry) is a motivated sign realised by a material form/signifier which is a set or a system of interacting geometric objects: points, lines and planes (Hilbert, 1894 as quoted in Mancosu, 2005, p. 14; Netz, 1999). Moreover, it conveys a (mathematical) meaning/signified. See Chapter 3.

Gesture: A gesture is a mode of representation and communication for a meaning-making process that is materialised by the movement of hands and fingers.

Meaning: Meaning is a social (and cultural) construct formed during a communicative act, i.e. a meaning-making process. As a result of that conceptualisation, Kress and Ogborn write, there is a need to conceive of:

> meaning not as simply and solely inherent in the system; meaning not as stable; not as a matter of correspondence; but meaning as the result of action and work; as dynamic; and meaning as the result of transformative work of socially formed and socially located individuals. (Kress & Ogborn, 1998, p. 7)

Meaning-making: Meaning-making is a social activity which occurs in social practices using different semiotic resources such as language, visual representations and gestures (Evans, Morgan, & Tsatsaroni, 2006; Kress & Van Leeuwen, 2001; Lemke, 2003). In other words:

Meaning making can be understood as the interaction between the socially situated interest of the sign maker and the potentials for meaning (what it is possible to mean) with the resources available to them and their realization in specific representational and communicational acts (signs). (Jewitt, 2003a, p. 39)

Mode: 'A mode is a socially and culturally shaped resource for making meaning.' (Bezemer & Kress, 2008, p. 171). Image, writing and gestures are examples of modes or semiotic resources for representation and making meaning. These modes/resources are regularised and organised through cultural and social practices and 'are what have been called 'grammars' traditionally' (Jewitt, 2003a, p. 40). In communication, people use different modes to make meaning (Bezemer & Kress, 2008; Jewitt & Kress, 2003; Kress & Van Leeuwen, 2001). This notion has been termed multimodality or the multimodal meaning making approach.

Multimodality (or multimodal approach): Because this concept is used widely and with different potential meanings (e.g. Arzarello, Paola, Robutti, & Sabena, 2009; O'Halloran, 2005, 2004c; Radford, 2009; Radford et al., 2009), I want to make it clear that I adopt the approach of Gunther Kress and his colleagues (Bezemer & Kress, 2008; Jewitt, 2006; Jewitt & Kress, 2003; Kress, 2003; Kress et al., 2005; Kress et al., 2001; Kress & Van Leeuwen, 2001, 2006; Mavers, 2009). Kress's approach takes into consideration all the different modes involved in representation and communication and 'treats [them] as equally significant for meaning and communication' (Kress & Jewitt, 2003, p. 2). One result, among many others, is that language is no longer considered the only or the central mode, monomodal (Kress et al., 2001), but rather is just one part of multiple modes in the act of communication and representation.

Discourse: Discourse is the socially and culturally constructed knowledge about reality. Kress & Van Leeuwen (2001) describe it as such:

> People often have several alternative discourses available with respect to particular aspect of reality. They will then use the one that is most appropriate to the interests of the communication situation in which they find themselves. (p. 21)

7.   Summary:

This current chapter set up a general background for communication and language and then explored the relationship between them and mathematics and mathematics education. I paid a lot of attention to the relationship between language and mathematics, and I explored various aspects of that relationship, starting from mathematics as language. I addressed the shortcomings of that approach, especially the way it avoids considering natural language in mathematics learning, focusing exclusively on the mathematical symbolism and specialist vocabulary. Then I revisited the notion of a mathematical register introduced by Halliday (1975), who highlighted the mathematical use of natural language. This notion has been further developed by Pimm (1987) and Morgan (1996b), who move to the concept of the discourse by conceiving of (doing) mathematics as social practice.

However, language alone, it was argued (e.g. Kress & Van Leeuwen, 2006), can only express part of the communicative act. The unified meaning may be expressed in the ensemble modes of communication and representation. Influenced by other disciplines such as multimodality social semiotics, research in mathematics education moved beyond the notion of discourse to the notion of multimodal discourse in which research about mathematics started to consider, in addition to language, other modes of communication and meaning-making such as diagrams and gestures. In the next chapter I present a review of these two modes as a justification for the current study.

3 Extending the semiotic landscape of mathematics: Diagrams and gestures

1.   Plan of the chapter:

As we see from the discussion about communication and representation in the previous chapter, a multimodal account is needed to examine mathematical discourse. While the previous chapter focused on the use of language in mathematical discourse, this chapter addresses additional modes that occur in that discourse, focusing on the diagrammatic and the gestural modes. In doing so, I intend to set the background for the use of diagrams in mathematics, mainly in geometry. I start with a historical account of the developmental use of diagrams, in which I present an overview of three eras: Babylonian-and-Egyptian mathematics, Greek mathematics and 'Modern Western mathematics'. At the end of the chapter I consider a third mode of representation and communication which has recently been the focus of research into teaching and learning mathematics, that is, gestures.

2.   Diagrams: from privilege to prejudice

The conclusion drawn from my review of the relationship between language and mathematics in the previous chapter is that mathematics is a multimodal discourse. The main argument in that claim is that language (spoken and written) is not the only mode to make meaning, that it expresses only part of the meaning-making process and that there are other modes of communication such as images (visual representations) and gestures (Kress & Van Leeuwen, 1996, 2006; Lemke, 1998a). In order to achieve successful communication, people use what they think is the apt mode to communicate (Kress & Ogborn, 1998). As a result, visual representations or gestures, for instance, contribute to the construction of the meaning together with language (written or spoken) and other modes, and, therefore, they should be taken into consideration when analysing any communicative act or text. Any mathematical text, within the lens of the multimodal social semiotic approach, is multimodal and has verbal and visual modes (Morgan, 2006; Veel, 1999).

Mathematics educators and semioticians (e.g. Duval, 2000; Morgan, 2006; O'Halloran, 2005) view mathematics as a multisemiotic (or multimodal) system, meaning that its discourse is formed or constructed through different semiotic systems: verbal language, algebraic notations[1], visual forms (diagrams, tables and graphs) and gestures. See Figure 2-1 for an illustrative example. Morgan (1995; 1996a; 1996b; 2006) has developed a linguistic approach to analyse the verbal mode of mathematical texts using Halliday's Systemic Functional Linguistics or Grammar, SFL (1985) (see below). She, moreover, argues that algebraic notations may be translated into words and analysed according to her approach. However, O'Halloran (1999) argues that algebraic notations, or, using her words, mathematical symbolism, have a different meaning potential, and she used SFL to develop a descriptive framework for them (as well as for visual forms). Since my interest is in visual representations, I will focus on diagrams in geometry.[2] Although Morgan (1996a; 2006) makes some notes about the role of visual representations in mathematical texts and provides some analysis, she comments that the analysis is not fully developed, and that more research and exploration are needed.

Seeing mathematics as a social practice, I think that there is a need for further investigation beyond the direct aspect of mathematics (teaching and learning, for example), to explore the (social) mathematical practice itself, in order to make meaning of, for instance, what the practice of mathematics is and how mathematicians conceive of mathematics. One aspect that is directly related to this inquiry is the use of diagrams. In the current study, I try to contribute to that endeavour by suggesting a descriptive framework for reading diagrams; I offer a grammar of diagrams.

In order to do that, I examine the status of diagrams in mathematics, taking geometry as a case study. My focus is limited to geometric diagrams rather than all the other visual forms such as graphs, charts, Venn diagrams or tables. I do this for two main reasons: first, because of the historical role of geometric diagrams in the history of mathematics, as I will show below, in which diagrams were considered at some point the 'hallmark' of mathematics (Netz, 1999); and, second, because of the significant

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[1] O'Halloran (2003) uses the term 'mathematical symbolism' to describe the same feature, while Morgan (2006 and in personal meetings) uses the term algebraic notations, which I will use also, to avoid confusion with the term 'symbol' used within semiotics.

[2] The other kinds of visual forms (such as the presentation of the texts: format, titles, labels, colours, etc.) will not be addressed in this study.

role diagrams play in doing mathematics (Maanen, 2006; Sfard, 1994) and in learning and teaching mathematics (Arcavi, 2003; Martin, 1971; Stylianou & Silver, 2004). O'Halloran (2005) has suggested descriptive tools for visual forms in mathematical texts including titles of tables and graphs.

In the following, I consider the history of changes in the use of geometric diagrams in mathematicians' practices, in which diagrams moved from a privileged status as a core element in mathematical texts to a position of disfavour. In parallel, I look at how research into mathematics education focuses on diagrams from a different viewpoint, namely the perspective of visualisation.

2.1   Diagrams in mathematical discourse:

> ... nothing must be assumed from this picture, every thing stated must be deduced from the Axioms laid down. ... If we illustrate our argument by figures, nothing save what is explicitly stated and deduced may be used from these figures. Theoretically, figures are unnecessary; actually they are needed as a prop to human infirmity. Their sole function is to help the reader to follow the reasoning; in the reasoning itself they must play no part. (Forder, 1927, pp. 42-43)

> Their [geometric diagrams'] function is merely to bring home my meaning to my hearers, and, if I can do that, there would be no gain in having them redrawn by the most skilful draughtsman. They are pedagogical illustrations, not part of the real subject matter of the lecture. (Hardy, 2004, p. 125)

Stated by two well-known mathematicians, these quotations, among many others (see below the quotations Mancosu (2005) derives from Hilbert and Pasch), represent the hitherto status of diagrams in mathematics. Both quotes view the role of diagrams as assistance or pedagogical illustration, **not** part of mathematics itself or not 'real' mathematics. Below, I discuss this view in more detail. This view is still valid today. In a personal communication (2008) about the current study, a mathematician commented on a written piece of mine which I sent to him, at his request:

> I only went through your pages quickly and see that you are working with a literature that I did not know and from a point of view that is quite different from mine and that of the other scholars working on Euclidean diagrams that I know. This makes your approach very interesting for me, though it seems to me, at first glance, that it should

be substantiated with a closer connection with *real* mathematical theories, practice, etc. (My emphasis.)

These quotes may offer a window into how mathematicians view the use of diagrams in mathematics and in solving mathematical problems. Dreyfus (1991), a mathematics educator, claims that mathematicians (and mathematics educators, see below) are to blame for the low status of diagram in mathematics:

> Mathematicians are not innocent of the fact that visual reasoning has a low status. Many indicators point to the fact that most mathematicians rely very heavily on visual reasoning in their work. But with few exceptions (...) these same mathematicians do their utmost to hide this fact. (p. 36)

This view has been shared among other studies in mathematics and mathematics education (e.g. Davis & Hersh, 1981; Sfard, 1994). Moreover, these studies consider diagrams to be part and parcel of mathematics and note that diagrams have been used heavily, historically and currently, in (teaching and learning) mathematics (e.g. Hadamard, 1945; Netz, 1998; Presmeg, 2006). The main questions are why and when the practice of hiding the use of diagrams or exhibiting prejudice against the visual part of mathematics developed. Was this practice common in the history of mathematics? If yes, why do mathematicians throughout history refrain from acknowledging their use of diagrams? And if no, when (and perhaps why) did this denial begin to manifest?

Although there are studies about the history of mathematics which show that diagrams have been used in ancient civilisations such as Old Babylon 4,000 years ago (e.g. Robson, 2008a) these questions remain unanswered. In a personal communication, De Young (personal communication, November 26, 2008) commented that the 'full history of diagrams (...) is far from being written yet'. Detailed answers to those questions are beyond the scope of the current study, and I will try to provide only general answers to these questions. In this section, I present a very broad historical overview of the use of geometric diagrams from the oldest-known mathematics, Babylonian and Egyptian, to Greek mathematics to 'modern Western mathematics' which is still dominant. I then move to look at the research about diagrams in mathematics education.

2.1.1 Geometric diagrams in Babylonian and Egyptian mathematics:

The Old Babylonian mathematics, dated around 2000 BCE, is considered the world's first 'pure' mathematics' (Robson, 2001, 2008b). There is much evidence that the Old Babylonian mathematics used diagrams (Friberg, 2007; Robson, 2008b). Robson (2008b, p. 45) claims that around a third of the available corpus of the Old Babylonian mathematical word problems centres around 2D or 3D diagrams, and that some of these problems are illustrated by diagrams. Neugebauer & Sachs (1945) and Friberg (2007) mention many examples of these diagrams such as triangles, trapeziums, circles, squares, etc.[3] 'YBC 7289, from the Yale Babylonian Collection, is one of the best-known Old Babylonian mathematical clay tablets' (Fowler & Robson, 1998, p. 366), and it may be used to illustrate how diagrams were used. The YBC 7289 (Figure 3-1, the diagram on the left is written in sexagesimal digits which are translated in the diagram on the right) shows a square with an approximation to the square root of 2. Three numbers are shown in the diagram: 30 on the side of the square and two numbers on the diagonals (1,24,51,10 and 42,25,35). If 1,24,51,10 is multiplied by 30, the result is 42,25,35. Thus, the number 1,24,51,10 is the approximation to the square root of 2.[4]
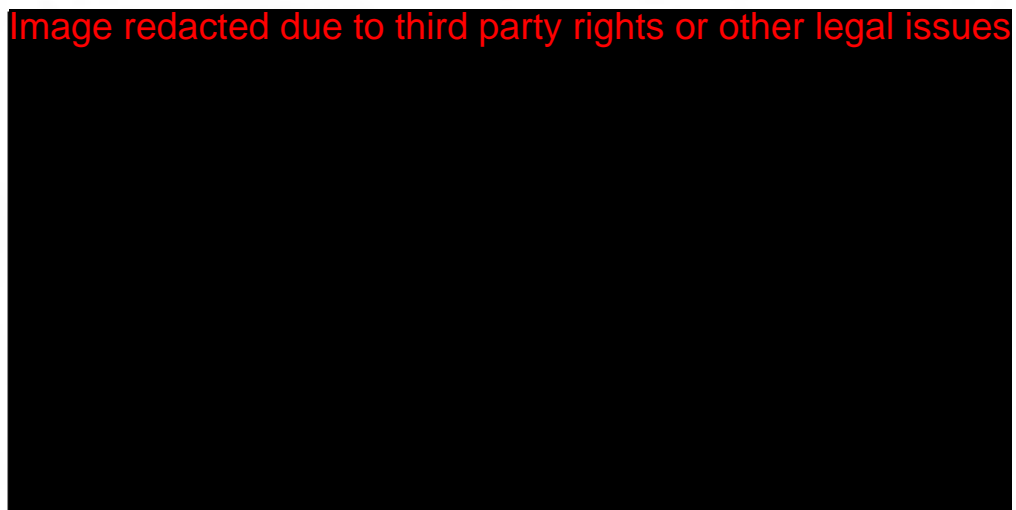
Figure 3-1 : YBC 7289 (The Yale tablet) - an approximation to the square root of 2

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[3] Friberg (2007) has commented on the translation issue from Old Babylonian mathematics and Greek and Latin mathematics. The Old Babylonian 'triangle', for example, 'was always specified in terms of the lengths of two or three of its sides' (p. 1) and not in terms of the vertices or angles as in Greek mathematics. Angle is a Greek concept lacking in Old Babylonian mathematics.

[4] These numbers are written in sexagesimal (base 60). For example, the number 30 here is 1/2 in the decimal system. Zero is represented by a space in the Babylon numeral system, a system that had no notation for zero. So 42;25 35 is 1/root(2) and 1;24 51 10 is root(2). For more details, see Fowler, D., & Robson (1998) and O'Connor & Robertson (2000).

Another example is the tablet, now in the British Museum, BM 15285 (Figure 3-2), which is a compilation of forty geometric problems or exercises, similar to a 'textbook' (Robson, 2008a). The tablet seeks to find the areas of diagrams within squares such as triangles and circles, drawn in a 30×50 cm tablet (Robson, 1999, 2008a, 2008b). Robson (2008b, p. 48) argues that '[n]one of the textual description [in this compilation] is complete in itself, in that the problems cannot be solved without reference to the image.' (See Robson, 2008a for more details.)
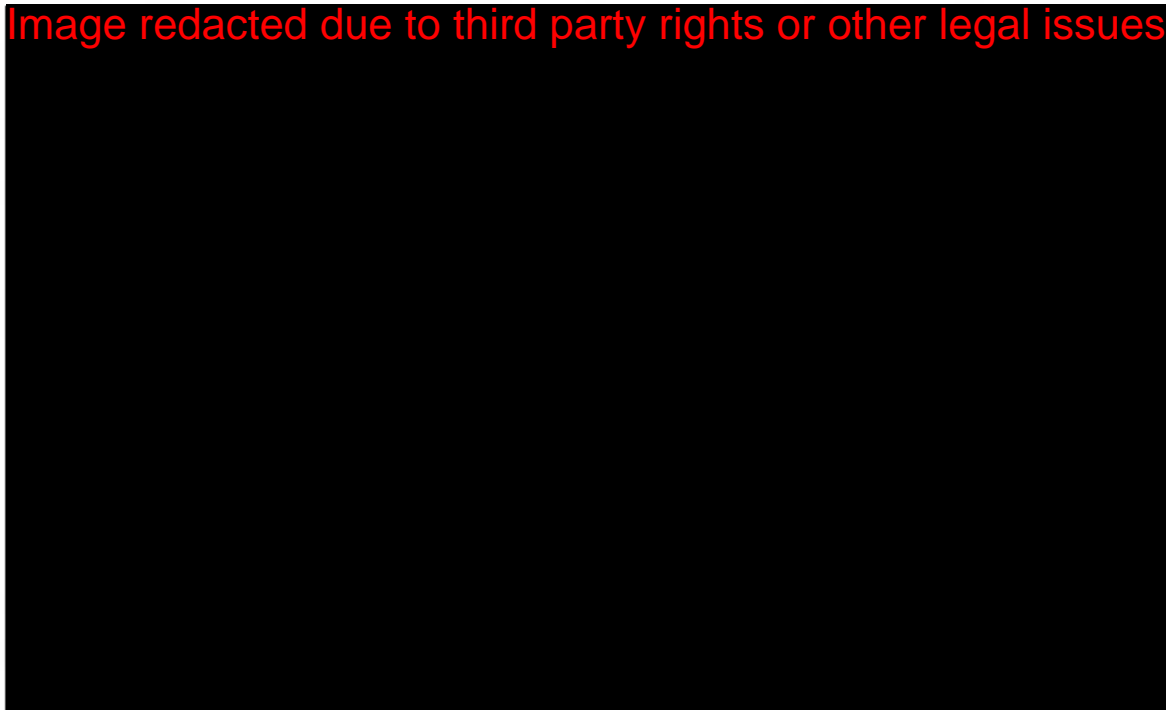
Figure 3-2: BM 15285: An Old Babylonian 'textbook' (Robson, 2008a)

Ancient Egyptian mathematics also used diagrams. The Moscow papyrus, around 1850 BCE, and the Ahmes (or Rhind) Papyrus, around 1650 BCE, are two famous examples of that mathematics (Boyer & Merzbach, 1989; Eves, 1969). While the former consists of 25 problems about volume, the latter contains 85 problems about fractions and areas and other mathematical elements. 'Twenty-six of the 110 problems in the Moscow and Rhind papyri are geometric' (Eves, 1969, p. 40). For example, a diagram (Figure 3-3) appears in Problem 14 in the Moscow papyrus, which looks an isosceles trapezium to calculate 'the volume of a frustum of a square pyramid six units high if the edges of the upper and the lower bases are two and four units respectively' (Boyer & Merzbach, 1989, p. 22).
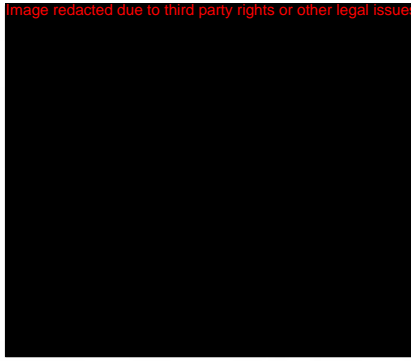
Figure 3-3: Problem 14 in the Moscow Papyrus
(from Boyer & Merzbach, 1989, p. 22)

There also are some geometric problems in the Ahmes Papyrus, as Boyer & Merzbach (1989, p. 20) state:

> Problem 51 of Ahmes shows that the area of an isosceles triangle was found by taking the half of what we would call the base and multiplying this by the altitude. Ahmes justified this method of finding the area by suggesting that the isosceles triangle can be thought of as two right triangles, one of which can be shifted in position, so that together the two triangles form a rectangle (…) "so as to make a rectangle". (…) In transformation such as these, in which isosceles triangles and trapezoids are converted into rectangles, we see the beginnings of a theory of congruence and the idea of proof in geometry, but the Egyptians did not carry this work further.

The Greeks, however, carried the 'work further', especially regarding the notion of proof. In the next section, I demonstrate how diagrams play an influential role in shaping that notion.

2.1.2   Diagrams in Greek mathematics

Geometric diagrams have had a unique status in Greek mathematics. The word 'diagram' itself was the symbol of mathematics, the metonym of mathematics, the hallmark of mathematical activity or of the mathematics itself (Netz, 1999). The modern word 'diagram' is derived from the Greek word *diagramma* which means 'figure marked by lines', but, Netz (1999, p. 35) continues, the Greek use of it is more complex. Plato, for example, used the word for mathematical activity, while Aristotle used it to mean mathematics. A third meaning for 'diagram' is 'a mathematical proposition'.

Due to the limitations of communication, long distances, the need to write letters, and the limitation of the available media (dusted surface, wax tablets, etc.), Netz (1999) argues that diagrams had to be drawn first and that Greek mathematicians would start doing mathematics by drawing a diagram. One main argument that Netz (1999) makes is that Greek mathematics relies heavily on diagrams or, more strongly, 'is visual rather than verbal' (p. 49) and, he continues, the 'argumentation [is] based on the diagram' (p. 57) and that the point of departure for the mathematical activity is the diagram.

If diagrams were so central to mathematics, what changed and when did it change, or when did mathematicians start denying the importance of diagrams? One direction to explore in order to find an answer is the way in which mathematicians conceive of mathematics and its nature. In his introduction about the nature of mathematics, Kline (1968) suggests that the interpretation that mathematicians made of the *Elements* (300 BCE) may be one of the reasons for such denial. Kline argues that mathematicians' interpretation of the *Elements* shaped their (and perhaps our) view of mathematics as an axiomatic and logical body of knowledge and led them to discard other views such as conjecture, intuition and creativity, a phenomenon which made the *Elements*, notwithstanding its value as 'an intellectual triumph' – 'a pedagogical misfortune'. Kline writes:

> The world did derive from this book the notion of mathematical proof, the logical organization of a body of mathematical knowledge, and, of course, invaluable information. But far too many intellectuals, including mathematicians, mistook the import of Euclid's work and formed a concept of mathematics that is too narrow. Mathematics, they concluded, was a purely logical development. It starts with axioms and definitions, which are explicitly stated at the outset, and proves deductively results about the mathematical concepts delineated in definitions. (p. 2)

The discussion about the nature of mathematics, 'ranging from axiomatic systems or heuristics for solving problems' (Dossey, 1992, p. 2) has relevance to philosophy mainly in the dialogue between Plato and Aristotle or Idealism and Realism in the fourth century BCE. Plato argued that mathematical objects 'exist' on their own, beyond the mind, in the Ideal world, and that the way to 'discover' those objects is through mental activity on their representations in the sensual world (Davis & Hersh, 1981; Dossey, 1992). In other words, the idealised mathematical objects exist in the ideal world, and what we deal with is their representations, including diagrams:

> The relationship between the real and the ideal is illustrated by the accompanying diagram (though, strictly speaking, we cannot draw the ideal objects on the right side of the diagram). (Davis & Hersh, 1981, p. 128)

In other words, diagrams or any sensual activity are not a trusted source for knowledge and, thus, have to be abandoned, or 'stripped away' as Aristotle described the process of idealisation of the abstraction (Davis & Hersh, 1981). See Chapters 5 and 7 in the current study for a more detailed discussion of mathematical objects.

Aristotle took a different position from his teacher Plato when he emphasised the role of senses, experimentation, observation and abstraction (Dossey, 1992, p. 40):

> Aristotle attempted to understand mathematical relationships through the collection and classification of empirical results derived from experiments and observations and then by deduction of a system to explain the inherent relationships in the data.

This distinction between the two ways of thinking about mathematics and mathematical objects is manifested in the work of other philosophers such as Francis Bacon and the French salon circle (Dossey, 1992). The work of Descartes, however, was a critical development that had a remarkable effect on 'Modern Western' mathematics.

2.1.3 Geometric diagrams in 'Modern Western' mathematics:

Descartes's philosophical project is based on the notion of Plato that senses are not a reliable source of knowledge, since they may deceive or mislead the perceiver. Mathematics, for example, has to be obtained by deduction from accepted axioms and not from experimentation. As a result, any trace of experimentation or human activity has to be removed from mathematical matters (Dossey, 1992; O'Halloran, 2005). In diagrams, for example, all human figures or physical contexts started to disappear with Descartes and Newton, beginning in the seventeenth century (O'Halloran, 2005). See Chapter 6 for a more detailed discussion of nominalisation in language and diagrams and the role of human beings. O'Halloran (2005) demonstrates this idea by presenting the evolution of the use of diagrams beginning in the sixteenth century which at first include human figures and then later only part of the human figure and finally no human figure at all (Figure 6-17). Descartes started to draw diagrams that include line segments, circles and curves focusing on

the spatial and temporal relationships between these entities. In order to explain these relationships and to solve the mathematical problems, he, and Newton and other mathematicians following him emphasised the mathematical symbolism (O'Halloran, 2005). Cartesian Geometry is a realisation of that approach.

This approach and the later development of different mathematical views in the 19th and the early 20th century such as logicism, intuitionism and formalism conceived of mathematics as abstract, formal, impersonal and symbolic (Morgan, 2001). As a result, the use of diagrams, or any other mode rather than the symbolic, has been resisted. Mancosu (2005, pp. 14-15) quotes Hilbert and Pasch respectively in commenting on this issue:

> A theorem is only proved when proof is completely independent of the diagram. The proof must call step by step on the preceding axioms. (Hilbert, 1894, 11)

> For the appeal to a figure is, in general, not at all necessary. It does facilitate essentially the grasp of the relation stated in the theorem and the construction applied in the proof. Moreover, it is a fruitful tool to discover such relationships and constructions. However, if one is not afraid of the sacrifice of time and effort involved, then one can omit the figure in the proof of any theorem; indeed, the theorem is only truly demonstrated if the proof is completely independent of the figure. (Pasch, 1882/1926, 43)

The main argument against the use of diagrams is that diagrams (or visual representations in general) are a) limited in representing knowledge and may be vulnerable to misuse (Shin, 1994); b) of an 'informal and personal nature' (Misfeldt, 2007); and c) unreliable and lack rigour (Kulpa, 2008). The reactions to the 'informality' of diagrams approach were varied in mathematics and mathematics education. Some tried to challenge the idea by showing that diagrams can be 'formal' systems and provide 'formal' or 'logical' proof, and they make it clear in their titles, for example: 'A formal system for Euclid's Elements' (Avigad, Dean, & Mumma, 2008), 'A diagrammatic formal system for Euclidean geometry' (Miller, 2001), and 'The logical status of diagrams' (Shin, 1994). It is not surprising to notice that these studies make use of algebraic notation or symbols to demonstrate the formality of diagrams. Some of these studies and others tried to construct computerised, computational or automated reasoning systems to prove that diagrams can be formal and can be used for reasoning (e.g. Barker-Plummer & Bailin, 1997; Barker-Plummer, Bailin, & Ehrlichman, 1996; Furnas, 1992).

To conclude, mathematicians appear to use diagrams to talk to themselves (i.e. thinking), and they sometimes use them for pedagogic purposes. They shy away from using them in proof. The distinction between mathematics and mathematics education with respect to the use of diagrams has widened so that now there is a discontinuity between the two disciplines, especially because mathematics educators consider communicating mathematics to be an essential part of mathematics. So diagrams might be assumed by mathematics educators to be essential in mathematics, even if they are not used in proof.

2.1.4   Geometric diagrams in mathematics education:

Research in mathematics education treats the use of diagrams or visual representations differently from the research in mathematics. Two aspects of that research are relevant to this discussion: visualisation and representation. The scholarship about visualisation and representation is informed by the cognitive approach which focuses on the individual's activity in constructing 'images' of mathematical objects. A third aspect which has been the focus of the recent research, as we shall see in reviewing the studies about the previous two aspects, is the view of mathematics as a cultural and social practice. This view is especially prominent in the social semiotics approach (Morgan, 1996b; Pimm, 1987; Radford, 2003) where representation (and communication) is the focus. In the following, I shall describe how research in mathematics education addresses visualisation and representation.

Visualisation in mathematics education. Researchers of mathematics education began to take an interest in the issue of visualisation during the 1970s and 80s, when constructivism started to arise as a new approach, in opposition to behaviourism. Two main reviews about visualisation and mathematics education have been conducted: Bishop's (1988) and Presmeg's (2006). While Bishop's (1988) review was limited because of the limited number of studies that had been conducted at that time, Presmeg (2006) analyses a 30-year history of research on visualisation and mathematics education from 1976 to 2006. She traces the development of the interest of the mathematics education community in visualisation through the proceedings of the Annual Conferences of the International Group for the Psychology of

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Mathematics Education (PME). The beginning of that interest in visualisation can be traced to the 12th PME in 1988, when Bishop presented his review. In the following I look at Presmeg' review of visualisation in general and, where appropriate, refer to the review of geometry conducted by Owens & Outhred (2006).

While her extensive review is chronological, the areas that Presmeg tries to articulate may be categorised by themes such as: visual imagery (and its relationship to visual thinking/reasoning), students' interaction with the visual representations, the role of technology/computer, curriculum and recent evolving aspects (such as gesture). Before I examine each of these aspects in a broad sense, I note that the term visualisation in Presmeg's review refers to the 'visual image in the person's mind' (Presmeg, 2006, p. 206), which is based on Piaget's work.

Visual imagery is a notion associated with visualisation, especially in geometry, as reviewed by Presmeg (2006) and others (e.g. Owens & Outhred, 2006), with a focus on reasoning and thinking, terms which have been linked to the category of visual imagery and are used interchangeably. In this category, the main interest was the visual reasoning/thinking in which studies focused on students' geometric or mathematical reasoning and on the stages (or levels) of the development of such thinking among students, using the cognitive approach mainly informed by the work of Piaget. For example, Van Hiele's theory was and still is a dominant framework for studying students' geometric reasoning (Owens & Outhred, 2006). Visual imagery was dominant in the 1970s and 80s.

Students' interaction with visual representations was investigated by other researchers. Presmeg focuses on Dreyfus's (1991) plenary session in PME 1991, 'On the status of visual reasoning in mathematics and mathematics education' (I revisit this issue below in more detail). Dreyfus's main argument is that, as the title suggests, students are reluctant to use visual representation in problem solving, which Presmeg challenges through her own research and others and suggests that there are other aspects which may influence that use such as the mathematical task itself or the sociocultural context.

Technology has a strong effect on teaching and learning in geometry and on visualisation. There is good evidence in studies about this issue, especially those of Presmeg (2006) and Owens & Outhred (2006) who analysed the use of Logo and

Cabri in developing visual imagery and learning geometry. The role of visualisation in curriculum development was an additional area of research. The last category to be explored is gestures, which Presmeg highlights as a new area in research about mathematics education, connected to visual imagery (e.g., the work of F. Arzarello and his colleagues). The issue of semiotic systems in visualisation was not considered in Presmeg's or in Owens & Outhred's review, since they only refer to two studies (Arzarello's work on gesture and Duval's paper in PME-24). This is understandable for two reasons: first, the research interest of both reviews; second, the field of research on semiotics in mathematics education is still 'young' (not to mention the research on visual representation). I will revisit this issue when considering representation and social semiotics below. In the meantime, a discussion relevant to my study is the work of Dreyfus and Eisenberg on visual representation.

Dreyfus and Eisenberg (Dreyfus, 1991; Eisenberg & Dreyfus, 1991) called attention to the avoidance of visualisation among students. They (1991) reviewed some studies (e.g. Vinner, 1989) about students' reluctance to use visual representations in solving mathematical problems. Some of the assumptions, Eisenberg and Dreyfus (1991) argue, about using visual representations are that 'thinking visually makes higher cognitive demands than thinking algorithmically' (p. 25). Moreover, they argue (pp. 30-31) that mathematicians and mathematics educators are to blame for the belief that mathematics is 'nonvisual':

> We in the academic community have only ourselves to blame for perpetuating this view of mathematics. ... This belief is deeply grounded in us, even among many who advocate visualisation.

This attitude among mathematicians, the anti-diagram attitude (Kulpa, 2008), was revisited by some studies. Although there is near consensus among mathematics educators that visual representation plays an important role in learning and teaching mathematics and problem solving (see for example, Elia & Philippou, 2004; Guzman, 2002; Presmeg & Balderas-Canas, 2001), mathematicians still deny their use of visual representations. A number of studies show that mathematicians use diagrams in their 'private' work, but they would 'hide this fact' (Dreyfus, 1991, p. 36), believing that 'diagrams must be swept to the side', especially when the work is to be 'formally' proven (Mann, 2007, p. 137) or published (Misfeldt, 2007).

Hadamard (1945), in his inquiry into the practice of mathematicians and their daily habits, asked many mathematicians about the kind of 'mental pictures' they use, including whether they use visual (images) or rely on other representations, such as auditory representations. Most described their daily habits in terms of visual images. Einstein, for instance, replied:

> The words or the language, as they are written or spoken do not seem to play any role in my mechanism of thought. The psychical entities which seem to serve as elements of thought are certain signs and more or less clear images which can 'voluntarily' reproduced and combined. (Hadamard, 1945, p. 142)

Hadamard (1945, p. 74) himself states:

> I insist that words are totally absent from my mind when I really think and I shall align my case with Galton's in the sense that even after reading or hearing a question, every word disappears at the very moment I am beginning to think it over; words do not reappear in my consciousness before I have accomplished or given up the research ... I behave in this way not only about words, but even about algebraic signs.

Moreover, Halmos (1985, p. 400) recognised the importance of images and visual representation when he states: 'To be a scholar of mathematics you must be born with ... the ability to visualize'.

To summarise, visual imagery, or 'image in the mind', is tantamount to visualisation which has been investigated in different aspects of mathematics learning and teaching such as mathematical (and geometric) thinking, proof and problem solving. Visualisation research in mathematics education has grown gradually from the psychological basis in the 1970s and 1980s that focused on mathematical thinking. In the 1990s, visualisation became a recognised field of research in mathematics education, where studies in students' learning, curriculum development and the role of technology have investigated mathematical visualisation.

Mainstream mathematicians, however, reject the use of visual reasoning, distancing images or diagrams from their work. This theoretical stance is the cause of students' reluctance to visualise, as argued by some researchers (Eisenberg & Dreyfus, 1991).

So far, I have not yet considered the concept of representation. One reason for the delay is the 'various meaning and connotations' which have been associated with representation, making it difficult to maintain 'an accurate definition' (Presmeg,

2006, p. 206). Indeed, Presmeg herself used the term 'inscriptions' instead of the term 'representations', but I choose the latter term for my upcoming discussion.

Representations and social semiotics. Instead of seeing 'meaning' as a pre-existing entity, social semiotics considers it as a social construct that is created during the act of communication and representation in meaning-making process (Evans et al., 2006; Kress, 1988b). Thus, studying communication and representation would enable us to analyse what meanings people are trying to make. There are a good number of studies about representation in mathematics education, which is why Radford (2003, p. 40) announced that the 'concept of representation has been one of the most talked about concepts over the last two decades in mathematics education'. Thus, it is not surprising that the National Council of Teachers of Mathematics (NCTM) included the concept of representation in its Standards for school mathematics and recommended that instructional programmes should (quoted from National Council of Teachers of Mathematics, 2000b):

- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena.

Furthermore, two consecutive Special Issues of the *Journal of Mathematical Behavior*, Numbers 1 and 2 of Volume 17, 1998, were devoted to exploring the concept of representation. Goldin & Janvier (1998) and Goldin (1998) summarise various interpretations and meanings of the concept of representation to include:

a. External physical embodiments which embody mathematical concepts such as number line, diagram, calculator or computer-based environment.

b. External linguistic embodiment that is represented verbally and semantically 'of the commonly shared language in which mathematical problems are posed and mathematics is discussed.' (Goldin, 1998, p. 285)

c. Formal mathematical constructs that can be represented through symbols but are still external to the individual. This includes games which can be related

to mathematical concepts, like the Tower of Hanoi, or can be represented by mathematical entities.

d. Internal cognitive representations, such as students' internal representation(s) of mathematical concepts such as area and function. These representations are inferred from behavior or introspection, describing some aspects of the processes of mathematical thinking and problem solving. (Goldin & Janvier, 1998, p. 2)

While these various meanings illustrate the previous comment made by Presmeg (2006) about representation, they can be categorised, as they are in some studies (e.g. Mesquita, 1998), into two broad categories: external and internal representations. However, some scholars argue that representations have pre-existing meanings, which students should uncover in order to 'get the full message', as suggested by this quote:

> the pictures and the diagrams used in mathematical text have their own conventions, which pupils need to learn to read. ... However, if he [the pupil] cannot read the graphic language of his mathematics book as the words and symbols, he will not be able to get the full message from the page. (Shuard & Rothery, 1984, p. 65)

There are other perspectives which challenged this 'decoding' notion such as constructivist perspectives (Cobb, Yackel, & Wood, 1992; Glasersfeld, 1991) and the social practice perspectives (social semiotics, for example) that conceive of mathematics as a social practice. Rather than considering that a reader/viewer is 'encoding-decoding' the message made by a producer, Kress (1997; 2003) argues that reading consists of making new signs, rather than trying to decode the original sign made by the author. The issue of meaning-making, together with communication and representation, have become the focus of the research in mathematics education, especially after social semiotics was adopted by some researchers in mathematics education (e.g. Chapman, 2003b; Morgan, 1996b; O'Halloran, 2005; Pimm, 1992; Radford, 2003).

Moreover, social semiotics would bring the 'representers and their intentions' (Pimm, 1990) or their interest back to the communicative act. Representation in social semiotics is a sign that 'focuses on what the individual wishes to represent about the thing represented' (Kress et al., 2001, p. 4). In other words, the sign-makers (re)present their interests about the thing represented, and they try to make those

representations, as closely as possible, match their 'intentions' or their experiences in a communicative act (Kress et al., 2001). Thus, representations are not merely encoding information but rather they are semiotic signs available for meaning-making. A geometric diagram, for example, is a representation of a mathematical activity or an object that its maker chooses to be the carrier of meaning.

Representations move beyond the linguistic monomodal approach, in which language is the dominant form of communication, to a multimodal approach which considers other modes of communication and representations (Jewitt, 2003a; Kress et al., 2001; Kress & Van Leeuwen, 2001). The multimodal approach, or multimodality, takes into consideration the different modes of representation in a communicative act such as visual representations, language and gestures (see Figure 2-1 for an example). In mathematics, for instance, Morgan (1996a; 1996b) considers the language aspects of mathematics texts and offers an analytic tool to read and analyse mathematical texts as signs. She (2001) conceives of mathematics as a social practice, a 'human activity', a sign which may be interpreted from the social semiotic point of view. O'Halloran (2005), moreover, offers other frameworks for reading other modes of communication in mathematics such as visual representations and symbolism.

Both Morgan and O'Halloran agree that, still, there is room to investigate other modes of representation and communication of mathematical discourse (Alshwaikh, 2009). Morgan (2006, p. 226) states that:

> [m]any mathematical texts also contain significant non-verbal components, including algebraic notation, diagrams, tables and graphs. Tools for the description of these components are less fully developed from a systemic functional perspective.

The current study is, therefore, another endeavour which focuses, to a large extent, on the geometric diagrams and, to a lesser extent, on gesture, offering a (sort of) grammar to read these diagrams and gestures. Grammar in social semiotics 'goes beyond formal rules of correctness', Halliday (1985, p. 101 as quoted in Kress & Van Leeuwen, 2006, p. 2) argues, and he continues:

> It is a means of representing patterns of experience ... It enables human beings to build a mental picture of reality, to make sense of their experience of what goes around them and inside them.

In other words, there is a relationship between the grammatical structure of a semiotic mode (language, diagrams or gestures) and the social experience or interaction with which human beings are involved, which is expressed linguistically (or visually or gesturally). Thus I seek a framework (or frameworks) to describe that relationship or, in other words, to construct a grammar which will enable me to articulate that description. Using the SFL, that grammar would include three functions: ideational (the way people describe their experience), interpersonal (the interaction between people) and textual (the ways that people arrange their description into coherent text).

To summarise, although mathematical diagrams are part and parcel of mathematics and were used in ancient civilisations such as Old Babylon four thousand years ago (Robson, 2008b) and Greek mathematics (Netz, 1999) and although there is near-consensus that diagrams are important in doing, learning and teaching mathematics and in visualisation, mathematical thinking and problem solving – the current mainstream trend among mathematicians is prejudiced against the use of diagrams or, more precisely, mathematicians 'deny' and hide their use of diagrams in their work (Dreyfus, 1991; Morgan, 2001). Mann (2007, 137) also states:

> When a mathematician explores new ideas or explains concepts to others, diagrams are useful, even essential. When she instead wishes to formally prove a theorem, diagrams must be swept to the side.

The traditional approach to diagrams or to visual representation within mathematics education, moreover, is that they encode information that students need to uncover in order to solve problems. In my study, however, I consider (geometric) diagrams as available resources for meaning-making and as a means for representation for students to communicate with each other or with themselves in order to convey specific meanings.

Furthermore, the multimodality approach focuses on the different modes of representation in constructing (mathematical) meaning which entails the need to look beyond language and diagrams to include other modes such as gestures. Thus, in the current study, I suggest a preliminary framework for the gestural mode. In the following section, I give a broad overview of recent studies which have considered gestures in learning and teaching mathematics.

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5. Gestures

In her commentary, '*What's all the fuss about gestures?*', on the research on gestures in *Educational Studies in Mathematics*, Sfard (2009) asks some questions:

> While reading the articles assembled in this volume, one cannot help asking *Why gestures?* What's all the fuss about them? In the last few years, the fuss is, indeed, considerable, and not just here, in this special issue, but also in research on learning and teaching at large. What changed? After all, gestures have been around ever since the birth of humanity, if not much longer, but until recently, not many students of human cognition seemed to care. (pp. 191, her emphasis)

Kress & Van Leeuwen (2001) answer that such questions were not being asked because:

> Language was (seen as) the central and only full means for representation and communication, and the resources of language were available for such representation. (...) And of course there *were* other modes of representation, though they were usually seen as ancillary to the central mode of communication. (2001, pp. 45, their emphasis)

In other words, what changed is our way of viewing and interpreting the act of communication and representation from monomodality to multimodality. However, although Kress & Van Leeuwen (2001; 2006) stress that language is no longer the dominant or the central mode and that 'language may now be 'extravisual" (2001, p. 46), many researchers (in mathematics education and other domains) who are interested in different modes still look at these modes from a linguistic point of view. Arzarello, Paola, Robutti, & Sabena (2009), for instances, consider gestures to be an 'extra-linguistic' mode of expression, Sfard (2009) considers gestures to have a linguistic counterpart that is utterance and addresses the relationship between the gesture and utterance, and McNeill (1985, p. 350) argues that gestures and speech are 'overt products of the same internal processes.'

One main challenge for the research about gesture is to investigate its role in learning and teaching mathematics (Radford, 2009): how to connect mathematics, which has been viewed for centuries as a 'mind-based' (Lakoff & Núñez, 1997) enterprise, with the movement of body, namely hands and fingers. Philosophically, seeing mathematics as a mental activity synchronises with Idealism, which perceives thinking as a purely mental activity, occurring in the mind and not in the physical experience. Mathematical thinking is no exception.

Radford (2009) presents a short account of the considerations of gestures, starting in the nineteenth century, when some studies (e.g. Cushing, 1892) argued that gesture precedes language, and continuing to the twentieth century, through the studies about the origin of language and its relationship to thinking. Since then, this relationship has dominated the research on about gesture. Some studies, Radford (2009) continues, suggest that gesture facilitates or can express what an individual cannot say verbally. Others consider gesture to be a 'window' for accessing thinking.

As an alternative to the-immateriality-of-thinking approach, Radford (2009) suggests a 'sensuous cognition' in which gestures are *genuine constituents* of thinking' (p. 113, his emphasis), not windows or facilitators. However, as Radford himself states, there remain unresolved problems in the relationship between gesture and its role in learning and teaching mathematics, because the research about gestures in mathematics education is still in its infancy (Radford et al., 2009). The Special Issue of the Educational Studies in Mathematics-ESM Journal (2009) volume 70, took the initiative to explore that role.

Some studies investigated the contribution of gestures, together with other semiotic systems (verbal and visual), to mathematics learning and teaching. Adopting the Peircean approach, Arzarello et al. (2009), for example, present the notion of 'semiotic bundle', 'speech, gestures, and inscriptions and their relationships' (p. 100), in order to explore how students make use of those systems in learning mathematics. They conclude that gesture may play two different roles; it supports students' thinking processes and, as a communicative act, constitutes an alternative way to express what students are *unable* to articulate in 'purely verbal or formal ways' (p. 107, my emphasis). The view taken in this study is that gesture is a part of an ensemble or a bundle of semiotic systems which together 'grasp' the 'mathematical ideas' or the 'correct scientific meaning' (p. 106).

Arzarello et al. also consider the relationship between the teacher's and students' use of gestures and claim that 'students are reactive to the teacher's gestures' (p. 107) and that:

> the teacher uses the same gestures as the students and rephrases their sentences using precise mathematical language. Doing so, he supports the students towards a correct scientific meaning. (p. 106)

This assumption that students and teachers make use of the same gestures was initially made by Morgan & Alshwaikh (2008) in a learning geometry experiment using a three-dimensional 'turtle world' (MaLT). They noticed, however, that students' meanings were different from those used by teachers. They distinguish between two types of gestures, imaging and imagining. While an imaging gesture is a construction of an image of the turtle path, imagining refers to the 'mental image of the desired outcome of turtle drawing' (p. 140). In other words, imaging and imagining may be seen as gestural realisations of the famous dichotomy in mathematics: process and product (Gray & Tall, 1994; Hersh, 1999; Sfard, 1994) or operational and structural (Sfard, 1991). Moreover, students used 'pointing' in referring to the position of the turtle but made use of the everyday discourse of pointing rather than the specialised directional pointing of MaLT.

The notion of pointing and sliding in gestures has been noticed by Bjuland, Cestari, & Borgersen (2007). They consider that students' mathematical reasoning may be realised through the gestural strategies of pointing and sliding in solving problems. However, Bjuland et al. (2007) and most of the studies to which they refer (e.g. Bartolini Bussi, 1998; Edwards, 2005; Nunes, 2004) deal with gesture as dependent on the verbal mode.

While it is true that the research about gesture in teaching and learning is still very young, there is a need to look at gesture as a distinctive mode on its own, which contributes to the construction of (mathematical) meaning (Morgan & Alshwaikh, 2008). This is in contrast to considering it as an accompaniment to the verbal, or any other mode, as do the studies in the Special Issue of ESM (2009) volume 70 (e.g. Arzarello et al., 2009; Sfard, 2009), and in contrast to considering it as a 'connection' or 'passage' between systems of representation, such as figure and Cartesian diagram (Bjuland et al., 2007).

A seminal work about gesture as a distinctive mode and about the relationship between gesture, language and problem solving is the work of Luis Radford (e.g. Radford, 2003, 2009; Radford et al., 2007). Radford et al. (2007), for example, conducted a careful analysis of students' interaction in solving a mathematical problem using different sources of data such as videotapes and written texts produced by the students. Adopting a semiotic-cultural theoretical framework (see Radford, 2003), they considered a 'multisemiotic data analysis' to analyse in detail

how the students solve that problem focusing on students' modes of communication (verbal and written words, gestures and rhythm) separately and simultaneously. Indexical gestures, for instance, were dominant in students' interactions, where students used their fingers to point to mathematical objects.

In analysing the rhythm, Radford et al. (2007) used a voice analysis software to follow the stress and intonation patterns of the utterances students made, and they integrated that analysis with analysis of the words and gestures students used. The fundamental conclusion from Radford's et al. (2007) experiment is three-fold: First, to understand students' mathematical thinking, there is a need for more attention to semiotic modes other than language. Second, gesture, as a semiotic mode, is a key element in students' mathematical experience that should be considered. Third, the ensemble of modes is crucial in understanding the way in which students experience mathematics.

While the above-mentioned studies used different theoretical accounts (for example, Radford's work uses a semiotic-cultural theoretical framework), in the current study, I adopt the multimodality social semiotics approach in which I try to look at gestures as a mode of representation and communication which needs, as do the diagrammatic and the verbal modes, a 'grammar' to read it. Gesture, in this sense, is a mode of representation and communication for meaning-making process that is materialised by the movement of hands and fingers.

7.   Summary

A background for the scene of communication and representation has been set up in the previous and the current chapters. While the previous chapter presented a general view about the verbal mode of communication and the relationship between language and mathematics, the current chapter extends the 'semiotic landscape' (Kress & Van Leeuwen, 2006) to include visual communication, namely the diagrammatic mode in mathematics. I presented in a detailed manner the status of diagrams in mathematics and the prejudice against the use of diagrams as a mode of communication and representation. The extension of the scope of communication, the multimodality social semiotics approach, led me to look at a third mode of representation and communication, namely gestures in mathematics teaching and learning.

A key finding so far from this review, which may be seen as a justification and a motivation for this study, is that I deal with diagrams (and visual representations) as social products, not as cognitive products or coding systems containing pre-existing meanings. In other words, a diagram is an essential part of mathematical discourse which needs to be considered in the construction of mathematical meaning. The same principle applies to gestures.

The main conclusion I would say in these two chapters (2 and 3) is that there is a need to investigate and to study the diagrammatic and gestural modes in mathematics discourse. That is what the current study attempts to do.

In the following chapters, I focus on the diagrammatic mode, followed by the gestural mode. The separation between diagrams and gestures is a pragmatic one, since I need to provide a detailed account of each of them before considering them together with the verbal mode in the multimodal analysis chapter.

4 Methodology - Aim and design of the study: An iterative approach

1.   Aim of the study:

The primary aim of this study is to develop a descriptive framework that can be used as a tool to analyse the role of diagrams in constructing mathematical meaning and, to a lesser extent, to develop a framework to analyse the role of gestures and the multimodal interaction between the verbal, diagrammatic and gestural. The previous chapters show that there is a need to create such tools. This endeavour is a contribution toward broadening the scope of understanding of mathematical communication by considering modes of communication other than the linguistic, namely the visual and the gestural.

A primary question arises: What do diagrams contribute to the construction of mathematical meaning? As shown in the literature, this question has been raised in different studies and in different contexts (Kress & Van Leeuwen, 2006; Lemke, 1998b), including in mathematics education (Morgan, 1996b; O'Halloran, 2005).[5]

In this chapter I set up the methodology I used (and developed) toward that aim. I start by presenting the iterative approach to develop the desired framework. The iterative process/approach is not unique to this study but is an accepted methodology within social science research, especially as a means of developing a theory. I explore the similarities and differences between the different methodological approaches that informed the current study, including grounded theory, action research and design research. Then, I address the data of the study, explaining the different types and how they were collected, exploring cultural and linguistic issues that arose and outlining the tasks of the study. I also address the sampling process used in this study and comment on the robustness of the suggested framework. In order to illustrate the iterative process used in this study, I then present briefly the story of the development of the overall framework, although I provide more details about the development of one version of the framework. I end the chapter with a discussion of ethical considerations and of the sampling process.

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[5] While the focus of this study is the visual form in mathematics and its role in constructing mathematical meaning, this study also considers gestures and offers a preliminary descriptive framework to 'read' these gestures in mathematical context.

2.   The iterative methodology

In order to answer the question, what do diagrams contribute to the construction of mathematical meaning, I want to *develop* a framework to analyse the role of diagrams in geometry – an analytical tool not only to describe the features of mathematical diagrammatic representations but also to offer possible reading(s) for these representations in mathematical discourse. The word 'develop' suggests change, trial and re-trial, or in other words – iterations. Iteration is a process of design, testing, revision and modification (Hjalmarson & Lesh, 2008; Pratt, 1998) – making it well-suited for developing a framework. Examples of the iterative methodology include the grounded theory method (Bryant & Charmaz, 2007; Glaser & Strauss, 1967/2006), action research (Kemmis & McTaggart, 1990; McNiff, 1997), and design research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; Kelly, 2004; Pratt, 1998), all of which informed the current study.[6]

Glaser & Strauss helped launch the Grounded Theory Method (GTM) in the 1960s with the publication of their seminal book, 'The Discovery of Grounded Theory' (Bryant & Charmaz, 2007), which challenged the practices then dominant in the social sciences within the United States. Later, each of the two sociologists worked individually to develop distinct approaches to GTM (Denzin, 2007). Despite the divergence, grounded theories continue to be defined as qualitative research methods to generate a theory from data (Glaser & Strauss, 1967/2006). Grounded theories are based on the interaction between the researcher and the data, in which theories are generated using an iterative process (Bryant & Charmaz, 2007). The process of generating a theory is summarised in the following steps: collecting data; finding accurate evidence using comparative analysis and coding to create categories and concepts until saturation is achieved; and verifying and generating the theory (Glaser & Strauss, 1967/2006). Dick (2005) provides an overview of this process as does the web site, Grounded Theory Online, http://www.groundedtheoryonline.com.

Glaser & Strauss (1967/2006) developed the iterative nature of the process of generating theory by drawing attention to the need to revisit the collected data or to collect new data in order to test or create categories. The iterative process is also apparent in the sampling aspect of grounded theory, described by Glaser and Strauss

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[6] There are other resarch methodologies/approaches in social science in which iterative processes are considered to be vital, such as the case study (Eisenhardt, 1989).

as aspiring to a point of 'theoretical saturation' in which a researcher will 'continually judge how many groups [of data] he should sample for each theoretical point' (Glaser & Strauss, 1967/2006, p. 61). Bryant & Charmaz (2007, p. 25) explain that '[t]heoretical concepts in GTM result from iterative processes of going back and forth between progressively more focused data and successively more abstract categorizations of them.'

Iterative process is also a feature of action research. 'Action research is an iterative process involving researchers and practitioners acting together on a particular cycle of activities, including problem diagnosis, action intervention, and reflective learning' (Avison, Lau, Myers, & Nielsen, 1999, p. 94). McNiff (1997) reviews the main approaches in action research starting from the seminal work of Kurt Lewin and the works of Lawrence Stenhouse, Stephen Kemmis, John Elliott, Dave Ebbutt and Jack Whitehead, as well as more recent developments of the theory. Lewin defined action research as an action-reflection cycle or spiral of steps, each of which has four stages or moments: planning, acting, observing, and reflecting. Kemmis & McTaggart (1990) applied action research in educational practice, focusing on learning, teaching and systems of the school. Action research in that sense uses action to reflect on one's own practice (i.e. that of a teacher or student) by first identifying 'thematic concerns' or 'defining the field of action', suggesting a plan, acting, reflecting through communication with the action research group and then suggesting a revised plan or re-plan (Kemmis & McTaggart, 1990).

In other words, in action research, the whole action-reflection cycle is iterated through reflection and re-planning conducted by the research group in order to improve the desired product of the research, namely the action intervention which has been determined as the goal at the beginning of the research. Recent advancement of the theory has moved to the notion of 'spiral of spirals' or 'generative action research' (McNiff, 1997, p. 45) which enable a teacher-researcher to use action research to consider a number of problems at the same time.

Some studies (e.g. Dick, 2007) have examined the similarities and differences between grounded theory method and action research. Although action research is concerned with action and change, both approaches, for example, seek to develop theory from specific evidence using iterative processes. One main difference is that action research is participatory (Kemmis & McTaggart, 1990), while participants in

grounded theory are informants and only rarely act as agents in developing the theory (Dick, 2007).

Like Kemmis's work in action research in educational settings, design or design-based research methods focus on designing learning environments and conducting educational research by continuous cycles of designing, applying, studying, refining and re-designing in classroom environments in order to link theoretical research with educational practice (Collins, Joseph, & Bielaczyc, 2004; The Design-Based Research Collective, 2003). In fact, Kelly (2004, p. 118) refers to design research as a 'new form of action research'. The term 'design experiments', introduced in 1992 by Ann Brown and by Allan Collins (Collins et al., 2004), emphasises the relationship between design-based research and action research by applying innovation, usually in a teaching/learning situation. The main feature of design research is the presence of a designed artifact, for example a concrete artifact such as a computer programme (Kelly, 2004) 'as well as less concrete aspects such as activity structures, institutions, scaffolds, and curricula' (The Design-Based Research Collective, 2003, pp. 5-6).

Cobb et al (2003) identify five 'crosscutting features' which are characteristic of design research or design experiments: developing theories about the learning environment; exploring possibilities for improving the educational environment; using prospective and reflective process; using iterative design; and making the developed theory practical for the purpose for which it was designed. The current study differs from design research in that it does not, for example, seek to develop theories about the learning environment, but it shares the iterative design feature identified by Cobb et al (2003).

These three approaches – grounded theory, action research and design research – informed the current study's development of the desired framework and also contributed to my own reflections as a researcher. The former is the aim of this study, while the latter is not, although my reflections on each iteration contributed to the development of its successor.

To achieve their aim of developing a theoretical framework to understand mathematical meaning-making processes, Noss & Hoyles (1996) suggest the notion of 'thinking-in-change'. They argue that researchers get to know 'things' (ideas, or mathematical objects such as integers) by 'acting on' them, in other words, by setting

them in motion and studying the change that happens. My journey in developing the framework underwent a similar process (Figure 4-1 shows a sketch of the iterative design in this study). After suggesting each version of the framework, a validation process was followed by applying the suggested framework to mathematical diagrams in textbooks or students' texts. Feedback from the application process was incorporated into amendments, creating a refined version of the framework. This process was repeated iteratively.
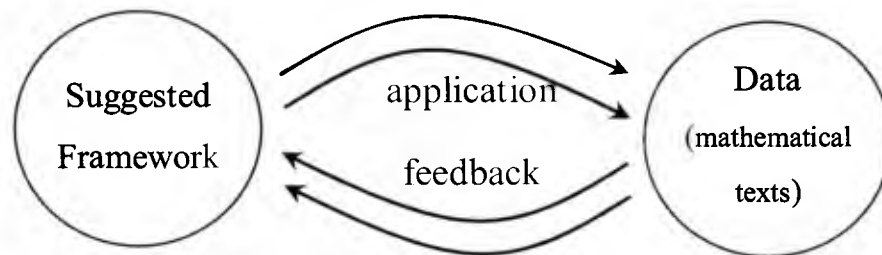
Figure 4-1: A sketch of the iterative design of the study

In other words, the iterated aspects in the current study were the suggestion of framework and the application of that suggested framework to the collected data. Interaction with the data was a central aspect in the iterative methodology of the current study. I collected new data based on the feedback produced from each cycle (see Figure 4-2) and then applied the suggested framework to the new (and old) data. The iterative process in the current study has similarities to the other three approaches: namely revisiting the collected data or collecting new data in light of feedback from each iteration as in grounded theory and iterating the whole cycle of investigation as in action research and design research. Moreover, as is the case for these three approaches, the purpose of iteration in the current study was to test the applicablity of the object of the research, which would be the suggested framework or categories in grounded theory; the action plan in action research; and the designed artifact in design research. Table 4-1 offers a comparison of the four methods.

The current study, however, differs from the other methods in that its output is a descriptive tool rather than a theory, as is the case in grounded theory and design research (see the discussion in Strauss & Corbin, 1998, pp. 18-19), or an action plan, as is the case in action research. I should also note that while in grounded theory and

the current study, researchers work on their own, action research adopts a participatory approach in which researchers and participants work together, and in design research, educators and designers work collaboratively.

Table 4-1: Comparison/contrast between grounded theory, action research, design research and the current study

	What is iterated	Purpose of iteration	Output/goal of the research	Relationship with participants
Grounded theory	Suggested categories: apply and revisit the collected data or collect new data.	Test or create categories, generate theory	Theory	Researchers work on their own
Action research	Action-reflection cycle: plan, act, observe, reflect, re-plan	Test the plan	Action intervention to improve educational practice (teaching/ learning situation)	Participatory: researchers and participants work together
Design research	The whole cycle: design an artifact, test the artifact, reflect, re-design	Test the designed artefact	Theories about the learning environment	Educators and designers work together
Current study	The whole cycle: new suggested framework, apply it to data, evaluate and get feedback	Test the applicability of the suggested framework in each cycle, re-suggest new version	Descriptive framework	Participants were source of data, they generated texts upon which I acted

Iterative design research, moreover, offers a suitable environment to explore thinking-in-change notions and conjectures (Cobb et al., 2003; Mor & Noss, 2008). It is particularly suited for the exploratory nature of my study, which brings a multimodal social semiotics account (Kress & Van Leeuwen, 2006) into mathematics education, in a context in which there is a lack of research about learning mathematics from a semiotic point of view (Morgan, 1996b; cf. Rotman, 1988). Furthermore, this study is exploratory, since it deals with two different cultures (in the UK and the OPT), two different languages and different scripts and writing

systems (English and Arabic). It explores how all these factors contribute to the construction of mathematical meaning – again in a context of a lack of research.

2.1   Iterations and cycles of the current study:

Pratt (1998) suggests a four-iteration framework to develop a computer-based tool to study young children's intuitive knowledge of randomness. Iteration 0 (Bootstrapping) is a 'guess' stage that aims to initiate the iterative design process. This guessing is indeed based on previous knowledge. However, Iteration 1 (Exploratory) is considered as a revised and refuted version of the previous Bootstrapping iteration, focusing more on the tool itself rather than its use. The main feature in Iteration 2 (Developmental) is that the tool is advanced enough to give researchers an understanding of its effect on children's use of it. In the final iteration, Iteration 3 (Analytical), no major modifications are needed, allowing researchers to focus on its impact on children's engagements.

Pratt's study has a different approach from mine. His study has two aims, namely developing a computer-based tool and in parallel studying the development of children's thinking. Moreover, his iteration is based on testing the tool with children. In contrast, my study focuses on developing a conceptual framework, and the iteration is based on testing the framework on mathematical texts. In my study, Bootstrapping was not iterative since it happened, as Pratt himself suggested, to initiate the process. Actually, the first two iterations, Bootstrapping and Exploratory, suggest exploration, which is exactly the case in my study. Thus I consider the four iterations in three cycles (see Figure 4-2).

- Iteration 0 – Bootstrapping: I suggested the first framework (Framework 0) based on my engagement with the literature. The idea was to make a start of the process, and thus this iteration was not tested on mathematical texts.

- Iteration 1 – Exploratory: After engaging with the literature (mainly Kress & Van Leeuwen, 2006; Morgan, 1996b; O'Halloran, 2005) and looking at many mathematics textbooks, I suggested an ad hoc outline of my expectation for the framework (Framework 0). This framework was applied to two mathematical texts (see Figure 4-6 and Figure 4-7). My aims were to explore the applicability of Framework 0 and, indeed, to get feedback for the next version of the

framework. As a result of the Exploratory iteration, the general characteristics of the framework started to become apparent (Framework 1).

- Iteration 2 – Developmental: This iteration was the fundamental and most significant and 'time-demanding' stage in the development of the framework where I started to see not just the 'outer shell' but the details of the framework as well. Having developed a new version of the framework (from Cycle 1), Cycle 2 began with thought, reflection and more engagement with the literature regarding Framework 1. In the Developmental iteration, I applied the framework to the previous two mathematical texts, student's texts from Cycle 2 (see section 3.3.2 & 3.3.3 below) and many different diagrams from different sources especially the Internet (see section 3.3.1 & 3.3.5 below). Framework 2 was the product of this iteration and Cycle 2.

- Iteration 3 – Analytical: I applied Framework 2 to the final data of the study in order to validate it (to test its applicability – see the following section) on one hand and, on the other hand, to get feedback for further development (refinement). As a result, Framework 3 appeared. The validation and refinement process showed that no further changes were needed, at least not in order to address satisfactorily the present data set.
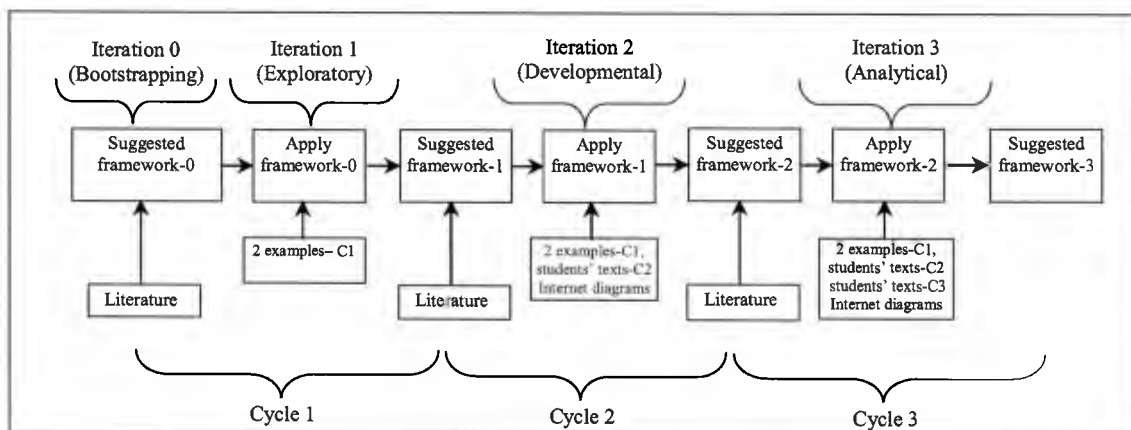
Figure 4-2: Cycles and iterations in the methodology

The main source of data for this iteration was the mathematical texts that students produced (see section 3.3.2 & 3.3.3 below) in response to the tasks of this study (see section 3.6 below). This iteration, moreover, was tested by the same two

mathematical texts from Cycle 1 in addition to diagrams from textbooks and the Internet (see section 3.3.1 & 3.3.5 below).

Although the process in Figure 4-2 is shown as linear, it is not. Another possible representation of the process would be a 'spiral' diagram that shrinks as the process advances toward other versions of the framework. In other words, Cycle 1 would have a wide base, reflecting the nature of exploring, while Cycle 2, as the next loop, would be narrower, and Cycle 3 would be still narrower but more focused as well. Each cycle ended with a new and refined version of the framework. For example, Cycle 1 ended when Framework 1 had been established. The same applies to Cycles 2 and 3. The 'final' suggested Framework 3 would need to be tested in the 'field,' perhaps with students, textbooks, teachers and the method of teaching, although such test is outside the scope of this study, as explained in section 4 of this chapter.

My engagement with the literature was continuous (see Figure 4-2), where I moved between the literature, frameworks and the applications of these frameworks. Obviously there should be a 'limit' or an end to this process for various reasons such as the aim of the study (as a doctoral thesis) and the time constraints. However, there were primary crucial 'events' in the process that 'pushed' the process and study toward that end. These are the iterations of the frameworks on specific data.

2.2   Validation of the framework:

Part of the iterative design methodology is to validate the tool/framework in order to test it and to get feedback for the next iteration (Pratt, 1998). Although the iterative design methodology and the notion of thinking-in-change have been used in designing computer-based tools/environment and students' learning (Noss & Hoyles, 1996; Pratt, 1998), my use of these notions was in a 'different' context and reflected a different approach. While I did not seek to develop a computer-based tool, I made use of the iteration notion focusing on the framework itself, rather than on another notion such as students' learning. Actually, the aim of the study informs the way in which a researcher makes decisions about the validation and refinement process.

In Pratt (1998), where the aim was twofold, namely designing a computer-based tool as well as students' use of that tool, the validation process of the tool took place with students and is evaluated based on their engagement with the tool. The students were

observed and were interviewed during and after their engagement with the tool. Each suggested version of the tool was tested with students in order to validate and refine it. In each iteration, 'opportunities and weaknesses' emerged as a result of students' engagement with the tool. Then, the following iteration took into consideration these opportunities and weaknesses and redesigned the tool for the next iteration.

In my study, on the other hand, my aim and focus was on the framework itself and its development. Consequently, I needed to validate the frameworks with texts rather than students. Each cycle in my study (where the suggested framework had been applied to various diagrams in textbooks and in the empirical data and diagrams available on the Internet, see Figure 4-2) underwent a careful process of validation in the journey through which the framework was developed.

Validation: this process has two primary aims: to check the applicability of the framework and to get feedback that leads to a new refined version of the framework. The validation process occurred by applying the new version of the framework to mathematical texts either from textbooks or empirical data collected in schools. While the application process is 'straightforward', the resulting feedback played a crucial role in refining and developing the desired framework. During and after the application process, I focused on the weaknesses or mismatches between the framework and diagrams in the data which indicated that the framework is not valid yet. Three criteria were considered in the validation process: accuracy, delicacy and inclusiveness. The first two criteria deal with specific examples of diagrams, looking carefully for the details. The other criterion, in contrast, considers a horizontal view of diagrams, i.e. a variety of diagrams across Euclidian geometry school (trying to include all diagrams in the framework). As a consequence, three different kinds of feedback were offered which suggest the need to refine/change the framework.

a.1. *Accuracy*: by this criterion I was seeking a framework that can describe a diagram accurately. By accuracy I mean that there is a clear match between the suggested categories in the framework and their physical realisations in the diagram. During my interaction with the diagrams, I studied the diagram itself and its potential mathematical meaning. At the diagram level, an accurate framework means that any suggested category in the framework describes exactly what it is intended to describe

in the diagram. If I want to call a diagram narrative, for example, it should clearly contain a physical realisation of the indicators that I identify as being determinative of the narrative category of diagram. Here I can recall a good number of examples in my first trials to develop the framework. For example, in Framework 1, I suggested, following Kress & Van Leeuwen (2006) that the presence of vector, or arrows, is the distinguishing feature between narrative and conceptual diagrams. However, in one of my engagements with the analysis of data, I considered dotted lines to be vectors. Later I decided that this was not accurate simply because dotted lines do not always tell the direction, the starting position or the final position.

Another two issues of accuracy were raised after I engaged with the data analytically: the presence of human agency and the classification and analytical structures in conceptual diagrams.

At the meaning potential level, the potential mathematical meaning should have its material realisations in the diagram and not be initiated (although it will be influenced) by prior knowledge of the context or the subject matter (school geometry). Again, here I can recall examples during the development journey. First of all, I recall the confusion between the different kinds of functions in the SFL approach, namely the ideational, interpersonal and textual functions in the first two versions of the framework. For instance, in Framework 0, the very first version, I wrote under the ideational function:

'The right angle sign ... raises the issue of using Pythagoras theorem'

This statement is inaccurate. The little square mark (or the right angle sign) does not necessarily urge the use of Pythagoras theorem. The statement proposes a link between labels presented in the diagram and prior knowledge of the field, without supporting such link by any visual structure of the diagram. Furthermore, while this statement was presented within the ideational meaning in the first framework, labels (such as this sign) in the final version are considered within the interpersonal meaning in the 'contact' category, where they – labels – either offer or demand 'something' from the viewer of the diagram.

a.2. *Delicacy*: In addition to establishing an accurate framework, I tried to offer a subtle, delicate one that would be able to distinguish carefully between different kinds of the same category, where they exist. Here I refer to the presence of vectors,

which have been suggested by Kress & Van Leeuwen (2006) to distinguish narrative images from conceptual images, where vectors express narrative. I adopted this approach at the beginning of the development journey of the desired framework. The use of arrows in mathematical texts is highly conventionalised, and it required a different interpretation than what Kress & Van Leeuwen have suggested. Therefore, I soon faced many examples of different kinds of arrows in mathematical texts that do not comply with this directionality feature. There are many different kinds of uses of arrows in geometry such as parallelism, Lines, Rays, and others, in addition to arrows that refer to mathematical action such as transformations (Figure 5-1).

For example, the arrow in Figure 4-8 expresses a directional feature with a potential meaning as an extraction of a triangle in order to apply Pythagoras theorem. However, a bidirectional arrow next to a side of a triangle (Figure 4-3a) may suggest a process of measuring the size of that side. These two kinds of arrows are also different from arrows that express geometrical characteristics such as parallel lines and a secant, as in Figure 4-3b.

Image redacted due to third party rights or other legal issues

Figure 4-3: Different types of arrows in geometry
Diagram a is taken from http://www.mathsisfun.com/triangle.html
Diagram b is taken from http://www.tutornext.com/help/parallel-lines-cut-by-a-transversal-worksheet [7]

This extensive engagement with diagrams has led me to rethink the directionality feature. As a result of that rethinking process, I have suggested *temporality* as a feature that distinguishes narrative from conceptual diagrams. Rather than considering the spatial features in diagrams (the presence of arrows as a directional

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[7] Instead of referencing every diagram in every figure separately, I present a list of references of all diagrams in all figures before the bibliography list. The references include the source of the diagrams (e.g. URL, 'Grade 7/Year 8', UK or OPT student textbook, 'name of student' which is a pseudonym). However, I may mention the reference in the same figure where appropriate. Where a diagram comes from student's work, I indicate whether it is typical (common) or unique. Typicality refers to the fact that the majority of students drew similar features of the diagram. See section 6 of this chapter 'Iterative sampling of data'.

feature), I suggest looking at the temporal feature of diagrams or, in other words, the sequence of time represented in diagrams through different visual marks. The presence of visual marks which convey sequence of time refers to the temporal factor which distinguishes narrative from conceptual diagrams. This development is discussed further in Chapter 5.

The case of arrows provides an example of the accuracy feedback discussed above and also raises the issue of delicacy, namely whether the framework is subtle enough to distinguish between different kinds of arrows.

a.3. *Inclusiveness*: Accuracy and delicacy feedback were related to the description of specific diagrams, to make sure that the framework is settled and valid. But what about *inclusiveness;* would the suggested framework be able to describe *any* diagram? I faced this question when I tried to apply the suggested versions of the framework to the empirical data and mathematical texts in textbooks, which showed me a range of examples that the initially suggested framework did not include. The shading aspect in narrative diagrams, for instance, was raised during my interaction with students' texts in the feedback phase, when I noticed that some students shaded their diagrams, while others did not. The framework I initially suggested was unable to describe this difference. Another example that demonstrates the inability of the initially suggested framework to include all examples was construction diagrams, meaning diagrams bearing the physical marks showing the construction process. I considered construction diagrams to be narrative diagrams, something I developed only in later iterations of the framework, as a result of my interaction with different mathematical texts in textbooks in order to address the inclusiveness issue. The first version of the framework did not include such diagrams, because construction diagrams did not appear in students' texts.

To sum up, as soon as I began applying the framework, I received different kinds of feedback that needed to be considered in the subsequent cycle of development. As a result, a 'refined' version of the framework emerged and was used in the following cycle/iteration. In that sense, iteration is a process of investigation, construction, validation, feedback and refinement.

I must say that the process of validation played not only a crucial role in developing the framework but also in the development of the whole study, especially the

consideration of arrows as expressing temporality instead of directionality. It is because of these kinds of feedback that I re-thought, re-tested, re-applied and re-suggested new versions and, thus, managed to develop the desired framework. All of these processes – thinking, testing, application and suggestion – needed data to address, and this is what I consider in the following section.

3.   Data of the study:

Data is the main focus in this section. I start with the reasons to collect data in order to validate the framework and, consequently, what types of data were needed to achieve that goal and others (see below). I then discuss the various sources of the data used in my study, how I collected them, the tasks of the study (as a means of collecting data) and sampling strategies.

3.1   Rationale:

Both the aim of this study and the theoretical orientation informed my choice of data. My aspiration is to construct a framework that is able to describe geometrical diagrams and analyse their role in constructing mathematical meaning. Thus different geometrical diagrams were needed from different sources such as textbooks, students' texts and the Internet to validate the suggested versions of the framework. Textbooks and the Internet are 'easily' available. The question thus became how to get students' texts about diagrams. This led me to the idea of having students work on specific tasks.

The theoretical orientation (from mathematics education and multimodal social semiotics) also contributed to my choice of data. I position myself among those who believe that mathematics is a social activity in which people communicate with each other, and that the meaning-making process is also a (social and cultural) communicative act. Understanding this communication and meaning-making exchange must take into consideration the need to understand the immediate situation of production of texts, the context of that situation, (i.e., in the current study, the context of learning, where students produce their mathematical texts for the aim of the study as discussed) and the broader context of culture (Morgan, 2006). This theoretical orientation thus contributed to the development of the framework in

offering potential interpretations of the framework either from a mathematical point of view or a social semiotics account. The straightforward example is the development of the ideational function in the framework, where mathematical activity and mathematical objects are the focus.

Furthermore, the issue of generalisability was also raised. Traditionally, generalisability 'is interpreted as comparability and transferability' (Cohen, Manion, & Morrison, 2007, p. 137), or, in other words, the ability to generalise findings to various populations, settings and cultures in order to achieve the 'scientific-ness' of research. As Smith (1975, p. 88) claimed, 'the goal of science is to be able to generalize findings to diverse populations and times' (quoted in Schofield, 2007, p. 182). The typical method to deal with generalisability used to be quantitative, in a statistical sense, where statistical tests or measures will be applied to data. Schofield (2007) claims that the generalisability issue is a goal in quantitative research, while, in contrast, qualitative researchers either reject this stance or give little attention to the generalisation issue, arguing that it is irrelevant to their goals. This stance changed in the 1970s.

Following (Schofield, 2007), Pratt (1998) suggested reconceptualisation of the term based on the research aims, methodology and context:

> Generalisability needs to be reconceptualised as 'fittingness', or 'translatability and 'comparability', or 'naturalistic generalisation'. By giving thick descriptions of the situation observed, it is possible to determine intuitively whether the description fits another situation or not. (Pratt, 1998, p. 110)

The term 'thick description' has been dealt with in research methods in many studies (see for example: Hammersley, 2008; Hammersley & Atkinson, 2007; Schofield, 2007). Schofield (2007) discusses the issue of thick description in the context of redefining generalisability and the need for thick description.

In his critical review of Geertz's notion of 'thick description', Hammersley (2008) allocated a chapter, 'On thick description: Interpreting Clifford Geertz', to discussing this term and the contribution of Geertz in developing the term, which he borrowed from the philosopher Gilbert Ryle. Thick description is a description of an act which 'does not just tell us *what* was done but *how* it was done. [In this sense, it is] a description of an action that involves at least one adverb' (Hammersley, 2008, p. 53, emphasis is in original). Geertz pointed out the role of context and background in

providing insight to researchers/ethnographers and enabling them to achieve thick description and, consequently, to be able to judge the issue of 'fit', which Schofield (2007) considers as generalisability.

Thus, we need to understand the context of research and consider background information which has implications for the required data of the study. Moreover, in my study, applying the suggested framework in different situations or contexts (either by conducting the study in different classes or schools or even different cultures and languages) will contribute to the generalisability of the study. I feel 'fortunate' to be able to do that in the Occupied Palestinian Territories (OPT) with some Palestinian students in the Arabic language and also with some British students in the UK, in the English language.

3.2   What are the needed data?

These three factors, the aim of the study, the theoretical orientation and the generalisation issue, have affected the type of data needed in order to meet these demands. The necessary data have to achieve three intertwined goals (see Table 4-2):

a. Validation of the framework: The idea is to apply the framework to different sets of data to test the applicability of the framework and to get feedback for the next iteration. As shown in Table 4-2, the source of data to achieve this purpose is mathematical texts to which different suggested versions were applied. Since the scope of my study is within school mathematics, the 'natural' way is to look at textbooks and students' texts. The Internet offers a wide range of geometrical diagrams which I needed to see and to which I needed to apply the framework. This is related to the issue of generalisability as discussed below.

b. Understanding the context of production of mathematical texts (context of situation and context of culture): Most of the data were intended to be collected in schools in the UK. I was not familiar with the UK context, including the classroom environment, students' interaction, modes of teaching, textbooks and the like. I needed to understand the context of production of mathematical texts in order to deepen my engagement with the data collected. To attain this purpose, I observed the whole class in 'normal' teaching/learning lessons, preferably, but not obligatorily, about geometry. I also video- or audio-recorded these lessons. In addition to the

observation, I collected students' texts after they solved the problems (see the tasks section in this chapter).

As for the OPT, I wanted, to some extent, to understand the immediate context of production of students' mathematical texts (the context of situation), although I undertook that inquiry differently from the inquiry in the UK, because I live within the broader Palestinian culture and therefore am more familiar with the context. This difference is reflected in the data collected in the OPT in comparison to the UK (see Table 4-3 in this chapter).

c. Generalisation: Collecting data in two different contexts (linguistically, socially and culturally) – in the UK (in English) and in the OPT (in Arabic) – offers a rich opportunity for the issue of generalisation in the sense that Pratt (1998) and Schofield (2007) discussed, especially the issue of thick description and fittingness. Besides the issue of generalisability, students' mathematical texts produced in this study (in two different languages and cultures) offered a chance for more insights into mathematical communication and discourse. These texts offer, for example, a chance to understand how mathematics is produced in different languages (English and Arabic), where each of them has its own tradition and writing systems (see the discussion of information value in Kress & Van Leeuwen, 2006). I discuss this issue in a separate section in this chapter (see below) and in more detail in Chapter 9.

Table 4-2: Purposes, sources and nature of data of the study

Purpose of data	Source of data	Nature/type of data
Validation of the framework	• Textbooks • Students' texts • Internet	Mathematical texts (written and visual)
Understanding the context	• Observation the whole class • Students' work in group problem solving	• Video and audio records • Field notes
Generalisation	• Different contexts, languages and cultures: applying the study in two different countries. • Different texts (textbooks, students' texts and texts in the Internet)	Mathematical texts in English and Arabic

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3.3 Sources of data:

The sources of the data needed to be responsive to the above-mentioned purposes and to be available within the context of the study (diagrams in geometry school in two different situations and languages). Thus, data in these sources could be divided into three categories based on the purpose they meet:

a) Data for validation of the framework. Since the validation needs to be applied to as many diagrams as possible (and to be generalised as well); textbooks, students' mathematical texts and Internet were good sources to get data which achieved this purpose. The second source, students' mathematical texts, was obtained by asking students to solve two geometrical problems (see tasks of the study).

b) Data for understanding the context of situation and the context of culture. This type of data could be gained from observation and field notes in the classroom while students learn in a 'normal' class and from observing students while they solved geometrical problems (Group problem solving).

c) Data for generalising the framework. This was achieved by examining a large number of texts and by applying the study in two countries with different cultures and languages. In other words, the need for generalisation justified collecting data in the UK and the OPT.

These categories are not distinct completely; they have common data which belong to both, such as students' mathematical texts which meet the need for validation and also for the generalisability of the framework. At the same time, the development of the framework created a need to have tasks for the study that students needed to complete while I observed them. Therefore, I describe these sources individually based on their connection with each other rather than the categories to which they belong.

3.3.1 *Textbooks:* this study focuses on geometrical diagrams within the context of school mathematics in order to suggest a framework to analyse the kind of meaning that these diagrams might offer in the construction of mathematical meaning. Hence, textbooks were a source for looking at how geometrical diagrams are (re)presented in mathematical texts and, consequently, to inform the development of the framework. On the other hand, these textbook were also texts to which to apply the framework.

The iterative design of the study made it possible for the textbooks to play these two roles. In the developmental role, textbooks offer a wide variety of representations of diagrams that helped in the construction of the suggested framework. For example, I noticed the construction structure of diagrams only in textbooks (and not in the tasks of the study). At the same time, textbooks played a role in the validity process of the framework, namely in the application process, where the framework was applied to different diagrams which in turn offered feedback for the development of the framework.

One difference between the two places in which this study took place (The UK and the OPT) is that in the OPT, one textbook is used for each grade throughout the 'territories', while in the UK various alternative 'textbooks' are available for schools to use. Most, if not all, of these textbooks are available on the Internet.

3.3.2 *Students' mathematical texts*: While textbooks 'represent' the 'official' discourse of mathematics, I wanted to see how students (re)present (and communicate) geometrical diagrams in their problem solving. Students' own mathematical texts play a different role from that of textbooks, though students are influenced by, and may adopt, to some extent, the 'official' discourse of school mathematics (see Morgan, 1996). In order to achieve this goal, i.e. to create students' texts, two geometrical problems (see section 3.6) were introduced for the students to solve in small groups (see section 3.3.3 below). After arriving at a solution as a group, students produced their individual solutions to the two tasks in written form. Approximately 350 texts were collected.

The large number of texts played a tri-fold role in the study, namely in the development of the framework, in its application, and in the analysis process. As with other texts, students' texts were a source of various kinds of diagrams addressing the same problems. This variety informed the development of different versions of the framework and also provided diverse material for applying the different versions of the framework. The third important role was in the analysis process, in which I applied the suggested framework in order to construct the potential meanings of these texts, both in the textual function of the framework as

well as in analysing the communication process and taking into consideration different modes of communication (language, diagrams and gestures).

3.3.3 *Group problem solving*: This source contributed to the purpose of creating mathematical texts as well as understanding the context of learning (contexts of situation). The idea here was to get three students to work together in solving two geometrical problems and to produce their individual solutions as described in the previous section 3.3.2. Because some research shows that heterogeneous students' abilities play an important role in creating effective communication of small groups of students (Curcio & Artzt, 1998), I consulted teachers for the selection of the groups to have the most effective communication. Consulting teachers also helped in obtaining a range of students' involvement and attitudes while they solved the tasks in small groups.

As in the observation (see section 3.3.4 below), this stage of data collection included a video-record of the small groups working and the collection of students' written texts produced after solving the two problems. Where allowed, I video[8]- or audio-recorded students' work in the small groups. Besides understanding the contexts of situation, video offers a chance to 'capture' all modes of representation and communication (such as the verbal, the diagrammatic and the gestural) that students use to solve the task. It enables us to view the data whenever we want and also to transcribe it for the analysis process (Jewitt, 2006).

3.3.4 *Observation*: As part of understanding the context (situation) within which students learn and do mathematics, observation of the class was conducted before the group problem solving took place. Where permitted, I video- or audio-recorded the whole class, especially in the early cycles (1&2). Another reason for the observation is to 'eliminate' the effect of my presence as an external factor by getting students to become familiar with it.

Field notes were taken as a result of this observation. These notes were very helpful, especially when the video camera was focused on specific events and could not 'capture' everything that happened or when I was not allowed to video record the

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[8] One school refused to allow use of the video camera to record students. I therefore audio-recorded the whole class and the groups' discussions.

class. Most of the field notes were written after the class, during my reflection on each event of data collection.

3.3.5 *Internet* or *diagrams-on-screen*: Another source of different geometrical diagrams was the Internet. This source contributed, as did other sources, to the development of the framework especially in suggesting different kinds of diagrams. At the same time, the diagrams on the screen provided an area for the application and testing of the framework (validation process).

Here I distinguish between two types of diagrams on the Internet (or diagrams on the screen): static and dynamic diagrams. Static diagrams-on-screen are similar to those on the page (paper, book, etc.) in the sense that they do not move physically and, thus, the suggested framework is applicable to these diagrams as well. In other words, this framework is applicable to 'any' 2D geometrical diagram, whether it is drawn on the page or on the screen, so long as these diagrams are not in motion.

Dynamic diagrams, on the other hand, are diagrams-in-motion in the material sense that we see them moving. Dynamic diagrams are beyond the scope of this study and, hence, I do not claim that the suggested framework is applicable to this type of diagram. More investigation is needed to explore this question.

3.4   Data collection:

Before collecting any data, the necessary preparations had been done, such as addressing ethical considerations (academic and legal aspects, including obtaining approval and signed consent – see section 4 in this chapter), contacting the schools and teachers and later conducting the study. However, there was a difference between my collection of data in the UK and in the OPT, because I was not able to travel to the OPT freely. Thus, I describe here the methods of data collection separately. Table 4-3 shows a detailed picture of the study's collected data in terms of type of data (Cycle 2 or Cycle 3), students' year or grade in school, place and date.

3.4.1   Data collection in the UK

Collecting data took place at different times (see Table 4-3). I will not refer to particular sets of data during my description, because the same procedures were

followed in each of the schools where I collected data. I observed and video-recorded the whole class, with the exception of one school which did not allow video recording but rather only audio recording. I recorded the classes for one to three lessons. Usually, I sat behind pupils at the back of the class, to minimise the effect of my presence and the camera. After the lesson, I wrote my notes, comments and impressions about the lesson, teacher and students. In general, and based on school demands, the teacher took responsibility for talking to the students and arranged any contact, needed documents or papers for my data, meaning that I rarely talked directly to students unless they asked me questions about the problems/tasks. The students were informed in advance about problem solving and group work, which took place in a lesson dedicated to that purpose. The teacher distributed pupils into different groups. Ideally, there were three students in each group to work on the problems, however, and because of the space available, we (the teacher and I) agreed to have three students in the group that was to be video- (or audio-) recorded and to have more than three students in the other groups.

I assigned a number for each group and asked the students to write the group number on their texts. The reasons were that, on one hand, I wanted to see the whole group work and, on the other hand, I needed an administrative tool to organise the collection of the texts. Each problem was addressed separately. The first problem was distributed, and students were asked to work together and then to write their individual solutions on separate plain paper, which were then collected. Only after that process was completed was the second problem distributed. The students were told that ten minutes were expected to be sufficient to discuss the problem, and five minutes were expected to be sufficient to write their own solutions, but students were allowed to take more time if they needed it. The students were allowed to ask me any questions while solving the problems.

Where it was possible, and in order to understand the context of learning and teaching in the UK, I observed many different lessons in different classes with the same teacher (not necessarily mentioned in Table 4-3). For example, in KNT School I observed lessons for Y10 (14 year-old) and Y11 (15 year-old), and Y10 in LON School as well.

3.4.2 Data collection in the OPT

Because of the political situation and the Israeli restrictions on our movement (as Palestinians), I could not conduct the data collection myself. Thus, with agreement of my supervisor, I asked a colleague/researcher to collect the data in the OPT. Every possible arrangement had been made to make sure that the same procedures were applied in schools in the OPT. I take full responsibility for the data collection in the OPT mentioned in this study.

The researcher in the OPT holds a Masters Degree in social science. She collected the data in the OPT in Cycles 2 and 3. In Cycle 2, I sent her the two problems (in Arabic) and a general letter about my research and its aim to the students whom she contacted and who agreed to participate in the study. The researcher in turn contacted the participants and arranged with them a location to conduct the data collection. Since the purpose of that data was to explore how the framework might be applied to some mathematical texts and to obtain students' mathematical texts, the collection of data took place in informal settings (students' houses and not classrooms). Five students in Grade 8 (13 year-old) participated in the study: two males working together and three females working together in different places and at different times. The researcher sent me all their solutions and the video record.

After applying the framework to the data from Cycle 2, I made slight amendments to one problem. In Cycle 3, my purpose was to apply the suggested framework to students' mathematical texts in a classroom. I contacted the school (a private one) and made the necessary arrangements with the principal of the school, explaining the aim of the study, determining in which grade to collect data and writing a letter to the parents. After that, the colleague researcher (the same one in Cycle 2) contacted the school and arranged a date to conduct the observation and the group work, while another colleague did the video record.

Here I must comment about students' texts produced in students' home and in the classroom. Although I think that that there was no difference between these two data in relation to the research aim in this study, I prefer to have data from the context of classroom, where students learn school mathematics and discuss it with each other, rather than students' houses. I collected data in students' homes to test if the tasks

were good tools for collecting data and if the translation (from English to Arabic) was appropriate.

The same procedures in collecting data in the classroom in the UK were conducted in the Palestinian school (observation, students' work in groups, the number of students in one group, the time needed, how to collect the texts, etc.). For the class observation, a video record of one of the sessions was sent to me.

All the arrangements had been done through intensive communication by all possible means such as phone, e-mail and mail.

Table 4-3: An overview of data of the study

Cycle	Date(s)	Class code	Observation record (min)		Group work record (min)		Number of students' texts*	General comments
			video	audio	video	audio		
Cycle 1	Not applicable							2 mathematical texts (English and Arabic)
Cycle 2	13,14,19,20 June 2007	LON1Y8	180		60		57	2 groups were recorded. The rest of the class worked in groups as well.
	21 March 2007	PAL1G8			60		10	Individuals and not class members. Recording took place in students' homes.
Cycle 3	27 February 2008	PAL2G8	45		30		37	
	20 Nov – 6 Dec 2007	LON2Y8	480		31		32	ReMath data.[9] same teacher in Cycle 1
		LON3Y7			19		54	2 hours observation
		LON4Y9					0	1 hour observation
	10-12 Dec 2007	KNT5Y7		100		26	55	Y7&Y9 have the same teacher
		KNT5Y8		50		21	56	
		KNT5Y9		100		29	53	
Total (recordings in minutes)			705	250	200	76	354	1231
Total (recordings in hours)			11.75	4.17	3.33	1.27		20.5

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[9] ReMath (Representing Mathematics with Digital Technologies) funded by the European Commission FP6, project no. IST4-26751. For more details see http://remath.cti.gr.

The codes in the tables are as follows

LON, KNT	Schools in the UK	Y8	12-13 years old
OPT	School/students in the Occupied Palestinian Territories	G8	13-14 years old

Comments on Table 4-3:

- The use of web-based geometrical diagrams is not mentioned in the table because it was continuous and not specified in time. I made use of the two texts mentioned in Cycle 1 throughout the study, therefore I do not specify a date.

- The teacher for LON School was the same in Cycle 2 and Cycle 3.

- KNT school did not allow the video record, and therefore all the data are audio recorded.

- The long duration of observation (480 minutes) is due to the fact that I observed that class as part of ReMath project. This data was used as background information to understand the context of learning for the group of students who participated in my study.

3.5 Two cultures and two languages: Students and schools in the study

As mentioned earlier in my discussion of generalisability as one motivation for collecting data (sections 3.1-3.3 and Table 4-2), I wanted to apply the framework to different contexts, and I chose to apply it in the UK and in the OPT. The data was collected in three different schools: two in the UK and one in the OPT. That means that my study includes two cultures and two languages, with many differences in writing system, traditions and practice. I deal with this issue in Chapter 9, but here I want to give a general description of the participants of the study and their schools.

3.5.1 Education in the UK:

Schools in the UK are either run by the state or are independent, depending on whether they are funded by the state. Their status affects the extent to which they are bound by legislation concerning curriculum. However, there are many different types

of state schools and independent schools (see www.direct.gov.uk for more details). State schools, funded by the government and following the National Curriculum, are either Mainstream (which, in turn, includes different types such as: Community schools, Foundation and Trust schools, Specialist schools, etc.) or State schools with particular characteristics (which, again, include different types of schools such as: Academies, Faith schools, Grammar schools, etc.). Independent schools, on the other hand, receive no funding from the government, and they are either public or private (though they are called private schools in general). Sometimes, the term 'private' is used to distinguish, among independent schools, between 'private' and 'public'. The name 'Public' schools is a remnant of the historical willingness of these schools to accept students from any background, if they can pass an entrance exam and pay the fees or are supported by a scholarship. Public schools are generally old and established, enjoy a high status and high academic standards, and are expensive. This is a closed and privileged group. On the other hand, anyone can set up a 'private' independent school, and the quality of these schools varies (Morgan, 2009, personal communication). See www.publicschools.co.uk for more details.

All state schools have to follow the National Curriculum which is organised in 'key stages'. There are five key stages (KS): Early Years Foundation Stage (for students under the age of five - 3&4 years old – which is the compulsory school age), KS1 (5-7 years-old or Year 1-Year 2), KS2 (7-11 years-old, Y3-Y6), KS3 (11-14 years-old, Y7-Y9), and KS4 (14-16 years-old, Y10-Y11). Students in the National Curriculum are assessed based on 'attainment targets' that they are 'expected' to reach in subjects. These targets are divided into eight levels (1 to 8). Each of these levels is described, as is 'exceptional performance' above level 8 for each subject in each key stage. 'For example, by the end of Key Stage 1, most children will have reached level 2, and by the end of Key Stage 2, most will be at level 4' (www.direct.gov.uk). Starting from Level 4, mathematics, in KS3, is designed around four categories: mathematical processes and applications, numbers and algebra, geometry and measures and handling data. For more details about the National Curriculum, see the website of Qualifications and Curriculum Authority, http://curriculum.qca.org.uk/index.aspx.

The students who participated in this study were in KS3, 11-13 years-old in secondary schools. School 1 (LON) is a state secondary school, 'a co-educational voluntary-aided comprehensive school for pupils aged 11-18 years', located in a

middle class area in London. The number of students is around 900. In this school, I collected data from three classes: two classes from Year 8 (LON1Y8 & LON2Y8) in different academic years and one class from Year 7 (LON3Y7). School 2 (KNT) is a selective grammar school outside London. In this school, I collected data from three classes: Year 7 (KNT5Y7), Year 8 (KNT5Y8) and Year 9 (KNT5Y9). This school specialises in mathematics and computing and was rated as 'Outstanding' in the Ofsted (Office for Standards in Education) inspection report in 2007. It is an average size school, teaching approximately 700 British girls, predominately White, from the top 25% of the ability range and coming from a wide geographical area. All the classes had different levels of attainment.

A typical mathematics lesson in schools is described by the UK Department for Education and Skills (Department for Education and Skills (DfES), 2001, p. 28) as follows: An oral and mental starter (about 5 to 10 minutes), the main teaching activity (25-40 minutes) and a final plenary to round off the lesson (5-15 minutes).

3.5.2   Education in the OPT:

Education for Palestinians was not designed and implemented by Palestinians until 1994, when the Palestinian National Authority was established (for more information, see Jerusalem Media & Communication Center, 2001; Palestinian Ministry of Education and Higher Education, 1996; Shakhshir Sabri, 1992). There are three types of schools in the OPT: Governmental (75%), UNRWA (United Nations, 13%) and Private (12%). The majority of students enroll in the governmental schools (70%) while around 11% of the total students enroll in the private sector (Palestinian Ministry of Education and Higher Education, 2007/2008), and the remainder study in UNRWA schools. There are two main cycles in the Palestinian educational system: Basic Cycle (which is divided into lower stage – Grades 1 to 4 – and upper stage – Grades 5 to 10 –) and Secondary Cycle (Grades 11-12, 17-18 year-old). Governmental schools provide education free of charge to all students up to Grade 12. UNRWA provides education up to Grade 9 free of charge as well, after which students have to move either to Governmental or Private schools. The Private schools charge fees, and although they have their own regulations, they

have to follow the general guidelines and teach the textbooks suggested by the Ministry of Education and Higher Education (MoEHE).

The participating students were in Grade 8: 13 year-olds in a middle-class Private school in the West Bank that charges a fee. The school provides classes for Kindergarten up to the final year of school (Grade 12) in two separate buildings: one for young children (KG-4, 188 children) and another for Grade 5 to Grade 12 (463 students). It has its own regulation for accepting new students and setting the school year calendar. This school adopted the British system for the year studied (08/09), dividing the school year into three terms. The school offers students the possibility of studying science and mathematics in English in addition studying these topics in Arabic from the National Curriculum. Beginning at Grade 9, for two years, the school offers the International General Certificate of Secondary Education (IGCSE) programme for its students, followed by A Level for another two years (Grades 11 & 12), which is equivalent to the Palestinian general certificate for secondary education (*Tawjihi*).

In the Palestinian education system, students learn together regardless of their achievement and assessment. The main mode of assessment is based on written exams that take place at the end of the school year. Because it is a private school, this school enjoyed a certain 'freedom' to assess its students based on its own criteria which include tests, quizzes, 'behaviour', participation in the classroom, etc. These criteria, however, cannot compete for significance with the 'official' assessment, the high school final exam given by the government – *Tawjihi* – which is required by universities in the OPT and abroad.

In a meeting I conducted with the head of the mathematics department in the school, he told me that the department had agreed on how to proceed in an ordinary class of 45 minutes duration: Preparation for the class (about 5-10 minutes) consisting of reminding the students of a previous lesson or prior knowledge in order to get them into the main lesson, lasting about 25-35 minutes. The lesson is concluded by general questions for assessment or feedback (5-10 minutes).

3.6 Tasks of the study

In order to have students' mathematical texts validate and refine the suggested framework, I selected two geometrical problems for the students to solve. The selection process was based on the following criteria:

1. The solution of each task should require drawing diagrams in order to solve the task. This criterion is essential since the framework needed to be applied to diagrams at the first instance.

2. Each task should address a different geometrical subject (for example, proof, measurement, construction, etc.). This criterion allows a variety of solutions and, hence, a rich opportunity for the validation and refinement of the framework.

3. The task should not be trivial, so that students would need to communicate and discuss it. The task should encourage communication and discussion among students.

I chose two problems that have been used in other studies: the Trapezoidal Field task and the Proof problem. In the following I describe each of them separately.

Task 1 - the Trapezoidal Field task (TF): Evans, Morgan & Tsatsaroni (2006) used a mathematical problem (Figure 4-4) to explore emotions in learning mathematics and power relationships among school students. The source of this problem is Santos & Matos (1998, p.111), as they mentioned. The problem says:

> Mr. Antonio's lawn is shaped like a rectangular trapezium: the bases are 16 and 24 meters long and the height (PL) is 10 meters (..). To water the lawn, Mr Antonio has two water 'sprinklers', one next to P, and one beside E. (..) How far must the sprinklers throw the water to irrigate the whole lawn? (Evans et al., 2006, p. 216)
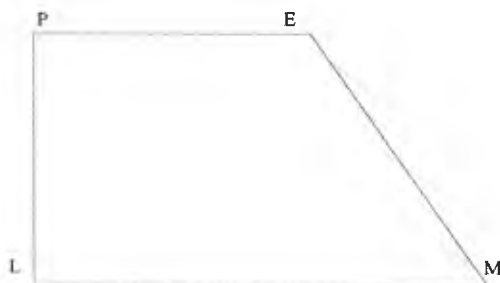

Figure 4-4: The Trapezoid Field task (TF)

Here are the reasons that selecting this task meets my criteria:

1. The description in Evans et al. (2006, p. 216) shows that this task meets criterion no. 1; students chose 'to use measurement rather than (Pythagorean) calculation.' In other words, students tried many drawings to determine the positions of the needed two points.

2. This task is different from task 2 (see below, the Proof task) in the purpose or the needed mathematical solution or activity. Solving the problem is expected to require the use of Pythagoras theorem or measurement. This meets criterion number 2.

3. The discussion described by Evans et al. (2006) and a look at the video (in Cycle 1) showed how students struggle to solve this problem. They discuss their solutions with each other and sometimes ask their teachers or even their colleagues in other groups. This meets criterion number 3.

This task (TF) is presented as a 'real-life' problem-type that asks students to find two distances or the value of the length of two segments. One possible solution is to find the radii of two circles constructed at E and P. The first one will be the radius EM. The circle with its centre at E and radius EM, will cross LM with a point, say N. The second needed radius will be PN. In order to find the length of each of these sides, Pythagoras theorem is an option. Alternative solutions might include measurement and construction using the notion of scale. One may construct an exact diagram with exact circles at E and P and then measure the needed lengths.

Task 2- the Proof problem (Pf): The second task (Figure 4-5) has been used in the Longitudinal Proof Project that aimed 'to understand further how students develop their competencies in mathematical reasoning over time, and how schools and teachers promote this development' (www.ioe.ac.uk/proof). This task was presented as a part of other problems that students need to solve on their own within a time limitation. It asks students to decide if they agree with the claim presented and to explain their decision.

This task also meets the criteria as follows:

1. The coding scheme and the results in Kuchemann & Hoyles (2006, pp. 588-589) suggest various students' responses to this task, especially drawing counter examples (Kuchemann & Hoyles, 2006, p. 590). This meets criterion number 1.

2. This task was different from task 1 (measurements, using Pythagoras theorem) because it asks for proof or investigation of a proffered claim. Therefore, it meets criterion number 2.

3. Kuchemann, the Project Research Officer in the Proof project, in a personal communication (23 February 2007) agrees that this problem does create communication and discussion between students. He himself used it with groups of students who interacted with each other trying to solve this problem. Moreover, the results of the Proof project show that this task was not trivial for students: almost half (46%) of Year 8 students answered this problem incorrectly or correctly but without explanation. Thus this task meets criterion number 3.
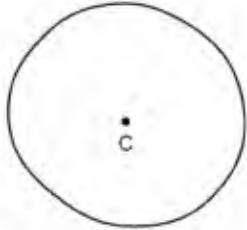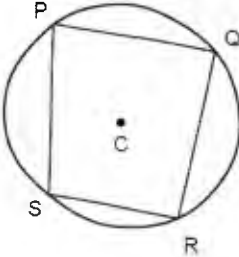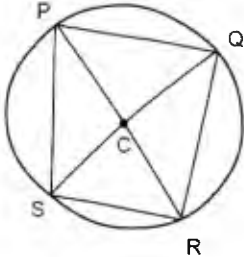
G1 Darren sketches a circle. He calls the centre C.

He then draws a quadrilateral PQRS, whose corners lie on the circle.

He then draws the diagonals of the quadrilateral.

Darren says

"Whatever quadrilateral I draw with corners on a circle. the diagonals will always cross at the centre of the circle".

Is Darren right?

Explain your answer.

Figure 4-5: The Proof task (Pf)

In comparison to task 1, this task is presented as a 'pure' mathematical one that asks students to decide if they agree with the claim presented or not and to explain their

answer. Again, there are different approaches to solving this problem. One possible idea is to try drawing different diagrams and to see if the claim works or not (trial and error or investigation process). Another approach is to find only one diagram where the claim does not work (proof by counter example). Actually, the latter solution to some extent also requires an investigation process in order to find the counter example. A third possible approach is to prove that the only quadrilateral that works in Darren's claim is a rectangle.

4. The development of the diagrammatic framework: A story inside a story

In this section, I present a general account of the story of the development of the diagrammatic framework. The first subsection (4.1) presents a brief story of the development of Framework 0 and Framework 1 followed by a detailed story about Framework 1 (section 4.2). Section 4.3 describes the movement from Framework 1 to Framework 2 and, finally, section 4.4 provides a brief story of the movement to Framework 3, the ultimate aim of the current study. At the end of each section, I reflect on how my thinking evolved and changed. In doing so, I try to summarize my thinking at each version of the framework while at the same time highlighting the main features which were the focus of the next developed version of the framework.

4.1   The beginning: Framework 0 and Framework 1

I started to develop the framework by reading and making some conjectures about the structure of the expected framework through the interaction with the literature. This is the Iteration 0 in Figure 4-2, or what Pratt (1998) called Bootstrapping iteration in order to initiate the process. Then I applied that framework to two mathematical examples in English and Arabic (Figure 4-6 and Figure 4-7), the '2 examples-C1' in Figure 4-2. Table 4-4 shows the application of Framework 0 to these mathematical texts. This interaction is Iteration 1, Exloratory, in Figure 4-2. The following two paragraphs are what I wrote at this beginning of the journey (lightly edited).

> *The diagram [in Figure 4-6] shows an equation of the circle when the origin is the centre of a circle, using Pythagoras theorem. Presenting the diagram in the centre of the page suggests the importance of this diagram*

and the demand of interaction. At the same time, students need to look at this diagram while they read the written text in order to follow what is being written and drawn.

[In Figure 4-7,] two types of visual representations are offered: a rhombus (two with different features) and a Venn diagram. The two rhombuses are presented to show the characteristics of the rhombus: the upper one accompanies the definition and the lower one is to be used for the proof of a theorem: 'the diagonals of a rhombus are perpendicular and they bisect each other'. Venn diagram is used here to indicate the relationships among geometrical shapes.
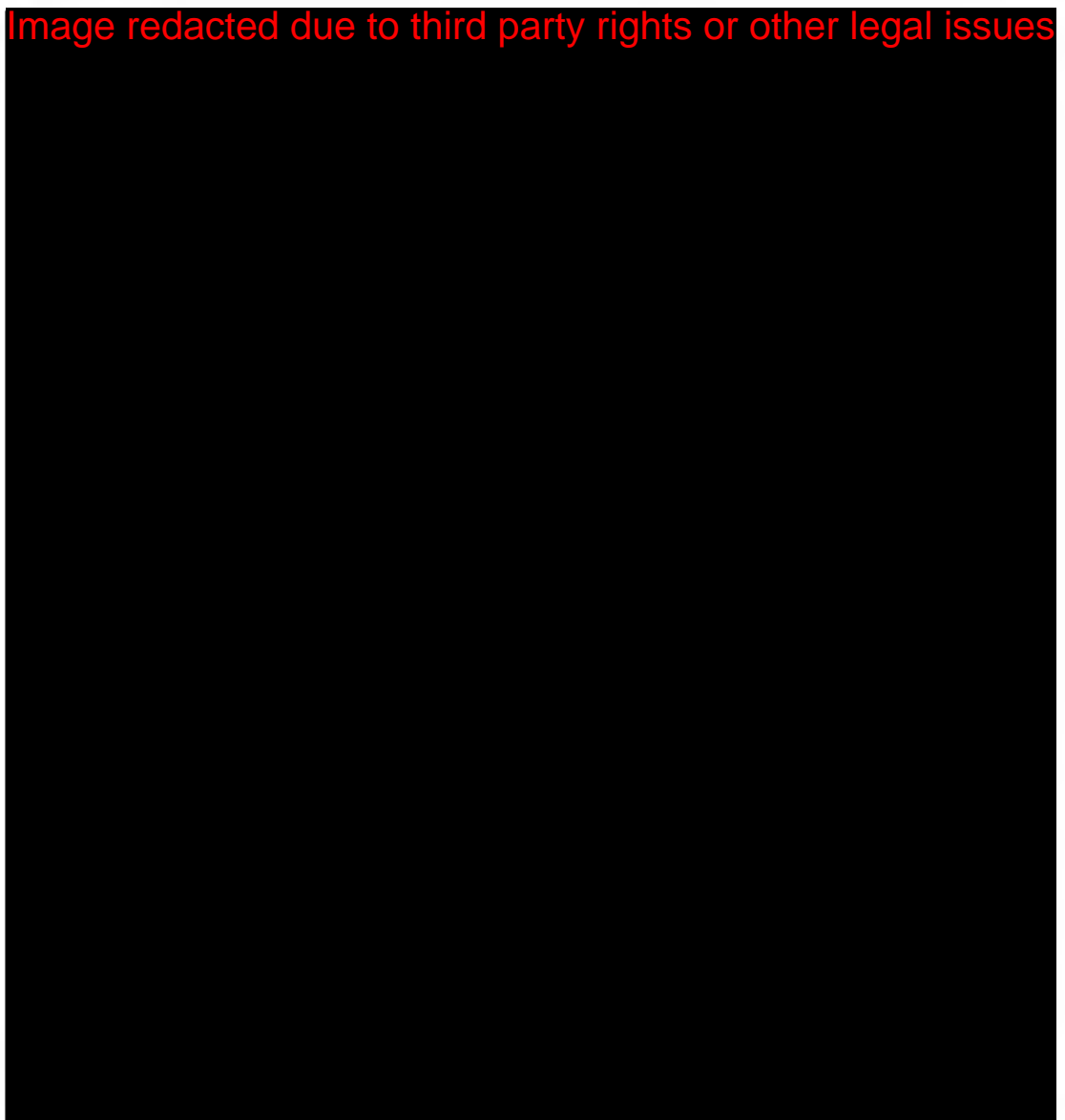

Figure 4-6: A mathematical text in English
(http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-circles-2009-1.pdf, p. 3)

Table 4-4: Preliminary framework: Framework 0 (an overview)
Applying the frameworks of Morgan (2006) and Kress & Van Leeuwen (2006) to specific examples

|	**Representational/ideational**	**Interactive/interpersonal**	**Compositional/textual**
	Nature of mathematics (image of mathematical activity)	The roles of the participants (relationships to each other and to the subject matter)	The role of diagrams/ shapes within the context of situation
Figure 4-6	• Generalisation: symbols -and not numbers- suggest generalisation and not a specific example. • Subject matter: the origin is the centre of a circle suggests the equation of circle. • Proof/human agent: the dotted segment (PN) suggests 'action' done by people towards proof. • The right angle sign (the angle PNO) raises using Pythagoras theorem.	• Participants as specialists: using general (not specific) diagrams suggests speciality among the participants. • Authority/membership/sol idarity: although the written texts use 'we', the diagram suggests the authority between the producer and the reader in presenting the details and highlights the perpendicular line. Reader/viewer's roles: students are expected to read and accept the generated ('proved') equation (passive role).	• Relationship to the page: The diagram is presented in the centre of the page, suggesting importance. • Relationship to the written text: the written text and the diagram together provide a proof.
Figure 4-7	• The object: the upper rhombus by itself is presented without symbols or numbers which emphasises the general image about mathematics (timeless and non-human (Morgan, 2001)). • Generalisation: The signs on sides and angles suggest general characteristics of rhombus (equal sides and opposite angles are equal) (not a specific example as the lower rhombus). • Subject matter: The absence of parallel signs on sides suggests that rhombuses are mentioned here as special cases of parallelograms (such as rectangles and squares) (see Venn diagram).	• Participants as specialists: the absence of symbols or numbers in the upper rhombus suggests speciality among the participants. • Authority/membership/sol idarity: the use of colour to show the characters tics of rhombuses (and the presenting example in the lower rhombus) suggest relationship of authority between the producer of the text and the reader. • Reader's/viewer's roles: students are expected to read and solve the problems based on what is presented for them (passive role).	• Relationship to the page: The diagrams are presented in the left-hand side of the page (Arabic) that suggests presenting new information. • Relationship to the written text: there is no reference to the upper rhombus in the text when presenting a definition, while the other two diagrams/shapes are mentioned in the text.
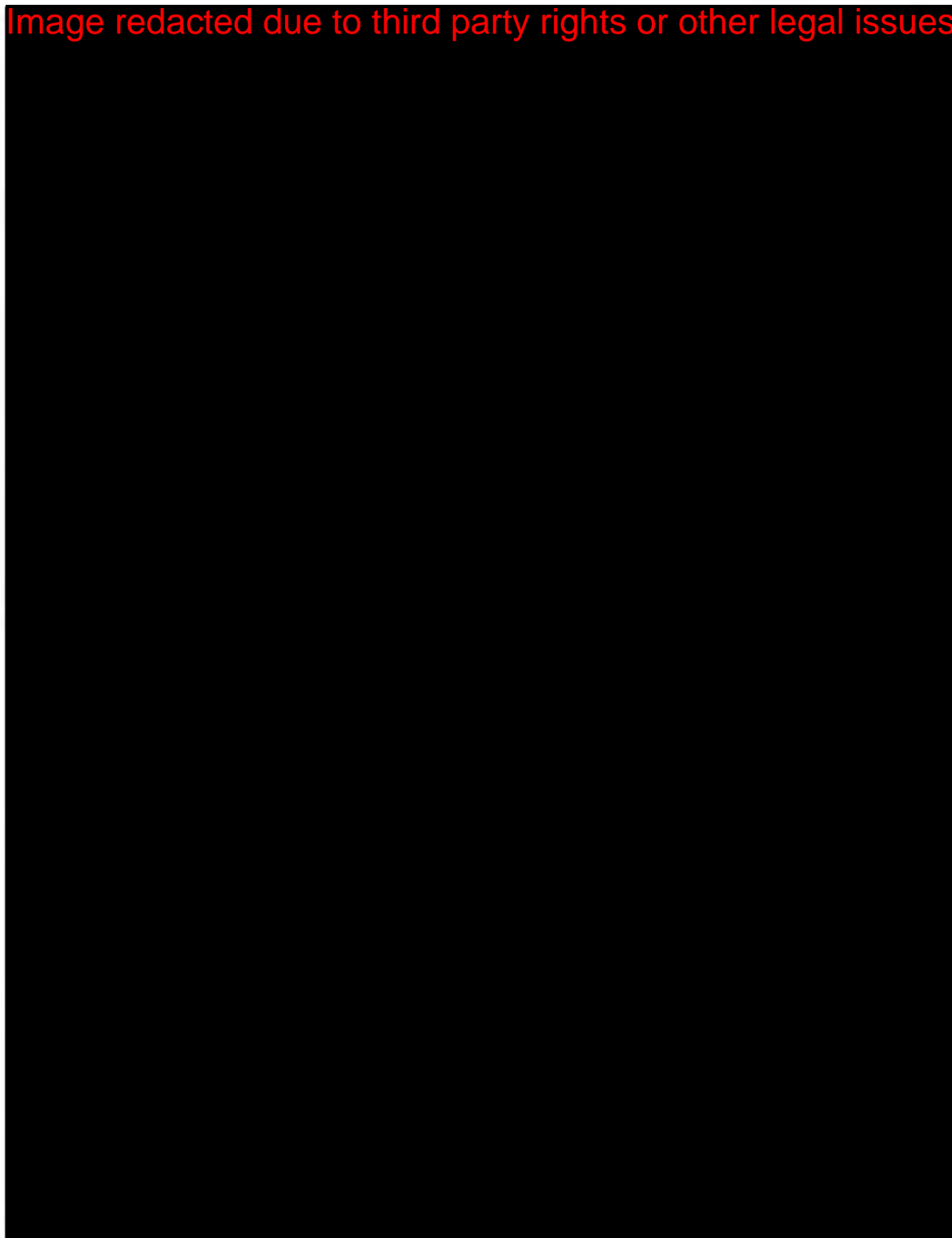
Figure 4-7: A mathematical text in Arabic
(http://www.pcdc.edu.ps/textbooks/math_G8_p2.pdf, p. 43)

<u>A reflection:</u> The first thing I can say is that social semiotics as a theoretical approach was not sufficiently clear for me, and, consequently, there was no clear distinction between the three functions (ideational, interpersonal and textual). Reading Table 4-4 and looking back now reveals an immature understanding of the role of each meaning in meaning-making process. Ideational meaning, for instance, includes different aspects, and even some (later identified) interpersonal aspects such as the right angle sign. My textual analysis focused on the position of diagrams in a mathematical text rather than the whole text and the internal relationships between its elements. Moreover, my thinking did not properly distinguish diagrams from the verbal text. In conclusion, I was looking at the surface of a mathematical text unequipped theoretically. I was at the beginning of the journey – on the periphery of the discourse. That meant more readings.

4.2   Framework 1

As a response to the feedback from the application process of Framework 0 to the two mathematical texts, I focused more on the literature of social semiotics especially on 'Reading Images' (Kress & Van Leeuwen, 2006) and 'Writing Mathematically' (Morgan, 1996b). Furthermore, and as a result of Cycle 1, I constructed a new version of the framework, Framework 1 (shown in Table 4-5) in which I made distinction between two types of diagrams, narrative and conceptual. Moreover, I suggested that the presence of a vector is the distinguishing feature between them. Here is what I wrote:

> *Since my focus is on the visual forms in the mathematical texts, my analysis will start by looking at these forms. The representational (ideational) meaning in diagrams is realised by determining the nature of the diagram: whether it is a narrative structure or conceptual structure. The main distinguishing feature is the presence of an action or not, that is, following Kress & Van Leeuwen (2006), **the presence of a vector**. Vectors might be a curved arrow, 'attenuated' vectors (dotted or solid line) or 'amplified' vectors. In both structures, we need to look at the types of processes and participants active in them. Based on the Hallidayan SFL, Kress & Leeuwen state that in narrative structure, the type of processes is that of 'happening',*

'doing' or 'going on', and the participants are active; they are carrying out the identified process. In mathematical discourse, these processes might be generalisation, measurement, naming, etc. In conceptual structures, no actions are being carried out; the participants are, thus, not active. There are three types of processes that represent participants 'in terms of their class, structure or meaning' (Kress & Van Leeuwen, 2006, p. 59): classificational, analytical and symbolic.

The distinction between narrative and conceptual structure of diagrams was made as a result of my interaction with the work of Kress in reading images and the work of Morgan in anaylysing the verbal mode in mathematical texts especially the notion of the represention of mathematics and mathematical activity/process. In the interaction with the data, I first applied this framework to the same previous two mathematical texts (Figure 4-6 and Figure 4-7), the '2 examples-C1' in Figure 4-2. This is Iteration 2 in Figure 4-2. The following paragraph is a sample of what I wrote at that stage of the study:

In her linguistics approach, Morgan (1996b) uses the transitivity system, based on Hallidayan's SFL, to examine the picture of the nature of mathematics and the mathematical activity presented in a mathematical text. That could be done by identifying the ongoing processes which are represented in a verbal text (or diagram). Example 1 (Figure 4-6) is an illustration of a narrative structure because of the presence of the dotted line PN which I considered as a vector. The dotted line is used, in geometry, to represent an action that has been done (or needs to be done) to solve a problem or to show how a problem is being solved. The PN line constructs a triangle with the x-axis, and this triangle is a right one (indicated by a small square at N). This right triangle suggests the use of Pythagoras theory. Thus a proof activity is going on. The participants in this activity are the points P, N, O and the right-angled triangle formed as a result of drawing the line PN. The ongoing mathematical activity presented in the diagram is showing or proving by Pythagoras theorem that $r^2 = x^2 + y^2$ which is the equation of a circle of radius r, and its centre is the origin.

Table 4-5: A preliminary suggested framework: Framework 1 (an overview)
Based on the frameworks of Morgan (2006) and Kress & Van Leeuwen (2006)

Representational/Ideational meaning Designing social actions & constructs	Interactive/Interpersonal meaning 'Designing the position of the viewer'	Compositional/Textual meaning Unity & Coherence
• **Nature/image of mathematics and mathematical activity** The picture of mathematics might be represented through the examination of types of processes and participants acting in them. This meaning (ideational) is realised by determining the nature of the diagram; whether it is a narrative structure or conceptual structure: * <u>Narrative structures</u>: (designing social actions) 'goings-on'- 'doing', 'happening', 'sensing', 'meaning' (vector: action) *Processes*: generalisations, measurements, naming, .. *participants*: active * <u>Conceptual structures</u>: (designing social constructs) *Processes*: • Classificational: classify • Analytical: part-whole • Symbolic: meaning/identity of participants *Participants*: not active	• **Roles and relationships between author/ producer and viewer** There are two kinds of participants in the (re)production of diagram: represented participants ('things' depicted) and interactive participants (real people, the producers and the viewers). Hence, there are three kinds of relations between these participants. These relations are realised by: * <u>Contact</u>: Does the diagram offer information not mentioned in the co-text? Is the diagram drawn 'differently' in a way that involves the viewer's attention? * <u>(Social) Distance</u>: personal, impersonal. (drawing the diagram neatly vs. roughly) * <u>Attitude/point of view</u>: involvement vs. detachment, relationships (power, equality). (specialty, certainty and authority) * <u>Modality</u> (design the reality/truth) (naturalistic vs. scientific modality). 'shared truths', 'imaginary we'– mathematical community.	• **Unity & Coherence** The way that elements are presented/ placed in a text contributes to its meaning. This textual meaning relates the ideational and interpersonal meanings together into a 'meaningful whole' or a message by: * <u>Information value</u>: 'placement of the elements': left-right, top-bottom, centre-margin. * <u>Salience</u>: 'eye-catching' or 'attract the viewer's attention': colour, size, perspective (foreground, background, overlap, appearance of human figure) * <u>Framing</u>: separation such as frame lines, white space, colour, etc. What message(s) does the whole/integrated mathematical text present? Examples: 'instructions for a calculation, argument, new mathematical concept or procedure, proof or a solution to a problem, story', etc.

However, I did not elaborate more on what meaning a vector conveys or how its presence may help me to 'read' the mathematical process that is going on. Instead, I considered the presence of the dotted line as a demand of the viewer to draw it in order to prove the equation. I recall, here, what I wrote about the presence of the dotted line:

> *The dotted line (PN) needs to be drawn in order to prove the equation. This suggests that a human agent exists, and, consequently, the image of mathematics is as a human practice rather than impersonal. The labelling process also emphasises this image; different kinds of labels are presented: measurements (r, x, y), names (O, N), variable (P(x, y)), or property (the right angle symbol at N).*

Looking now at this analysis reveals that there is confusion about the relationship between the engagement with the diagram and the presence of the human agent. It seemed to me that at that time, I thought that if there is a need to draw a diagram, and that a student or a mathematician is asked to draw it, then this means human agency. Later, I realised that this is not the case. Human agency must be presented within the diagram in order for us to say that there is human agency. This presentation needs to be explicitly represented, such as by a hand drawing a line or measuring an angle using a protractor, etc. In linguistic texts, the human agency is exhibited by using pronouns such as 'we', as in Figure 4-6.

Furthermore, at that stage of analysis I looked at the use of symbols on the diagram instead of specific numbers to express the length of sides or points. I considered the use of symbols (such as x, y, r) instead of specific numbers to name the points and the sides to suggest a general principle, such as representing the equation of a circle where the centre is the origin, as opposed to a specific example.

> *In example 2, [Figure 4-7], there are three shapes; two rhombuses and a Venn diagram. The upper rhombus and Venn diagram are conceptual structures (not narrative) since no vector or directional component is presented in them. They are, respectively, symbolic and classificatory structures. 'Symbolic processes are about what a participant means or is' (Kress & Van Leeuwen, 2006, p. 105) and they are either attributive (showing 'a property or attribute of an object') or identifying/suggestive*

(showing 'an identity between two objects') (Morgan, 1996b, p. 81). The upper rhombus is an example of identifying symbolic structure. The diagram has only labels, and it might be replaced by a statement such as: this diagram is a rhombus. This statement, following Hallidayan SFL, is an identifying relational process. Actually the written text beside the diagram is a definition of the rhombus (I translate): 'it is a parallelogram in which two adjacent sides are equal (and this means that all sides of the rhombus are equal)'. That written text includes identifying and attributive relational processes. The Venn diagram is a classificatory structure presenting relationships between rhombuses, parallelograms and quadrilaterals.

As for the lower rhombus, here what I wrote at an earlier stage:

The lower rhombus, on the other hand, is a narrative structure with dotted lines (which represent its diameters) that need to be formed in order to solve the problem. In this case, human agency is clearly needed; therefore, the mathematical activity is portrayed as human-made.

Again, looking at this analysis now, I can observe the same approach mentioned in example 1. There is no elaboration regarding the existence of the dotted line and what type of (mathematical) process it represents. Moreover, there is confusion between the ideational and the interpersonal meanings regarding the issue of the demanded action of drawing the line from the viewer.

Later, I applied this version of the framework to students' mathematical texts from the first cycle data collection, the 'students' texts-C2' in Figure 4-2, and here is an extract of what I wrote:

Following Morgan (2006, p. 231), the first step in analysing the representation of the nature of mathematics 'is to look at the objects represented in the text and the processes they are involved in and to identify who are the actors in those processes.' The main mathematical activity and processes used in students' communication are measurements, where students use compasses and rulers to draw diagrams and to write 'scale=1cm:2m'. This type of processes suggests that mathematics is constructed by doing.

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I also applied the suggested framework to some diagrams from the Internet as shown in Iteration 2 in Figure 4-2.

A reflection:

The first idea that comes to me in reading this table is that the use of the conventional terms of social semiotics started to emerge, though in a very naïve way. Now I identify unsettled elements which needed further investigation.

One of the main improvements was the use of directionality, the presence of a vector, as the distinguishing feature between narrative and conceptual structures of diagrams. Although the use of directionality was very naïve and 'straightforward', it contributed effectively to the development of the framework and to my personal development as a researcher. It helped me to 'dig in' and to work closely with mathematical diagrams and written texts.

For instance, although I had identified the rhombus in the Arabic text as a narrative structure because of the dotted lines in it, I could not answer my question to myself: so what? What is the narrative or the story behind it? In other words, there was a need for interpretation and meaning making of that story, and that meaning has to be mathematical. As I will present in the following stage of development, my notion of directionality was challenged when applied in geometry. The main challenge to that notion was the different mathematical meanings of arrows as they appear in geometry.

4.3   Framework 1 → Framework 2

As a result of the interactions with the collected data in Cycle 2, the main challenge I faced was the different meaning of arrows in the geometry context. While Kress & Van Leeuwen (2006) suggest that the presence of an arrow indicates the directionality feature, and hence the structure of a diagram will be narrative, arrows in geometry have different mathematical meanings, such as parallelism. For example, looking at the accompanying diagrams to the definitions of line or angle in geometry such as diagrams in Figure 5-2a&b raised questions, such as what action is happening here? Such questions led me to see the different uses of arrows in

geometry. I present them in Figure 5-2, as taken from the different sources of data used in the current study: Internet, mathematics textbook and mathematical students' texts collected for this study.

As a result, there was a need to look for a different distinguishing feature between the narrative and conceptual structures of diagrams. This led me to suggest the *temporal* factor as a distinguishing feature. See Chapters 5 and 6 for more details.

By temporal factor I mean that there is a representation, within the diagram, of the time sequence in drawing the diagram, or there is a timeline one can follow to 'read' or to make sense of that sequence. In other words, time has elapsed (and sometimes is still elapsing) and this 'time' is seen or observed in the diagram in different ways. In this sense I distinguish four structures of narrative diagrams: directional, dotted, shaded and construction. These structures have been modified and extended to five structures as in the following stage. Table 4-6 shows an overview of the new version of the framework.

Table 4-6: Reading geometrical shapes: Framework 2 (an overview)

Representational/Ideational meaning Designing mathematical activities & objects	Interactive/Interpersonal meaning 'Designing the position of the viewer'	Compositional/Textual meaning Unity & Coherence
• **Nature/image of mathematics and mathematical activity** This meaning (ideational) is realised by determining the nature of the diagram; whether it is a narrative structure or conceptual structure: * Narrative structures: • Directional structure • Dotted structure • Shaded structure • Construction structures * Conceptual (Symbolic) structures: • Symbolic Suggestive • Symbolic Attributive	• **Roles and relationships between author and viewer** The realisations are: * Contact: • Labelling (notations, specific quantities or measurements, variable names) • Dotted line • Shading * (Social) Distance: neat vs. rough diagrams * Attitude/point of view: involvement vs. detachment, relationships * Modality (design the reality/truth) (naturalistic vs. scientific modality).	• **Unity & Coherence** There are two levels to look at this meaning: 1) design/organisation of the text: • Information value • Salience • Framing 2) the relationship between the visual and the verbal: • Inclusion structures • Distinct structures (Linked-distinct, Implicit-link distinct)

4.4   Framework 2 ➔ Framework 3

My switch to the temporality feature, rather than directionality, had to be tested on different diagrams, including the first two examples (Figure 4-6 and Figure 4-7), students' texts, textbooks and the Internet. (This is Iteration 3 in Figure 4-2 where different sources of data were tested: 2 examples-C1, students' texts-C2, students' texts-C3 and Internet diagrams respectively.) I also identified a new category within the narrative structure, the sequence of diagrams in which I noticed the temporal factor as an indicator of narrative. One reason which led to this identification is the way in which the participant British students worked on Task 2 of the study. The common practice among them was to draw at least two diarams showing their process of investigating the claim presented in the task. Figure 6-12 and Figure 9-7 show some examples of such investigation which required rethinking of Framework 2 that could analyse each diagram but not the sequence of diagrams.

The switch, moreover, encouraged me to move forward with the framework in which I suggested other aspects of interpretation within the interpersonal and the textual. I have to mention here that the communication I made with mathematicians, mathematics educators and colleagues from different areas, through personal meetings and professional discussions, enriched my research and widened the scope of it. Table 4-7 shows the 'final' product of this development journey.

A reflection: The main achievement in this last stage is the synthesis I had made and the construction of my own terms. I refer here to the temporality feature. After applying the directionality feature as in 'Reading Images', I realised the significance of the different use of directionality in geometry, in which arrows have different and domain-specific meanings, such as parallelism. I also realised the significance of the abstract nature of geometric diagrams, as distinct from the images for which Kress and Van Leeuwen (2006) developed their framework. Actually, one of the main reasons I looked at the temporal issue is my understanding of the narrative term. I understand narrative in relation to storytelling or as a sequence of events happening in time. These stories or events are about social practices. My argument here is that time is represented in geometric diagrams, and that this temporality phenomenon is the distinguishing feature between narrative and conceptual diagrams.

Another consequent development was the need to suggest subcategories within the narrative and the conceptual, because I found different styles of practices such as dotted lines, arrows which do not signify parallelism, shading, etc.

Two issues arose which I would like to mention here. The first is my observation of the extensive use of gestures among students while they were solving the tasks of the study. This led me to consider the gestural mode of representation and communication and, later, to develop a preliminary framework to read gestures. The framework was limited to ideational meaning because of time constraints. The second issue is my struggle with the notion of modality. It took me a long time to make sense of it and to suggest modality markers/cues for diagrams (Chapter 8).

Table 4-7: A 'final' version of the suggested framework for reading diagrams

Ideational (Representational) meaning designing mathematical activities & objects	Interpersonal (Interactive) meaning designing the position of the viewer	Textual (Compositional) meaning Unity & Coherence
• **Nature/image of mathematical activity** This meaning is realised by determining the structure of the diagram; whether it is narrative or conceptual: * Narrative structures: • Arrowed • Dotted • Shaded • Sequence of diagrams • Construction * Conceptual structures: • Classificational • Identifying (indexical & symbolic) • Spatial (positional & size)	• **Roles and relationships between author and viewer** The realisations of this meaning are: * Contact: • Demand diagrams • Offer diagrams (labels & colour) * (Social) Distance: • Neatness (neat vs. rough diagrams) • Labels (general vs. specific) • Colour and arrows and words * Modality • Diagrammatic modality markers (abstractness, natural or contextual, label, additional information, neatness).	• **Unity & Coherence** This meaning takes into consideration the whole text (mainly the visual and the written). The realisations of this meaning are * Information value • Left and right (given and new) • Top and bottom (ideal and real) • Centre and margin * Salience • Colour, size, perspective, position * Framing • separation (frame lines, white space, colour) vs. connection (visual links, lack of framing)

5.   Ethical considerations

Ethical issues should be taken into consideration in all stages of any educational research (and of course in other types of research): the design, data collection, analysis and publication (Jewitt, 2006). This study follows the British Educational Research Association (BERA, 2004) guidelines for educational research, imposing three main responsibilities on any researcher: responsibility to participants, responsibility to sponsors and responsibility to the community of educational researchers. All ethical requirements and regulations of the Institute of Education/University of London have been met by this study, including an outline of the proposal and the Ethics Approval for Student Research Projects in IoE.

Special attention to ethical considerations has been paid in my research because it included video recording in schools. Therefore, I went through the process, governed by BERA and by law, to fulfil ethical and legal requirements. A consent form was sent to the teacher who contacted students' parents or guardians via the school, informing them about the study and requesting consent for participation. In schools where video recording took place, all parents gave permission for their children to participate in the study and to be observed. A few parents asked that the identity of their students remain anonymous during any screening of the video or use of the images in academic contexts (conferences, presentations, papers, etc.). There is only one parent who asked not to use any still images of the child in academic contexts.

As I mentioned earlier, data collection was different in the UK and the OPT. Although there are no 'regulations' to conduct research in the OPT, I decided to follow the same BERA guidelines there. In the two occasions of collecting data, I sent a letter to the parents or guardians of participants, asking their permission to use the data and explaining the aims of the study. In Cycle 3 of collecting data, I sent the letter to the principal of the school who told me that she informed the parents of the students and that there is no need to sign any consent letters.

6.   Iterative sampling of data: Robustness of the framework

The nature of the aim of the study (developing a framework) and the iterative methodology used in the study required a non-traditional approach to looking at the data and making use of it for a number of reasons. First, the process of collecting

data was continuous. The continuity of collecting data demanded continuity of looking at the data while validating and refining the framework rather than considering a representative sample (in a statistical sense). Second, the data was collected from different sources that offered different types of data, mainly data from two different languages. Third, the corpus of the data was open and growing throughout the study. I started the validation and the refinement with two mathematical texts in Cycle 1, followed by students' texts in Cycle 2 and, later, by extra students' texts in Cycle 3. In addition, I drew on web-based diagrams or diagrams on screen. Thus, and in order not to miss any 'different' or 'unusual' texts, I needed to develop a 'new' way to deal with this data in order to achieve the goal of the study.

I collected 354 students' texts in three different schools and in two languages and with two different tasks. Because the large number of texts makes it difficult to analyse each one in depth, I chose to sample the texts. Rather than using representative-statistical sampling methods, I selected the examples I use in this study as illustration based on a scale consists of two points: typical (or common) and unique (or uncommon). By typical/common (see the 'commonness' sampling strategy below) I mean that the majority (more than 50%) of students' mathematical texts included features similar to the feature that I highlight in the specific figure. Figure 6-14, for instance, is a typical example of how students included the (bidirectional) arrows in their diagrams. Unique or uncommon, in contrast, means that a very small number of students' mathematical texts included the feature discussed. See the 'uniqueness' sampling strategy below and Figure 4-8 which shows how a student uses an arrow to show a process of happening.

I used an iterative sampling strategy which consisted of three interrelated strategies:

a.   *Screening*: the idea behind this sampling strategy is to enable me to look at as many texts as possible and, thus, it enabled me look through all the texts in the collected data regardless of language, task, year of study, class or groups of students. In general, this strategy used 'bootstrapping' and exploration (Pratt, 1998) aiming to initiate the sampling process and to make sense of the data. This screening strategy was a foothold for the subsequent two strategies as part of the sampling process. Moreover, in the screening strategy, I applied the suggested framework in a scanning-like way, and, at the same time, I was open to seeing 'different' examples.

b.   *Commonness*: As a result of the screening strategy, I identified some general commonalities among diagrams/texts. This strategy aimed to establish general categories of data by engaging more closely with the data to identify common characteristics among texts. This strategy is akin to the developmental iteration in Pratt (1998) and in this study's application of the suggested version of the framework applied to mathematical texts. I tried to sample students' texts based on language, year of study and task, in order to identify some patterns or common characteristics. For example, in their solution of the tasks, students started their solutions by writing, and they separated diagrams from writing. I could identify types of diagrams as narrative or conceptual.

c.   *Uniqueness*: The idea here was not to miss any significant or different text (diagram) that would contribute to the development of the framework. While identifying general common properties among the categories in the previous strategy, I identified some texts which I could not locate among these general categories which raised the issue of uniqueness. For example, the vast majority of students drew only one diagram to solve the TF-task, and that diagram was either narrative or conceptual, depending on the presence of any of the identified indicators. However, one student, Carly, took a different approach to solving the task (Figure 4-8): she drew two diagrams with an arrow between them.
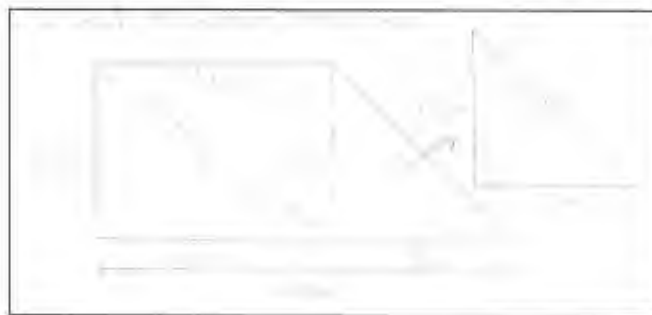
Figure 4-8: Carly's TF diagram (Year 9, unique)

This iterative sampling strategy, I argue, is more comprehensive than representative sampling in which the researcher adopts either a quantitative approach, collecting a number of data and using statistical methods in order to generalise from them, or a qualitative approach, determining 'critical' or 'interesting' events and building the conclusion from them. The contribution of iterative sampling is twofold. First, it enabled me to screen *all* data and students' mathematical texts and, consequently, to

identify common properties among them while at the same time identifying unique texts. Second, while proceeding through this iterative sampling process, I applied the suggested framework and got feedback at the same time. In other words, applying the three strategies (screening, commonness, and uniqueness) minimised the risk of 'missing' some texts that would contribute to the development of the framework.

These twofold aims contributed to the robustness of the framework, which, I claim, has been tested with sufficient examples and is appropriate for analysing any example from within Euclidean geometry. I note two central aspects of the framework which support my claim of robustness: first, the way in which I used the data to derive and test the framework; second, the fact that the framework withstands the diverse examples I have selected to illustrate it. For instance, *screening* students' texts, I noticed a *common* practice, namely that most of the students drew three examples with a circle in each to argue against Darren's claim. A different example (*uniqueness*) was drawn by a student who drew only one circle, including three examples.

The iterative sampling, thus, is a powerful strategy. The applicability strategy I used in this study was an added value to the robustness of the framework. In order to select examples to illustrate the framework, I had to go beyond the collected data (students' mathematical texts) and move to the textbooks and web-based diagrams. For example, the issue of 'sequence of diagrams' (in narrative diagrams discussed in chapter 6) was a result of the interaction between iterative sampling and application processes used in this study.

6.1   *Why sample at all? Comments on the analysis in the study*

This study is methodological and theoretical, in which I seek to develop a framework. As I have shown in various sections of this chapter, it required iteration, validation, refinement, and the sampling strategy for treating its data. I do not intend to write an analysis in a separate chapter in a 'traditional' way, but rather, following Kress & Van Leeuwen (2006), I incorporate the analysis into the chapters of the study.

The point I want to make here is that while I use a wide range of diagrams and texts, I could not include all the diagrams and texts that I screened. That is a primary

reason for using a sample of diagrams and texts. In addition to the iterative sampling strategy, the (iterative) analysis was based on group work in which students were video-/audio-recorded; the 'significant different' texts; and the framework itself. In other words, I examined all the students' texts from the viewpoint of the framework, focusing on features relevant to the three aspects (ideational, interpersonal and textual) such as: the presence of vector(s), dotted lines or shading; rough or neat diagrams; the notations in the diagram; the position of the diagram; and its relation to the text.

To sum up, in this chapter, I set up the methodology used and developed in this study as well as the context and data collected in the study. I also mentioned the sampling strategies used in the study for the purpose of analysis, which will be explored further in subsequent chapters analysing the suggested framework and its main categories, the functions of diagrams. Before introducing each of those functions, I want to tell the story, in a concise way, of how I developed the framework. That story is the subject of the next chapter.

5 From directionality to temporality: Development of the diagrammatic framework

1.   Plan of the chapter:

In this chapter I highlight the critical events and issues in developing the framework: the distinction between narrative and conceptual in general and in mathematics and mathematics education in particular, the transition from directionality to temporality (as the feature distinguishing between narrative and conceptual) and the analysis stage in which I applied the framework to students' mathematical texts. In doing so,[10] I recall some general points from relevant literature, especially from the field of multimodality (Kress & Van Leeuwen, 2006) and mathematics education (Morgan, 1996b; Sfard, 2008). The intention behind this summary is to establish the landscape for the rest of the study, especially the four coming chapters where I address each of the main aspects (functions) of the suggested diagrammatic framework separately.

2.   Narrative versus conceptual:

Mathematical activities and mathematical objects in mathematical discourse

In their interaction with their environment (and with each other), people create different means (modes) to represent their experience in the world, in order to make sense of their environments and to act on them and on others (Halliday, 1985). Acting on environments and other people involves, beside representation, communication. Moreover, the resulting objects of these actions are part and parcel of all these processes (actions, representation and communication). As people represent and communicate, they make use of the resources available in different ways based on their culture or the shared meanings among specific communities of practice. This difference is realised in people's texts, whether these texts are written, verbal, visual or gestural. The 'common' activity is that people tell stories, or narratives, about their experiences in order to organise and make sense of them (Mor

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[10] While presenting these three critical events/aspect, I do not arrange them in any particular order (chronologically or order of importance). They have been developed in interrelated activities (communication with others and engagement with the data).

& Noss, 2008). 'We live in a sea of stories,' as Bruner (1996, p. 147) states. He continues:

> ... it is very likely the case that the most natural and the earliest way in which we organize our experience and our knowledge is in terms of the narrative form. (p. 121)

As Young & Saver state, 'narrative is the inescapable frame of human experience' (quoted in Healy & Sinclair, 2007, p. 5). For Bruner, a story has two sides: a sequence of events that convey the meaning and the narrator's 'implied evaluation of the events recounted' (p. 121). In other words, a 'narrative is the coherent sequencing of events across time and space' (Boles, 1994), where the narrator tries to present the story in a coherent way to convey the meaning of the experience. This presentation or organisation of the events, whether it is written or verbal or gestural, involves the use of multiple modes of representation and communication. In this sense, 'narratives are central to meaning-making' process (Healy & Sinclair, 2007, p. 5).

However, people tell their stories in different ways. Mathematicians, for example, tell their stories differently than historians do. Mainstream mathematicians conceive of mathematics as 'abstract, formal, impersonal and symbolic' (Morgan, 2001, p. 169). Moreover, some studies reject the whole idea of mathematics as narrative. Solomon & O'Neill (1998), for example, state that 'mathematics cannot be narrative for it is structured around logical and not temporal relations' (p. 217). The studies about mathematicians' narratives about their practice in mathematics, however, revealed other accounts of how mathematicians do mathematics (Burton, 2007; Burton & Morgan, 2000; Misfeldt, 2007; Sfard, 1994). There is one common activity among these narratives: mathematicians' personal or private work becomes a different story when it gets to be published or public. All the 'private' processes (thinking, scribbling, visualising, imagining, drawing, gesturing and the like), which would reveal the 'human' nature of doing mathematics, should not be presented or shown to the public and, somehow, must 'disappear' (see Chapter 2). Not only the author herself/himself but also the imagined reader must be hidden. Davis & Hersh (1981, p. 36), in 'The Ideal Mathematician', describe this trend among mathematicians: 'His writing follows an unbreakable convention: to conceal any sign that the author or the intended reader is a human being.'

Mathematicians achieve this goal of obscuring the human agency *linguistically*, by nominalisation, namely transforming the verb (action/process) into a noun (object) and *visually*, by drawing abstract diagrams without any human figures (see Kress & Van Leeuwen, 2006; Morgan, 1996b). Sfard (1994; 2008) calls this the reification process – transition of process into object. It might be reasonable to say that this process is one reason for the 'stereotype', common among laypersons and learners, that mathematics is irrelevant to reality. The reification process has been studied in mathematics education from different perspectives (for example, Morgan, 1996b; O'Halloran, 2005; Sfard & Lavie, 2005) but with the common goal of answering key questions: how do students learn mathematics, and how should we teach mathematics?

Research in mathematics education reveals that mathematicians adopt two prevailing approaches to conceive of mathematics: the narrative approach, which tends to expose the personal aspects of mathematicians' work, and the conceptual approach, which presents mathematics as a set of concepts and 'hides' the personal aspects of mathematicians' work (cf. Burton, 2007; Healy & Sinclair, 2007; Schiralli & Sinclair, 2003). It is the latter that is the dominant approach among mathematicians (and possibly among most teachers and learners). However, researchers who conceive of mathematics as a social practice, and the present study adopts this stance, have a duty to reveal the social (and, consequently, the narrative) aspects of mathematicians' practice and of mathematical activities in general (see for example Burton & Morgan, 2000). Morgan (1996b; 2001) shows that spoken and written language in mathematical texts carries social meanings, and she offers an analytical framework to analyse these texts. In particular, Morgan (2001) challenges the conceptual approach which denies the personal narrative and asks whether the personal narrative could:

> ... rather than being seen as less mathematical, be seen as examples of different mathematical genres, expressing different aspects of mathematics for different purposes, for example, to instruct a student or to display the way in which the mathematician discovered the phenomenon. Why should we privilege the formal (non-redundant, timeless, non-human, context-independent) text as more mathematical? (p. 171)

She provides an example to show that there is a story even behind the very symbolic mathematical form. The product *(a+ib+jc)(x+iy+jz)* did not equal *ax-by-cz+i(ay+bx)+j(az+cx)+ij(bz+cy)* before Hamilton.

Furthermore, Sfard (2008) shows that there are always stories behind any mathematical object. She offers many examples to demonstrate her claim, including the following:

> ... the object we use to refer to as "number five" arises from sets of objects which, when counted, lead to the final number word *five*. This happens in two steps. First, the term *five fingers* is used to reify the process of counting the fingers of one's hand, the phrase *five apples* comes to replace the discursive process of counting apples up to five, etc. This assignment reifies the process of counting in that noun phrase *five apples* replaces the processual description which says, "When I count these apples, I invariably end with the word *five*." At a later point, the discursive object "number five" arises when we decide to use the common name *five* to name all the instances of "five *somethings*". (p. 171, italics in origin)

Other studies go beyond the language in mathematical texts and analyse other modes of communication such as the symbolic, visual and gestural modes. O'Halloran (2005) offers two descriptive frameworks for the symbolic and the graphical representations. *Educational Studies in Mathematics* has dedicated a special issue to gestures in mathematics (2009, vol. 70, issue 2). This current study, focusing on the diagrammatic mode, is an additional endeavour toward creating more constructions about the social aspects of mathematical activities and practice. In this study, I consider the duality of (re)presenting mathematical activities and distinguish between narrative diagrams (which 'expose' the mathematical activity) and conceptual diagrams (which present the mathematical objects).

3.   From directionality to temporality:

3.1   Directionality: the presence of vectors or arrows:

In *Reading Images*, Kress & Van Leeuwen (2006) consider the presence of arrows as the distinguishing feature between narrative representations and conceptual representations:

> The hallmark of a narrative visual 'proposition' is the presence of a vector: narrative structures always have one, conceptual structures never do.' (p. 59)

Two aspects here establish my departure point in looking at geometric diagrams: the narrative/conceptual dichotomy (which I dealt with in the previous section) and the presence of vector (directionality) as a distinguishing feature between the two representations. Thus I started the journey by focusing on the ideational (representational) function of diagrams as suggested in *Reading Images* without considering the 'unique' status of mathematics as a special (social) practice which has its own traditions and conventions. This approach worked well when the distinction between diagrams was 'straightforward' as in the following examples (Figure 5-1) where one 'easily' notices the presence of the arrow:
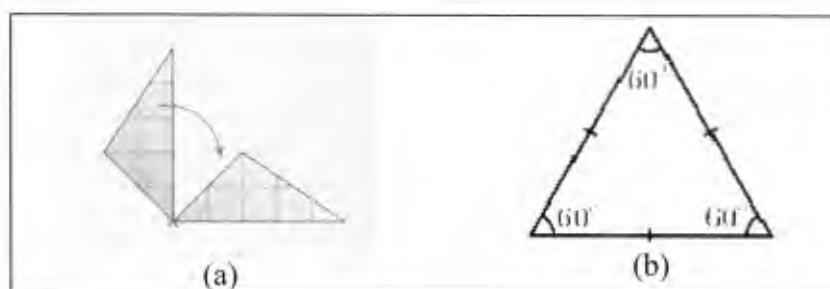

Figure 5-1: Narrative and conceptual diagrams based on the directionality characteristic

However, arrows have special meanings in mathematics, particularly in geometry, such as parallelism. Figure 5-2 provides examples of different uses of arrows in geometry.
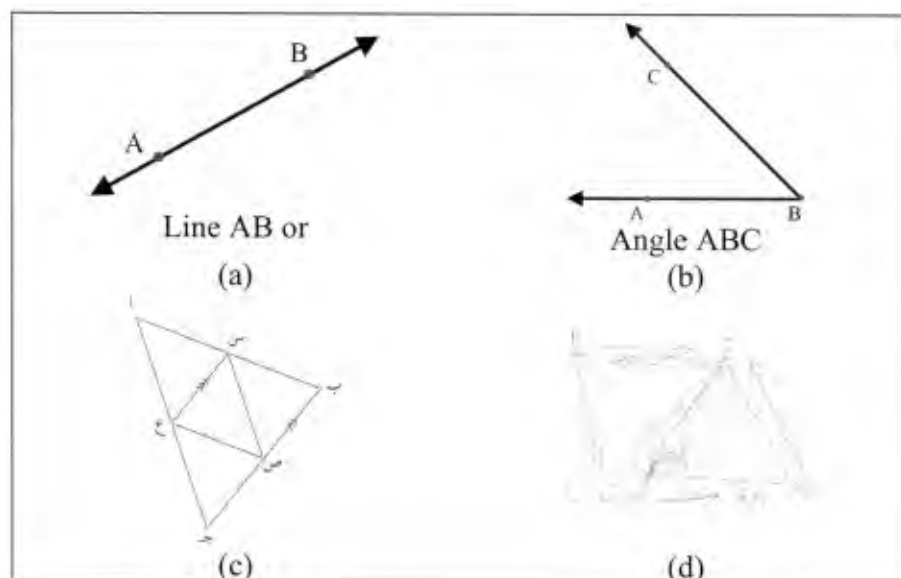

Figure 5-2: Different possible uses of arrows in geometry

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Arrows in geometry are used to define some basic geometric entities such as lines (Figure 5-2a) and angles (Figure 5-2b). There are other conventional uses of arrows such as labeling the sides of a diagram with small, similar arrows to show the parallelism property (Figure 5-2c) or using numbers to indicate specific quantities that give the measurements of the sides of the diagram (Figure 5-2d). See Chapter 8 for more details.

Although there were examples that confirmed my thoughts about the suggested criterion (the presence of arrows) as in Figure 5-2a&b, others, such as Figure 5-2c&d forced me to revise and think more. Figure 5-2d adds a different challenge, since it has arrows, but they are not used to show direction in the sense of the suggested distinguishing feature (starting and ending points) but rather they denote a measuring process. I had to revise and think more and, hence, one of the most exciting and enjoyable parts of the journey had started!

3.2   Temporality: the story behind dotted lines in mathematics

Why do mathematicians draw dotted lines? See examples in Figure 5-3. This question guided my inquiry in this journey. When I asked this question to colleagues or mathematicians, the answers included the use of dotted lines for highlighting and for expressing uncertainty ('not sure yet'), in addition to the other conventional uses in mathematics such as the symmetry line or line of reflection[11]. I still remember the use my mathematics teacher in primary school made of dotted lines in geometry lessons. He would first write the 'template' for solving problems in geometry: the given information, the problem or the definition of the goal (what we need to prove/calculate, etc) and the proof in a style very similar to Greek mathematics (see Netz, 1999). Most of the time, he would leave a space between the definition of the goal and the proof. This space, as I later learned, was left for needed actions (constructions) to do the proof. Alternatively, my mathematics teacher's question would mention that an action has been done in order to prove something (geometric relations, or finding the value of a side or an angle, etc.). Most of these constructions were represented by dotted lines and were completed later. In other words, these

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[11] Another use is to show 'unseen' parts of a three-dimensional shape. However, I don't consider 3D shapes in my study, which is limited to two-dimensional Euclidean geometry.

actions are time-dependent; they occur in a sequence of time. Thus, the notion of temporality emerged.

For example, AM and BC, in Figure 5-3d, have been extended so that they meet at the point H and constitute a triangle HMC. From this action (extension) it is expected that one needs to prove geometrical relations between the 'new' entity or any of its elements and the 'original' diagram or any of its elements. Actually this diagram is taken from a Palestinian mathematics textbook for Grade 8 (Part 2, p. 42) and it asks students to prove that BC=CH. It also may ask students to prove that AM=MH or that the two triangles are congruent, etc.

Figure 5-3: Different uses for dotted lines in 2D geometry

This story on its own did not help me at the beginning of my journey (it is reasonable, however, that there would be a delay, since I had been looking at dotted lines in diagrams for many years without recognizing their significance!). What made the difference was my communication with others during formal and informal meetings and discussions. This interaction between my experience (as a learner and as a researcher) and communication led to the notion of temporality. Diagrams either have temporal aspects, or they do not. If they have a temporal aspect, then one can 'unpack' events within the diagrams and possibly reconstruct the events in a time sequence. In other words, these diagrams tell stories! Stories include participants and activities or actions. Participants and activities in mathematics should be mathematical! If diagrams have no temporality, they present a 'thing' or an 'object' that has no action. This object, in mathematics, should, again, be mathematical.

Having established the notion of temporality in diagrams that include dotted lines, I began to ask many questions: what about diagrams which don't have dotted lines, is

it possible to identify the temporal aspect in them? Could elements other than dotted lines be identified as narrative structured-diagrams?

The reasonable next step is to revisit the diagrams with arrows (vectors). The arrow in Figure 5-1a shows starting and end positions. It shows an action in which a triangle moves around a point from left to right with a right angle (90°). This movement is represented by an arrow. In the mathematical context, this action is a mathematical activity called rotation (it would occur in geometry lessons about Transformations). Figure 5-1b has no actions since no indicator of temporality is represented, and hence it represents a geometrical object (triangle) on its own.

After that, I looked for other indicators in diagrams and activities in geometry in textbooks, the Internet and students' texts. I identified another three indicators (beside the previous two described, directional and dotted): shading, sequence of diagrams, and construction. In short, shading is an indicator for temporality since the shading process will take place only after one part of the diagram has been drawn. The temporality, in sequence of diagrams, is represented by the arrangement of diagrams in time sequence. Construction activity has a long history in geometry and in teaching and learning geometry, and the main indicators or traces that 'tell' about construction activity are the construction signs, the most common of which are small arcs drawn by the compass or the 'extra' segments resulting after connecting different points. These indicators will be discussed in more detail in Chapter 6.

Looking back now at this stage of development, I think that my identifying the presence of temporal factors in diagrams evidences the originality of this study, as is expected for a PhD thesis (see Dunleavy, 2003), as well as my critical engagement with the literature in the field of study.

4.   Mathematics as a form of communication:

So far, I have focused on the ideational function of diagrams as visual representations. But there are additional functions that diagrams, as visual representations, play, including the interpersonal and the textual, or, in the terms used by Kress & Van Leeuwen (2006), the interactive and the compositional, respectively. In order to avoid burdening this study with a long story, I now give just a 'flavour' of the challenges posed by the multiple functions and how I tackled them.

In the interpersonal function, I focused on how the social relationship between the author and the viewer is constructed in geometric diagrams. Specifically, I focused on contact (labels), social distance (the type of diagrams, neat or rough) and modality (abstract diagrams versus naturalistic diagrams). Labels, for example, are one way to make a connection with the viewer by offering information to the viewer or demanding something from her or him. Considering the textual function posed a different challenge, because it invites the written part (and possibly other parts) of the text into the analysis. Thus, one feature of the textual function is the relationship between the diagrammatic and the verbal modes in the text.

As was the case for most of the events in this study, and consistent with its methodology, I addressed the challenge of the multiple functions of diagram by considering the work of other researchers, engaging in direct communication with colleagues, and developing my own thinking. The notion of communication and representation as social activities, in addition to the notion of mathematics as a form of communication (Pimm, 1987; Rotman, 1988; Sfard, 2008), and similar studies that adopted social semiotics in mathematics, namely those of Morgan (1996b) and O'Halloran (2005), suggested possible directions.

Figure 5-4 shows a few examples of the interpersonal and the textual. I address these issues more fully in Chapters 8 & 9 respectively. This study has developed a framework to analyse the role of diagrams in the construction of mathematical meaning, which I will present in the following section.

Furthermore, I considered the gestural mode of communication where I argue that gestures, like language and diagrams, contribute to the construction of mathematical meaning. This argument is presented in Chapter 10, where I offer a preliminary framework to analyse the role that gestures play.

I have already presented in brief a final version of the suggested framework to analyse the diagrammatic mode in Table 4-7 (Chapter 4). Applying this framework more widely, to additional texts and by additional researchers, may (or may not) result in adding more features to it and, I hope, to refining it. The following four chapters are dedicated to delivering the details of this framework. I present each function separately and devote two separate chapters (6 & 7) to the ideational (representational) function.
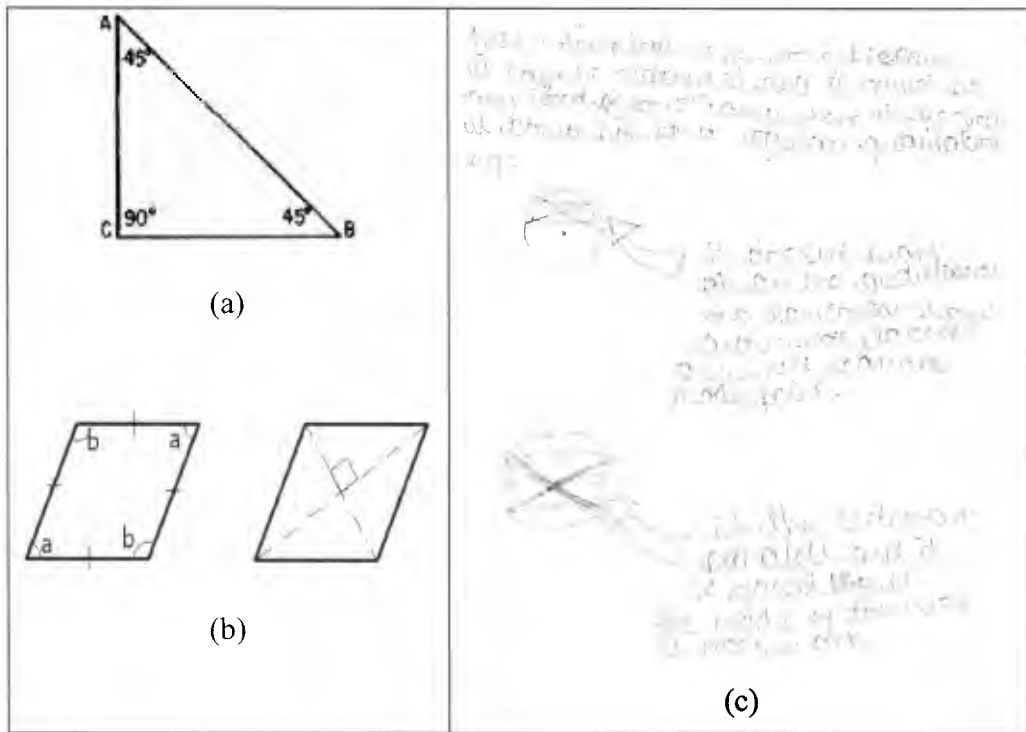
(a)

(b)

(c)

Figure 5-4: Interpersonal and textual functions

6 Narrative diagrams: Designing mathematical activity

1. Plan of the chapter:

Having identified temporality as a distinguishing feature of narratives in diagrams in the previous chapter, I try in the introduction to this chapter to characterise the difference between images and geometric diagrams, moving to the notion of narrative structure in geometric diagrams as the first aspect of the ideational function of the suggested framework. I identify five types of structures and provide examples to illustrate them: arrow, dotted, shaded, sequence of diagrams and construction.

At the end of the chapter, I discuss the issue of the human role in mathematics as presented by diagrams. I argue that this role has traditionally been eliminated in service of philosophical stances. Descartes, for instance, claimed that mathematics or mathematical facts have their own pre-existence, a claim that led people to conceptualise mathematics or mathematical objects as independent entities, as discussed in the next chapter.

2. Introduction:

The picture shown in Figure 6-1 was taken in the Occupied Palestinian Territories (OPT) in 2005 (reproduced from the British Broadcasting Corporation - BBC website). It shows two Israeli soldiers stopping Palestinian school children trying to reach their school. The written caption suggested by the BBC does not reflect the details of the picture but rather reads: 'Dozens of the schoolchildren tried to burst through the checkpoint, but soldiers warned them to stop'. First of all, the caption says 'Dozens of the schoolchildren' which suggests many children as if there is a 'threat' in some way or another, and we read that these children 'tried to burst through the checkpoint'. But the picture discloses that the school children are few, five or six students (at least that is the number shown in this picture, while there may be more out of shot). The caption says 'through the check point', but the picture does not show a checkpoint where one would expect a process of checking, but rather a barrier, a kind of blockade which prevents schoolchildren from going to their school.
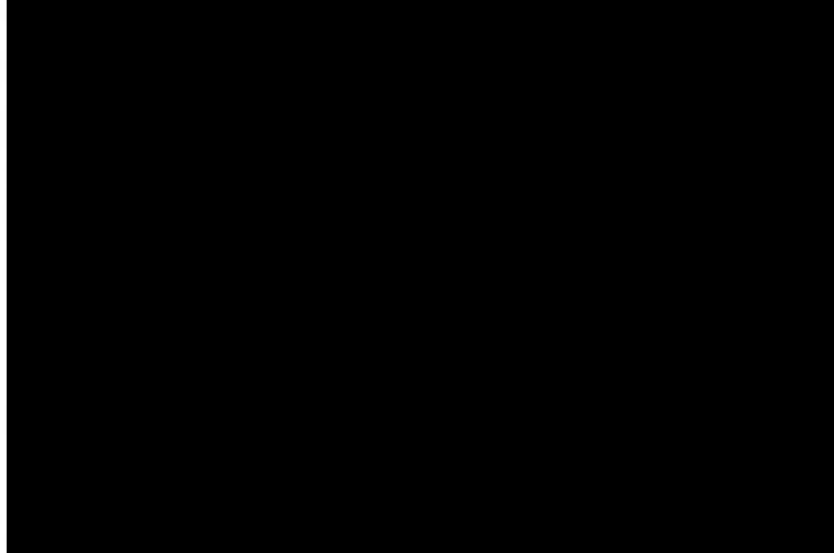
Figure 6-1: Israeli soldiers threatening Palestinian schoolchildren
'Dozens of the schoolchildren tried to burst through the checkpoint, but
soldiers warned them to stop.'
(http://news.bbc.co.uk/2/hi/in_pictures/4464332.stm)

Moreover, the caption also says '...but soldiers warned them to stop' without saying how. The picture does. The soldiers hold guns and point them at the students. The caption does not say how the school children look (angry, afraid, etc.) or how the soldiers behave. In the picture, one can see that some children are afraid, and even raise their hands as a sign of 'surrender', while others are challenging the soldiers. The soldiers, on the other hand, are not 'warning' the school children using their hands or bodies, but rather aim guns at the children, which the caption does not mention.

While it is not my purpose to discuss how the BBC adopts a stance in its written caption, I want to discuss how pictures or images can show 'actions'. In *Reading Images* (2006), Kress & Van Leeuwen discuss the issue of visual (narrative) representation as ideological, when they analyse the 'The British used guns' picture according to processes and participants. This picture, originally taken from 'Our Society and Others' by Oakley, M. et al. (1985) (as quoted in Kress & Van Leeuwen, 2006), shows the technology used by the British, as the original source called them, against the Aboriginal people in which two British soldiers with guns 'stalk' a group of Aboriginal people sitting around a fire. To demonstrate their idea, Kress & Van Leeuwen (2006) use a schematic diagram Figure 6-2 for that picture (p. 49) and state that:

the two men (the participants from which the vector emanates) have the role of *Actor*, and the Aborigines (the participants at which the vector points) have the role of *Goal* in a structure (…) as something *done by* an Actor *to* a Goal. (pp. 50, italic in origin)
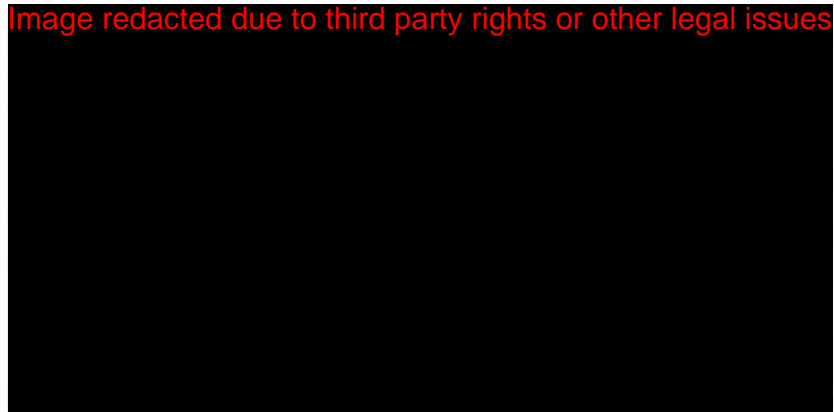
Figure 6-2: Schematic figure for 'The British used guns' picture (Kress & Van Leeuwen, 2006, p. 49)

In other words, the meaning that Kress & Van Leeuwen constructed in Figure 6-2 is that of an action by one set of participants upon other participants, referring to the vector as process and the people as participants – borrowing terms (Actor, Goal, etc.) from Halliday's functional linguistics.

In a similar way, the schematic figure for the 'Israeli soldiers threatening Palestinian schoolchildren' (Figure 6-3) shows a process with participants, Actor and Goal. The arrow originates from the side of the Israeli soldiers, suggesting that they are the Actor and, in contrast, the Palestinian schoolchildren are the Goal to which the arrow points.
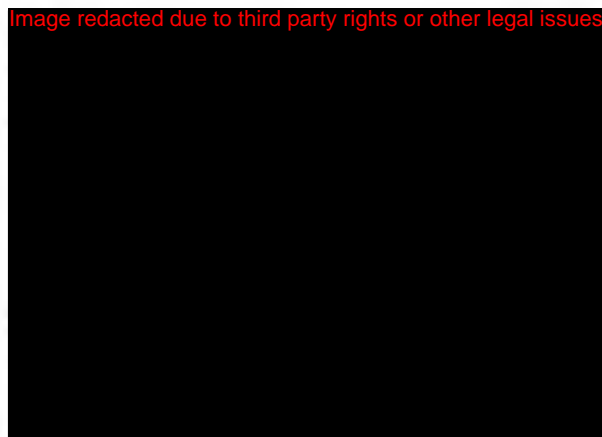
Figure 6-3: Schematic figure for the 'Israeli soldiers threatening Palestinian schoolchildren' picture

What happens if the participants are replaced, as in Figure 6-4, which is taken from a Year 9-student's mathematical text (Carly) responding to the Trapezium Field (TF) problem in this study (see Chapter 4 for more details)? The arrow here is different from the ones in Figure 6-2 and Figure 6-3, although it still suggests an action.

While Kress & Van Leeuwen (2006) focus on actions distinguished by directionality, my focus is on actions distinguished by temporality re-presented in diagrams. The arrows in Figure 6-2 and Figure 6-3, for example, depict the action as a directional spatial relationship from actor to goal. However, the arrow in Figure 6-4 indicates a 'before – after' temporal relationship between the original figure and the part of the figure that results from the action – possibly the action of 'extraction' in this case.
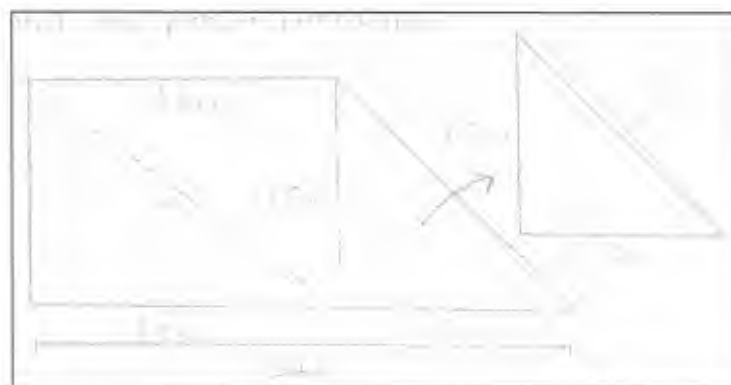
Figure 6-4: Carly's TF diagram (Year 9, unique)

As I argued in the previous chapter, there is a temporal order in any narrative (by definition), and any action takes place in time. In other words, the participants in Figure 6-2 and Figure 6-3 are not temporally arranged but rather co-exist throughout the action (or at least until actually shot), and the image 'freezes' a single instant. On the other hand, in Figure 6-4 the passage of time is actually represented in the diagram. The triangle is only brought into existence (or at least discovered) by the action. The diagram as a whole thus depicts the action in terms of a relationship between 'before' and 'after' states rather than between co-temporal participants.

3.   Narrative Diagrams:

In Chapter 4, I presented an overview of the suggested framework to analyse potential role(s) of geometric diagrams in constructing mathematical meaning. In the

rest of this chapter, I consider the ideational function of the suggested framework, focusing on the narrative structure of diagrams. In the next chapter I contrast this narrative structure of diagrams with conceptually-structured diagrams.

Kress & Van Leeuwen (2006) determine the presence of a vector to be that feature representing an action or directionality. They consider directionality to be the distinguishing feature of the narrative structure of images or pictures. These vectors could be represented by:

> depicted elements that form an oblique line, often a quite strong, diagonal line, as in 'The British used guns' [Figure 6-2], where the guns ... form such a line. The vectors may be formed by bodies or limbs or tools 'in action' ... . A road running diagonally across the picture space, for instance, is also a vector, and the car driving on it an 'Actor' in the process of 'driving'. (Kress & Van Leeuwen, 2006, p. 59)

They extend their argument to include diagrams and state, on the same page, that:

> [i]n abstract images such as diagrams, narrative processes are realized by abstract graphic elements — for instance, lines with an explicit indicator of directionality, usually an arrowhead.

In mathematical (geometrical) discourse, however, arrows have conventional meanings such as parallelism, defining geometric entities or the value of lengths of sides (see Figure 5-2). This means, as a consequence, that there is a need for a 'different' distinguishing feature of arrows, which led me to think about *temporality*. In narratives, people organise their actions or stories in spatial arrangements and/or in time sequences. Since directionality as defined by Kress and Van Leeuwen (2006) refers to the spatial arrangements, my thought went to time arrangements, i.e. a diagram representing a time sequence.

Why did Kress and Van Leeuwen choose directionality rather than temporality? Directionality has the advantage (at least in non-mathematical images) of being unambiguously present in the image, while temporality is more difficult to detect. Indeed, I have had to identify a range of indicators showing temporality. Moreover, the directionality indicator focuses on the relationship between the participants in the narrative (one acting upon the other) rather than on the temporal nature of the action. Narratives involve participants but also involve actions that take place in time. Kress and Van Leeuwen's framework focuses on the agent-object relationship between the participants, while the suggested framework in the current study focuses on the action. It is thus different but equally valid in terms of the nature of narrative.

Furthermore, I argue that temporality is more relevant to analysing mathematical diagrams.

Figure 6-5, for example, shows a diagram being moved five units to the right and three units downward. The movement of the shape is indicated by arrows, i.e. arrows have been used to represent movement or change in position. Moreover, arrows represent an order in which sub-actions were performed – to the right *before* down. In other words, this action takes place not only in space but also in time, and time is represented in a before-after relationship. This action is a mathematical one known as translation (one type of geometrical transformation).

Image redacted due to third party rights or other legal issues
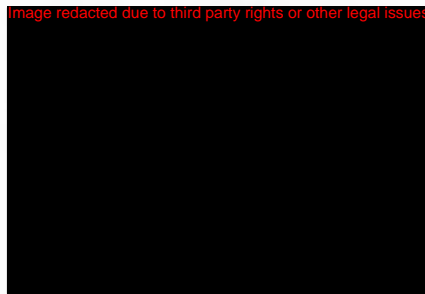
Figure 6-5: Translation process
(Allan, Williams, & Perry, 2005a, p. 130)

Figure 6-6 illustrates a different way of representing temporality. In mathematics, and geometry specifically, one conventional use of dotted lines is to indicate lines/features be added *afterwards* – after drawing the given information (see Chapter 5). In other words, dotted lines are *added* to enable someone to do a mathematical activity, e.g. prove a theorem (see for example the proof of propositions XVI & XVII in Loomis, 1861). The *afterwards* suggests that time has passed after drawing the original shape (the star in this example) and *before* drawing the dotted lines. Hence there is a representation of time in the diagram realised by the dotted lines and, as in the previous example, there is a representation of a mathematical activity that is ongoing. Actually this diagram demonstrates a proof that the vertex angles of a pentagram sum to 180°.
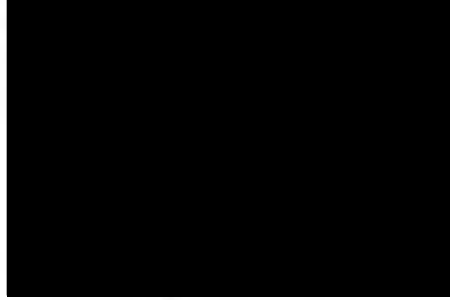
Figure 6-6: The vertex angles of a pentagram sum to 180° (Nelson, 1993, p. 14)

The last two examples prompt the notion of *temporality* as a distinguishing feature of the narrative structure of geometric diagrams. By temporality I mean that there is a representation, in the diagram, of the time sequence followed in drawing the diagram and in doing mathematics, or, in other words, there is a timeline one can follow in order to 'read', to make sense of or to unfold that sequence. This timeline is represented or can be observed in the diagram in various ways. I distinguish five structures of narrative diagrams: arrowed, dotted, shaded, sequence of diagrams and construction.

3.1. Arrowed diagrams

In Figure 6-5, arrows show what precedes what in terms of time, and, hence, an action meaning is realised (translation in this example). In geometry, arrows are either conventional non-action arrows such as parallel lines or arrows with action meaning such as transformation. The focus of the current study is on the latter, where I identify two types of structures of diagrams in which arrows carry an action meaning and show the temporal order in which this action occurs:

3.1.1. Movement/change of position: the arrows here represent an action (mathematical activity) such as transformation (translation, rotation, reflection and enlargement), folding, etc. In Figure 6-7, the arrows show a temporal sequence in each diagram where the starting and ending positions are clear.
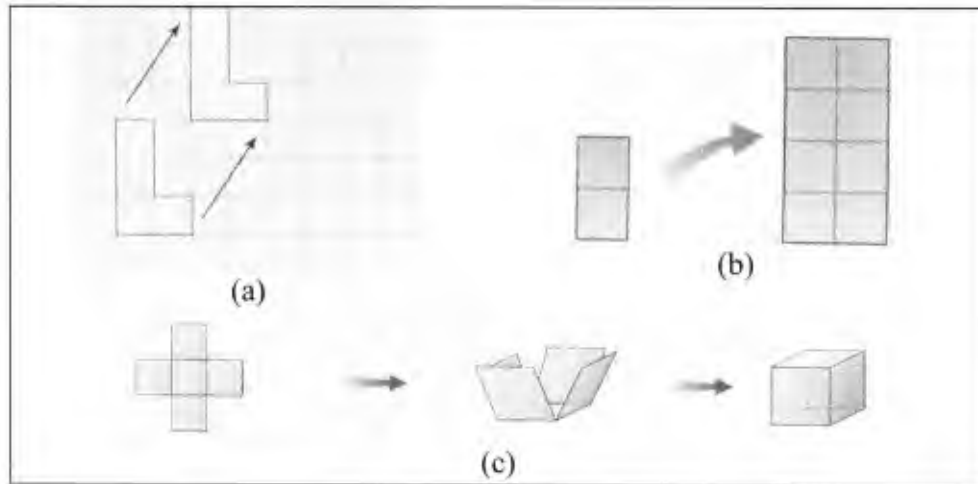
Figure 6-7: Arrowed diagrams

3.1.2 Measurement: The temporal order is represented by arrows *added* close to the sides or angles of a diagram to specify their size (Figure 6-8). This temporal order suggests that an action is happening, or a mathematical activity is going on, such as measuring the length of a side or the value of angle, which could be compared with a simple line indicating the size of a side or an angle. If one wants to describe this action in language, one possible way is to say: if you measure the side PL, you will find that its length is 10m. This is similar to measuring the length of a guitar in Figure 6-9.
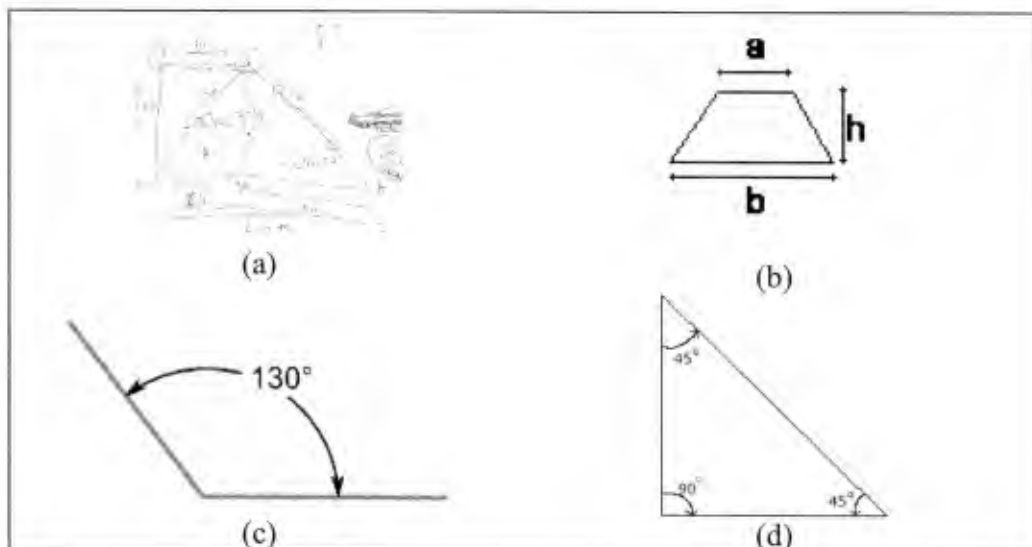
Figure 6-8: Measurement diagrams

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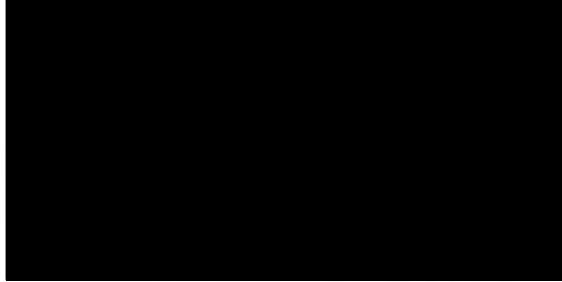
Figure 6-9: Measuring the length of a guitar
(http://www.mathsisfun.com/geometry/transformations.html)

3.2   Dotted (dashed) diagrams:

In these diagrams, the temporality is presented by dotted lines which suggest an action has been performed on the diagram to solve the problem. Again, in the geometry context, dotted lines which imply a timeline are used in different forms. The focus of the study is on diagrams that represent actions where a dotted line shows additional action such as constructing a perpendicular line from a vertex of a trapezium to its base to calculate the area or to find the value of another side (Figure 6-10a) or drawing a reflected image (Figure 6-10b).

(a) dotted line follows drawing the trapezium   (b) dotted lines show reflection process

(c)

(d) Proof of the Exterior Angle Theorem
(the exterior angle of a triangle is equal to the sum of the two interior opposite angles)
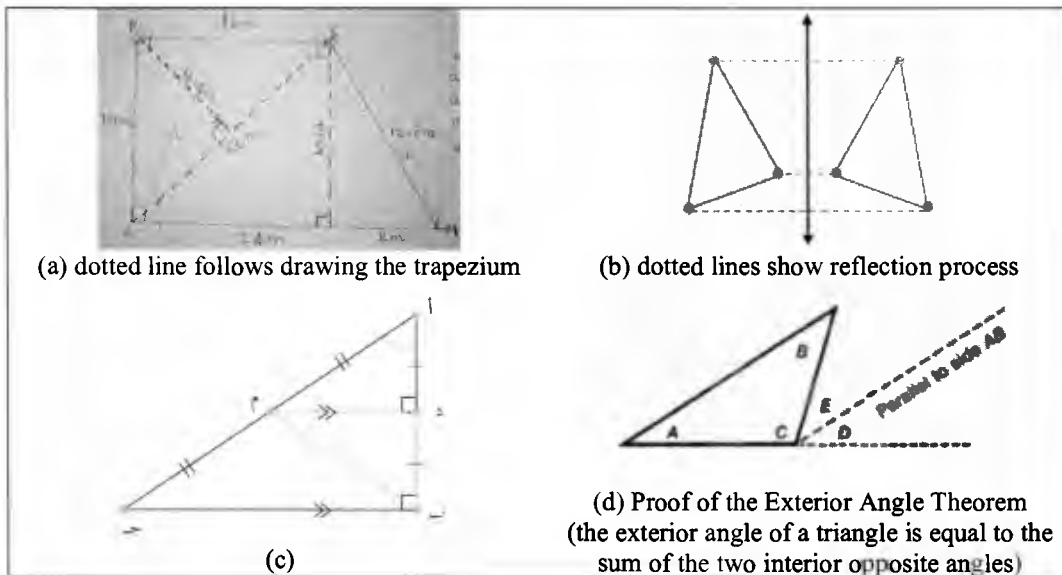
Figure 6-10: Dotted diagrams

3.3   Shaded diagrams:

In geometry some structures include shaded areas in order to: i) distinguish between original objects and images (or between the original position of an object and the new position as in the case of translation); ii) show a plane of symmetry; or iii) show

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the intersection between two planes (the last two cases are in 3D geometry which is beyond the scope of the current study). In the case of the reflection example, Figure 6-11b, the shading (colour in this example) is used to distinguish between the original triangle and its image. There is thus a sequence in drawing the diagram which suggests the temporality. A colour might be used to distinguish the object and image instead of shading.

a) colouring (shading) the image of the triangle follows drawing the original triangle

b) the colour shows translation action
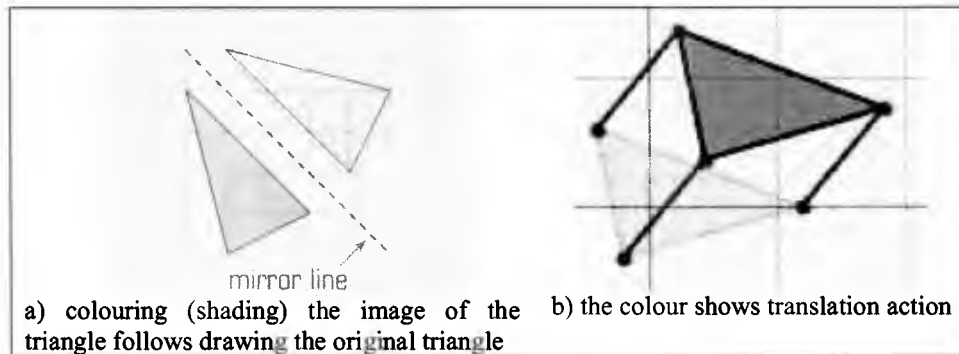
Figure 6-11: Shaded diagrams

3.4   Sequence of diagrams:

The temporality, here, is represented by the spatial arrangement (left-to-right, top-to-bottom) of diagrams in a time sequence. These diagrams are mostly used in proof in geometry as in Figure 6-12. Figure 6-12a shows a solution given to Task 2 in this study, in which the student presented three diagrams to show her proof, starting with two examples which do not agree with the claim and ending with the third diagram which agrees with the claim. The sequence of the three diagrams (from top to bottom) suggests temporality. Figure 6-12b shows a proof of the area of triangle where the sequence is realised by arrows, while Figure 6-12c presents a proof from left to right.

(a) Sequence of diagrams (from top to bottom) shows a student's response to a claim: 'Whatever quadrilateral I draw with corners on a circle, the diagonals will always cross the centre of the circle.' (Year 9, Sara, typical)

(b) 'Why the area of a triangle equals half of bh (A= ½b×h)?'
(http://www.mathsisfun.com/triangle.html)

(c) 'A square is equal to a parallelogram of equal base and height' (Maanen, 2006)

Figure 6-12: Sequence of diagrams

3.5 Construction diagrams:

In construction diagrams, temporality is realised by construction marks; the common signs are small arcs drawn by the compass or 'extra' segments resulting from connecting different points (as at the point C & O in Figure 6-13a&d). The mathematical activity here, as the name suggests, is construction. What is unique about this type of diagram is that the process or the mathematical activity is still ongoing; the diagram (or the mathematical object) is not ready yet but rather is 'under construction', as is indicated or realised by the construction 'traces' in the diagram. Figure 6-13c is a solution to the problem: Construct a triangle ABC in which BC = 3.5cm, CA + AB = 10cm and ∠B=60°.

In the following section, I elaborate on the analysis of the role of these diagrams in constructing mathematical meaning. Specifically, I want to look at these diagrams as semiotic representation of mathematical activity and at the role of humans in doing mathematics based on the suggested framework.

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(a) Constructing a 30°- 60°- 90° triangle given the hypotenuse

(b) Bisecting an angle

(c) Constructing a triangle with specific properties

(d) Constructing a circle through 3 points
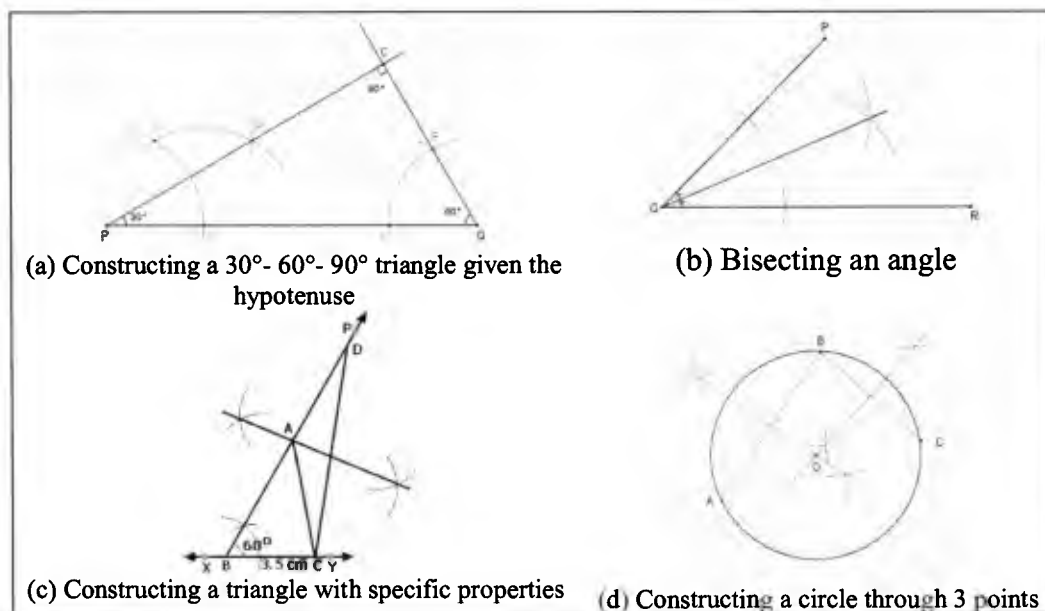
Figure 6-13: Construction diagrams

4. Narrative diagrams as representation of mathematics and mathematical activity: Three examples

In order to look at narrative geometric diagrams, one has to consider a few questions based on Halliday's social theory of communication (Halliday, 1985) and Morgan's linguistic approach to mathematical texts (Morgan, 1995). Halliday argues that in verbal interaction (written or spoken), the three interrelated meanings we try to make (ideational, interpersonal and textual) are realised through the means of representation. Morgan (1995; 1996a; 2006) suggested a linguistic approach to look at mathematical texts, using specific questions whose answers would offer potential mathematical meaning. With respect to the ideational meaning, Morgan focused on the representation of mathematics and mathematical activity on one hand, and the role of human beings in mathematics on the other hand. Or, as Burton & Morgan (2000, p. 435) put it, 'is the focus on mathematics as a product of human mathematicians or an autonomous mathematical system, or is it on the process of doing mathematics?' Thus my focus within the ideational function will be on the type of processes that are going on (happening), i.e. the mathematical activity, the participants (actors) in those processes, and human agency.

In diagrams, the structure of the diagram realises the ideational meaning in Halliday's SFL in which there is representation of *doing*, where people/participants who do

mathematics (mathematicians, teachers, students, etc.) represent their (mathematical) experiences, which I call narrative diagrams. These can be contrasted with conceptual diagrams in which the representation presents participants in 'timeless essence' (Kress & Van Leeuwen, 2006).

Narrative diagrams, I argue, represent the mathematical activity that is happening at the point of producing the diagram and/or when a reader makes sense of the diagram by reconstructing the reasoning. I present now three examples from the data collected in the current study to illustrate how the previous categories are helpful in presenting a picture of the mathematical activity represented in geometric diagrams. The first two examples present generally the typical answers students provided for the solution of the two tasks of the study (Chapter 4): Trapezium Field (TF) and Proof (Pf). The third example is presented to illustrate how the suggested framework may be used to analyse 'different' or unique answers.

The diagram in Figure 6-14 is taken from a text by Claire, a Year 9 student, responding to task TF in the current study. The bi-directional arrows next to the sides and the dotted lines are all indicators of narrative structure. These bidirectional arrows suggest mathematical actions of measuring (16 cm, 10 cm, etc.) lengths of sides (*if you measure the side PE, you will find it is 16 cm*). The dotted lines (produced from E, first, and then from P to the lower base) were drawn to achieve a conventional situation, namely to use Pythagoras theorem, to find the length of the hypotenuses (labelled x) or to calculate the area of the triangles labelled A & B.
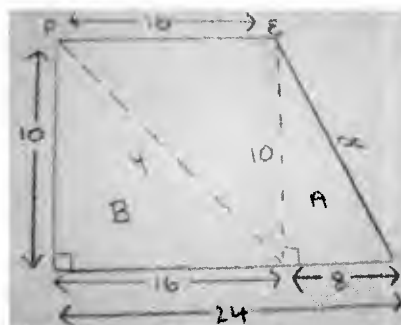
Figure 6-14: Claire's TF solution text (Year 9, typical)

In the second task (Pf), Sarah, Year 9, drew three diagrams, Figure 6-15, in sequence, to show her proof. The sequence indicates the temporal order (from top to bottom) and the mathematical activity is an investigation process to check whether

Darren's claim (whatever quadrilateral I draw with corners on a circle, the diagonals will always cross the centre of the circle) is right or wrong by giving three different examples. Each example shows whether or not the diagonals of a quadrilateral cross the centre of the circle, provided that the presumed conditions are valid. Sarah highlighted the cross point of the diagonals and the centre of the circle, so her challenge has thus become to show whether these two points meet or not in order to show her agreement (or disagreement) with Darren.
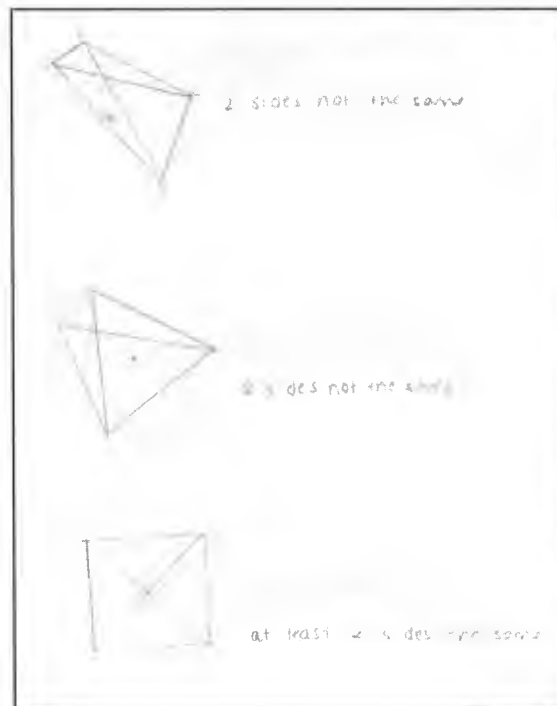
Figure 6-15: Sarah's solution to task 2 (Year 9, typical)

Instead of drawing three distinct diagrams in response to the same task (Pf), Mandy, a Year 8 student, drew one diagram showing her investigation process using colours: one circle and three coloured quadrilaterals (Figure 6-16). Actually this is similar to what Sarah did in her investigation process in the previous example. The temporal order is presented by the use of colour (though Mandy could have drawn her diagram using dotted lines or shading). As did Sarah, Mandy wanted to show whether the crossing point of the diagonals meets the centre of the circle or not. I wonder if this diagram would be considered as an investigation without the suggested framework.

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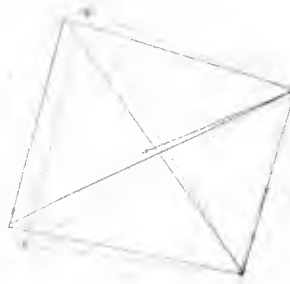
Figure 6-16: Mandy's diagram for task 2 (Year 8, unique)

5. Mathematics: a product of human mathematicians or an autonomous mathematical system? - The role of human agency

In the previous section I discussed the type of processes that are going on, i.e. the mathematical activity represented in diagrams and the participants (actors) in those processes. In this section I deal with these two categories from another point of view: how these processes are expressed in diagrams indicating the role of human agency in doing mathematics.

In English, perhaps in other European languages, and certainly in Arabic, verbs are used to express processes or actions, while nouns tend to express the participants (people, objects, abstract notions and concepts) that take part in processes (Halliday & Martin, 1993; Hodge & Kress, 1993). For instance:

*If we **add** all three angles in any triangle we get 180°.*

or

*This shape **rotates** 1/2 turn.*

The study of scientific writing in Western culture, however, reveals that other modes of expression occur in the scientific discourse (e.g. Halliday, 2004; Halliday & Martin, 1993; Martin & Veel, 1998). Influenced by the ancient Greek scientists, and starting from Isaac Newton onwards, Halliday & Martin (1993) argue that scientific writers extensively make use of nominalisation, especially in mathematics and science. Nominalisation is a grammatical metaphor in which verbs are turned into nouns, or actions are transformed into objects such as:

*The **sum** of the angles of a triangle is 180°.*

or

*The **rotation** is a turn.*

However, we don't simply change words or vocabularies (*add* to *sum*; *rotates* to *rotation*); instead, we change the grammar (this is the reason for calling this process a grammatical metaphor): the processes *add* and *rotates* have been turned into nouns, *sum* and *rotation*. Since we are accustomed to seeing the world using verbs for actions and nouns for people, this change of grammar presents a different view of the world and also creates new objects, and, consequently, we have to reconstruct our image of the world. This happened with Newton and other scientists when they tried to construct and present new knowledge as a cause or effect – 'this event caused that event' (Halliday & Martin, 1993, p. 81). For specific examples of how Newton used nominalisation, see Gerstberger (2008). In mathematics, for instance, Morgan (1996b, p. 82) derives an example of the use of nominalisation in how a GCSE student describes a number pattern ended by the creation of a new object which may 'have properties of its own and be seen to change.'

Another important effect of nominalisation is to obscure the human agency in the process. The participants *we* and *shape*, in the previous two examples, have been removed. Morgan (1996b) discussed nominalisation and its role in analysing mathematical texts in relation to the picture of the nature of mathematics and mathematical activity:

> the use of, for example, *rotation* or *permutation* without any indication that these processes are actually performed by anyone fits in with an absolutist image of mathematics as a system that exists independently of human action. (p. 82, her emphasis)

In other words, scientific and mathematical writing have concealed the role of human beings as agents in doing science and mathematics, as if this knowledge is 'objective'. What about diagrams, do they also obscure human agency?

5.1   Geometrical diagrams and the role of human beings:

As in written scientific texts where nominalisation is common, diagrams 'do something similar ... [they] turn 'process' into 'system' – or something ambiguously between' (Kress & Van Leeuwen, 2006, p. 62). Unlike natural images or pictures where human participants appear, abstract diagrams are like nominalised writing

where the human agency is concealed. But what is the history of this practice, and is it 'similar' to that of written scientific texts?

O'Halloran (2004b; 2005) presents an historical account of, among other things, the presence of human figures in mathematical texts, marking the appearance of *La Nova Scientia* (The New Science) in 1537, written by Tartaglia, as the beginning of the Renaissance and moving forward in time. In the past, human figures, physical objects and the contexts of doing mathematics were depicted in mathematical texts. However, this depiction changed as the human body and the physical context began to be eliminated (to the point where only parts of the human body were participating, e.g. hands and eyes), eventually disappearing entirely and leaving only the mathematical objects (circles and lines) remained. Figure 6-17 shows this historical development.

Image redacted due to third party rights or other legal issues

Figure 6-17: Elimination of the human figure in mathematical discourse (O'Halloran, 2005, 43-45)

Interestingly, this change occurred, again, with Descartes and Newton (see the previous discussion about nominalisation in language). In short, Descartes' philosophical project is known by the separation (or distinction) between mind and body (*Cogito, ergo sum* or 'I think, therefore I am') and by the assertion that true knowledge can only be reached by 'geometrical-style' reasoning where conclusions can be drawn based on simple and indubitable truths, like axioms in geometry (Skirry, 2008). In other words, the realm of mind (the immaterial) is where the true knowledge resides and the realm of body (the material), in contrast, is where senses operate. Descartes believed that because our senses sometimes deceive us, they cannot be a reliable source of knowledge and, hence, any sensual form of knowledge has to be rejected. And since he considered that the presence of human bodies and physical contexts belonged to that sensual realm, he removed them from mathematical texts, together with language, which, he believed, belongs to the

common-sense and material world (O'Halloran, 2005). Symbols (and diagrams), in contrast, belong to the realm of the mind. This separation and distinction mark a critical stage in the development of (all) modern Western sciences that continues until now, as O'Halloran argues.

Further Descartes' philosophical project, the 'growing significance of the role of mathematical symbolism' at that time enabled Descartes to advance his geometrical project: constructing geometry from segments of lines, circles and curves using algebraic notations or, in other words, the algebraisation of geometry (O'Halloran, 2005) (For more details about the process of algebraisation of geometry, see (Mancosu, 1996). The argument behind this project is that symbols have some advantages over language. They are more economical than words; the symbolic relationship is dynamic, whereas the linguistic is static. Also, symbols have no ambiguity or confusion (see Descartes' example comparing the word 'cube' and $2a^3$ (O'Halloran, 2005, pp. 53-54)). As a result of these advantages, symbols – as a semiotic tool – became the centre of mathematics, and diagrams became a companion and an aid to symbols.

Although O'Halloran's discussion is to some extent detailed, it does not exhaust the story of the presence and removal of human figures and physical context from mathematical diagrams. Diagrams have always been part and parcel of mathematics, and they have a history which warrants further investigation, especially from the social semiotics point of view, although some research efforts have been made from other perspectives (e.g. Maanen, 2006; Miller, 2001; Netz, 1999; Robson, 2008b; Shin, 1994).

One point that O'Halloran's discussion misses is that even though mathematicians did their best to remove the traces of human beings and/or physical context, I would argue that they still represent their mathematical activities in the diagrams they present (as I already presented in this chapter), simply because these diagrams are social and cultural activities! However, they sometimes 'succeed' in their attempts not to 'leave' any traces or representations of these activities (see the next chapter).

While it is hard to find human figures and physical contexts depicted in contemporary mathematical texts, some texts continue to use them. Figure 6-18 is taken from the Palestinian textbook for Grade 7 (12-13 year-olds) in applications

about congruent triangles, similar triangles and Pythagoras theorem in the geometry unit.

Figure 6-18: Applications (of congruent triangles, similar triangles and Pythagoras theorem) show human figures, physical objects and context in a Palestinian textbook (Grade 7, part 2)

Figure 6-19 shows other examples where human figures, physical objects and context are depicted together with geometrical diagrams.
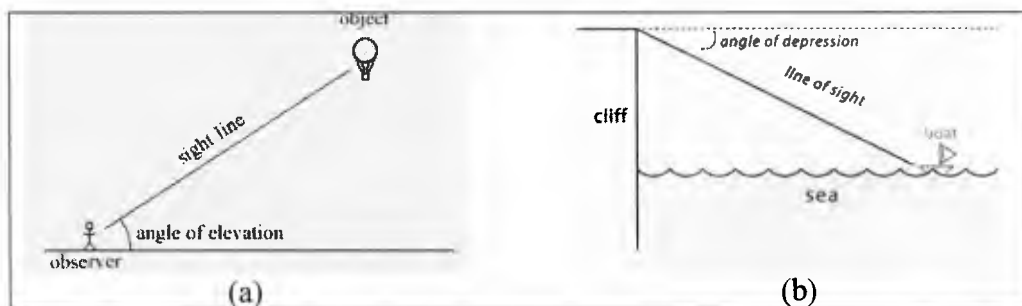
Figure 6-19: Human figure, physical objects and context represented in diagrams

However, it seems that this use is limited to problems which require the use of specific geometric concepts as congruent triangles, similar triangles and Pythagoras theorem. One reason might be the practical nature of problems which involve people or the physical contexts in trigonometry in order to measure the angles and sides of triangles in practical contexts. It is not common to draw human figures or physical contexts in mathematical texts. None of the student participants in the current study included any human figure in their texts in response to the two tasks, although the context of Task 1 (TP) relates to the 'real world' and does not include even a single

diagram! This may be because of the way that mathematics is perceived in general as impersonal and formal (Morgan, 2001).

However, in their solution of task 1, students drew diagrams which included 'traces' of physical objects (sprinklers) – shown either by a small circle at the top of two vertices of the trapezium (all diagrams in Figure 6-20) or by words (Figure 6-20d) or both (Figure 6-20d). More interesting is that some students did depict the water traces during their discussion of the task (Figure 6-20a&c) using lines or arrows[12], but these traces were either totally removed from their final diagrams (Figure 6-20b, same group) or changed to geometrical lines (Figure 6-20d, same group). Again, one possible reason for such practice is the perception (on the part of the students) that mathematical activity (and mathematics itself) should be presented as formal and abstract (Morgan, 2001).
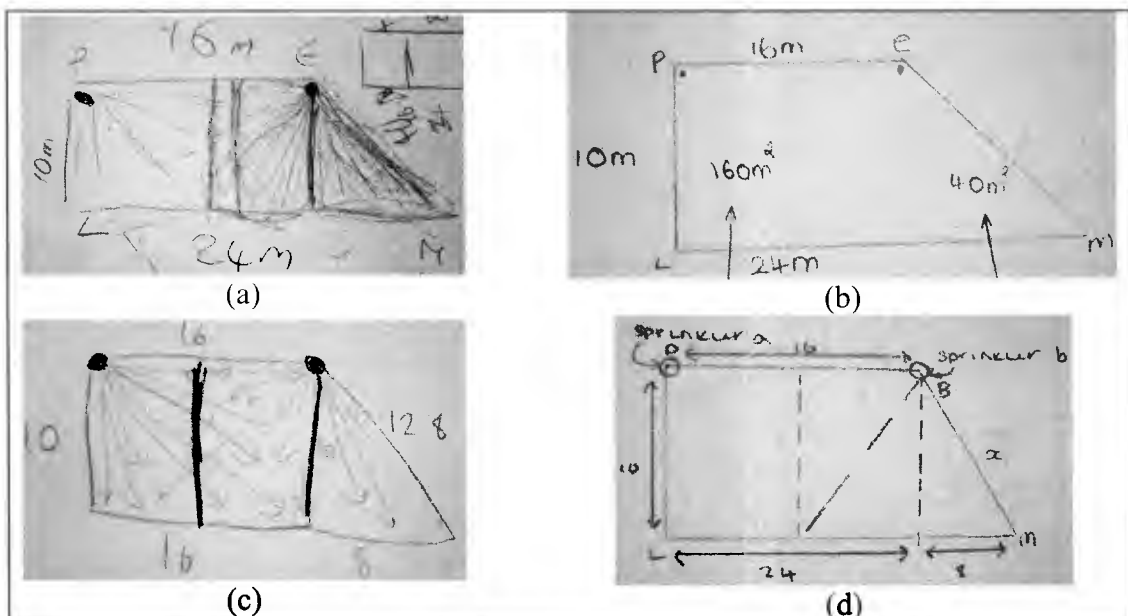
Figure 6-20: The draft and the final diagram for two students

This should not be surprising! There is consensus among researchers in mathematics education and mathematics that mathematicians 'deny' their use of diagrams (Dreyfus, 1991). For instance, Pimm (1990) quotes P. Hammer's aphorism that 'the

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[12] These arrows raise an issue about the activities happening within mathematical representation and communication: whether they are 'mathematical' or not. One possible way of looking at this issue is to consider that 'anything' that happens within a mathematical discourse or practice becomes an object of mathematics, meaning that the sprinklers are mathematical objects, and therefore they participate in mathematical activity. In that sense, for that student, the mathematical activity includes the context of the problem or, in other words, the context is part of the activity, and it is modelled by these arrows.

most neglected existence theorem in mathematics is the existence of people'. Davis & Hersh (1981) share this attitude in their construction of a portrait of 'The Ideal Mathematician' who should follow 'an unbreakable convention: to conceal any sign that the author or the intended reader is a human being' (p. 36). This practice indeed has consequences. Morgan (1996b, p. 15), for instance, argues that nominalisation in mathematical texts 'increases the impersonal effect, strengthening the impression that it is these process-objects that are the active participants in mathematics rather than the human mathematicians'. The implications of this practice for the processes of learning, teaching and doing mathematics in the classrooms and for the design of textbooks are addressed in Chapter 11.

As a result of such practice and of the success of not leaving any traces of human agency in doing mathematics, mathematical objects come to life. These objects are the focus of the following chapter, where I present conceptual diagrams in contrast to narrative diagrams.

7 Conceptual diagrams: Designing mathematical objects

1. Introduction:

The two diagrams in Figure 7-1 represent the Proof of the Exterior Angle Theorem (the exterior angle of a triangle is equal to the sum of the two interior opposite angles).[13] They look 'similar' from that point of view! However, I argue, they are different in their representation and communication.
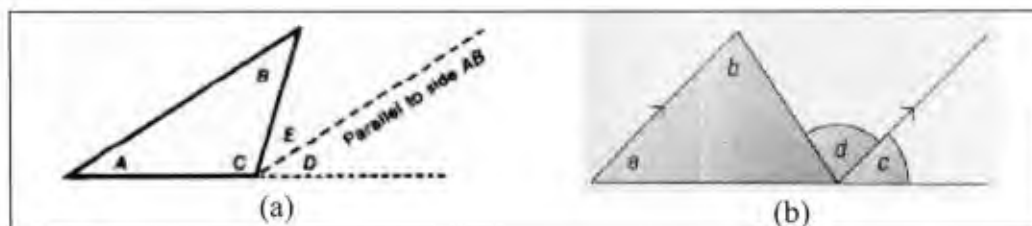
Figure 7-1: Two different diagrams represent the 'same' theorem

Beside the words 'Parallel to side AB' in Figure 7-1a, the extended lines are expressed in dots, while they are solid lines in Figure 7-1b. This difference (dots instead of solid or vice verse) is not an arbitrary or a random change. Rather it is motivated by the diagram-maker's interest at the point of representing the diagram (Kress et al., 2001). Figure 7-1a (which is the same as Figure 6-10d) represents an action, a mathematical activity that is happening (Proof of the Exterior Angle Theorem). Figure 7-1b, in contrast, shows a product of that action, an object – a mathematical one. In other words, if we want to put Figure 7-1a in words, we may say:

> *If you want to prove the Exterior Angle Theorem, you need to extend the base AC of the triangle and then from C construct a parallel side to AB* (Then move on the symbolic proof, see below.)

But if we want to say the 'same' thing about Figure 7-1b, we can say:

> *Since ∠c = ∠a .... (Corresponding angles)*
>
> *And ∠d = ∠b .... (Alternate angles)*

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[13] The diagram a in Figure 7-1 is also used to prove that the sum of the three angles of any triangle is 180°.

Then, $\angle c + \angle d = \angle a + \angle b$ .... (Adding the two equations)

Or, after naming the third angle in the triangle – say f – we can use the-notion-of-the-sum-of-the-angles of a triangle and the measure of the straight angle (see http://en.wikipedia.org/wiki/Exterior_angle_theorem):

$180° = \angle a + \angle b + \angle f = \angle c + \angle d + \angle f$

And hence, $\angle c + \angle d = \angle a + \angle b$

Or, *the measure of an exterior angle of a triangle is equal to the sum of two angles of that triangle that are not adjacent to the exterior.*

In the first linguistic instance concerning Figure 7-1a, the verbs (prove, extend, construct) refer to (mathematical) actions, while the second instance concerning Figure 7-1b does not mention any action but rather a relational process – see below. In contrast to the narrative diagram in Figure 7-1a (see the Chapter 6), the diagram in Figure 7-1b is conceptual, i.e. it represents an object – a mathematical object.

2. But how do mathematical objects come into being, and where do they come from?

Talking about mathematical objects necessitates a visit to the field of philosophy of mathematics especially concerning the 'existence' of these objects such as Platonism, Formalism, constructivism, postmodernism and semiotics. Mathematical objects were one of the topics discussed in the philosophy of mathematics along with the nature of mathematics, mathematical knowledge, history of mathematics, mathematical practice and mathematical discourse, and they are also of interest to those who care about mathematics education because of their link to learning, teaching and doing mathematics. In other words, mathematical objects and the other topics listed above lie in an area which links mathematics and mathematics education (see for example, Davis & Hersh, 1981; Ernest, 1994; Hersh, 1999; Sfard, 1998; Tymoczko, 1998).

One each extreme of this issue are Platonists and Formalists. Platonists argue that mathematical objects exist (live?) in a separate realm, and some of these Platonists even call this realm 'mathematical reality', as Hardy, for example, states:

> I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations', are simply our notes of our observation. (Hardy, 2004, pp. 123-124)

Formalists, on the other hand, argue that such objects do not exist, and that mathematics is about axioms, definitions and theorems (i.e. formulae) which consist of nothing but symbols (Davis & Hersh, 1981).

A sociolinguistic and social semiotics point of view conceives of mathematical objects as a creation of the human activity of doing mathematics and communicating in language. Halliday (2003, p. 140) expresses that:

> The 'content' of mathematics does not exist in the material world; it is created by the activity of mathematics itself and consists of ideal objects like numbers, square roots and triangles.

Most of the recent studies which dealt with mathematical objects faced this sort of question about the nature of mathematical objects and their role and relationships to each other (e.g. Confrey & Costa, 1996; Dörfler, 2002; Dubinsky, 1997; Godino & Batanero, 1998; Sfard, 1994). Although the authors of these studies adopt a range of theoretical perspectives on what this role is, they focus on the role of language in constructing mathematical objects. Conceiving of mathematics as a social practice entails that mathematical objects are products of this practice. Thus, mathematical objects are constructs of the communication among mathematical communities, where mathematicians communicate mathematics. And since communication is inevitably multimodal (Kress & Van Leeuwen, 2001), mathematical objects are (re)presented in the different modes of communication. My plan is to demonstrate how mathematical objects come to being in the language of mathematics and in diagrams and to analyse the difference between these two modes in representing mathematical objects.

Before moving on, I must say that my interest is in the representation of these objects and in the way mathematicians communicate about (with) them. Thus, my discussion is not about whether these objects exist or not but rather how mathematicians represent mathematical objects when they practice mathematics (learn, teach, do, write, etc.). In saying so, I hope that my position is already clear regarding the issue of the 'existence' of mathematical objects when I state that mathematics is a social activity.

2.1 Mathematical objects as discursive constructs: the role of metaphor

In his discussion about the relationship between mathematics and language, Halliday (1975) introduces the notion of a register and a 'mathematics register' (see Chapter 2). He describes the ways in which this mathematics register is developed using language, such as reinterpreting existing words, creating/inventing new words, borrowing from another language or creating 'locutions' (technical terms such as *right-angled triangle*). In other words, developing a register entails developing new meanings, new words, new objects and new structures. Halliday (1975) explains:

> the development of a new register of mathematics will involve the introduction of new 'things-names': ways of referring to new objects or new processes, properties, functions and relations.' (p. 65)

Analogy and metaphor are 'powerful linguistic' tools in 'creating new meanings' (Pimm, 1987, p. 93). The notion of metaphor and its role in the creation of a mathematics register is investigated by Pimm (1987). Taking the stance that 'mathematics is a language' in a metaphoric way, he distinguishes between two types of metaphors: extra-mathematical metaphors and structural metaphors. Extra-mathematical metaphors borrow words from everyday life in order to explain mathematical objects or processes, e.g. *a diagram is a picture*. Structural metaphors, in contrast, occur within mathematics itself, e.g. *spherical triangles*. These two metaphors play a crucial role in developing the mathematics register, since new words and, hence, new meanings are being added to the register. As a result of extra-mathematical metaphors, mathematical concepts become objectified, and mathematicians deal with them as objects. For instance, the metaphor '*a diagram is a picture*' may lead to talk about the diagram as a picture by describing it (e.g. large, nice) and the depicted 'things' in it (e.g. longest). Moreover, and as a result of the structural metaphors, the notion of triangle, as in *spherical triangles*, needs to be reconstructed or extended based on the 'new' analogy between the planer and the spherical context to enable mathematicians to talk about *congruent spherical triangles*, for instance.

Another major linguistic means for objectification is nominalisation. Halliday (1975; 1985) refers to nominalisation as a 'grammatical metaphor' in which a transformation of the verb (action/process) into a noun (object) occurs. I have already shown in Chapters 5 & 6 how the nominalisation process is responsible for the creation of

objects. In the context of mathematics, nominalisation creates mathematical objects (see the example about number five in Sfard, 2008, p. 171). There is a considerable number of studies showing that mathematicians *talk about* mathematical objects as if they exist independently (e.g. Charles, 2009; Hersh, 1999; Sfard, 1994, 2008; Sierpinska, 1996). The language of mathematics, or the mathematics register according to Halliday (1975; 1978), makes extensive use of nominalisation. This nominalisation feature, or reification, Sfard (1994; 2008) argues, is what makes the metaphor comes to life.

Furthermore, Sfard (1994) claims that 'mathematical objects' is not just a metaphor but a 'leading' one in mathematics[14]:

> If the meaning of abstract concepts is created through the construction of appropriate metaphors, then metaphors, or the figurative projections from the tangible world onto the universe of ideas, are the basis of understanding. ... [Moreover,] the leading type of sense-rendering metaphor in mathematics is the metaphor of an ontological object. (p. 52)

For mathematical objects to be created, the reification process needs another process, alienation, which removes any traces of any human agency, i.e. obscuring the agency, and presents the object in an impersonal way (Morgan, 1996b; Sfard, 2008). Another powerful feature of nominalisation is the ability to create new objects which may have properties of their own and become the focus of the author (Morgan, 1996b). As a result of nominalisation and metaphor as linguistic features, the mathematics register is extended, and mathematical objects are constructed.

2.2   Geometric (mathematical) objects in diagrams:

The notion of metaphor entails analogy (Kress & Van Leeuwen, 2006; Pimm, 1987; Sfard, 1997) where we use objects from everyday life and from mathematics to (re)present mathematical objects. Language has been established over time as the central mode of communication, but this will not last forever (Kress et al., 2001) because of the 'new media age' (Kress, 2003) which makes other modes, such as images, available for communication. It is true that we still describe images and diagrams or any action using language, but sometimes the order of importance is

[14] See the critique (Confrey & Costa, 1996) of the notion of 'object as a central metaphor in advanced mathematical thinking' and the responses to this critique (Dubinsky, 1997; Tall, 1997).

reversed, and a diagram may be 'worth ten thousand words' (Larkin & Simon, 1987). One main difference between language and diagrams is that language has a specific reading path that is temporal, while diagrams have spatial arrangements with no specific reading paths (Kress, 2003), although the presence of an arrow in narrative diagrams does direct the reader to one potential reading path among others. Geometric (mathematical) diagrams, like any other diagram or image, show the 'whole' scene, and the viewer/reader is free to choose what to look at and where to start from. Thus any transformation or transduction (Kress, 2003) between modes of communication is not possible without changing or losing meanings. Nevertheless, we try our best to convey the meanings we want to make. One way to do so is using analogy and metaphor when we *talk* about diagrams and objects, using language.

The situation becomes very different if we, in contrast to using language, *only look* at these objects in diagrams. We *see* objects and, therefore, we need not use the metaphor or analogy to objectify them. They are already objects. They are totally human made and human invention, as Netz (1999, p. 60) puts it when he describes the diagram as a metonym of Greek mathematics:

> The mathematical diagram did not evolve as a modification of other practical diagrams, becoming more and more theoretical until finally the abstract geometrical diagram was drawn. Mathematical diagrams may well have been the first diagrams. The diagram is not a representation of something else; it is the thing itself. It is not like a representation of a building, it is like a building, acted upon and constructed.

The above mentioned sentence -'*They are already objects*'- is a metaphor at the linguistic level, but not so at the visual level. While I agree that images convey metaphor, and we interpret them in language or other modes[15], I think that geometric diagrams are different. Geometric diagrams do not represent anything but themselves, as Netz (1999) above argues. Moreover, geometric objects themselves became a source of metaphor within mathematics itself and beyond. Pimm (1987), as presented earlier, identifies metaphors within mathematics itself, namely structural metaphors such as *spherical triangles* or *a complex number is a vector*. '*Squaring the circle*' is a metaphor within mathematics but is also used metaphorically beyond

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[15] See for example Forceville (1996, p. 109) in which a visual metaphor is discussed, such as an image of a man wearing a suit with a shoe instead of a tie: 'the metaphor can be verbalised as SHOE IS TIE'.

mathematics to denote 'hopelessness', 'meaninglessness' or 'trying to solve the unsolvable' in reference to impossibility.

As in language, the human agency in doing mathematics is also suppressed in diagrams. Mathematicians suppress human agency by using nominalisation or passive voice in language (Morgan, 1996b), on the one hand, and by eliminating human figures, physical objects and context from mathematical diagrams, on the other (O'Halloran, 2005).

Having established mathematical objects, I now explore the main issues they raise: identity, which is related to the existence issue; attributes; and relationships between objects. I now turn back to the conceptual diagrams which represent mathematical objects, illustrated by examples. I end this chapter with some thoughts about narrative and conceptual diagrams and the relationship between them.

Conceptual diagrams:

In this section I present how objects and the relationships between them are identified using the language of Halliday's Systemic Functional Grammar supplemented by the visual grammar of Kress & Van Leeuwen (2006). Applying these two approaches, together with the linguistic approach to mathematical texts (Morgan, 1996b), to geometric diagrams guided my thinking in proposing relational processes in identifying geometric objects and the relationships between them.

Rather than showing actions, geometric conceptual diagrams show the products of actions, geometric objects. These diagrams often 'show' information about the geometric objects such as identity and/or attributes and relationships between these objects (or, using geometry terms, axioms, definitions and theorems). In contrast to Figure 7-1a which depicts action, Figure 7-1b shows two parallel segments and four labelled angles that signal the Exterior Angle Theorem.

Halliday (1985) contrasts material processes, as processes of doing and actions, with relational processes, which are processes of being. Taking, for example, the phrase, 'Sarah is wise', the 'central meaning of clauses of this type is that something is' (Halliday, 1985, p. 112). Furthermore, he claims that relational processes tend to be the major processes found in mathematical and scientific writing (Halliday, 2004). Halliday distinguishes three categories of relational processes: intensive,

circumstantial and possessive, each of which has two modes, attributive and identifying. In the attributive mode, an attribute is qualified to an entity while in the identifying mode, in contrast, 'one entity is used to identify another' (Halliday, 1985, p. 113). The distinction between attributive and identifying processes is based on reversibility; identifying processes are reversible, while attributive processes are not. This is one main reason to refer to the identifying process by the equal sign '=' (cf. Morgan, 1996b).

In general, attributive processes are about the attributes of objects. In the phrase, 'a is an attribute of x', 'a' is the Attribute, and 'x' is the Carrier. In '*a rectangle is a parallelogram*', for instance (see Table 7-1), *rectangle* is the Carrier, *is* is the relational (attributive) process, and *parallelogram* is the Attribute. This relational attributive process is intensive or qualitative ('x is a'), while other types may be circumstantial ('x is at a' in terms of time or place, as in *a triangle inside a circle*) or possessive ('x has a', as in $\triangle ABC$ *has two equal sides*).

The other mode of relational processes is identifying, for example 'stating an identity between two objects' (Morgan, 1996b, p. 81). The function of this mode is to identify one object by another, as in, 'a is the identity of x', where 'a' is called identifier and 'x' is the identified. The phrase, '*Any four-sided polygon is a quadrilateral*', for instance (see Table 7-1), is an intensive relational process where *Any four-sided polygon* is the identifier, *is* is the identifying relational process, and *quadrilateral* is the identified. Table 7-1 summarises these relational processes with examples from geometry.

Table 7-1: Relational processes suggested by Halliday (1985)

type \ mode	Attributive 'a is an attribute of x' or 'x has attribute a'	Identifying 'a is the identity of x'
Intensive ('x is a')	a rectangle is a parallelogram	any four-sided polygon is a quadrilateral
Circumstantial ('x is at a')	A triangle inside a circle	The orthocentre of a triangle is the point where the three altitudes of a triangle intersect
Possessive ('x has a')	$\triangle ABC$ has two equal sides	A triangle with two equal sides is an isosceles

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While narrative diagrams are distinguished by the presence of temporality, where narrative processes are happening in time sequence, and participants are represented as actively engaging in that process as doers (see Chapters 5 & 6), conceptual diagrams are distinguished by the absence of any temporal feature; they represent participants in a 'timeless essence'. As discussed in the previous chapter, the elimination of human figures, physical objects and context in mathematical diagrams plays the 'same' role in concealing human agency as nominalisation plays in language (see for example, O'Halloran, 2005). The suppression of human agency in doing and constructing mathematics (Morgan, 1996b) eliminates the mathematical doing itself. When diagrams do not show (narrative) actions, they show mathematical objects.

Kress & Van Leeuwen (2006), adopting Halliday's social semiotics theory, contrast narrative processes, 'presenting actions and events, processes of change, transitory spatial arrangements', with conceptual processes which represent participants in 'more generalized and more or less stable and timeless essence, in terms of class, or structure or meaning' (p. 79). They consequently distinguish between three types of conceptual processes: classificational, analytical and symbolic.

In classificational processes, participants are connected to each other in a taxonomy 'kind of' relation. This can be a covert taxonomy, in which participants are represented in a compositional symmetry, or an overt taxonomy, such as tree structures (e.g. family tree) in which participants are represented in a hierarchical order. Analytical processes represent participants in terms of a part-whole, 'part of', and relation. 'Symbolic processes are about what a participant *means* or *is*' (Kress & Van Leeuwen, 2006, p. 105 italic in origin). This distinction is based on the participants (Carrier and Symbolic Attributes). Either there are the two kinds of participants, where one (the Attribute) 'defines' the identity of the other (the Carrier), or there is only one participant, the Carrier, who possesses self-contained qualities or identity.

Kress & Van Leeuwen compare their categories to those of Halliday's in language. Visual classification and analytical processes are akin to, respectively, linguistic intensive and possessive attributive processes; symbolic attributive is akin to the

identifying process; and the symbolic suggestive would be akin to Existential processes. For more details, see (Kress & Van Leeuwen, 2006, Chapter 3).

3. Relational processes in diagrams

While in language we speak about mathematical objects, give them identities, and talk about their attributes, features and relationships, in diagrams we 'see' all of these and, hence, I argue, many things change. Besides the processes of metaphor, analogy and nominalisation on which I commented above, geometric objects are seen in 'specific' positions with specific sizes and, hence, relations between them are also identified. We no longer talk about a point or a segment inside a triangle (e.g. the orthocentre or altitude), but we see its position (and its size, if given, and if not, we can compute it). In other words, relations between geometric objects are structured by the discourse itself. I refer here to the fact that relations between geometric object have to involve what the discourse of geometry is about, such as spatial relations (position and size). Axioms, definitions and theorem are also mathematical relations between geometric objects and should therefore also be considered. This is one reason that we need to search for other categories of relational processes in diagrams. I noted additional reasons in my discussion of the difference between language and visual representations.

While I borrow many of the categories of relational processes from Halliday and Kress & Van Leeuwen, in analysing geometric diagrams, I again borrow Morgan's (1996b) linguistic tool to analyse mathematical texts. The key questions to determine whether the diagram is conceptual are: what are the mathematical objects represented within the diagram and what are the characteristics/features of these objects? A further question that should be added concerns the relations between these objects.

Furthermore, there is a need for constructing relational categories for geometric diagrams which is informed by the work of Halliday and Kress & Van Leeuwen. Although both Halliday's categorisation of relational processes and Kress & Van Leeuwen's approach inform my thinking about relational processes in conceptual diagrams, they cannot offer a sufficient account of relations between geometric objects in diagrams. Halliday's SFG is about language, while the visual grammar suggested by Kress & Van Leeuwen needs to be contextualised in geometry. The

issue of identity and attributes is an illustrative example of the need for such categorisation.

Every (mathematical) object has two aspects: a carrier and a number of attributes. In language, these aspects are presented in statement about object. For example, in *an equilateral triangle has three equal sides*, *'equilateral triangle'* is the carrier and *'equal sides'* is an attribute. This distinction is possible because the grammatical components are discrete and combined linearly; one component precedes the other in a temporal sense. In diagrams, however, components are arranged spatially and are seen together, i.e. carrier and attributes are shown at the same time. In addition, the carrier is the object itself that we see (Netz, 1999), and its attributes are its properties which may be highlighted through different means, including labels, colours, words and arrows.

Furthermore, the relations between a geometric object and its attributes are so interrelated and intricate, that it becomes difficult to separate them. Sometimes the attributes of an object may be used to identify the object itself (Pimm, 1987) or the reverse. For instance, labelling one angle in a triangle as a right angle suggests the identity of that triangle as a right triangle, or identifying a diagram as an equilateral (or isosceles) triangle suggests its attribute as having three (or two) equal sides. In other words, labels *identify* components of diagram, and the identified nature of these components comprises the *attributes* of the whole diagram.

As a result of the need for a contextualised categorisation of relations in geometric diagrams, I have identified three kinds of relations in geometric diagrams. They are: classificational processes, identifying processes (which are of two types; indexical and symbolic); and spatial processes (which are of two types; positional and size processes).[16] These processes will be the focus of the rest of this chapter.

3.1. Classificational processes:

These processes relate participants to each other in an 'of the same kind' relationship, a taxonomy where one participant, or sets of participants, will play the role of Subordinate with respect to the other participant, the Superordinate (Kress & Van

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[16] The analytical processes suggested by (Kress & Van Leeuwen, 2006) could also occur in mathematical diagrams such as pie diagrams or statistical graphs, although they do not occur often within the scope of my research – (2D) geometry in school.

Leeuwen, 2006, p. 79). They are akin to intensive processes in language, such as 'x is a member of the class a'. The visual classificational processes in geometric diagrams may be realised in sets or class structures, such as in Figure 7-2, showing the relationships between different polygons. The information in the diagram may be expressed verbally as, 'squares are rhombuses', meaning that 'squares are members of the class of rhombuses', or 'trapezia are members of the class of quadrilaterals'.
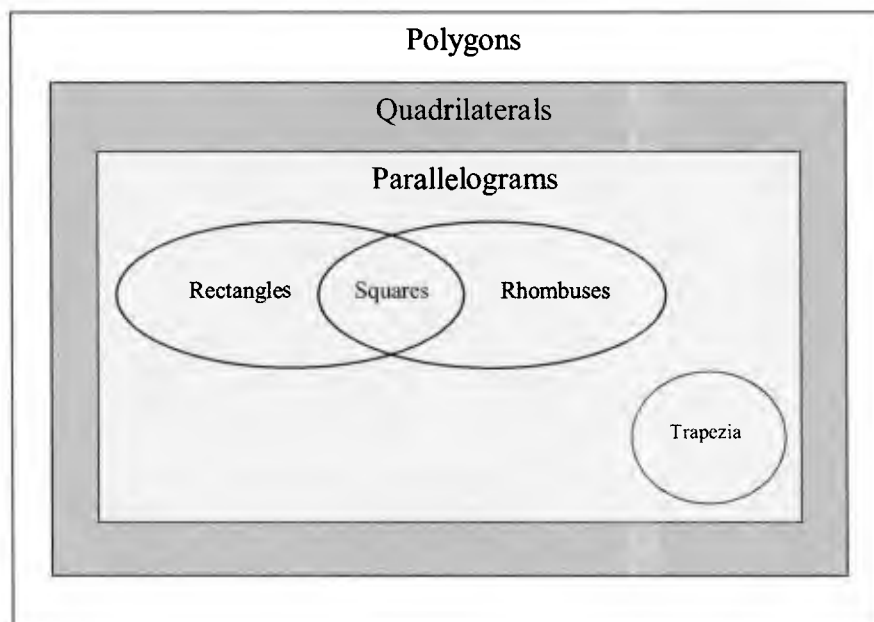
Figure 7-2: The relationships between different polygons
(Palestinian textbook, Grade 8-part 2, p. 30, translated and redrawn by the author)

Other classificational processes may be realised in vertical hierarchical tree structures in which participants at the same level, the Subordinates, represent the same kind of relation with respect to the previous vertical category, the Superordinate. Figure 7-3, for instance, shows the relationship between different quadrilaterals, where parallelograms and isosceles trapezia are in the same level representing the Superordinate Trapezium in the upper level. The arrows in this figure are of 'the same kind of' relation, and hence they designate a conceptual structure, rather than a temporal narrative structure.

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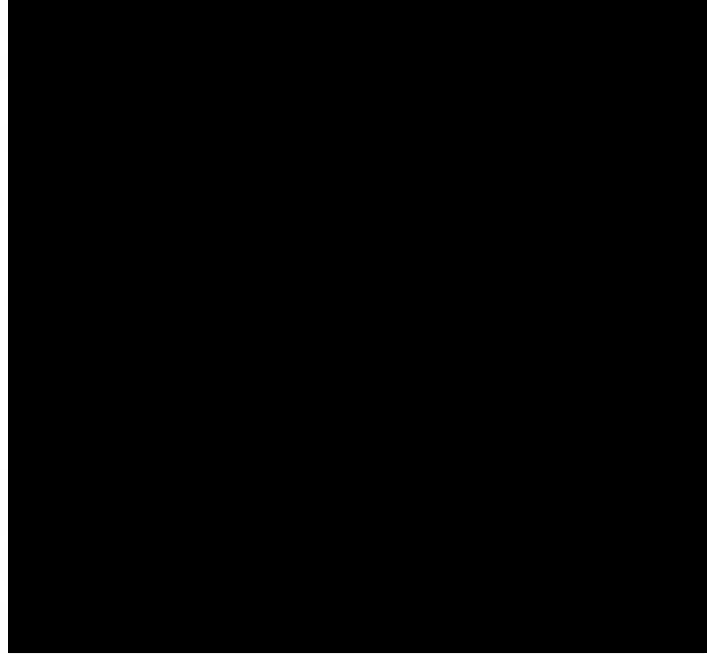
Figure 7-3: The relationships between different quadrilaterals
(http://www.mathsisfun.com/quadrilaterals.html)

3.2. Identifying processes

As mentioned earlier, conceptual diagrams are about mathematical objects and their identities and attributes. In such diagrams, identity and attributes of mathematical objects are derived from the object itself or from relations with other objects. In geometry, we identify these qualities/attributes using two different 'visual' tools: indexical and symbolic.

Before moving on, I want to clarify a few points about these visual tools. The Peircean (Peirce & Buchler, 1955) distinction between three kinds of signs, based on the relations to their object, is well-known. An icon is a sign which has its own characteristics or properties and signifies its object by virtue of similarity to (one of) these properties. In other words, the iconic sign is not influenced by the objects it represents, but rather it shares some common properties with these objects. Illustrative examples are portraits, images and diagrams. Index, on the other hand, is a sign that has a direct relationship with (and is affected by) its object. Good examples of this kind are: the index finger, the barometer and letters on geometric diagrams. Finally, symbols are related to their objects by conventions. 'All words, sentences, books, and other conventional signs are Symbols' (Peirce & Buchler, 1955, p. 112).

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Iconic signs in relational processes are not relevant to this study, because I look at relations between parts of the diagram and not at the diagram as a sign. In other words, I do not look at the diagram as an icon and then use that icon in another relationship. A diagram, in the sense it is used in this study, is an icon of geometry in the same sense that algebraic notations/expressions are icons of algebra. Thus, I exclude iconic signs, leaving two kinds of identifying processes: indexical and symbolic.

Indexical processes:

Indexical processes refer to the use of indices in identifying a geometric object or (one of) its attributes. Peirce identifies three distinguishing features of indices:

> first, that they have no significant resemblance to their objects; second, that they refer to individuals, single units, single collection of units, or single continua; third, that they direct the attention to their objects by blind compulsion. (Peirce & Buchler, 1955, p. 108)

In geometric diagrams, the use of letters and arrows represents indexical processes, since each of them meets Peirce's distinguishing features (see below). Letters, for instance, refer to points or lines in Figure 7-4 and, similarly, an arrow points to a specific component of a diagram in Figure 7-8.

a. Lettered diagrams: Peirce himself uses letters in geometrical diagrams as an example of indices. Lettering diagrams is one of the most common features in geometry. Netz (1999) considers 'lettered diagrams' to be one of two tools, beside language, which shaped Greek mathematical deduction. Letters on Greek geometrical diagrams are not merely letters, they are also the objects. Usually, capital letters are used to denote points, and, hence, lines, angles, shapes and planes could be identified (named) by those letters, as in Figure 7-4. In terms of relational processes, these letters identify objects. For instance, AB represents a line AB, or ABC represents a triangle. Letters are also used to refer to these objects in the written text, but this is a textual function to be discussed in a separate chapter.
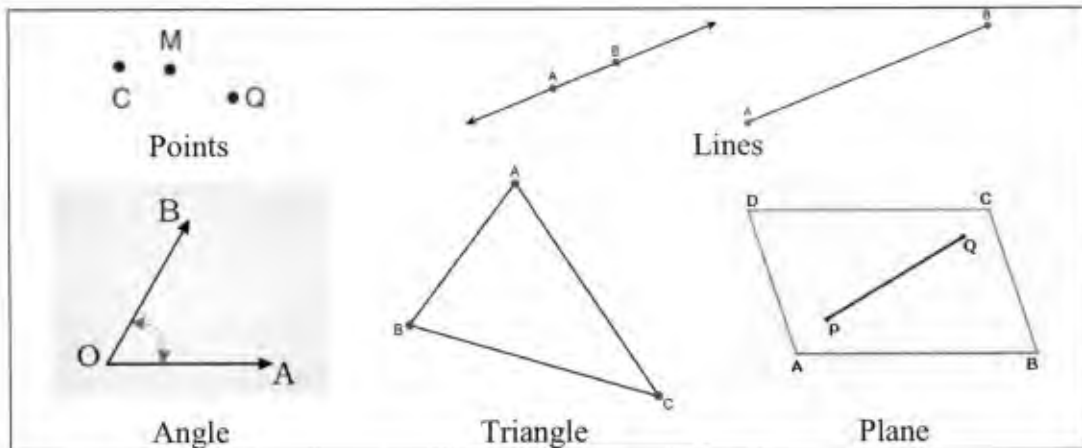
Figure 7-4: Lettered diagrams with capital letters

Indexical letters may sometimes be small letters used to identify attributes of a diagram such as the size or value of lines, angles and areas (see Figure 7-5 for different examples). These small letters are also indices used to refer to size relationships (see below).
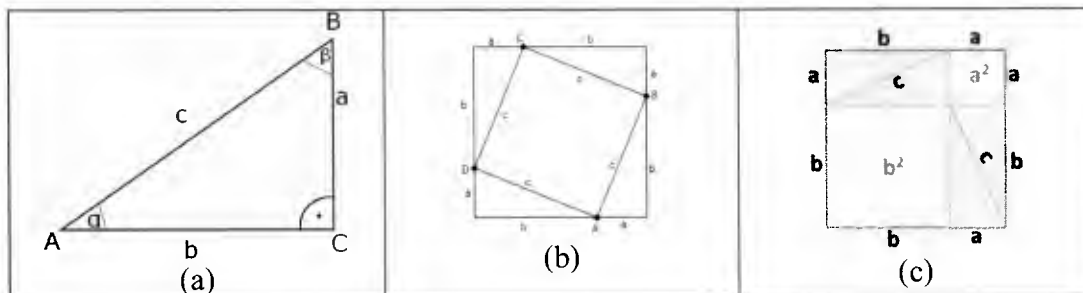
Figure 7-5: Lettered diagrams with small letters

b. Arrows: Arrows (or lines) may also be indexical signs in geometrical diagrams. They are usually joined either with words or with numbers to refer to the object they identify. Unlike the arrows discussed in Chapter 6, these arrows do not represent action or narrative because they in some way come from 'outside' the diagram, referring to a specific part of the diagram in order to identify it with words. Moreover, the presence of the arrow is the reason we consider these diagrams to represent indexical processes. Indexical arrows come in two modes: attributive, when they refer to specific parts of the diagram (Figure 7-6) or identifying when they refer to the whole diagram (parallelogram and $\triangle$ABC in Figure 7-7), although the identifying mode has fewer examples than the attributive.

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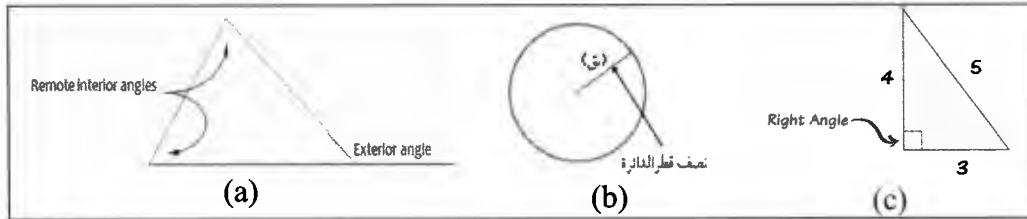
Figure 7-6: Indexical attributive arrows
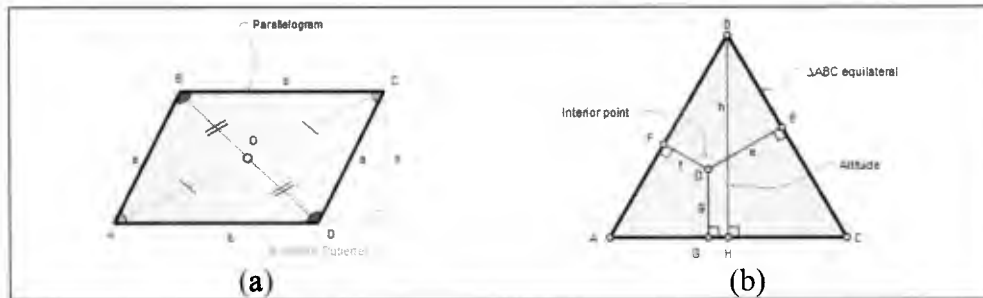
Figure 7-7: Indexical identifying arrows

Some geometric diagrams present different types of arrows simultaneously, including narrative type arrows (indicating measurement of length) as well as indexical ones. Figure 7-8, for instance shows a diagram presented by a participant student in the current study using a narrative type arrow to indicate the measurement of the side LM and two indexical attributive arrows indicating the measure of two angles. The other diagram is taken from the Internet indicating the measurement of the side AB and two indexical attributive arrows, one of which indicates some feature of the triangle ACB, while the other indicates the name of the side AB as Hypotenuse.
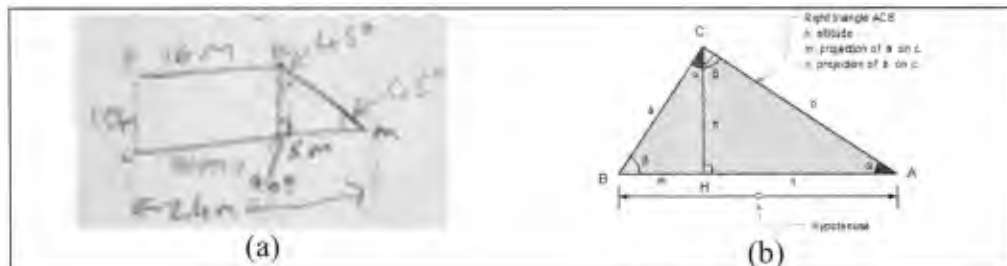
Figure 7-8: Narrative and indexical arrows

Symbolic processes:

Symbolic signs have convention-based relationships with their objects. According to Peirce, words are the first thing to consider as symbolic signs. The word itself is a signifier, and its meaning is the signified (Hodge & Kress, 1988). In other words, symbolic process makes a statement about what the object means or is (Kress & Van Leeuwen, 2006). I have introduced how letters and arrows, as indexical processes, are used to identify some parts of the diagram or the entire diagram. In symbolic processes, the identification is done by words.[17] Words can identify parts of the diagram, through attribution (Figure 7-9) or the whole diagram through identification (Figure 7-10).
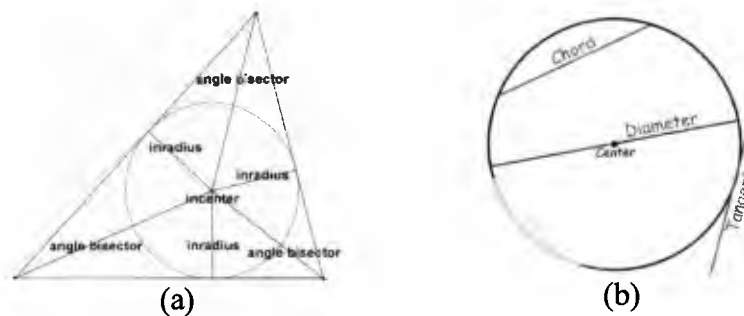
Figure 7-9: Symbolic attributive
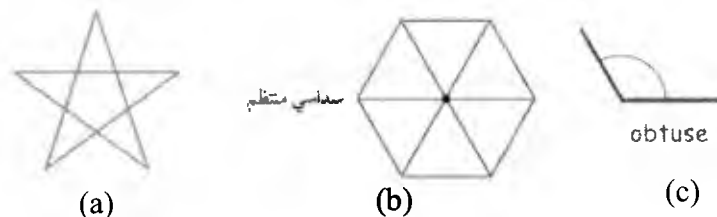
Figure 7-10: Symbolic Identifying

As mentioned in the indexical processes, if arrows are joined by words (Figure 7-6 and Figure 7-7), or the other way around, I consider them to be indexical, because of the role of the arrow. While arrows with words can also be used to identify the whole diagram, there are fewer examples of this practice.

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[17] Labels and colour could also be used here as symbolic processes. The main reason not to include them here is that both refer to the size (value) of parts of diagrams. Therefore, I consider them in the size relationship within the spatial relation (see below).

3.3 Spatial relations

The previous two kinds of processes may be categorised as 'naming' processes, whether the whole diagram is classified or named, as, for example, whether or not it is a parallelogram, or whether a specific name or feature is attributed to parts of the diagram. In spatial processes, on the other hand, we focus more on the details of the diagram and how they express relations between geometric objects. Generally speaking, mathematical objects in Euclidean geometry are: points, lines (segments), angles and shapes (actually angles and shapes are inherently relations between points and lines or segments). Table 7-2 summarises these relations (represented by *):

Table 7-2: Relations considered among geometric objects

	Point	line	angle	shape
point	*	*	*	*
line		*	*	*
angle			*	*
shape				*
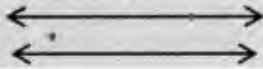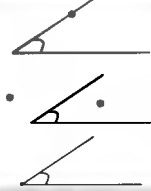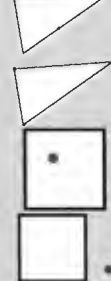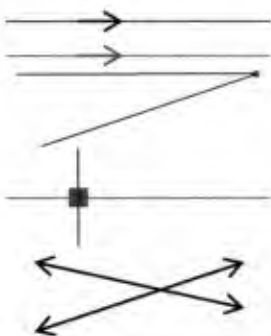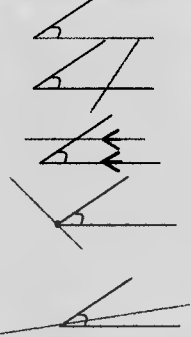
Relations in the shaded cells are of the same type of the other corresponding cells (reflexive relations). For example, the relation between points and lines is the same as the relation between lines and points, so I don't put a star (*) in the latter corresponding cell. Thus, ten relations are identified in spatial processes. These types of relations are of two types: positional and size (measuring). The claim here is that an object has two parameters (position and size) that are necessary to *see* its details and its relationships with other objects. In the following section, I address each parameter, illustrated by several examples.

3.3.1 Positional relations
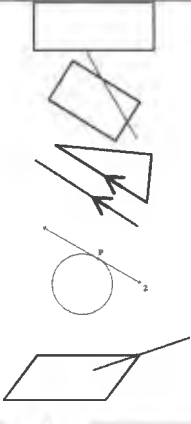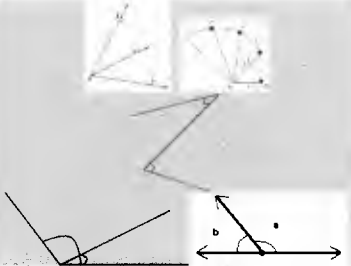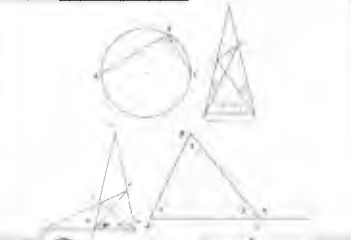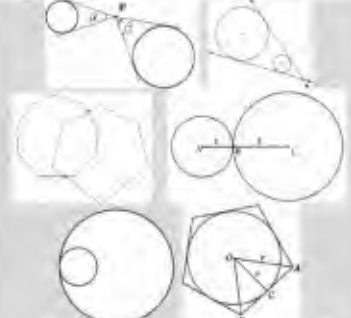
The spatial relations are based on the position of objects. In language, words which describe these relations are *coincide, lie on, distinct, interior, exterior, inside, outside, parallel, perpendicular, intersect, tangent, touches, pass, contain* and the

like. They may be akin to circumstantial relational processes in language. There are ten types of positional relations, which are summarised in Table 7-3.

Table 7-3: Positional relations in diagrams

Name of relation	Verbal description	Visual examples
(P1) Point & Point relations (PP):	a. Coincident (same point) b. Distinct (not coincident)	
(P2) Point & Line relations (PL):	a. Lies on it (collinear) b. Does not lie (distinct)	
(P3) Point & Angle relations (PA):	a. Lies on one of its sides (collinear) b. Does not lie on any side (interior & exterior) c. Vertex	
(P4) Point & Shape relations (PS):	a. Lies on one of its sides/circumference b. On/at vertex c. Inside the shape (e.g. centroid, orthocentre, incentre) d. outside the shape (e.g. circumcentre)	
(P5) Line & Line relations (LL):[18]	a. Coincident b. Parallel c. Concurrent (touch on one point) d. Perpendicular (orthogonal) e. Intersect (not perpendicular)	
(P6) Line & Angle relations (LA):	a. One of its sides (or extensions) b. Intersect with one of its sides (or extensions) c. Parallel to one of its sides d. Touches its vertex through the vertex but outside the angle e. Passes through the vertex (e.g. bisector) through the vertex and inside the angle	

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[18] Because the focus of the current study is 2D Euclidean geometry, skew lines are not considered.

(P7) Line & Shape relations (LS):	a. One of its sides (or extensions)	
	b. Intersect with (at least) one of its sides (or extensions)	
	c. Parallel to (at least) one of its sides	
	d. Touches one of its vertices 'outside' (e.g. tangent)	
	e. Passes through one of the vertices (e.g. bisector, altitude, median) 'inside the angle'	
(P8) Angle & Angle relations (AA):	a. Share a vertex	
	b. Share a side	
	c. Share both a vertex and a side	
(P9) Angle & Shape relations (AS):	a. One of its angles (inside)	
	b. Outside (e.g. exterior angle, equal one of its angles)	
(P10) Shape & Shape relations (SS):	a. Distinct	
	b. Intersect (share points or sides)	
	c. Contains (inclusion, inscribed)	

There are, indeed, diagrams which may include different relations, such as a point lying on a line, an angle and a shape (PLa, PAa, Pac, PSa, PSb). Moreover, it is possible for more than one relation to coexist between the same pair of objects in some cases, such as SSb and SSc.

An example:

The diagram in Figure 7-11 contains many geometric objects such as points (A, B, C, O), lines (radii, AB, AC, CB, OC, etc.), angles (e.g. AOB) and shapes (circle, pentagon, triangles).
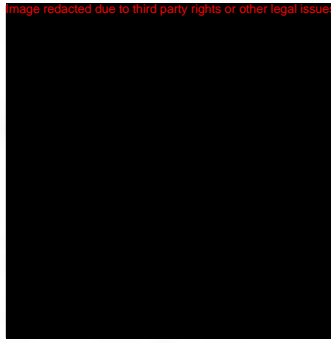
Figure 7-11: Positional relations: An illustrative example
http://etc.usf.edu/clipart/36700/36715/pentcirc_36715.htm

The following relations may be identified in this diagram:

Name of relation	Examples in the diagram
(P1b) Point & Point (PP):	O & A are distinct
(P2a) Point & Line (PL):	B & C lie on the same line
(P3c) Point & Angle (PA):	O is the vertex of $\angle$AOB
(P4a) Point & Shape (PS):	C lies on the side AB, and it is a tangent point to the circle
(P5d) Line & Line (LL):	OC$\perp$AB
(P6a) Line & Angle (LA):	OA & AB are sides of $\angle$OAB
(P7a&d) Line & Shape (LS):	AB is a side in the pentagon (P7a) AB is a side in $\triangle$OAB (P7a) AB is a tangent to the circle (P7d)
(P8a) Angle & Angle (AA):	$\angle$AOC &$\angle$BOC share the vertex O
(P9a) Angle & Shape (AS):	$\angle$AOB is a central angle in the circle
(P10c) Shape & Shape (SS):	The circle is inscribed in the pentagon

3.3.2 Size processes

Spatial relations are also based on the size of objects. In language, words which describe these relations are *equal, greater than, less than, congruent, similar, same as* and the like. Not all the ten spatial relations mentioned earlier appear in size processes. It is expected that special relations that include points (PL, PA and PS) do not appear in size processes, because points have no size and, consequently, no size-based comparison is applicable. The remaining seven spatial relations are divided into two categories: relations between the same types of objects (and, hence, a comparison relation between them is established) and relations between different types of objects (and, hence, a measurement-based relation between them is established).

Comparison-based size relation:

This type of relation occurs between the same types of geometric object: line-line (LL), angle-angle (AA) and shape-shape (SS). In geometry, it most often takes the form of propositions such as definitions and theorems. For instance, the Triangle Inequality Theorem, 'the sum of the lengths of any two sides of a triangle is greater than the length of the third side', relates the three sides of a triangle together (LL). Pythagoras theorem is another example in which two types of relationships may be expressed (LL and SS). Moreover, the Exterior Angle Theorem in Figure 7-1 also shows a relationship between angles (AA).

I consider all such relationships as 'special' in Table 7-4. This type of relationship is only possible to identify if one has specific forms of knowledge beyond the conventions. In other words, there are no indicators that can be applied objectively by anyone having knowledge of the conventions, like the equal length markers that indicate congruency (S1a or S3a&b in Table 7-4). One reason to consider them here is to illustrate that sometimes there is a need for some mathematical knowledge in order to read and analyse mathematical diagrams.
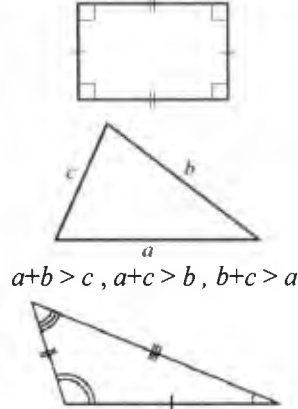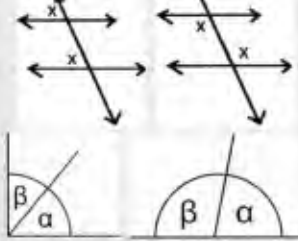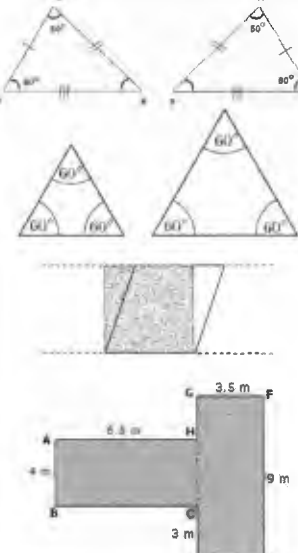
Measurement-based size relation:

This relation occurs between different types of geometric objects such as line-angle (LA), line-shape (LS) and angle-shape (AS). An additional relation, which is the relation between two points, is also considered, because the length of a line segment is defined by the distance between two points (see Table 7-5). As a result of the interaction between geometric objects in measurement-based size relation, a value (most often, numerical) is suggested for these objects. This value could be that the distance between two points is referred to as the length of a segment line, or that the area of a parallelogram is calculated by multiplying the lengths of its base and height. In general, these relationships are either (a) given in a problem in writing or visually, or (b) could be calculated by geometric formulae of measurements (similar to the 'special relationships' mentioned in the comparison-based size relation).

Measurements in Euclidean geometry take four forms based on the units of measurement: linear, angular, square (area), cubic (volume). As mentioned earlier in footnote number 18, the current study focuses on 2D Euclidean geometry; hence,

cubic measurement will not be considered in detail. Thus, I will consider the first three types of measurement which embody the four size relationships (PP, LA, LS, AS).

Table 7-4: Comparison-based size relations in diagrams

Name of relation	Verbal description	Visual examples
(S1) Line & Line relations (LL):	a. Equal (e.g. square, isosceles, equilateral) b. Special relationships (e.g. Triangle Inequality Theorem, Pythagoras Theorem) c. None of the above	 $a+b > c$, $a+c > b$, $b+c > a$
(S2) Angle & Angle relations (AA):	a. Congruent (equal, same) (alternate -Z-, correspondence -F-, or just marked to be the same) b. Special relationships such as complement, supplement, the Exterior Angle Theorem c. None of the above (greater or smaller)	
(S3) Shape & Shape relations (SS):	a. Congruent b. Similar c. Special relationships such as 'a square is equal to a parallelogram of equal base and height', Pythagoras theorem. d. None of the above, e.g. the area of a polygon is equal to the sum of areas of its component shapes, or an area of a shape is greater (smaller) than another.	

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Linear measurements: This type of measurement refers to the length of a line segment (PP relationships) or perimeter (LS) – Figure 7-12. Length is defined as the distance between two points, or an interaction between two points (PP), and as a result a (numerical) value is suggested for that length.

The perimeter of a polygon is the total distance around the polygon, and it thus embodies relationships between lines and shapes (LS). Another name for perimeter is circumference, which is suggested by the geometry of a circle and may be defined as the total 'distance around a circle or a closed curve'. The circumference of a circle is calculated by the formula $2\pi r$ (r: radius).
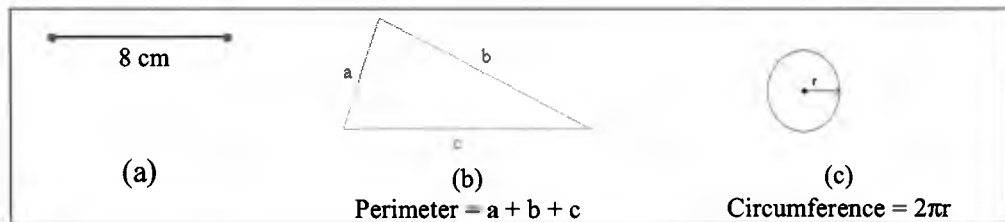
Figure 7-12: Linear measurement-based size relation

Pythagoras theorem has a special status here, since it embodies relationships between the sides of a triangle (LS) which indicate linear measurements, and, at the same time, it embodies relationships between shapes (SS) as mentioned in comparison-based size relationships. Moreover, it indicates square measurements (see below) in which a relationship between shapes is established.

Angular measurements: This type of measurement is found in the geometry of circles, where the main difference from linear measurement is the presence of arcs rather than line segments. Besides arc length, the most well-known object measured by this type is the angle.

Arc length is measured by multiplying the length of the radius and the measure of angle which is measured in radians (Figure 7-13a). 'An angle is formed by two rays with a common endpoint' (http://www.icoachmath.com/SiteMap/Angle.html) and is measured either by degrees (in which a circle is divided into 360 degrees) or radians. 'One radian is the angle subtended at the center of a circle by an arc that is equal in

length to the radius of the circle' (http://en.wikipedia.org/wiki/Radian), i.e. 1 radian = $(180/\pi)°$ or about 57.3° .
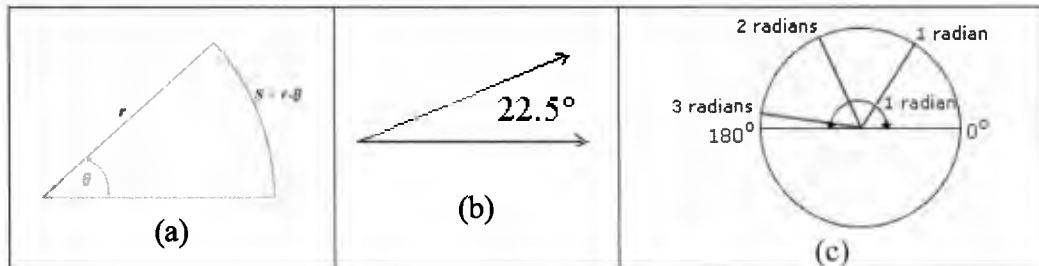
Figure 7-13: Angular measurement-based size relation

Area (square) measurements: This type of measurement focuses on 2D closed polygon and curves. For regular closed polygons (e.g. square or rectangle), area is defined as the number of square units that cover these diagrams. The area of irregular polygons can be found by dividing them into different standard polygons (Figure 7-14).
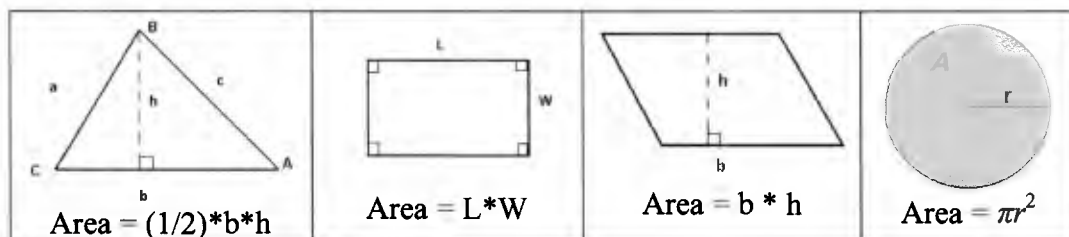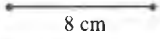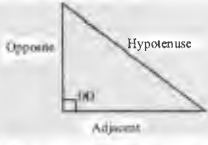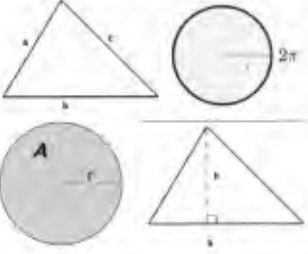
Figure 7-14: Area measurement-based size relation

All diagrams in Figure 7-14 are examples of relationships between (areas of) shapes and sides, i.e. Lines and Shapes (LS). However, these diagrams may also embody relationships between lines, angles and shapes (LA, AS). For instance, the area of a triangle may be calculated in terms of angles as follows:

Area = (1/2)*b*c sinA = (1/2)*a*c sinB = (1/2)*a*b sinC

A summary of measurement-based size relationships in diagrams is shown in Table 7-5:

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Table 7-5: Measurement-based size relations in diagrams

Name of relation	Verbal description	Visual examples
(S4) Point1 & Point2 relations (distinct PP):	The measure of a line is (equals) the distance between two points.	
(S5) Line & Angle relations (LA):	The measure of an angle is computed by trigonometry	
(S6) Line & Shape relations (LS):	a. The measure of the **perimeter** or **circumference** of any polygon (circle) has a formula that links to lengths of sides or radius b. **Area** has specific formula. There are several ways to compute the area using lengths of sides. Note: Volume is also a measure-formula, though it is not considered in this study.	
(S7) Angle & Shape relations (AS):	a. The sum of angles in a polygon (=180(n-2))	
	b. Area	

An **example:** According to the suggested measurement-based size relationships, the following relations may be identified in the diagram in Figure 7-15:
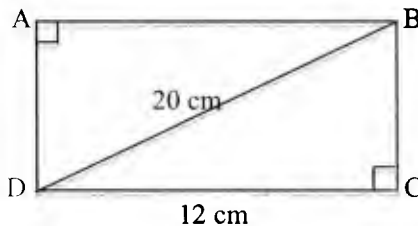
Figure 7-15: Measurement-based size relations: An illustrative example
(Drawn by the author)

Name of relation	Examples in the diagram
(1) Comparison-based size relation	
(S1b) Line & Line (LL)	Pythagoras theorem, BC=AD, AB=DC
(S2a) Angle & Angle (AA)	$\angle A = \angle C$
(S3a) Shape & Shape (SS)	Pythagoras theorem, $\triangle ABD \cong \triangle CBD$
(2) Measurement-based size relation	
(S4) Point1 & Point2 (distinct PP):	CD=12 cm
(S5) Line & Angle (LA)	cosBDC=12/20 → $\angle BDC$=53.1°
(S6a) Line & Shape (LS)	Perimeter of ABCD =AB+BC+CD+DA =12+16+12+16=56 cm [since BC=sqrt(400-144)=16 cm by Pythagoras Theorem]
(S7b) Angle & Shape (AS)	Area of ABCD=12*16=192

4. Labels and colour as relational (size) processes:

Size relations are expressed in diagrams (not in the written texts) by conventional symbols which are labels (marks or numbers) or by colour.

Labels: This is one of the most conventional forms in mathematics and geometry where specific visual 'marks' are used to denote size relational processes such as (in)equality or parallelism. The little square mark, for example, denotes the size of an angle (=90°). Labels which denote size relations come in different forms; special marks (e.g. /, //, ||) letters or numbers (or a combination of letters and numbers). See Figure 7-16. Letters refer to the object (as a name or as a value) next to them: a point, a side, an angle or a part of the shape. Similarly, numbers come in two types: they can denote value or they can name an object. Most often, numbers present the value of the object, e.g. the length of a segment, the measure of an angle, the measure of a circumference or the area. Labels offer information about relations between parts of the diagram. See the next chapter on interpersonal function.
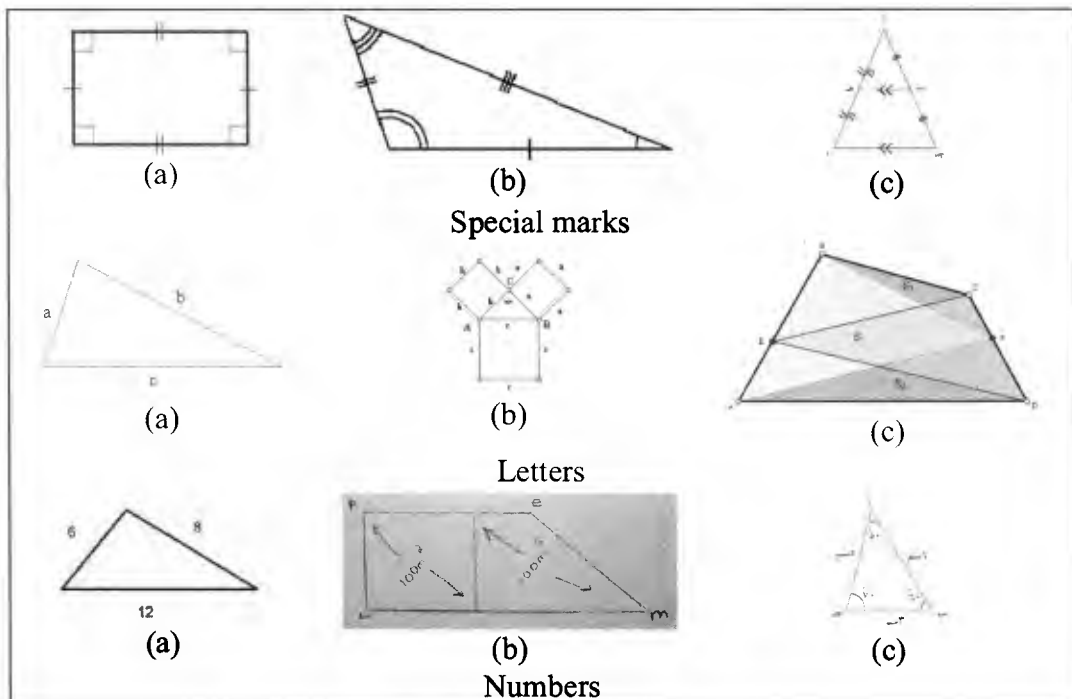
Figure 7-16: Labels as size relational processes

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Colour: Colour is also used to denote size relational processes in geometric diagrams, although written books, the traditional means of communication and representation, provide few examples in comparison to the screen, the new media of production (for more elaboration, see Kress, 2003). The data collected from students in schools are of the traditional type, and I found few examples in which students used colour. As mentioned in the discussion of methodology in Chapter 4, the web-based diagrams, or diagrams-on-screen, were one of the sources of the current study, and they made extensive use of colour. In other words, most of the coloured diagrams presented here are taken from the Internet.

The common feature shown by colour in diagrams is equality: equal sides (e.g. the opposite sides of a parallelogram are of equal length), equal angles (e.g. alternate interior angles) or equal areas (e.g. Pythagoras Theorem). See Figure 7-17.
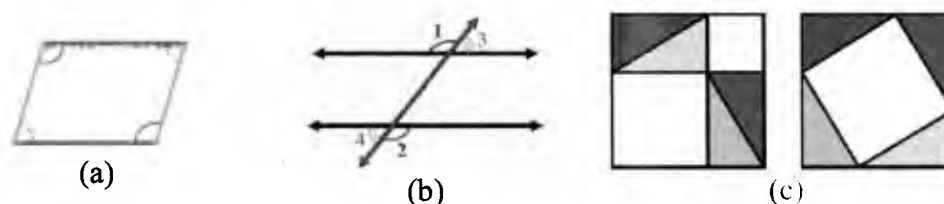
Figure 7-17: Colour as size relational processes

5. Definitions and theorems as interaction between spatial relations: the discourse level

Euclidean geometry is an axiomatic system based on axioms which are relationships defined between geometric objects that were agreed on 'without' a proof. One of the powerful features of axiomatic systems is the ability to create new objects based on 'existing' ones using relations. In other words, Euclidean geometry is an autopoietic system, 'a system that produces the things it talks about' (Sfard, 2008, p. 161). Historically, the creation of the new objects is based on construction using tools (compass and edge) to execute the new relations *on* the 'old' objects. Thus, the story of any object is, to a certain extent, a creation and a proliferation of 'old' objects. Actually if one wants to follow how every object was created in EG (Euclidean Geometry), one will arrive at the basic objects (or more primitive ones), definitions and axioms.

When mathematicians construct a new object, they will apply new relations to it, and, again, new objects will appear. What may be confusing is the endless number of objects which can be created using relations and the need to distinguish between them or to refer to them during the creation of new objects. I refer here to the notion of naming and the need for it. In geometry, some objects have specific position and/or size properties that led mathematicians to give them names to identify and refer to them as a shortcut that eliminates the need to list their features. The following example illustrates how the new objects are constructed through/in/by a diagram where mathematicians give names and, moreover, how the process of construction and definition can be observed in this diagram by using the suggested system of analysis (Figure 7-18).

I see the 'angle bisector' as a 'new' object created by the relations between other more 'primitive' objects. A line (segment) is *at the vertex and inside* (P6e) one angle, which is *one of the angles of* (P9a) a triangle, such that the two angles formed at this vertex are *equal* (S2a). I refer to this line segment as 'angle bisector' [this name is not part of the diagram but introduced into the description for ease of reference]. There are three such angle bisectors *inside* (P7e) the triangle. A single point *lies on* (P2a) all three angle bisectors. This point is named 'incentre', which is part of the diagram, indicating a symbolic process.
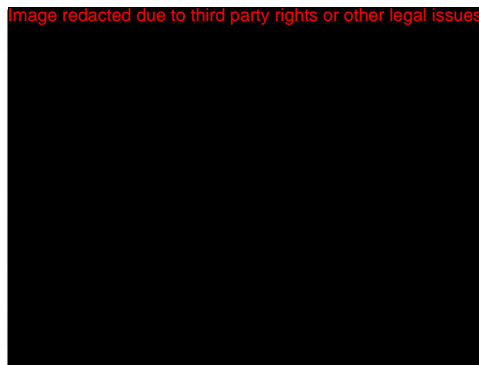
Figure 7-18: Angle bisector and incentre as new created objects
(http://www.mathopenref.com/triangleincenter.html)

Theorems and proofs may also be seen as instantiations of spatial relational processes. One of the famous theorems in geometry is Pythagoras Theorem ('in any right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides', see www.pythagorastheorem.co.uk). Figure 7-19 shows a

diagrammatic or visual proof (know as proof without words) for that theorem which is based on positional and size-relational processes. The basic idea of the proof is that all the shapes in both sides of the bidirectional arrow are *included* (P10b) in one single universal square where all of these shapes *share* sides with each other (e.g. the yellow square in the left side diagram shares its sides with the surrounding triangles). This means that the two diagrams in both sides are *congruent* (S3a) or in other words, the total area of the left side is equal to the total area of the right side. Actually the size relational processes in this theorem are based on the *congruence* of triangles that is realised by the same colour and labels. As a result of the two relational processes and by removing the four congruent triangles, the yellow squares in both sides are equal.
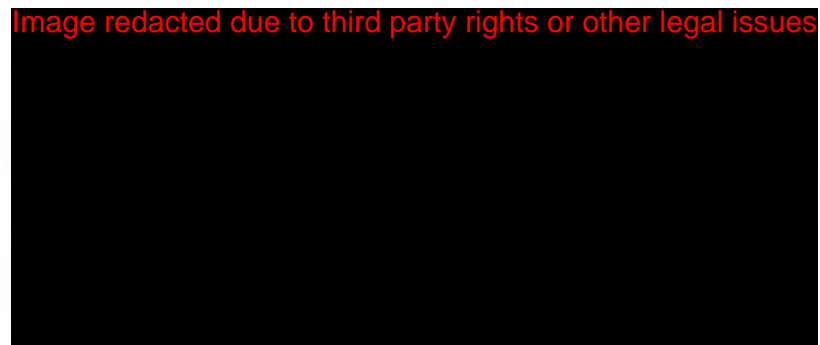
Figure 7-19: Proof without words of Pythagoras Theorem as relational processes
(http://www-users.math.umd.edu/~gfleming/JIM/PtPww/PtPwwFrame.html)

6. Relational processes in students' mathematical texts:

In the mathematics texts that students produced in response to the tasks of the current study, they used these different types of relational processes, with one main exception: the classificational. The absence of the classificational process is not a surprise, since this kind of relation has few examples in geometry and is most commonly used to 'show' students a wider view of relations between diagrams.

The context of the tasks affects the way students present mathematical objects in their diagrams and the relationships between these objects (Morgan, 2006) (see Chapter 4). Each task belongs to a different mathematical genre which influences the way students approach it, as demonstrated in their texts. Students used identifying processes in both tasks in the suggested categories, indexical (Figure 7-20) and symbolic (Figure 7-21), to identify objects and to show their attributes. For instance,

in the two tasks, students commonly used identifying processes such as arrows (or lines without arrows) together with words to identify objects such as the crossing point in Figure 7-20a or arrows with numbers to identify the area of specific parts of the diagram as in Figure 7-20b. In their texts, students commonly used letters to identify objects in diagrams.

Figure 7-20: Indexical relations in students' diagrams

Using words was common as well in students' texts in identifying symbolic processes as suggested by the framework. This was especially prevalent in the solutions to task 1 such as using the word 'sprinkler' in Figure 7-21a (written in Arabic at the top of left side of the diagram) and less common in task 2, although Figure 7-21b provides an example of using the word 'Centre'.

Figure 7-21: Students' symbolic diagrams

Furthermore, students also made use of spatial relationships, positional and size, in their texts. Task 1 (TF), for example, asks students to find distances, so students used measurements and, hence, size relational processes, especially the measurement-based relations. These relations dominated many of the texts, such as the diagram in Figure 7-20b, which shows both linear measurements (e.g. the length of PE = 16m –

S4 in Table 7-5) as well as area measurements (two areas in the trapezium: one equals 160 metre squared and the other is 40 metre squared – S6b in Table 7-5).

Task 2, in contrast, solicited investigation of a geometric proposition, asking students to agree or disagree or to (dis)prove, and, hence, positional and comparison processes were more dominant. For instance, in Figure 7-22, the quadrilateral is inscribed (P10c in Table 7-3) by the circle and the centre of the circle, and the crossing point of the diagonals of the quadrilateral are coincident (P1a in Table 7-3). Comparison-based relations were also found in students' diagrams, as in Figure 7-22 where the opposite sides of the quadrilateral are equal (S4a in Table 7-4). Finally, students used labels and colour in their investigation to solve Task 2 as shown in Figure 7-22, in which visual marks are used to signal the equality of the opposite sides of the quadrilateral.
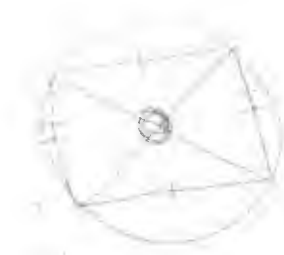
Figure 7-22: Labels in students' diagrams
(Year 7, Gillian, not common)

7. A summary: Reflective remarks on narrative and conceptual diagrams

This chapter focuses on 'timeless' geometric diagrams in which no actions could be identified. It is presented in contrast to narrative diagrams where mathematical activities could be identified through the reading suggested by the framework. Conceptual mathematical diagrams (re)present objects and the relations between these objects. After summarising how mathematical objects are constructed verbally and visually, I have identified the kinds of relations between those objects as represented in diagrams. Following Halliday (1985), Kress & Van Leeuwen (2006) and (Morgan, 1996b), I have identified four types of relational processes: classificational, identifying, positional and size. I presented them in Figure 7-23.
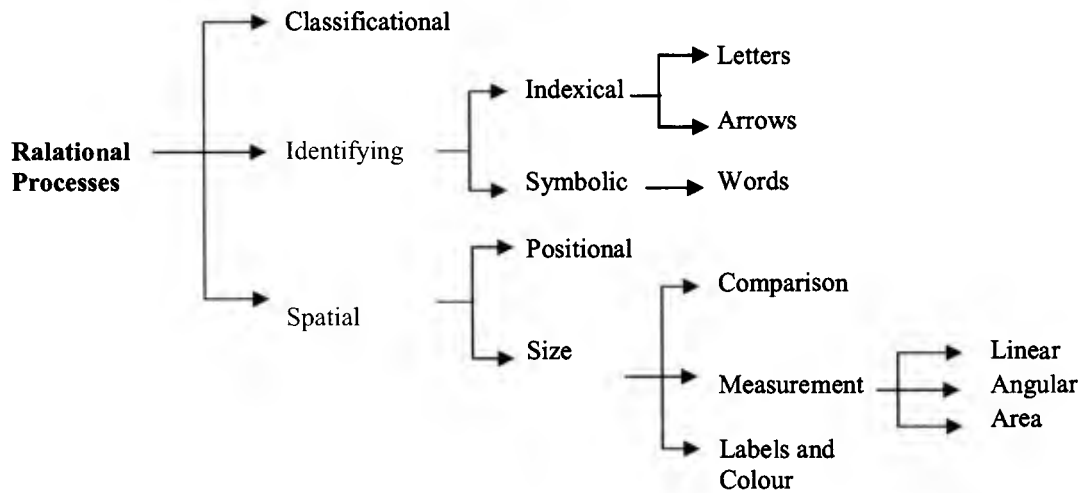
Figure 7-23: A summary of relational processes in geometric diagrams

Before moving to the next metafunction of diagrams as a mode of communication in mathematical discourse, there are two issues I want to highlight in regard to the distinction between narrative and conceptual diagrams offered in the previous chapter and the present one. These are: mixed diagrams and ambiguity in diagrams.

Mixed diagrams:

Although I tried to distinguish (and I identified distinguishing features of) diagrams as either narrative or conceptual, geometric diagrams do not come as 'purely' narrative or 'purely' conceptual. Diagrams are often mixed; they have both narrative and conceptual features. As a general 'rule', diagrams with a temporal factor are narrative, and the others are conceptual. Diagrams in Figure 7-24 are narrative (dotted lines, bidirectional arrows) but they also have conceptual features: identifying processes (letters and arrows) and size relation (numbers as values).
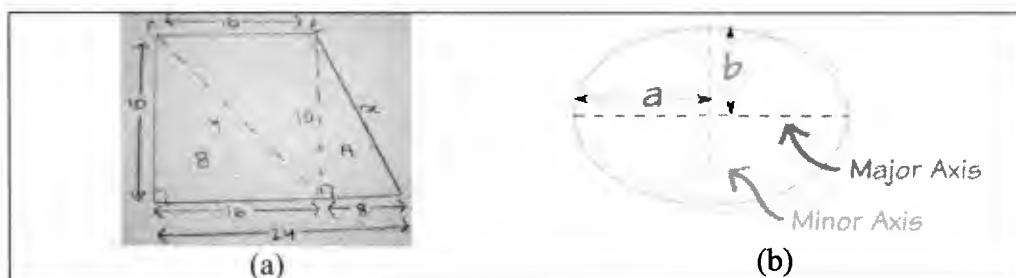
Figure 7-24: Mixed diagrams

This is not a defect in the suggested framework. Rather, it may be seen as another demonstration of the famous dichotomy in mathematics education research, namely the dichotomy between process and object in comparison to narrative vs. conceptual under different names such as 'operational vs. structural' (e.g. Sfard, 1991) or 'process vs. object' (Hersh, 1999; Sfard, 1994) or 'procept' (Gray & Tall, 1994).

This comment raises the issue of ambiguity, which is my next point in this reflection.

Ambiguity in diagrams:

The issue of 'mixed' diagrams (narrative with conceptual features or conceptual with narrative features) raises the notion of ambiguity. Are these diagrams ambiguous? While I think that I make it clear that there is no ambiguity in distinguishing between narrative and conceptual diagrams, ambiguity may be present in diagrams as in language. Ambiguity in mathematics education has been investigated in several studies in connection with language (Barwell, 2005; Barwell, Leung, Morgan, & Street, 2002; Pimm, 1987). Rather than conceiving of ambiguity as an obstacle to mathematics learning, some studies suggest that ambiguity in mathematics, or even in other disciplines, may contribute to students' understanding, argumentation and meaning-making process (Morgan, 2004; Street, 2005). To make it clearer, I am not saying that the determination of whether a diagram is narrative or conceptual is unambiguous. Rather, I tend to agree that diagrams, like any other form or mode of communication, contain ambiguities. Perhaps one of the 'justifications' offered by those who oppose the use of diagrams is that students would deduce 'wrong' information from diagrams because of these ambiguities. While a more detailed inquiry into ambiguity is outside the scope of this study, it indeed warrants further investigation.

8 Diagrams as interaction: Designing the position of the viewer

1. Plan of the chapter:

I have already introduced one function that any text would fulfil according to the systemic grammar model (Halliday, 1985), namely the ideational function. This function represents the way we experience the world or the way we construct the world according to our experiences. In geometry, this function is realised in two types of diagrams, narrative and conceptual, distinguished by the presence of temporality. Representation is interwoven with communication (Kress et al., 2001). In other words, when we start to represent, we immediately engage with other 'imagined' people, such as the audience, readers or viewers. As a result, we enter into an 'imagined' relationship with that audience – a social relationship.

A question struck me as I looked at the data: why do some students draw neat and accurate geometric diagrams, while others draw rough diagrams? This question guided my thinking about the interpersonal meaning in diagrams, as I will discuss in section 4 of this chapter. One way of thinking about this question is to think about viewers, or, to be precise, how the authors of the diagrams represent and construct the position of viewers and what possible social relations between authors and viewers may be read into the diagrams.

This chapter will address the indicators of such relations in geometrical diagrams. I will explore not just the (social) relationship between the author and the viewer but also the roles of each as constructed in diagrams. This social relationship and the role of authors and viewers are, following Kress & Van Leeuwen (2006), realised by contact (section 3), social distance (section 4) and modality (section 5).

This interpersonal meaning together with the ideational function will be combined in a specific way, according to the author's interests, to produce a text. The discussion about those texts will be the focus of the next chapter.

2.   Introduction:

The two images in Figure 8-1 present the same laundry detergent. A close look at the ad reveals that the same woman appears in both images. Indeed, the images themselves are identical, but one of them has been manipulated. But why does the woman dress, or more precisely, why has she been dressed, differently? This ad is a commercial one (laundry detergent) which aims to 'convince' customers to buy its product. But customers have different views and beliefs, and one way to persuade them to buy the product is to present the laundry detergent as if it belongs to them and to their beliefs. First of all, in both images, a woman, not a man, is doing the washing, which is consistent with mainstream social traditions in Palestinian society concerning the role of women as sisters, wives and mothers (e.g. Abu Ghazaleh, 1998; WCLAC, 2001). The woman in the two images, furthermore, gazes at the viewer as if speaking to them, creating a kind of close social relationship with them (Kress & Van Leeuwen, 2006). The main difference is that in the left image, the woman is not veiled, while in the right image, she wears a veil. Actually, these two advertisements appeared in two different Palestinian cities in the West Bank in the OPT.
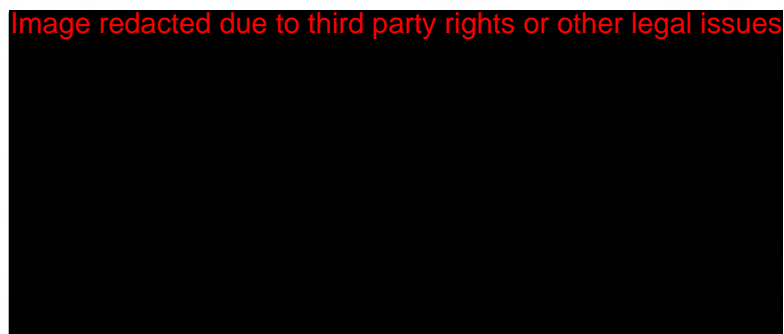
Figure 8-1: Laundry detergent advertisment

The point I want to make here concerns the relationship between producers of images and their audience. An image is a motivated sign in which the producers take into consideration their audience by engaging with them in an 'imaginary' social relationship (Kress & Van Leeuwen, 2006) through different realisations in images. The gaze of the represented participants, for instance, suggests a demand from

viewers, as in Figure 8-1, in which the woman may be saying 'trust me, I experienced it', and the presence of clean towels is, somehow, a proof.[19]

The consideration of audience is, however, common in other modes of representation and communication. In his description of 'How to write mathematics', for instance, Halmos states:

> The basic problem in writing mathematics is the same as in writing biology, writing a novel, or writing directions for assembling a harpsichord: the problem is to communicate an idea. To do so, and to do it clearly, you must have something to say, and you must have *someone to say it to*, you must organize what you want to say, and you must arrange it in the order you want it said in, you must write it, rewrite it, and re–rewrite it several times, and you must be willing to think hard about and work hard on mechanical details such as diction, notation, and punctuation. (Halmos, 1970, p. 124, my emphasis)

Halmos, here, refers to the interaction between the writer (author) of a text and imaginary readers whom the author takes into consideration in writing a text: who they are (students, colleagues, or general audience) and how I should approach them (should I provide information or ask questions, and if so, what type of information or questions?). In other words, authors enter into social relationships with their audience, and these relationships are represented in the produced text. This chapter seeks to show how authors or producers of different geometric diagrams approach that *'someone'*, the reader or the viewer, although not through words but rather visually.

Something similar happens in diagrams. Consider the following two diagrams (Figure 8-2) focusing on the question mark (?):[20]

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[19] The accompanying written text in Arabic reads (in rhymed colloquial): 'Where have you been all this time... others ruin the colours'. The word 'colour' is written in different colours.

[20] I think that one need not understand Arabic to work out what the diagram in Figure 8-2a demands in general (not necessarily solving the problem). In any event, the Arabic text reads 'The area of the square [is] 36 cm²'.
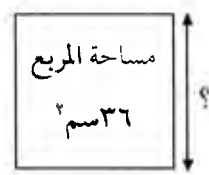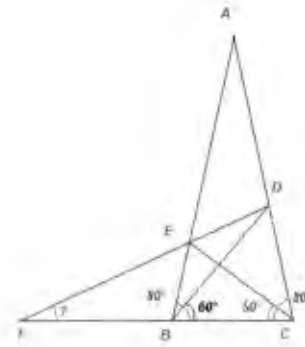
Figure 8-2: Question marks in diagrams

There are differences between the two diagrams (e.g. the information they provide, the number of shapes in each of them, the colour) but they share the presence of a question mark. As I will argue in this chapter, in the context of geometry, these question marks *ask for* something from an 'imagined' viewer, for example, the measure of the square's side in Figure 8-2a and of ∠EFB in Figure 8-2b. The imagined viewers in both diagrams, however, differ in the knowledge of geometry they presumably have to solve the problem. In diagram a, the geometry knowledge expected of the viewer (student?) is limited to knowing the formula for finding the area of the square and the characteristics of square, while in diagram b, the viewer is also expected to know facts about the isosceles triangle, that the sum of the internal angles of a triangle equals 180°, and the exterior angle formula. What I focus on here is the notion of the 'other' and its realisation in diagrams.

In written language, as previously mentioned, Halliday (1985) developed his SFL account to read such relations by focusing on the clause. In the following, I present Halliday's account of 'clause as exchange' in written and spoken texts which has been adopted for visual forms by Kress & Van Leeuwen (2006) in *Reading Images* and for written mathematical texts by Morgan (1996b). The main argument is that in the act of representation and communication, the author produces an image, for example, to convey a meaning. While doing so, a social relationship is constructed between the author and the viewer, and this relation is realised by different indicators, namely contact, (social) distance and modality.

3.   Diagrams as contact:

In his SFL approach, Halliday (1985) suggests that the nature of dialogue, spoken or written, between speaker/writer and listener/reader takes two forms: *giving* and *demanding*. Either the author gives something to the reader (e.g. information) or demands something (e.g. answer a question). Halliday identifies four functions of a clause based on the commodity being exchanged and the form of exchange. Table 8-1 presents these functions.

Table 8-1: Giving or demanding, goods–&–services or information (taken from Halliday, 1985)

Commodity exchange / Role in exchange	Goods–&–services	Information
Giving	Offer	Statement
Demanding	Command	Question

He termed the role of the clause in the exchange of information and goods–&–services as Proposition and Proposal, respectively. My interest will be in proposition, since this study is about geometrical diagrams, in which information is the only form of exchange that occurs. Proposition refers to the exchange of information, in which people can decide whether to accept or to challenge the statements and questions.

These propositions are, in English and Arabic, realised by the linguistic system of mood and the order of its components, the subject (which is a nominal group) and the finite element (which is part of a verbal group). Thus the general grammatical category for exchanging information is the indicative mood. Within the indicative, information takes the form of a statement (subject before finite) and, hence, it is declarative, or it takes the question form (finite before subject) and, hence, it is interrogative. Furthermore, the interrogative mood is either a yes/no interrogative (polar questions) or a WH–interrogative (content question) (Halliday, 1985). Offer information is realised by declarative mood, while demand information, in contrast, is realised by interrogative mood.
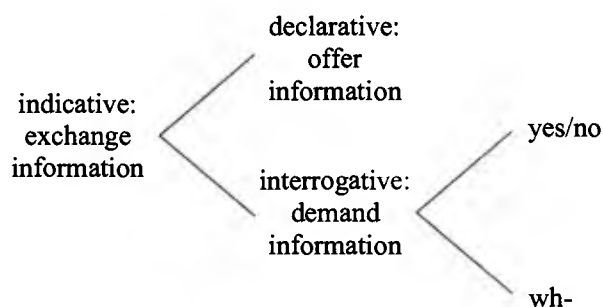
Figure 8-3: Indicative exchange information

In *Reading Images,* Kress & Van Leeuwen (2006), adopting the SFL approach, have suggested a similar approach. A Demand image 'wants something from the viewers – wants them to do something' (p. 118) while an offer image, in contrast, "offers' the represented participants to the viewer as items of information, objects of contemplation, impersonally, as though they were specimens in a display case' (p. 119). They distinguished between the two types of images by the presence of contact (a gaze) in demand images in which depicted participants look (gaze) at the viewers, as the woman does in Figure 8-1, while in offer images there is no such contact. Kress & Van Leeuwen were describing the represented participants, people or things, in images, while diagrams, in contrast, usually offer information. 'Demand diagrams' are less common. They also claim that there are contexts, such as school textbooks, which use a combined form of offer and demand. Geometric diagrams, I argue, most often use this combined form. But let me first introduce each form (demand and offer diagrams) separately.

3.1 Demand diagrams:

Interpersonal contact in geometric diagrams is either of offer form or of demand form. Either the author of a diagram offers 'something' to the viewer, and in scientific texts the offer is primarily information, or the author demands 'something' from the viewer, for example to answer a question whose presence needs to be reinforced verbally or by a conventional visual sign such as a question mark (Kress & Van Leeuwen, 2006).

In geometry, the main conventional means of contact between the author and the viewer is labelling, although contact can be made in different ways, such as by using colour. 'Geometric demand labels' are either realised directly by question mark or

indirectly by unknown quantities or variable names, both of which come as letters. Both diagrams in Figure 8-2, for instance, are direct demand diagrams with direct question marks. The viewer is expected to know that a specific mathematical action is needed to be done such as 'find the value of'. Figure 8-2–a asks for the length of the side, while Figure 8-2–b asks for the measure of ∠EFB.

The reason to distinguish between direct and indirect demands is that in direct labels, the presence of a question mark suggests directly what the demand is, as in Figure 8-2 and Figure 8-4. In indirect labels, on the other hand, no question mark is presented, and, instead, the diagram contains unknown quantities or variable names, most often in the form of letters. Viewers may find the value of the unknown quantity or the variable, however, they may not be sure if this is the problem or the question needed to be answered.

3.1.1   Question mark –a direct demand

Question mark is a conventional visual sign which carries the meaning of demanding something from the viewer. As said earlier, the viewer is expected to find the value of the marked part of the diagram as in Figure 8-4, in which the question mark suggests or demands solving a problem such as finding the value of a side of a triangle, the measure of an angle or the area of a shape.
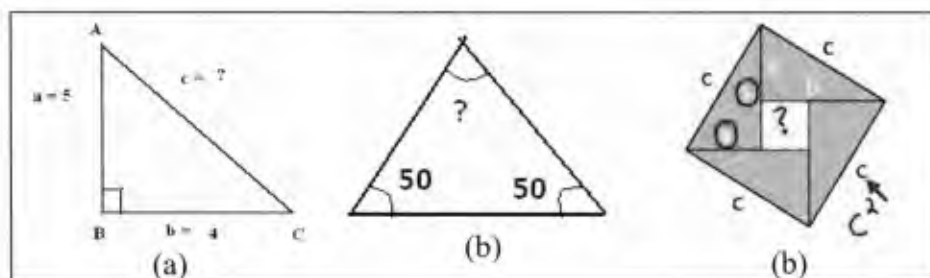
Figure 8-4: Question mark as demand

3.1.2   Indirect demand

Indirect demands take different forms such as unknown quantities or variables which may take the form of letters or (nominal) numbers to ask for the value of a specified side, angle or area. These diagrams present one unknown quantity or multiple

unknown quantities instead of a side, an angle or an area (or even without any label), and the viewer is expected to find the value of that unknown (see Figure 8-5).
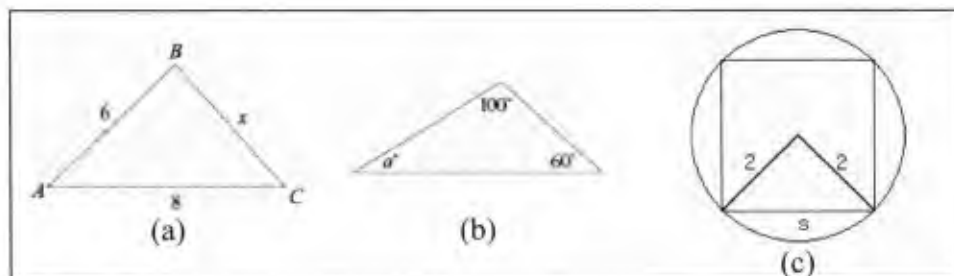
Figure 8-5: Indirect demand labels

However, the demand is not necessarily direct or 'self–contained', especially if information is presented in unknown values. For instance, all the presented measurements of the diagram in Figure 8-6 are shown in letters except the size of the angle ACB and the angle AHC. While it is possible for a person who is familiar with geometry to infer some geometric relationships within the diagram (using Pythagoras theorem, for example), it is not clear what is required. The demand will be apparent when the diagram is situated within a text with accompanying verbal demands which is provided to the right of the diagram.
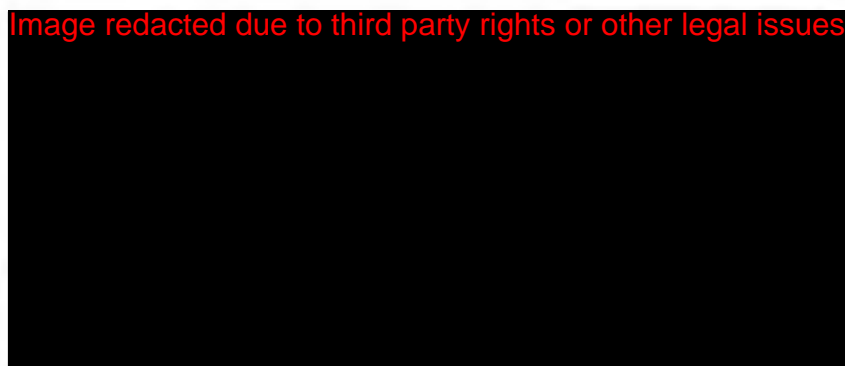
Figure 8-6: Diagram and the accompanying verbal text pose the problem
(http://gogeometry.com/problem/p266_right_triangle_altitude_geometric_mean.htm)

3.2 Offer diagrams

In general, and as shown in the previous section, all geometric diagrams or visual representations offer information even when they demand something from the viewer (Kress & Van Leeuwen, 2006). In Figure 8-2a, for example, the shape is a square

whose area is given. Offer diagrams offer information to the viewer about geometric objects (properties and relationships between them) without asking that any action be taken. In other words, I consider any diagram that makes a demand to be a 'demand diagram' (even though it also makes an offer), while a diagram that does not make a demand is classified as an 'offer diagram'.

In the previous chapter (on conceptual diagrams), I discussed the different types of relationships between mathematical objects (classification, identification and spatial relationships). While all of these relational processes do offer information about geometric objects, not all of them contribute to the interpersonal meaning, where the interaction happens between the author of the diagram and the viewer. Classificational processes, for instance, give information about a diagram in relationship with other diagrams. A square, for example, could be classified as a rectangle and as a rhombus. The interpersonal function of diagrams focuses on how the author takes the viewer into consideration in producing the diagram. A diagram itself shows the attributes and the identity of an object, and it highlights these qualities for the viewer in different ways, which have an interpersonal function.

Labels, for example, establish relationships (identifying and spatial) between objects in diagrams and, at the same time, visually highlight these relationships to the viewer. Figure 8-7 shows two examples of how labels do that. The author of these diagrams has the choice either to mention these qualities (equality and parallelism) in written language or to show them visually. This choice has interpersonal meaning in which the author chooses how to present the information to a specific audience (colleagues, teachers, students, etc.).
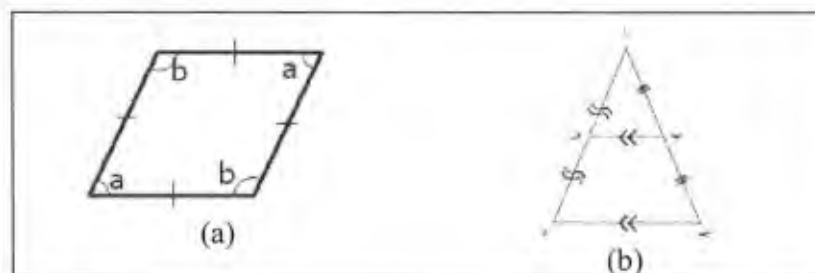
Figure 8-7: Labels as interpersonal aspects and relational processes

Authors of diagrams can also make choices about colour, arrows and words. Moreover, labels realise a social relationship between the producer and the viewer (see below).

3.2.1   Labels as offer

In geometry, labelling is one of the most conventional forms, where labels are given to the components of shapes or diagrams: the vertices, the sides, the angles and parts of the diagram. They express either a) geometric relationships such as equality, parallelism, similarity or b) specific quantities. Labels come in different forms such as special marks or symbols, letters and nominal numbers. I addressed these qualities of labels in section 4 of the previous chapter in my discussion of spatial relations (labels and colour as relational processes). Now I deal with the two types of labels.

3.2.1.a Labels offer geometric relationships

In Figure 8-8, all labels are presented to show properties and geometric relationships in diagrams.
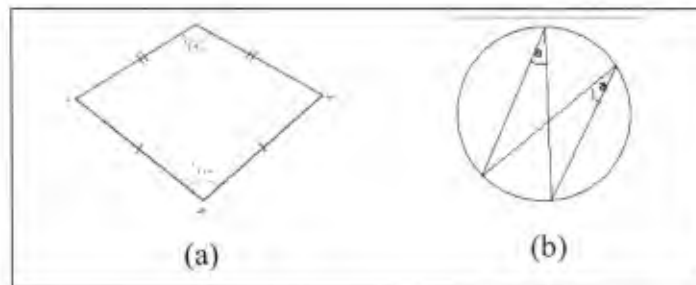
Figure 8-8: Labels as geometric relationships

3.2.1.b Labels as specific quantities (as seen in Figure 8-9)
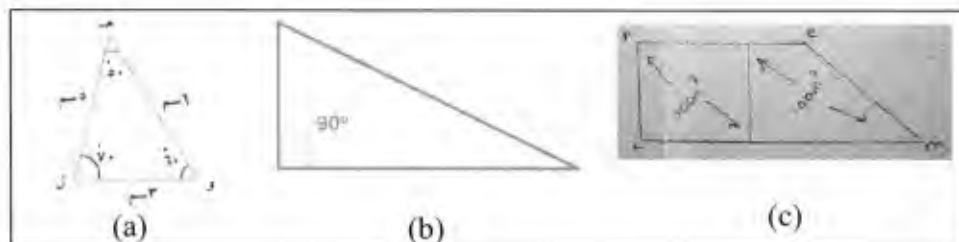
Figure 8-9: Labels (numbers) as specific quantities

3.2.2 Colour as offer

Colour also is used to offer geometric relationships, though it is limited to equality: equal sides, equal angles or equal areas. I dealt with colour in the previous chapter in section 4. Figure 8-10 shows some examples.
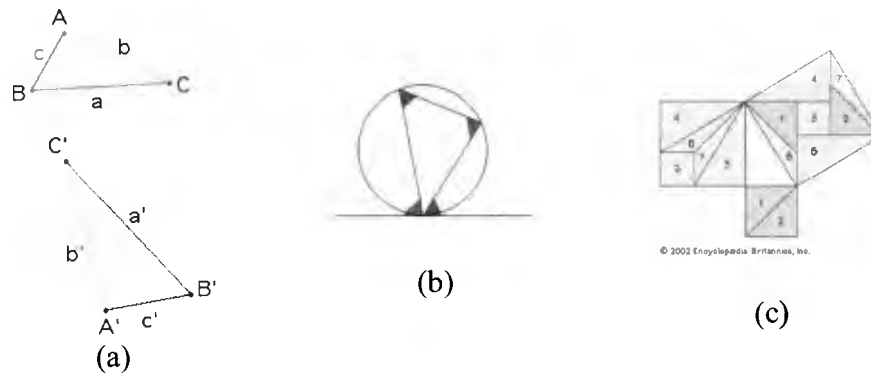
Figure 8-10: Colour as offer

3.2.3 I also dealt with arrows and words in identifying processes in the relational processes in the previous chapter, where I distinguished between indexical processes and symbolic processes (section 3.2). Examples of arrows and words as offer are presented in Figure 8-11.

Figure 8-11: Arrows and words as offer

4. Social Distance

Another aspect of the relationship between represented participants and viewers is the social distance and how it is represented in the image or diagram. Kress and Van Leeuwen (2006) argue that the distances that people keep from each other depend on their social relations. They identified five types of distance–based relations: close

personal, far personal, close social, far social and public. Each of these relations is defined based on how personal issues are discussed and how intimate or 'formal', or even strange, the relation would be. At a closer personal distance, for example, where people have an intimate relationship, the distance would be at which 'one can hold or grasp the other person' (Kress & Van Leeuwen, 2006, p. 124). At a public distance, in contrast, the distance would be such as to keep people in a formal relationship with each other. A similar practice occurs in television interviews. 'Close–up' frames of an interviewee convey an intimate relationship with the viewer, while the 'set–up' frame for an expert would keep him or her at a distance from viewers.

Kress and Van Leeuwen (2006) use physical distance as an expression of social distance (close distance, middle distance and long distance), but this distinction is not relevant in geometric diagrams, because these diagrams are not physical objects to be seen, and hence it is not possible to determine physical distance. Instead, I suggest that social distance is realised by the degree of 'neatness' of the diagram (Morgan, 1996b), labels, colour, arrows and words.

4.1 Neatness of diagrams

4.1.1 Neat diagrams:

In producing diagrams, authors (mathematicians, teachers, students, etc.) draw accurate (or rough) diagrams according to the interest of the author, the context and the audience. A neat diagram 'indicates that the text is formal and that there is some distance in the relationship between the author and the reader' (Morgan, 1996b, p. 91), and may be considered as an expression of respect for the viewer. In a school context, students or authors may want to show their teachers or assessors that they care and are trying to solve a problem, within the context of an authority relationship (see below). Alternatively, the issue of neatness may be related to the attitude towards mathematics (see below), where mathematics is considered to be a discipline of accuracy. The distinction between neatness as respect or as precision depends on the context in which the diagram is produced. The first two diagrams in Figure 8-12 show two typical neat diagrams drawn by participants in the current study. The third is taken from a textbook.
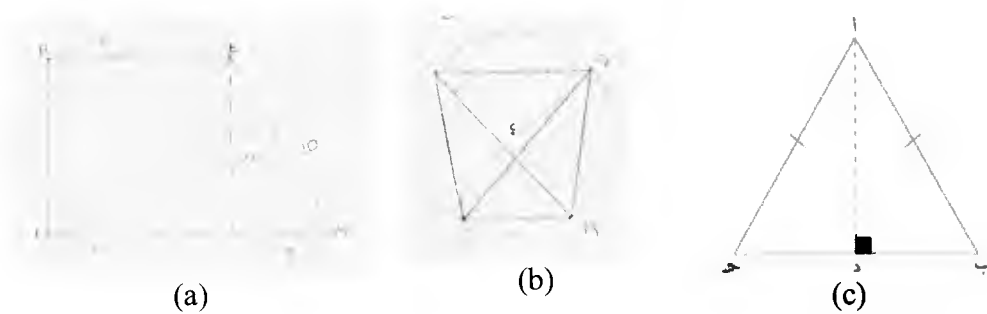
Figure 8-12: Neat diagrams

4.1.2 Rough diagrams:

Usually, when two colleagues are in informal geometric communication, participants would draw inaccurate and hand-drawn diagrams, or rough diagrams, to show their trials in solving problems. Such rough diagrams suggest a close personal distance, an intimate relation between the author and the viewer. These diagrams may be drawn for the authors themselves, as a 'private' drawing for personal use, while they work alone (Misfeldt, 2007; Morgan, 1996b). The first two diagrams in Figure 8-13 were drawn by students who participated in the current study, trying to solve the same problems addressed by the diagrams in Figure 8-12. The third is taken from the Internet.
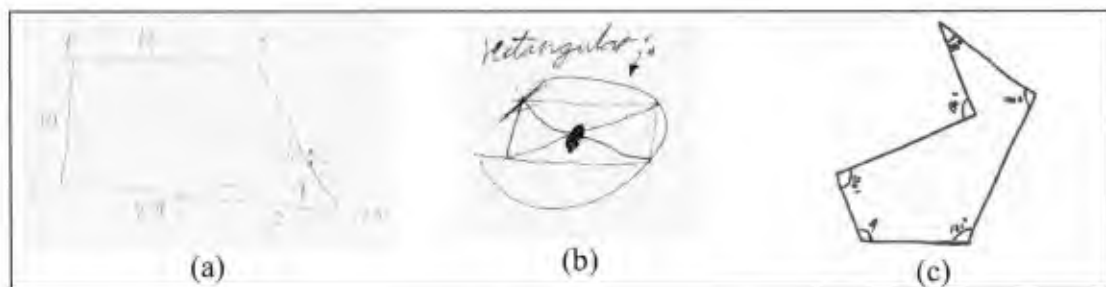
Figure 8-13: Rough diagrams

4.2   Labels: general vs. specific

A further feature which may contribute to the social relationship is the type of labels (offer and demand) used in diagrams. The general–type, or variables, of these labels suggests that they are used to introduce definitions or qualities of these diagrams. This practice often occurs in school textbooks. In other words, presenting labels in a general form suggests an authority who says what is the definition of a

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parallelogram, as in the parallelogram in Figure 8-7 where the characteristics of the parallelogram are presented in general–type. This approach is clearer in demand labels than in offer labels, because an author would explicitly present question marks or unknown quantities to be found or worked out by the viewer. This issue is related to the conventional view about mathematics as an abstract subject, and hence the author claims his or her membership in the mathematical community (Burton & Morgan, 2000) (see below).

On the other hand, and because of the current mathematical mainstream understanding which privileges general, abstract and formal prepositions (e.g. Davis & Hersh, 1981) over specific examples, labels which specify values or quantities, as in Figure 8-14, suggest lower authority.
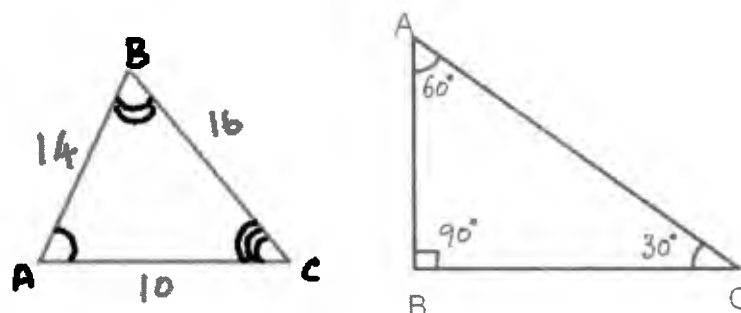
Figure 8-14: Labels express specific quantities

Another possibility is that a student drew these diagrams in order to solve specific problems, and s/he is showing them to the teacher/assessor.

4.3   Colour, and arrows and words:

A formal or an intimate relationship, however, is not the only type of social distance that can exist between an author and the viewer of a diagram. As is true between any two people, between the author and the viewer of a diagram there may exist a power relationship, in which either authors have power over viewers or the other way around. Kress & Van Leeuwen (2006) distinguish between power relationships in face–to–face communication and mediated communication. They derive the relationship between teachers and pupils as an illustrative example of the former, where the lack of reciprocity limits the choices available to each party in the verbal

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interaction; teachers can demand 'goods–&–services' from the pupils using imperatives, while pupils, on the other hand, can demand only by asking questions 'politely'. Mediated communication (e.g. writing) also reveals power relationships in a 'similar' way. The author is absent, and, hence, reciprocity is absent. Power relationship is realised by the use of the second–person pronoun you and imperatives such as calculate, show, prove, etc. (Kress & Van Leeuwen, 2006; Morgan, 1996b).

In images, one way in which power relationships are realised is by the system of perspective; eye–level angle (equality), high angle (power of the viewers over the represented participants) and low angle (power of the represented participants over the viewers). Perspectives are not included in the current study, where the focus is on 2D Euclidean geometry (and not 3D where perspectival images would be found) except for the eye–level angle. All 2D geometric diagrams are of the eye–level angle type. Does that mean that power relationships cannot be found in those diagrams? Not necessarily. Diagrams in Figure 8-11 may be seen as examples of power relationships between authors and viewers in which the author presents information to the viewer in a teacher–pupil–like relation; these are the 'Remote interior angles' or 'the altitude', or 'the interior point' of a triangle in Figure 8-11.

In a similar way, students may use arrows and words to construct both kinds of relationships. Arrows in Figure 8-11, for instance, suggest an authority (e.g. teachers) showing these qualities to students. It is possible for students to use such forms in authoritative teacher–texts in order to make a power claim: "I am presenting this as the solution, and I am certain enough to make a claim that it is correct (by using colour or arrows and words). For example, Mandy, a participant student in the current study, showed her solution to task 2 using colours as in Figure 6-16. The common solution among the (English) participant students is to draw three examples, each of which includes a circle and an inscribed quadrilateral and shows whether the crossing point of the diagonals of the quadrilateral meets the centre of the circle or not (see Figure 6-15, as an example). Instead of showing her attempts in that common way, Mandy used colour to refer to each attempt.

5.   Modality

Modality refers to how reality and truth are represented in communication, or, in other words, what authors would use to show the degree of certainty and truth of their statements or propositions about the world. Such certainty or truth is social in nature, in that it is constructed by a specific social group in meaning–making processes, as Kress & Van Leeuwen (2006) put it:

> Reality is in the eye of the beholder; or rather, what is regarded as real depends on how reality is defined by a particular social group. (p. 158)

In other words, certainty and reliability are social values and conventions set by a specific social group in order to judge or trust any given piece of information proposition or how people in that social group may or should give information. In mathematical discourse, abstraction is highly valued, and the 'more abstract approach is likely to be judged by teacher/assessor to demonstrate a higher level of mathematical thinking' (Morgan, 1996b, p. 92). Thus abstractness would be a distinctive feature of mathematical texts, a feature that participants in a social group would trust to some extent in order to act. Kress & Van Leeuwen (2006) refer to such features as modality markers or cues.

5.1   Modality markers:

In language, modality markers are auxiliary verbs such as *may* and *must* and their adverbs such as *possible* and *certain* or adjectival phrases such as 'I am sure that ..' (Burton & Morgan, 2000; Kress & Van Leeuwen, 2006; Morgan, 1996b). Halliday (1985) suggests a sort of a scale to identify the degree of certainty, where the value of modality runs from low modality to high modality as in Table 8-2:

Table 8-2: Modality values in propositions

Value of modality Possibility	Low modality	Median modality	High modality
Probability	Possible	Probable	Certain
Frequency (usuality)	Sometimes	Usually	Always
Modal verbal operators	can, may, could, might	will, would, should, is to, was to	must, ought to, need, has to, had to

Kress and Van Leeuwen (2006) distinguish between two different points of view which represent reality. Naturalistic point of view (art, for example) suggests that

what we see with our 'naked eye' in reality is the reference point for considering a visual representation to be real (high modality) or not (lower modality). Scientific point of view (science or mathematics, for instance), in contrast, considers the naturalistic view to be superficial and considers reality to be too sophisticated for the naked eye to capture. Reality, thus, from a scientific view, has to be more 'general and regular', or, in other words, abstract.

They (Kress & Van Leeuwen, 2006) suggest modality markers in visual representation according to naturalistic and scientific coding orientation. From the naturalistic point of view, they identified eight modality markers according to colour (three markers: saturation, differentiation, and modulation), context (absence of background vs. articulated and detailed background), representation (abstract vs. pictorial), depth (presence of perspective), illumination (presence vs. absence of light) and brightness (degree of black, white and grey). In scientific coding orientation, they identified four markers: technological (effectiveness of the visual representation), sensory (colour as a source of pleasure), abstract (personal vs. general, signalling 'eliteness') and common sense naturalistic (shared culture).

In both naturalistic and scientific approaches, modality varies from lowest to highest degrees according to scales running from the degrees of the use (or the appearance) of these markers. In colour saturation, for instance, 'a scale running from full colour saturation to the absence of colour; that is, to black and white' (Kress & Van Leeuwen, 2006, p. 160), the highest modality would be at a point which is close to full saturation, while the lowest will be at the black and white point ('not natural'). However, at the full colour saturation point where the colour is 'more than real', modality will be low since it – the picture, for example – contains such heightened colours, that it does not look real. It is quite the opposite in the scientific coding orientation. The highest modality would be at full abstraction, while the lowest modality would be at no abstraction.

Diagrammatic modality markers:

I have already shown in a previous chapter (6) how the authors of geometric diagrams have eliminated human figures and physical context. In mathematics, it is not common to use naturalistic modality in (modern) texts, meaning that one rarely

uses photographs or draws pictures to solve a mathematical problem. Actually, the dominant values and beliefs among mathematicians are that mathematics is abstract, formal, impersonal and symbolic. School mathematics is no exception. Hence, schematic or abstract diagrams are considered 'more' mathematical within the discourse of mathematics. Taking this practice and the 'anti–diagram' attitude (see Chapter 3) into consideration, I suggest five modality cues/markers in geometric diagrams. See Figure 8-15 in which the high modality represents or closely approximates the mainstream stance towards mathematics among mathematicians and others:
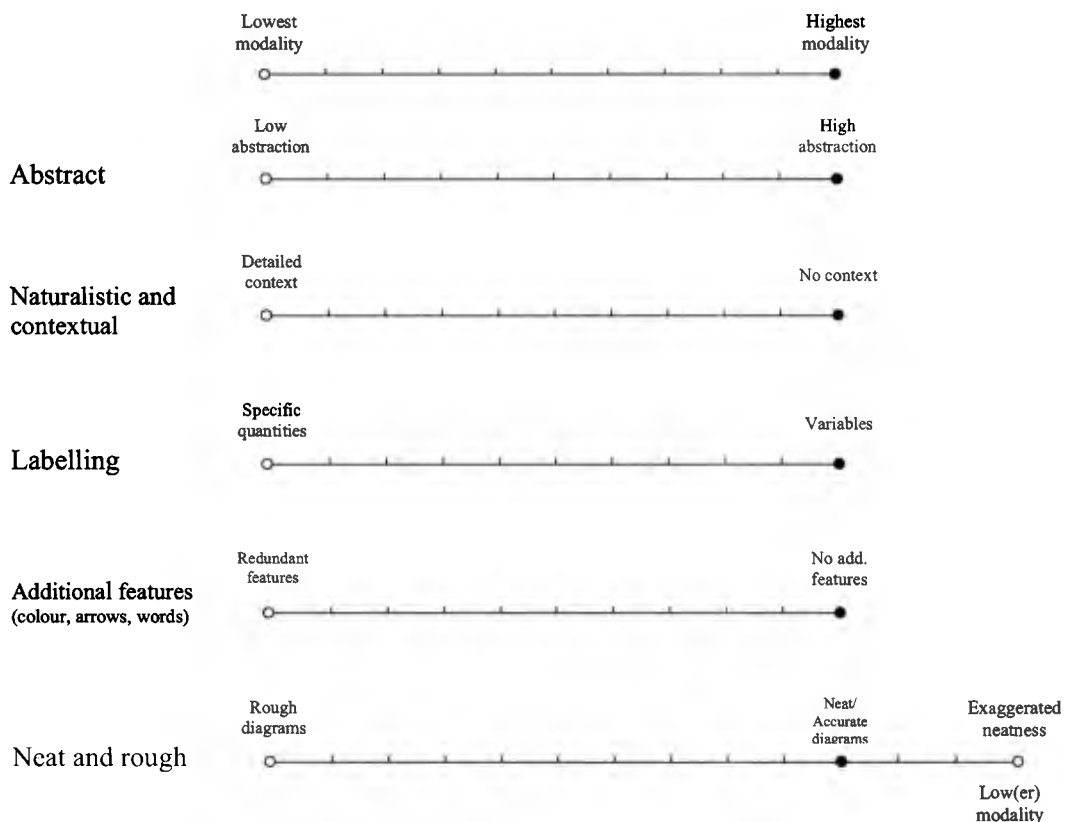
Figure 8-15: Modality markers and values in geometric diagrams

(1) Abstract diagram: a scale with an end expresses 'low abstraction' (lowest modality), and another expresses 'high abstraction' (highest modality). Low abstract diagrams may show concrete or practical activity (e.g. personification, embodiment) while high abstract diagrams are distinguished by the absence of any practical activity or perspective and have only geometric objects (Morgan, 1996b). See diagram b in Figure 8-16.

(2) Naturalistic or contextual diagrams: a scale running from 'detailed context' (lowest modality) to 'no context' which is similar to full abstraction (highest modality). See diagram a in Figure 8-16.

(3) Labelled diagrams: a scale running from labels denoting specific quantities (lowest modality) to variable labels (highest modality) with a middle point (lower modality), when a diagram has both kinds (see Morgan, 1996b, pp. 91-92, 158). See Figure 8-17. In the personal use of diagrams, if no labels have been used in the solving problem process, then a node of 'no labels' should be added at the high modality end of the labelling spectrum.

(4) Additional information in diagrams: some diagrams use/have complementary features such as colour, arrows and words to identify geometric objects and relationships between them. The criterion here is the effectiveness of these features: whether they add geometric information about the represented objects (identity or relationships) or not. The scale runs from redundant additional features (lowest modality) to 'no additional features' with the highest modality. In the middle of this scale, some of these features contribute to identifying geometric objects or indicating geometric relationship (e.g. equality) and, hence, the modality will be higher. See Figure 8-10 and Figure 8-11.

(5) Neat and rough diagrams: a scale runs from rough diagrams (lowest modality) to neat and accurate diagrams (highest modality). Morgan (1996b) claims that the extra care taken in drawing very neat and accurate diagrams may be assessed as a 'waste of time', thus according it a low(er) modality. This is akin to the colour saturation suggested by Kress & Van Leeuwen (2006), when a picture with exaggerated colours may look 'more than real'. See Figure 8-12 and Figure 8-13. In the personal use of diagrams, it seems that the 'neat and rough' visual marker may indicate the opposite of what I stated above; some mathematicians would use rough diagrams in their personal attempts in problem solving. Therefore, the scale may run from the use of neat diagrams as low modality to the use of rough diagrams as high modality.

Figure 8-16: Two diagrams of a pile of rods

(a)   (b)
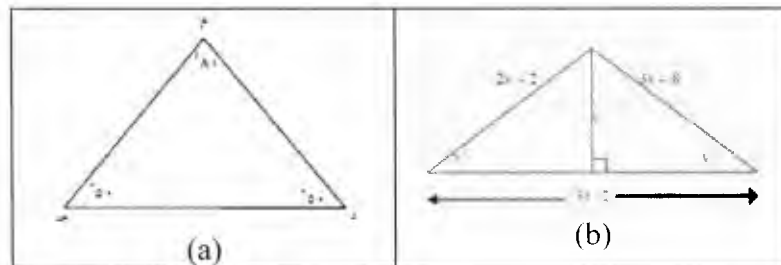
Figure 8-17: Labels: Specific quantities and variables

It is worth mentioning how these markers are relevant to the assessment process. Morgan & Watson (2002) identify six 'resources' available for teachers to assess students' mathematical texts, including teachers' beliefs about mathematics, their mathematical knowledge and how this knowledge can be communicated. Setting aside any predisposition against the use of diagrams in doing mathematics, teachers/assessors would mainly assess students' diagrams based on these cues, explicitly or implicitly. Morgan & Watson explore how this use of cues may affect the assessment of the students' mathematical texts in general. Morgan (1996b) derives an example of how a naturalistic diagram (lower modality) by a student named Sandra 'so strongly presents her work as being of a very concrete nature' (p. 156), that it was assessed as 'low level' by some teachers. In other words, low modality diagrams may cause 'lower' assessment. However, assessment criteria are not always consistent. Concerning the case of neat diagrams (high modality), Morgan (1996b, p. 91) states:

> a very neat diagram or set of diagrams which is read as background 'working out' may be judged to be a 'waste of time', while a very rough diagram which is read as forming part of an explanation may lead to judge the author to be lazy or careless.

The issue of assessment brings back the issue of relationships between authors, viewers (or readers/assessors) and subject matter (or diagrams/texts) in the modality

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of diagrams as interpersonal meaning. Authors (or students) who use mathematical conventions express their views of (attitudes towards) mathematics as well as their claim to be part of the mathematical community, i.e. their mathematical identity. Burton & Morgan (2000) studied, among other issues, how identity is constructed as an authority in mathematics and how claims to membership in the mathematical community are constructed through language. They claim that the naïve assumption that mathematics is about certainty led to the widespread use of some words in mathematicians' writings such as 'clearly' and 'obvious'. Such terms and other phrases, however, indicate authors' claims to positive authority.

The claim to membership in the mathematical community may be found, moreover, in the use of imperatives (such as consider, suppose, define, let x be) and of specialist mathematical vocabulary (Morgan, 1996b). Citation of one's own work and that of others is also an indicator of membership in the mathematical community (Burton & Morgan, 2000). In diagrams, as in written language, there is 'conventional and specialist mathematical vocabulary' to be used, although such vocabulary is, or must be, visual. Diagrammatic modality markers offer 'similar' criteria to reading authors' claims to membership in the mathematical community. Using the higher modality markers, such as abstract markers, less additional information and more general type of labels in diagrams, is a realisation of that claim to membership in the mathematical community.

6.   Summary:

I have discussed how relationships between the author, the viewer and geometric objects depicted in a diagram (represented participants) are realised in diagrams through contact, social distance and modality. In general, the 'not–geometric object' characteristics of any diagram (i.e. not points, lines, angles, shapes) contribute to the interpersonal meaning and may be organized into categories: (1) general reference refers to the appearance of the diagram (neat or rough), and (2) particular reference refers to specific characteristics of any geometric object in the diagram (namely labels, colour, and arrows and words). While the former contributes mainly to determining the social distance, the latter contributes to the three realisations (contact, social distance and modality). Other not-geometric object characteristics

such as titles of diagrams were not considered in the current study, although O'Halloran (2005) included them in her suggested framework for visual forms.

Equal and power relationships were discussed in this chapter by offering some visual marks that realise these relationships, such as labels. I moreover identified diagrammatic modality markers in diagrams which refer to the mathematical practice, conventions and values in mathematical discourse and how assessment may take place.

Having established how social relationships between the author and the viewer of geometric diagrams are realised (in this chapter) and how mathematical activity is represented in the previous two chapters, I turn now to look at the whole mathematical text, including diagrams and verbal content. In other words, I turn to consider the third meaning suggested by the SFL (Halliday, 1985), that is, the textual meaning.

9   Visual cohesion: The textual meaning

1.   The plan of the chapter:

The previous three chapters discuss two main kinds of meaning in diagrams as representation and communication: the ideational (narrative and conceptual) and the interpersonal. There is a third meaning, the textual, which is a vehicle for expressing these two kinds of meaning and relates them to each other. This chapter is about this meaning and its realisations in geometric diagrams.

As in the previous three chapters, and in order to set the context of diagrams, I start this discussion by giving a general background of how the textual function is realised in language and written texts as argued by Halliday (1985). According to Kress & Van Leeuwen (2006), the textual, or the compositional, meaning is realised through three systems: information value, salience and framing not only in images but also in composite or *multimodal* texts (written text and images). I will consider each of these systems in mathematical texts (Morgan, 1996b) followed by an illustrative analysis of two mathematical texts.

2.   Introduction:

According to the Hallidayan SFL, the way in which a text is organised as a coherent and meaningful message contributes, in addition to the ideational and the interpersonal meanings, to the textual meaning of that text. 'The textual meaning is the internal organisation of this [the ideational and the interpersonal meanings] as a message with the focus on what is demanded, together with its relation to the preceding text through presuppositions' (Halliday, 2002, pp. 199-200). There are two types of features which contribute to the construction of this meaning: structural and cohesive. Thematic structures (themes) and information structure are the focus of the structural features, while cohesive devices are the focus of creating coherence.

Theme refers to the concern of the message, and it is distinguished by different ways in different languages. In English (and in Arabic actually), for instance, theme is indicated by the position of the words in the clause – it comes first. Halliday (1985, p. 36) states, the theme 'is what the message is concerned with: the point of departure

for what the speaker is going to say.' The choice of information, what to put first and at the end, also contributes to the textual meaning. The Given-New informational structure presents the way in which an argument or a narrative proceeds.

Cohesive devices also contribute to the textual coherence. They are of four types: reference, ellipsis, conjunction and lexical cohesion (Halliday, 1985). Briefly, reference refers to the grammatical way of repeating what has happened earlier or what is going to happen later in the text (Thompson, 2004). There are three main types of cohesive reference: the third-person personal pronoun (he, she, it, him, and her); demonstratives (this, that, these, those, here, there); comparative (the same, another, similar, likewise, differently, such, more). Ellipsis is also a cohesive device that may be used to avoid repetition of a clause. This can be done either by 'missing out' the repeated element, ellipse proper, or by substitution, by using a linguistic token instead of repeating. Conjunction is used to join or combine two textual elements into a coherent unit, and it may be realised by the presence of propositions: *of, by, for, with*; conjunctions: *and, but, or, nor, although, because*; and conjunctive adjunct: *moreover, however, alternatively, therefore* (Thompson, 2004). Lexical cohesion, finally, can be achieved through the selection of items to replace other related and preceding items by two means: repetition and synonymy (Halliday, 1985).

Adopting the SFL approach in her linguistic approach to mathematical texts, Morgan (1996b) suggests three tools for the textual analysis of a mathematical text. These are: thematic progression, the way in which reasoning is expressed and the overall structure of the text (p. 96). Since deductive reasoning is highly regarded in mathematics, it is expected to be thematised. The way in which the reasoning theme is expressed helps determine the type of mathematical texts. Logical reasoning themes (e.g. *Hence, Therefore, etc.)*, for example, contribute to the coherence of the text and suggest text as a deductive argument. Temporal themes, such as *First, Next, Then, etc.*, in contrast, construct a story (Morgan, 1996b, p. 87).

The way in which reasoning is expressed in mathematical texts is realised through the use of different grammatical components such as conjunction (e.g. because, so); nouns (the reason is...); verbs (X causes Y) or prepositions (by, because of). Conjunctions and juxtaposition may also express causality among statements in

mathematical writing, and, because of the high regard enjoyed by deductive reasoning in mathematics, cause-result order is expected to be dominant.

Finally, Morgan (1996b) considers the overall structure of the text as an important feature of the textual meaning of the mathematical text. Special attention is given to the lay-out methods such as labelling, paragraphing or any other devices which indicate a flow or a change in the content or the style and which require attention of reader. These styles of writing may (or may not) meet the conventional styles recognised by mathematical discourse, but they will affect the meanings that the reader constructs when reading the text.

In other words, the position of textual elements in mathematical texts affects the meaning of a text. While that position is expressed temporally, in visual representation the composition is spatial. Kress & Van Leeuwen (2006) focus on this visual element, namely on the arrangements of components in the visual mode (images or diagrams) and in multimodal texts and on how cohesion is constructed (and realised) in visual or multimodal texts.

3.   The meaning of composition:

Kress & Van Leeuwen (2006) suggest three interrelated systems in investigating how a text is organised or arranged in order to communicate a coherent and meaningful message. These are:

1. *Information value*: the placement of elements in different 'zones' of a text (left-right, top-bottom, centre-margin) suggests different information values (given-new, ideal-real, centre and margins).

2. *Salience*: making some elements more salient ('eye-catching' or 'attracting the viewer's attention') than others contributes to their importance in the text. This can be done in different ways such as: colour, size, perspective, position.

3. *Framing*: separation (such as frame lines, white space, and colour) or connection (visual links and lack of framing) between elements in the text affects the unity of the message as a coherent unit of information.

Geometric (and mathematical) texts are inherently multimodal, where different modes of representation and communication are used such as verbal language,

algebraic notations and visual forms. The way in which the elements of a text are placed or arranged as a coherent and meaningful unity contributes to its meaning or at least has a meaning potential. In the following, I look at the realisations of these 'three principles of composition' in geometric texts in mathematics. Although I look separately at each principle, these principles are related and occur simultaneously.

Given and New: the information value of horizontal structure

In language, information is presented in a sequential or temporal structure in a way that signifies the interest of the author. In English, for example, 'before' and 'after' in speech are 'transcoded' as 'left' and 'right' in writing (Van Leeuwen, 2005). In Arabic, in contrast, this arrangement would be the other way around (right to left). The departure point of the message in English, thus, would be something common or known to the reader or agreed upon between authors and readers. That is the Given part of a message, while the end of the message, in contrast, would be what the author wants to prove or argue, i.e. the New part of the message which is not yet agreed upon by the reader. 'The exterior angle of a triangle' part of The Exterior Angle Theorem, for instance, is Given, meaning it contains common or shared knowledge between the author and the reader that would be used to prove the New part of the message, 'is equal to the sum of the two interior opposite angles'. Note that the Given here is the theme of the message or what the message is concerned with.

The exterior angle of a triangle   is equal to the sum of the two interior opposite angles

Given →   New

The sequential information structure in language is akin to the horizontal structure in visual representation, images or diagrams (Kress & Van Leeuwen, 2006). The Given in Figure 9-1 is the triangle ABC with specific properties (AB=CD=BC, $\angle A=\alpha$, $\angle B=x$, $\angle C=2\alpha$), and it is placed on the left side of the page. The New, in contrast, is placed on the right and poses the problem: prove that $x=120°-2\alpha$. The given part of the problem, the message, is not questioned or problematised. It is a part common to the author and the viewer of the diagram, something agreed upon. The new part, in turn, is something questioned which needs to be proven. The arrow connects the given and the new parts of the message, serving as an integrating device.
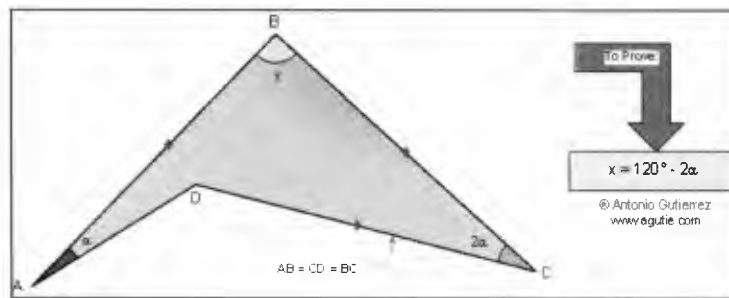
Figure 9-1: Given and New in geometric problems

In Arabic, where the writing is from right to left, the Given-New structure may be applied as well. Figure 9-2 is taken from a Palestinian geometry textbook, Grade 8, in which all the right parts are given and the left parts are new. The given part is verbal and offers information about the mathematical problem. The new part, on the other hand, is visual and may be problematised or debated; for instance, someone may draw it differently. I translated the first problem in Figure 9-3.
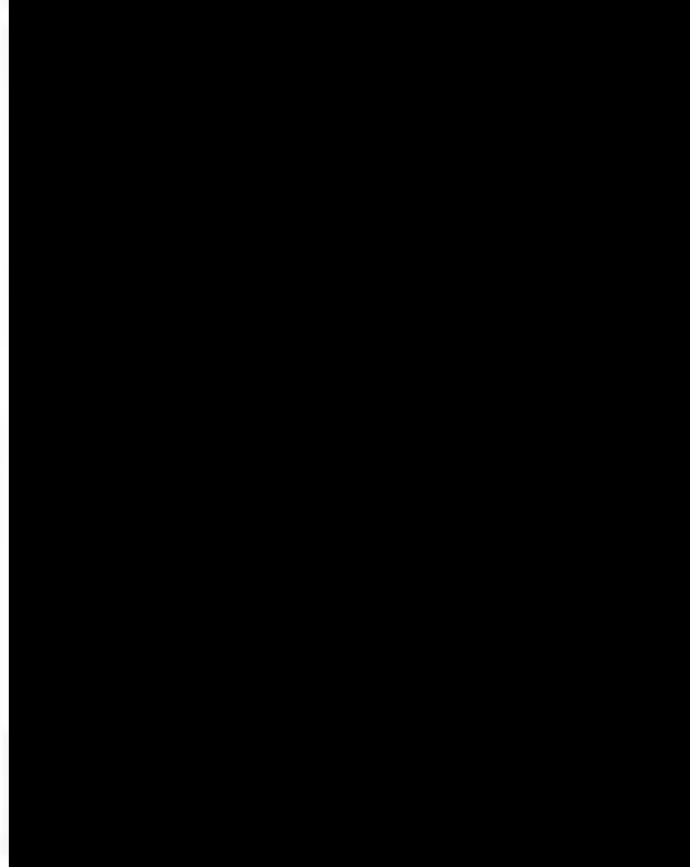
Figure 9-2: Given and New in an Arabic geometric text (Grade 8, part 1, p. 54)

In the next equilateral triangle ABC, AB=6 units. Find the following and justify:

a) The length of AC, the length of BD
b) ∠BAD, ∠CAD
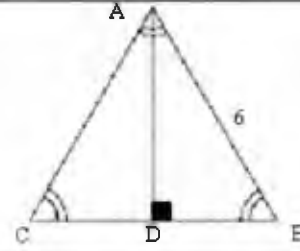
Figure 9-3: The first problem in Figure 9-2 (translated from Arabic)

This is also the case in pictures as in Figure 8-1 in the previous chapter. The laundry detergent itself is given, and its result, the new part, is shown on the left side of the image. The message the producer of the image wants to convey is something like: this laundry detergent is efficient (given), and here is the evidence (new).

In texts, Kress & Van Leeuwen (2006) argue, the New part can be Given for the next New in what they called 'cumulative Given-New structure' which may occur in language, in speech and writing, and in visual representations. Cumulative Given-New structure in writing occurs in a sequential (horizontal and vertical) form as in Figure 9-4.

Image redacted due to third party rights or other legal issues

Figure 9-4: Cumulative Given-New structure (Kress & Van Leeuwen, 2006, p. 185)

In geometry, ongoing verbal, visual or multimodal texts often take the form of a proof of a theorem or a solution of a problem and are presented in a way similar to the cumulative Given-New structure. Figure 9-5 shows how the New part becomes the Given for the information in a visual geometric text through the use of arrows and repetition as an integrating and cohesive device. The arrow connects Given 1 and New 1 in the first line. The repetition of the light-yellow colour in every diagram connects Given 1 and New 1, and it connects the first and second lines.

Figure 9-5: Cumulative Given-New structure in diagrams
(http://www.mathsisfun.com/triangle.html)

Ideal and Real: the information value of vertical structure

Again, in the left image of the 'laundry detergent advertisement' in Figure 8-1 in the previous chapter, the upper section has words that rhyme in Arabic ('what took you so long … others ruin the colours') as if the woman in the picture is speaking to the liquid itself. The lower section of the advertisement contains images of the liquid itself, the woman and a word (the name of the liquid in Arabic). In comparison with the horizontal structure (Given-New), this structure is vertical and contains fewer connections or less ongoing movement, or as Kress & Van Leeuwen (2006, p. 186) state:

> there is a sense of contrast, of position between the two [sections]. The upper section tends to make some kind of emotive appeal and to show us 'what might be'; the lower section tends to be more informative and practical, showing us 'what is'.

This structure is highly regarded in mathematics. Geometric theorems most often are placed at the top of mathematical texts, where they identify general attributes and relationships, something like an *ideal* situation, while their proofs are placed on the

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lower section showing how these general statements 'come to earth' or become *real* by, for instance, presenting examples. In Palestinian geometric textbooks, a typical design of a subject or a lesson would present the theoretical section, the ideal, (definitions or theorems) on the top of the text, at the beginning of the lesson, followed by an illustrative example or more, then some practice-and-drill problems followed by exercises or homework (see Figure 9-6).

Figure 9-6: 'Interesecting chords' lesson in a Palestinian textbook (Grade 9, part 1, pp. 70-71)

This practice was also dominant in participant students' texts that responded to the second task (Pf), as in Figure 9-7. The students privileged the verbal mode of representation and communication. Their texts would begin with writing a definition, a theorem or a general statement (the Ideal) followed by evidence which demonstrates that general statement.
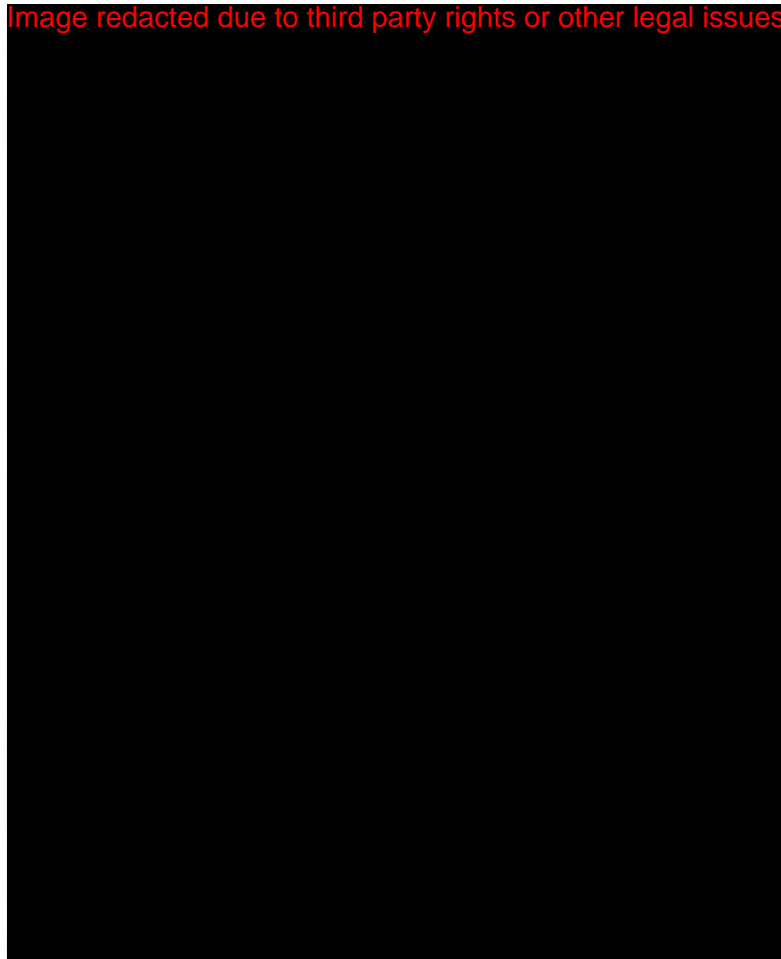
Figure 9-7: Verbal ideal in Ideal-Real structures in a participant's text (Year 8, Wendy, typical)

However, this practice may be contested by other authors or producers. Figure 9-8 shows a text that begins with visual examples and proceeds with the investigation of the argument towards a general statement, thus reversing the order of the Ideal and the Real sections. The author of that texts starts with diagrams, where the Ideal (or the theme, the concern of the message) is an investigation of the suggested geometric problem occupying half of the page, followed by a general verbal conclusion that is verified by a diagram.

The authors of the last two examples present the text as a message in different ways in terms of the 'thematic progression', 'the way in which reasoning is expressed' and 'the overall structure of the text', all of which together are realisations of the textual meaning and which construct a (verbal) mathematical text (Morgan, 1996b, p. 97). While I consider these issues in more detail in section 4 of this chapter in analysing two examples, the Ideal-Real structure presents the theme of the overall text when one reads or views it as a vertical structure. Moreover, authors of these two texts

express reasoning in different ways. In the text in Figure 9-7, reasoning takes the form of cause-result, while in Figure 9-8 it takes the form of result-cause.

Image redacted due to third party rights or other legal issues
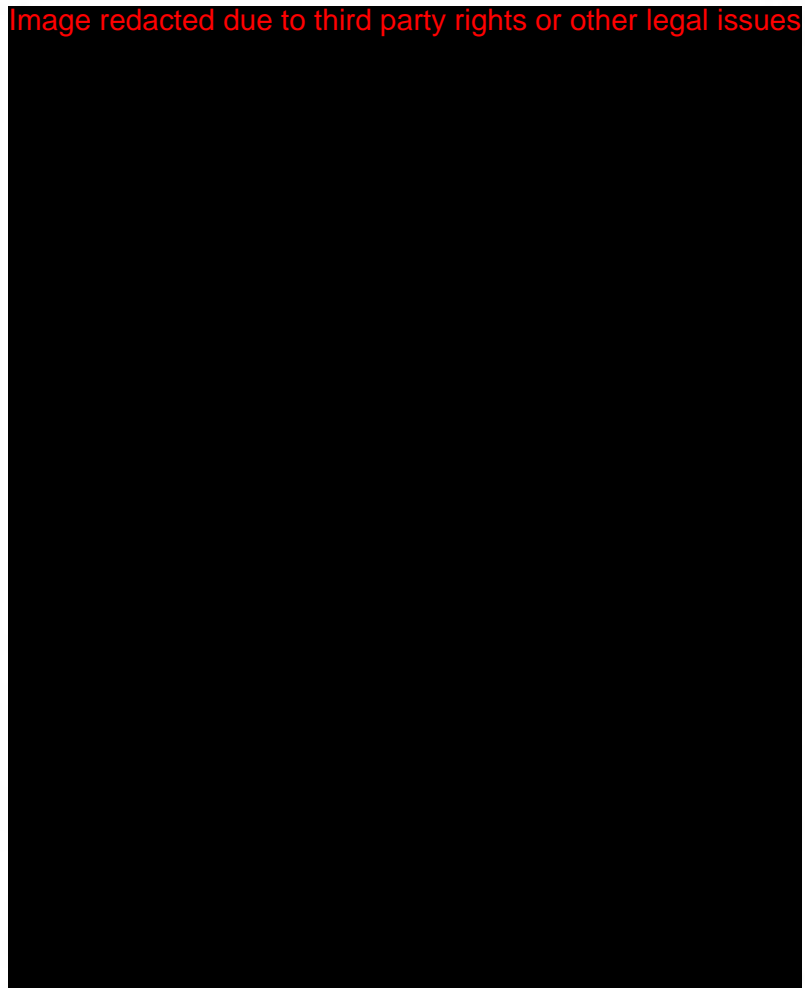
Figure 9-8: Visual ideal in Ideal-Real structures in a participant's text
(Year 8, unknown name, unique)

The information value of centre and margin

Besides the horizontal and vertical structures, information may also be structured through use of the centre and margin in visual composition. As Kress & Van Leeuwen (2006) comment, this composition is uncommon in Western visualisation but is important in Asian designs. In an ancient Chinese geometric text (100 BCE), *Zhoubi Suanjing* (*Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*), a proof of Pythagoras theorem, or *Gougu* theorem, is presented in Figure 9-9 which is a reproduction of the original text that can be found in Swetz & Katz (2009, p. 38):
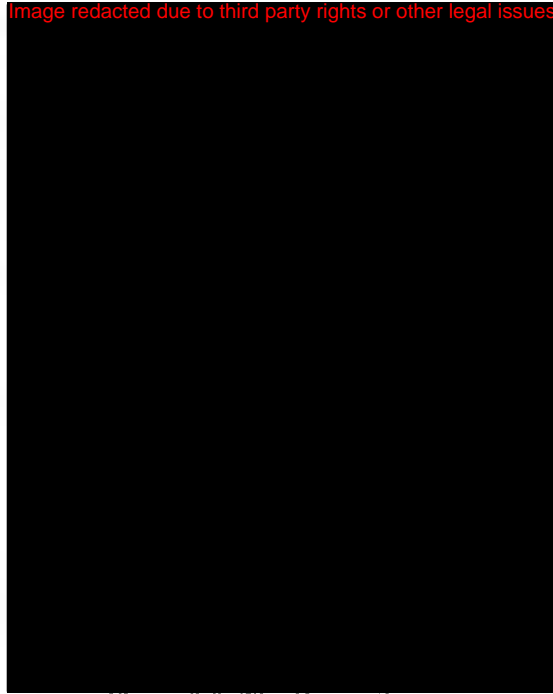
Figure 9-9: The Gougu theorem
(http://kaleidoscope.cultural-china.com/en/135Kaleidoscope5303.html)

The centre-margin composition may also be seen in a late 14th century manuscript of Euclid's *Elements* in Latin translation (Swetz & Katz, 2009, p. 44) in Figure 9-10. Four propositions from Book I are shown in writing in the centre surrounded by diagrams in margins. The element of the text which is placed in the Centre 'is represented as the nucleus of the information to which all other elements are in some sense subservient' (Kress & Van Leeuwen, 2006, p. 196). In other words, the focus of the message would be the centre element, while the marginal elements would be dependents.

This structure, furthermore, is similar to what Kress & Van Leeuwen (2006) referred to as triptych structure, where the page is divided into three sections. They show how triptychs combine Given-New, Ideal-Real and Centre-Margin structures. The structure of triptychs 'can be either a simple and symmetrical Margin-Centre-Margin structure [as in Figure 9-10] or a polarized structure in which the Centre acts as a Mediator between Given and New or between Ideal and Real' (p. 199) as in magazines and newspaper layouts.
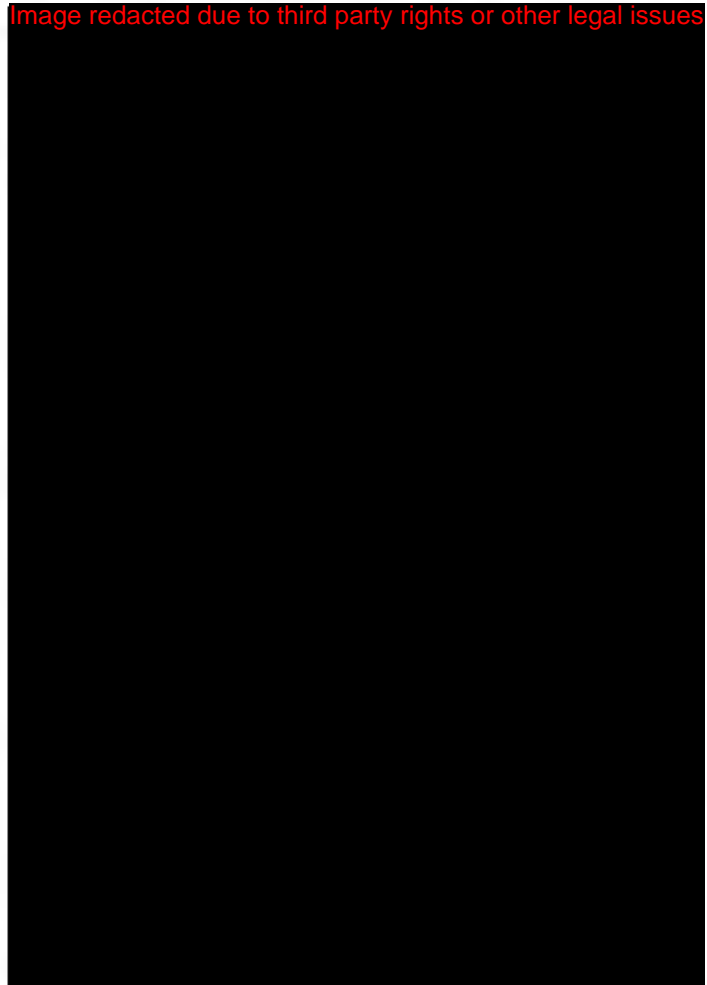
Figure 9-10: Centre-Margin composition in Euclid's Element
http://mathdl.maa.org/images/cms_upload/0800907159170.jpg

In conclusion, Kress & Van Leeuwen (2006) summarise 'the dimensions of the visual space' in Figure 9-11 in which they present the different structure of the information value in visual representations.
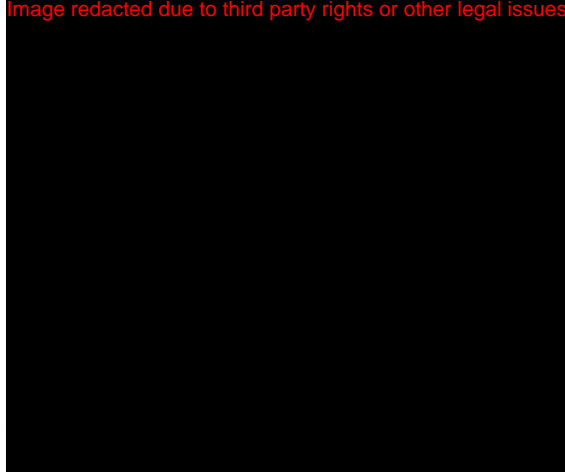
Figure 9-11: The dimensions of the visual space
(Kress & Van Leeuwen, 2006, p. 197)

Salience:

So far I have discussed how the position of elements in a text affects their information value in order to construct a coherent message. Salience may also suggest different values among the elements of the message, the value of importance and attention of the viewer. Thus making one element more salient than the others suggests that that element is more important and is worthy of more attention than others. An element may be more or less salient than another element, or both may be equally salient. In other words, the Given may be more important than the New, the New may be more important than the Given, or both may be equally important (Kress & Van Leeuwen, 2006).

In visual representation, salience occurs as a result of 'complex interaction' between a number of factors such as: size, focus, tonal contrast, perspective (foreground, background) and placement in the visual space (Kress & Van Leeuwen, 2006, p. 202). All these factors are related to the notion of *visual weight* and *balance* in visual art (Van Leeuwen, 2005), and, consequently, within the notion of salience, the 'heavier' the element, the greater its salience. Kress & Van Leeuwen (2006) and Van Leeuwen (2005) claim that balance in composition, together with rhythm in temporally integrated texts, play the crucial role in the aesthetic pleasure of texts. Balance is achieved when all the aforementioned visual factors cooperate to create a balance point, irrespective of whether that point lies in the centre of the composition. Therefore, introducing a heavy element would affect that balance, making the heavy element more salient.

Not all of these factors are relevant in geometric diagrams, but there are some 'visual cues' which are related to these factors and can distinguish salience in diagrams such as size, labels, colour, arrows, intensity of lines (dotted, thin or bold) and the placement of the visual space, as Kress & Van Leeuwen (2006) have suggested. Colours in Figure 9-1, for instance, draw attention to the figure itself and to the angles A, B and C. Arrows in Figure 9-1 & Figure 9-5 also attract viewers' attention. Furthermore, diagrams in the left side text in Figure 9-7 have salient features such as the thickness of lines in the first (upper) diagram and the size of the centre of the circle in the second (lower) diagram.

In multimodal mathematical texts, the placement of diagrams may be considered as a visual cue by which salience can be judged. Diagrams in Figure 9-6, for instance, occupy different positions in that text and, consequently, have different information value, which I have already discussed.

It is worth commenting that although the balance notion as a salient cue is important in visual art, the composition of elements in a text is not just about aesthetic pleasure or attracting the viewer, but it also arranges these elements into a meaningful and coherent text in order to communicate. Together with the informational value discussed in the previous sections, salience can show what message a text is trying to communicate, or, in other words, what a textual meaning may be. However, there is a third aspect of how a compositional meaning may be realised: framing.

Framing:

Textual aspects are about information in a text: information value, which is related to the way information is positioned, and information importance, which is related to salience. Framing, as the term itself indicates, deals with the way in which information is disconnected or separated or, alternatively, connected or joined together as a unit. Like salience, visual framing is a matter of degree, i.e. elements of a text may be strongly or weakly framed (Kress & Van Leeuwen, 2006). There also are some visual cues that help us judge disconnected and connected framing.

The framing of an element suggests a disconnected or a separated unit of information that is created through different visual cues such as: frame lines (thick or thin framing), empty or white space between elements, discontinuities in colour or any other form of perceptual discontinuity (Jewitt & Oyama, 2001; Kress & Van Leeuwen, 2006). The theorem in the right-side text in Figure 9-6, for instance, is framed and separated from the rest of elements. Similarly, texts in Figure 9-7 are divided into three different pieces or 'identities' of information through separation achieved by an empty space between them.

The absence of framing signals a 'group identity', a unit of information or similarities between the elements. Connectedness can be realised by simply doing the opposite of what is done to realise discontinuity: through vectors that connect elements, through similarities of colour and forms and through the absence of frame lines or empty

space (Jewitt & Oyama, 2001; Kress & Van Leeuwen, 2006). Figure 9-5 is an example of connected information where vectors, colour and a similar form (triangles) contribute to the construction of a coherent text. First, there is no framing in it. Second, the arrows connect the first and the second diagram in the first line. Moreover, the use of the same colour connects the four diagrams as belonging to the same unit of information (proposition).

4. The creation of mathematical texts

The main focus in the previous part of the section was on visual cohesion in texts, meaning how a text is arranged spatially on different sections of a page. We saw that some elements will be on the left (Given in the Western culture) or on the right (New), or in reverse order in right-to-left languages such as Arabic, others will be on the top (Ideal) or the bottom (Real), and other elements will be in the Centre and/or in the Margins. We also saw that salience contributes to the information importance of the elements, while framing creates a sense of discontinuity or connectedness between elements of a text and constructs the separation or the unity of elements.

The discussion so far has drawn on the work of Kress & Van Leeuwen (2006) which focuses on visual cohesion and visual representations. Halliday's SFL, in contrast, as presented in the introduction section above, focuses on verbal cohesion and verbal representation. In addition to the several examples that have been presented illustrating information value, this section focuses on how elements of information, whether visual or verbal, are linked to each other in (mathematical) composite or multimodal texts. In doing so, the two parts — the visual and the verbal – discussed so far will offer an analytic tool to analyse the textual meaning, which is the concern of this chapter. The analytic tool I will construct, moreover, will be used to analyse two examples at the end of this chapter.

Information linking: the relationship between the visual and the verbal

Adopting Halliday's SFL, Van Leeuwen (2005) suggests that there are two main relationships between two items of information: elaboration and extension. In elaboration, an item 'elaborates on the meaning of another by further specifying or describing it. ... [in other words, one] provides a further characterization of one is

already there, restating it, clarifying it, refining it, or adding a descriptive attribute or comment' (Halliday, 1985, p. 203). In other words, there can be different methods of elaboration such as reformulation, exemplification, specification, summary and correction (Van Leeuwen, 2005). In extension, an item 'extends the meaning of another by adding something new to it. What is added may be just an addition, or a replacement, or an alternative' (Halliday, 1985, p. 207).

According to Kress & Van Leeuwen (2006), the approach of Roland Barthes represents the classic semiotic work in the relationship between the verbal and the visual, or the word and image. The two main types of relationships which Barthes considered are anchorage and relay (Table 9-1). He argued that anchorage is a specification relationship in which the image comes first, and the word comes to specify the image or to 'fix' it. Relay, in contrast, is a complementary relationship between the image and word in which each of them contributes to the meaning of the message. In other words, anchorage is akin to the elaboration relationship, while relay is akin to the extension relationship.

There are other types of elaboration, such as illustration, in which the text comes first and the image makes it more specific, and explanation, in which the word paraphrases the image or vice versa. The extension relationship has other forms as well, such as similarity, in which a word is similar to the image or vice versa, and contrast, in which a word contrasts the image or vice versa.

Table 9-1: Overview of visual-verbal relatioship (Van Leeuwen, 2005)

Type of relation	Subtypes	Meanings
Elaboration	Specification	The image makes the word more specific (illustration)
		The word makes the image more specific (anchorage)
	Explanation	The word paraphrases the image (or vice versa)
Extension	Similarity	The content of the verbal text is similar to that of the image
	Contrast	The content of the verbal text contrasts with that of the image
	Complement	The content of the image adds further information to that of the verbal text, and vice versa (relay)

Although Kress & Van Leeuwen (2006) draw on Barthes' notions of anchorage and relay and incorporate these into their scheme of verbal-visual relationships, they oppose his privileging of words over images. They make their theoretical position clear about the visual representation, the verbal representation and the relationship between them by stating in a number of places, including in the following quote, that:

> the visual component of a text is an independently organized and structured message, connected with the verbal text, but in no way dependent on it – and similarly the other way around. (p. 18)

The relationships between the visual and the verbal will be illustrated in the following analysis of two mathematical texts that the participant students in the current study produced in response to Task 2 (Pf, see Chapter 4). The first one is in English produced by a student in Year 8. The second is an Arabic text produced by a student, Sami, in Grade 8. In each example I will look how a mathematical text is constructed through three interrelated analytic tools or on three levels: 1) the visual analysis suggested by Kress & Van Leeuwen (2006) as mentioned earlier in this chapter, namely, how information is arranged in the text (the value of information, salience and framing); 2) the verbal analysis suggested by Morgan (1996b) as presented in the beginning of this chapter, namely, thematic progression, the way in which reasoning is expressed and the overall structure of the written text; and 3) the interaction between the visual and the verbal as presented in this part of the chapter, namely, the relationship between diagrams and words. The analysis will consider all of these tools together, as interrelated tools toward the creation of a mathematical text. Before presenting the analysis, I would like to recall Task 2 of the current study (Pf) which asks the students to investigate the claim made by a student, Darren: 'Whatever quadrilateral I draw with corners on a circle, the diagonals will always cross at the centre of the circle.' The two examples show how the participant students respond to this task focusing on the diagonals of the quadrilateral and the circle of the centre.

Example 1 (Figure 9-12):

The verbal parts of the text in Figure 9-12 read as follows:

I think that he is wrong because it says he sketched it and it could be any sized [size] or anywhere in the circel [circle]. it could not be a regular quadrilateral.

eg [e.g.]:

[diagram]   it doesn't work as the quadrilateral is a nonregular shape and is not placed anywhere near the middle point.

[diagram]   if the sides are parallel and it is placed in the middle of the circle it does work.

Looking at the overall structure of the text, we might note that it is divided into three horizontal equal sections; the upper section is verbal, and the second and third sections are multimodal, including a diagram on the left and writing on the right. Each of these sections is disconnected from the other by white space constructing a frame around each of them. The upper verbal section represents the Ideal part of the text, the theoretical part of it. The author of that text thematises her position towards Darren's claim: *'I think that he is wrong'*. The use of *because* suggests that a reasoning-type text is (will be) presented.

In the next student text to be considered, the author provides different examples to demonstrate the theoretical position expressed in the Ideal section. The second example is placed in the Real section, on the bottom of the page contrasting the Ideal position and showing that Darren's claim may work *if* some conditions are achieved. The first example, on the other hand, is in the centre of the text, playing the role of 'Mediator' between the two stances of the Ideal and the Real.

In the Ideal section, the author's position is thematised, or it takes the position of the Given in the statement, the departure point about which all agree, while the reason for that position (*because* ...) is presented as New that can be problematised and challenged. The New needs more investigation, and this is exactly what the author does by giving two examples to illustrate her point.

Image redacted due to third party rights or other legal issues
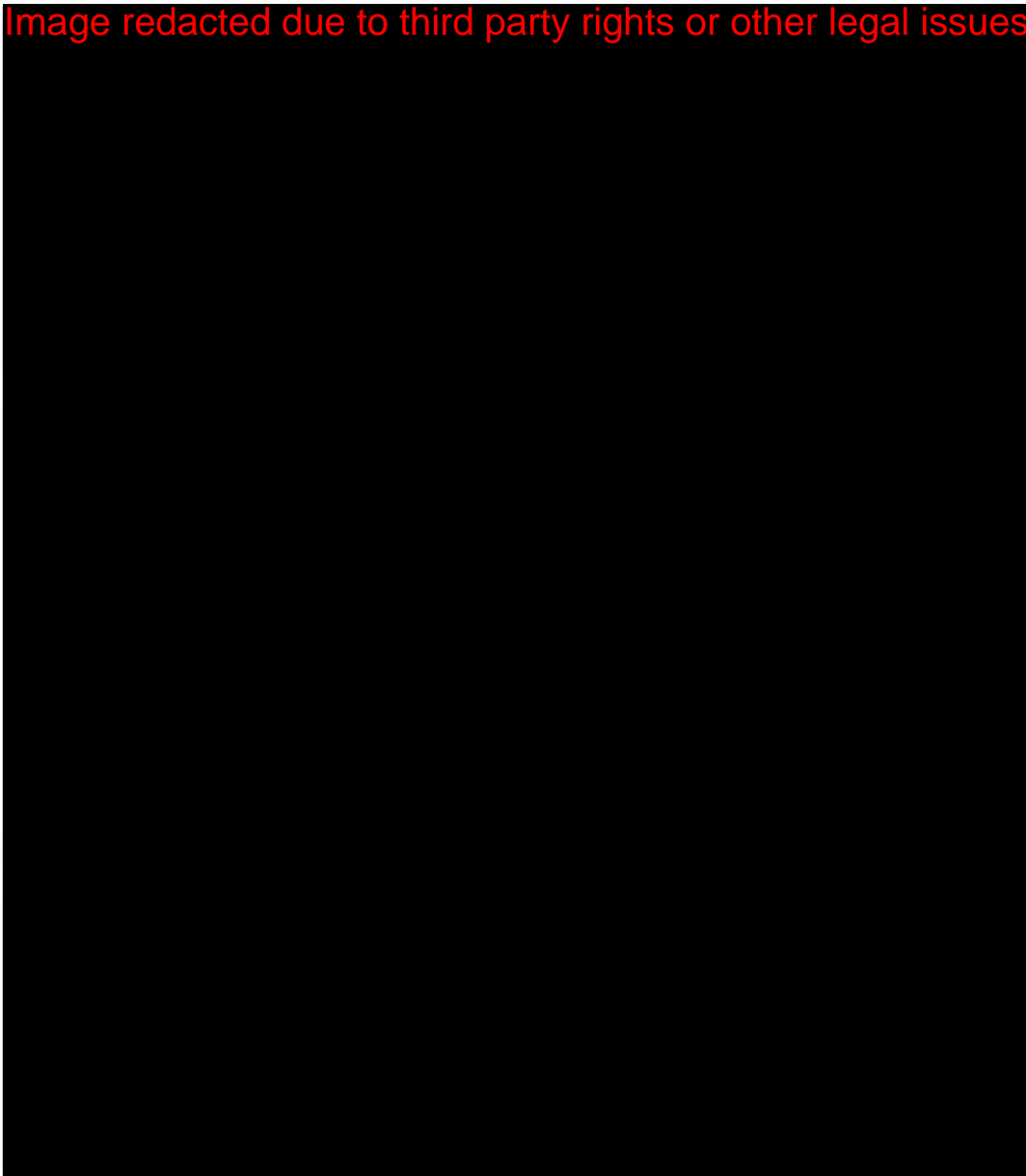
Figure 9-12: An English investigation text of Task 2 (Year 8, unknown name)

The way in which the first example is presented suggests that this example is a meaningful message in itself; it is separated from the rest of the text by a white space (framing), but at the same time it is a coherent text and constructs a unit of information realised by the vector between the diagram and the verbal. The diagram, the Given element, shows that the diagonals of the quadrilateral do not cross the centre of the circle, which is salient in the circle. This situation is taken as granted, as a departure point from the diagram in order to present the content of the verbal. The verbal, the New element, 'paraphrases' the diagram. The entire content of the verbal text is presented in the diagram, establishing an (elaboration) explanation

relationship between the verbal text and the diagram. The author emphasises this relationship using the direction of the vector.

In other words, the diagram and the verbal text construct a coherent message that says: here is an example which shows that the theoretical position presented in the previous paragraph (the Ideal), Darren's claim, does not work in this case, and that consequently – his claim is wrong.

Similarly, the student uses white borders around the second example to present it as a separated message, but a coherent one as well, by drawing the vector which connects the diagram and the verbal text. However, it contrasts the theoretical position in the Ideal part and shows the status of Darren's claim *if* some conditions are realised. Therefore, the content of the diagram and the content of the verbal text in this example are presented differently. The salient part in the diagram (the Given element) is the thick lines representing diagonals of the quadrilateral which cross the centre of the circle. Moreover, the diagram together with the verbal text show some details of that quadrilateral, including equality labels and *parallel* signs which suggest a specific type of quadrilateral, a regular quadrilateral, which is used to contrast the statement in the Ideal section.

In this example, to conclude, the analysis of this text shows how the arrangement or the composition of the elements of a text contributes to the textual meaning of a mathematical text through the progression of the theme or the expression of reasoning and 'the overall structure of the text' (Morgan, 1996b, p. 97). A final comment on the two examples presented here is about the direction of the arrows that connect the verbal and the visual parts. While it is expected that the reader/viewer would read the text from left to right, the vector in both examples emanates from the verbal toward the diagram, suggesting another direction for reading the text, in which the Given is the written text, and the New is the diagram.

Example 2 (Figure 9-13):

The text in Figure 9-13 consists of two sections, a diagram and a verbal text reading
(I translate from Arabic, which is written right to left):

> You conclude that the above diagram was quadrilateral and its vertices were
> on the circumference of the circle, but its diagonals have not crossed the
> centre, therefore what Reem [the Arabic substitute for Darren] said is wrong.

The overall structure in which the text is presented may be seen as a triptych in
which a diagram and a verbal text (with an arrow to the right) are presented in the
central part of that triptych, and the other two parts are white spaces. This structure
signals the coherence of the message presented in the text, as we will see below. The
diagram, the Ideal part, is salient in position and size. It is the point of departure of
the text; it is the concern of the message which shows the relationship between the
quadrilateral and the circle. The diagram itself presents an example which meets the
conditions suggested in the task but does not agree with the result that Darren/Reem
concludes: the diagonals of the quadrilateral do not cross the centre of the circle. The
author expresses this by emphasising the two points (the centre of the circle and the
crossing/intersection point of the diagonals of the parallelogram) in the diagram,
thereby making them salient. Moreover, the diagram occupies nearly two-thirds of
the space of the text. This, again, signifies the important value of this information in
comparison to the verbal, which occupies one-third of the text.

The theme in the verbal reveals the concern of this piece of information, namely the
conclusion which may be made according to the diagram (see the discussion about
the arrow between the diagram and the verbal text below):

The written text starts by saying:

> *you conclude that* [in Arabic it is one word, *Tastantij*]

then paraphrases the content of the (*above*) diagram and present this content as Given

> *the above diagram was quadrilateral and its vertices were on the*
> *circumference of the circle, but its diagonals have not crossed the*
> *centre,*

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in order to make the conclusion, the New

therefore what Reem [Darren] said is wrong.

The main message that the verbal text says, is, in other words, something like: because of what is given in the diagram, Reem's [Darren's] claim is wrong.

The overall structure of the text contributes to the coherence of the message as a unit of information. The diagram and the verbal text are connected by an arrow and by the way in which the verbal is presented. The arrow emanates from the right as a continuation of the diagram, similar to an *'if ..., then ...'* statement, which suggests a causal relation between the diagram and the verbal text. Thus, the arrow connects the diagram and the verbal as a unit of information that should be considered together. Moreover, the way in which the verbal text is written contributes to the unity of the message as well. The verbal text is arranged in an arrow-like way, with wide lines at the bottom that become narrower, closer to the diagram. In other words, the verbal text itself is linked to the diagram. This, indeed, adds to the centrality of the diagram, as discussed earlier, in this text to presenting the solution of the task.
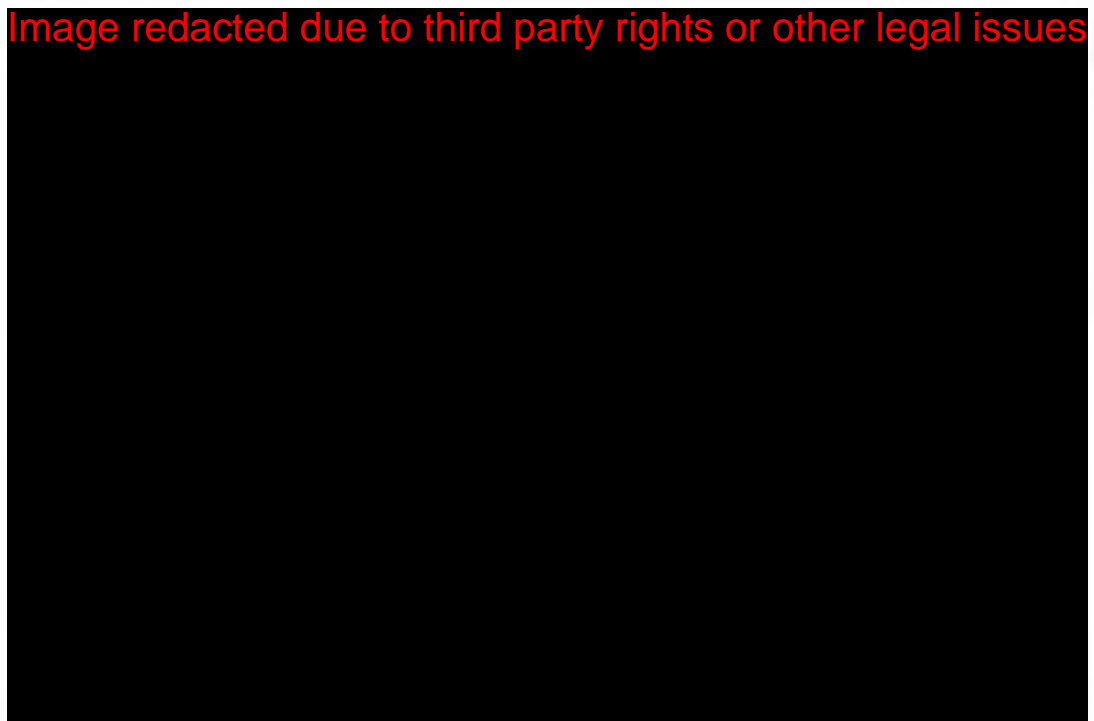
Figure 9-13: An Arabic investigation text of Task 2 (Grade 8, Sami)

It is worth comparing the two examples, since each of them is presented in a different language and comes from a different culture. In doing so, I present an

illustration of the difference rather than generalising, which would require further investigation and examination. Example 1 presents its argument in result-cause order from the beginning of the text, *I think that he is wrong because*, while Example 2 presents its conclusion at the end of the text, in a cause-result order, *thus what Reem [Darren] said is wrong*. Because of the high status of deductive reasoning in mathematics, we would expect to see more emphasis on the cause-result order in mathematical texts realised through analysis of the textual meaning, especially regarding the expression of reasoning in the text. Morgan (1996b) argues that deductive reasoning has high status in mathematics, creating a tendency for doers of mathematics to privilege cause-result order. In a personal communication, however, she says that theoretical knowledge, in which result-cause order is privileged, is also highly regarded by mathematicians. Therefore, one may expect to see both orders but possibly in different contexts and, hence, possible differences between spoken and written responses, school versus research mathematics, or different kinds of school curriculums. This might be the basis for a hypothesis to account for differences in the balance between the two orders across cultures (Morgan, 2010, personal communication).

This indeed has consequences for the way in which the texts would proceed. Example 1 presents its argument as an investigation process, using more words and more diagrams than Example 2, which presents its argument as a product. The former discusses two different possibilities to illustrate its stance and the latter presents one possibility that is a counter example.

These two issues, the order of the argument and the way in which a text proceeds, may signal the way in which mathematics is perceived or constructed in different educational systems (or cultures). Most of the participant Palestinian students presented their texts as products in cause-result order with one diagram and few words, as in Example 2, while most of the English students presented their texts as investigation or trial and error processes with more diagrams and words. Does that mean that mathematics in the OPT, educationally and culturally, is perceived differently from the English context? The educational system in the UK began officially encouraging an investigation attitude toward mathematics twenty years ago 'through the implantation of the curriculum development of 'coursework' as a component of the General Certificate of Secondary Education (GCSE) examination'

(Morgan, 1996b, p. 1). It is only in recent years (since 1999) that Palestinians started to write their own textbooks (see Chapter 4). Because Palestinians have lived under so many different foreign authorities, each of which had its own educational system (such as Ottoman, British, Jordanian, Egyptian, and Israeli occupation), the view of mathematics is isolated from the living reality in the OPT (Fasheh, 1997). Palestinian textbooks are content-focused, and there is little research about the social aspect of mathematics. Generally, the situation has not changed since 1999, and mathematics is still perceived as a formal, definite, impersonal and symbolic subject.

5. Summary

This chapter dealt with the third meaning a text may fulfil in order to construct a meaningful message, namely textual. The textual meaning is realised differently through visual representation and communication and verbal representation and communication, respectively. As was the case for the other meanings discussed in the previous chapter, I took as the departure point the Hallidayan SFL framework and then moved to the work of Kress and Van Leeuwen (2006) about visual representation, while at the same time considering Morgan's linguistic framework (1996b) to read mathematical texts.

Two main issues were discussed in the chapter: the meaning of composition and the creation of mathematical text. The meaning of composition dealt with the information arranged in texts spatially: left to right, top to bottom, centre and margin, salience and framing. The discussion focused on how these arrangements contribute to the meaning potential of the elements of a text.

I addressed the creation of mathematical texts in the last part of this chapter, in which I highlighted the relationship between the visual and the verbal followed by an analysis of two multimodal mathematical texts produced by participants in the current study. In this particular part, the work of Morgan (1996b) informed my attempts to construct an analysis of the two examples in the last section of this chapter. I have looked in detail at the elements which have been thematised, either the diagram or the verbal, and also where these elements are placed in the 'visual space' suggested by Kress and Van Leeuwen (2006). I also looked at the way in which reasoning is expressed in the two examples. One main difference I have

identified between the English text and the Arabic text is that the former used result-cause order, while the latter used the cause-result order. In other words, the framework suggested by the current study enabled me to make distinctions between different kinds of multimodal mathematical texts that have significance beyond the texts themselves.

This analysis, however, focuses only on the textual meaning of the diagrammatic mode, which does not occur in isolation from the other two meanings, the ideational and the interpersonal. The next chapter considers this challenge.

10 Multimodal communication and representation: An analysis

1. Plan of the chapter:

The last four chapters (6-9) discussed the role of the diagrammatic mode in constructing mathematical meanings. While Chapters 6-8 focused on the ideational and the interpersonal meanings realised in diagrams alone, the previous chapter, Chapter 9, dealt with the textual meaning in mathematical written texts, considering the verbal mode as well as the diagrammatic. Mathematical communication and representation, however, most often occur in a context wider than just the written text, such as oral discussion, for example, which involves more than two modes. In this chapter, therefore, I look at how three mathematical modes of communication and representation may interact to construct mathematical meanings. These modes are: the diagrammatic; the verbal; and the gestural. Since the first two modes have already been developed (the diagrammatic in this study and the verbal in Morgan (1996), both adopting SFL approach), there is a need to develop a framework for the gestural mode. This will be the first step of this chapter.

The second step will be analysing an episode of the participant students in the current study. In doing so, I aim to achieve two goals: the direct one is to demonstrate how both of the frameworks suggested by this study, the diagrammatic and the gestural, may be applied in a wider context of communication and representation. The indirect goal is to show the complexity of mathematical communication which happens in a class and, consequently, to argue that more attention is needed to the way in which students interact in, and with, mathematics.

2. Introduction: the multimodal nature of communication

So far I have dealt with the diagrammatic mode through the previous chapters in order to develop a framework for reading geometric diagrams that is similar to the way in which other studies make use of the verbal mode to read mathematical texts using the SFL approach (e.g. Morgan, 1996b). However, 'communication is always and inevitably multimodal' (Kress, 2005, p. 5), where meaning is constructed through

different modes occurring simultaneously. In other words, the ensemble of modes (multimodality) of representations and communication, rather than one mode (usually the verbal, spoken or written), is actually the carrier of the unified meaning (Kress et al., 2001; Lemke, 1999). Thus, the way in which modes are combined or interact and their contribution to the unified meaning should be considered. Furthermore, because people communicate to effect change on their social worlds, these different modes of communication are also functional (Morgan, 2006).

Mathematics, as a socio-communicative practice, is a multimodal discourse, as I argued in this study and elsewhere (see Chapter 2 in this study and Alshwaikh, 2009), where different modes of communication take place including language, diagrams, gestures and algebraic notations. These different modes may offer different meanings, or they may convey one set of meanings (Kress & Van Leeuwen, 2006).

As a result, we must consider gestures in analysing the communicative act in mathematics discourse as well as in other contexts. That is what I intend to do in the following section. My plan is to develop a means of analysing the contribution that gesture makes to the overall meaning, just as we have a means of analysis for the verbal and diagrammatic forms.

3.   Gestures as a mode of communication

During my iterative watching of the video records of students' communication about the two geometric tasks of this study, I noticed their extensive use of gestures, such as using their fingers to point to specific parts of a diagram (the length of a side, for instance), their hands to draw diagrams or to indicate some aspect of the problem or solution (showing the traces of the water in Task 1) and/or their use of artefacts (ruler, compass, pencil, etc.). This led me to try to develop a preliminary analytic framework (see Table 10-1) using a social semiotics approach, as I did with the diagrammatic mode. Because of the time constraints inherent in a PhD study, I have developed only one aspect of the intended framework for gestures, which is the ideational, hoping to pursue this endeavour more fully later. Furthermore, I must admit that I am including only a limited variety of gestures in the suggested

framework, which I consider to be only the first step in developing a more detailed framework.

Similar to the diagrammatic framework developed in the previous chapters of the current study, the suggested gestural framework distinguishes between two types of gestures: narrative gesture and conceptual gesture. The main distinction is the movement of fingers or hands. If there is a movement, a dynamic motion in producing the gesture, then I consider the gesture to be narrative, in which a story is being told by showing the structure of a constructed product. Conceptual gestures, in contrast, lack that action or motion and appear static, in order to refer to a 'presupposing' (Haviland, 2000) or pre-existing object.

While the suggested framework is preliminary and limited, this distinction between narrative and conceptual is consistent with the suggested framework for the diagrammatic mode and may help offer a coherent analysis for multimodal communication. Other distinctions in the studies about gestures (see Chapter 4) adopt the Peircean classification of signs; iconic, indexical and symbolic (e.g. McNeill, 1985; Streeck, 2008). Among these classifications, the indexical gesture is the most recognized type, discussed prominently in research about communicative practices in people's daily lives (e.g. Haviland, 2000) or in specialised communication such as mathematics learning (e.g. Radford et al., 2007). Streeck (2008) identifies 12 methods or heuristics of depictive gestures, Bjuland, Cestari, & Borgersen (2007) found *sliding* and *pointing* are prominent in students' interactions while solving mathematical tasks, and Morgan & Alshwaikh (2008) distinguished between two strategies, *imaging* and *imagining*, that students used in the context of a teaching experiment to explore three-dimensional shapes. Wherever they were applicable to the set of data I collected, these studies informed the suggested distinction in the current study.

In the following, I consider each of the gestures with illustrative examples, and then present a multimodal analysis for an episode from the study, taking into consideration three modes of representation and communication: verbal, diagrammatic and gestural. Before doing that, however, I want to provide additional justification for the distinction between narrative and conceptual gestures. In the diagrammatic framework, the distinction between narrative and conceptual diagrams was based on the presence of temporal factors. The challenge here is that gestures

occur in time. That challenge informed my thinking in the distinction between narrative and conceptual gestures, and I found the movement (or motion) to be a useful distinction.

Thus the distinction is based on whether there is an action represented in the gesture. Since my interest is the content of the represented action, I make a further distinction, looking closely at whether the action represents a process or an object. Thus, if a student repeatedly traces an imagined mathematical object (a segment line, for instance) I would consider that action to represent both object and process. I would say that at the time the object was depicted with no motion, the gesture was conceptual. At the moment the gesture starts to express motion, it becomes narrative.

3.1   Narrative gestures:

Narrative gestures are distinguished by the presence of the *movement* of fingers or hands (or artefacts). I have identified three types of processes based on the activity they indicate: drawing, symbolic and modelling.

3.1.1   Drawing/sliding:

This type of gesture is made by moving one's index-finger (or an artefact, a pen for example) forward and backward along a line (or other parts of a diagram such as an arc). The forward-and-backward motion suggests a mathematical meaning of measuring the length of that line. Ruth, for instance, in Figure 10-1, reading from left to right and top to bottom, moved her pen several times over the side of a trapezium (I added arrows to show how she moved her pen) in a gesture that appears to refer to the length of that side. Usually the single indexical gesture is accompanied by verbal demonstratives such as *this* and *that*.

Figure 10-1: Drawing/sliding as a narrative action

Two fingers (most often the index will be one of them), furthermore, may be used to embody an arc or a curve rather than a straight line. Lionel, for example, referred to the radius of a circle using two fingers; one (the index) represents the centre, while the other (thumb) was moving, representing an arc (Figure 10-2).

Figure 10-2: Two fingers drawing a radius

Sliding gestures are akin to material processes in the transitivity system which, among other processes, realise the ideational meaning in language (Halliday, 1985), and, at the same time, are akin to the (measurement) arrowed diagram, embodied by the presence of bidirectional arrows, in the ideational (representational) meaning of diagrams. Measuring the length of a side is an example (see Figures 6-8&9 in Chapter 6) of this type of gesture, in which the bidirectional arrow may be (or may not be) accompanied by words (*measure, the length of the side*), as an embodiment of a (narrative) sliding gesture.

3.1.2 Symbolic:

A second type of narrative gestures is distinguished by the use of symbolic gestures or processes. According to the Peircean classification, symbols are signs that are related to their objects by conventions (Peirce & Buchler, 1955) such as words. A symbol refers to what the object is or means (Kress & Van Leeuwen, 2006). For example, the word 'triangle' is a symbol of an object called triangle; the triangle word itself is a signifier, and the shape triangle is the signified (meaning). An example of a symbolic gesture is when two fingers or hands are *moving* while keeping a fixed distance between them. The gesture itself (moving the two fingers of hands) is a signifier, and the (mathematical) meaning is parallelism (verbally, something like 'two lines never meet') which is a geometric property. The gesture may be made using two fingers from the same hand (Figure 10-3a), using one finger from one hand (Figure 10-3b) or using two hands (Figure 10-3c).

(a)　　　(b)　　　(c)
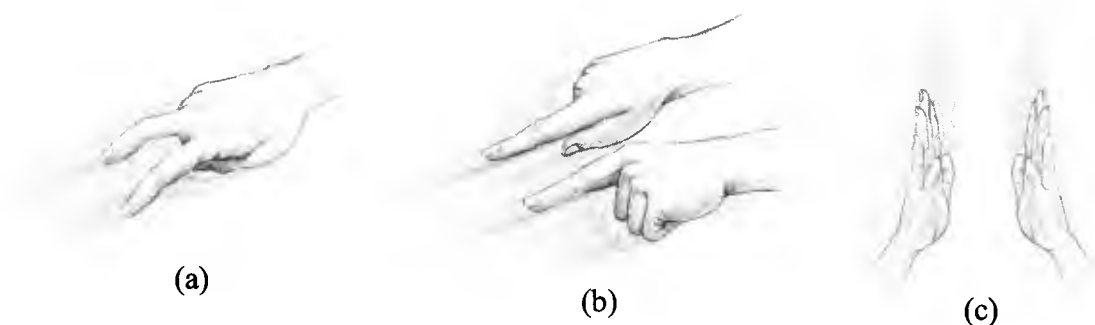
Figure 10-3: Symbolic gestures showing parallelism

3.1.3 Modelling:

Sliding may also construct images or diagrams when a finger or hand has a starting and ending point, a closed area. Modelling may be indicated by the use of one or two fingers or hands. Richard, for example, used his pen to refer to a triangle in his discussion about Task 1 (see line 10 in Table 10-2 and Figure 10-11). Another student, Timmy, in a different episode, made use of his right and left index fingers to draw a figure that he couldn't name. He put his two index fingers in contact as a starting point, moved each of them apart in a straight line (one to the left and the other to the right), slid each of his two fingers downward, and then slid them toward each other again (one to the right and one to the left) until they met, thus 'closing' the diagram (Figure 10-4).

Starting points
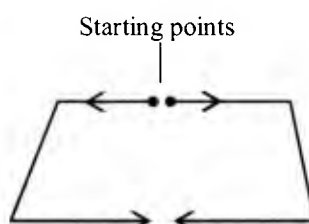
Figure 10-4: Modelling a diagram by two fingers or two hands

3.2 Conceptual gestures:

Conceptual gestures are distinguished by the lack of motion. It is interesting to note that when the movement in a gesture ceases or is frozen, a narrative gesture becomes conceptual, and instead of referring to mathematical action, it refers to pre-existing mathematical objects. As I have done for narrative gestures, I have identified three types of processes occurring in conceptual gestures: pointing/indexical (referring to

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part or the whole of the diagram), bounding (referring to a fixed length) and still-modelling (using a hand or hands to refer to a diagram).

3.2.1 Pointing/indexical:

In general, an indexical gesture is distinguished by the use of the finger (especially the index finger) or hand(s) to direct the attention of the viewer/or interlocutor toward something. As Haviland (2000) states:

> [p]ointing seems a straightforward matter: you stick your finger out in the appropriate direction, perhaps saying some accompanying words, and your interlocutors follow the trajectory of your arrow-like digit to the intended referent. (p. 14)

But pointing is not always simple and straightforward, especially when there is a need to specify where exactly to direct the attention, in order to be accurate. Some studies report the use of index/pointing gestures in mathematics. Bjuland, Cestari, & Borgersen (2007), for instance, distinguished between students' use of pointing and sliding as prominent gestures in solving mathematical tasks. Moreover, Radford, Bardini, & Sabena (2007) identify several indexical gestures in students' algebraic generalisation.

I use the indexical/pointing gesture to refer to the use of a finger (index) or hand(s) to point at an 'presupposed' (Haviland, 2000) object. In the mathematical context, this gesture is realised by using the (index) finger (or an artefact such as a pen) to point at a diagram or part of it. This is manifested by pointing one finger either to the whole diagram or to specific parts of it (Figure 10-5). The gesture is usually accompanied by the use of demonstratives such as *this* and *that*.

Figure 10-5: Pointing at the whole diagram or parts of it

3.2.2 Bounding:

Bounding is indicated by the use of two fingers or two hands to embody a fixed length or distance encircled between the two fingers or hands (see the schematic drawings in Figure 10-6). I note that bounding is distinguishable from the symbolic gesture indicating parallelism described in section 3.1.2, which is a narrative gesture, by the absence of motion. In Figure 10-6, Lara uses her two fingers to make the gesture of bounding, in order to refer to the length of EM in Task 1.

Figure 10-6: A fixed distance (length) between two fingers/hands

This type of gesture is visually similar to the way in which the length of a segment is shown traditionally in mathematical texts (Figure 10-7), where fingers or hands are represented either by two points or two little vertical signs.
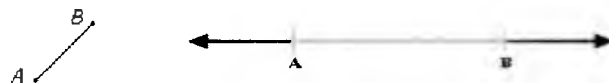
Figure 10-7: Length of a segment as shown visually in mathematics texts

3.2.3 Still-modelling:

Modelling in narrative gesture took place when students constructed an image of a diagram by sliding two fingers or hands and drawing an image of a diagram. Still-modelling, on the other hand, is modelling without motion, in which the gesturer indicates a diagram using fingers or hands. A student, Lionel, held up two fingers (the index and the thumb) in both hands referring to a square (Figure 10-8) that had been presented earlier in his discussion with his colleagues about Task 2 in the current study.

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Figure 10-8: A 'gestured' square

As a summary, narrative gestures are realised by the action of doing rather than referring to an object, while conceptual gestures, in contrast, refer to the mental image of an object, rather than indicate an action. As in the diagrammatic and the verbal modes, mathematical activity may be represented in the gestural mode. Table 10-1 summarises the types of gestures, the processes they involve, and how they are realised to achieve ideational meaning in the mathematical context.

Table 10-1: Ideational meaning in gestures

Gesture type	Type of process	Realisation	Types of mathematical meaning
Narrative gesture: Dynamic: ongoing action	• Drawing/sliding	• One finger[21] or two slide (forward and backward) over a specific side of a diagram embodying an action of measuring the length of the side (Figure 10-1, Figure 10-2).	• Measuring
	• Symbolic	• Two fingers/hands moving while keeping a fixed distance between them indicating parallelism (Figure 10-3).	• Indicating a property (parallelism, perpendicular)
	• Modelling	• Drawing an image of a diagram using finger(s) or hand(s), Figure 10-4.	• Drawing a diagram
Conceptual gesture: Static: no action	• Pointing	• Pointing with one finger, usually, accompanied by the use of demonstratives *this* and *that* (Figure 10-5).	• Part of or the whole diagram
	• Bounding	• Fixing two fingers/hands with fixed distance (Figure 10-6).	• Length or distance
	• Still-modelling	• Holding fingers/hands presenting a diagram, creating an imaginary diagram (Figure 10-8)	• Diagram

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[21] Artefacts may be used instead of fingers.

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4. Multimodal communication: an illustrative analysis

So far I have discussed (the ideational meaning of) the gestural mode of representation and communication and what relationships may be constructed between the three different modes. In this part of the chapter, my plan is to illustrate that interaction by analysing an episode of students' communication in which the students try to solve the Task 1 (TF) in the current study. My goals, in doing so, are, first, to illustrate the applicability of the suggested framework (the gestural) and, second, to present an analytic (and transcription) tool which considers the three modes together (see also Morgan & Alshwaikh, 2009).

Because my example is illustrative, I will be selective in presenting an extract from students' communication in solving the tasks. I searched for an extract which includes the three different modes: the verbal, the diagrammatic and the gestural. However, the issue of presenting an extract or episode in the communicative act raises the methodological issue of transcription: how to transcribe, which mode to start with, and how to present the transcript, all of which involve decisions that require adopting a theoretical position (Kress et al., 2001). Different studies have dealt with multimodal transcription within the social semiotics approach (e.g. Kress et al., 2001; Mavers, 2009; Norris, 2004; O'Halloran, 2004c). Norris (2004) suggests a step-by-step guide for multimodal transcription:

> First, we complete a transcript for each communicative mode; then combine two or more; and finally, combine all of our individual transcripts to present a complete transcript. (p. 66)

The representation of the complete transcript, furthermore, is a matter of theoretical stance, too. Kress et al. (2001, p. 37) argue that the 'decision of what form to represent information in [written only or in written and visual] depended on the intensity of the information and the focus of the analysis.' In other words, multimodal transcripts provide a closer look at the data and a thick description of each mode and, moreover, of the interaction between modes. Jewitt (2006) describes different ways of organising transcripts such as a 'play script' in which language is the main mode, or a 'musical score' in which each mode resides in a row of the score, or organising each mode in a separate column based on time.

Table 10-2 is an example of the last option of multimodal transcript, and it may take another form such as starting with the gestural mode or the diagrammatic rather than

the verbal. While most transcription presented in studies would start with the written or verbal mode, Mavers (2009), for example, presented her data starting with the images with which students engaged when they draw and write. This stance, placing one of the modes in the foreground and focusing on it and its meaning potential, may fragment the text and the text's potential meaning (Jewitt, 2006). Therefore, Jewitt (2006) suggests another complementary approach, which is taking the three interrelated metafunctions (ideational, interpersonal and textual) as a starting point for analysis. I do this in an illustrative example below.

The 'process of transcribing multimodal data is extremely complex' (Norris, 2004, p. 64). In order to reproduce Table 10-2, it took days of watching the video data in order to select a salient episode of just 34 seconds. Then, I watched and listened to each mode in isolation: I listened to the conversation only, watched the images alone, then the gestures, and then the three modes together. I then produced schematic drawings to illustrate the gestures before finally deciding how to present the data. This process has some limitations, such as the technical problems that may arise (audibility, pausing exactly at the required moments, the quality of the images, etc.) and the synchronisation of the three modes in order to construct an ensemble meaning.

The analysis done by Radford et al. (2007) may overcome the shortcomings I have mentioned. Radford and his colleagues used complex tools to collect and analyse the data, including three or four video cameras, transcription, written texts and audio analysis. Furthermore, they performed an audio analysis in order to analyse the tone of the sounds that accompanied the gestures. These technological tools undoubtedly facilitate a rich analysis. Radford and his colleagues presented their analysis following the conventional form, in which they recorded the words spoken, interspersed with a description of the gestural mode in square brackets. They separately presented the images.

I adopted a different transcription process, reproduced in Table 10-2, which I believe facilitates the multimodal analysis by presenting the different modes (verbal, gestural, and visual) in an integrated form. This process has been used by Mavers (2009) and Kress et al.(2001), as well as Morgan & Alshwaikh (2008; 2009) in the ReMath project.

Table 10-2: Multimodal transcript of students' geometric activity

Line	Time	Stdnt	Verbal	Gesture/ Diagram		Comments
				In words/pictures	Schematic drawing	
1	00.31	Richard	What's your opinion Ruth?			Turning the page towards her side. (The diagram in the problem is shown in the previous column.)
2	0.32	Ruth	What opinion?			
3	0.33-0.42	Ruth	I think we know the length			• May be referring to PL [not seen in the video]
			or whatever	Moving her pen next to EM		• May be referring to EM since she moved her pen to it [not clear in the video]
			… do we know the width of it?	Pointing at LM		
4	0.43	Richard	Yeah			
5	0.44	Ruth	Do we know the length of it?	Sliding her pen in the same direction of PL (top to bottom direction)		
6		Richard	Yeah			
7	0.47-0.50	Ruth	So I think ... so we need to find out what this ... this ... this			• Sliding her pen over the side EM, moving from E to M repeatedly for few times (Figure 10-1). Every 'this' may refer to the route E-M-E
8	0.51	Richard	Yeah ... or maybe we make a square	Moving his pen over the diagram as follows: Pointing: 1-2 Sliding 2-3-4 Pointing 5 Sliding 5-6-7 (see Figure 10-10)		• The sequence of numbers indicates the sequence of gestures. • I think that the points 4&5 are supposed to represent the same point that Richard wanted when he said 'square'. • The point 7 is coloured to be distinguished from the point 2. Similarly point 9. • Richard used sliding to indicate an imaginary image.

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9		Ruth	Yeah			
10	0.53	Richard	and then that ...	Sliding 7-8-9 (see Figure 10-11)		
	0.54		and then we use that ...	Pointed at MLP and a point after E to indicate the whole diagram (trapezium plus the imaginary triangle) (see Figure 10-12)		The imagined triangle is the one that may be constructed by the points 5-6-7, 7-8-9 or 16-17-13
	0.56		and then we take away how many metres and centimetres...	sliding between 10-11-12 (see Figure 10-13)		Referring to the triangle (see the previous comment).
11	0.59-1.05	Richard	is that it?... yeah?... we have to draw on the sheet			

An illustrative episode: The multimodal dimensions of students' geometric activity

The context - Three students (Richard and Caroline sitting next to each other and Ruth sitting in front of them) are supposed to work together to solve Task 1 (see Chapter 4) by first discussing it, agreeing on the solution and then writing their solutions individually. This episode occurred in the first minute of the discussion, when Richard initiated the dialogue by reading the problem and then asked Ruth her opinion. The ellipsis (...) in Table 10-2 refers to a pause that lasts less than a second.

The mathematical task - The problem in Task 1 asks the students to find the distance needed for the two sprinklers to throw water in order to water the lawn (see Chapter 4). In other words, students need to find two lengths/distances or radii of the two circles whose centres are E & P. However, most of the solutions presented by the students focused on finding the length of the side EM or the area of trapezium. These two types of solutions were dominant in students' interactions as well as in their written texts as presented in the analysis below. While my goal is to present and analyse students' interaction to solve this task, I think that the presentation of the task itself may influence that interaction. The text is presented in words followed by a

diagram that illustrates the verbal by presenting a specific example. While the main mode here is the verbal, the verbal and the diagrammatic modes working together introduce the problem, where students have to read them together.

Since the ideational meaning is the only meaning that has been developed in the gestural mode, I will limit my analysis to the ideational meaning in each mode separately and in the interaction between the three modes. Specifically, the image of the mathematical activity will be the focus of this illustrative analysis. This analysis, indeed, requires looking at the kind of processes, the participants and the role of human agency (see Morgan, 1996 and Chapters 6 & 7 of the current study).

The image of mathematical activity, I argue, is represented in (or *within*) each mode and in the interaction *between* the three modes. The original diagram presented in the problem (see line 2 in Table 10-2) is conceptual, meaning that it represents mathematical objects with spatial relational processes among them but does not suggest any action by or on these objects. As these processes are not specified in the diagram, one has to refer to the verbal text in order to recognise the spatial relations. For example, the size of the bases or height of the trapezium is only mentioned in the verbal text, an elaboration (anchorage) relation.

Students' interaction – an analysis of this episode reveals two main events; setting the goal of the task and suggesting a solution. The contrast between the two events suggests that the students differ in their understanding of the goal of the task. This difference is indicated in the students' words and gestures. While Ruth in the first seven lines suggests that the goal of the task is to *find out* the length of EM, saying in line 7 'So I think ... so we need to find out what this...this...this' accompanied by a drawing/sliding gesture by moving her pen over the side EM (Figure 10-1), Richard, on the other hand, suggests a different route for the solution. His words and gestures were different from Ruth's. In line 8, he suggests to *make* a square and continues his suggestion to the end of the episode, accompanied by gestures, suggesting to solve the problem/task using the area concept. Actually his words were very limited, and he used the word *that* three times. The use of *this* and *that* in Ruth and Richard's utterances is impossible to follow without observing the gestures made over the diagram simultaneously (the three modes). These instances are good illustrations of the point I want to make: each mode offers different meaning potential, and there is a

need to look at the ensemble of modes in order to make meaning (Kress & Van Leeuwen, 2001; Radford et al., 2007).

The drawing/sliding gesture that Ruth signalled suggests a measurement process of the side EM. Richard, on the other hand, made modelling gestures in order to show his interlocutor the diagrams he refers to. Although it is not entirely clear why Richard introduces the square, it seems that he wanted to find the area of the trapezium by adding an imaginary triangle to the trapezium to get a rectangle (that seemed an easier way to calculate its area) and then subtract the area of the triangle from the area of the rectangle to get the required area. Actually, Richard wrote this in the verbal part (Figure 10-9) of his final text:
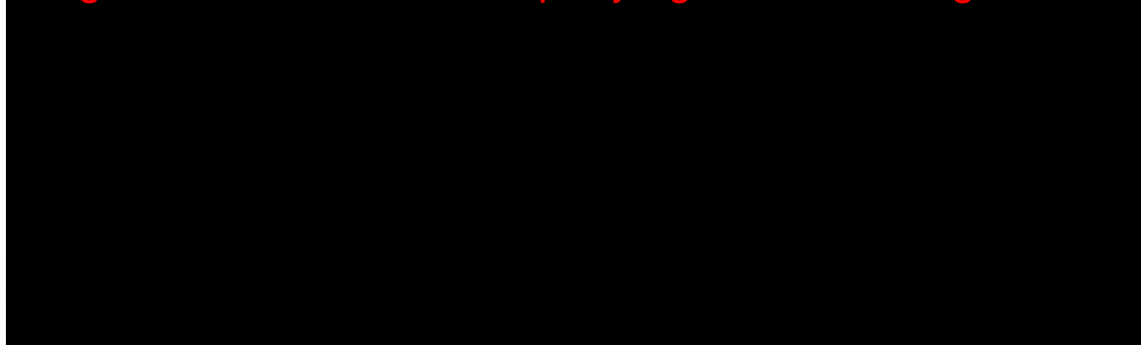
Figure 10-9: The verbal part of Richard's final text

However, Richard made different gestures, not all of which are purposeful to achieve his goal. Richard went through the following procedures in order to find the solution:

(8@0.51[22])   *Yeah... or maybe we make a square* [The square is constructed through four points as in the schematic figure, as recorded in the transcript and as constructed by the points 1, 2, 3 and 5 in Figure 10-10]
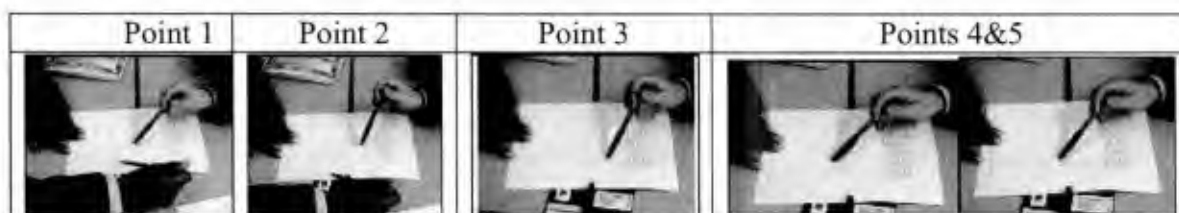
Figure 10-10: A square

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[22] Line 8 at time 0.51

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As I indicated in the transcript, points 4 and 5 are supposed to be one point (M) since Richard referred to a square. What is not clear is his gesture going back from point 5 to 3 to 2 (5-6-7 in the schematic drawing) referring to a triangle as in the following procedure. One can see the relevance of this gesture when looking at his final text and noticing the square and the triangle (see below).

(10@0.53) *and then that* [a triangle is constructed through his gestures moving between the points 7-8-9 in the schematic drawing in the transcript and in Figure 10-11]

| Point 7 | Point 8 | Point 9 |

Figure 10-11: A triangle

Now, Richard made a modelling gesture of the triangle he wanted according to his plan. This triangle, together with the trapezium presented in the given diagram, form a rectangle whose area is known to Richard. His gesture refers to the trapezium, and the triangle is presented in the next gesture.

(10@0.54) *and then we use that* ... [referring to the whole diagram in the written part of his final text and in Figure 10-12]

Here again, Richard highlighted the same triangle to which he pointed in every step of his suggestion of the solution, points 16-17-13.

Point 13	Point 14	Point 15

Point 16	Point 17	Point 13
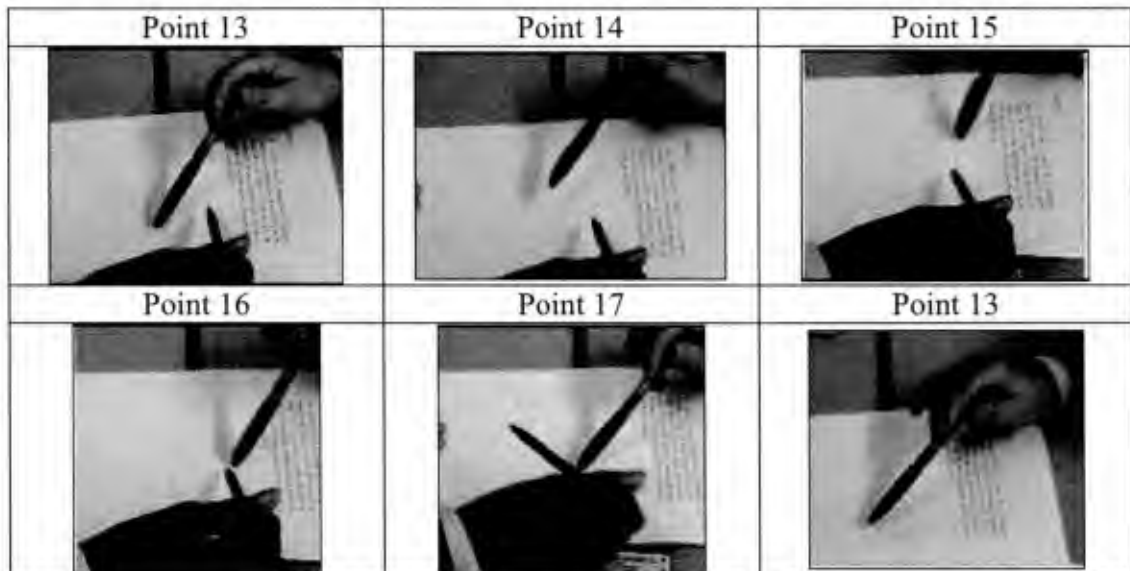
Figure 10-12: A trapezium and a triangle

After setting the scene for his plan, Richard now presented the last step in his solution in the following gesture.

(10@0.56)   *and then we take away how many metres and centimetres...*

The 'take away' suggests the subtraction process that Richard refers to in order to find the area he planned to find from the beginning. This taking away process is related to the triangle to which he insisted on referring most of the time, as in 5-6-7, 7-8-9, 16-17-13 and finally 10-11-12, as in Figure 10-13.

Point 10	Point 11	Point 12
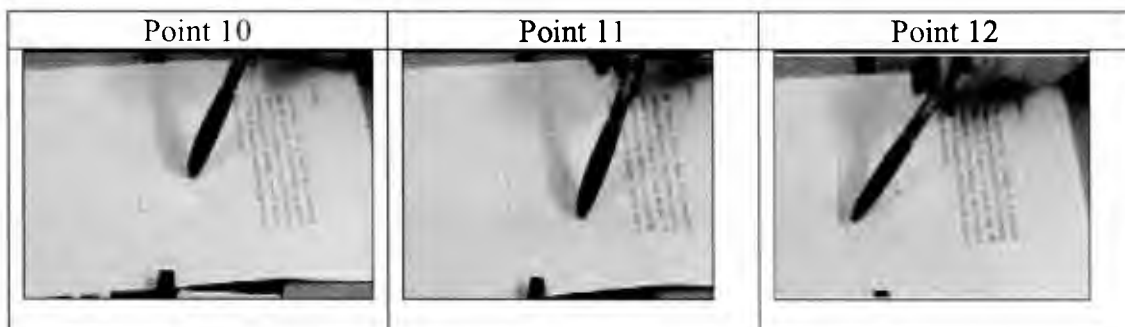
Figure 10-13: Another triangle

It is interesting to notice Richard's final text (Figure 10-14) and to notice the triangle (shaded, suggesting a narrative diagram, see Chapter 6) in his diagram, which is presented first. In his diagram, the triangle to which Richard gestured most of the

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time (see his gestures in lines 8 and 10) is shaded, suggesting that it has been added afterwards. The 'metres and centimetres' in his utterance indicates the notion of the area of the triangle and the 'square'[23] to which he gestured at line 8. The triangle appears in his final text in the calculation he did to find the area. It is, moreover, interesting to note that Richard's final text follows the pattern observed by Radford et al. (2007), namely that Richard's final text provides a history or narration of his attempts to solve the problem (see below).

Figure 10-14: Richard's final text for Task 1

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[23] Actually this diagram is not a square but rather a rectangle. See the 'comments' below.

<u>Comments</u>:

- *Multimodal demonstratives*:

It seems that the use of demonstratives may be inevitable in communication, especially when considering the different modes. In analysing the communication in the previous episode, I want to extend what Rowland (1992) suggested about *Pointing with Pronouns* in which he focused on '*it*'. Rowland's 'it' is not accompanied by any other sign but 'points' to an entity that has no physical presence in the setting. On the other hand, it could be argued that the demonstrative plus the combination of diagram and gesture do in some sense point beyond what is already present in the setting.

Rather than describing *it* as a linguistic pointer which occurs in 'maths talk', I would refer to demonstratives, *this* or *that*, as *multimodal pointers* which occur in multimodal-mathematical communication when gestures and diagrams are considered. The '*this... this ... this*' in line 7, for example, refers to a specific entity on the diagram using a specific gesture. Moreover, the two '*thats*' in line 10 are also multimodal pointers, where the reference in each is shown by the accompanied modes, the gestural and the diagrammatic. It seems that the entities being discussed have a significant role in communication, where geometric entities are the focus of the discussion which have physical presence on papers as diagrams, unlike the entities in Rowland's examples, which have no physical presence. This presence makes the communicative act meaningful when a reference is made to any part of that diagram or gesture, such as by saying 'this' or 'that'.

Rowland argues that the use of demonstratives contributes to the efficiency of communication as a shortcut or linguistic economy, facilitating the flow of communication rather than 'interrupting' the idea of the speaker, especially when the idea is still 'unseen', as in '*that*' in line 10 where the square or the triangle are not named yet. The unnamed concepts or diagrams may be another interpretation for the use of demonstratives. If students do not know the name of a diagram, they may use *this* or *that* to refer to it. The ability to consider the different modes of communication together, I argue, is another advantage of the multimodal analysis that I discuss in the next point.

- *Students' meaning-making:*

The multimodal analysis may offer a chance to 'uncover' or interpret some of students' understandings about some concepts (such as area, circumference, etc.) and what learning difficulties students might have.

Although his solution is not what the problem asks for and even has mistakes in finding the area of trapezium, the calculations that Richard wrote in his final text also contain traces of his process in solving the problem, which is synchronised with his gestures. Instead of writing his solution as: The required area = the area of the rectangle (trapezium and the added triangle) − the area of the added triangle, or simply finding the area of the trapezium, he divided the trapezium into two rectangles (he called the smallest a 'square' in his gesture at line 8) and found the area of each (Figure 10-15), which recalls the gestures he made (lines 8 & 10@0.54). Then he added the two areas, subtracted the area of the added triangle and claimed that the result is the required area.

The area of the greater rectangle is supposed to be 160, and the area of the smaller one (which Richard called a square) is supposed to be 80. The fact that he repeated them suggests an alternative conception or understanding of the concept of area and circumference. Later, he subtracted the area of the triangle:

Line 8, *we make a square*

(10@0.56) *and then we take away how many metres and centimetres ...*
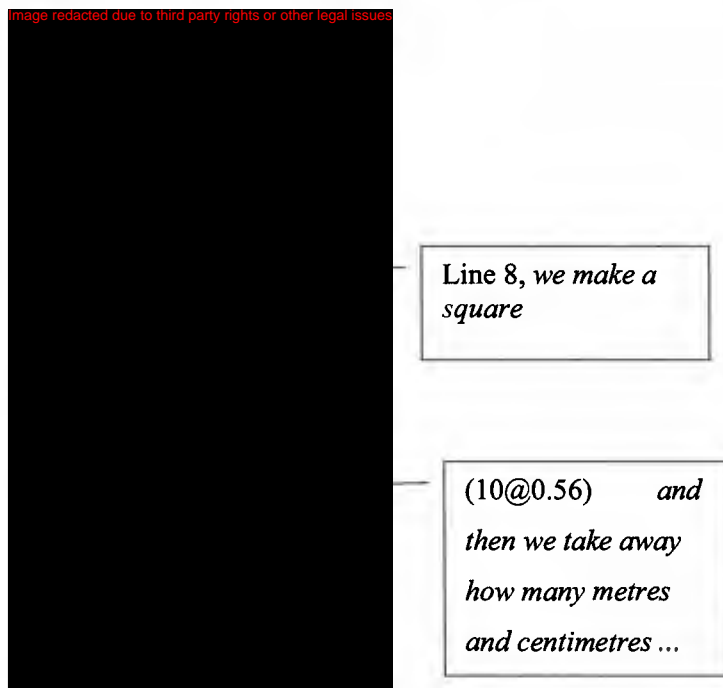
Figure 10-15: Calculating the required area and the connected gestures

- *Multimodal transcripts:*

'Transcription is a theory-laden practice' (Kress et al., 2001, p. 33). The way in which a transcript is organised is theoretically motivated. To put the modes in a table next to each other rather than in a hierarchical order, for example, may indicate that they occur together. There are some researchers who tried to develop accounts for multimodal transcription (e.g. Baldry, 2004; Jewitt, 2006; Kress et al., 2001; Morgan & Alshwaikh, 2009) and answers to the questions that arise, such as what to put first or second and how many details to include, answers which are influenced by the interest of the transcribers, the research concern or/and the focus of analysis.

I have tried to emphasise the nature of information that I was able to add to the analysis by taking into account both the gestural and the diagrammatic modes separately as well as the interaction between them. While this attempt was complex and driven by the aim of my research, the multimodal transcript offers a thick descriptive tool (Jewitt, 2006; Pratt, 1998) for the analysis.

5. Summary

The aim of this chapter was to analyse mathematical communication and representation taking into consideration three modes; verbal, diagrammatic and gestural. To do so, I presented a preliminary framework to analyse, ideationally, the gestures used by students while solving mathematical problems. This framework needs, however, more development, in order to take account of the interpersonal and the textual functions of gestures. Then I demonstrated the multimodal analysis by selecting an episode from the set of data collected for the current study.

That analysis aimed to show the applicability of the suggested frameworks (the diagrammatic and the gestural) and, moreover, to show that the consideration of additional modes of representation and communication would give a more detailed picture of students' meaning-making processes in learning mathematics. The illustrative example presented demonstrates how the multimodal analysis offers a deep look into meaning making processes that would not available by conducting the 'traditional' form of analysis. This last claim needs further exploration, which is beyond the scope of the current study. The implications of the suggested frameworks and the multimodal analysis will be the focus of the next chapter.

11 Conclusions and implications of the study

1.   Plan of the chapter:

My plan, in this final chapter, is to present the main findings of the study and to discuss its implications and its limitations. I will first present findings from the review of the literature and then from the study itself, before addressing some of the limitations of the current study and pointing out what it has not achieved. Then, I will discuss the implications of the study on three levels: its theoretical implications, its contribution to research and its implications for practice and for the development of practice. On this third level, I suggest some possibilities for use of the suggested frameworks. I finish this chapter by offering suggestions for future research.

2.   Theoretical research

Most mathematicians exhibit prejudice against the use of mathematical (geometric) diagrams. The main arguments which back this stance are: that diagrams are unreliable and lack rigour; that the potential for misuse, for example, by inferring non-proven statements, is great, because diagrams by their nature may 'deceive the senses' and are limited in representing knowledge; and that diagrams are of an informal and personal nature (Dreyfus, 1991; Netz, 1999; Shin, 1994). Consequently, diagrams are not considered to represent the 'real' mathematics (see Chapter 3 for more details).

Diagrams, however, throughout the history of the development of mathematics, have not always been disfavoured. A study of the mathematics of the Old Babylonian and Egyptian civilisations reveals that diagrams were a significant element of mathematical texts (Robson, 2008b). Moreover, Greek mathematicians gave such considerable attention to diagrams that at some point the diagram was the 'hallmark' of mathematics (Netz, 1999). Diagrams played a critical role in shaping Greek mathematics and, consequently, the most distinguished notion in mathematics – the proof. It was only in the 17th century that mathematicians, influenced by the work of Descartes, began avoiding the use of diagrams and developing the current form of mathematics as 'abstract, formal, impersonal and symbolic' (Morgan, 2001).

The avoidance of diagrams finds its way into different mathematics contexts, including didactic ones in which students became reluctant to use diagrams or visual representations in their solutions, as shown by some studies (e.g. Dreyfus, 1991). Recently, and thanks to the development of the discipline of mathematics education, the 'stereotype' of mathematics as a symbolic and formal field has been challenged, mainly by scholars researching the language of (and in) mathematics and of teaching and learning mathematics. As researchers began to explore the interaction between sociolinguistics (Halliday, 1978) and mathematics education (Austin & Howson, 1979; Pimm, 1987), they began to consider a more detailed analysis of mathematical texts, in which the notion of communication was central.

A major development in this process of placing communication at the centre of research into mathematics education took place when scholars, especially Morgan (1995; 1996a; 1996b), adopted the Hallidayan SFL approach in their work on teaching and learning mathematics (see a review of Morgan's work in Pimm & Wagner, 2003). Seeing mathematics as a social semiotic practice is the main assumption which underlies Morgan's work. She offers a linguistic approach:

> to provide a means of identifying and interpreting features of the [written mathematical] texts that are likely to be of significance to the mathematical and social meanings constructed in the interaction between writers/speakers and readers/listeners. (Morgan, 2006, p. 226)

As social semiotics developed, it addressed the multimodal characteristics of the interaction/communication between writers/speakers and readers/listeners, the participants in the communicative acts. Social semioticians, especially Kress and his colleagues (Kress & Van Leeuwen, 2001, 2006), began to challenge the dominance of language as the main mode of communication and extended social semiotic theory to include other modes of communication. This has led to the emergence of the social semiotics multimodality approach. Multimodality, briefly, gives equal attention to other modes of communication and representation such as images and gestures.

The emergence of multimodality created a need to extend Morgan's work. Mathematics discourse is multimodal where different modes of communication coexist, (Morgan, 2006; O'Halloran, 2005), and, thus, it has to consider not only the language (written or spoken) but also the other modes such as diagrams and gestures. However, descriptive tools for such modes are not yet fully developed (Morgan, 2006). This study set out to offer such tools.

3.   The current study

Having explored the relevant theoretical research, the aim of the current study is to construct an analytic framework for the diagrammatic mode that will offer a means to interpret and read geometric diagrams, in order to broaden our understanding of the role of mathematical visual representations in the construction of mathematical meaning. To a lesser extent, this study also addressed the gestural modes of representation and communication.

Therefore, the focus of the current study centred on the mathematical texts collected for the purpose of the study, mainly students' texts produced in response to two mathematical tasks and also other available texts such as textbooks and diagrams from the Internet. In order to build the framework, I developed analytic and interpretative tools to identify significant features of the mathematical diagrams and gestures. These tools are based on the visual analysis derived from Kress and Van Leeuwen's (2006) visual grammar and Morgan's linguistic approach to mathematical texts (Morgan, 1996b). Both of these approaches adopt Halliday's systemic functional linguistics (SFL) which offers a grammar for reading the relationship between language and social structure (Halliday & Hasan, 1985) and argues that any human act fulfils three essential functions: ideational, interpersonal and textual.

In order to achieve the aim of the study, an iterative design (Pratt, 1998) methodology was used. Iterative design is a process of investigation, construction, validation and refinement. In short, I started with a very general framework based on the literature, applied it to mathematical texts from the collected data, tested it to get feedback, refined the version and then suggested a 'new' version. The new version, in turn, went through the same method of application and feedback, and so on. In total, I have suggested three versions of the diagrammatic framework. An 'at a glance' version of the diagrammatic framework is presented in Table 11-1 (see Chapter 5 for more details). Chapter 4 provides more details about the methodology of the study, including the journey of the development of the framework.

Table 11-1: An overview of the suggested framework for reading geometric diagrams

Type of meaning	Diagram type	Type of process/visual mark	Types of potential mathematical meaning
Ideational: Nature/image of mathematical activity	**Narrative diagram:** Distinguished by the presence of temporal factors	• Arrowed	• Transformation • Measurement
		• Dotted	• Constructing perpendicular or parallel line
		• Shaded	• Reflection
		• Sequence of diagrams	• Proof
		• Construction	• Construction
	Conceptual diagram: (Relationships between objects) Distinguished by the absence of temporal factor	• Classificational ('of the same kind' relation)	• Relationships between shapes (e.g. 'squares are rhombuses')

		• Identifying
		  o Indexical processes (letters, arrows)
		  o Symbolic processes (words)
			• Identify objects or refer to size relationships.
			• Arrows or words: attributive (parts of diagram); or identifying (refer to whole diagram)

		• Spatial
		  o Positional (10 relations)
		  o Size (7 relations)
			• Examples: Line&Line (perpendicular, parallel), Shape&Shape (congruent, similar)

Interpersonal: Relationships between author and viewer	**Contact**	• Demand diagrams (action required)	• Labels: Question mark, unknown quantity
		• Offer diagrams (no action required)	• Information about objects (properties and relationships).
	(Social) Distance (Relationships between producer and viewer of diagram)	• Neat vs. rough diagrams	• Neat: formal & distance • Rough: close personal distance • Labels, colour, arrows, words: authority
		• Labels (general, specific)	
		• Colour, arrows, words	
	Modality	• Diagrammatic modality markers (5 markers/cues)	• Examples: abstract, naturalistic & contextual diagrams
Textual: Unity & Coherence	**Information value**	• Left-to-right, top-to-bottom, centre and margin	• Thematic progression (e.g. deductive argument) • Expression of reasoning (e.g. cause-result) • Overall structure of the text (e.g. proof)
	Salience	• Importance and attention	
	Framing	• Frame: disconnected	
		• No frame: connected, unit	

The data needed for the iterative design were of three types: (a) data for validation of the framework such as diagrams from textbooks, students' mathematical texts and Internet, (b) data for understanding the context of situation and the context of culture, such as observation and field notes in the classroom, and (c) data for generalising the

framework, which was achieved by the large number of texts and by applying the study in two different cultures and languages, namely the UK and the Occupied Palestinian Territories (OPT).

The development of the diagrammatic framework was informed by all these sources of data together with the literature and the two different contexts and cultures. The main aims behind these different types of data were the validation and the generalisation of the framework.

Since the aim and focus of this study were the intended framework itself and its development, the framework needed to be validated with texts rather than with students. In order to do that, the suggested framework was applied to various diagrams in textbooks and in the empirical data and diagrams available on the Internet. The validation process had two main aims: to check the applicability of the framework and to get feedback that led to a new refined version of the framework. Moreover, three criteria were suggested to validate the framework: accuracy, delicacy and inclusiveness. The framework was considered to be accurate if there was a clear match between the suggested categories in the framework and their realisations in the diagram. A delicate framework is one that would be able to distinguish between different items of the same category. Inclusiveness means that the framework can address 'all' diagrams across the Euclidian geometry school.

In this study, I adopt the reconceptualisation of the principle of 'generalisation' offered by Pratt (1998) and Schofield (2007) to mean 'fittingness' or 'natural generalisation', where a 'thick description' is provided, and the researcher can decide intuitively whether the data is generalisable or not based on the research aims, methodology and context. The framework met these criteria. Moreover, the application of the suggested framework in different situations/contexts, by conducting the study in different classes and schools in two different languages and cultures, English in the UK and Arabic in the OPT, contributed to the generalisability of the study.

3.1   Mathematics as a social practice concerned with communication

The theoretical background underpinning this study, which informed the suggested frameworks, is that doing mathematics is a social practice (Morgan, 1996b; Pimm,

1987). This approach challenges the dominant image of mathematics which considers mathematics to be a 'universal' subject concerned with procedures and rules. Within this 'traditional' image, diagrams are presented as a means to assist problem solving or to clarify the verbal or the symbolic part of the mathematical text. The social perspective enabled me to make use of the SFL approach toward the diagrammatic mode (and the gestural as well). In other words, the diagrammatic framework presupposes that there is a relationship between the structure of diagrams and social structure or human experience.

Adopting the Hallidayan SFL, the multimodality approach (Kress & Van Leeuwen, 2006) and Morgan's linguistic approach to mathematical texts (Morgan, 1996b) offered together a means to construct the types of meanings of the structure of the diagrams and, moreover, the possible mathematical meaning. In other words, these two approaches enabled me to look at the way that the picture of the mathematical activity is presented in mathematical diagrams (and gestures), the social relationship between the producer of the diagram and the reader/viewer is constructed, and the elements of the text are arranged (see Table 11-1).

3.1.1 The structure of diagrams and the social structure

The picture of mathematical activity: The development of the framework of the diagrammatic mode revealed that the use of geometric diagrams is consistent with the philosophical tension between divergent ways of looking at mathematics – as a process of doing versus a set of concepts – and the role of human beings in the construction of mathematics (see the discussion about this issue in, for example, Solomon & O'Neill (1998) and Morgan (2001)). As a result, I have distinguished between two structures of geometric diagrams: narrative and conceptual.

Narrative diagrams tell a story that indicates what kind of mathematical activity is going on (measurement or proof, for instance) and how it proceeds. They are distinguished by the presence of *temporal* factors which are represented by visual cues or indicators such as arrows, dotted lines, shading, sequence of diagrams and construction marks.

Conceptual diagrams present mathematical objects in timeless essence. They are distinguished by the absence of temporality. The suggested framework offers details

about the relational processes of conceptual diagrams such as classificational processes, identifying processes (indexical and symbolic) and spatial relations (position and size). In other words, these different subtypes of conceptual diagrams can be distinguished according to the relations between the geometric objects presented in the diagrams.

The role of human beings in the construction of mathematics has been discussed in Chapter 6 of this study. Historically, the image of human beings and traces of their physical contexts have been removed from mathematical texts. This is related to the construction of the hitherto attitude towards mathematics as 'objective' knowledge that people (mathematicians) discover. Again, the suggested framework claims that although mathematicians 'succeeded' in concealing the role of human beings in the construction of mathematics, narrative diagrams play a significant role in revealing the 'humanistic' or 'subjective' origin, because of the temporality present in these diagrams.

The social relationship between the author of the text and the reader/viewer:

A social relationship is constructed between the author and the viewer, and it can be analysed via different indicators, namely contact, (social) distance, and modality. The framework offers an analytic tool to describe the *contact* between the author and the viewer of a diagram. It distinguishes between demand and offer diagrams and explores contact through visual indicators such as labels, variable names and colour in diagrams, which play a similar role as the gaze plays in images (Kress & Van Leeuwen, 2006).

Demand diagrams ask for something to be done by the viewer. A direct demand can be realised by the presence of a question mark demanding a solution or asking the viewer to find the value of the marked part, for example. Indirect demands have visual indicators such as unknown quantities or variables (letters or numbers) asking for the value of a specified side, angle or area. Offer diagrams offer information to the viewer about geometric objects (properties and relationships between them) without asking that any action be taken. The visual indicators for offer diagrams are labels, colour, arrows and words.

The *social distance* constructed between the author and the viewer is realised through the neatness aspect of diagrams (neat diagrams vs. rough diagrams) and labels, colour, arrows and words. While a neat diagram 'indicates that the text is formal and that there is some distance in the relationship between the author and the reader' (Morgan, 1996b, p. 91), rough diagrams suggest a close personal distance, an intimate relationship between the author and the viewer.

Modality refers to how reality and truth are represented in communication, or, in other words, what authors would use to show the degree of certainty and truth of their statements or propositions about the world. I have identified five diagrammatic modality markers such as abstractness of diagrams, naturalistic or contextual diagrams, labelled diagrams, additional information in diagrams, and neatness of diagrams.

The arrangement of the text:

Kress & Van Leeuwen (2006) argued that the value of the information, salience, and framing are three elements in the arrangement of a text which contribute to its meaning and coherence. In other words, the position that an element occupies in the 'visual space' offers a meaning potential. Following this stance, the suggested framework provides an analytic tool to describe the components of the whole mathematical text, including the way in which its elements (verbal and visual) are organised and the relationship between them.

The information value of the horizontal structure (Given and New) and the vertical structure (Ideal and Real), as well as placing information in the centre or margin of mathematical texts, were discussed fully in Chapter 9 using Morgan's linguistic approach (Morgan, 1996b). When analysing a mathematical text, there are two levels at which to view the text: internal and external. The internal level addresses the internal features of the text: the theme and the way in which reasoning is constructed. The external level looks at the structure of the text as a whole.

The analysis of the textual meaning provides a means to describe and interpret the practice in mathematical texts. For example, analysis of some texts showed that the vertical structure is highly regarded in mathematics, where the theoretical part will be

idealised and located in the upper section, while the real part, in contrast, will be introduced at the lower section of the text.

An examination of the whole text must incorporate its verbal mode (if any). I addressed the relationship between word and image based on the work of Halliday (1985) and Barthes, as discussed by Van Leeuwen (2005) and Kress & Van Leeuwen (2006). I identified two such types of relationships and their subtypes: elaboration, whose subtypes include specification and explanation; and extension, whose subtypes include similarity, contrast and complement. The significance of these types is the means they offer to describe and analyse multimodal texts that include, at least, the verbal and the diagrammatic modes.

3.2   Mathematical Arabic and English texts.

The study used Arabic data either from students' texts produced in response to the tasks of the study or from the Palestinian school mathematics textbook. While I am neither a linguist nor an expert in either of the two languages, I have a sufficient user's knowledge of Arabic, being a native speaker, and good use of English. There are similarities between the structures of the two languages. For example, the verb in both languages refers to the deed, the action. Actually the word that stands for the verb in Arabic, *fe'l* (فعـل), is used to mean 'to act'. In a broader sense, the clause in Arabic starts either with the verb or with the noun, which means that the theme may vary according to the interest of the sayer/writer, as in English. However, the two languages also differ, for example regarding the arrangement of words and the direction of writing and reading. A standard Arabic clause would start with the verb rather than the noun, unlike in English.

A strong evidence of the power (robustness issue discussed in Chapter 4) of the suggested diagrammatic framework was its ability to distinguish between different kinds of texts, which has significance beyond the texts themselves. The way in which mathematics is perceived or constructed in an educational system (culture) may be realised through the order of the argument and the way in which a text proceeds. The Palestinian participant students tend to present their texts as products in cause-result order, with one diagram and few words. In contrast, the English students presented their texts as investigation or trial and error processes with more diagrams and

words. Can we read 'something' about the mathematics in the OPT or in the UK, educationally and culturally? As I mentioned, the English educational system, for the past 20 years, has encouraged an investigation attitude towards mathematics, while for historical reasons that I have described, the view toward mathematics in the Palestinian educational system is isolated from the living reality in the OPT, and Palestinian mathematics textbooks are content-focused. Unlike in the UK, in the OPT, mathematics is still perceived as a formal, definite, impersonal and symbolic subject.

Finally, we should be careful not to draw too many conclusions from the study of the cultural differences, because the study was not about the way Palestinian and British students present their arguments, and the generalisability of the above-mentioned observation has not been established. However, the potential cultural differences between learning mathematics in the UK and the OPT warrant further investigation, and the analytic framework suggested by this study provides a tool with which to conduct such investigation.

3.3 The ideational meaning of gestures:

As a result of the iterative watching of the video records of students' communication about the two geometric tasks of this study and the observation of their extensive use of gestures, I have developed a preliminary analytic framework for gestures using a social semiotics approach, as I did with the diagrammatic mode. That framework, however, is initial and limited to analysing the ideational meaning of gestures. Table 11-2 provides a 'glance' at that framework.

I have distinguished between narrative gestures and conceptual gestures. Narrative gestures are distinguished by the presence of the *movement* of fingers or hands (or artefacts). Similar to narrative diagrams, they refer to a mathematical activity that is taking place. I have identified three types of processes based on the activity they indicate: drawing, symbolic and modelling.

Conceptual gestures are distinguished by lack of motion. Like conceptual diagrams, they refer to pre-existing mathematical objects. I have identified three types of processes based on the concept they indicate: pointing/indexical (referring to a part

of or the whole diagram), bounding (referring to a fixed length) and still-modelling (referring to modelling a diagram by one or two hands).

While the suggested framework is preliminary and nascent, this distinction between narrative and conceptual is consistent with the suggested framework for the diagrammatic mode, and it may contribute to the coherency of the analysis of the multimodal communication.

Table 11-2: An overview of the suggested framework for reading ideational meaning in gestures

Gesture type	Type of process	Types of potential mathematical meaning
Narrative gesture: Dynamic: ongoing action	• Drawing/sliding	• Measurement
	• Symbolic	• Property (parallelism, perpendicular)
	• Modelling	• Drawing a diagram
Conceptual gesture Static: no action	• Pointing	• Part of or the whole diagram
	• Bounding	• Length or distance
	• Still-modelling	• Diagram

3.4   Multimodal analysis:

Having suggested two frameworks to read diagrams and gesture, I analysed a very short episode of participant students as an illustrative example for a possible multimodal analysis taking into consideration three modes of communication: verbal, diagrammatic and gestural.

The aim of the multimodal analysis was to show the applicability of the suggested frameworks (the diagrammatic and the gestural) and, moreover, to show that the consideration of more modes of representation and communication would give a richer picture of students' meaning-making processes in learning mathematics.

The illustrative example presented demonstrates how the multimodal analysis offers an in-depth analysis of meaning making processes that would not be available by doing the 'traditional' form of analysis. The latter issue, however, warrants further exploration, which is not within the scope of the current study.

## 4.	Limitations of the current study

The aim of this study was to develop a framework to analyse the diagrammatic mode in mathematics discourse and, to a lesser extent, the gestural mode. The study was mostly dedicated to the former. However, I wish to highlight three different aspects of the limitations of this study:

- Aspects of the framework(s) that I have worked on and included in my analysis but that still require further development.

- The generalisability of the framework beyond geometry

- Issues that arose which are outside the scope of the original aims of the study but which could be explored by an extension of the methodology.

### 4.1	Aspects of the framework(s) requiring further development

During the development of a framework to analyse the role of diagram in the construction of mathematical meaning, the interpersonal meaning was particularly difficult to construct, especially the social relationship between the producer of the diagram and the viewer. One reason is that the adopted approach, the visual grammar of Kress & Van Leeuwen (2006), focuses on images, not abstract diagrams. Many of the visual indicators suggested by that approach, such as the use of physical distance as an expression of social distance, were not relevant to geometric diagrams. While the neatness and roughness elements identified by the visual grammar approach offer good tools for analysing diagrams, the other visual indictors such as labels, colour, arrows and words require additional refinement in order to be more effectively applied to diagrams.

The significance of labels and colour was considered in the analysis of all three meanings: the ideational meaning in conceptual diagrams, the interpersonal meaning, and the textual meaning as expressed by salience and framing. While this issue needs more exploration, in thinking about it, I do not see it as a deficit in the framework. The major visual indicators in geometric diagrams may be limited in terms of their types (labels, words, arrows, colours and combinations between them) and, therefore, it may make sense to consider them in the analysis of all kinds of meanings. What

does need more thinking is the mathematical meaning potential for the different uses of these indicators.

Another aspect which the current study aimed to achieve, though to a lesser extent than was the case for the diagrammatic framework, was a framework to analyse the role of gestures. The preliminary suggested gestural framework clearly requires more investigation. First, the suggested analysis of the ideational meaning presented in this thesis should be developed. Second, the gestural framework should be developed for the other meanings, interpersonal and textual. There are, however, studies which may assist in that development, most prominently the work of Luis Radford (Radford, 2003, 2009; Radford et al., 2007), among others.

4.2   Generalisability of the framework beyond geometry

There were different sources of data for this study (textbooks, Internet, students' written texts) which enabled me to generalise the framework to geometric diagrams within Euclidean (2D) geometry. While most of the suggested features may be generalised to 3D geometry, the ability to generalise the suggested framework beyond geometry requires more work. One aspect of the framework, for example, which cannot be generalised to 3D geometry, is construction diagrams, which are limited to 2D diagrams. This issue, nevertheless, does not jeopardise the robustness of the framework. I have pointed in different places (mainly in Chapter 4 and in the current chapter) to evidence of the robustness of the framework.

A similar warning should be issued about the gestural framework, to which I add the comment that the gathered data were not intended to be used for consideration of the gestural mode of communication. It was only after observing the students' heavy use of gestures during their communication in solving the tasks of the study that the significance of the gestural mode became apparent to me.

4.3   Issues outside the scope of the study

In addition to the large number (around 350) of students' written texts which informed the development of the diagrammatic framework, the Internet was another way to look at different geometrical diagrams, which include what I term 'dynamic

diagrams on the screen'. By dynamic, I mean diagrams that move, in the physical sense. This kind of diagram was beyond the scope of the current study. In other words, the suggested framework for the diagrammatic mode is applicable to 2D geometric diagrams whether they are drawn in print or on the screen, as long as these diagrams are not in motion. While moving diagrams call for further research and investigation, such research and investigation could be facilitated by extending the methodology of developing the framework.

5.   Implications

The current study offers two suggested frameworks to describe and analyse the role of diagrams and gestures in mathematical communication and representation. It furthermore suggests a multimodal analysis approach to look at ensemble modes of communication and representation (verbal, diagrammatic and gestural) simultaneously. These contributions have three different implications: theoretical implications, possible contribution to research (as a tool for analysis and interpretation) and possible contribution to practice and to the development of practice (for teachers and textbook writers).

5.1   Theoretical implications

Halliday & Hasan (1985) argue that language is a social semiotic system of meaning, and together with other systems of meaning, it constitutes human culture. This study suggests that diagrams, as visual representations, are part of these systems of meaning. In other words, producing diagrams is a social activity 'similar' to writing or gesturing, though offering different possible meanings.

Within the diagrammatic mode, different structures can be distinguished one from the other by the element of temporality, the significance of which has been identified in this study. Narrative diagrams, as suggested in this study, display temporal elements such as dotted lines or arrows, and, in doing so, they tell a story. Stories include participants and activities or actions. Participants and activities in mathematics should be mathematical! If diagrams have no temporality, they present a 'thing' or an 'object' that has no action. This object, in mathematics, should be, again, mathematical.

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In that sense, diagrams are functional. They do something within a context, tell a story for example. Moreover, diagrams, as a result of this study, fulfil the three metafunctions suggested by the Hallidayan SFL.

This study is concerned with the relationship between the structure of diagram and the social structure. The approach to analysing diagrams and constructing meaning starts by looking at visual indicators in diagrams as suggested by this study, recognising the structure and, later, deriving potential meaning, considering the context in which the diagrams were produced. This analysis provides a powerful tool in revealing the role of human beings in doing mathematics in narrative diagrams. I would argue that even when mathematicians obscure the human action in mathematics, as they often do, they still represent their mathematical activities in the diagrams they present, simply because these diagrams are social and cultural activities. Sometimes they even 'declare' the presence of activities in diagrams by showing some human figures or physical context, as in Figure 6-18. However, they sometimes 'succeed' in avoiding 'leaving' any traces or representations of these activities and, hence, they present mathematical objects in conceptual diagrams.

A related aspect is the pedagogical consequences of that development. I find myself asking questions which may reveal the way people construct mathematics and hence the social nature of mathematics. For example, why was Descartes' approach so successful in changing the way mathematicians viewed the use of diagrams in geometry? How did Descartes' approach influence other areas of mathematics? How has the change in the way diagrams are viewed led to the construction of the current dominant view of mathematics? While it is true that these questions belong to the field of philosophy of mathematics, the answers directly affect the way we conceive of mathematics, which is an area of research in mathematics education worthy of investigation. In other words, the question of how the theoretical approaches to mathematics adopted by mathematicians find their way to school mathematics is worth exploring.

What is mathematical literacy?

The development of the frameworks and the multimodal analysis raise a challenge about literacy in mathematics for mathematics education. The challenge is to answer

the question, what does it mean to be literate in mathematics? This is a twofold challenge for mathematics education; first, literacy, traditionally, means reading and writing, for which the page is the dominant medium, but now, with the development of technology, mathematics teaching and learning takes place on the screen, not just on the page. Second, mathematics is an area which is shaped and distinguished by the use of symbols and 'very little writing' (Morgan, 1995).

These challenges, however, have been tackled from different theoretical perspectives especially from the sociolinguistic, social semiotic and multimodality approaches. The relationship between writing and learning mathematics, for instance, has been the focus of the Writing-to-learn movement as a development of the Writing Across the Curriculum approach which spread in the UK and the United States in the 1970s and 1980s (Bazerman et al., 2005; Morgan, 1995). Bazerman et al. (2005) review the state-of-art of the Writing-to-learn approach in mathematics among many other disciplines. Most of the studies about Writing-to-learn mathematics, including Emig's work in the 1970s and Connolly & Vilardi's work in 1989 – *Writing to learn mathematics and science* – show to some extent that writing support learning.

On the first issue, the relationship between language and literacy, I note that the view of literacy as just reading and writing has been challenged. Street (2005), for example, argues that there is a need to extend the traditional perspective which reduces language and literacy to rules and products in order to see them as processes and social practices. He also calls for considering other modes of communication such as images and gestures. This call has been the focus of the work of scholars adopting the multimodality approach, which argues that the notion of literacy as reading and writing has changed as a result of the shift from page to screen and that literacy is no longer just a matter of language but also of different modes of communication which take place simultaneously, such as images and gestures (Kress, 2003).

In the current digital era, reading and writing take on new meanings and new 'shapes'. For example, digital books, web pages and chat rooms impose new ways of reading, writing and learning. A substantial body of research has adopted a variety of views of the relationship between technology and learning (mathematics), but more exploration is still needed to consider different modes of communication, such as visual communication, gestures and gaze (Jewitt, 2003a, 2003b, 2006; Kress, 2003).

For instance, visual representations as a mode of communication need to be read in a 'different' way. *Reading Images* (Kress & Van Leeuwen, 2006), for example, offers one way to do so. In other words, the new developments make it necessary to explore further how to read visual representations (as a mode of communication either in digital or non-digital forms) and their relationship to the learning process.

While there is a recognition that mathematics textbooks use images, pictures, graphs and diagrams to offer alternative ways to communicate mathematical concepts (Campbell, 1981; Shuard & Rothery, 1984) and that reading visual representations inside mathematical texts is as important as reading and writing the natural language itself, especially when teaching young children (e.g. Clements, 1999; Noss & Hoyles, 1996), the dominant attitude toward visual representations is that they are necessary to enable students 'to get the full message' (Shuard & Rothery, 1984).

Morgan (1995, p. 222) challenges the notion of 'getting' the full message or meaning intended by the author, arguing that a 'reader, rather than gaining direct access to the author's meaning, constructs her own meaning from the text.'

5.2   Contribution to research:

The suggested diagrammatic (and gestural) framework offers a tool for analysis and interpretation of mathematical diagrams. One of the justifications for focusing on diagrams in this research is that some studies reported reluctance among students to use visual representation in problem solving (Eisenberg & Dreyfus, 1991). Eisenberg and Dreyfus (1991) provided a cognitive interpretation for that reluctance, noting that 'thinking visually makes higher cognitive demands than thinking algorithmically' (p. 25). The suggested framework, however, offers a different interpretation, a social one, derived, for instance, by analysing the way diagrams are represented in textbooks and in teachers' practices. We can imagine what conceptions about diagrams or even mathematics students might construct if they would learn, as some already have, from a textbook produced by Dieudonne in the 'new maths' period in France (as quoted in Mesquita (1998, p. 184)):

> I had allowed myself not to include any figure in the text, perhaps only to see that we can entirely dispense with them very well; but this is an omission, and my readers will fill in the gap of my text.

Even if textbooks include diagrams, their use of and attitude toward diagrams have an effect on students' approach to mathematics. The influence of mathematics textbooks on positioning the mathematics learner has been the focus of Herbel-Eisenmann & Wagner (2007) who argue that obscuring human agency, nominalisation for example, in doing mathematics presents mathematical activity as if it occurs of its own accord. This issue has also been discussed in Morgan's work (e.g. Morgan, 1996b). Similarly, I would argue, the presentation of diagrams without any trace of human agency may contribute to the notion that mathematics is an autonomous system which, in turn, serves the absolutist perspective that mathematics exists somewhere, and students need to discover it. In order to conceive of mathematics as a social practice, there is a need to change that representation in class practice (learning and teaching) and in the textbooks.

The multimodal analysis used in the current study may contribute to the study of the way in which mathematics is constructed in schools. Mathematics is a social practice shaped by the different modes of communication and representation, and, thus, this analysis calls attention to the need to take these modes into consideration. The analysis of the role of the diagrammatic and the gestural modes may contribute to the study of the construction of school mathematics. The ideational meaning in the diagrammatic mode, for example, offers a tool for analysing the image of mathematical activities presented in mathematics textbooks. This analysis is strengthened by the other analysis offered by the gestural mode suggested here and the verbal mode suggested by Morgan (1995). Furthermore, the multimodal analysis is additional evidence of the robustness of the diagrammatic framework, as discussed in Chapter 4.

5.3   Implications for practice and the development of practice

The way in which textbooks and teachers make use of diagrams may contribute to a better understanding of the way in which students make their own mathematical meaning while solving mathematical problems. The framework suggested for reading diagrams offers textbook writers/designers and teachers ways of reviewing and planning their work when they wish students to encounter particular kinds of mathematical meanings.

One of the main contributions of the framework, I think, is that it draws the attention of these groups to the need for developing different views of mathematics. Rather than presenting a limited variety of diagrams in textbooks, the framework distinguishes between narrative and conceptual diagrams and, thus, encourages a richer image of mathematics to be made available for students. Therefore, textbook writers/designers may need to review the types of diagrams presented in textbooks to encourage students to engage with different kinds of mathematical meanings.

I would also recommend that mathematics teachers use a variety of diagrams in their practice and that they encourage students to do so in their solutions. The framework suggests planning for the use of diagrams so that the students will be aware of the implications of the different diagrams with which they interact.

Similarly, the suggested gestural framework calls attention to the significance of the gestures used by teachers or represented in textbooks. The framework distinguishes between narrative and conceptual gestures, each of which offers different views about mathematics with which teachers and textbook writer/designers may wish to familiarise students.

6.   Suggestions for future research

While developing the diagrammatic mode made use of a large number of geometric diagrams, the scope of the study has been limited to 2D geometry in order to build the case about the diagrammatic mode of communication and the need for considering it in mathematics discourse. Thus extension of the suggested diagrammatic mode needs careful further investigation. However, I think that there are some features that may be considered to be essential in developing a general framework for mathematical (beyond geometrical) diagrams. I would start with the temporality issue as a distinguishing feature between narrative and conceptual structures of diagrams. The dotted line feature is also another element that would probably be relevant in other mathematical texts. Moreover, some of the considered features are related to a specific context such as the construction structure.

An interesting aspect that this study did not set out to address but has nonetheless explored is the historic development of the use of diagrams within mathematics discourse. Although there are considerable and important studies that have looked at

diagrams within the history of mathematics such as those by Netz (1999), O'Halloran (2005) and Robson (2008b), still the 'full history of diagrams (...) is far from being written yet' as De Young commented (personal communication, November 26, 2008). A specific question that arises, after I made a short historic presentation of the use of diagrams in the history of mathematics, is when was the turning point of prejudice against the use of diagrams? This question is especially interesting, because we know from the literature that diagrams were used throughout the different civilisations, such as Old Babylon and Ancient Egypt and Greece. Although the current study has indicated that the turning point may have been the algebraisation of geometry and the work of Descartes followed by Newton, the question requires further investigation.

One consequence of Descartes's algebraisation is a transition from a geometric mode of thinking to an algebraic one (Mancosu, 1996) in which the dominant image of mathematics is formal and symbolic. The current study has not investigated algebraic notations (symbolism) or the symbolic mode, which have not been fully developed in the field of mathematics education. O'Halloran (2005), however, has suggested a theoretical framework to read mathematical symbolism and claims that the symbolic mode has developed a new and different meaning potential. I want to suggest, however, that still there is a need to develop a framework that is consistent with the two suggested frameworks here, namely to consider the narrative/conceptual aspect of the intended framework. This, I argue, is more consistent with the SFL approach, the visual grammar suggested by Kress & Van Leeuwen (2006) and the linguistic approach to mathematical texts suggested by Morgan (1996b), all of which aspire to a comprehensive framework that takes into consideration the different modes of communication and representation.

The multiple modes of mathematical discourse have been analysed in one of the chapters of this thesis (Chapter 10) by providing an illustrative example analysing three modes of communication. While I did not initially plan to explore the multimodal analysis, during my research, I gained some insights into how it would look in mathematical texts. However, the multimodal analysis still needs more development, and there is a need for further study of the ensemble modes in the representation and communication of mathematics discourse.

The two suggested frameworks, finally, together with the multimodal analysis method developed in this study, provide a set of tools which deepen and extend our understanding of the multimodal nature of mathematical communication in mathematics learning and teaching and provide a critical perspective on the way in which mathematics is constructed in curriculum and school practice.

References of figures

Figure 5-1
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Figure 5-2
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      the author)
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      author)
Diagram c: Palestinian textbook, Grade 8 (part 2, p. 36)[24]
Diagram d: Year 8, Debbie, typical

Figure 5-3
Diagram a: Year 8, Wendy, typical.
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Figure 5-4
Diagram a: http://www.lexington.k12.il.us/teachers/menata/MATH/geometry/
      trianglesdd.htm
Diagram b: http://www.mathsisfun.com/quadrilaterals.html
Diagram c: Year 8, Patricia, unique.

Figure 6-7
Diagram a: UK textbook (Allan, Williams, & Perry, 2004, p. 246)
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Figure 6-8
Diagram a: group's work (Lallie, Miranda & Laurinda), Year 8, typical
Diagram b: http://www.mathsisfun.com/geometry/transformations.html
Diagram c: http://www.mathsisfun.com/obtuse.html
Diagram d: http://math.about.com/library/blmoreonangles.htm

Figure 6-10
Diagram a: Year 9, Hailey (typical)
Diagram b: http://www.tutorvista.com/math/solve-triangle-transformation
Diagram c: Palestinian textbook, Grade 8 (part 2, p. 60)
Diagram d: (Stein, 1999, p. 73)

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[24] All the Palestinian textbooks can be viewed and downloaded via: http://www.pcdc.edu.ps/
textbooks/index.htm. The number of page(s) refers to the page of the book in the pdf file and not to
the page of the file itself.

Figure 6-11
Diagram a: UK textbook (Allan et al., 2005a, p. 130)
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Diagram a: http://korn19.ch/coding/triangle/
Diagram b: http://commons.wikimedia.org/wiki/
File:Pythagorean_Theorem_Proof.png
Diagram c: http://math4allages.wordpress.com/2010/02/03/pythagorean-theorem/

Figure 7-6
Diagram a: http://rchsbowman.wordpress.com/2010/09/23/geometry-notes-exterior-
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Diagram b: Palestinian textbook, Grade 7, part 2, p. 59
Diagram c: http://www.mathsisfun.com/right_angle_triangle.html

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Diagram b: http://www.gogeometry.com/problem/
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Diagram b: http://www.mathsisfun.com/geometry/circle.html

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Figure 7-13
Diagram a: http://faculty.matcmadison.edu/kmirus/textbooks/804380text/
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Diagram b: http://www.tutorvista.com/math/measurement-of-angle
Diagram c: http://www.tutorvista.com/topic/3pi-2-radians

Figure 7-14
Area of triangle, rectangle & parallelogram:
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Diagram a: http://www.mathsisfun.com/quadrilaterals.html
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Diagram a: Grade 8, Sami, typical
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    author)

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Diagram a: http://www.tutorvista.com/topic/pythagorean-theorem-tutor
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Figure 8-5
Diagram a: http://www.tutorvista.com/topic/pythagorean-theorem-tutor
Diagram b: http://www.sparknotes.com/testprep/books/gre/chapter2section3.rhtml
Diagram c: http://www.algebra.com/algebra/homework/Surface-area/Surface-
    area.faq.question.197702.html

Figure 8-7
Diagram a: http://www.mathsisfun.com/quadrilaterals.html
Diagram b: Palestinian textbook, Grade 7 part 2, p. 52

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Diagram a: Palestinian textbook, Grade 8, part 1, p. 49
Diagram b: http://www.mathsrevision.net/gcse/circle_theorems.php

Figure 8-9
Diagram a: Palestinian textbook, Grade 7 part 2, p. 38
Diagram b: http://www.bbc.co.uk/schools/ks2bitesize/maths/shape_space/
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Diagram c: Year 9, group's text (Sandra, Colleen & Lillian), uncommon

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Diagram b: http://www.mathsrevision.net/gcse/circle_theorems.php
Diagram c: http://a.parsons.edu/~lik43/sackboys/?p=208

Figure 8-11
Diagram a: http://gogeometry.com/problem/
           p221_viviani_theorem_equilateral_triangle.htm
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Diagram a: Year 8, Group text (no names), common
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Diagram a: Year 8, Homer, common
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Diagram a: Palestinian textbook, Grade 8 part 1, p. 48
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           88557.html

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