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EDITORIAL

Research on the reasoning, teaching and learning of probability and uncertainty

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Abstract In this editorial, we set out the aims in the call to publish papers on informal statistical inference, randomness, modelling and risk. We discuss how the papers published in this issue have responded to those aims. In particular, we note how the nine papers contribute to some of the major debates in mathematics and statistics education, often taking contrasting positions. Such debates range across: (1) whether knowledge is fractured or takes the form of mental models; (2) heuristic or intuitive thinking versus operational thinking as for example in dual process theory; (3) the role of different epistemic resources, such as perceptions, modelling, imagery, in the development of probabilistic reasoning; (4) how design and situation impact upon probabilistic learning.

Keywords Probability · Sampling · Confidence intervals · Modelling · Risk

Probability, statistics, data handling and stochastics are different names for what is taught at school level in various countries. Factually we find an overlap in these topics but we recognise that different names often carry different meanings and philosophies or fundamental ideas of the subject (Burrill & Biehler, 2011). For this special issue, we were particularly interested in the role of probability related to data analysis and we invited scholars to submit papers addressing at least one of the following three subthemes influencing the current debate:

(i) We were interested in studies of informal inference, including sampling distributions and routes to formal inference. We recognized that informal inference might be seen either as an essential skill for the statistically literate citizen or as the root to a sophisticated understanding of formal inference. Either way, we saw probabilistic thinking as an essential component in making judgements about the reasonableness of patterns and trends identified in data, especially in an era when recent developments have focussed on data handling, to some extent marginalising the role of probability (Konold & Kazak 2008; Borovcnik, 2011).

(ii) Randomness and variation in the short and long term in data sets generated by experiments have been the foci of several studies over the last two decades (recent examples include: Abrahamson, 2006; Paparistodemou et al., 2008; Pratt & Noss 2002). Given the importance of this central idea to probabilistic reasoning, we invited contributions that would report on recent studies in this particular area.

(iii) Finally, we were interested in studies of modelling and risk, including theoretical, experimental and subjective probability. In a sense this theme anticipated where we might see new developments in research on uncertainty. As technological tools become ever more sophisticated, there are new possibilities for introducing modelling into younger students' thinking. It has been argued (Burrill & Biehler; 2011; Konold et al., 2007) that, whereas typical school curricula focus on classical probability as exemplified by spinners, coins and dice,

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> professional statisticians and workplace applications are much concerned with probabilities that can only be estimated in frequentist or subjective terms. We invited studies that might be researching new opportunities in areas such as risk-based decision making.

In this short editorial, we introduce the reader to nine papers, published in response to this invitation. In fact, many of the papers of this special issue use designs that essentially rely on the use of technology, particularly for simulating stochastic processes and analysing the data produced (see Biehler et al. 2012 for more details on technology use in probability and statistics). Most of the papers touch on more than one of these topics. The papers which mainly focus on (i) above are those by Ben-Zvi et al. and Pfannkuch et al.; those that contribute mainly to (ii) above are by Schnell and Prediger, Chernoff, Prodromou and Abrahamson; finally, Eichler and Vogel speak primarily to (iii) above, as well as Pratt et al. and Garfield et al., though the latter fits well into (i). In our commentary below, we will discuss the relevance of the specific papers to these topics. The papers often take different positions in some of the key ongoing debates in mathematics and statistics education. In this editorial, we look across the papers and highlight those debates, beginning with the nature of mathematical or statistical abstraction.

Since Piaget & Inhelder's seminal work (1975) on the origin of chance, there has been particular interest in how notions of randomness and key probabilistic concepts such as the Law of Large Numbers develop. Indeed, one account of why probability is regarded as a difficult concept is based around their assertion that probability is dependent on: (i) randomness, which at earlier stages cannot be accommodated within operational thinking; (ii) proportional reasoning; (iii) combinations and permutations; according to Piagetian theory, (ii) and (iii) require formal operational thinking, the organism's highest cognitive achievement. While stage theory approaches are perhaps now less popular in modern research, constructivism continues to have a strong influence on research into the development of mathematical and statistical abstraction and this is evident in many of the papers in this issue where echoes of assimilation, accommodation and reflective abstraction can be discerned.

A common interest in recent decades has been to identify what experiences during early years of schooling might facilitate learning probability. In particular there has been a focus on how learners might respond to experiencing situations where outcomes are uncertain. Fischbein (1975) argued that primary intuitions for the stochastic could be nurtured through systematic schooling, allowing cognitive development well before probability is invented through formal operations. Indeed, researching how experience shapes intuitions could be seen as lying at the heart of several papers in this issue (in particular, those by Schnell & Prediger, Eichler & Vogel, Ben-Zvi et al. and Pratt et al.).

In this issue, Schnell and Prediger refer to vertical and horizontal development. While horizontal development describes conceptual change that extends across changing contexts for the same concept, vertical development focuses on students' development of new conceptions, which substitute initial conceptions. Insofar as a vertical development might be seen as abstraction to 'higher' concepts and horizontal might be seen as generalising across contexts. Pratt and Noss (2002) have argued that development occurs essentially through a broadening of the contextual neighbourhood by bringing *more* context into meaning-making and that abstraction *away* from context is an illusion. However, in their study, Schnell and Prediger adopted the notion of 'construct' from the model of abstraction in context proposed by Schwarz et al. (2009) to explore students' development of thinking about randomness in the short and long term in terms of horizontal and vertical development. They propose four categories of microprocesses that contribute to vertical and horizontal lines of conceptual change. Under the vertical line of conceptual change, they include actions such as 'refining or broadening the scope of applicability', which Pratt and Noss had seen as a key aspect of the micro-evolution of knowledge. However Pratt and Noss made no distinction between these actions and those categorized as horizontal by Schnell and Prediger, such as 'building complementary contexts' and 'transferring constructs to different settings'. The question remains whether there is theoretical leverage in distinguishing between 'broadening the scope of applicability' and 'transferring constructs to different settings'. It may help to resolve this difference in the future by referring to recent work on transfer, in particular the notion of 'transfer in pieces' (Wagner, 2006), which argues that there are different types of transfer.

In contrast to the theory developed by Schnell and Prediger, Eichler and Vogel's paper in this issue uses mental models as its theoretical lens. They argue against Piagetian stage theory and propose an alternative in which mental models are seen as representations of an entire situation, rejecting the use of semantic representations of isolated propositions.

The Schnell and Prediger and Eichler and Vogel positions stand either side of a hotly contested debate (diSessa, 2008). On the one hand, conceptual change is seen as the gradual alignment of knowledge that begins in a fragmented state. On the other hand, relatively large and coherent conceptual models of the behaviour of phenomena replace prior models through substitution. Nevertheless, a major contribution of Eichler and Vogel in this issue is that they consider how young students, prior to explicit teaching on probability, make sense of uncertain situations where the sample space is not countable. This stands in contrast to most research on probabilistic reasoning that has focused on coins, spinners and dice, in which it is possible to identify a sample space of countable and equally likely outcomes. Indeed, Pratt (2011) has urged more research in this area to inform the development of new curricula that escape the limitations of approaches that focus narrowly on increasingly irrelevant artefacts such as coins, spinners and dice.

Eichler and Vogel argue that young students might not even recognize that situations could be seen as random and so seek in their method to avoid leading them to any such conclusion. They offer a range of situations to which their subjects have to respond. They suggest that students cannot be ascribed general levels of cognition since their levels are in fact dependent on the complexity of the specific tasks. Such a conclusion might easily have been drawn via other theoretical positions but what is novel in this study is that they analyse the complexity of their situations according to how visible are the objects and data and whether the tasks demand an inference to be made.

Schnell and Prediger created situations in which the students experienced in a direct way the uncertainty apparent in the experiments they conducted. Similarly, Prodromou in this issue reports on pre-service teachers, who reason about connections between experimental and theoretical probability in the light of experience during a course that involved manipulation of material and virtual objects. We have also discussed how Eichler and Vogel's tasks urged the students to imagine scenarios that might stimulate mental models. However, Abrahamson in this issue questions whether abstracting from experience is the only, or even the best, approach for accepting the logic of combinatorics. He emphasises how experience can be misleading, such as when students do not discern Head/Tail and Tail/Head as distinct events when flipping a pair of coins. In other words, probability in its formal sense is unlikely to be accommodated, in Piagetian terms, until students are able to construct a combinatoric approach to probability. Abrahamson argues that perceptual reasoning provides an alternative to experimentation as an epistemic resource to ground the logic of combinatorial analysis. A pedagogy based on building upon perceptions, exploiting carefully customised event spaces, might effectively supplement experience.

We should stress that much of Abrahamson's work (2007) has in fact focussed on providing experiences as a basis for students to make connecting bridges between different epistemologies for probability and it is clear that the position taken in this paper is one in which he reminds the reader that such an approach should not exclude methods based on the perceptions.

We note the potential for modelling approaches in relation to Abrahamson's position since engagement with the modelling cycle demands the construction of a first model, which may well be based on initial perceptions, which will then be modified iteratively by experiences of running that model and comparing the outcomes with expectations. Garfield et al. report in this issue on their novel design for a tertiary level course, based on the assumption that modelling phenomena is key in developing statistical thinking. Although statistical models lie at the heart of the discipline, courses at either school or college level have not generally been built around modelling as the central activity. Garfield et al. chose Tinkerplots to be the main modelling tool used by the students. They designed open-ended tasks that would motivate students to create and evaluate stochastic models. For example, the students have to decide whether multiple playlists from an iPod shuffle were in fact randomly generated, as claimed by the manufacturer.

A third way is proposed in this issue by Pfannkuch et al. Rather than focussing on perceptions or modelling, they emphasise imagery. Their methods of course recognise the power of initial perceptions and how modelling might be utilised to enhance reasoning but Pfannkuch et al. promote the use of imagery in designing a teaching pathway that begins with elementary understanding of sampling at about 14 years of age and culminates in a sound appreciation of confidence intervals at tertiary level. The authors argue in the paper that the pathway should lead to appreciation of confidence intervals rather than significance testing. Technology is an important tool since some of the methods such as bootstrapping confidence intervals are not feasible without technology use but, throughout the development, the aim, whether achieved through the use of technology or not, is to facilitate the construction of powerful images for key statistical concepts.

The reports by Abrahamson, Garfield et al. and Pfannkuch et al. offer insights into how perception, modelling, simulation and imagery might provide valuable epistemic resources for the construction of probabilistic concepts. Pfannkuch et al. argue that the conceptual pathway towards confidence intervals needs to begin at an early age, referring in fact to age 14 years. Ben-Zvi et al. report in this issue on a study, which draws on all of these epistemic resources but at a much younger age, 11 years. Again deploying Tinkerplots, their study builds on earlier work into growing samples (Bakker & Frederickson, 2005) whereby students begin with small data sets and are encouraged to make inferences about the class as a whole. These initial perceptions are expected to be insufficient or inaccurate when the data about the whole class are analysed. In a similar way the data for the class are then used to make inferences about a larger group such as the whole year, and so on. The use of Tinkerplots allows the students to create images of the data and so draw inferences in the spirit of exploratory data analysis (EDA). They report on the increasingly sophisticated expressions of uncertainty that emerge as the students continue to grow their samples.

The role of the growing samples task is key to the development of the probabilistic language that Ben-Zvi et al. report. Indeed, the criticality of task design and situation is a theme that runs through all of the above papers. Schnell and Prediger carried out a microanalysis of students playing a race game and uncovered highly situated microprocesses. Eichler and Vogel's analysis picked out the visibility of the objects and data as critical elements in determining the level of complexity of a task. Abrahamson reports on how opportunities to experience phenomena might at least be supplemented by tasks that exploit students' perceptions. Garfield et al. invented new modelling tasks in order to implement a course that might stimulate the growth of statistical thinkers. Pfannkuch et al. reported on a series of design principles that they feel should underpin the design of a conceptual pathway that aims to facilitate stochastic knowledge and ultimately understanding of confidence intervals.

The report in this issue by Pratt et al. follows this theme. Recognising the opportunities and needs for teaching and learning of risk, they have built a microworld to research mathematics and science teachers' knowledge of risk, situated in this case in a fictitious young woman's dilemma as to whether to have an operation that might cure her back condition. They search for a way of accounting for the decisions that the teachers made, and for the deep use of personal experience and other contextual factors that seemed to shape the decision-making process. Their review of the literature led them to the Priority Heuristic (Brandstätter et al., 2006), which is a recently published description of how people make decisions. The Priority Heuristic is part of the school of thought, led by Gigerenzer et al. (1999), which asserts that heuristics are natural and effective ways of making decisions and, within any constrained situation, may prove more effective than methods dependent on formal logic. In their study, Pratt et al. found that the teachers' decision making did often conform to the logic of the Priory Heuristic but that important insights could be gained by careful analysis of the apparent discrepancies.

Gigerenzer has been a critic of the seminal work on heuristics by Kahneman et al. (1982), which has been highly influential on research into probabilistic thinking. More recently, Kahneman has recognised that a weakness in the original heuristics and biases research was its lack of a theoretical underpinning, and so has realigned the original analysis by bringing it into the broad field of research on dual-process theories (Kahneman & Frederick, 2002), so that their research on heuristics and biases is seen as part of System 1, the fast intuitive dimension of decision making and reasoning. In contrast System 2 relates to a slower, more reflective, analytic type of reasoning. In this issue, Chernoff criticises how, although heuristics such as representativeness have been an ongoing reference point to researchers in statistics education in recent decades, they have been slow to adopt the freshly aligned perspective on that original research. In particular, Chernoff draws on the notion that, when people are confronted with a difficult question, and perhaps have few resources (for example time and tools) to analyse that question, their 'fast' response will often be to replace the difficult question with an easier one, a process referred to as 'attribute substitution'. Chernoff's study uses that notion to analyse responses of prospective mathematics teachers to a task in which they decide which of two answer keys to a multiple choice quiz is more likely.

With the incorporation of heuristics into System 1 of dual process theory, attribute substitution is seen as a fast response to some demand, much as Fischbein's primary and secondary intuitions. The subjects in the range of studies presented in this issue are encouraged, whether through drawing on modelling, perceptions, simulations or imagery, to mix fast intuitive responses with more careful analytic thinking, the hallmark of System 2, accessible through the sort of formal operational thinking that Piaget envisaged. Research will need to continue its efforts to understand what sorts of pedagogic mix between intuitive System 1 and formal analytic System 2 thinking might be most influential in supporting the development of probabilistic reasoning.

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