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Reflections and Transformations: A Mathematical Autobiography

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My talk at the ICME Congress in 2004, and this paper, serve as a modest attempt to offer heartfelt thanks to the mathematical and mathematics education communities for choosing me as the first recipient of the Hans Freudenthal Medal, an honour that belongs to my colleagues and collaborators as much as to myself, and which represents a step forward for our mathematical community much more than for me personally.

In this paper, I map out some of the challenges facing mathematics education that I have tried to address during my professional life to date, and that I intend to continue to work on in subsequent years. How to exploit the potential of new technologies for rigorous rather than superficial engagement with mathematics, either face-to-face or at a distance? How better to design and operationalise teacher development programmes that foster a professional culture that is subject-specific, relevant to teaching and learning and is sustainable over time? How to help employees, in workplaces that have been transformed by the presence of computers, come to terms with the growing obsolescence of craft skills and the emergence of new mathematically-based skills relevant to their new situation? How to find ways to engage in genuine collaborations that respect and learn from the diversity of different cultures, curricula and research paradigms? How further to develop knowledge in mathematics education that derives from theoretical effort, yet which is sufficiently robust to have policy and practical implications? And finally – perhaps most crucially – how to ensure that students of all ages come to recognise that mathematics is about structures and relationships with an internal coherence, an elegance and an aesthetic, and that an exploration of these structures and the development of mathematical habits of mind (Goldenberg et al, 1998) need be neither boring nor irrelevant. As an aside, and before addressing any of these challenges in any depth, I should add that I am committed to rigorous and systematic research methods in the field of

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mathematics education and have made efforts over many years to become familiar with a range of research methods (qualitative, quantitative and mixed) to be used as appropriate to the goals of the study in hand.

In this paper, I sketch my personal response to some of these challenges and draw together some overarching themes that have underpinned my work. I want to state from the outset that any success achieved in my endeavours, has been the result of working in an environment that is both challenging and supportive, that prioritises research and scholarship while acknowledging the importance of practice, and - even more crucially – of enjoying collaborations with brilliant and diverse groups of people, many of whom are co-authors of papers arising from our joint work[1]. In particular, I must acknowledge from the start the huge contribution to most if not all of my work of Professor Richard Noss, with whom I have collaborated for many years.

I have approached this chapter as a selective summary of my research efforts. I present brief descriptions of different research strands and some projects within each, and give references to relevant papers so readers can follow up the work if they wish. I rarely refer to work of researchers who have not been part of my research teams, in order to achieve a coherent and relatively succinct overview. I hope this will not be interpreted as ignoring the work of others: appropriate references are made in the papers to which I refer.

[1] I note here the organisations with which I am – or have been – involved since 1970.

- Research projects based at the Institute of Education, University of London, UK that have included teams of researchers and teachers.
- The British Mathematics Education Community, including specifically the British Society for Research into Learning Mathematics (BSRLM) and the Joint Mathematical Council of the U.K. (JMC).
- International Group for the Psychology of Mathematics Education (PME), International Congress on Mathematics Education (ICME) and ICMI Study groups.
- Logo Mathematics Education Community, a group of researchers in mathematics education who carried out research, during the 1980s and 1990s, on how Logo could be used as a tool to explore mathematics.
- Basic Components of Mathematics Education for Teachers (BACOMET), a small group of scholars who worked together from 1980 to 2004 and produced four edited volumes.
- The Advisory Committee on Mathematics Education (ACME), established in 2002 in England as a single voice for the mathematics community. I was a founder member.

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FOSTERING ENGAGEMENT AND RESPECTING RIGOUR THROUGH DESIGN EXPERIMENTS

A major objective in my teaching and research has been to find ways to motivate students to engage in mathematical thinking. I recall in my early teaching that the worst moments were when students complained: 'it's boring'. In my PhD, I chose to investigate students' affective responses to mathematics, being convinced that we must find ways for all students to benefit from mathematics learning (Hoyles, 1982). Although not continuing this theme as an explicit research focus, it was undoubtedly its motivational potential alongside the possibility of enhancing accessibility that led me to work with Logo in the early 80's. I still believe that the potential of using software to motivate rigorous engagement with mathematics should not be underestimated – although much still needs to be done to realise this potential in practice, in schools, colleges and universities. Some of these early ideas concerning students' work with Logo were explored with my colleague, Ros Sutherland, and described in for example, Hoyles & Sutherland (1989) and Hoyles (1985). In that corpus of early work, we noted the importance of students being able to build their own models through Logo programming, which served as a medium to construct and express their evolving mathematical ideas in collaborative, long-term projects.

Thus my early Logo work was focused on ways to motivate and engage students in mathematics, and the role of the teacher in promoting this engagement. Gradually my research took on a more conceptual focus, in collaboration with Ros Sutherland and Richard Noss, through investigating new representations for mathematical objects, such as variables, ratio and functions (see, for example, Hoyles & Noss, 1986; Sutherland & Hoyles, 1986). This strand of work, shared among a group of researchers who became known as the LogoMathematics community, culminated in an edited volume (Hoyles & Noss, 1992a). Increasingly, I recognised that the public character of the screen on which the Logo programs were written could serve as a window on a range of issues that shaped mathematics learning: for example, on the way students were construing their mathematics, on how mathematics was discussed, on how teachers intervened in the process of learning, on gendered ways of working and on teachers' beliefs about mathematics teaching and learning (Hoyles, 1988; Hoyles et al, 1991; Hoyles, 1992).

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During this time, I became increasingly involved in what would now be called design research; in particular in designing microworlds around particular mathematical topics (see for example, Hoyles et al, 1989). The research evolved, in collaboration with Richard Noss and Lulu Healy, into designing and building programming tools (in Logo and later in other software) through which students could investigate deep and challenging mathematical ideas in a playful way, by building, expressing and debugging their ideas and by making sense of, and possibly arguing over, the computer feedback. As part of this research, we designed activities in which the tools could be used to maximise the potential for learning mathematics; for example, where students would invariably be faced with surprising feedback that would challenge them to reflect on any 'incorrect' mathematical assumptions they might have made, or would foster collaboration and the sharing of perspectives in order to support alternative approaches (Hoyles & Noss, 1992b). We iteratively tested the designs through extensive qualitative and sometimes quantitative evaluations. One of our most successful microworlds was called Mathsticks (Noss et al, 1997), designed to help students construct meanings for algebraic generalisations by forging links between the rhythms of their actions (the enactive mode), the graphical output (iconic mode) and the corresponding symbolic (Logo) representation, which served as a language to describe the patterns generated. The results touched on a key insight that has been at the core of my subsequent work: namely that at the heart of mathematics learning is first, the medium of expression, and second, the ability to coordinate different mathematisations of any situation, possibly using different media and different metaphorical or real connections and networks. I extended my design research by moving from a focus on tasks to the creation of activity sequences, embedding computer use, whose goal was, for example, to foster collaborative work (Hoyles et al, 1992; Healy & Hoyles, 1999), or to provoke students to explain and prove their conjectures in the domains of number/algebra and geometry (Hoyles, 1997; Hoyles, 2001; Healy & Hoyles 2001).

Following these studies, my research moved outside conventional classroom environments and in doing so I had to face another design challenge: how to to exploit the potential of informal learning for mathematics. In the Playground Project (1998-2001), Richard Noss and I set out to design systems and activity structures where mathematics learning was the outcome of a synergy of building, sharing and playing, face-to-face and at a distance. We started this work with two premises. First, when playing games, children necessarily follow rules passed down from above by the game designers. And second, there was a range of

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strategies that we knew could promote reflection in computer-based environments on underlying structures, relationships and rules, namely, strategies stimulating discussion, prediction and explanation in and across multiple forms of expression, planning for cognitive and socio-cognitive conflict, and orchestrating their resolution through debugging and the exploitation of feedback. What we did not know, of course, was how far these strategies could be incorporated into game-building and game-playing without destroying the game, and how all this could be achieved in a distributed community of young students collaborating over the web.

The Playground Project aimed to place children in the role of producers as well as consumers of games, so our overriding design criterion was to build a computational environment in which children were able to program their games in ways that were inspectable, modifiable and shareable over the web. The project achieved some success in terms of long term engagement in the development of a joint product (the game) during which time students engaged with issues of inference and the meaning of rules (see for example, Noss et al, 2002a). We also were able to identify and describe the practical challenges of this work, which provoked us to revisit our theoretical ideas on microworlds and tool use (Hoyles et al, 2002a).

In a later project, WebLabs (http://www.weblabs.eu.com/), we are extending our design work still further and shifting its focus to iteratively building tools and activity sequences in which students, in different sites across Europe, program models of their mathematical and scientific knowledge and then share, discuss and modify the models through a web-based system, WebReports. We try to ensure that the potential of collaboration is exploited in all its forms, by including asynchronous discussion and exchange around WebReports as part of any activity sequence, alongside synchronous interchange, both face-to-face and at a distance. We also aim for a more explicit promotion of learning mathematics through the processes of modelling and sharing, collective reflection and participation in a joint enterprise. We do not underestimate the challenge in this approach, but are heartened by the productive interchanges we have recorded and analysed that have demonstrated conceptual change and the overcoming of known mathematical obstacles (see, for example, Simpson et al, 2005). Inevitably, much is still to be done and the work has, once again, stimulated more theoretical reflection in two directions. First, design issues raised by this new kind of microworld involving collaboration over the web alongside the constructive use of tools. And second, the roles teachers need to

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play in building the socio-cultural mathematical norms of a distributed community, where students are expected to post their products on the web, and expected to comment on, modify and discuss the contributions of others.

Before ending this section, I wish to suggest that design research may have an important role in mathematics education, beyond the experiment and its evaluation. Design research should perhaps play a more explicit role in a teacher's everyday practice? Teachers not only shape the culture in the classroom through interactions at a meta-discursive level, but also (possibly in collaborative teams with researchers) can play an active role in shaping the tasks and activities at an object level, evaluating students' tool-mediated responses and generating the norms of language and communication. Design research could also play a more central part in programmes of teacher development, as, again, it necessarily focuses attention on the mathematical knowledge that teachers need to operationalise in their classroomas. We need of course to work together to build a robust theoretical basis for research in all its forms, and it is to theory development that I now briefly turn.

WORKING ON THEORY ALONGSIDE DESIGN

I first became involved in theoretical work in the early '80s, partly stimulated by the need for more clarity about the meaning of a microworld, partly provoked by the complexity of the task of integrating computers in teaching and learning mathematics and the need to better understand the trajectories of students' mathematics learning, and partly through my early involvement in BACOMET (Hoyles, 1991; Hoyles, 1993; Hoyles, 2005). With the benefit of hindsight, I now recognise that in my early Logo research, I may not have given sufficient attention to the complexities underlying the introduction of microworlds into institutionalised mathematics teaching, even microworlds that had been carefully designed in terms of computer tools, sequenced activities and the teacher's role. These complexities include a recognition of the ways the 'computer' shapes mathematical knowledge and the interactions between learners and between learners and teachers, and crucially, how computers frame the language in which mathematics is expressed and the meanings of 'doing maths' and communicating mathematically. Moreover, I have come to recognise that if these ways of doing and communicating mathematics are not legitimised, computer use will inevitably be marginalised in mathematics classrooms or defined as 'learning about computers' rather than learning mathematics.

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Reflecting on my early aspirations, I am struck how similar they are to much of the rhetoric around computer use that we hear in 2004, and that despite our energetic, enthusiastic and disciplined research with teachers and students, rather little has changed in mathematics classrooms that are not supported by research or development teams. Certainly the technology has changed and there is now a plethora of excellent software available with which to explore and to build mathematics. There is also a growing corpus of fine research in the field of computers and mathematical learning: I will mention just one source with which I have been associated, the International Journal of Computers for Mathematical Learning: there are, of course, many more. But rather little has altered in the way mathematics is conceived, the way it is taught, the hierarchies in place and who is and is not judged as mathematically competent. In the '80's, I was filled with huge optimism that computers (alongside teachers) would change mathematics teaching, making it more exciting and more inclusive. I did not recognise adequately the complexity of using computers for mathematics learning; complexity in terms of curriculum and pedagogy and access to hardware and software, but more profoundly, complexity around epistemology, conceptual understanding and how to 'transfer', to relearn in new contexts, the mathematical knowledge developed through interaction with computers.

In our early Logo research, Richard Noss and I noted how students' work with Logo served as a window on their evolving mathematical knowledge, knowledge which not only shaped the Logo language and programs they chose to use, but also was shaped by Logo, the medium through students largely expressed their mathematics. We described this process as one of 'situated abstraction' (see Hoyles & Noss, 1992a), and elaborated this notion, along with that of webbing – the connections made during this conceptual process - in our book (see Noss & Hoyles, 1996a). Underpinning both notions is the recognition of the crucial role that symbolic tools play in mediating mathematical expression and mathematical communication.

As I argued in Hoyles (2002), discussion of tool mediation as a unit of analysis is all too often missing in mathematics education research. The reference for this remark was the mediation by computer tools, but is not necessarily limited to computer interaction. Written production or verbal remarks are as much shaped by the medium as by computers; it is simply a matter of our familiarity with the outputs that renders the medium invisible. I discern the beginnings of a convergence of views in recognising the centrality of tool mediation in mathematics education (see the overview of uses of digital technology in Hoyles & Noss, 2003,

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the discussion in Hoyles et al, 2004, along with the recent edition of Educational Studies in Mathematics, edited by Nemirovsky & Borba, 2004). It may be that establishing and elaborating a tool-mediation focus to mathematics education research could help to build bridges between the individual and the social, as it places the spotlight on the expression of mathematics and on the communication of mathematical ideas within and between communities. Additionally this focus, while keeping mathematics at the centre of inquiry, foregrounds the role of the design of activities, the design or choice of the tools or sign systems that introduced to foster mathematics learning and, of course, requires investigation of the transformative potential of these tools (see Hoyles, 1995, and more recently, Kaput et al, 2002; Hoyles & Noss, 2003). Further work in this direction is planned in an international setting in the forthcoming ICMI Study that I am co-chairing with Jean-Baptiste Lagrange. As part of its remit around digital technologies and mathematics teaching and learning, this Study will consider how cultural factors impinge on computer use, in different phases of education, both within and outside educational institutions by particularly incorporating analyses of the situation in developing countries.

Clearly there will be a continual need to revisit this agenda as technology changes. But at the core must be the theoretical resources, necessary to optimise the chances of sustained, systemic strategic change for the benefit of mathematics teaching and learning. We know now that the use of computers to motivate investigation of mathematics – even if undertaken after careful design experiments as we did in the early Logo research – may not result in any long-term change and improvement. We have long recognised that computers could not achieve anything on their own and that ultimately, despite the best tools and resources, "It's down to the teachers". But we need nonetheless, to elaborate and refine what this might mean in different circumstances and with different mathematical goals, and so be better placed to anticipate the challenges that inevitably must be faced. Design research has a growing theoretical base alongside the development of tool-rich activity sequences (see, for example, diSessa & Cobb, 2004). But due research attention must also be paid to how to 'transfer ownership' of the design innovation beyond its initial sites, and to how to do this without trivialising the innovation and losing sight of its essential mathematical goals[2] (see, for

[2] Computer-catalysed innovations all too often are trivialised: take for example what happened so often with Logo, which simply became a tool to draw a square!

example, my work with Teresa Smart around the professional development of mathematics teachers in London, Smart & Hoyles, in preparation).

RESEARCHING STUDENTS' CONCEPTIONS OF PROOF AND REASONING

As well as researching the use of computers in mathematics education, another major strand of my research, that links with the agenda of investigating 'engagement with rigout' has been around the investigation of students' conceptions of proof. My motivation once again grew out of my teaching. Why was it that students did not feel the need to prove, the need to struggle to ensure their arguments were water-tight - something I had always so enjoyed myself. At the time of my first proof project with Lulu Healy, the mathematics curriculum in England had undergone considerable change and we set out to investigate the effects of this change on mathematical reasoning. Given that the English National Curriculum was statutory, there was consistency in the intended curriculum across the country, so it was reasonable to adopt a methodology to investigate students' responses to proof tasks that comprised a large national paper-and-pencil survey, followed by classroom observations and interviews with teachers and students. This project entailed adopting (for me) new research techniques, quantitative statistical analyses alongside qualitative studies: for example, we used multilevel modelling of data to identify predictors of success in proof and to identify schools and students with exceptional success, and case study to elaborate and contextualise possible reasons for this success.

The multilevel modelling analyses showed consistently that students' successes in constructing proofs, and their choices of arguments that best exemplified their approaches, were strongly influenced by mathematical attainment as measured by standardised national tests, but were never determined by this factor alone: students' views and evaluations of proofs, their gender, and their experiences of the curriculum, all exerted significant influences on responses. Our findings also suggested that classroom climate might be influential, in that we consistently found that students in classes in which a larger percentage of students were to be entered for the most challenging assessment at age 16 years, produced better responses from their students than equivalent students (that is, students matched in terms of all other predictors), in classes with a smaller percentage. This finding is particularly interesting, not

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least as it has been replicated in my later research. We found that students revealed many of the problems identified in previous research on proof, but we also found, through this large-scale research, new factors that seemed to frame their responses. For example, students simultaneously held two different conceptions of proof; those about arguments they considered would receive the best mark and those about arguments they would adopt for themselves. We also noted that students tended to confer status on proofs because of factors quite apart from their generality or logical nature, such as the presence of named geometrical facts or relationships, or the inclusion of algebra (see for example, Healy & Hoyles, 2000).

Of course, longitudinal studies were needed to draw out any causal links between the factors identified and the outputs measured. I therefore was fortunate enough to be able to follow up this first proof project with a longitudinal study, the Longitudinal Proof Project (1999-2003), this time in collaboration with Dietmar Küchemann. The project analysed students' mathematical reasoning over time, focusing on students aged from 13 to 15 years old. Again mixed methods of analysis were used (see Hoyles et al, in press, for a discussion on the strengths and challenges of mixed methods). We carried out an annual written survey of high-attaining students from randomly selected schools within nine geographically diverse English regions. In the first year (June 2000) 3000 students, aged 13, from 63 schools were tested in number/algebra and geometry. The same students were tested again in 2001. Some of the questions were from the previous test, others were new or slightly modified questions. The same students were tested in a similar way in June 2002. Case studies and interviews of selected students and teachers were also carried out every year. Findings from this longitudinal study confirmed many of the findings of the earlier cross-sectional study, but also produced new insights, for example concerning the sustainability of mathematics learning over time (or lack of it), and how the introduction of new curriculum content can have unanticipated effects on students' proof responses. The challenge for teachers and researchers is therefore to develop sustained programmes of activities, in which new ideas are introduced into the curriculum, that build connected and layered mathematical knowledge and ways of explaining and proving, rather than simply replace old content or modes of expression with new ones (Küchemann & Hoyles, in preparation).

I will end this section with a remark that could not be presented as a research finding but captures one general lasting impression I have of my work on proof. It is that I have been

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continually amazed at the originality of so many of the students' proofs and explanations, proofs that would perhaps not be judged as adequate mathematically (they might not display a fluency with mathematical language and procedures for example), but nontheless proofs that somehow displayed mathematical integrity and mathematical creativity. How can we ensure we do not suppress this creativity, this student voice, in our quest for rigour?

WINDOWS FROM THE WORKPLACE

I now move on to the last strand of my research, which is how mathematics is used in the workplace. In this corpus of work, we continued to refine our notions of situated abstraction and webbing, as we were constantly forced to address issues of 'transfer' (or lack of transfer) and tool mediation: why was it that workers so often could not use the mathematics they were supposed to know, or paradoxically, why could they clearly display mathematical competence yet deny it or fail to answer correctly any traditional mathematics questions? To investigate these questions I began to research how mathematics was used in an investment bank (Noss & Hoyles, 1996b). This work triggered so many new challenges and insights, that we continued with longer studies of nurses and pilots (in collaboration with Stefano Pozzi; for overviews of this work, see Hoyles et al, 1998; Noss et al, 2000), and then of workers without professional qualifications in a range of sectors (see Hoyles et al, 2002b).

Studies of the workplace raise methodological challenges: what to observe, for example, as mathematics is so often invisible to practitioners. Our findings consistently showed that what we termed the 'visible' mathematics of a practice was almost invariably associated with routine activities, often involving measurement and recording, the use of algorithms to find unknowns from one or more known quantities, or the communication of results so as to inform action or decision. However, we were also aware of what we would describe as mathematical activity, which tended to be invisible to practitioners and managers alike. For example, we noted how practitioners often used a range of apparently idiosyncratic strategies finely-tuned by the tools available for solving particular problems in specific circumstances. Yet despite their specificity, we could discern sound mathematical models underpinning practitioners' strategies, many of which only came to light at times of disruption to routine or the need to communicate to others. For example, in the nurses' study, we identified largely unarticulated ways that expert nurses undertook the calculation of drug dosages on the ward and noted how

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they actually used a range of correct proportional reasoning strategies based on the invariant of drug concentration to calculate dosage on the ward, rather than the single taught method they described outside of the practice. These strategies were tied to individual drugs, specific quantities and volumes of drugs, the way drugs are packaged and the organization of clinical work (see Hoyles et al, 2001). We used data from the nursing study to elaborate the notion of situated abstraction as an analytical tool to understand nurses' conceptions of the intensive quantity of drug concentration, which we argued, was webbed to the mathematics of ratio and proportion as well as to the contextual artefacts and procedures of the practice. We also noted the fragility of the nurses' knowledge when they were no longer able to coordinate – to web - their mathematical knowledge with their professional expertise (see Noss et al, 2002b).

In our later work, in collaboration with Phillip Kent and Arthur Bakker, Richard Noss and I are specifically investigating the shifts of perspective in practice, the 'transfer' of mathematical ideas across boundaries in technology-rich workplaces seeking to improve the efficiency of their production processes, to increase the quality of their products, or the return on their financial products (see http://www.ioe.ac.uk/tlrp/technomaths/). In this drive to improve, abstract computer-based models are developed of the work process, and employees at all levels have to develop new skills that we are seeking to describe: for example, systematic and precise measuring and data entry techniques, monitoring systems against targets, and interpreting and communicating progress in relation to targets. Increasingly therefore workers are involved in interpreting, manipulating and communicating numerical, symbolic and graphical information.

Situated abstraction and webbing have again proved to be useful analytical tools, enabling us to appreciate the challenge of meaningful mathematical communication at work: the former offers a tool for valuing and making sense of what is mathematically understood by people in work, and the latter helps us understand how the pragmatic, specific and concrete connect with the theoretical, general and abstract. At the same time, we are developing design experiments that aim to foster this communication, using constructionist computers tools and activities: thus bringing several strands of my research together.

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CONCLUSIONS AND FUTURE CHALLENGES

I now briefly summarise what I see as some of the key challenges for the mathematics education community today, both old and new. They include to foster engagement with mathematics and promote legitimate diversity without forfeiting rigour; to achieve a more robust understanding of the complexity of introducing computers, particularly in collaborative endeavours; to develop short, medium and long-term innovations and implementation strategies; to devise systematic methodologies to design and evaluate innovations over time; to investigate (theoretically and practically) 'crossing boundaries', that is the processes by which knowledge has to be re-learned in new contexts and in order to address new audiences (e.g. employers, employees and policy makers), and to seek to exploit learning in informal as well as formal settings. Outside the constraints of 'school', we might be able to foster 'real' engagement with mathematics, without the need to 'please the teacher', while at the same time face new challenges about what is actually being learned, and how can this be aligned with instiutional modes of mathematical expression.

I have sketched my personal response to some of these challenges in a form that is loosely autobiographical. I have always loved mathematics and taken pleasure in the struggle to solve problems and to explain solutions – at first to myself, but later to others when I became involved in education. I have enjoyed the challenge of moving into an 'unknown territory', whether it be the design of activities with new technologies in teaching and learning mathematics, the critical adoption of different research paradigms to investigate new research questions, or the identification and fostering of mathematical skills required in a range of workplaces. I have welcomed the testing experience of addressing new audiences or making new connections – both can be catalysts to develop novel modes of communication and to challenge hidden assumptions. I have tried to address the general public in the popularisation of mathematics (Hoyles, 1990); employers, in attempts to communicate research results and to re-negotiate with them the skills they require at work; and researchers in countries with different traditions in mathematics and schooling, so that we can better learn from each other. Most recently, I am facing a new challenge, a new boundary to negotiate, in working with Government in the U.K. and seeking to convince politicains of the strategic importance of mathematics; politicians who may be driven by a different agenda from mine but one that would benefit from evidence derived from research. In this endeavour, I am fortunate to be

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able to draw on the rich and diverse results of mathematics education researchers from all over the world, who I thank again for the award of the Hans Freudenthal medal.

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