Efficient Allocations with Moral Hazard and Hidden Borrowing and Lending: A Recursive Formulation*

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Abstract

We propose a tractable recursive framework to study the optimal allocation of consumption and effort in a dynamic setting with moral hazard where agents have secret access to the credit market or to storage. The recursive structure is based on a generalized first order approach, whose validity must be verified *ex-post*. Thanks to the recursive formulation of the optimal contract, the verification procedure turns out to be numerically parsimonious as it can be performed using standard dynamic programming techniques with only one endogenous state variable: The agent's level of assets. We study the performance of our ex-post verification test in practice by solving numerically three representative infinite horizon examples.

Keywords: Moral hazard, hidden savings, efficiency, recursive contracts, first order approach, ex-post verification.

JEL Classification: C61, D82, H21.

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1 Introduction

In this paper, we consider an environment where risk averse individuals have random income and can borrow and lend at a given risk-free interest rate. Their asset and consumption decisions are private information. Moreover, each individual can affect future income realizations through his effort decision, which is also non-monitorable. The efficient allocation in such setting is derived by solving the problem of a risk-neutral planner who provides incentive compatible insurance contracts based only upon income histories.

A large literature studied optimal long-term insurance contracts under moral hazard assuming that agents cannot borrow or save.[1] Rogerson (1985a) shows that preventing the agent from entering the asset market is critical - even in the presence of liquidity constraints - since in the optimal contract, the agent is actually willing to save. It can easily be shown that when the planner can perfectly control the agents' asset holdings, and can contractually restrict their acquisition of additional assets and liabilities, the efficient allocation is the same as the one where the agents have no access to the credit market. In many situations, however, the planner cannot have perfect control over the agents' wealth and consumption. This is true for instance when there are hidden storage or investment opportunities. In transitional and developing countries, agents often use foreign currency, gold or some other forms of storage of value for self insurance. These forms of asset accumulation are typically not observable by the government. There are also cases where agents can have secret access to domestic or foreign accounts and credit lines. Therefore, relaxing the assumption of perfect observability and contractability on agents' asset holdings and analyzing the resulting optimal allocation is a potentially very valuable exercise from both the theoretical and applied point of view.

Introducing hidden/anonymous asset accumulation (or hidden storage) into the dynamic moral hazard model raises important methodological complications as the problem fails to have a recursive structure, at least in the usual sense. Fudenberg *et al.* (1990) provide characterizations of efficient allocations in a wide class of dynamic environments where agents' preferences over continuation contracts are common knowledge after any history. Since the level of wealth typically affects the agents' attitude toward risk, hidden borrowing and lending leads to a violation of the common

[1] This literature includes, among many others, Townsend (1982), Rogerson (1985a), Spear and Srivastava (1987), Green (1987), Phelan and Townsend (1991). Atkeson (1991) analyses a moral hazard model with borrowing and lending and default. His model differs crucially from ours since he assumes that asset holdings are observable.

knowledge of preferences assumption.[2] In spite of that, by using a generalized *first order conditions approach*, we are able to formulate the problem within the dynamic programming framework. In order to keep track of the marginal value of wealth, together with the agent's expected discounted lifetime utility, we introduce the agent's marginal utility of consumption as an additional endogenous state variable. Intuitively, the recursive formulation can be obtained because the first order approach allows us to write the problem in terms of equilibrium values alone. Then, incentive compatibility guarantees that common knowledge of preferences is maintained along the equilibrium path.

Virtually all existing literature justifies the use of the first order approach by showing analytically that - given the optimal contract - the agent's problem is globally concave. For static moral hazard models or for dynamic models with linear taxation without moral hazard this is a perfectly viable procedure.[3] In Ábrahám and Pavoni (2007a) (AP) we also follow this route, in a two period moral hazard framework with hidden savings, and provide sufficient conditions for concavity of the agent's problem. Unfortunately, the derivation of (not too restrictive) analytical sufficient conditions for global concavity becomes difficult (perhaps impossible) in a general multi-period setting such as that analyzed here.[4] In this paper, we propose to use a numerical approach, which can be easily extended to a richer class of models with incentive constraints. First, using the first order conditions approach, we solve the (relaxed) problem recursively. Then we take advantage of the dynamic programming formulation and develop a numerical procedure to *verify ex post* whether the obtained allocation is in fact incentive compatible. We allow the agent to re-maximize his lifetime utility taking the optimal (relaxed) transfer scheme as given. We then check whether the optimal value of the re-maximization problem coincides with that implied by the optimal contract.

Notice that the equality between the value of the re-maximization problem and that delivered to

[2] Fernandes and Phelan (2000) and Doepke and Townsend (2006) propose a way to solve recursively an even wider class of problems than that analyzed by Fudenberg *et al.* (1990). Unfortunately, their methods are not viable when hidden actions belong to a continuum as it is the case for saving decisions in our model. Cole and Kocherlakota (2001b) extend the Abreu *et al.* (1990) framework to a wide class of dynamic games. None of their extensions, however, is of any use for us, since our law of motion for bond holdings does not satisfy the 'full support' assumption required there. Finally, Hagedorn *et al.* (2007) provide a recursive formulation for a repeated moral hazard model with adverse selection problem in the first period. They only consider observable asset accumulation.

[3] See, e.g., Rogerson (1985b) and Jewitt (1988) for the analytic first-order approach in the static moral hazard model; and Chang (1998) and Phelan and Stacchetti (2001) for a similar analytic approach in optimal linear taxation models without moral hazard.

[4] According to our knowledge, there have been no previous attempts in deriving analytical conditions for the first order approach in discrete time dynamic moral hazard models with hidden savings. Williams (2003) proposes an interesting and tractable model in continuous time. However, the conditions for concavity based on the Hamiltonian obtained there are not satisfied in a context where there is a linear intertemporal transfer technology such as that assumed here for savings.

the agent by the optimal contract is both necessary and sufficient for the validity of the first order approach.[5] Thanks to the fact that the optimal contract takes a recursive from, the verification procedure turns out to be numerically parsimonious, since the agent's re-maximization exercise requires the solution of a simple dynamic programming problem with only one endogenous state: The agent's level of assets. The latter is an attractive property of this procedure in terms of applicability.

In order to study the effectiveness of the ex post verification procedure in the presence of approximation errors, we study three infinite horizon examples. For our first example, Kocherlakota (2004) shows analytically that the first-order approach is not applicable. In this case, our numerical procedure finds sizeable discrepancies between the optimal value of the re-maximization problem and that implied by the optimal contract. In particular, these deviations are on average two to three magnitudes higher than the numerical precision of the procedure. In other terms, the agent finds it profitable to deviate from the 'relaxed' optimal policy for every initial state, implying that the relaxed policies are not incentive compatible and hence the first-order approach is not valid. In the second example, we rely on the closed form solution for the CARA utility case to show, analytically, that the agent's problem is concave in the constrained efficient allocation. In this case, the ex post verification procedure always finds deviations well bellow numerical precision, confirming the validity of the first-order approach.

The third example considers a case where from AP we know that the agent's problem is concave in the two period model, but we are unable to show it analytically for the general multiperiod case. This example turns out to be an intermediate case. Our numerical procedure detects profitable deviations of the agent from the relaxed optimal policy for some initial life-time utilities and find no such deviations for others. We found this property of the numerical approach appealing compared to the analytical approach, because in applications, we typically use particular parametrizations (usually obtained by calibration) and we restrict the initial state using some economic argument (e.g., the value of the outside option). The standard analytical approach in contrast, implicitly checks whether the agent's problem is concave for all parametrizations and for all initial states. Our results show that the analytical approach can sometimes be more restrictive than necessary from the point of view of the considered application.

Our results also indicate that the crucial step in the numerical procedure is the approximation

[5] Clearly, the first-order condition approach can be valid in many cases where the analytical approach would not allow for it.

of the optimal policies, because for any given level of approximation, the ex post verification procedure can determine whether the approximated policy is incentive compatible or not with high confidence. However, it can happen that rough approximations of the relaxed optimal problem deliver approximated solutions which are incentive compatible, while more accurate approximations (and presumably the true solution) of the problem are not incentive compatible. Since, an accurate approximation of the optimal allocation is the main objective of any numerical procedure anyway, in this sense, the ex post verification stage does not seem to pose any additional challenge in terms of numerical accuracy.

The way we use the first order approach together with the marginal utility of consumption as state variable resembles that adopted in the Ramsey taxation literature by Kydland and Prescott (1980), Chang (1998), and Phelan and Stacchetti (2001). For those models with linear taxation and no moral hazard however the first order approach can easily be justified analytically. One key methodological contribution of this paper is to show an important complementarity between the recursive formulation and the first order approach which allows the formal study of dynamic incentive models where the global concavity of the agent's problem cannot be guaranteed or verified analytically. In an independent work, Werning (2001-2002) develops a similar recursive formulation for the dynamic moral hazard model with hidden savings. This work is simultaneous to ours, but Werning does not formally address the issue of the validity of the first order approach.

Because of the methodological problems we mentioned above, the remaining few papers that analyze dynamic moral hazard with non-monitorable asset holdings use particular models and study specific issues. Allen (1985) and Cole and Kocherlakota (2001a) (ACK) study the effect of secret asset accumulation in a hidden information moral hazard model. In Allen's framework, the agent is allowed to both borrow and lend and the set of incentive compatible contracts turns out to be a singleton: The zero-transfers contract. Cole and Kocherlakota consider an economy with hidden storage (agents can only save). They show that although the set of incentive compatible contracts is very large, whenever the return to storage is not too low the efficient allocation is equivalent to a self-insurance equilibrium. In our model with action moral hazard, the constrained efficient allocation does differ from (i.e., it is welfare improving with respect to) self insurance. In a two period principal agent relationship, Bizer and DeMarzo (1999) show that hidden access to the credit market reduces total welfare with respect to the no asset market case. They focus on the possibility of increasing welfare by allowing the entrepreneur to default on the debt. We study the general model where default is not allowed. Bisin and Rampini (2006) study the effect of bankruptcy provision, in a two period model similar to that of Bizer and DeMarzo, where agents have hidden access to *insurance contracts* and can default on the principal insurer as well. In addition to no-default, we do not allow agents to secretly trade assets other than a risk free bond. Chiappori et al. (1994) and more recently Park (2004) analyze the optimal contract with discrete effort. They find that - under some conditions - a renegotiation-proof contract always implements the minimum level of effort. We consider a continuous-effort model, where the planner can commit not to renegotiate the contract ex post. In our framework the optimal allocation of effort is non-degenerate. Kocherlakota (2004) characterizes the optimal UI transfer scheme in a two-output moral hazard model with hidden savings, where agents' preferences are linear in effort, and effort affects linearly job-finding probabilities. We provide a framework to characterize the optimal contract in a general specification of the model, whenever the incentive constraint can safely replaced by the first order conditions of the agent. Finally, Golosov and Tsyvinski (2007) study competitive equilibria with hidden information moral hazard and hidden asset accumulation with endogenous interest rate. They show that if only a risk-free bond is traded then the competitive allocation is generally inefficient. This paper studies the constrained efficient allocation (in a small open economy), where the return on assets/storage is exogenously given. Our ex post verification approach can be easily extended to endogenous interest rates as far the agent takes the return on savings as given.

The paper is organized as follows. In the next section, we present our environment and define constrained efficiency. The recursive formulation and the ex-post verification procedure are presented in Section 3. In Section 4, we study the numerical implementation of the recursive formulation and the verification procedure. Section 5 concludes.

2 Environment and Constrained Efficiency

Environment Consider a small open economy consisting of a large number of agents that are ex-ante identical, and who each live $T \leq \infty$ periods. Each agent is endowed with a private stochastic production technology which takes the following form. There is a finite set $Y = \left(y^1, ..., y^N\right)$ of possible output levels, with $y^i < y^{i+1}$. At each period t, the realization $y_t \in Y$ is publicly observable; however, the probability distribution over Y is affected by the agent's unobservable effort level e, which we assume to belong to a bounded interval $E = [0, e_{\max}]$. The conditional probabilities over Y are defined by the publicly known continuous functions[6] $p_i(e) = \Pr\left\{y = y^i \mid e\right\}$. Hence, agents are subject to idiosyncratic risk, and we assume time independent conditional distribution of income.[7,8] Similarly to most of the dynamic moral hazard literature, we assume *full support, i.e.* $p_i(e) > 0$ for all $i = 1, 2, ...N$ and $e \in E$. The history of public outcomes up to period t will be denoted by $h^t = (y_1, ..., y_t)$.[9]

Agents are allowed to buy and sell a risk-free bond which pays a constant interest rate $r \geq 0$. Their asset holdings are private information, and we assume that each agent is born with no wealth ($b_0 = 0$).[10] Note that, since agents can trade only risk free bonds, asset markets are incomplete. Therefore, we can expect that a social planner could increase overall welfare by providing additional insurance. In this setting, a constrained efficient allocation can be computed by solving the problem of a benevolent planner whose aim is to reallocate resources optimally in order to insure agents, subject to the feasibility and incentive constraints which will be specified below.

An *allocation* (or social contract) in this economy is a contingent plan

$$\mathcal{W} := (\boldsymbol{\tau}, \boldsymbol{\sigma})\,; \text{ with } \boldsymbol{\tau} := \left\{\boldsymbol{\tau}_t(h^t)\right\}_{t=1}^T, \text{ and } \boldsymbol{\sigma} := \left\{\mathbf{e}_t(h^t), \mathbf{b}_t(h^t), \mathbf{c}_t(h^t)\right\}_{t=1}^T,$$

where $\boldsymbol{\tau}_t(h^t)$ represents the transfer the individual receives in period t, $\mathbf{e}_t(h^t)$ the implemented effort, $\mathbf{b}_t(h^t)$ the bond holdings and $\mathbf{c}_t(h^t)$ the agent's consumption level as a function of the realized history h^t. Note that we assume that agents can only be distinguished through their output histories. In this sense, since all individuals are ex ante identical, we restrict ourselves to symmetric allocations.

To simplify the analysis, we separate the planner's transfer plan, $\boldsymbol{\tau}$, from $\boldsymbol{\sigma}$, the components of the allocation under the agent's control. The metaphor used in contract theory is that the planner proposes $\boldsymbol{\sigma}$, and the plan will be implemented by appropriately designing the transfer scheme $\boldsymbol{\tau}$.

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[6] We assume continuity of the vector function $\mathbf{p} : E \to \Delta^N$, where $\Delta^N = \left\{x \in \Re^N \mid x \geq 0, \ \sum_i x_i = 1\right\}$.

[7] For the variable y we will use, interchangeably, the terms output and income. In the first interpretation, we stress the fact that agents have access to a stochastic production technology. Viewing y as income, emphasizes more that agents are facing idiosyncratic risk (with an endogenous distribution).

[8] Notice, that this model can be naturally extended to allow for persistence in idiosyncratic shocks by defining $p_{ij}(e) = \Pr\left\{y_{t+1} = y^i \mid e, y_t = y^j\right\}$.

[9] Since the only other publicly observable variable is the planner's transfer, and the planner has full commitment, without loss of generality we can restrict public histories to be histories of income realizations alone (see Pavoni, 1999, for details).

[10] Note that if the initial distribution of assets were not degenerate we would also face an adverse selection problem and, in period zero, the planner would propose a menu of long term contracts in order to screen agents with different b_0's. We do not consider this here, however, since both agents' saving decisions and the interest rate are deterministic, there will be no role for revelation games about wealth after the first period.

At period t, each agent receives a transfer payment $\tau_t = \boldsymbol{\tau}_t(h^t)$ from the planner, contingent on the realized history h^t. Given today's income level y_t, transfer τ_t, asset level b_{t-1} and the continuation plan $\boldsymbol{\tau} \setminus h^t$, the agent chooses consumption $c_t \geq 0$ and bond holdings b_t subject to the following budget constraint:

$$\mathbf{c}_t\left(h^t\right) + \mathbf{b}_t\left(h^t\right) = y_t + \boldsymbol{\tau}_t\left(h^t\right) + (1+r)\mathbf{b}_{t-1}\left(h^{t-1}\right). \tag{1}$$

We impose the general condition $\mathbf{b}_t\left(h^t\right) \geq -\mathbf{B}_t\left(h^t\right)$ for all h^t on asset holdings, where $\mathbf{B} := \left\{\mathbf{B}_t\left(h^t\right)\right\}_{t=1}^{T}$ is an exogenously given plan of borrowing constraints such that $\mathbf{B}_t\left(h^t\right) \geq 0$ for all h^t, and $\mathbf{B}_T\left(h^T\right) = 0$.[11,12] At the beginning of each period, the agent also decides the effort level $e_t \in E$, which affects the stochastic output realization $y_{t+1} \in Y$, leading to next period output history $h^{t+1} = \left(h^t, y_{t+1}\right)$. This sequence of events continues until period T is reached.

Agents have intertemporally additive separable von Neumann-Morgenstern utility function. The continuation plan $\mathcal{W} \setminus h^t = \left\{\boldsymbol{\tau}_\tau(h^\tau), \mathbf{e}_\tau(h^\tau), \mathbf{b}_\tau(h^\tau), \mathbf{c}_\tau(h^\tau) \setminus h^t\right\}_{\tau=t}^{T}$ from node h^t generates the following expected discounted utility at time $t \geq 1$:

$$\mathbf{U}_t^T\left(\mathcal{W}; h^t\right) = \mathbf{U}_t^T\left(\boldsymbol{\tau}, \boldsymbol{\sigma}; h^t\right) = \mathbf{E}\left[\sum_{\tau=t}^{T} \beta^{\tau-t}\, u\left(\mathbf{c}_\tau(h^\tau),\ \mathbf{e}_\tau(h^\tau)\right) \ \mid\ \alpha \setminus h^t\right],$$

where $\alpha \setminus h^t = \left\{\mathbf{e}_\tau(h^\tau) \setminus h^t\right\}_{\tau=t}^{T}$ denotes the implemented effort plan from history h^t onward, $\mathbf{E}$ is the usual expectation operator, and $\beta \in (0,1)$ is the discount factor. We assume that the choice of $\mathcal{W}$ is implicitly restricted in such a way that both the expectation and the (possibly infinite) summation are well defined. We also assume u to be real valued and continuous; strictly increasing, strictly concave in c and decreasing in e.

To be feasible, an allocation $\mathcal{W}$ must be deviation-proof in all components of $\boldsymbol{\sigma}$, that is, in effort e, bond holdings b, and consumption c. Hence, we say that the allocation $\mathcal{W}$ is *sequentially incentive compatible* if, for any history h^t, we have

$$\mathbf{U}_t^T\left(\boldsymbol{\tau}, \boldsymbol{\sigma}; h^t\right) \geq \mathbf{U}_t^T\left(\boldsymbol{\tau}, \widetilde{\boldsymbol{\sigma}}; h^t\right), \qquad \text{for all } \widetilde{\boldsymbol{\sigma}} \in \boldsymbol{\Sigma}\left(\boldsymbol{\tau}, \mathbf{B}\right), \tag{2}$$

[11] The latter requirement is the usual condition to exclude Ponzi Games in finite time horizon models. In the infinite horizon version of the model ($T = \infty$), we require $\mathbf{B}$ to satisfy the (equivalent) minimal condition that $\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^T \mathbf{B}_T(h^T) \leq 0$ almost surely for all histories.

[12] The enforceability of the repayment of debt obtained through anonymous credit lines is an important and delicate issue, which is common to many environments and that we do not address here. The most skeptical approach would require $\mathbf{B}_t\left(h^t\right) \equiv 0$, which corresponds to a situation of pure storage.

where $\boldsymbol{\Sigma}(\boldsymbol{\tau}, \mathbf{B})$ contains all feasible continuation plans of actions given $\boldsymbol{\tau}$ and $\mathbf{B}$. In particular, if $\widetilde{\boldsymbol{\sigma}} \in \boldsymbol{\Sigma}(\boldsymbol{\tau}, \mathbf{B})$ then $\widetilde{\boldsymbol{\sigma}} \setminus h^t = \left\{ \widetilde{\mathbf{e}}_\tau(h^\tau), \widetilde{\mathbf{b}}_\tau(h^\tau), \widetilde{\mathbf{c}}_\tau(h^\tau) \setminus h^t \right\}_{\tau=t}^{T}$ represents a contingent plan such that for all histories h^τ following node h^t we have: $\widetilde{\mathbf{c}}_\tau(h^\tau) \geq 0$, $\widetilde{\mathbf{e}}_\tau(h^\tau) \in E$, $\widetilde{\mathbf{b}}_\tau(h^\tau) \geq -\mathbf{B}_\tau(h^\tau)$, and - given the transfer plan $\boldsymbol{\tau} \setminus h^t$ - the budget constraint (1) is satisfied at h^τ. We denote *the set of incentive feasible allocations* as

$$\Omega = \left\{ \mathcal{W} : \text{for all } h^t \text{ satisfies } \mathbf{c}_t(h^t) \geq 0,\ \mathbf{e}_t(h^t) \in E,\ (1),\ (2), \text{ and } \mathbf{b}_t\left(h^t\right) \geq -\mathbf{B}_t\left(h^t\right) \right\}.$$

Constrained Efficiency For technical tractability, we define the optimal contract as the one that maximizes the planner's net returns (or minimizes costs), subject to incentive feasibility and to the social restriction that each individual must receive at least an expected discounted utility level U_0. We will then choose U_0 so that the planner's expected discounted returns equal zero. The planner is represented by a risk neutral principal who faces the same interest rate as the agents, and whose net return at node h^t induced by the continuation plan $\mathcal{W} \setminus h^t$ is

$$\boldsymbol{\Pi}_t^T(\mathcal{W}; h^t) = \mathbf{E}\left[\sum_{\tau=t}^{T} \frac{(1+r)^t}{(1+r)^\tau}(-\boldsymbol{\tau}_\tau(h^\tau)) \ \mid\ \alpha \setminus h^t\right].$$

Given the social restriction U_0, the $T-$horizon planner's problem can then be formulated as follows:

$$\sup_{\mathcal{W} \in \Omega} \boldsymbol{\Pi}_0^T(\mathcal{W}); \text{ s.t. } \mathbf{U}_0^T(\mathcal{W}) \geq U_0, \tag{3}$$

where

$$\boldsymbol{\Pi}_0^T(\mathcal{W}) = \sum_{i=1}^{N} p_i^0 \boldsymbol{\Pi}_1^T(\mathcal{W}; y^i), \text{ and } \mathbf{U}_0^T(\mathcal{W}) = \sum_{i=1}^{N} p_i^0 \mathbf{U}_1^T(\mathcal{W}; y^i)$$

for some initial distribution $\mathbf{p}^0$. We postpone the issue of existence until Section 3.

3 Recursive Formulation and Ex-post Verification

It should not be difficult to see that condition (2) defines a complicated set of constraints: already in the two period version of the model, the number of constraints is a bidimensional continuum. Perhaps more importantly, there is no tractable way of writing this problem recursively in its original form. Along the lines of Spear and Srivastava (1987) and Green (1987), Abreu *et al.* (1990) show that when agents' preferences over continuation contracts are common knowledge after any history, the efficient allocation can be characterized by using a one dimensional state variable: the agent's continuation utility. Notice that the assumption of common knowledge of preferences is not satisfied in our framework. Essentially, the possibility of hidden asset accumulation introduces an adverse selection problem in each period since agents with different asset levels respond differently to the contract. Fernandes and Phelan (2000) and Doepke and Townsend (2006) show that this adverse selection problem can be resolved by using one state variable for each agent type. According to this approach, in our framework with a continuum of possible asset levels, the number of types explodes and the relevant state becomes a *function*, i.e. an infinite dimensional object. These infinities pose obvious computational difficulties that make this approach infeasible in practice within our model (for a similar discussion see Phelan and Stacchetti, 2001; and Kocherlakota, 2004).

In this section, we adopt a generalized first order approach which solves both aforementioned problems (the numerosity of the incentive constraints and the tractability of the recursive formulation). First we assume the validity of the first order approach, and present a tractable recursive formulation. Second, we take advantage of the dynamic programming framework and develop a numerical procedure to verify ex-post whether our assumption on the sufficiency of first order conditions is valid.

3.1 The First Order Conditions Approach

>From now onward, we assume that both u and $\mathbf{p}$ are differentiable. The adoption of the first order approach means that the set of constraints described in (2) are replaced by the agent's corresponding first order conditions along the optimal path. Using the budget constraint (1) to eliminate the planner transfers τ_t, and assuming interiority with respect to e,[13] for any $h^t \neq h^T$ the agent's first order conditions become

$$e: \qquad -u_e'(\mathbf{c}_t(h^t), \mathbf{e}_t(h^t)) = \beta \sum_i p_i' \left(\mathbf{e}_t(h^t)\right) \mathbf{U}_{t+1}^T(\mathcal{W}; (h^t, y^i)); \tag{4}$$

[13] We will never consider the possibility that the upper bound on E is binding. However, we might easily allow for a more general formulation of the following form:

$$u_e'(\mathbf{c}_t(h^t), \mathbf{e}_t(h^t)) + \beta \sum_i p_i' \left(\mathbf{e}_t(h^t)\right) \mathbf{U}_{t+1}^T(\mathcal{W}; (h^t, y^i)) \leq 0$$

with equality if $\mathbf{e}_t(h^t) > 0$. Moreover, notice that when $T < \infty$ we must impose the obvious corner solution $\mathbf{e}_T(h^T) = 0$.

and

$$b: \quad u_c'(\mathbf{c}_t(h^t), \mathbf{e}_t(h^t)) \geq \beta(1+r) \sum_i p_i(\mathbf{e}_t(h^t)) u_c' \left(\mathbf{c}_{t+1}(h^t, y^i), \mathbf{e}_{t+1}(h^t, y^i)\right). \tag{5}$$

Notice, that the asset level b_t does not enter any of these constraints. This is essentially due to fact that the agent faces the same interest rate as the planner does.[14] Now, define the set of social contracts satisfying these first order conditions as

$$\Omega_{FOC} = \left\{\mathcal{W} : \text{for all } h^t \text{ satisfies } \mathbf{c}_t(h^t) \geq 0,\ \mathbf{e}_t(h^t) \in E;\ (1), (4) \text{ and } (5)\right\}.$$

With some abuse of notation, we will also consider contract continuations $\mathcal{W} \setminus h^s$ belonging to the set Ω_{FOC}. In these cases, the restrictions are obviously only related to histories after node h^s.

3.2 The Recursive Problem

In this section, we focus on the infinite horizon case, and in order to simplify notation, we do not report the superindex $T = \infty$. The 'true' value function of the relaxed problem is thus defined as follows:

$$V_{foc}^*(U_0) = \sup_{\mathcal{W} \in \Omega_{FOC}} \mathbf{\Pi}_0(\mathcal{W}) \tag{6}$$
$$\text{s.t. } \mathbf{U}_0(\mathcal{W}) \geq U_0.$$

We argued above that when consumers can secretly accumulate assets, the state variable used by the standard recursive contracting literature - the agent's lifetime utility U - is no longer sufficient to describe the constrained efficient allocation. We will see that recursivity can be recovered by complementing lifetime utility with an additional endogenous state: the marginal utility of consumption $u_c'(c, e) = x$. From the Euler equation, it is clear that by affecting the marginal value of wealth $\beta \mathbf{E}\left[(1+r)\, u_c'\left(c_{t+1}, e_{t+1}\right)\right]$ the planner can fully control agents' asset decisions at the margin. The adoption of $u_c'(c, e)$ as a state variable becomes now natural since in our framework both r and β are exogenously fixed at a constant level.

Our relevant state space is hence bidimensional.[15] However, before applying the recursive

[14] Notice that condition (5) itself is a relaxed version of the true first order condition of the agent, since it does not state that for $\mathbf{b}_t\left(h^t\right) > -\mathbf{B}_t\left(h^t\right)$ the Euler equation must be satisfied with equality. This - and the absence of the transversality condition for $T = \infty$ - are further dimensions along which we relax the incentive constraint.

[15] Notice, that in our repeated framework, the exogenous state y does not affect the space of endogenous states U and x.

techniques of Stokey *et al.* (1989) (SLP) to our problem, we have to consider the possibility that the feasibility correspondence can be empty for some combinations of the states. To overcome this complication, we could basically follow two alternative procedures. The first possibility is the use of infinite penalizations. This option basically sets the value of the planner to minus infinity for each combination of states (U, x) which cannot be implemented by any incentive feasible contract.[16] We choose to follow a second approach, which typically delivers a continuous value function. This second procedure is divided into two main steps. It first derives the state space (or domain restriction) M^* : the set of combinations (U, x) for which a relaxed incentive compatible contract delivering lifetime utility U, and a marginal utility x to the agent in period zero does exist. In the second step of the procedure, the problem is solved using usual recursive techniques.

At each node h^t, the (time invariant) set of relaxed incentive feasible endogenous states is formally defined as follows:

$$M^* = \left\{(U, x) \in \Re^2;\ \exists\ \mathcal{W} \setminus h^t \in \Omega_{FOC}\ ;\ u_c'\left(\mathbf{c}_t(h^t), \mathbf{e}_t(h^t)\right) \leq x,\ \text{and}\ \mathbf{U}_t(\mathcal{W}; h^t) = U\right\}.$$

It is easy to see that M^* is non empty.[17] Given M^* we have the following:

Proposition 1 *Given an initial distribution* $\mathbf{p}^0$*, and an ex-ante expect discounted utility level* U_0 *guaranteed to the agent, the value function* V^*_{foc} *solves*

$$\begin{aligned} V^*_{foc}(U_0) \quad &= \quad \sup_{\substack{(x^i, U^i) \in M^* \\ i\,=\,1,\ldots,N}} \sum_i p_i^0 V\left(y^i, U^i, x^i\right) \qquad (7) \\ s.t. \quad &: \quad \sum_i p_i^0 U^i \geq U_0, \end{aligned}$$

where $V : Y \times M^* \to \Re$ *is a solution to the following functional equation*

$$\begin{aligned} V\left(y, U, x\right) \quad &= \quad \sup_{\substack{(x^i, U^i) \in M^* \\ e \in E, c \geq 0}} y - c + \frac{1}{1+r} \sum p_i(e) V\left(y^i, U^i, x^i\right) \qquad (8) \\ & \quad s.t. \end{aligned}$$

[16] See Rockafellar (1975), and Rustichini (1998).

[17] For example, the full insurance contract with $\mathbf{e}_t(h^t) \equiv 0$ is always feasible. In this case, for any given $x \in \Re_{++}$ the corresponding utility level is obtained by $U = \frac{u(g(x),0)}{1-\beta}$, where g is the consumption component of the inverse of the marginal utility when $e = 0$.

$$
\begin{aligned}
U &= u(c,e) + \beta \sum_i p_i(e) U^i & (9)\\
-u_e'(c,e) &= \beta \sum_i p_i'(e) U^i & (10)\\
u_c'(c,e) &\geq \beta(1+r) \sum_i p_i(e) x^i & (11)\\
x &\geq u_c'(c,e). & (12)
\end{aligned}
$$

Conversely, if a bounded function V defined on $Y \times M^*$ satisfies the functional equation (8)-(12) and M^* is compact then a solution to (6) exists and

$$
V_{foc}^*(U_0) = \max_{(x^i,U^i)\in M^*} \sum_i p_i^0 V(y^i, U^i, x^i), \quad s.t. \quad \sum_i p_i^0 U^i \geq U_0. \tag{13}
$$

The result in the second part of the proposition implies existence of a solution to the original problem any time the first order approach is valid. We are able to show existence of a solution to the relaxed problem (6) as long as u is bounded with bounded derivatives (so that M^* is a compact set). This is so since the continuity of u and $\mathbf{p}$ imply that V is a continuous function. Moreover, V is weakly increasing in x and constant for all values for which constraint (12) does not bind. Finally, from (11), the choice of low values for x^i is always feasible. Hence x^i can without loss of generality be chosen so that to satisfy (12) with equality.

Let us now briefly turn to the domain restriction M^*. An argument similar to that of Abreu *et al.* (1990) implies that the set M^* can be derived by starting from a sufficiently large set $\overline{M} \supset M^*$, and computing the largest fixed point of the following operator:

$$
\mathbf{F}(M) = \left\{(U,x) \in \Re^2;\ \exists \left(U^i, x^i\right) \in M,\ e \in E \text{ and } c \geq 0\ ;\ \text{(9)-(12) are satisfied}\right\}. \tag{14}
$$

It turns out that $\mathbf{F}$ is monotone[18] and maps closed sets into closed sets.[19] Moreover, since the sequence $M_n = \mathbf{F}^n\left(\overline{M}\right)$ is monotone, it must converge to the set $M_\infty = \lim_{n\to\infty} \mathbf{F}^n \overline{M} = \cap_{n=1}^{\infty} M_n$, which is closed as an intersection of closed sets. It can be shown that if $\overline{M}$ is chosen sufficiently large, we have $M_\infty = M^*$ since the sequence converges to the largest fixed point of the operator $\mathbf{F}$. It can be easily shown that when u is bounded with bounded first derivatives, M^* is compact. In the finite horizon version of our model, the domain restriction sets can be similarly

[18] That is, for any two sets $M, M' \subseteq \Re^2$ if $M \subseteq M'$, then $\mathbf{F}(M) \subseteq \mathbf{F}(M')$.

[19] Notice that the constraint set is formed by either equalities or weak inequalities.

computed following a backward procedure.[20]

3.3 Ex-post Verification Procedure of the First Order Approach

Obviously, any interior contract $\mathcal{W}$ that is incentive feasible according to (2) - i.e. $\mathcal{W} \in \Omega$ - is such that $\mathcal{W} \in \Omega_{FOC}$. For the purposes of this paper, we should look at conditions under which an optimal solution $\mathcal{W}^*$ to the *relaxed* planner's problem (6) is such that $\mathcal{W}^* \in \Omega$. In that case, the solution to problem (6) satisfies incentive compatibility, hence we have actually derived the efficient contract. In more detail, since $\Omega \subset \Omega_{FOC}$ implies that the value V^*_{foc} is (weakly) larger than the value of problem in (3) and since the value associated to any feasible contract is obviously lower than the optimal value of the problem (among all feasible contracts), $\mathcal{W}^* \in \Omega$ implies the claim.

A direct application of this argument is the basis of our verification of the first order approach. After computing the optimal contract according to (6), the procedure allows the agent to *re*-maximize his lifetime utility by choosing effort, consumption, and bond holdings taking the optimal (relaxed) transfer scheme as given. Then we check whether the optimal decisions of this re-maximization problem coincide with those implied by the relaxed optimal contract, i.e. whether $\mathcal{W}^*$ is actually an incentive compatible allocation.

Notice that for any given transfer scheme $\boldsymbol{\tau} = \left\{\boldsymbol{\tau}_t(h^t)\right\}$, and the sequence of borrowing constraints $\mathbf{B}$ the agents' incentive constraint (2) is described by the following (re)maximization problem

$$U_0^R(\boldsymbol{\tau}, y) = \sup_{\boldsymbol{\sigma}} \mathbf{E}_0 \left[\sum_{t=1}^{\infty} \beta^{t-1} u(c_t, e_t) \mid \boldsymbol{\alpha}\right] \tag{15}$$

$$\begin{aligned} \text{s.t.} \quad \mathbf{c}_t\left(h^t\right) + \mathbf{b}_t\left(h^t\right) &= y_t + \boldsymbol{\tau}_t(h^t) + (1+r)b_{t-1};\ \mathbf{c}_t\left(h^t\right) \geq 0,\ \mathbf{e}_t\left(h^t\right) \in E; \\ \mathbf{b}_t\left(h^t\right) &\geq -\mathbf{B}_t(h^t) \text{ for all } h^t; \text{ with } y_1 = y \text{ and } b_0 = 0, \end{aligned}$$

where each consumer chooses contingent plans $\boldsymbol{\sigma}$ of effort, consumption and bond holdings, and $\boldsymbol{\alpha}$ is the effort plan implied by $\boldsymbol{\sigma}$. The basic idea of the verification procedure is to allow the agent to solve the above remaximization problem, assuming that the transfer scheme is that implicitly defined in

[20] The set M_T for the $T-$horizon problem can be computed by applying the same map $\mathbf{F}$ we defined above as follows $M_T = \mathbf{F}(M_{T-1}) = \mathbf{F}^{T-1}(M_1)$, where M_1 represents the set of states attainable in a one-period problem:

$$M_1 = \left\{(x, U) : u'_c(c, 0) \leq x,\ U = u(c, 0),\ c \geq 0\right\}.$$

Proposition 1. The first order approach is valid if the agent chooses the same consumption, effort and bond holdings as the one implied by the initial solution.

Proposition 2 *Assume* M^* *is compact and* V *is bounded. Let* $U^*(U_0)$ *be the expected discounted lifetime utility level obtained by the agent according to (7) with* $\mathbf{p}^0$ *degenerate at* y*, and* $\boldsymbol{\tau}$ *be the associated transfer scheme. The first order condition approach is justified if and only if* $U_0^R(\boldsymbol{\tau},y) = U^*(U_0)$*.*

The proof of Proposition 2 uses the fact that the relevant deviations will always increase the agent's lifetime utility. Since it is always true that $U_0^R(\boldsymbol{\tau}, y) \geq U^*(U_0)$, the task of comparing two allocations is in fact greatly simplified as one only need to compare two real numbers: $U_0^R(\tau, y)$ and $U^*(U_0)$.[21] However, since $\boldsymbol{\tau}$ is a history dependent stochastic process (i.e. an infinite dimensional object), the verification procedure seems to still require a formidable task. The key advantage of our procedure comes from the observation that the recursive formulation implies that past history can be summarized by the states U_t, x_t.[22] In particular, by using the budget constraint and normalizing bond holdings to zero in each period, the disposable income coincides with the policy function for consumption:

$$y_t + \tau_t = c(U_t, x_t).$$

In turn, because of the dynamic programming framework, the states evolve according to the following *time invariant* policy rules:

$$U_{t+1}^i = f(U_t, x_t, y_{t+1}^i) \tag{16}$$

$$x_{t+1}^i = h(U_t, x_t, y_{t+1}^i),\ i = 1,\ 2,\ ...,\ N. \tag{17}$$

As a consequence, the re-maximization problem (15) can be written in recursive form as follows:

$$J(U, x, b) = \max_{c^R \geq 0, b^R \geq -B(U,x), e^R \in E} u(c^R, e^R) + \beta \sum_i p_i(e^R) J(U^i, x^i, b^R) \tag{18}$$

[21]Consistently with our recursive approach based on continuation utilities, we disregard payoff-equivalent deviations.

[22]Since the problem is a repeated one, the relevant policies do not depend on y_t either.

$$\begin{aligned} \text{s.t.} \quad c^R + b^R &\leq c(U,x) + (1+r)b \\ U^i &= f(U,x,y^i) \\ x^i &= h(U,x,y^i), \end{aligned}$$

where $-B(U,x)$ is an appropriately defined lower bound for assets that replicates $\mathbf{B}$. Therefore, given that the policies c, f, and h (i.e. the transfer scheme τ) are exogenous rules for the agent at this stage, the problem to be solved is very similar to that of self insurance, where the only endogenous state variable is the level of bonds b. According to Proposition 2, the first order approach is verified if we have $J(U^*(U_0), x^*(U_0), 0) = U^*(U_0)$, where $(U^*(U_0), x^*(U_0))$ are the initial values for the states derived in the period zero maximization problem (7), with $\mathbf{p}^0$ degenerate at the initial level of income y.

In the next section, we explain in detail the numerical implementation of the verification procedure using three infinite horizon examples.

4 Numerical Implementation

In this section, we show how to implement the procedure described above, numerically. First, we explain the three main steps (finding the domain restriction, solving for the relaxed optimal contract and ex post verification) of the procedure and then we provide some carefully chosen infinite horizon examples. In all of the three examples, we can use previous literature to obtain conjectures about the validity of the first-order condition.

For expositional convenience, we describe the numerical procedure for the case where $N = 2$, hence $Y = \{y^l, y^h\}$ with $y^h > y^l$. This implies that we can define the probability shifting functions as $p(e) = \Pr(y = y^h \mid e)$ (implying that $1 - p(e) = \Pr(y = y^l \mid e)$). We will also assume that $\beta(1+r) = 1$. Finally, we will restrict ourselves to cases where the Euler equation (11) and the promise keeping constraint with respect to x (12) are satisfied with equality.

4.1 Domain Restriction

The first step of our numerical procedure is to compute the domain restriction M^*. In order to construct M^*, we used a modification of the algorithm proposed by Chang (1998).[23] In the $N = 2$

[23] We should mention that Judd *et al.* (2003), and Cronshaw (1997) provide different numerical techniques to compute sets analogous to M^*. Note, however, that as opposed to our procedure, their methodology works only if

case, the domain restriction set M^* is the set of (U,x) couples such that there exists $(U^l,x^l)\in M^*$, $(U^h,x^h)\in M^*$, $e\in E$ and $c\geq 0$ such that (19)-(22) are all satisfied:

$$U = u(c,e)+\beta\left(U^l+p(e)(U^h-U^l)\right) \tag{19}$$

$$-u'_e(c,e) = \beta p'(e)(U^h-U^l) \tag{20}$$

$$x = x^l+p(e)(x^h-x^l) \tag{21}$$

$$x = u'_c(c,e) \tag{22}$$

where equations (19)-(22) are the $N=2$ counterparts of equations (9)-(12). First, note that in our environment with a fixed interest rate, consumption is not bounded above. This implies that the set M^* may be unbounded as well. Therefore, in order to develop a numerically feasible procedure, we restrict consumption such that $c\in C:=[\underline{c},\overline{c}]$ with $0<\underline{c}<\overline{c}$. Then we define the initial state space M_0 compatible with this choice, as $M_0=[\underline{x},\overline{x}]\times[\underline{U},\overline{U}]$, where $\underline{x}:=\min_{c\in C,e\in E}u'_c(c,e)$, $\overline{x}:=\max_{c\in C,e\in E}u'_c(c,e)$, $\underline{U}:=\left(u(\underline{c})-v(\overline{e})\right)/(1-\beta)$ and $\overline{U}:=\left(u(\overline{c})-v(0)\right)/(1-\beta)$, where $\overline{e}:=\max(E)$. By construction, M_0 turns out to be a compact set. In order to find the domain restriction we need to apply the set valued operator $\mathbf{F}$ defined by (23) on M_0 iteratively until we obtain a fixed point:[24]

$$M_t=\mathbf{F}(M_{t-1})=\left\{(U,x)\in M_{t-1};\ \exists\left(U^l,x^l\right)\in M_{t-1},\ \left(U^h,x^h\right)\in M_{t-1},\ e\in E,\ c\in C;\ \text{(19)-(22) hold}\right\}. \tag{23}$$

A natural way to implement the above operator would be to replace M_0 with a two dimensional rectangular grid and iterate on (23) until convergence on this discrete set. We denote this two-dimensional grid as $\widetilde{M}_0=\{U_k\}_{k=1}^n\times\{x_j\}_{j=1}^m$, where x_j ranges between $\underline{x}$ and $\overline{x}$ and U_k ranges between $\underline{U}$ and $\overline{U}$. Note that conditions (19)-(22) impose 4 constraints on our 6 endogenous variables $\left[U^l,x^l,U^h,x^h,c,e\right]$. Since $\widetilde{M}_{t-1}$ is a discrete set, for a given (U,x), it is not always possible to find any $e\in E$, $c\in C$, $(U^l,x^l)\in\widetilde{M}_{t-1}$ and $(U^h,x^h)\in\widetilde{M}_{t-1}$ satisfying (19)-(22) exactly.[25] This property is solely due to the discrete nature of the set and not due to the actual feasibility of these allocations. For this reason, we had to modify the algorithm defined by (23). First, we

M^* is convex. Following Abreu *et al.* (1990) one can easily show that M^* is convex whenever income shocks are extracted from an atomless distribution. In our model, shocks belong to a finite support so M^* is in general, if not typically non-convex.

[24]Since we generate a decreasing sequence of (nested) sets, we are allowed to search (U,x) bundles only within the set M_{t-1}. This speeds up the algorithm.

[25]This is typically the case when $u(c,e)$ is additively separable because (22) uniquely defines c.

construct $K(\widetilde{M})$ as a set containing $\widetilde{M}$ defined as follows. For a given $\widetilde{M}$ and U_k, we can define $x_{\max}\left(U_k;\widetilde{M}\right) = \max\left\{x_j : (U_k, x_j) \in \widetilde{M}\right\}$ and $x_{\min}\left(U_k;\widetilde{M}\right) = \min\left\{x_j : (U_k, x_j) \in \widetilde{M}\right\}$ as the maximal and minimal marginal utility in $\widetilde{M}$ for a given U_k. Then we can define $\tilde{x}_{\max}$ and $\tilde{x}_{\min}$ as continuous functions mapping $[\underline{U},\overline{U}]$ into $[\underline{x},\overline{x}]$ for a given $\widetilde{M}$, where the values between grid points are determined by linear interpolation. Then, $K(\widetilde{M})$ is given by

$$K(\widetilde{M}) = \left\{(U,x) \in M_0 : \tilde{x}_{\min}(U;\widetilde{M}) \leq x \leq \tilde{x}_{\max}(U;\widetilde{M})\right\}.$$

and we use the following operator instead of (23)

$$\widetilde{M}_t = \widetilde{\mathbf{F}}(\widetilde{M}_{t-1}) = \left\{ \begin{array}{c} (U,x) \in \widetilde{M}_{t-1};\ \exists \left(U^i, x^i\right) \in K\left(\widetilde{M}_{t-1}\right)\ i = l,h \text{ and } e \in E\ ,\ c \in C \\ \text{such that (19)-(22) are satisfied and} \\ \forall i\ \exists \left(U^i_k, x^i_j\right) \in \widetilde{M}_{t-1} \text{ s.t. } \left\|U^i - U^i_k\right\| \leq \varepsilon^U_k \text{ and } \left\|x^i - x^i_j\right\| \leq \varepsilon^x_j \end{array} \right\}.$$

The modified operator selects (U,x) tuples only if there exist continuation values $\left(U^i, x^i\right)$ satisfying constraints (19)-(22) and they are contained in the union of closed rectangles centered on the grid points of $\widetilde{M}_{t-1}$ with height and width given by ε^U and ε^x. The set $\widetilde{\mathbf{F}}(\widetilde{M}_{t-1})$ excludes (U,x) bundles for which there is no such solution of (19)-(22) that has a close enough point in the discrete set $\widetilde{M}_{t-1}$. In the implementation of the procedure, we set ε^U_k and ε^x_j as one half of the distance between grid points implying that the 'interior' of the set is covered completely. The advantage of this approach is that no feasible point will be excluded from the interior of the set $\widetilde{M}_t$ due to the discrete nature of $\widetilde{M}_{t-1}$. However, letting (U^i, x^i) take all possible values in the rectangle around points on the frontier of $\widetilde{M}_t$ can lead to the inclusion of tuples which are clearly not feasible.[26] The restriction on the frontier of $K(\widetilde{M}_{t-1})$ given by $\tilde{x}_{\min}$ and $\tilde{x}_{\max}$, however, provides a solution for this problem.

Further, as the grid size (m and n) goes to infinity this procedure will approximate the domain restriction arbitrarily well. Given the monotonicity properties of $\widetilde{\mathbf{F}}$, our approximation of the domain restriction is the largest fixed point of this modified operator, and it can be obtained by applying the operator until convergence starting from a sufficiently large grid $\widetilde{M}_0$.

[26]Take for example $U = \underline{U}$, in this case, it is easy to see that the only feasible allocation is given by $U^h = U^l = \underline{U}$, $x^h = x^l = \overline{x}$, $c = \underline{c}$, and $e = 0$.

4.2 Solving the Relaxed Problem by Value Function Iterations

Now we will turn to the solution of the relaxed problem described by equations (8) to (12). We solve this problem by value function iterations, where we approximate the value function $V(y, U, x)$ with a continuous function on the two-dimensional grid $\widetilde{M}^{*}$. Continuity is achieved by linear interpolation between grid points. More precisely, consistently with the derivation of the domain restriction, the endogenous states can take values from $K(\widetilde{M}^{*})$. Here a few observations are worth noting. First, this approach is compatible with the computation of the domain restriction because, there we also had to allow $\left(U^{l}, x^{l}\right)$ and $\left(U^{h}, x^{h}\right)$ take values from the set $K(\widetilde{M}^{*})$. Recall, that that whenever preferences are additively separable in c and e, conditions (19)-(21) impose 3 constraints on 5 endogenous variables $\left[U^{l}, x^{l}, U^{h}, x^{h}, e\right]$, hence $(U^{l}, x^{l}) \in \widetilde{M}_{t-1}$ and $(U^{h}, x^{h}) \in \widetilde{M}_{t-1}$ cannot typically hold together and hence interpolation is necessary. Second, the domain restriction sets turned out to be without 'holes',[27] therefore interpolation within the borders is a well-defined procedure. Third, note that linear interpolation using the closest four neighbor in the grid $\widetilde{M}^{*}$ is only straightforward in the interior of the set. Around the frontier, however, we cannot guarantee that all these neighbors are included in $\widetilde{M}^{*}$. In these cases, we use the closest three neighbors and interpolate using the perpendicular distance from these points. Finally also notice, that the value function is linear in y therefore we can recover $V(y^{h}, U, x) = V(y^{l}, U, x) + \left(y^{h} - y^{l}\right)$.

Therefore for the practical implementation we need to solve the following dynamic programming problem

$$V^{\tau+1}(y, U, x) = \sup_{\substack{\left(x^{i}, U^{i}\right) \in K(\widetilde{M}^{*}) \\ e \in E, c \in C}} y - c + \beta\left((1-p(e)) V^{\tau}\left(y^{l}, U^{l}, x^{l}\right) + p(e) V^{\tau}\left(y^{h}, U^{h}, x^{h}\right)\right) \quad (24)$$

$$s.t.\ (19)-(22),$$

where superindex τ refers to the τ-th iteration.

Also notice that if $u_c(c, e)$ is invertible in c for a given e then by fixing a particular couple $\left(e, x^{l}\right)$ we can solve analytically for the remaining 4 endogenous variables using the constraints (19)-(22) for any $(U, x) \in M$. Therefore, we solve (24) by maximizing with respect to couples $\left(e, x^{l}\right)$.

[27] Clearly, a discrete set contains several holes. What we mean here is the discrete analogous of connecteness for set of real numbers. Formally, if a point in the original grid is not in $\widetilde{M}^{*}$ it cannot be that moving far enough in all four (horizontal and vertical) directions one is able to find at least one point in $\widetilde{M}^{*}$ in all such directions.

4.3 Implementing the Ex-post Verification Procedure

We can now use the numerical solution of the value function and optimal policies to implement the verification procedure of the first order condition approach described in Section 3.3. Recall, that the validity of this approach is a prerequisite of our recursive reformulation of the constrained efficient problem.

We first obtain the optimal continuos policy rules for next period promised utility and promised marginal utility of consumption ((16) and (17)) together with $c(U, x)$ over the grid $\widetilde{M}^*$ (here we again use linear interpolation in order to find the policy rules between grid points). Then we define an interval of admissible asset levels $G = [0, \bar{b}]$ and solve (18) with value function iterations, where we use linear interpolations over $\widetilde{M}^* \times \widetilde{G}$, where $\widetilde{G}$ is a discrete grid of q points defined on G. Linear interpolations are particularly useful at this stage because polynomial interpolations are not very reliable around the borrowing limit, because polynomials do not approximate well the steep initial segment of the re-maximization value function J. Notice that by fixing the grid $\widetilde{G}$ for b to the singleton $\{0\}$ we could test whether the agent has incentives to deviate when he can choose only his effort level, as consumption is determined by the budget constraint.[28]

Specifically, according to (18), we need to iterate on the following functional equation until we find a fixed point (superindex τ refers to the τth iteration) :

$$J^{\tau+1}(U, x, b) = \max_{b^R \in G, e^R \in E} u\left(c(U, x) + (1 + r)b - b^R, e^R\right) + \beta \sum_{i=1}^{2} p_i(e^R) J^\tau\left(f(U, x, y^i), h(U, x, y^i), b^R\right), \quad (25)$$

with $(U, x, b) \in \widetilde{M}^* \times \widetilde{G}$, and the first guess of the value function is given by

$$J^0(U, x, b) = \max_{b^R \in G,\ e^R \in E} u\left(c(U, x) + (1 + r)b - b^R, e^R\right) + \beta \sum_i p_i(e^R) f(U, c, y^i).$$

Note that the domain restriction, the solution of the relaxed problem and the solution of re-maximization problem are all approximated using a similar methodology. Since, there are approximation errors, it seems that we cannot expect $U_0^R(\tau, y) = U^*(U_0)$ to hold exactly. However, assume that the first-order approach is actually valid in our problem. In this case, the first-order conditions given by (19)-(22) also characterize the solution to the ex post verification problem for $b = 0$. Note that the allocation derived in Step 2 satisfies these conditions as well. Then, if the computation

[28] This test, however, could never yield a violation of the first-order condition approach, because one can show that the agent's problem is concave in effort alone under all of our parametrizations.

of the optimal contract and the re-maximization problem has the same precision then, in the case where the first-order approach is valid, we should observe a discrepancy between $U_0^R(\tau, y)$ and $U^*(U_0)$ which is smaller the convergence criterion for the fixed point of (25). Hence, in this case, the approximation errors will influence only how well the value function and the optimal policies are approximated and not the ex post verification procedure. Of course, when searching for the solution of (25) we need to make sure that we find a global optimum. This is particularly important because we know that the agent's problem may not be concave. For this reason, we use a two dimensional grid search method which guarantees that we find a global optimum.

In order to assess the performance of our verification procedure in practice (e.g., in presence of approximation error), let's assume for simplicity[29] that $n = m = q = N$, that is, all grids have the same number of points. Also let's define $D_N(U_0) := \frac{J(U^*(U_0), x^*(U_0), 0)_N - U^*(U_0)_N}{\left|U^*(U_0)_N\right|}$, where the N subscripts reflect that these figures were calculated using a grid size of N and $(U^*(U_0), x^*(U_0))$ is the solution of (13). We divided the absolute deviation $\Delta_N(U_0) := J(U^*(U_0), x^*(U_0), 0)_N - U^*(U_0)_N$ by $|U^*(U_0)_N|$ in order to get a discrepancy measure which is independent of the particular model specifications. For a given convergence tolerance level $\epsilon > 0$ in (25), we expect that $\lim_{N\to\infty} \Delta_N(U_0) \in [\Delta - \epsilon, \Delta + \epsilon]$, where $\Delta \geq 0$ is the 'true' discrepancy. Intuitively, if the first-order approach is justified ($\Delta = 0$) we should get that $\Delta_N(U_0) \leq \epsilon$. On the other hand, we reject the validity of the first-order approach if $\Delta_N(U_0) > \epsilon$. First of all, if $\Delta_N(U_0) > 0$ only because of approximation errors along the procedure, we might reject models falsely where the first-order approach is actually verified. We argued above that given our approximation procedure, this is not a likely outcome. Another potential problem arises when $0 < \Delta \leq 2\epsilon$, that is when the first-order approach is not valid, but the 'true' discrepancy is very small. In this case, our approach may falsely accept the validity of the first-order approach. There are two main ways to face this problem. First of all, by choosing ϵ to be a small number the probability of this event can be minimized. Further, whenever $\Delta_N(U_0) \leq \epsilon$, one can check whether the agent's optimal choices in the remaximization problem are different from those prescribed by the relaxed optimal problem (i.e. whether $\tilde{b}(U, x, 0) \approx 0$ and $\tilde{e}(U, x, 0) \approx e^*(U, x)$ where $\tilde{b}$ and $\tilde{e}$ are the optimal asset and effort choices of problem (25), and e^* is the optimal effort choice of the relaxed problem). In our examples, in all the cases where we had $\Delta_N(U_0) \leq \epsilon$, we find that the agent's optimal decisions were practically identical of the optimal allocation of the relaxed problem.

[29] More generally, we can assume that $n = \kappa_n N$, $m = \kappa_m N$ and $q = \kappa_q N$, that is all the grids as scaled by N, with fixed proportions.

4.4 Examples

In this section we implement the three steps of the verification procedure we described above for three different examples with varying grid sizes. Essentially, these examples are only different in the particular functional forms $u(c, e)$ and $p(e)$ take. Also, examples 1 and 2 are such that the validity of the first-order approach can be conjectured with high confidence from previous literature or from our own derivation, respectively. Therefore they provide a nice 'testing' environment for our recursive approach with ex post verification.

4.4.1 Example 1: The Linear-Linear Case

This example was studied by Kocherlakota (2004) who shows that the first-order approach is not valid in this environment. The utility function is additively separable in c and e, and *both* the cost function of effort and the probability shifting functions are linear:[30]

$$u(c, e) = \frac{c^{1-\sigma}}{1-\sigma} - \eta e, \qquad p(e) = e.$$

For this specification, we will follow the general procedure explained above, the only difference is that because of the additive separability of the utility function, condition (22) fully determines c. This implies that, in this case (and in Example 3 below), we can use c as a state variable instead of x without a loss of generality. Moreover, it is easy to derive that, due to linearity, conditions (19) and (20) imply that whenever effort is positive we have that

$$U^l = \left(U - \frac{c^{1-\sigma}}{1-\sigma}\right)/\beta \text{ and } U^h = \left(U - \frac{c^{1-\sigma}}{1-\sigma}\right)/\beta + \frac{\eta}{\beta},$$

that is continuation life-time utilities are solely determined by the state (U, c) and independent of effort. When the agent is required to exert zero effort then U^l takes the same form and any $U^h \leq \left(U - \frac{c^{1-\sigma}}{1-\sigma}\right)/\beta + \frac{\eta}{\beta}$ is incentive compatible. This result makes the calculation of the domain restriction and the optimal (relaxed) optimal policies significantly easier.

[30] See Mitchell and Zhang (2007) for a similar linear-linear formulation where the utility of consumption takes a CARA form.

4.4.2 Example 2: Exponential Utility

In this case, the utility function and the probability shifting functions are taking the following functional forms:

$$u(c,e) = -\exp\{-(c-e)\} \text{ and } p(e) = 1-\exp\{-\rho e\}.$$

In Appendix B, we argue that, for any level of life-time utility there is only one compatible value of marginal utility of consumption. Therefore, we do not have to calculate numerically the set $\widetilde{M}^*$ for this specification. One can show that for this example, we have: $u(c,e) = (1-\beta)U$, hence $c = -\log(-U(1-\beta)) + e$; $u'_c(c,e) = -u$, hence $x = (1-\beta)U$; and $u'_e(c,e) = u(c,e)$, hence $u'_e(c,e) = U(1-\beta)$.

The above results allow us to collapse several constraints, making the problem much simpler. Let $\hat{V}(U,y) \equiv V(U, -(1-\beta)U, y)$, then we have:

$$\begin{aligned} \hat{V}(U,y) &= \sup_{\substack{\{U^i\}_{i=l,h}\in[\underline{U},\overline{U}]^2 \\ e\in E}} y + \log(-U(1-\beta)) - e + \beta\left[p(e)\hat{V}\left(U^h,y^h\right) + (1-p(e))\hat{V}\left(U^l,y^l\right)\right] \\ U &= p(e)U^h + (1-p(e))U^l, \\ -(1-\beta)U &= p'(e)\beta\left(U^h - U^l\right). \end{aligned}$$

The above constraints imply two simple expressions for continuation utilities:

$$U^l = U\left[1 + \frac{p(e)(1-\beta)}{\beta p'(e)}\right] \text{ and } U^h = U\left[1 - \frac{(1-p(e))(1-\beta)}{\beta p'(e)}\right].$$

We can hence solve the above problem by maximizing the objective function only with respect to e using the above values of U^h and U^l.

For this specification, we show analytically in Appendix B, that the first-order approach is valid when c and consequently U is unbounded. Obviously, because of computational feasibility we need to assume that life time utilities and consequently consumption are contained in a compact set. Our proof cannot be directly extend to this case lacking a closed form solution, but as we will see below, our numerical procedure verifies the first-order approach in this case as well.

4.4.3 Example 3: The "Concave" Case

In this example, the utility function is additive separable again but the cost function of effort is strictly convex and the probability shifting function is strictly concave. In Ábrahám and Pavoni

(2007a), we provide analytical conditions under which the agents problem is concave in the optimal contract when $T = 2$. It is shown there that the first-order approach is valid if the utility from consumption has the non-increasing absolute risk aversion property, $v(e)$ is convex and $p(e)$ satisfies some strong concavity condition. Unfortunately, the proof there does not generalize easily for $T > 2$. In this example, we check the validity of this case for $T = \infty$ using the ex post verification procedure. In this example, we use a probability shifting function which satisfy the strong concavity conditions on $p(e)$ and a quadratic cost function for effort:

$$u(c, e) = \frac{c^{1-\sigma}}{1-\sigma} - \eta e^2, \qquad p(e) = 1 - \exp\{-\rho e\}.$$

Further, since $v'(0) = 0$ while $p'(0) = \rho > 0$, we can expect interior solutions for effort.

5 Numerical Results

Parametrization We used parameters for all three examples such that in the full information case, where both effort and asset accumulation are observable, the probability of low and high outcome is equal to 1/2. This implies that, in the absence of information problems, the three examples are observationally equivalent. This choice makes the parametrizations of the three examples comparable. Table 1 provides the parameter values we use for the simulations (recall that $\beta(1 + r) = 1$).

Table 1: Parameter Values

	β	σ	ρ	y^l	y^h	η
Example 1	0.99	2	-	0.1	100	0.0399
Example 2	0.99	-	0.02	0.1	100	-
Example 3	0.99	2	1.0	0.1	100	0.0144

We also set the lower an upper bounds on consumptions as $\underline{c} = y^l = 0.1$ and $\bar{c} = y^h = 100$.

5.1 Ex Post Verification Results

In this section, we evaluate the ex post verification procedure for all the three examples for different grid sizes. For all cases, we have used the same grid for asset levels $\widetilde{G}$ with $q = 20$ unequally spaced grid points and with upper bound on asset holdings given by $\bar{b} = y^h$. Recall, that we interpolated the ex post verification value function between grid points also for asset levels.

Table 2 below summarizes our results. Note that ϵ, the convergence criterion for the ex post verification procedure was 0.001 in all the calculations. Given that in the (relaxed) optimum the agent is supposed to receive life-time utility U_0, recall that we have defined $\Delta(U_0)$ as her absolute ex post deviations. Hence, whenever in the first two columns we find deviations below 0.001, we should be able to claim that the first-order approach is verified.

Table 2: Ex Post deviations in the different examples

	$\max_U(\Delta(U))$	$E[(\Delta(U)]$	$\max_U(D(U))$	$E[(D(U)]$
Example 1, N=50	2.5236407	0.2254422	28.20688%	2.613894%
Example 1, N=100	2.1105425	0.5456722	53.45429%	21.38107%
Example 2, N=50	0.0000762	0.0000099	0.00008%	0.00001%
Example 2, N=100	0.0001068	0.0000114	0.00011%	0.00001%
Example 3, N=50	0.0025912	0.0002271	0.10592%	0.00900%
Example 3, N=100	0.1210131	0.0224213	4.28606%	0.72686%

Let's consider first Example 1. For this example, the deviations are always above the tolerance level, and typically they are sizeable. In the case of our finer grid, this is true for all initial life-time utilities. Therefore, in this case, we confirm the analytical results and reject the validity of the first-order approach with high confidence.

As expected, the results are dramatically different for Example 2. Note that both the average and maximum ex post deviations in this case are always well below ϵ. Therefore, in this case, we can be completely confident about the validity of the first-order approach. There is no particular pattern how the magnitude of the deviations varies with grid size but it should not be surprising, since all these values are at least a magnitude below the convergence criterion and hence we cannot differentiate any of them from zero. Finally, recall that in Appendix B, we show the validity of the first-order approach analytically for the case when consumption and life-time utility is unbounded. The results in this table indicate that the fact that we made life-time utility and consumption bounded for computational reasons, does not influence the validity of the first-order approach in this case.

We believe that the above results have a general message: when the first-order approach is valid approximation errors play smaller role at the ex post verification stage. As we explained in Section 4.3, if a given *approximate* solution to the relaxed problem (24) is incentive compatible, then conditions (20)-(21) provide sufficient characterization of the ex post verification problem. Since, these conditions are satisfied exactly (subject to rounding errors) when we calculate the optimal policies and we use the same degree of approximation in both procedures, we should not expect to see any significant deviations. The approximation of the ex post verification value function J over asset levels plays only a small role here, because if the allocation is actually incentive compatible (due to our normalization) the agent will uniformly choose zero assets along the whole equilibrium path.

Figure 1 Here

If we consider Example 3, we obtain a more complex picture. First of all, Figure 1 shows for $N = 100$ that for certain initial life-time utilities the ex post verification procedure detects small deviations (below the tolerance level) while for some other initial utilities it detects deviations which are above the tolerance (from Table 2, we know that the maximum is 0.12 which is about 4.3% in relative terms).[31] This result is in contrast with Example 1, where we find sizeable deviations everywhere over the whole state space (even the minimum deviation is a magnitude above the tolerance level). This result implies that we cannot justify the first-order approach *globally* for Example 3.[32]

>From a practical point of view, these results have the following interesting implication. In applications, we typically interested in some particular parametrization (usually obtained by calibration) and we can pin down initial life-time utilities by using some economic considerations (e.g. the outside option of the agent or by a zero surplus condition on the planner/principal). In this sense, analytical (global) sufficient conditions can be unnecessarily restrictive for any particular application. For example, in our example, if we take the value of autarky (self-insurance) as the initial utility ($U_0 = -12.3$), then we find deviations below the tolerance level, while if we take the initial life-time utility which makes the planner's surplus equal to zero ($U_0 = -3.05$) the discrepancies are quite large (see, Figure 1). Therefore whether we have found the true constrained efficient allocation seems to depend on which initialization makes sense in the particular application.

However, one has to be cautious with this case, because some extra checks might be necessary. First, the agent may not find it profitable to deviate in an initial state U_0, but there might be future

[31] We have only plotted the value of the ex post discrepancy $\Delta(U_0)$ for Examples 1 and 3 because, for both grid sizes, the deviations for Example 2 were always well below the tolerance level.

[32] Notice however, that the dynamic programming formulation of the ex post verification problem guarantees that there are no profitable deviations along *the whole equilibrium path* starting from those U_0's where we have not found sizeable ex post deviations.

contingencies ((U, x) tuples) with positive probability originating from this node where she would deviate from the relaxed optimal contract. If these probabilities are relatively small compared to the gains in those contingencies, we might have that $\Delta_N(U_0) \leq \epsilon$.

This does not contradict the fact that the requirement for the validity of the first order approach is only the (period zero) condition $U^R(\boldsymbol{\tau}, y) = U^*(U_0)$.[33] When we find that $\Delta_N(U_0) \leq \epsilon$ is only valid for a subset of the domain, one might however say that the first-order approach is justified only in a weak sense. In order to verify the first order approach in a stronger sense, one needs to show that there is no future node which can be reached with positive probability, such that the agent has positive utility gains for deviating. When we checked this for Example 3 ($N = 100$)[34], we found that when $\Delta_N(U_0) \leq \epsilon$, for some U_0, the agent indeed finds it profitable to deviate in some future contingencies which occur with low probability, while for other initial states it was not the case. Note that this latter result does not mean that there are some levels of life-time utility which cannot be reached from some initial U_0. It rather implies that the *combinations* of life time utility and marginal utility of consumption such that the agent has positive gains from deviating from the relaxed optimal plan are not reached by the agent with positive probability starting from this initial life-time utility level. Controlling for the absence of profitable deviations for all possible continuation histories is a procedure computationally very demanding, implying the possibility of further numerical errors. In contrast, when $\Delta_N(U, x) := J(U, x, 0)_N - U \leq \epsilon$ holds for the whole domain $(U, x) \in \widetilde{M}^*$ as in Example 2, we can always verify the validity of the first-order approach in a stronger sense.

Figure 2 Here

A second property of the ex post verification approach seems to be less attractive. In the case of Example 3, whenever we use rougher grid size we always find smaller deviations for a given U_0. This is apparent when we compare Figure 1 to Figure 2 where we plot absolute ex post deviations for Examples 1 and 3 for $N = 50$. Moreover, for the rougher grid, the magnitude of these deviations can be actually below the tolerance level. This is particularly important for Example 3, where, for most utility levels, the deviations are below the tolerance level and for the points between -3 and

[33] As it is shown in Proposition 2, theoretically, sequential incentive compatibility and period zero incentive compatibility coincide. However, in the presence of numerical errors, this is not necessarily the case. We can have that $U^R(\boldsymbol{\tau}, y) - U^*(U_0) \leq \varepsilon$ and, at the same time at future nodes with small probability, the discrepancy between $U_t^R(\boldsymbol{\tau}, y) - U_t^*(U_0; h^t)$ is large, and, in particular, it is well above ε.

[34] For computational tractability, we checked this at all the nodes only up to 10 periods into the future.

-2, where the deviations are significant, they are hardly visible (see in Table 2 that the maximum is 0.0026). In the case of Example 1, this happens only for high lifetime utilities.[35] This implies that for the rougher grid size, we can find an approximation of the relaxed optimal allocation which is actually incentive feasible for a given U_0. However, when we increase the precision of the approximation of the relaxed problem, we detect some positive deviations for the same U_0. In other words, rougher grids may deliver solutions of the relaxed problem which allow for smaller or no deviations of the agent. It hence seems to be key to find as good approximation of the relaxed problem as possible. Then, the design of the ex post verification procedure is going to tell us whether the given approximate solution of the relaxed problem is incentive compatible or not with high confidence. However, the best possible approximation of the optimal allocation is always desirable, because typically the optimal policies are the main object of investigation. In this sense, the ex post verification procedure does not impose any additional requirement on the precision of the approximation. Importantly, when we can be reasonably certain that we found a precise enough approximation of the (relaxed) optimal policies, our results suggest that we can be also reasonably certain that the ex post verification will give us the right answer about the incentive compatibility of this allocation.

Finally, this example highlights the fact that the time horizon of a problem can be also important for the applicability of the first-order approach. AP show analytically that for $T = 2$ the agent's problem is concave in the optimal allocation with the specification of Example 3. Here we have learnt that this result does not generally extend to the infinite horizon case.

6 Conclusions

Relaxing the assumption of perfect observability and contractability on agents' asset holdings in the dynamic moral hazard model is a potentially very valuable exercise from both the theoretical and applied point of view. However, the introduction of hidden assets introduces serious methodological complications. In this paper we propose a way of solving the methodological problems.

We show that, by using a generalized first order condition approach, the model can be solved within the recursive contracts framework where, together with the promised lifetime utility, we used the agent's marginal utility of consumption as an additional endogenous state variable. The

[35] For low life-time utilities, the ex post deviations tend to be higher with the rougher grid. However, in these cases, the size of these deviations is several magnitudes above the tolerance level for both grid sizes.

recursive formulation also permits a parsimonious numerical ex-post verification procedure of the first order approach. We study the performance of our verification procedure in practice by solving three infinite horizon examples numerically. We find that the procedure never rejects models that should be trusted while for too coarse grids it might fail to detect the lack of incentive compatibility of the 'true' relaxed optimal contract.

The general model and methodology studied in this paper can be applied to study the qualitative and quantitative characteristics of optimal policies such as unemployment insurance and welfare programs. For example, in Ábrahám and Pavoni. (2007b), we study the optimal unemployment insurance schemes in the presence of hidden asset holdings and displacement risk. We compare our results with those previously obtained in the literature, where the complication of the model forced the authors to consider suboptimal transfer schemes (e.g. Hansen and İmrohoroglu, 1992; and Abdulkadiroğlu et al. 2002).

In an extended version of this paper (Ábrahám and Pavoni, 2006), we study some of the main qualitative properties of the efficient allocation. We find that hidden asset accumulation changes dramatically the intertemporal properties of the key variables. We focus on the case with additive separable preferences in c and e. The optimal allocation under hidden savings displays (on average) increasing consumption and lifetime utility, key properties of the self insurance allocation. This leads to an intertemporal path of effort (production) and asset holdings that are also strikingly different from standard moral hazard models (with observable assets). The intertemporal discrepancies with respect to self insurance are also important but essentially of quantitative nature. The source of the main discrepancy is consumption smoothing. By decreasing the level of idiosyncratic uncertainty the agent faces, the planner reduces the precautionary motive for savings, and makes the intertemporal path of consumption flatter than that in self insurance. This implies a relatively flat intertemporal path of effort and asset holdings as well. In this sense, the forces operating in the hidden asset moral hazard economy place the optimal intertemporal allocation of consumption and effort in-between self insurance and pure moral hazard.

The framework we develop in this paper can be fruitfully used to study the optimal (private) contract in several other moral hazard problems where hidden savings may be relevant (e.g., long-term employment and compensation contracts, corporate loans and managerial contracts), in a systematic way. More generally, the ex post verification procedure we propose in this paper could be easily applied to a richer set of models. For example, in the recursive formulation of Fernandes and Phelan (2000) and Doepke and Townsend (2006), the set of incentive compatibility constraints easily becomes very large. The computational burden could be reduced significantly by imposing only a carefully chosen (and economically guided) subset of the constraints, and verify ex post whether the obtained efficient allocation is in fact incentive compatible. Another application could be to Ramsey optimal taxation models, when it is difficult to guarantee global concavity of the household's program.[36] In this case, our ex post verification procedure would be equivalent to solving the relevant competitive equilibrium taking the tax rate processes as given.

[36] We have situations in mind where there is a discrete labor market participation decision, or where the planner imposes income tax schemes with some degree of regressivity.

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Appendix A: Proofs

Proof of Proposition 1 In what follows, without loss of generality we will consider contracts where $\mathbf{b}_t\left(h^t\right) \equiv 0$. Notice that to each contract $\mathcal{W}$, we can associate the states as follows:

$$
\begin{aligned}
u_c'\left(\mathbf{c}_t\left(h^t\right), \mathbf{e}_t(h^t)\right) &= \mathbf{x}_t(h^t) \quad (26) \\
\mathbf{U}_t\left(\mathcal{W}; h^t\right) &= \mathbf{U}_t(h^t). \quad (27)
\end{aligned}
$$

>From the definition of lifetime utility $\mathbf{U}_t\left(\mathcal{W}; h^t\right)$, (27) satisfy the following version of (9)

$$
\mathbf{U}_t(h^t) = u\left(\mathbf{c}_t\left(h^t\right), \mathbf{e}_t(h^t)\right) + \beta \sum_i p_i\left(\mathbf{e}_t\left(h^t\right)\right) \mathbf{U}_{t+1}\left(h^t, y^i\right); \quad (28)
$$

and - by construction - (26) guarantees the sequential version of (12)

$$
u_c'\left(\mathbf{c}_t\left(h^t\right), \mathbf{e}_t(h^t)\right) \leq \mathbf{x}_t(h^t). \quad (29)
$$

Moreover, if the resulting plan of states is generated by an incentive feasible contract $\mathcal{W} \in \Omega_{FOC}$ then by definition $\mathbf{c}_t\left(h^t\right) \geq 0$ and $\mathbf{e}_t\left(h^t\right) \in E$ and it must satisfy the following sequential version of (10), (11) and the domain restriction

$$
\begin{aligned}
-u_e'\left(\mathbf{c}_t\left(h^t\right), \mathbf{e}_t(h^t)\right) &= \beta \sum_i p_i'\left(\mathbf{e}_t\left(h^t\right)\right) \mathbf{U}_{t+1}\left(h^t, y^i\right) \quad (30) \\
u_c'\left(\mathbf{c}_t\left(h^t\right), \mathbf{e}_t(h^t)\right) &\geq \beta(1+r) \sum_i p_i(\mathbf{e}_t(h^t)) u_c'\left(\mathbf{c}_{t+1}(h^t, y^i), \mathbf{e}_{t+1}(h^t, y^i)\right) \quad (31) \\
\left(\mathbf{x}_t(h^t), \mathbf{U}_t(h^t)\right) &\in M^* \quad (32)
\end{aligned}
$$

The domain restriction is satisfied since whenever $\mathcal{W} \in \Omega_{FOC}$, the contract in period zero generates incentive compatible continuations $\mathcal{W} \setminus y^i$.

Denote by $\mathcal{S} = \{\mathbf{x}_t(h^t),\ \mathbf{U}_t(h^t)\}_{t=1}^{\infty}$ the contingent plan of states and by $\mathcal{M}^*$ the set of contingent plan of states and contracts that satisfy all such constraints, i.e.

$$
\mathcal{M}^* = \left\{(\mathcal{S}, \mathcal{W}) : \text{ for all } h^t\ \mathbf{c}_t\left(h^t\right) \geq 0,\ \mathbf{e}_t\left(h^t\right) \in E,\ \text{(28)-(29)-(30)-(31)-(32)}\right\}.
$$

We have just shown that for each $\mathcal{W} \in \Omega_{FOC}$ we can find a pair $(\mathcal{S}, \mathcal{W}) \in \mathcal{M}^*$. We now want to show the converse. Notice first that from the definition of M^*, for each continuation $\mathbf{U}_t(h^t)$ utility in $(\mathcal{S}, \mathcal{W}) \in \mathcal{M}^*$ there is an incentive contract $\mathcal{W}$ such that $\mathbf{U}_t\left(\mathcal{W}; h^t\right) = \mathbf{U}_t(h^t)$. As a consequence, (28) together with the domain restriction guarantee that we generate well defined values for the agent. Moreover, (30) together with the domain restriction implies that effort incentive compatibility (4) is satisfied at each node. In order to show that (11) implies that the saving incentive compatibility is satisfied we need to show that (29) and (31) together guarantee (5). This is the case, because for any two consecutive periods at node h^t we have

$$\begin{aligned} u_c'\left(\mathbf{c}_t\left(h^t\right), \mathbf{e}_t\left(h^t\right)\right) &\geq \beta\left(1+r\right)\sum_i p_i\left(\mathbf{e}_t\left(h^t\right)\right)\mathbf{x}_{t+1}\left(h^t, y^i\right) \\ &\geq \beta\left(1+r\right)\sum_i p_i\left(\mathbf{e}_t\left(h^t\right)\right)u_c'\left(\mathbf{c}_{t+1}\left(h^t, y^i\right), \mathbf{e}_{t+1}\left(h^t, y^i\right)\right) \end{aligned}$$

which is just (5). By the above argument, problem (6) can be equivalently written as

$$\begin{aligned} V_{foc}^*(U_0) = \sup_{(\mathcal{W},\mathcal{S})\in\mathcal{M}^*} & \sum_i p_i^0 \mathbf{\Pi}_1(\mathcal{W}; y^i) \\ \text{s.t.} & \sum_i p_i^0 \mathbf{U}_1(\mathcal{W}; y^i) \geq U_0. \end{aligned}$$

It is now a straightforward application of the Bellman principle to show that the true value of the problem V_{foc}^* can be decomposed as follows:

$$\begin{aligned} V_{foc}^*(U_0) &= \sup_{(x^i,U^i)\in M^*} \sum_i p_i^0 W_{foc}^*(y^i, x^i, U^i) \\ \text{s.t.} &: \sum_i p_i^0 U^i \geq U_0, \end{aligned} \tag{33}$$

where, for $\left(x^i, U^i\right) \in M^*$ and $y^i \in Y$, we have

$$\begin{aligned} W_{foc}^*(y^i, x^i, U^i) &= \sup_{(\mathcal{W},\mathcal{S})\ \setminus\ y^i\ \in\mathcal{M}^*} \mathbf{\Pi}_1(\mathcal{W}; y^i) \\ \text{s.t. } \mathbf{U}_1(y^i) &= U^i;\ \mathbf{x}_1(y^i) = x^i. \end{aligned}$$

The proof of this last statement requires to verify for the last problem the properties for the sup operator which defines the original function $V_{foc}^*(U_0)$. To show that it is an upper bound notice that since any pair $\left(\bar{\mathcal{W}}, \bar{\mathcal{S}}\right) \in \mathcal{M}^*$ such that $\sum_i p_i^0 \mathbf{U}_1(\bar{\mathcal{W}}; y^i) \geq U_0$ induces continuations $\left(\bar{x}^i, \bar{U}^i\right) \in M^*$, it follows that $\bar{\mathbf{U}}_1(y^i) = \bar{U}^i$; $\bar{\mathbf{x}}_1(y^i) = \bar{x}^i$ and that $W_{foc}^*(y^i, \bar{x}^i, \bar{U}^i) \geq \mathbf{\Pi}_1(\bar{\mathcal{W}}; y^i)$. We must hence have that $\sup_{(x^i,U^i)\in M^*}\sum_i p_i^0 W_{foc}^*(y^i, x^i, U^i) \geq \sum_i p_i^0 W_{foc}^*(y^i, \bar{x}^i, \bar{U}^i) \geq \sum_i p_i^0 \mathbf{\Pi}_1(\bar{\mathcal{W}}; y^i)$ for all such pairs. We now want to show that it is the least upper bound. Take any $\varepsilon/2 > 0$. By the definition of sup in (33) there exists a set of pairs $\left(\bar{x}^i, \bar{U}^i\right) \in M^*$ $i = 1, 2, ...N$ such that $\sum_i p_i^0 \bar{U}^i \geq U_0$ and $\sup_{(x^i,U^i)\in M^*}\sum_i p_i^0 W_{foc}^*(y^i, x^i, U^i) < \sum_i p_i^0 W_{foc}^*(y^i, \bar{x}^i, \bar{U}^i) + \varepsilon/2$. Moreover, for any i and $\left(\bar{x}^i, \bar{U}^i\right) \in M^*$ there exists a pair $\left(\bar{\mathcal{W}}, \bar{\mathcal{S}}\right) \setminus y^i \in \mathcal{M}^*$ such that $\bar{\mathbf{U}}_1(y^i) = \bar{U}^i$; $\bar{\mathbf{x}}_1(y^i) = \bar{x}^i$ and

$W^*_{foc}(y^i, \bar{x}^i, \bar{U}^i) < \mathbf{\Pi}_1(\bar{\mathcal{W}}; y^i) + \varepsilon/2$. All this implies that for any $\varepsilon > 0$ there exists $\left(\bar{\mathcal{W}}, \bar{\mathcal{S}}\right) \in \mathcal{M}^*$ such that $\sum_i p_i^0 \mathbf{U}_1(\bar{\mathcal{W}}; y^i) \geq U_0$ and $\sup_{(x^i,U^i)\in M^*} \sum_i p_i^0 W^*_{foc}(y^i, x^i, U^i) < \sum_i p_i^0 \mathbf{\Pi}_1(\bar{\mathcal{W}}; y^i) + \varepsilon$.

Following a line of proof similar to that we used above and using again standard arguments (e.g., using a direct application of Theorem 4.2 of SLP) it is easy to show that the 'interim' value function W^*_{foc} solves the functional equation (8)-(12). The key is to realize that the restriction $\mathcal{M}^*$ can be equivalently written recursively by using the following state dependent correspondence

$$\Gamma(x, U) = \left\{ \left(c, e, \left\{U^i, x^i\right\}_{i=1}^N\right) : \ c \geq 0, e \in E, \left(U^i, x^i\right) \in M^* \text{ for all } i \text{ (9)-(10)-(11)-(12)} \right\}$$

which is straightforward to verify directly from the definition of $\mathcal{M}^*$.

The converse is standard. When V is bounded we can use Theorem 4.3 of SLP to show that $V = W^*_{foc}$. In this case, since M^* is compact, we can apply Theorem 4.6 of SLP and prove that V is continuous. Notice that we can use the theorem of the maximum (e.g. see Theorem 3.6 in SLP) despite that the restriction on $c \geq 0$ is unbounded above. This is so since the objective function is *coercive* in c and this allow us to restrict the choice of c to compact sets only.[37] The maximum theorem hence guarantees that the policy correspondence is non empty for any $(y, x, U) \in Y \times M^*$, and existence of an optimal plan can be shown by repeatedly applying (any selection of) the policy. **Q.E.D.**

Proof of Proposition 2 Consider the case where p^0 is degenerate at y, and let U_0 the utility to be delivered to the agent in period zero. We denote by $\mathcal{W}^*_{foc}(U_0)$ the relaxed optimal contract (which exists by Proposition 1), and by $V^*(U_0, y)$ the value of the true optimal (fully incentive compatible) contract associated to problem (3). In terms of the proposition, $\mathbf{U}^*(U_0) = \mathbf{U}_1^T\left(\mathcal{W}^*_{foc}(U_0), y\right) \geq U_0$. Clearly if $U_0^R(\boldsymbol{\tau}, y) > U^*(U_0)$ there exists a feasible deviation hence $\mathcal{W}^*_{foc}(U_0)$ cannot be optimal since it is not incentive feasible, i.e. $\mathcal{W}^*_{foc}(U_0) \notin \Omega$.

To show the converse, notice first that - since $\Omega \subset \Omega_{FOC}$ - $\mathbf{\Pi}_1^T\left(\mathcal{W}^*_{foc}(U_0), y\right) \geq V^*(U_0, y)$. Hence, whenever $\mathcal{W}^*_{foc}(U_0) \in \Omega$ then $\mathcal{W}^*_{foc}(U_0)$ is optimal. Moreover, since obeying to the proposed contract is feasible for the agent we always have $U_0^R(\boldsymbol{\tau}, y) \geq U^*(U_0)$. We hence have to show that $U_0^R(\boldsymbol{\tau}, y) = U^*(U_0)$ implies that $\mathcal{W}^*_{foc}(U_0) \in \Omega$, i.e. it is sequentially incentive compatible.

[37] Since V is bounded we have coercivity: $\lim_{c\to\infty} \ f(c) = y - c + \frac{1}{1+r}\sum p_i(e) V\left(y^i, U^i, x^i\right) = -\infty$. Coercivity and continuity guarantee that the objective function f has compact upper sections:

$$U(\alpha) = \{c \geq 0 : \ f(c) \geq \alpha\},$$

and we can without loss of generality focus on the compact sets $U(\alpha)$ for c.

If at node h^t, $W^*_{foc}(U_0)$ is not incentive compatible, there must be a feasible deviation strategy $\bar{\sigma}$ such that $U_t^T\left(\boldsymbol{\tau}^*_{foc}, \bar{\boldsymbol{\sigma}}; h^t\right) > U_t^T\left(\boldsymbol{\tau}^*_{foc}, \boldsymbol{\sigma}^*_{foc}; h^t\right)$. Suppose that we construct a new plan $\bar{\sigma}^*_{foc}$ for the agent as follows. Assume the agent behaves as suggested by the contract both before period t and after period t in all nodes but h^t; while at node h^t he follows plan $\bar{\sigma}$. This plan is clearly available to him in the remaximization problem. It must hence be that $U_0^R(\boldsymbol{\tau}, y) \geq U_1^T\left(\boldsymbol{\tau}^*_{foc}, \bar{\boldsymbol{\sigma}}^*_{foc}; y\right) > U^*(U_0)$ where the last inequality is due to the assumption that $\beta > 0$ and the full support assumption, which implies that h^t is reached with positive probability. **Q.E.D.**

Appendix B: The Exponential Utility Example

Closed form for the planner's problem Consider problem (8)-(12), with $N = 2$ and $T = \infty$, where the utility function and the probability of the high state are as follows:

$$u(c, e) = -\exp\{-(c - e)\} \text{ and } p_h(e) = p(e) = 1 - \exp\{-\rho e\}, \ \rho > 0. \tag{34}$$

We will furthermore assume that the discount factor is not too low, that is $\beta > \frac{1}{1+\rho}$. Note that in our simulations we set $\beta = .99 > \frac{1}{1+\rho} = \frac{1}{1.02} \cong .98$. In what follows we will show that: *When facing the optimal contract in the re-maximization stage, the agent optimally decides to follow the planner original recommendations. In other terms, for this specification of the model we are entitled to use the first order approach when solving for the optimal contract.*

The proof of the claim will be done in several steps. First, we will be able to derive a recursive closed form for the planner's problem. This will provide us with an analytical expression for the optimal policy, which will in turn allow us to obtain a closed form for the agent's re-maximization problem as well. The analytical expression for the agent's re-maximization problem will take a recursive form along the lines of our J function in Section 4.3.

Several things are peculiar of this parametric formulation of the problem. First, we can allow c to become negative.[38] Second, $u_c'(c, e) = -u_e'(c, e) = -u(c, e) = \exp\{-(c - e)\}$. The Euler equation (for $\beta(1 + r) = 1$) together with the law of iterated expectations imply $x_t = u_c'(c_t, e_t) = E_t[u_c'(c_{t+k}, e_{t+k})]$ $\forall k \geq 1$. The life-time utility of the agent is then given by

$$U_t = \mathbf{E}_t\left[\sum_{k=0}^{\infty} \beta^t u(c_{t+k}, e_{t+k})\right] = \sum_{k=0}^{\infty} \beta^t \mathbf{E}_t u_c'(c_{t+k}, e_{t+k}) = -\frac{u_c'(c_t, e_t)}{(1-\beta)} = -\frac{x_t}{(1-\beta)}.$$

[38] Notice however that in order for $p_i(e)$ $i = h, l$ to be probabilities we require $e \geq 0$.

That is, we must have $u = (1-\beta)U$ and $x = -u$ (see also Werning, 2002). This allows us to define a new function $\hat{V}(U,y) := V(U, -(1-\beta)U, y)$ that solves the following Bellman (functional) equation:

$$
\begin{aligned}
\hat{V}(U,y) &= \sup_{e\geq 0,(u,U^i)\leq 0} y - e + \ln(-(1-\beta)U) + \beta \sum_i p_i(e)\hat{V}\left(U^i, y^i\right) \\
s.t.\ u &= (1-\beta)U; \\
u &= \sum_i p_i(e)(1-\beta)U^i; && (\delta) \\
-u &= \beta \sum_i p_i'(e)U^i; && (\mu) \\
U &= u + \beta \sum_i p_i(e)U^i. && (\lambda)
\end{aligned}
$$

It is easy to see that the first and the last constraint impose the same restrictions on the choices.[39] We can hence simplify the expression for the functional equation by eliminating the variable u, its restrictions and constraint (δ), so that to obtain the following formulation of the value function $\hat{V}$:

$$
\begin{aligned}
\hat{V}(U,y) &= \sup_{e\in E,U^i\leq 0} y - e + \ln(-(1-\beta)U) + \beta \sum_i p_i(e)\hat{V}\left(U^i, y^i\right) && (35) \\
& \quad s.t. \\
-(1-\beta)U &= \beta \sum_i p_i'(e)U^i && (\mu) \\
U &= \sum_i p_i(e)U^i. && (\lambda)
\end{aligned}
$$

We guess that the value function takes the following form

$$\hat{V}(U,y) = A(y) + \frac{1}{1-\beta}\ln(-U),$$

where, obviously, $A(y) = \hat{V}(-1,y)$. In order to verify that the above expression represents the correct value function, suppose first that we are at $U_0 = -1$. Let $\left\{\bar{e}_t\left(h^t\right), \bar{c}_t\left(h^t\right)\right\}_{t=0}^{\infty}$ be the optimal plan of effort and consumption for a contract delivering $U_0 = -1$ to the agent. Now, assume that we increase utility from -1 to $U > -1$ by keeping exactly the same effort and increasing the consumption of the agent by θ in each

[39] Given $u = (1-\beta)U$, we can rewrite (λ) so that it takes exactly the same form as (δ).

history. It is easy to see that such plan is incentive compatible. In order to obtain θ note that

$$-1 = \mathbf{E}_0 \left[\sum_{t=1}^{\infty} \beta^{t-1} - \exp\{-\bar{c}_t + \bar{e}_t\} \right].$$

Hence θ is defined by the following relationship:

$$U = \mathbf{E}_0 \left[\sum_{t=1}^{\infty} \beta^{t-1} - \exp\{-(\bar{c}_t + \theta) + \bar{e}_t\} \right] = \exp\{-\theta\} \mathbf{E}_0 \left[\sum_{t=1}^{\infty} \beta^{t-1} - \exp\{-\bar{c}_t + \bar{e}_t\} \right].$$

That is $-\exp\{-\theta\} = U$ and therefore $\theta = -\log(-U) > 0$. This implies that the cost of this perturbation for the planner is $\frac{-\log(-U)}{1-\beta} > 0$, which implies

$$\hat{V}(U, y) \leq \hat{V}(-1, y) + \frac{1}{1-\beta} \ln(-U). \tag{36}$$

Now, if we use the same argument to reach utility level $U_0 = -1$ from an initial $U < -1$, we have from (36) that

$$\hat{V}(-1, y) \leq \hat{V}(U, y) + \frac{1}{1-\beta} \ln\left(-\frac{1}{U}\right) = \hat{V}(U, y) - \frac{1}{1-\beta} \ln(-U).$$

Since U was chosen arbitrarily, both (36) and the above inequalities are true for all U. Combining the two expressions we have the desired expression

$$\hat{V}(U, y) = \hat{V}(-1, y) + \frac{1}{1-\beta} \ln(-U).$$

Interestingly, the function is very similar to that obtained in the literature for the moral hazard model with no access to the credit market (e.g., Green, 1987). One can however show that the latter differs from ours.[40] More precisely, note that for a given U,without loss of generality, we can describe the optimal policies $U\left(U, y^i\right)$ as

$$U\left(U, y^i\right) = \gamma^i U,$$

where $\gamma^i \geq 0$ are multiplicative constants.

[40] Details are available upon request.

Note now that both constraints of the problem can be written as

$$\sum_i p_i(e)\gamma^i = 1 \text{ and } \qquad (\lambda')$$
$$\sum_i p_i'(e)\gamma^i = -\frac{1-\beta}{\beta}, \qquad (\mu')$$

that is, they do not depend on U. By combining these two conditions with our specification for p in (34), one can easily verify[41] that since $\gamma^h = 1 - \frac{1-\beta}{\beta\rho}$ we need $\beta > \frac{1}{1+\rho}$ in order to guarantee $\gamma^h > 0$.[42]

By using the properties of the logarithm it is easy to see that the planner objective function is additive separable in U on the one hand, and γ^i $i = 1, \ldots N$ and e are independent of U on the other hand. This implies that the whole constrained maximization, hence its solution, does not depend on U.[43] We can finally show that the constant $A(y)$ is implicitly defined by

$$A(y) = \hat{V}(-1, y) = y - e^* + \ln(1-\beta) + \beta\bar{A} + \frac{1}{1-\beta}\sum_i p_i(e^*)\ln\left(\gamma^i\right),$$

where $\bar{A} = \sum_i p_i(e^*) A\left(y^i\right)$, and (with some abuse of notation) we use γ^i for their optimal values. For the optimal value of effort, we use e^*, and we assume that the parameters of the model (in particular the levels of income) are such that $e^* > 0$.

The Agent's Re-Maximization Problem: A Closed Form Consider now the problem faced by the agent in the ex-post re-maximization stage. We will show that at this stage the agent's unique optimal decision is to follow the planner recommendations on effort and asset holdings. We will use a recursive approach, which is the analytical analogous to our verification procedure of Section 3.3.

[41] The incentive compatibility constraint implies

$$\rho \exp\{-\rho e^*\}\left(\gamma^h - \gamma^l\right) = -\frac{1-\beta}{\beta}$$

and

$$\sum_i p_i(e)\gamma^i = \gamma^h - \exp\{-\rho e\}\left(\gamma^h - \gamma^l\right) = 1.$$

[42] This is the only purpose for our initial assumption on β. This parametric requirement is due to the particular timing we use throughout the paper. In particular, such restriction is not required when a timing à la Werning (2002) is adopted instead.

[43] The intuition is that the absence of wealth effects implies that the incentive structure is exactly the same for any level of promised utility, up to a scaling variable given by U.

The guess for the agent's value function when facing the optimal contract is (recall that $\frac{1}{1+r} = \beta$) :

$$J(U, b) = U \exp\{-(1-\beta) b\},$$

where b is the level of assets and U is the lifetime utility according to the optimal contract. Note that - as we expect - for $b = 0$ we have $J(U, 0) = U$. To verify our guess, notice that plugging in our policies for the optimal contract we derived above, the value function solves

$$J(U, b) = \max_{b', e} \quad -\exp\{-c^*(U) + e\} \exp\{-b + \beta b'\} + \beta \sum_i p_i(e) \gamma^i U \exp\{-(1-\beta) b'\},$$

were $c^*(U)$ is the net transfer the agent receives from the planner when the history of shocks implies state U. First of all, we need to verify that our guess for J is correct. We will do it by using as candidate policies the solution to the first order conditions with respect to b' and e of the agent's problem. We will then show that the agent's problem is globally concave so that the choices for e and b' are the (unique) optimal ones for the agent. The proof becomes complete by showing that the recommendations of the planner are the ones that solve the first order conditions for the agent.

The first order condition with respect to b' is

$$-\beta \exp\{-c^*(U) + e - b + \beta b'\} - \beta(1-\beta) \sum_i p_i(e) \gamma^i U \exp\{-(1-\beta) b'\} = 0. \tag{37}$$

The first order condition with respect to e is as follows

$$-\exp\{-c^*(U) + e\} \exp\{-b + \beta b'\} + \beta \sum_i p_i'(e) \gamma^i U \exp\{-(1-\beta) b'\} = 0. \tag{38}$$

We now show that the optimal effort recommendation of the planner for e^* and $b' = b$ are solving the above conditions, when the agent faces the optimal contract.[44] Since the planner recommendation in the optimal contract is $e_t \equiv e^*$ and $b_t^* \equiv 0$ we will be done since the agent starts with zero assets in period zero.

If we use $(\lambda') : \sum_i p_i(e^*) \gamma^i = 1$ and the $-\exp\{-c^*(U) + e^*\} = (1-\beta) U$ condition (37) evaluated

[44]The intuition for this result is as follows. The absence of wealth effects implies that for any level of assets b, the agent will supply exactly the same level of effort and will consume the annuity of her financial wealth $(1-\beta) b$, hence keeping b constant. This additional consumption will increase the agent's utility by $\exp\{-(1-\beta) b\}$ every period in addition to the average utility delivered by the contract given by $(1-\beta) U$ in each period.

at $e = e^*$ becomes

$$\beta(1-\beta)U\exp\left\{-b-\beta b'\right\} - \beta(1-\beta)U\exp\left\{-(1-\beta)b'\right\} = 0,$$

which - for $\beta \in (0,1)$ - can be satisfied only when $b' = b$, since $U \neq 0$. Similarly, the first order condition for e evaluated at $b' = b$ and $e = e^*$ becomes the identity

$$(1-\beta)U\exp\{-(1-\beta)b\} - (1-\beta)U\exp\{-(1-\beta)b\} = 0,$$

where we used again the fact $-\exp\{-c^*(U)+e^*\} = (1-\beta)U$, and condition (μ'): $\sum_i p_i'(e^*)\gamma^i = -\frac{1-\beta}{\beta}$. This condition is not surprisingly the same as the first order condition for b'. It is now easy to verify, by plugging our optimal solutions into the Bellman equation which defines J, that our guess is correct. Finally, one can use known verification theorems to show that J is the true value function for the agent problem.[45]

Recall that we are entitled to use the first order conditions to derive the policy functions only when the maximization problem defining the Bellman functional equation is concave. What is hence left to demonstrate is the concavity of the agent's problem for all U and b. Global concavity entitles us to obtain the optimal policies for the agent's problem by only looking at first order conditions. Since we have shown that the planner recommendations are solving the agent's first order conditions in the re-maximization problem, global concavity will imply that the use of the first order conditions of the agent in place of the incentive constraint in the planner's problem of the previous section was actually justified as the planner's recommendations according to the relaxed problem are optimal for the agent when facing the optimal (relaxed) contract.

We now compute the Hessian matrix $H(e,b';b)$. From (37), the second derivative with respect to b' is

$$Q\left(e,b';b\right) := -\beta^2\exp\{-c^*(U)+e\}\exp\left\{-b-\beta b'\right\} + \beta(1-\beta)^2\sum_i p_i(e)\gamma^i U\exp\left\{-(1-\beta)b'\right\}.$$

Notice that we can write

$$-\exp\{-c^*(U)+e\} = (1-\beta)U\exp\{e-e^*\}, \tag{39}$$

so $Q(e,b';b)$ becomes $\beta^2(1-\beta)U\exp\{e-e^*\}\exp\{-b-\beta b'\}+\beta(1-\beta)^2U\exp\{-(1-\beta)b'\}\sum_i p_i(e)\gamma^i < 0$. The last inequality is obtained from the observations that $U<0$ and $\gamma_i \geq 0$, and the non-negativity of

[45] Further details can be made available upon request.

the exponential function.

Now, since the problem is smooth, both cross derivatives can be obtained - for example - by taking the derivative of (37) with respect to e. We hence have

$$\begin{aligned} C\left(e, b'; b\right) \quad : \quad &= \beta \exp\left\{-c^*\left(U\right) + e\right\} \exp\left\{-b - \beta b'\right\} - \beta\left(1-\beta\right) \sum_i p_i'(e)\gamma^i U \exp\left\{-\left(1-\beta\right) b'\right\} \\ &= \; -\beta\left(1-\beta\right) U \exp\left\{e - e^*\right\} \exp\left\{-b - \beta b'\right\} - \beta\left(1-\beta\right) U \exp\left\{-\left(1-\beta\right) b'\right\} \sum_i p_i'(e)\gamma^i, \end{aligned}$$

where we have used (39) to show the equality in the second line.

Finally from (38), the second order condition with respect to e is as follows

$$\begin{aligned} L(e, b'; b) \quad : \quad &= -\exp\left\{-c^*\left(U\right) + e\right\} \exp\left\{-b - \beta b'\right\} + \beta \sum_i p_i''(e)\gamma^i U \exp\left\{-\left(1-\beta\right) b'\right\} \\ &= \; \left(1-\beta\right) U \exp\left\{e - e^*\right\} \exp\left\{-b - \beta b'\right\} + \beta U \exp\left\{-\left(1-\beta\right) b'\right\} \sum_i p_i''(e)\gamma^i, \end{aligned}$$

where we have again used (39). Recall that from the incentive compatibility, for our parametrized model with two income levels we obtain $\gamma^l > \gamma^h$, we have for all e

$$\sum_i p_i(e)\gamma^i = \gamma^h - \exp\left\{-\rho e\right\}\left(\gamma^h - \gamma^l\right) > 0, \tag{40}$$

and

$$\sum_i p_i''(e)\gamma^i = -\rho^2 \exp\left\{-\rho e\right\}\left(\gamma^h - \gamma^l\right) = -\rho \sum_i p_i'(e)\gamma^i. \tag{41}$$

Since $U < 0$ the last conditions implies $\sum_i p_i''(e)\gamma^i > 0$, hence $L(e, b'; b) < 0$ for all e, b, and b'.

We have a 2×2 Hessian matrix. Recall that a sufficient condition for a 2×2 symmetric matrix to be negative definite is that it has positive determinant and a negative trace. Since both elements of the trace are negative we only need to show that the Hessian has a positive determinant. The determinant for H is:

$$\det\left(H\left(e, b'; b\right)\right) = Q\left(e, b'; b\right) L(e, b'; b) - C\left(e, b'; b\right) C\left(e, b'; b\right).$$

Notice first that in both Q, L and C we can collect U. Since $U^2 > 0$, This implies that we can compute $\det\left(\frac{H(e,b';b)}{U^2}\right) := \det\left(\hat{H}\right)$ instead, where we have omitted the arguments e, b, b'. Moreover, we can simplify the analysis further by denoting

$$\begin{aligned}\Gamma &: = (1-\beta)\exp\{e-e^*\}\exp\left\{-b-\beta b'\right\} > 0 \\ \Xi &: = \beta\exp\left\{-(1-\beta)\,b'\right\} > 0, \\ X &: = -\exp\{-\rho e\}\left(\gamma^h-\gamma^l\right) > 0.\end{aligned}$$

Using these simplifications and (41), we have

$$\det\left(\hat{H}\right) = \left(\beta^2\Gamma + (1-\beta)^2\,\Xi\sum_i p_i(e)\gamma^i\right)\left(\Gamma+\rho^2\Xi X\right) - \left(\beta\Gamma - \rho\,(1-\beta)\,\Xi X\right)^2.$$

Using now (40), we have $\sum_i p_i(e)\gamma^i > X$, so, since $\Gamma + \rho^2\Xi X > 0$ we have

$$\det\left(\hat{H}\right) > \left(\beta^2\Gamma + (1-\beta)^2\,\Xi X\right)\left(\Gamma+\rho^2\Xi X\right) - \left(\beta\Gamma - \rho\,(1-\beta)\,\Xi X\right)^2.$$

It is now easy to see that since $\beta \in (0,1)$ and $\Xi, X > 0$, we have

$$\left(\beta^2\Gamma + (1-\beta)^2\,\Xi X\right)\left(\Gamma+\rho^2\Xi X\right) > \beta^2\Gamma^2 + \rho^2\,(1-\beta)^2\,\Xi^2 X^2 > \left(\beta\Gamma - \rho\,(1-\beta)\,\Xi X\right)^2$$

hence $\det\left(\hat{H}\right) > 0$ as desired. Q.E.D.

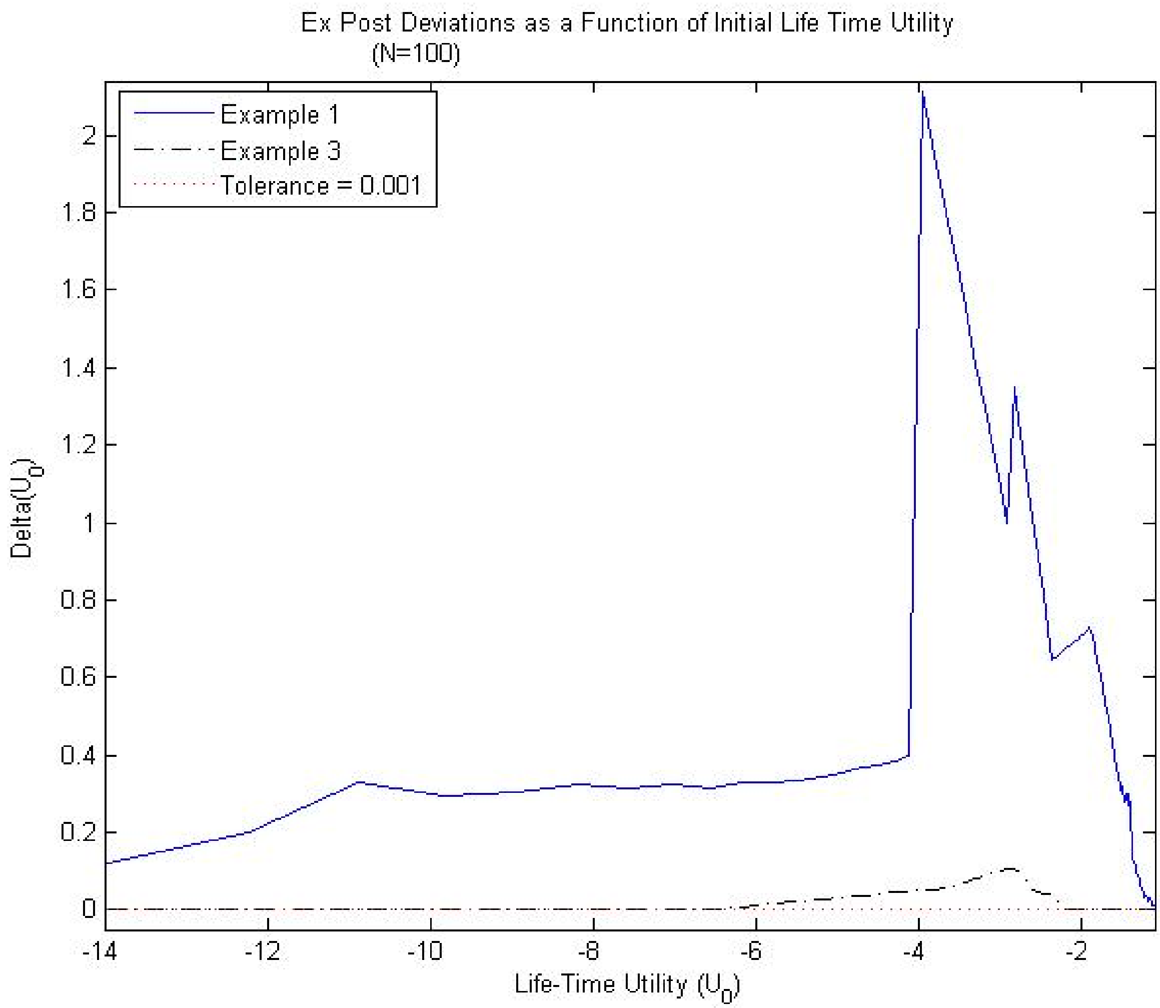

Figure 1: Ex Post Deviations ($\Delta(U_0)$) for Examples 1 and 3 (N=100)

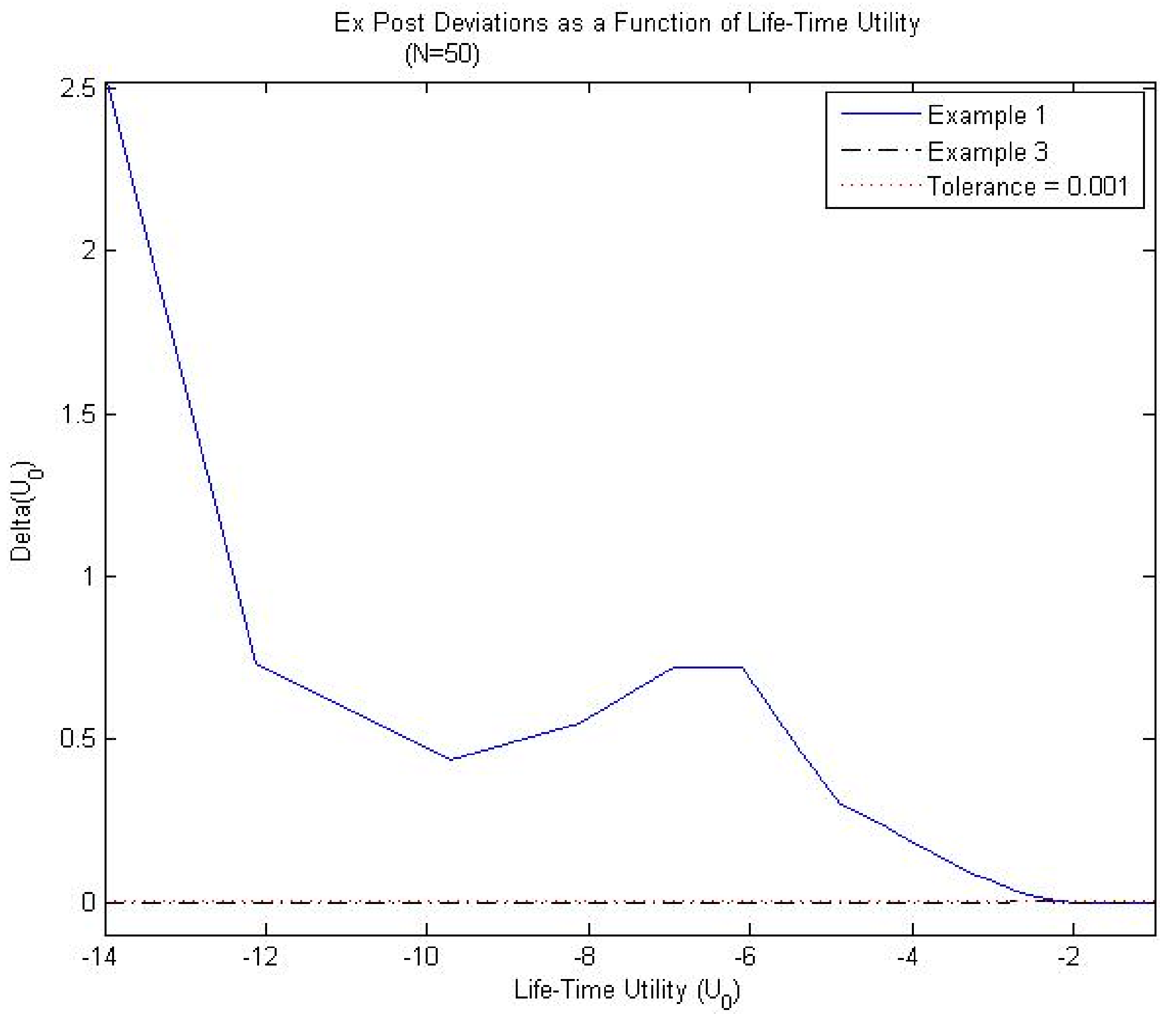

Figure 2: Ex Post Deviations ($\Delta(U_0)$) for Examples 1 and 3 (N=50)