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WHEN DOES THE BOMBIERI–VINOGRADOV THEOREM HOLD FOR A GIVEN MULTIPLICATIVE FUNCTION?

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Abstract

Let f and g be 1-bounded multiplicative functions for which $f * g = 1_{.=1}$. The Bombieri–Vinogradov theorem holds for both f and g if and only if the Siegel–Walfisz criterion holds for both f and g, and the Bombieri–Vinogradov theorem holds for f restricted to the primes.

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1. Introduction

1.1. Background and the main result. Given an arithmetic function f, we define, whenever $(a, q) = 1$,

$$\Delta(f, x; q, a) := \sum_{\substack{n \leqslant x \\ n \equiv a \pmod q}} f(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \leqslant x \\ (n,q)=1}} f(n),$$

which, as a varies, indicates how well $f(.)$ is distributed in the arithmetic progressions mod q. In many examples it is difficult to obtain a strong bound on $\Delta(f, x; q, a)$ for arithmetic progressions modulo a particular q but one can perhaps do better on 'average'. We therefore define the following:

The Bombieri–Vinogradov hypothesis for f**:** *For any given* $A > 0$ *there exists a constant* $B = B(A)$ *such that*

$$\sum_{q \leqslant \sqrt{x}/(\log x)^B} \max_{a:\ (a,q)=1} |\Delta(f, x; q, a)| \ll_A \frac{x}{(\log x)^A} \tag{1.1}$$

for all $x \geqslant 2$.

The Bombieri–Vinogradov hypothesis for f formulates the idea that f is well distributed, on 'average', in arithmetic progressions with moduli q almost as large as $\sqrt{x}$. It also directly implies that f is well distributed in arithmetic progressions with small moduli. In particular the following is an immediate consequence:

The Siegel–Walfisz criterion for f**:** *For any given* $A > 0$*, any* $x \geqslant 2$*, and any* $(a, q) = 1$ *we have*

$$|\Delta(f, x; q, a)| \ll_A \frac{x}{(\log x)^A}. \tag{1.2}$$

In this article we focus on 1-bounded multiplicative functions f; that is, those f for which $|f(n)| \leqslant 1$ for all $n \geqslant 1$. We define

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad -\frac{F'(s)}{F(s)} = \sum_{n=2}^{\infty} \frac{\Lambda_f(n)}{n^s},$$

for $\mathrm{Re}(s) > 1$. The function $\Lambda_f(.)$ is supported only on prime powers; we restrict attention to the class $\mathcal{C}$ of multiplicative functions f for which

$$|\Lambda_f(n)| \leqslant \Lambda(n) \quad \text{for all } n \geqslant 1.$$

This includes most 1-bounded multiplicative functions of interest, including all 1-bounded completely multiplicative functions. Two key observations are that if $f \in \mathcal{C}$ then each $|f(n)| \leqslant 1$, and if $f \in \mathcal{C}$ and $F(s)G(s) = 1$ then the multiplicative function g whose Dirichlet series is G also lies in $\mathcal{C}$. Here g is the convolution inverse of f; that is, $(f * g)(n) = 1$ if $n = 1$, and 0 otherwise.

Define $\mathcal{P}$ to be the set of primes, so that the arithmetic function $f \cdot 1_{\mathcal{P}}$ is the function f but supported only on the primes. The classical Bombieri–Vinogradov theorem is, in our language, the Bombieri–Vinogradov hypothesis for $1_{\mathcal{P}}$. The Bombieri–Vinogradov hypothesis holds trivially for the corresponding multiplicative function $1(.)$. Many of the proofs of the Bombieri–Vinogradov theorem (for example, those going through Vaughan's identity) relate the distribution of $1_{\mathcal{P}}$ in arithmetic progressions to the distribution of $\mu(.)$ in arithmetic progressions; here μ denotes the Möbius function, the convolution inverse of the multiplicative function 1. This is the prototypical example of the phenomenon we discuss in this article.

Our main question here is to address for what f does the Bombieri–Vinogradov hypothesis hold? Evidently the Siegel–Walfisz criterion must hold for f, but what else is necessary? In [4, Proposition 1.4], we exhibited an $f \in \mathcal{C}$ for which the Siegel–Walfisz criterion holds, and yet (1.1) fails for any $A > 1$ and any B. The key feature in our construction of f was that the Bombieri–Vinogradov hypothesis did not hold for $f \cdot 1_{\mathcal{P}}$. As we now see in our main result, this is also a necessary condition:

THEOREM 1.1. *Suppose that* $f, g \in \mathcal{C}$ *with* $F(s)G(s) = 1$.

(a) *If the Bombieri–Vinogradov hypothesis holds for both* f *and* g *then the Bombieri–Vinogradov hypothesis holds for* $f \cdot 1_{\mathcal{P}}$*; and*

(b) *If the Bombieri–Vinogradov hypothesis holds for* $f \cdot 1_{\mathcal{P}}$ *and the Siegel–Walfisz criterion holds for* f *then the Bombieri–Vinogradov hypothesis holds for* f.

Since $g \cdot 1_{\mathcal{P}} = -f \cdot 1_{\mathcal{P}}$, this can all be expressed more succinctly as follows:

Suppose that $f, g \in \mathcal{C}$ with $f * g = 1_{.=1}$. Then
The Bombieri–Vinogradov hypothesis holds for both f and g if and only if The Bombieri–Vinogradov hypothesis holds for $f \cdot 1_{\mathcal{P}}$, and the Siegel–Walfisz criterion holds for both f and g.

This kind of 'if and only if' result in the theory of multiplicative functions bears some similarity to (and inspiration from) (1.4) and Theorem 1.2 of [6], and much of the discussion there.

2. More explicit results

Theorem 1.1 is not as powerful as it looks at first sight since it is of little use if one wishes to prove (1.1) for a function f whose definition depends on a particular x (as the hypothesis of Theorem 1.1 makes assumptions for all x). In this section we will give uniform versions of both parts of Theorem 1.1.

The Bombieri–Vinogradov hypothesis for $f \in \mathcal{C}$ fails if f is a character of small conductor (for example $f(n) = (n/3)$), or 'correlates' with such a character; that is, the sum

$$S_f(x, \chi) := \sum_{n \leqslant x} f(n)\overline{\chi}(n)$$

is 'large'. We can take such characters into account as follows: Given any finite set of primitive characters, Ξ, let Ξ_q be the set of characters mod q that are induced

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by the characters in Ξ, and then define

$$\Delta_{\Xi}(f, x; q, a) := \sum_{\substack{n \leqslant x \\ n \equiv a \pmod q}} f(n) - \frac{1}{\varphi(q)} \sum_{\chi \in \Xi_q} \chi(a) S_f(x, \chi).$$

Note that $\Delta(f, x; q, a) = \Delta_{\{1\}}(f, x; q, a)$.

2.1. Precise statement of Theorem 1.1(b) and B–V for smooth-supported f.

Our uniform version of Theorem 1.1(b) is the following result, from which Theorem 1.1(b) immediately follows.

THEOREM 2.1. *Fix $A \geqslant 0$, $B > A + 5$, $\varepsilon > 0$, and set $\gamma = 2A + 6 + \varepsilon$. Given $x \geqslant 2$, let $Q = x^{1/2}/(\log x)^B$ and $y = x/(\log x)^{\gamma}$. Suppose that $f \in \mathcal{C}$, and assume that*

$$\sum_{q \leqslant Q} \max_{a:\ (a,q)=1} |\Delta(f \cdot 1_{\mathcal{P}}, X; q, a)| \ll \frac{X}{(\log x)^A \log(x/y)} \tag{2.1}$$

for all X in the range $y \leqslant X \leqslant x$; and that

$$|\Delta(f, X; q, a)| \ll \frac{X}{(\log x)^{A+2B}},$$

whenever $(a, q) = 1$, for all X in the range $x^{1/2} \leqslant X \leqslant x$. Then

$$\sum_{q \leqslant Q} \max_{(a,q)=1} |\Delta(f, x; q, a)| \ll \frac{x}{(\log x)^{A-1}}.$$

Remark. Theorem 2.1 is stronger the larger we can take y (and thus the smaller we can take γ), since that reduces the assumptions made of the form (2.1). We have been able to take any $\gamma > 2A + 6$ in Theorem 2.1. In Section 8, we will show that we must have $\gamma \geqslant A - 3$. It would be interesting to know the optimal power, γ, of $\log x$ that one can take in the definition of y. □

An integer n is *y-smooth* if all of its prime factors are $\leqslant y$. In [4] we proved the Bombieri–Vinogradov hypothesis for y-smooth-supported $f \in \mathcal{C}$ satisfying the Siegel–Walfisz criterion, provided $y \leqslant x^{1/2-o(1)}$. For arbitrary $f \in \mathcal{C}$, we may therefore use this result to obtain the Bombieri–Vinogradov hypothesis for the f-values restricted to y-smooth n, and need a different approach for those n that have a large prime factor (that is, a prime factor $>y$). See also [2] for similar results that are nontrivial for much smaller values of y.

First though, we have been able to develop a rather different method based on ideas of Harper [5] to significantly extend our range for y.

THEOREM 2.2. *Fix $A \geqslant 0$, $\varepsilon > 0$, and set $B = A + 5 + \varepsilon$, $\gamma = 2A + 6 + \varepsilon$. Given $x \geqslant 2$, let $Q = x^{1/2}/(\log x)^B$ and $y = x/(\log x)^\gamma$. Let $\mathcal{A}$ be the set of all primitive characters of conductor at most $(\log x)^B$. If $f \in \mathcal{C}$ is supported only on the y-smooth integers then*

$$\sum_{q \leqslant Q} \max_{(a,q)=1} |\Delta_{\mathcal{A}}(f, x; q, a)| \ll \frac{x}{(\log x)^A}.$$

Proving this result takes up the bulk of this paper, and occupies Sections 5 and 6. In Section 7 we deduce Theorem 2.1 from Theorem 2.2.

2.2. Precise statement of Theorem 1.1(a). It is considerably easier to prove the converse result, since one can easily express $f \cdot 1_{\mathcal{P}}$ in terms of a suitably weighted convolution of f and g. Theorem 1.1(a) follows from the next result, taking $\Xi = \{1\}$.

THEOREM 2.3. *Given $f \in \mathcal{C}$, define $g \in \mathcal{C}$ to be that multiplicative function for which $F(s)G(s) = 1$. Fix $A, C \geqslant 0$ and $\varepsilon > 0$. Let x be large, let $2 \leqslant Q \leqslant x^{1/2}/(\log x)^{5A/2+5C/4+7/2}$ and let Ξ be a set of at most $(\log x)^C$ primitive characters. Suppose that the following properties hold for both $h = f$ and $h = g$: (i) If $(a, q) = 1$ then*

$$\Delta_\Xi(h, X; q, a) \ll \frac{X}{(\log x)^{14A+7C+35}}$$

for all X in the range $x^{0.4} \leqslant X \leqslant x$; (ii) The B–V type result

$$\sum_{q \leqslant Q} \max_{a:\ (a,q)=1} |\Delta_\Xi(h, X; q, a)| \ll \frac{X}{(\log x)^{A+C/2+\varepsilon}}$$

holds for all X in the range $x/(\log x)^{6A+7C/2+10+\varepsilon} \leqslant X \leqslant x$. Then

$$\sum_{q \leqslant Q} \max_{a:\ (a,q)=1} |\Delta_\Xi(f \cdot 1_{\mathcal{P}}, x; q, a)| \ll \frac{x}{(\log x)^A}; \tag{2.2}$$

and the same holds with $f \cdot 1_{\mathcal{P}}$ replaced by $g \cdot 1_{\mathcal{P}}$.

The range $x^{0.4} \leqslant X \leqslant x$ appearing in property (i) above can be reduced to $x^{1/2}/(\log x)^{15A+15C/2+36} \leqslant X \leqslant x$, but we are not concerned about this detail. We will prove Theorem 2.3 at the end of Section 4, after establishing a result about convolutions in Section 3.

3. The algebra of the Bombieri–Vinogradov hypothesis

We will establish the next main result using ideas from [3, Section 9.8] (which has its roots in [1, Theorem 0]). We will sketch a proof of a modification of [3, Theorem 9.17], being a little more precise and correcting a couple of minor errors. We assume that f and g are arithmetic functions, with $|f(n)|, |g(n)| \leqslant 1$ for all n, but not necessarily multiplicative. Let Ξ be a set of primitive characters. For $h \in \{f, g\}$ we assume that if $(a, q) = 1$ then

$$\Delta_{\Xi}(h, N; q, a) \ll \frac{H(N)N^{1/2}}{(\log N)^A} \tag{3.1}$$

for some $A \geqslant 0$, where $H(N) := (\sum_{n\leqslant N} |h(n)|^2)^{1/2}$. In [3] they have $\Xi = \{1\}$ throughout but the modifications for arbitrary Ξ are straightforward. The first key step is (9.73) in [3]. One can be a bit more precise (for example, by choosing C there more precisely) and show that if ψ is a character mod q with $q \leqslant (\log N)^A$, but $\psi \notin \Xi_q$, and if (3.1) holds, then for any positive integer m we have

$$\sum_{\substack{n\leqslant N\\ (n,m)=1}} h(n)\psi(n) \ll q^{1/3}\tau(m)\frac{H(N)N^{1/2}}{(\log N)^{A/3}}. \tag{3.2}$$

Now for $M, N \geqslant 2$ let $f_M(m) = f(m)$ if $m \leqslant M$ and $f_M(m) = 0$ if $m > M$, and define g_N similarly. Assume that $M \leqslant N$ and that (3.1) holds for $h = g_N$. Theorem 9.16 of [3] then becomes, (Note that in the statement of Theorem 9.16 of [3], the authors claim to have obtained a power of $\log MN$ in the denominator of the third term of the upper bound whereas we only claim a power of $\log N$, as in their proof. This makes no difference here since we added the hypothesis that $N \geqslant M$, but they did not have this hypothesis in [3].)

$$\begin{aligned}\sum_{q\leqslant Q} \max_{a:\ (a,q)=1} &|\Delta_{\Xi}(f_M * g_N, MN; q, a)| \\ &\ll \left(Q + \sqrt{N}(\log Q)^2 + \frac{\sqrt{MN}(\log Q)^2}{(\log N)^{A/7}}\right)F(M)G(N),\end{aligned} \tag{3.3}$$

after replacing (9.73) in [3] by our (3.2) and choosing the parameter F in their argument to be $F = (\log N)^{A/7}$.

We now assume that (3.1) holds (In [3, Theorem 9.17], the authors only assume (3.1) for $h = g$. This is because of their overoptimistic error term in Theorem 9.16. The correction seems to force one to assume that (3.1) holds for $h = f$ as well.) for both $h = f$ and $h = g$ for any N in the range $\sqrt{x} \leqslant N \leqslant x$ and some $A \geqslant 35$. For $C \geqslant 0$ define

$$(f * g)_C(r) := \sum_{\substack{mn=r\\ m,n\leqslant x/(\log x)^C}} f(m)g(n).$$

Our version of [3, Theorem 9.17] states the following:

LEMMA 3.1. *Let the notations and assumptions be as above. Let* $A \geqslant 35$, $B = A/7 - 2$ *and* $C = 2B + 2$. *Then*

$$\sum_{q \leqslant Q} \max_{a:\ (a,q)=1} |\Delta_{\Xi}((f * g)_C, x; q, a)| \ll (1 + |\Xi|)^{1/2} \frac{x}{(\log x)^D} \tag{3.4}$$

for $Q = \sqrt{x}/(\log x)^B$ *with* $D = (B - 3)/2$.

Proof. The proof involves how to partition the values of m, n in the sum defining $(f * g)_C$, so as to apply (3.3). Using the trivial bounds $F(M) \leqslant \sqrt{M}$ and $G(N) \leqslant \sqrt{N}$, (3.3) now reads

$$\sum_{q \leqslant Q} \max_{a:\ (a,q)=1} |\Delta_{\Xi}(f_M * g_N, MN; q, a)| \ll \sqrt{Nx}(\log x)^2 + \frac{x}{(\log x)^B} \tag{3.5}$$

when $\sqrt{x} \leqslant N \leqslant x$, $M \leqslant N$, and $MN \ll x$. Let $\Delta = (1 + |\Xi|)^{-1/2}(\log x)^{-D-2}$. With two applications of (3.5), we obtain (3.5) with f_M replaced by $f_M - f_{(1-\Delta)M}$:

$$\sum_{q \leqslant Q} \max_{a:\ (a,q)=1} |\Delta_{\Xi}((f_M - f_{(1-\Delta)M}) * g_N, MN; q, a)| \ll \sqrt{Nx}(\log x)^2 + \frac{x}{(\log x)^B} \tag{3.6}$$

when $\sqrt{x} \leqslant N \leqslant x$, $M \leqslant N$, and $MN \ll x$.

We apply (3.5) and (3.6) as follows for those $m \leqslant \sqrt{x}$ (an analogous construction works for those $n \leqslant \sqrt{x}$, as well as for any overlap). For $j \geqslant 0$, set $M_j = (\log x)^C (1 - \Delta)^{-j}$ and $N_j = x/M_j$. Let J be the minimal integer for which $M_j \geqslant \sqrt{x}$ so that $J \asymp (\log x)/\Delta$. We apply (3.5) with $M = M_0$, $N = N_0$, and apply (3.6) with $M = M_j$, $N = N_j$ for $1 \leqslant j \leqslant J$. The total contribution here is

$$\begin{aligned} &\ll \sum_{j=0}^{J} \left((1 - \Delta)^{j/2} \frac{x}{(\log x)^{C/2-2}} + \frac{x}{(\log x)^B} \right) \ll \frac{x}{\Delta (\log x)^{B-1}} \\ &\ll (1 + |\Xi|^{1/2}) \frac{x}{(\log x)^D}, \end{aligned}$$

where the last inequality follows from our choice of D and Δ.

For each integer $m \in ((1 - \Delta)M_j, M_j]$ we have missed out the values of n in the range $(N_j, x/m]$. There are $\leqslant N_j \Delta/(1 - \Delta)$ such values of n, for each of the $\leqslant \Delta M_j$ values of m in the interval, a total of $\ll \Delta^2 x$ pairs for each j. Using the identity

$$\Delta_\Xi((f*g)_C, x; q, a) = \sum_{\substack{mn\leqslant x \\ m,n\leqslant x/(\log x)^C \\ mn\equiv a \pmod q}} f(m)g(n) - \frac{1}{\varphi(q)}\sum_{\chi\in\Xi_q}\chi(a)\sum_{\substack{mn\leqslant x \\ m,n\leqslant x/(\log x)^C}} f(m)g(n)\overline{\chi(m)\chi(n)},$$

we see that the total contributions to the left hand side of (3.4) from those $m\in((1-\Delta)M_j, M_j]$ and $n\in(N_j, x/m]$ are bounded by

$$\begin{aligned}
&\ll \sum_{q\leqslant Q}\Bigg(\max_{a:\,(a,q)=1}\sum_{\substack{(1-\Delta)M_j<m\leqslant M_j \\ N_j<n\leqslant x/m \\ mn\equiv a \pmod q}} 1 + \frac{|\Xi_q|}{\varphi(q)}\sum_{\substack{(1-\Delta)M_j<m\leqslant M_j \\ N_j<n\leqslant x/m}} 1\Bigg) \\
&\ll \sum_{q\leqslant Q}\left[\Delta M_j\left(1+\frac{\Delta N_j}{q}\right)+\frac{|\Xi_q|}{\varphi(q)}\Delta^2 x\right]. \\
&\ll \Delta M_j Q + \sum_{q\leqslant Q}\Delta^2 x\left(\frac{1}{q}+\frac{|\Xi_q|}{\varphi(q)}\right) \ll \Delta M_j Q + (1+|\Xi|)\Delta^2 x\log x.
\end{aligned}$$

Summing over $0\leqslant j\leqslant J$, we see that the total contribution from the mixed pairs m, n is

$$\begin{aligned}
&\ll \Delta Q\sum_{0\leqslant j\leqslant J} M_j + J(1+|\Xi|)\Delta^2 x\log x \ll Qx^{1/2} + (1+|\Xi|)\Delta x(\log x)^2 \\
&\ll (1+|\Xi|^{1/2})\frac{x}{(\log x)^D},
\end{aligned}$$

by the choice of D and Δ. This completes the proof of Lemma 3.1. □

Of course we are really interested in $f*g$ not $(f*g)_C$, so now we study the difference: For $y = x/(\log x)^C$ we have

$$\begin{aligned}
&\Delta_\Xi(f*g, x; q, a) - \Delta_\Xi((f*g)_C, x; q, a) \\
&\quad = \sum_{\substack{m\leqslant(\log x)^C \\ (m,q)=1}} f(m)(\Delta_\Xi(g, x/m; q, a/m) - \Delta_\Xi(g, y; q, a/m)) \\
&\quad\quad + \sum_{\substack{n\leqslant(\log x)^C \\ (n,q)=1}} g(n)(\Delta_\Xi(f, x/n; q, a/n) - \Delta_\Xi(f, y; q, a/n)),
\end{aligned}$$

and so

$$
\begin{aligned}
\sum_{q\leqslant Q}\max_{a:\ (a,q)=1}|\Delta_\Xi(f*g,x;q,a)| &\leqslant \sum_{q\leqslant Q}\max_{b:\ (b,q)=1}|\Delta_\Xi((f*g)_C,x;q,b)|\\
&+\sum_{m\leqslant(\log x)^C}\sum_{q\leqslant Q}\max_{c:\ (c,q)=1}|\Delta_\Xi(g,x/m;q,c)|\\
&+\sum_{m\leqslant(\log x)^C}\sum_{q\leqslant Q}\max_{d:\ (d,q)=1}|\Delta_\Xi(g,y;q,d)|\\
&+\sum_{n\leqslant(\log x)^C}\sum_{q\leqslant Q}\max_{e:\ (e,q)=1}|\Delta_\Xi(f,x/n;q,e)|\\
&+\sum_{n\leqslant(\log x)^C}\sum_{q\leqslant Q}\max_{k:\ (k,q)=1}|\Delta_\Xi(f,y;q,k)|.
\end{aligned}
$$

The first term above is handled by Lemma 3.1. We assume that for $h=f$ and $h=g$ we have

$$\sum_{q\leqslant Q}\max_{a:\ (a,q)=1}|\Delta_\Xi(h,X;q,a)| \ll \frac{X}{(\log x)^D\log\log x}$$

for all X in the range $y\leqslant X\leqslant x$, so the above becomes

$$\ll (1+|\Xi|)^{1/2}\frac{x}{(\log x)^D}+\sum_{m\leqslant(\log x)^C}\frac{x/m}{(\log x)^D\log\log x}\ll(1+|\Xi|)^{1/2}\frac{x}{(\log x)^D}.$$

We now summarize what we have proved.

PROPOSITION 3.2. *Fix $D\geqslant 0$ and let $A=14D+35$, $B=2D+3$ and $C=4D+8$. Let x be large and let Ξ be a set of primitive characters. Let f and g be given arithmetic functions with the following properties: Taking $h=f$ or $h=g$ we have that (i) Each $|h(n)|\leqslant 1$; (ii) If $(a,q)=1$ then*

$$\Delta_\Xi(h,X;q,a)\ll\frac{X}{(\log x)^A}$$

for all X in the range $x^{1/2}\leqslant X\leqslant x$; (iii) The B–V type result

$$\sum_{q\leqslant Q}\max_{a:\ (a,q)=1}|\Delta_\Xi(h,X;q,a)| \ll \frac{X}{(\log x)^D\log\log x}$$

holds for all X in the range $x/(\log x)^C\leqslant X\leqslant x$, where $2\leqslant Q\leqslant\sqrt{x}/(\log x)^B$. Then

$$\sum_{q\leqslant Q}\max_{a:\ (a,q)=1}|\Delta_\Xi(f*g,x;q,a)|\ll(1+|\Xi|)^{1/2}\frac{x}{(\log x)^D}.$$

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4. Proof of the uniform version of Theorem 1.1(a)

In this section we deduce Theorem 2.3 from Proposition 3.2, using the identity $\Lambda_f = g * f \log$. We start with two lemmas passing between f and $f \log$.

LEMMA 4.1. *Fix $A \geqslant 0$. Let f be a given 1-bounded arithmetic function, let $(a, q) = 1$, and suppose we are given a set of primitive characters Ξ. If*

$$\Delta_\Xi(f, X; q, a) \ll \frac{X}{(\log X)^A} \tag{4.1}$$

for all X in the range $x/(\log x)^A \leqslant X \leqslant x$, then

$$\Delta_\Xi(f \log, X; q, a) \ll \frac{X}{(\log X)^{A-1}} \tag{4.2}$$

for $X = x$. Conversely if (4.2) holds for all X in the range $x/(\log x)^A \leqslant X \leqslant x$, then (4.1) holds for $X = x$.

Proof. Let $F(n; q, a) = 1_{n\equiv a \pmod q} - (1/\varphi(q)) \sum_{\chi\in\Xi_q} \chi(a\bar{n})$, so that we have $\Delta_\Xi(f, x; q, a) = \sum_{n\leqslant x} f(n)F(n; q, a)$. Moreover

$$\Delta_\Xi(f \log, x; q, a) = \sum_{n\leqslant x} f(n)F(n; q, a) \log n = \int_1^x \log t \, d\Delta_\Xi(f, t; q, a),$$

by the usual technique of partial summation, so that

$$\begin{aligned} \Delta_\Xi(f \log, x; q, a) - \Delta_\Xi(f \log, X; q, a) &= \Delta_\Xi(f, x; q, a) \log x \\ &\quad - \Delta_\Xi(f, X; q, a) \log X - \int_X^x \Delta_\Xi(f, t; q, a) \frac{dt}{t}, \end{aligned} \tag{4.3}$$

where $X = x/(\log x)^A$. By (4.1) the three terms on the right hand side above are all $\ll x/(\log x)^{A-1}$. By trivially bounding each $|f(n)|$ by 1, we obtain

$$\begin{aligned} |\Delta_\Xi(f \log, X; q, a)| &\leqslant \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi\notin\Xi_q}} \left| \sum_{n\leqslant X} f(n)(\log n)\overline{\chi(n)} \right| \\ &\leqslant X \log x \leqslant \frac{x}{(\log x)^{A-1}}. \end{aligned}$$

This yields the first part of the lemma. For the second part we begin with the analogous identity

$$\Delta_\Xi(f, x; q, a) = \int_2^x \frac{1}{\log t} \, d\Delta_\Xi(f \log, t; q, a),$$

and the proof proceeds entirely analogously. $\square$

LEMMA 4.2. *Fix $A, C \geqslant 0$. Let f be a given 1-bounded arithmetic function, $2 \leqslant Q \leqslant x/(\log x)^{A+C/2+1}$, and suppose we are given a set of primitive characters, Ξ, containing $\leqslant (\log x)^C$ elements. If*

$$\sum_{q\leqslant Q} \max_{a:\ (a,q)=1} |\Delta_\Xi(f, X; q, a)| \ll \frac{X}{(\log X)^A} \tag{4.4}$$

for all X in the range $x/(\log x)^{A+C/2+1} \leqslant X \leqslant x$ then

$$\sum_{q\leqslant Q} \max_{a:\ (a,q)=1} |\Delta_\Xi(f \log, X; q, a)| \ll \frac{X}{(\log X)^{A-1}} \tag{4.5}$$

for $X = x$. Conversely if (4.5) holds for all X in the range $x/(\log x)^{A+C/2+1} \leqslant X \leqslant x$ then (4.4) holds for $X = x$.

Proof. As before we use (4.3) but now with $X = x/(\log x)^{A+C/2+1}$. This implies that

$$\begin{aligned}\sum_{q\leqslant Q} \max_{a:\ (a,q)=1} |\Delta_\Xi(f \log, x; q, a)| &\leqslant \sum_{q\leqslant Q} \max_{b:\ (b,q)=1} |\Delta_\Xi(f \log, X; q, b)| \\ &\quad + \sum_{q\leqslant Q} \max_{c:\ (c,q)=1} |\Delta_\Xi(f, x; q, c)| \log x + \sum_{q\leqslant Q} \max_{d:\ (d,q)=1} |\Delta_\Xi(f, X; q, d)| \log X \\ &\quad + \int_X^x \sum_{q\leqslant Q} \max_{e:\ (e,q)=1} |\Delta_\Xi(f, t; q, e)| \frac{dt}{t}.\end{aligned}$$

By (4.4) the last three terms are

$$\ll_A \frac{x}{(\log x)^{A-1}} + \int_X^x \frac{dt}{(\log t)^A} \ll_A \frac{x}{(\log x)^{A-1}}.$$

Now, by trivially bounding each $|f(n)|$ by 1, we obtain

$$\begin{aligned}|\Delta_\Xi(f \log, X; q, b)| &\leqslant \sum_{\substack{n\leqslant X \\ n\equiv b \pmod q}} \log n + \frac{1}{\varphi(q)} \sum_{\chi\in\Xi_q} \sum_{\substack{n\leqslant X \\ (n,q)=1}} \log n \\ &\ll (1 + |\Xi_q|) \frac{X \log X}{q},\end{aligned}$$

and so

$$\begin{aligned}\sum_{q\leqslant Q}\max_{b:\ (b,q)=1}|\Delta_{\Xi}(f\log,X;q,b)| &\leqslant \sum_{q\leqslant Q}(1+|\Xi_q|)\frac{X\log X}{q}\\ &\ll X(\log x)^2+\sum_{\chi\ (\mathrm{mod}\ r)\in\Xi}\ \sum_{\substack{q\leqslant Q\\ r|q}}\frac{X\log X}{q}\\ &\ll\left(1+\sum_{\chi\ (\mathrm{mod}\ r)\in\Xi}\frac{1}{r}\right)X(\log x)^2\ll X(\log x)^{C/2+2}\ll_A\frac{x}{(\log x)^{A-1}}.\end{aligned}$$

which yields the first part of the lemma. The proof of the second part is again analogous. □

Proof of Theorem 2.3. Let $(\log x)^{-1}f\log$ denote the function $n\to(\log x)^{-1}f(n)\log n$. By the B–V type assumption on f, the first part of Lemma 4.2 (with A replaced by $A+C/2+\varepsilon$) implies that

$$\sum_{q\leqslant Q}\max_{a:\ (a,q)=1}|\Delta_{\Xi}((\log x)^{-1}f\log,X;q,a)|\ll\frac{X}{(\log X)^{A+C/2+\varepsilon}},$$

for all X in the range $x/(\log x)^{5A+5C/2+9}\leqslant X\leqslant x$. The same holds with $(\log x)^{-1}f\log$ above replaced by g by hypothesis.

By the S–W type assumption on f, the first part of Lemma 4.1 (with A replaced by $14A+7C+35$) implies that if $(a,q)=1$ then

$$\Delta_{\Xi}((\log x)^{-1}f\log,X;q,a)\ll\frac{X}{(\log X)^{14A+7C+35}},$$

for all X in the range $x^{0.45}\leqslant X\leqslant x$. The same holds with $(\log x)^{-1}f\log$ above replaced by g by hypothesis.

The coefficients of $-F'(s)/F(s)=G(s)\cdot(-F'(s))$ yield the identity $\Lambda_f=g*f\log$. Therefore by Proposition 3.2 applied to the 1-bounded functions g and $(\log x)^{-1}f\log$ (with D replaced by $A+C/2$), we obtain

$$\sum_{q\leqslant Q}\max_{a:\ (a,q)=1}|\Delta_{\Xi}(\Lambda_f,X;q,a)|\ll\frac{X}{(\log X)^{A+C/2}}(1+|\Xi|)^{1/2}\ll\frac{X}{(\log X)^{A}},$$

for all X in the range $x/(\log x)^{A+C/2+1}\leqslant X\leqslant x$.

The contribution of the prime powers p^k with $k\geqslant 2$ does not come close to the upper bound, and so

$$\sum_{q\leqslant Q}\max_{a:\ (a,q)=1}|\Delta_{\Xi}((f\cdot 1_{\mathcal{P}})\log,X;q,a)|\ll\frac{X}{(\log X)^{A}},$$

for all X in the range $x/(\log x)^{A+C/2+1} \leqslant X \leqslant x$. Finally the second part of Lemma 4.2 implies (2.2). □

5. Factorizing smooth numbers and using the large sieve

We develop an idea of Harper [5] to prove the following result, which will yield rich consequences in the next section.

PROPOSITION 5.1. *Let $2 \leqslant y \leqslant x$ be large. Let completely multiplicative $f \in \mathcal{C}$ be supported on y-smooth integers. Let $2 \leqslant D, Q \leqslant x$. Then*

$$\sum_{q \leqslant Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \mathrm{cond}(\chi) > D}} |S_f(x, \chi)| \ll \left(Qx^{1/2} + x^{7/8} + \frac{x}{D}\right)(\log x)^5 + (xy)^{1/2}(\log x)^3.$$

To prove this we begin by proving a marginally weaker result (weaker in the sense that one only saves $(x/y)^{1/4}$ from the trivial bound instead of potentially saving $(x/y)^{1/2}$ in Proposition 5.1).

PROPOSITION 5.2. *Let $2 \leqslant y \leqslant x$ be large. Let completely multiplicative $f \in \mathcal{C}$ be supported on y-smooth integers. Let $2 \leqslant D, Q \leqslant x$. Then*

$$\sum_{q \leqslant Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \mathrm{cond}(\chi) > D}} |S_f(x, \chi)| \ll x^{1/2}\left(Q + (xy)^{1/4} + \frac{x^{1/2}}{D}\right)(\log x)^5.$$

The extra assumption of complete multiplicativity in Propositions 5.1 and 5.2 allows us to easily separate variables in a double sum (see (5.5) below).

Proof. Suppose that $\chi \pmod q$ is induced by $\psi \pmod r$. Let $h(.)$ be the multiplicative function which is supported only on powers of the primes p which divide q but not r, and then let $h(p^k) = (g\overline{\psi})(p^k)$ where g is the convolution inverse of f. Then $f\overline{\chi} = h * f\overline{\psi}$ and so

$$S_f(x, \chi) = \sum_{m \geqslant 1} h(m) S_f(x/m, \psi).$$

As each $|h(m)| \leqslant 1$ we deduce that

$$|S_f(x, \chi)| \leqslant \sum_{\substack{m \geqslant 1 \\ M | q \\ (M, r) = 1}} |S_f(x/m, \psi)|,$$

where $M=\prod_{p|m} p$. Now, we wish to bound

$$\sum_{q\leqslant Q}\frac{1}{\varphi(q)}\sum_{\substack{r|q\\ r>D}}\ \sum_{\substack{\psi\ (\mathrm{mod}\ r)\ \text{primitive}\\ \chi\ \text{induced by}\ \psi}}|S_f(x,\chi)|$$

which, writing $q=rMn$, is

$$\leqslant\sum_{D<r\leqslant Q}\ \sum_{\substack{\psi\ (\mathrm{mod}\ r)\\ \psi\ \text{primitive}}}\ \sum_{\substack{m\geqslant 1\\ (M,r)=1}}|S_f(x/m,\psi)|\sum_{rMn\leqslant Q}\frac{1}{\varphi(rMn)};$$

and this is

$$\ll(\log Q)\sum_{m\leqslant x}\frac{1}{\varphi(M)}\sum_{D<r\leqslant Q}\frac{1}{\varphi(r)}\sum_{\substack{\psi\ (\mathrm{mod}\ r)\\ \psi\ \text{primitive}}}|S_f(x/m,\psi)|. \tag{5.1}$$

Let us first dispose of large values of m. For fixed m, the sum over r and ψ can be bounded using Cauchy–Schwarz by

$$\left(\sum_{r\leqslant Q}\ \sum_{\substack{\psi\ (\mathrm{mod}\ r)\\ \psi\ \text{primitive}}}\frac{1}{r\varphi(r)}\right)^{1/2}\left(\sum_{r\leqslant Q}\frac{r}{\varphi(r)}\sum_{\substack{\psi\ (\mathrm{mod}\ r)\\ \psi\ \text{primitive}}}|S_f(x/m,\psi)|^2\right)^{1/2}.$$

The sum in the first bracket is $\ll\log Q$, and the sum in the second bracket is $\ll(x/m+Q^2)(x/m)$ by the large sieve. Thus the contributions to (5.1) from those $m\geqslant M_0$, where $M_0=x/y$, is

$$\begin{aligned}&\ll(\log Q)^{3/2}\sum_{M_0\leqslant m\leqslant x}\frac{1}{\varphi(M)}\left(\frac{x}{m}+\frac{Qx^{1/2}}{m^{1/2}}\right)\\ &\ll x^{1/2}\left(\frac{x^{1/2}}{M_0^{0.9}}+\frac{Q}{M_0^{0.4}}\right)(\log Q)^{3/2}\ll(x^{0.1}y^{0.9}+Qx^{0.1}y^{0.4})(\log Q)^{3/2},\end{aligned}$$

where the second line follows from the first by using Rankin's trick and the convergence of the Dirichlet series $\sum_m 1/(\varphi(M)m^\sigma)$ for any $\sigma>0$. This contribution is acceptable.

Now fix $m\leqslant M_0$, and write $X=x/m$ so that $X\geqslant x/M_0=y$. Set $V_0=(X/y)^{1/2}$ so that $V_0\geqslant 1$. For every y-smooth integer $V_0<n\leqslant X$, we have a unique factorization $n=uv$ with the properties that

$$P_+(u)\leqslant P_-(v),\quad v>V_0,\quad v/P_-(v)\leqslant V_0.$$

This can be achieved by putting prime factors of n into v in descending order, until the size of v exceeds V_0 for the first time. Thus

$$S_f(X,\psi) = S_f(V_0,\psi) + \sum_{\substack{V_0<v\leqslant yV_0\\ v/P_-(v)\leqslant V_0}} \sum_{\substack{u\leqslant X/v\\ P_+(u)\leqslant P_-(v)}} f(uv)\overline{\psi}(uv). \tag{5.2}$$

The u-summation has length at least $X/yV_0 = V_0$, which explains the choice of V_0.

Each $|S_f(V_0,\psi)| \leqslant V_0$, and so the contribution of these terms to (5.1) is $\ll QV_0$ for fixed m, giving in total

$$\ll Q\log Q \sum_{m\leqslant M_0} \frac{1}{\varphi(M)}\left(\frac{x/m}{y}\right)^{1/2} \ll \frac{Qx^{1/2}\log Q}{y^{1/2}},$$

since $\sum_m 1/(\varphi(M)m^{1/2}) \ll 1$. This is again acceptable.

To analyze the double sum over u, v, we first dyadically divide the ranges of $u, v, P_+(u), P_-(v)$. For parameters U, V, P_+, P_- (which can all be taken to be powers of 2) satisfying

$$U, V \leqslant X/V_0, \quad V > V_0, \quad UV \leqslant X, \quad 2 \leqslant P_+, P_- \leqslant y, \quad P_+ < 2P_-, \tag{5.3}$$

consider the double sum

$$\sum_{\substack{V_0<v\leqslant yV_0\\ V\leqslant v<2V\\ v/P_-(v)\leqslant V_0\\ P_-\leqslant P_-(v)<2P_-}} \sum_{\substack{u\leqslant X/v\\ U\leqslant u<2U\\ P_+(u)\leqslant P_-(v)\\ P_+\leqslant P_+(u)<2P_+}} f(uv)\overline{\psi}(uv). \tag{5.4}$$

For the moment, let us pretend that the 'cross conditions' $uv \leqslant X$ and $P_+(u) \leqslant P_-(v)$ are not there (for example when $4UV \leqslant X$ and $2P_+ \leqslant P_-$), so that the variables u, v are completely separated and (5.4) takes the form

$$\left(\sum_{U\leqslant u<2U} a(u)\overline{\psi}(u)\right)\left(\sum_{V\leqslant v<2V} b(v)\overline{\psi}(v)\right), \tag{5.5}$$

for some $|a(u)| \leqslant 1$ and $|b(v)| \leqslant 1$ (which depend on f but not on ψ). By Cauchy–Schwarz and then the large sieve inequality, we obtain

$$\begin{aligned}\sum_{R<r\leqslant 2R} \frac{1}{\varphi(r)} \sum_{\substack{\psi \pmod r\\ \psi \text{ primitive}}} \left|\sum_{U\leqslant u<2U} a(u)\overline{\psi}(u)\right|\left|\sum_{V\leqslant v<2V} b(v)\overline{\psi}(v)\right| \\ \ll \frac{1}{R}(U^{1/2}+R)(V^{1/2}+R)(UV)^{1/2}.\end{aligned} \tag{5.6}$$

Summing this up over R's in the interval $[D, Q]$ and the various possibilities given by (5.3), we get

$$\ll \left(\frac{X\log X}{D} + \frac{X}{V_0^{1/2}}\log Q + QX^{1/2}\log X\right)(\log y)^2$$
$$\ll \left(\frac{X\log X}{D} + X^{3/4}y^{1/4}\log Q + QX^{1/2}\log X\right)(\log y)^2.$$

Taking $X = x/m$, and summing this over m weighted by $1/\varphi(M)$, we have a contribution

$$\ll \left(\frac{x}{D} + x^{3/4}y^{1/4} + Qx^{1/2}\right)(\log x)^4,$$

to (5.1), as $Q, y \leqslant x$, which is again acceptable.

To deal with the restrictions $uv \leqslant X$ and $P_+(u) \leqslant P_-(v)$ (when necessary), we use Perron's formula in the form

$$\mathbf{1}_{uv\leqslant X} = \frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT} \frac{\widetilde{X}^{s_1}}{u^{s_1}v^{s_1}} \cdot \frac{\mathrm{d}s_1}{s_1} + O(x^{-4})$$

with $\widetilde{X} = \lfloor X\rfloor + 1/2$ and $T = x^5$; and

$$\mathbf{1}_{P_+(u)\leqslant P_-(v)} = \frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT} \frac{(P_-(v)+1/2)^{s_2}}{P_+(u)^{s_2}} \cdot \frac{\mathrm{d}s_2}{s_2} + O(x^{-4}).$$

For example, when $UV \asymp X$ and $P_+ \asymp P_-$, we can write (5.4) using the above applications of Perron's formula as

$$\int_{1/2-iT}^{1/2+iT}\int_{1/2-iT}^{1/2+iT}\left(\sum_{U\leqslant u<2U} a(s_1, s_2; u)\overline{\psi}(u)\right)\left(\sum_{V\leqslant v<2V} b(s_1, s_2; v)\overline{\psi}(v)\right)\frac{\mathrm{d}s_1\mathrm{d}s_2}{s_1 s_2}$$
$$+ O(x^{-3}), \tag{5.7}$$

where $a(s_1, s_2; u)$ is supported on those u with $P_+ \leqslant P_+(u) < 2P_+$ and takes the form

$$a(s_1, s_2; u) = f(u) \cdot \frac{U^{s_1}}{u^{s_1}} \cdot \frac{P_+^{s_2}}{P_+(u)^{s_2}},$$

and $b(s_1, s_2; v)$ is supported on those v with $V_0 < v \leqslant yV_0$, $v/P_-(v) \leqslant V_0$, $P_- \leqslant P_-(v) < 2P_-$ and takes the form

$$b(s_1, s_2; v) = f(v) \cdot \frac{\widetilde{X}^{s_1}}{U^{s_1}v^{s_1}} \cdot \frac{(P_-(v)+1/2)^{s_2}}{P_+^{s_2}}.$$

Note that $|a(s_1, s_2; u)| \ll 1$ and $|b(s_1, s_2; v)| \ll 1$. Thus we can treat the integrand of (5.7) just as we did (5.5). We have two extra powers of $\log T$ which come from integrating $ds_1 ds_2/|s_1 s_2|$, and we can absorb the errors coming from the $O(x^{-3})$ in (5.7) since they are negligible. We have only one power of $\log y$ arising from the dyadic dissection of P_- and P_+. We therefore obtain the upper bound

$$\ll \left(\frac{X}{D} + X^{3/4}y^{1/4} + QX^{1/2}\right)(\log X)(\log T)^2 \log y$$

in total for a given m. Summing over m gives a similar contribution to last time (but now with an extra factor of $\log x$), which is equally acceptable.

Finally we have to account for the cases where $4UV < X$ and $P_+ \asymp P_-$, and where $UV \asymp X$ and $2P_+ < P_-$. Following the same methods precisely we obtain the same bounds. This completes the proof. $\square$

Proof of Proposition 5.1. Let g be the completely multiplicative function for which $g(p) = f(p)$ when $p \leqslant 2\sqrt{x}$, and $g(p) = 0$ for $p > 2\sqrt{x}$. We apply Proposition 5.2 to obtain

$$\sum_{q \leqslant Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \mathrm{cond}(\chi) > D}} |S_g(x, \chi)| \ll \left(Qx^{1/2} + x^{7/8} + \frac{x}{D}\right)(\log x)^5,$$

an acceptable bound. We may clearly assume that $y > 2\sqrt{x}$ since otherwise $f = g$.

Now $S_f(x, \chi) = S_g(x, \chi) + S_h(x, \chi)$ where $h(n) := f(n) - g(n)$. If $h(n) \neq 0$ then $n = up$ for some prime p in the range $2\sqrt{x} < p \leqslant y$, and integer $u < \frac{1}{2}\sqrt{x}$, so that $h(n) = f(u)f(p)$. We proceed *analogously* to the proof of Proposition 5.2, though now U and V are restricted to the range $U \leqslant \frac{1}{4}\sqrt{x}$ and $2\sqrt{x} < V \leqslant y/2$, while P_+ and P_- are irrelevant and removed from the argument, so things are significantly simpler. We therefore obtain, for an element of our dyadic partition, the same upper bound (5.6). Summing now over our range for U and V with $UV < X/4$ we obtain the upper bound

$$\ll \frac{X \log X}{D} + X^{1/2}y^{1/2} + Xx^{-1/4} + QX^{1/2} \log X.$$

Finally summing up over m with $X = x/m$ gives an upper bound of

$$\ll \frac{x(\log x)^2}{D} + x^{1/2}y^{1/2} \log x + Qx^{1/2}(\log x)^2.$$

For the cases in which $UV \asymp X$ we obtain the same upper bound times $(\log T)^2 \ll (\log x)^2$. Our claimed result follows. $\square$

6. Consequences of Proposition 5.1

In this section we deduce Theorem 2.2. First we establish a version of Proposition 5.1 for $f \in \mathcal{C}$ that may not be completely multiplicative.

COROLLARY 6.1. *Let $2 \leqslant y \leqslant x$ be large. Let $f \in \mathcal{C}$ be supported on the y-smooth integers. Let $D \leqslant x^{1/3}$ and $Q \leqslant x/D^2$. Then*

$$\sum_{q \leqslant Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \operatorname{cond}(\chi) > D}} |S_f(x,\chi)| \\ \ll \left(\left(Qx^{1/2} + x^{7/8} + \frac{x}{D} \right) (\log x)^5 + (xy)^{1/2} (\log x)^3 \right) (\log D)^2.$$

Proof. Let f^* be the completely multiplicative function obtained by taking $f^*(p) = f(p)$. Let $g(p^k) = f(p^k) - f(p) f(p^{k-1})$, so that g is supported only on powerful integers, and $f = g * f^*$. We deduce that

$$S_f(x,\chi) = \sum_{n \leqslant x} g(n) \overline{\chi}(n) S_{f^*}(x/n, \chi),$$

and so

$$|S_f(x,\chi)| \leqslant \sum_{n \leqslant x} |g(n)|\, |S_{f^*}(x/n,\chi)|.$$

Summing over all $q \leqslant Q$ we obtain

$$\sum_{q \leqslant Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \operatorname{cond}(\chi) > D}} |S_f(x,\chi)| \leqslant \sum_{n \leqslant x} |g(n)| \sum_{q \leqslant Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \operatorname{cond}(\chi) > D}} |S_{f^*}(x/n,\chi)|.$$

Set $N = D^2$. For the sum over $n \leqslant N$ we use Proposition 5.1, as $D, Q \leqslant x/N$, to obtain the upper bound

$$\ll \sum_{n \leqslant N} \frac{|g(n)|}{\sqrt{n}} \left(\left(Qx^{1/2} + x^{7/8} + \frac{x}{D} \right) (\log x)^5 + (xy)^{1/2} (\log x)^3 \right).$$

Since each $|g(p^k)| \leqslant 2$ and g is only supported on the powerful, we deduce that

$$\sum_{n \leqslant N} \frac{|g(n)|}{\sqrt{n}} \leqslant \prod_{p \leqslant \sqrt{N}} \left(1 + \frac{2}{p} + \frac{2}{p^{3/2}} + \cdots \right) \ll (\log N)^2.$$

It remains to deal with $n > N$: The argument near the beginning of the proof of Proposition 5.2 which gave a bound for $m \geqslant M_0$ can be adjusted here to give a bound when $m \geqslant 1$, so that

$$\sum_{q \leqslant Q} \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \mathrm{cond}(\chi) > D}} |S_{f^*}(x, \chi)| \ll (x + Qx^{1/2})(\log Q)^{3/2}.$$

Therefore the sum over $n > N$ is

$$\ll \sum_{N < n \leqslant x} |g(n)| \left(\frac{x}{n} + Q\left(\frac{x}{n}\right)^{1/2} \right) (\log Q)^{3/2}$$
$$\ll \frac{x}{\sqrt{N}} (\log N)(\log x)^{3/2} + Qx^{1/2}(\log x)^{7/2}.$$

Taking $N = D^2$ we obtain the claimed result. □

Proof of Theorem 2.2. We take $D = (\log x)^B$ in Corollary 6.1, noting that

$$|\Delta_{\mathcal{A}}(f, x; q, a)| \leqslant \frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \mathrm{cond}(\chi) > D}} |S_f(x, \chi)|$$

by the definition of $\mathcal{A}$. Thus Corollary 6.1 implies that the quantity above is

$$\ll \left(\frac{x}{(\log x)^{B-5}} + \frac{x}{(\log x)^{\gamma/2-3}} \right) (\log D)^2.$$

This is $\ll x/(\log x)^A$ since $B > A + 5$ and $\gamma > 2A + 6$. □

7. Proof of the uniform version of Theorem 1.1(b)

In the section we deduce Theorem 2.1 from Theorem 2.2. We begin by extending Theorem 2.2 to all $f \in \mathcal{C}$.

COROLLARY 7.1. *Fix $A \geqslant 0$, $B > A + 5$ and $\gamma > 2A + 6$. Given x, let $Q = x^{1/2}/(\log x)^B$ and $y = x/(\log x)^\gamma$. Let $\mathcal{A}$ be the set of all primitive characters of conductor at most $(\log x)^B$. Suppose that $f \in \mathcal{C}$, and assume that*

$$\sum_{q \leqslant Q} \max_{a:\ (a,q)=1} |\Delta_{\mathcal{A}}(f \cdot 1_{\mathcal{P}}, X; q, a)| \ll \frac{X}{(\log x)^A \log(x/y)}$$

for all X in the range $y \leqslant X \leqslant x$. Then

$$\sum_{q \leqslant Q} \max_{(a,q)=1} |\Delta_{\mathcal{A}}(f, x; q, a)| \ll \frac{x}{(\log x)^A}.$$

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Proof. Let $f_y(p^k) = f(p^k)$ if $p \leqslant y$, and $f_y(p^k) = 0$ otherwise. If $f(n) \neq 0$ but $f_y(n) = 0$ where $n \leqslant x$, then n has a prime factor $p > y$, which can only appear in n to the power one, and so we can write $n = mp$ with $f(n) = f(m)f(p)$. Moreover $m = n/p < x/y$. Therefore

$$\begin{aligned}\Delta_{\mathcal{A}}(f - f_y, x; q, a) &= \sum_{\substack{m \leqslant x/y \\ (m,q)=1}} f(m)\Bigg(\sum_{\substack{y<p\leqslant x/m \\ p\equiv a\overline{m} \pmod q}} f(p) \\ &\qquad - \frac{1}{\varphi(q)} \sum_{\chi \in \Xi_q} \chi(a)\overline{\chi(m)} \sum_{\substack{y<p\leqslant x/m \\ (p,q)=1}} (f\overline{\chi})(p)\Bigg) \\ &= \sum_{\substack{m \leqslant x/y \\ (m,q)=1}} f(m)(\Delta_{\mathcal{A}}(f \cdot 1_{\mathcal{P}}, x/m; q, a\overline{m}) \\ &\qquad - \Delta_{\mathcal{A}}(f \cdot 1_{\mathcal{P}}, y; q, a\overline{m})).\end{aligned}$$

Summing this up over $q \leqslant Q$, and as each $|f(m)| \leqslant 1$, we deduce that

$$\begin{aligned}\sum_{q \leqslant Q} \max_{(a,q)=1} |\Delta_{\mathcal{A}}(f, x; q, a)| &\leqslant \sum_{q \leqslant Q} \max_{(b,q)=1} |\Delta_{\mathcal{A}}(f_y, x; q, b)| \\ &\quad + \sum_{m \leqslant x/y} \sum_{\substack{q \leqslant Q \\ (q,m)=1}} \max_{(c,q)=1} |\Delta_{\mathcal{A}}(f \cdot 1_{\mathcal{P}}, x/m; q, c)| \\ &\quad + \sum_{m \leqslant x/y} \sum_{\substack{q \leqslant Q \\ (q,m)=1}} \max_{(d,q)=1} |\Delta_{\mathcal{A}}(f \cdot 1_{\mathcal{P}}, y; q, d)|.\end{aligned}$$

We bound the first term by using Theorem 2.2, and the other terms using the hypothesis to get an upper bound

$$\ll \frac{x}{(\log x)^A} + \sum_{m \leqslant x/y} \frac{x/m}{(\log x)^A \log(x/y)} \ll \frac{x}{(\log x)^A},$$

as claimed. □

To use Corollary 7.1 we need the following result, which follows immediately from the proof of Proposition 3.4 of [4].

Lemma 7.2. *Fix $A, C \geqslant 0$. Let $f \in \mathcal{C}$ be such that*

$$|\Delta(f, X; q, a)| \ll \frac{X}{(\log x)^{A+C}},$$

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whenever $(a,q)=1$ *for all* X *in the range* $x^{1/2} < X \leqslant x$. *Suppose that* Ξ *is a set of primitive characters, containing* $\ll (\log x)^C$ *elements. Let* $Q \leqslant x$, *and for each* $Q < q \leqslant 2Q$ *let* $a_q \pmod q$ *be a residue class with* $(a_q,q)=1$. *Then*

$$\sum_{Q<q\leqslant 2Q} |\Delta_\Xi(f,x;q,a_q)| \ll \frac{x}{(\log x)^A}$$

if and only if

$$\sum_{Q<q\leqslant 2Q} |\Delta(f,x;q,a_q)| \ll \frac{x}{(\log x)^A}.$$

Proof of Theorem 2.1. The hypothesis of Corollary 7.1 holds, and so

$$\sum_{q\leqslant Q} \max_{(a,q)=1} |\Delta_{\mathcal{A}}(f,x;q,a)| \ll \frac{x}{(\log x)^A}.$$

Note that $|\mathcal{A}| \asymp (\log x)^{2B}$ so we may take $C=2B$ in Lemma 7.2 for each dyadic range of q to deduce our result. $\square$

8. f which satisfy the Siegel–Walfisz criterion but not the Bombieri–Vinogradov hypothesis

In this section we justify the remark following the statement of Theorem 2.1. The proof of Theorem 2.1 allows us to replace (2.1) in the hypothesis by

$$\sum_{q\leqslant Q} \max_{a:\ (a,q)=1} |\Delta(f\cdot 1_{\mathcal{P}},X;q,a) - \Delta(f\cdot 1_{\mathcal{P}},y;q,a)| \ll \frac{X}{(\log x)^A} \tag{8.1}$$

for all X in the range $y \leqslant X \leqslant x$.

The construction we shall give is similar as the one from [4], and to verify that this example satisfies the Siegel–Walfisz hypothesis we need to assume (1.5) of [4]), which is an (unknown) uniform form of the prime number theorem in arithmetic progressions. Let $y = x/(\log x)^\gamma$ and select any Q in the range $x^{1/3} < Q \leqslant x^{2/5}$. Let $\mathcal{P}$ be the set of primes p in the range $y/2 < p \leqslant y$ for which there exists a prime $q \in (Q,2Q]$ that divides $p-1$. We will work with the completely multiplicative function f, defined as follows:

$$f(p) = \begin{cases} 0 & \text{if } p \leqslant 2(\log x)^\gamma \text{ or } y < p \leqslant x;\\ -1 & \text{if } p \in \mathcal{P};\\ 1 & \text{otherwise.}\end{cases}$$

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Since f is supported only on y-smooth integers, (8.1) trivially holds for all X in the range $y \leqslant X \leqslant x$. From now on we follow the arguments of section 8.2 of [4], and assume the conjecture (1.5) of [4]. First we may deduce that f satisfies the Siegel–Walfisz criterion in the hypothesis of Theorem 2.1. Now note that

$$f(n) = |f(n)| - 2 \cdot \mathbf{1}_{\mathcal{P}}(n)$$

for each $n \leqslant x$. Thus

$$\Delta(f, x; q, 1) = \Delta(|f|, x; q, 1) - 2\Delta(\mathbf{1}_{\mathcal{P}}, x; q, 1).$$

Note that $|f|$ is the indicator function of the set of y-smooth integers with no prime factors $\leqslant 2(\log x)^{\gamma}$. It is straightforward to establish that this set has level of distribution $x^{1/2-\varepsilon}$. Thus

$$\sum_{Q<q\leqslant 2Q} |\Delta(f, x; q, 1)| \geqslant 2 \sum_{Q<q\leqslant 2Q} |\Delta(\mathbf{1}_{\mathcal{P}}, x; q, 1)| - O\left(\frac{x}{(\log x)^{\gamma+3}}\right).$$

On the other hand, we have

$$\Delta(\mathbf{1}_{\mathcal{P}}, x; q, 1) = \sum_{\substack{y/2<p\leqslant y \\ p\equiv 1 \pmod q}} 1 - \frac{\#\mathcal{P}}{\varphi(q)},$$

for prime $q \in (Q, 2Q]$, where, by the definition of $\mathcal{P}$, we are able to extend the range for the first summation from $p \in \mathcal{P}$ to all primes in $(y/2, y]$. By the Brun–Titchmarsh inequality, we have $\#\mathcal{P} \ll y/(\log x)^2$. Thus

$$|\Delta(\mathbf{1}_{\mathcal{P}}, x; q, 1)| \gg \frac{y}{\varphi(q)\log x} - O\left(\frac{y}{\varphi(q)(\log x)^2}\right) \gg \frac{y}{\varphi(q)\log x}.$$

Summing this over all primes $q \in (Q, 2Q]$, we obtain

$$\sum_{Q<q\leqslant 2Q} |\Delta(f, x; q, 1)| \gg \frac{y}{(\log x)^2} - O\left(\frac{x}{(\log x)^{\gamma+3}}\right) \gg \frac{x}{(\log x)^{\gamma+2}}.$$

Therefore if this is $\ll x/(\log x)^{A-1}$, we must have $\gamma \geqslant A - 3$, as claimed in the remarks following Theorem 2.1. In fact, since our example f is supported only on y-smooth integers, the same remark regarding the dependence of the optimal γ on A applies to Theorem 2.2 as well.

9. Further thoughts

Arguably the most intriguing issue is to try to improve the exponent, γ, of the logarithm in the definition of y in Theorem 2.1. We have shown that $A-3 \leqslant \gamma(A) \leqslant 2A+6+\epsilon$; one might guess that the optimal exponent has $\gamma(A) = \kappa A + O(1)$ for all $A \geqslant 0$.

The bound $\kappa \leqslant 2$ on the coefficient κ is a consequence of the $(xy)^{1/2}(\log x)^{O(1)}$-term in the upper bound in Proposition 5.1. If one can replace this term by $y(\log x)^{O(1)}$ then $\kappa = 1$ follows. Extending the idea in the proofs of Propositions 5.1 and 5.2, one can restrict attention to a much smaller class of f: Those completely multiplicative $f \in \mathcal{C}$ that are supported only on the primes $\leqslant 2(\log x)^{\gamma}$, and the primes in $(y/2, y]$.

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