The elliptic Cauchy problem revisited: Control of boundary data in natural norms

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Abstract

In this note we prove error estimates in natural norms on the approximation of the boundary data in the elliptic Cauchy problem, for the finite element method first analysed in *E. Burman, Error estimates for stabilized finite element methods applied to ill-posed problems. C. R. Math. Acad. Sci. Paris 352 (2014), no. 7-8, 655659.*

Résumé

Dans cette note nous montrons des estimations d'erreur pour l'approximation d'éléments finis des données sur le bord d'un problème de Cauchy elliptique. Ces résultats complètent l'analyse d'erreur de la méthode d'éléments finis proposée dans *E. Burman, Error estimates for stabilized finite element methods applied to ill-posed problems. C. R. Math. Acad. Sci. Paris 352 (2014), no. 7-8, 655659.*

1. Introduction

We consider the numerical approximation of the following linear elliptic Cauchy problem. Let Ω be a convex polygonal (polyhedral) domain in $\mathbb{R}^d$, $d = 2, 3$, and consider the equation

$$\begin{cases} -\Delta u = f, \text{ in } \Omega \\ u = g \text{ and } \partial_n u = \psi \text{ on } \Gamma \end{cases} \tag{1}$$

where $\Gamma \subset \partial\Omega$ denotes a simply connected part of the boundary and $f \in L^2(\Omega)$, $\psi \in H^{\frac{1}{2}}(\Gamma)$ and $g \in H^{\frac{3}{2}}(\Gamma)$. Introducing the spaces $V = H^1(\Omega)$, $V_g := \{v \in H^1(\Omega) : v|_\Gamma = g\}$ and $W := \{v \in H^1(\Omega) :$

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$v|_{\Gamma'} = 0\}$, where $\Gamma' := \partial\Omega \setminus \Gamma$ and the forms $a(u,w) := \int_\Omega \nabla u \cdot \nabla w \, \mathrm{d}x$, and $l(w) := \int_\Omega f w \, \mathrm{d}x + \int_\Gamma \psi w \, \mathrm{d}s$ equation (1) may be cast in the abstract weak formulation, find $u \in V_g$ such that

$$a(u,w) = l(w) \quad \forall w \in W, \tag{2}$$

where $a : V \times W \mapsto \mathbb{R}$ and $l : W \mapsto \mathbb{R}$.

It is well known that the Cauchy problem (1) is not well-posed in the sense of Hadamard. If $l(w)$ is such that a sufficiently smooth, exact solution exists, conditional stability estimates can nevertheless be obtained [1].

In a series of papers [3,4,5,6] we have developed a method, regularised using techniques from stabilised finite element methods that can be analysed using such conditional stability estimates. The stability estimate referred to was a simplified form of a detailed estimate derived in [1], that we recall here.

Assume that the linear form $l(w)$ is such that the problem (2) admits a unique solution $u \in V_g$. Define the following dual norm on l, $\|l\|_{W'} := \sup_{\substack{w \in W \\ \|w\|_W = 1}} |l(w)|$. Consider the functional $j : V \mapsto \mathbb{R}$. Let $\Xi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a continuous, monotone increasing function with $\lim_{x \to 0^+} \Xi(x) = 0$. Let $\epsilon > 0$.

$$\boxed{\text{Assume that there holds } \|l\|_{W'} \leq \epsilon \text{ in (2) then, for } \epsilon \text{ sufficiently small, } |j(u)| \leq \Xi(\epsilon).} \tag{3}$$

For the example of the Cauchy problem (1), it is known [1, Theorems 1.7 and 1.9] that if (1) admits a unique solution $u \in H^1(\Omega)$, a conditional stability of the form (3), (here neglecting geometric factors) with $0 < \epsilon < 1$, holds for

$$j(u) := \|u\|_{L^2(\omega)},\ \omega \subset \Omega : \operatorname{dist}(\omega, \partial\Omega) =: d_{\omega,\partial\Omega} > 0 \text{ with } \Xi(x) := C_{u\varsigma} x^\varsigma,\ C_{u\varsigma} > 0,\ \varsigma := \varsigma(d_{\omega,\partial\Omega}) \in (0,1) \tag{4}$$

and for

$$j(u) := \|u\|_{L^2(\Omega)} \text{ with } \Xi(x) := C_u(|\log(x)| + C)^{-\varsigma} \text{ with } C_u, C > 0,\ \varsigma \in (0,1). \tag{5}$$

Note that to derive these results $l(\cdot)$ is first associated with its Riesz representant in W (c.f. [1, equation (1.31)] and discussion.) The constant $C_{u\varsigma}$ in (4) grows monotonically in $\|u\|_{L^2(\Omega)}$ and C_u in (5) grows monotonically in $\|u\|_{H^1(\Omega)}$.

The above discussion however is incomplete, since it makes no mention of control of the solution on the boundary. Indeed in [1, Equation (1.25)] the following bound is required

$$\|g\|_{H^{\frac{1}{2}}(\Gamma)} + \|\psi\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \eta \tag{6}$$

for some $\eta > 0$, that should be added in the last equation in (3) in the form

$$|j(u)| \leq \Xi(\epsilon + \eta). \tag{7}$$

This omission may seem innocent, since the solution in [4,5,6] was assumed to be zero on the Cauchy boundary, and control of the boundary flux is built into the method. Indeed it follows from the analysis that

$$\|h^{\frac{1}{2}}(\partial_n u_h - \psi)\|_{L^2(\Gamma)} \leq Ch|u|_{H^2(\Omega)} \tag{8}$$

if we assume that there are no perturbations in data. The bound needed to satisfy (6) would be

$$\|\partial_n u_h - \psi\|_{H^{-\frac{1}{2}}(\Gamma)} \leq Ch|u|_{H^2(\Omega)}. \tag{9}$$

This does not follow from (8) and standard techniques to prove that the continuous $H^{-1/2}$-norm is bounded by the discrete counterpart, typically leading to

$$\|\partial_n u_h - \psi\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C\|h^{\frac{1}{2}}(\partial_n u_h - \psi)\|_{L^2(\Gamma)} + C\|u - u_h\|_{H^1(\Omega)},$$

fail due to the ill-posed character of the problem, since the last term of the right hand side does not necessarily converge. Naively bounding the $H^{-\frac{1}{2}}(\Gamma)$-norm by the $L^2(\Gamma)$-norm on the other hand leads to an estimate which is suboptimal by $\mathcal{O}(h^{\frac{1}{2}})$. The aim of the present note is to present an approach to prove the optimal bound applicable in all the methods [4,5,6] and also include the case of non-zero Dirichlet data. In the following we assume that (1) admits a unique solution $u \in V_g \cap H^2(\Omega)$.

2. Finite element discretization

Let $\mathcal{K}_h$ be a shape regular, conforming, subdivision of Ω into non-overlapping, quasi uniform triangles κ. The family of meshes $\{\mathcal{K}_h\}_h$ is indexed by the mesh parameter $h := \max(\operatorname{diam}(\kappa)) < 1$. Let $\mathcal{F}_I$ be the set of interior faces $\{F\}$ in $\mathcal{K}_h$ and $\mathcal{F}_\Gamma, \mathcal{F}_{\Gamma'}$ the set of element faces of $\mathcal{K}_h$ whose interior intersects Γ and Γ' respectively. Each interior face has a fixed but arbitrary normal n_F and the normal associated to faces on the boundary is defined as the outward pointing normal. We assume that the mesh matches the boundary of Γ so that $\mathcal{F}_\Gamma \cap \mathcal{F}_{\Gamma'} = \emptyset$. Let X_h^1 denote the standard finite element space of continuous, affine functions. Define $V_h = W_h := X_h^1$. Let $i_h : H^2(\Omega) \mapsto V_h$ denote the standard nodal interpolant, for which the following interpolation estimate holds

$$\|u - i_h u\|_\Omega + h\|\nabla(u - i_h u)\|_\Omega \leq Ch^2 |u|_{H^2(\Omega)}.$$

We may then write the finite element method: find $(u_h, z_h) \in V_h \times W_h$ such that,

$$\left.\begin{aligned} a_h(u_h, w_h) - s^*(z_h, w_h) &= l_h(w_h) \\ a_h(v_h, z_h) + s(u_h, v_h) &= s(u, v_h) \end{aligned}\right\} \text{ for all } (v_h, w_h) \in V_h \times W_h, \tag{10}$$

where $l_h(w_h) := l(w_h) - \int_\Gamma g \partial_n w_h \, \mathrm{d}s$,

$$a_h(v_h, w_h) := a(v_h, w_h) - \int_\Gamma v_h \partial_n w_h \, \mathrm{d}s - \int_{\Gamma'} w_h \partial_n v_h \, \mathrm{d}s.$$

In order to include the Dirichlet data in a straightforward manner we here use the Nitsche type imposition of the boundary conditions introduced in [5,6], that is the reason for the appearance of the boundary terms in the forms a_h and l_h.

A possible choice of stabilization operators for the problem (1) are

$$s(u_h, v_h) := \sum_{F \in \mathcal{F}_I \cup \mathcal{F}_\Gamma} \int_F h_F [\partial_n u_h][\partial_n v_h] \, \mathrm{d}s + \sum_{F \in \mathcal{F}_\Gamma} \int_F h_F^{-1} u_h v_h \, \mathrm{d}s, \quad \text{with } h_F := \operatorname{diam}(F) \tag{11}$$

and

$$s^*(z_h, w_h) := a(z_h, w_h) + \int_{\Gamma'} h^{-1} z_h w_h \, \mathrm{d}s \;\text{ or }\; s^*(z_h, w_h) := \sum_{F \in \mathcal{F}_I \cup \mathcal{F}_{\Gamma'}} \int_F h_F [\partial_n z_h][\partial_n w_h] \, \mathrm{d}s + \int_{\Gamma'} h^{-1} z_h w_h \, \mathrm{d}s \tag{12}$$

where $[\partial_n u_h]$ denotes the jump of $\nabla u_h \cdot n_F$ for $F \in \mathcal{F}_I$ and when $F \in \mathcal{F}_\Gamma \cup \mathcal{F}_{\Gamma'}$ define $[\partial_n u_h]|_F := \nabla u_h \cdot n_{\partial\Omega}$. Observe that by definition the right hand side of the second equation of (10) is

$$s(u, v_h) = \sum_{F \in \mathcal{F}_\Gamma} \int_F h_F^{-1} g v_h + h_F \psi \partial_n v_h \, \mathrm{d}s.$$

Using a Poincaré inequality on discrete spaces [2] the following bound holds for some $c_p > 0$

$$c_p h \|u_h\|_{H^1(\Omega)} \leq s(u_h, u_h)^{\frac{1}{2}} \tag{13}$$

and therefore the triple norm defined by $|||u_h, z_h|||^2 := s(u_h, u_h) + s^*(z_h, z_h)$ is a norm on $V_h \times W_h$. The following error estimate was shown in [6, Lemma 1], independent of the stability of the problem (1).

Lemma 2.1 *Let $u \in V \cap H^2(\Omega)$ be the solution of* (1) *and $(u_h, z_h) \in V_h \times W_h$ the solution of* (10) *then there holds*

$$|||(u-u_h, z_h)||| \leq Ch|u|_{H^2(\Omega)}$$

and $\|u-u_h\|_{H^1(\Omega)} \leq C\|u\|_{H^2(\Omega)}$.

Proof. The first inequality follows from [6, Lemma 1], with a minor modification to account for the Dirichlet data g. For the second observe that by the Cauchy-Schwarz inequality and the discrete Poincaré inequality (13),

$$\|u-u_h\|_{H^1(\Omega)} \leq \|u-i_h u\|_{H^1(\Omega)} + \|u_h - i_h u\|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)} + c_p^{-1} h^{-1} s(u_h - i_h u, u_h - i_h u)^{\frac{1}{2}}.$$

After an additional triangle inequality

$$s(u_h - i_h u, u_h - i_h u)^{\frac{1}{2}} \leq |||(u - i_h u_h, 0)||| + |||(u-u_h, 0)|||$$

the claim now follows from the approximation estimate $|||(u - i_h u_h, 0)||| \leq Ch|u|_{H^2(\Omega)}$ and the a priori error estimate on $|||(u-u_h, z_h)|||$.

Consider now the error equation, for all $w \in W$,

$$a(u-u_h, w) = \langle r(u_h), w\rangle_{(W',W)} \tag{14}$$

where

$$\langle r(u_h), w\rangle_{(W',W)} = (f,w)_\Omega + \langle \psi, w\rangle_\Gamma - a(u_h, w).$$

It was shown in [6, Theorem 1] that for $g = 0$,

$$\|r\|_{W'} \leq Ch\|f\|_{L^2(\Omega)} + |||(u-u_h, z_h)||| \leq Ch(\|f\|_{L^2(\Omega)} + |u|_{H^2(\Omega)}). \tag{15}$$

With Lemma 2.1 and equation (15) conditional error estimates were derived in [6, Theorem 1] using the conditional stability (3), but omitting the condition (6).

The objective in the next section is to show how the bound

$$\|h^{-\frac{1}{2}}(u_h - g)\|_{L^2(\Gamma)} + \|h^{\frac{1}{2}}(\partial_n u_h - \psi)\|_{L^2(\Gamma)} \leq Ch|u|_{H^2(\Omega)} \tag{16}$$

implied by Lemma 2.1, leads to (9), for a related perturbed approximation $\tilde{u}_h$ that is sufficiently close to u_h, or including the Dirichlet data,

$$\|\tilde{u}_h - g\|_{H^{\frac{1}{2}}(\Gamma)} + \|\partial_n \tilde{u}_h - \psi\|_{H^{-\frac{1}{2}}(\Gamma)} \leq Ch|u|_{H^2(\Omega)}.$$

This is then used in the analysis to show that the approximation error satisfies the bound (6).

3. Boundary error estimates in natural norms

As was pointed out already in [5,6] the error equation (14) can be written using any perturbation, $\tilde{u}_h$ of u_h that is sufficiently close to u_h to be controlled using the triple norm. For instance when nonconforming approximation is used [5], so that $V_h \not\subset V$ $\tilde{u}_h$ is some discrete interpolant of u_h in $V \cap V_h$. Herein we will use this idea to create a $\tilde{u}_h$ that has a suitable oscillating property of the flux error. Indeed drawing on ideas from [7, Lemma 4.1 and Remark 1] we divide Γ into N_Γ shape regular triangular subdomains $\mathfrak{F}_i$, $i = 1, \ldots, N_\Gamma$ each containing an agglomeration of element faces. The boundary of $\mathfrak{F}_i$ does not need to coincide with the boundary element edges, but the diameter of $\mathfrak{F}_i$ is proportional to h, $\mathrm{diam}(\mathfrak{F}_i) = C_{\mathfrak{F}} h$, for some fixed $C_{\mathfrak{F}} > 0$ that we may choose. For each $\mathfrak{F}_i$ we assemble all elements with one face entirely contained in $\mathfrak{F}_i$ and their nearest neighbours among the interior elements into patches $\mathfrak{P}_i \subset \overline{\Omega}$ such that $\mathfrak{P}_i \cap \Gamma \subset \mathfrak{F}_i$. By construction the patches also have diameter $\mathcal{O}(h)$. On each subdomain $\mathfrak{F}_i$ we define the following local projection onto a piecewise constant $\pi_0 w|_{\mathfrak{F}_i} = \text{meas}_{d-1}(\mathfrak{F}_i)^{-1} \int_{\mathfrak{F}_i} w \, \mathrm{d}s$. Then, following [7, Lemma 4.1], provided each $\mathfrak{F}_i$ contains a sufficient number of surface elements, i.e. the constant $C_{\mathfrak{F}}$ is taken large enough, we may construct a function φ_i, whose support is contained in $\mathfrak{P}_i$ such that, given $v_i \in \mathbb{R}$,

$$\pi_0 \partial_n \varphi_i|_{\mathfrak{F}_i} = v_i, \qquad \|\nabla \varphi_i\|_{\mathfrak{P}_i} \leq C \|h^{\frac{1}{2}} v_i\|_{\mathfrak{F}_i}. \tag{17}$$

The constant C and the size of $C_{\mathfrak{F}}$ only depends on the shape regularity of the mesh. Then we construct our $\tilde{u}_h$ as

$$\tilde{u}_h := u_h + v_\Gamma \text{ with } v_\Gamma := \sum_{i=1}^{N_\Gamma} \varphi_i \tag{18}$$

where the coefficients v_i in the definition of φ_i are fixed by the relation,

$$\int_{\mathfrak{F}_i} (\psi - \partial_n \tilde{u}_h) \, \mathrm{d}s = 0, \text{ implying that } v_i := \pi_0(\psi - \partial_n u_h)|_{\mathfrak{F}_i}. \tag{19}$$

The following bounds hold for the perturbation error introduced.

Lemma 3.1 *Let $u_h \in V_h$ and let $\tilde{u}_h$ be constructed using* (18)-(19) *then there holds*

$$\|h^{-\frac{1}{2}}(\tilde{u}_h - u_h)\|_\Gamma + \|h^{\frac{1}{2}} \partial_n(\tilde{u}_h - u_h)\|_\Gamma \leq C \|h^{\frac{1}{2}} \pi_0(\psi - \partial_n u_h)\|_\Gamma$$

and $h^{-1}\|\tilde{u}_h - u_h\|_\Omega + \|\nabla(\tilde{u}_h - u_h)\|_\Omega \leq C\|h^{\frac{1}{2}} \pi_0(\psi - \partial_n u_h)\|_\Gamma$.

Proof. By the definition of $\tilde{u}_h$ and using elementwise trace inequalities $\|u_h\|_{\partial K} \leq C h^{-\frac{1}{2}} \|u_h\|_K$ we have

$$\begin{aligned}\|h^{-\frac{1}{2}}(\tilde{u}_h - u_h)\|_\Gamma + \|h^{\frac{1}{2}} \partial_n(\tilde{u}_h - u_h)\|_\Gamma + \|\nabla(\tilde{u}_h - u_h)\|_\Omega &= \|h^{-\frac{1}{2}} v_\Gamma\|_\Gamma + \|h^{\frac{1}{2}} \partial_n v_\Gamma\|_\Gamma + \|\nabla v_\Gamma\|_\Omega \\ &\leq C(\|h^{-1} v_\Gamma\|_\Omega + \|\nabla v_\Gamma\|_\Omega) \leq \|\nabla v_\Gamma\|_\Omega.\end{aligned}$$

The last inequality was obtained by applying a Poincaré inequality locally on every patch $\|v_\Gamma\|_{\mathfrak{P}_i} \leq C\|h \nabla v_\Gamma\|_{\mathfrak{P}_i}$. Using the second inequality of (17) and the definition of v_i, (19) we conclude

$$\|\nabla v_\Gamma\|_\Omega^2 \leq C \sum_{i=1}^{N_\Gamma} \|h^{\frac{1}{2}} v_i\|_{\mathfrak{F}_i}^2 = C \sum_{i=1}^{N_\Gamma} \|h^{\frac{1}{2}} \pi_0(\psi - \partial_n u_h)\|_{\mathfrak{F}_i}^2 = C \|h^{\frac{1}{2}} \pi_0(\psi - \partial_n u_h)\|_\Gamma^2$$

To estimate the $H^{-1/2}$-norm of the perturbed flux error, $\psi - \partial_n \tilde{u}_h$ we observe that by the construction of $\tilde{u}_h$ there holds, for all $w \in H^{\frac{1}{2}}(\Omega)$,

$$(\psi - \partial_n \tilde{u}_h, w)_\Gamma = \sum_{i=1}^{N_\Gamma} (\psi - \partial_n \tilde{u}_h, w - \pi_0 w)_{\mathfrak{F}_i} \leq C \|h^{\frac{1}{2}}(\psi - \partial_n \tilde{u}_h)\|_\Gamma \|w\|_{H^{\frac{1}{2}}(\Gamma)}, \tag{20}$$

where we used the approximability properties of the piecewise constant functions on the shape regular triangular surface subdomains $\mathfrak{F}_i$ (see for instance [8, Theorem 10.2].) Hence, taking the supremum over (non-zero) $w \in H^{\frac{1}{2}}(\Gamma)$ in (20) we obtain

$$\|\psi - \partial_n \tilde{u}_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \|h^{\frac{1}{2}}(\psi - \partial_n \tilde{u}_h)\|_\Gamma \leq C \|h^{\frac{1}{2}}(\psi - \partial_n u_h)\|_\Gamma + C \|h^{\frac{1}{2}}(\partial_n u_h - \partial_n \tilde{u}_h)\|_\Gamma.$$

It follows by Lemma 3.1 that

$$\|\psi - \partial_n \tilde{u}_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \|h^{\frac{1}{2}}(\psi - \partial_n u_h)\|_\Gamma. \tag{21}$$

Considering now the Dirichlet condition we have, since $u = g$ for the exact solution and by applying the inverse inequality $\|v_h\|_{H^{\frac{1}{2}}(\Gamma)} \leq C \|h^{-\frac{1}{2}} v_h\|_\Gamma$, and a global trace inequality

$$\|\tilde{u}_h - g\|_{H^{\frac{1}{2}}(\Gamma)} \leq \|\tilde{u}_h - i_h u\|_{H^{\frac{1}{2}}(\Gamma)} + \|u - i_h u\|_{H^{\frac{1}{2}}(\Gamma)} \leq \|h^{-\frac{1}{2}}(\tilde{u}_h - i_h u)\|_\Gamma + C\|u - i_h u\|_{H^1(\Omega)}. \tag{22}$$

For the first term in the right hand side we observe that

$$\|h^{-\frac{1}{2}}(\tilde{u}_h - i_h u)\|_\Gamma \leq \|h^{-\frac{1}{2}}(\tilde{u}_h - u_h)\|_\Gamma + \|h^{-\frac{1}{2}}(u_h - i_h u)\|_\Gamma \leq C\|h^{\frac{1}{2}}\pi_0(\psi - \partial_n u_h)\|_\Gamma + Ch|u|_{H^2(\Omega)} \quad (23)$$

where we used Lemma 3.1 and Lemma 2.1 in the last estimate. We conclude that

$$\|\tilde{u}_h - g\|_{H^{\frac{1}{2}}(\Gamma)} \leq C\|h^{\frac{1}{2}}\pi_0(\psi - \partial_n u_h)\|_\Gamma + Ch|u|_{H^2(\Omega)}. \quad (24)$$

We summarize the above results in a Lemma

Lemma 3.2 *Let u_h be the solution of* (10) *and let $\tilde{u}_h$ be the perturbed solution of equation* (18) *then there holds*

$$\|\tilde{u}_h - g\|_{H^{\frac{1}{2}}(\Gamma)} + \|\psi - \partial_n \tilde{u}_h\|_{H^{-\frac{1}{2}}(\Gamma)} \leq Ch|u|_{H^2(\Omega)}$$

and

$$\|r(\tilde{u}_h)\|_{W'} \leq Ch(\|f\|_\Omega + |u|_{H^2(\Omega)}).$$

Proof. The proof of the first inequality is a consequence of the inequalities (21) and (24) and (16). The second inequality, which implies that the perturbed error $u - \tilde{u}_h$ satisfies the equivalent of (15), is straightforward to show since in this case

$$\langle r(\tilde{u}_h), w\rangle_{(W',W)} = (f, w)_\Omega + \langle \psi, w\rangle_\Gamma - a(\tilde{u}_h, w) = \underbrace{(f, w)_\Omega + \langle \psi, w\rangle_\Gamma - a(u_h, w)}_{I} + \underbrace{a(u_h - \tilde{u}_h, w)}_{II}.$$

Term I is bounded similarly as in [6, Theorem 1], with some minor modifications due to the non-zero boundary data g. Indeed Galerkin orthogonality yields in this case, for some H^1-stable approximation $w_h \in W_h$ of w,

$$I = (f, w - w_h) + \langle \psi, w - w_h\rangle_\Gamma - a(u_h, w - w_h) - \int_{\Gamma'} w_h \partial_n u_h \text{ d}s - s^*(z_h, w_h) - \int_\Gamma (u_h - g)\partial_n w_h \text{ d}s.$$

The last term on the right hand side is the contribution due to the non-homogeneous Dirichlet conditions and we bound it using a Cauchy-Schwarz inequality, a trace inequality, the stability of w_h, and equation (16)

$$\int_\Gamma (u_h - g)\partial_n w_h \text{ d}s \leq \|h^{-\frac{1}{2}}(u_h - g)\|_\Gamma \|w\|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}\|w\|_{H^1(\Omega)}.$$

For term II we proceed by the Cauchy-Schwarz inequality, Lemma 3.1 and (16)

$$a(u_h - \tilde{u}_h, w) \leq \|\nabla(u_h - \tilde{u}_h)\|_\Omega \|w\|_{H^1(\Omega)} \leq C\|h^{\frac{1}{2}}\pi_0(\psi - \partial_n u_h)\|_\Gamma \|w\|_{H^1(\Omega)} \leq Ch|u|_{H^2(\Omega)}\|w\|_{H^1(\Omega)},$$

which completes the proof.

We finally give a proof of the conditional error estimate using the conditional stability (3)-(5), (7) and the data condition (6).

Theorem 3.3 *Let u be the solution of* (2) *having the conditional stability* (4) - (5) *under the conditions* (3) *and* (6). *Let u_h be the solution of* (10). *Then there holds, with $\varsigma \in (0, 1)$*

$$\|u - u_h\|_{L^2(\omega)} \leq Ch^\varsigma \text{ and } \|u - u_h\|_{L^2(\Omega)} \leq C_u(|\log(Ch)| + C)^{-\varsigma}.$$

Proof. Let $\tilde{u}_h$ be defined by (18). Observe that by Lemma 3.1 and Lemma 2.1 $\|u_h - \tilde{u}_h\|_{L^2(\Omega)} \leq Ch^2|u|_{H^2(\Omega)}$. It follows by the triangle inequality that it is enough to prove the bound for $\tilde{e} = u - \tilde{u}_h$. We see that $\tilde{e}$ is a solution to (2) with the Dirichlet data, $\tilde{e}|_\Gamma$, the Neumann data $\partial_n \tilde{e}|_\Gamma$ and the righ hand side, $\langle r(\tilde{u}_h), w\rangle_{(W',W)} := (f, w)_\Omega + \langle \psi, w\rangle_\Gamma - a(\tilde{u}_h, w)$. By the first inequality of Lemma 3.2 we see that (6) holds with $\eta = Ch$. By the second inequality of Lemma 3.2 we see that (3) holds with $\epsilon = Ch$. To conclude we observe that using the second inequality of Lemma 2.1 and equation (16),

$$\|\tilde{e}\|_{H^1(\Omega)} \leq \|u - u_h\|_{H^1(\Omega)} + \|\tilde{u}_h - u_h\|_{H^1(\Omega)} \leq C\|u\|_{H^2(\Omega)} + C\|h^{\frac{1}{2}}\pi_0(\psi - \partial_n u_h)\|_\Gamma \leq C(1 + h)\|u\|_{H^2(\Omega)}$$

showing that $\|\tilde{e}\|_{H^1(\Omega)}$ is bounded by a constant independent of h. It follows by the conditional stability estimate that $\tilde{e}$ satisfies the required error bounds and the proof is complete.

3.1. *Application to other methods*

The above argument may be applied also to the higher polynomial order case of [6] and the nonconforming method of [5]. Observe that the arguments in the latter already relies on the construction of an H^1-conforming approximation $I_{cf} u_h$, where I_{cf} is the interpolation operator using local averaging in each node of the discontinuous finite element solution. The perturbation error is estimated in the triple norm in a similar way as above. Then the above argument can be applied, constructing $\tilde{u}_h$ in (18) with $I_{cf} u_h$ in the place of u_h. The estimate (20) is then obtained for $\tilde{u}_h$. We once again need to estimate the perturbation error $\tilde{u}_h - u_h$. This is made in two steps using the intermediate function $I_{cf} u_h$. The difference $\tilde{u}_h - I_{cf} u_h$ is estimated as above. Then in a second step u_h is added and subtracted to create residuals in u_h and all other terms on the form $I_{cf} u_h - u_h$ which is bounded as in [5] using the penalty term on the solution jumps that is part of the triple norm in this case.

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