CHAPTER 2

The Pythagoreans: number and numerology

ANDREW GREGORY

There is a common perception of the ancient Greek thinker Pythagoras (*c.* 570–*c.* 495 BCE) as a mathematician and geometer, famed for his discovery of Pythagoras' theorem. Pythagoras has also been seen as a pioneer in the application of mathematics to music theory, as a champion of the importance of mathematics in understanding the cosmos, and as the originator of the idea of music of the heavenly spheres.

Modern scholarship over the last 40 years has developed a rather different picture of Pythagoras as someone interested in the fate of the soul after death, an expert on religious ritual, perhaps a shaman, a wonder worker, and founder of a religious sect. While the common perception of Pythagoras is certainly in need of some modification, these two pictures are not necessarily incompatible.

Older views of the relation of religion and science may have seen religion and science as incompatible or in some form of inevitable conflict with each other. Dating from the late nineteenth century, these views are known as the 'conflict' theory. More modern views recognize that science and religion can interact in many ways and that for many thinkers, there is compatibility between their scientific and religious views. While not denying there can at times be conflict between science and religion, the modern 'complexity' view also allows there to be other relations, such as support, symbiosis, compatibility or indifference depending on specific circumstances.

The major problem in developing an accurate picture of Pythagoras is that Pythagoras himself wrote nothing, and if his contemporaries wrote anything about him nothing of this has survived. It may be that those who associated with Pythagoras deliberately kept silent about his key views. What we know of Pythagoras comes from much later sources, many of which are unreliable.[1] There was an unfortunate tendency after Plato (428/427–348/347 BCE) and Aristotle (384–322 BCE) for Pythagoras to be built up as a semi-divine or a divinely inspired figure and visionary. Often the views of later thinkers were attributed to him, especially those of the later Pythagoreans, and Pythagoras was also credited with originating aspects of Plato's metaphysics and cosmology. The 'Pythagorean question' is that of the extent to which we can reconstruct the historical views of Pythagoras from the information we have available.[2]

The key turning point in modern studies of Pythagoras has been Walter Burkert's *Lore and Science in Ancient Pythagoreanism*.[3] Burkert analysed the available evidence and concluded that to find out about Pythagoras, we must look to the earliest and least corrupt sources, which essentially means looking at the evidence of Plato and Aristotle. It is this move in what we see as reliable evidence that has effected the shift away from the view of Pythagoras as the master mathematician towards Pythagoras as the religious leader.

One thing to emphasize early on is that Pythagoreanism was never a tight body of doctrine or a rigid system of beliefs. There was a great diversity of views among Pythagoreans on issues of religion, the nature of numbers and the application of numbers in our understanding of the cosmos. We know this from the fragments that have survived from thinkers such as Philolaus and Archytas and the reports of Plato and Aristotle.[4]

Pythagoras and the early Pythagoreans

There is very little that we can say for certain about Pythagoras. Pythagoras was born on the Greek island of Samos *c.* 570 BCE and died *c.* 490 BCE. Around 530 BCE he relocated to Croton in southern Italy, which became a centre for the Pythagoreans. It is said that Pythagoras travelled widely in his youth, to Egypt and other parts of Africa, to Babylonia and possibly even to India. When Pythagoras was considered an important mathematician it was speculated that he got at least some of his mathematical knowledge from his travels to Egypt. Nowadays his travels are seen as an attempt to gain knowledge of various esoteric religious cults in these places.

In what follows, I am going to begin by looking at Pythagoras and the various issues concerning mathematics and religious practice that relate to him. There will be little here about any specific god. The Pythagoreans were not monotheists and their religious practices centred around how to be pure in this life and how best to pass into the next life here on earth rather than religious worship. I will also look at some followers of Pythagoras who are important in the history of mathematics for various reasons – Hippasus, Philolaus, and Archytas – as well as two groups of Pythagorean followers – the *acousmatikoi* (the listeners) and the *mathematikoi* (the learners) – who had rather different attitudes to the Pythagorean tradition. Finally, I will look at Plato, who while not a Pythagorean himself, was clearly influenced by Pythagorean ideas. How Plato treats those ideas can also throw some light on explaining what the Pythagoreans may have been trying to do with those ideas.

Pythagoras' theorem

Did Pythagoras discover what we now know as Pythagoras' theorem? This now seems unlikely though it is possible that either Pythagoras or another early Pythagorean made some sort of contribution. It is not unusual to find discoveries or inventions credited to the ancient Greeks when they merely improved on something that had been invented or discovered earlier. The Archimedes screw, a device for raising water, was used much earlier by the Babylonians but was named after Archimedes who made significant improvements to the efficiency of the device.

Much depends here on exactly what we mean by 'discover' when we ask if Pythagoras discovered Pythagoras' theorem. Pythagorean triples, which are integer lengths for right angled triangles that conform to Pythagoras' theorem, were known a long time before Pythagoras and were well known to the Babylonians. The simplest example here is 3, 4, 5 where $3^2 + 4^2 = 5^2$; other examples are 5, 12, 13 and 8, 15, 17 and 7, 24, 25 (there were many more known in antiquity). The Babylonians though, as far as we are aware, did not have a general expression for Pythagoras' theorem nor did they have a proof of it. The Babylonians were in many ways excellent mathematicians but tended to restrict themselves to the practical application of mathematics rather than investigate the abstract or concern themselves with proofs. It is also unlikely that Pythagoras provided a proof of the theorem. If he did, we do not know the nature of the proof and it is quite early in the history of Greek mathematics for the concept of proof. There are other things that Pythagoras may have done such that his name became associated with the theorem. It is possible that he formulated the theorem in an abstract, general manner which perhaps had not been done before, perhaps he produced a significant diagram, or perhaps he simply celebrated someone else generating the proof. The often repeated story that Pythagoras sacrificed oxen on discovering the theorem does not look reliable, as the Pythagoreans were vegetarians and also believed that the human soul survived death and was reincarnated, possibly in humans, possibly in animals (see below).

Pythagoras is not given the credit for a proof of Pythagoras' theorem, nor seen as an important mathematician or geometer, by either Plato or Aristotle. Nor is Pythagoras seen as a significant contributor to mathematics or geometry by early histories of Greek mathematics.

It is significant that while both Plato and Aristotle talk of presocratic natural philosophy, they do not give Pythagoras any important role in this.[5] Plato, who says remarkably little about Pythagoras himself, says that:

> Such was Pythagoras, who was particularly beloved in this way, and his followers have a reputation for a way of life they call Pythagorean even down to this day.[6]

The picture of Pythagoras and the Pythagorean way of life that emerges from looking at the evidence in Plato and Aristotle is of someone whose key beliefs were in the immortality of the soul and reincarnation and whose expertise was in the fate of the soul after death and in the nature of religious ritual. Pythagoras' major achievements are seen as the advocacy and the founding of a way of life based on stringent dietary regulations, strict self-discipline, and the keen observance of religious ritual. Pythagoras, or perhaps the early Pythagoreans, may have contributed something to our understanding of right angled triangles, but it is unlikely that this is the outright discovery or proof of what we now know as Pythagoras' theorem.

Metempsychosis

The idea that the soul survives the death of the body and then can reincarnate, either in another human body or in an animal body, is known as metempsychosis.

We have reasonably solid evidence that this was indeed Pythagoras' view. Diogenes Laertius, an ancient doxographer, tells us that:

On the subject of reincarnation, Xenophanes tells a tale which begins: Now I turn to another account and I will show the way. He says this about Pythagoras: Once he passed a young dog which was being mistreated, and taking pity he said: 'Stop, do not beat it, that is the soul of a man who was my friend, I recognised it when it cried aloud.'[7]

There is, though, some consensus that this is a significant move away from the Homeric conception of the fate of the soul, which was rather bleak. The standard passage for comparison in Homer is where Achilles says:

I would rather be above ground still and labouring for some poor and portionless man, than be lord over all the lifeless dead.[8]

We have very little definite information about the nature of metempsychosis. One problem is that we have very little on Pythagoras' account of the soul and we do not know if the entire soul or only part of it was supposed to transmigrate. We have nothing at all on the nature of the actual transmigration, of how the soul moved from its previous host body to the next host body. We do not know if every soul underwent transmigration, we do not know the extent of how many living things could participate (animals other than dogs, plants?), and we do not know if there was eventually an escape from the sequence of transmigration, either by death of the soul or escape to some heaven or state that did not involve embodiment.[9]

Shamanism?

It has been suggested that either Pythagoras was a shaman, or that what he did was related to shamanism. A shaman is someone who enters into a state of altered consciousness (perhaps induced by drugs, meditation or repetitive music/dance) and then claims to be able to commune with or perhaps in some manner affect or control the souls of the dead. The social anthropologist Shirokogoroff, who was one of the first to investigate the shaman of the Siberian Tungus people, said that:

In all Tungus languages this term (saman) refers to persons of both sexes who have mastered spirits, who at their will call and introduce these spirits into themselves and use their power over the spirits in their own interests, particularly helping other people, who suffer from the spirits; in such a capacity they may possess a complex of special methods for dealing with the spirits.[10]

The notion of a trance, or some form of ecstatic state, leading to access to a spirit world is the key part of shamanism. There is, though, no reliable evidence that Pythagoras entered trances or ecstatic states and the notion of entering a spirit world is contrary to the principles of metempsychosis. If souls do not enter into some sort of afterlife, but transmigrate to other bodies, what spirit world is there for Pythagoras to enter via some form of ecstatic state? It is perhaps significant that within shamanism proper there is no trace of any view like metempsychosis.

How to live better

Pythagoras was most famous in the ancient world for specifying how to live better. So we can find Isocrates saying that Pythagoras:

More conspicuously than others paid attention to sacrifices and rituals in temples.[11]

We know little of precisely what Pythagoras prescribed here, only that he paid keen attention to these matters. One part of this better way of life was vegetarianism, although it is not clear whether the Pythagoreans were outright vegetarians or only refused to eat certain types of meat. The evidence here is confused, some saying that Pythagoras would not even go near butchers and hunters, others saying that Pythagoras would not eat some parts or types of animals but would eat others. It may well be that the vegetarianism was related to the belief in metempsychosis with the ban on eating certain animals related to which animals were able to partake in metempsychosis. The Pythagoreans were also forbidden from eating beans. The reason for this may be simple and crude – that flatulence is not very helpful if people, either individually or in a group, are meditating and attempting to reach some higher plane of consciousness. Alternatively, it has been suggested that this is related to shamanism as some shamans refuse to eat beans.[12]

Certainly for some Pythagoreans there was a ban on suicide, again perhaps related to the issue of metempsychosis and the best way to enter the next life. Theories of this type often held that what you did in this life determined the nature of your next life and that there was a hierarchy of incarnations. It also seems that some Pythagoreans believed the soul to be in some sense a harmony or attunement. The nature of the soul and its fate after death are an important theme in Plato's Phaedo where some Pythagorean ideas are discussed. The question of how to live for these Pythagoreans was then one of how to bring one's soul into better harmony or attunement. This brings us back to number again as the Pythagoreans are associated with the idea that we can express musical harmony in terms of number.

Mathematics and music

Pythagoras is sometimes credited with the first application of mathematics to music theory. The general idea is straightforward. If we have a stringed instrument, we get a certain note when that string is played 'open'. If we alter the effective length of the string, we can get different notes. The discovery attributed to Pythagoras is that using ratios of simple integers to determine where to stop the string, we can produce harmonious notes. So a ratio of 2:1 will produce an octave, while 4:3 will produce a musical fourth, and 3:2 will produce a musical fifth. We have no direct evidence that Pythagoras discovered this and there is no application or development of this discovery which is attributed to Pythagoras himself. As we will see later, both Philolaus and Archytas, followers of Pythagoras, made considerable use of this insight in developing musical theory. There are many tales about Pythagoras' discovery, but all of these are apocryphal and often physically impossible. Figure 2.1 manages to give five physically impossible ways for this discovery to have been made. Treating these clockwise, starting from the top left:

1. One popular tale had Pythagoras discovering the musical ratios when passing a blacksmith's shop and supposedly noticing that different sized hammers produced different notes. However, the weight of hammer has no direct relationship to the note produced when it hits something. Try it yourself if you like!
2. The size or weight of a bell has no direct relationship to the note it will produce when struck.
3. The amount of water in a glass has no direct relationship to the note produced when the glass is struck.
4. Tensioning strings with different weights looks the most plausible of the methods proposed here. However, changing string tension by using differing weights again does not produce the required relationship to the pitch of the open string (frequency varies in proportion with the square root of the tension). It is only by stopping strings that the required ratios are generated.
5. There is of course a relationship between length of pipe and note produced, but once again it is not the relationship shown here.

Figure 2.1 A late medieval woodcut from Franchino Gafurio's *Theorica musice* (1492) giving five apocryphal representations of how Pythagoras linked music and mathematics.

That someone among the Pythagoreans discovered these ratios – or more likely, in discussion with musicians realized the significance of these ratios – is beyond doubt. As we will see, some Pythagoreans made major contributions to music theory. However, there is no direct evidence that Pythagoras himself had anything to do with this. It is very likely that the discoveries of the later Pythagoreans were attributed to Pythagoras himself in some of the later sources.[13]

Tetraktys

It is likely that the idea of the tetraktys can be traced back to Pythagoras. The tetraktys is the first four integers and their sum is the Pythagorean perfect number, 10 (1 + 2 + 3 + 4). There are records of a Pythagorean oath as:

No, I swear by he who gave to our heads the tetraktys,

The origin and root of immortal nature.[14]

The first four integers were arranged in this manner to form the tetraktys shown in Figure 2.2.

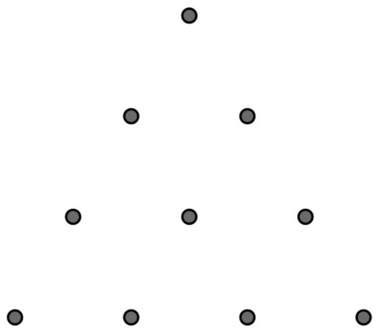

Figure 2.2 The tetraktys links visually the first four integers and the Pythagorean perfect number 10.

The Pythagoreans were quite keen on representing numbers in this way and, as we shall see later, they were also keen on using representations like this in order to understand the relations between numbers. The first four integers and the Pythagorean perfect number 10 feature prominently in Pythagorean thought. Some Pythagorean theories of music derive ratios related to musical notes which use only these first four integers. There is a Pythagorean cosmology where it is supposed that there are 10 (the perfect number) objects orbiting around a central fire.

A world of number?

It has sometimes been said that the Pythagoreans considered the world to be constituted out of numbers. It has never been entirely clear how to visualize this theory, but there is one contrast that may throw some light on this. While the Pythagoreans have been said to have an arithmetical cosmology, Plato has been said to have a geometrical cosmology. That is, while the Pythagoreans considered the world about us to be constituted from numbers, Plato considered it to be constituted from shapes.[15] So for Plato there were two fundamental triangles, which formed either a more complex triangle or a square, which in turn formed three dimensional shapes: tetrahedron, octahedron or icosahedron from the complex triangles or a cube from the squares. These shapes were fire, air, water, and earth respectively.

Philosopher of Science Karl Popper has commented that one of Plato's main contributions is that:

> Ever since Plato and Euclid, but not before, geometry (rather than arithmetic) appears as the fundamental instrument of all physical explanations and descriptions, in the theory of matter as well as cosmology.[16]

So one might say that since Plato we have thought in terms of geometrical shapes for the fundamental particles that make up our world. We have thought in terms of spherical atoms and when atoms were discovered to be comprised of smaller particles, we have thought in terms of spherical electrons, protons, and neutrons. It might also be said that the twentieth century has re-instated a more Pythagorean picture, as with the advent of quantum mechanics we now think of electrons in terms of wave or probability functions rather than in terms of shapes.

Modern scholarship on the Pythagoreans has moved on slightly though.[17] It is remarkably difficult actually to find any Pythagorean who explicitly advocated the idea that the world about us is indeed constituted from numbers. There is little evidence for this prior to Aristotle, who tells us that:

> Contemporaneously with these philosophers and before them, the so-called Pythagoreans, who were the first to take up mathematics, not only advanced this study, but also having been brought up in it they thought its principles were the principles of all things. Since of these principles numbers are by nature the first, and in numbers they seemed to see many resemblances to the things that exist and come into being.[18]

Aristotle also says that:

> The Pythagoreans believed in one kind of number, the mathematical. They hold that it is not separate, but sensible substances are constituted out of it. They construct the whole heaven out of numbers, not abstract units, but units which have size. However, on the subject of how the first extended one is constructed, it is likely that they are in difficulty.[19]

There are many ways though in which one might think that numbers are important in giving an account of the world without actually believing that the world is literally constituted from numbers.

Pythagoras' powers?

There are many tales of strange powers and deeds associated with Pythagoras. Some mentioned by Aristotle are that he was seen in two different places at exactly the same time, that one of his thighs was golden, and that when crossing the River Kosas the river spoke to him and many people heard this.[20] Other tales have Pythagoras prophesying the coming of a white, female bear, killing a dangerous snake by biting it, and to have prophesied to his followers approaching political strife.[21] We have no direct evidence of Pythagoras making any of these claims himself nor do we know his attitude to any of these claims.

What we make of these tales is open to debate. One view is that it is not surprising that these sorts of tales were attributed to a secretive, charismatic religious leader in antiquity and we need not take them too seriously. A second view is that these tales are in some way symbolic and have considerable significance in terms of magic, ritual, and access to the realm of the dead, and they fit into a broader pattern of such stories.[22]

Acousmatikoi and *mathematikoi*

There were two groups of immediate followers to Pythagoras', the *acusmatikoi* and the *mathematikoi*, the listeners and the learners. The classic statement of the division between the followers of Pythagoras is given by classics scholar F.M. Cornford, who says:

> Tradition points to a split between the Acousmatics, who may, perhaps, be regarded as the 'old believers' who clung to the religious doctrine, and the Mathematici, an intellectualist or modernist wing, who, as I believe, developed the number doctrine on rational, scientific lines, and dropped the mysticism.[23]

However, it is doubtful that such a bipolar split can be justified given more modern historiographies and it is more likely that there was a much wider spectrum of views, including these two wings but also those who brought together the religious, magical, and scientific aspects. Pythagoreanism was more of a broad church where some may have felt happy with a mix of what Cornford categorizes here as religious and scientific views. Cornford wrote this in the 1920s when ideas of an inherent conflict between religion and science, and indeed magic and science, were much more prevalent than they are today. So too ideas of a linear progression for humanity from magic to religion to science were more prevalent.

Hippasus and √2

One of the more colourful stories relating to the early Pythagoreans is that of Hippasus and the irrationality of the square root of 2. We know very little of Hippasus, other than that he was associated with the Pythagoreans and lived in the fifth century BCE. According to some tales, he may have discovered the irrationality of $\sqrt{2}$. A rational number is one that can be expressed as the ratio of two integers. An irrational number is one that cannot. It is said that having discovered the irrationality of $\sqrt{2}$, Hippasus also made this generally known and was then drowned at sea. It is believed that irrational numbers were discovered around this time, though in fact we have very little evidence on this. The irrationality of several square roots was certainly known to Plato, as is clear in his *Theaetetus*. How and why Hippasus was drowned at sea, if that indeed was his fate, is the subject of several stories. There is a good deal of variety and contradiction among these stories and it is difficult tell which, if any one of them, is true. The basic idea is that the discovery of the irrationality of $\sqrt{2}$ was in some way embarrassing and that Hippasus' death by drowning was in some way a punishment for either discovering or divulging the irrationality of $\sqrt{2}$. The embarrassment to the Pythagoreans is supposed to be that their belief in a cosmos comprised of numbers, where all relations should be expressible as the ratio of two integers, was compromised by the discovery of the irrationality of $\sqrt{2}$. How much of a problem that was for the Pythagoreans will depend on how much they were committed to the idea of a cosmos comprised of numbers. So either for the discovery of the irrationality of $\sqrt{2}$, or for divulging this knowledge contrary to Pythagorean principles of secrecy, Hippasus was in some way put to death by drowning by the Pythagoreans. Some tales have Pythagoras ordering this, some have Hippasus discovering the irrationality of $\sqrt{2}$ while on a sea voyage and being thrown overboard by his Pythagorean fellow travellers. Other versions of the tale have the gods as implicit in Hippasus' death by drowning, or the revealed secret being how to construct a dodecahedron inside a circle. Yet more versions have Hippasus merely being expelled from the Pythagorean brotherhood.

There are issues with this story apart from the multiple and conflicting versions of it. First, why should the Pythagoreans find the discovery of the irrationality of $\sqrt{2}$ so embarrassing? Only if they were committed to the idea of the world being comprised of integers and all relations being expressible in terms of ratios of those integers would this be problematic. However, as we have seen, there is very little evidence to tie the Pythagoreans to such a belief.

Pythagorean numerology

The Pythagoreans are supposed to have had an interest in numerology. There is a need here to be careful about what sort of numerology is ascribed to which

Pythagoreans, as there are many types of numerology and as we have seen Pythagoreanism was quite a diverse phenomenon encompassing a good many different ideas and attitudes. The first thing to say is that while there is something in the modern world called 'Pythagorean Numerology', as far as we know this sort of numerology was not practised by the Pythagoreans. The basic idea of modern Pythagorean numerology is that we can tell something about someone's character or fate by substituting numbers for the letters of their name (a = 1, b = 2, etc.) and then manipulating those numbers, along with the numbers of their birthday, to reach a single figured integer. So for simplicity, let us take someone called Aaron Abbs, born 01/02/2000.

$$\text{Aaron} = 1 + 1 + 18 + 15 + 14 = 39$$
$$3 + 9 = 12$$
$$1 + 2 = 3.$$
$$\text{Abbs} = 1 + 2 + 2 + 19 = 24$$
$$2 + 4 = 6.$$
$$01/02/2000 = 1 + 1 + 2 = 4.$$

Aaron Abbs then has key numbers of, 3, 6, and 4. If 3 = motivated, 6 = strong, and 4 = artistic, Aaron Abbs is motivated, strong, and artistic. The process of manipulating the numbers here is entirely arbitrary and can be made more complex and the interpretation of the key numbers made more complex to give the numerologist an air of expertise or mystery, but the basic principles remain the same. However, much as modern numerologists would like to give their practice ancient authority or mystique by linking it to early Pythagoreans, they were simply not numerologists of this sort.[24]

This is not to say that the Pythagoreans were uninterested in what we might term numerology. This is probably better put by saying that the Pythagoreans were interested in all aspects of numbers and their properties. They did not really have a distinction between a purely mathematical property of a number and numerology, which is not surprising in their historical context. Indeed, it is not as easy as it might seem, even today, to give a watertight definition of what is mathematical and what is numerological. Modern philosopher David Stove has commented that:

> No one actually knows, even, what is wrong with numerology. Philosophers, of course, use numerology as a stock example of thought gone hopelessly wrong, and they are right to do so; still, they cannot tell you what it is that is wrong with it. If you ask a philosopher

this, the best he will be able to come up with is a bit of Positivism about unverifiability, or a bit of Popperism about unfalsifiability.[25]

While it is easy to rule out the 'Pythagorean' numerology we have seen above, the fact is that numbers do have some interesting properties and it is not so easy to draw the line between the mathematical and the numerological – and it certainly would not have been easy for the Pythagoreans.

If we want a slightly surprising property of numbers, which was certainly known to the Pythagoreans, there is the fact that, loosely put, the sums of odd numbers give the square numbers. Put more precisely, the sum of the first n odd numbers = n^2 So:

$$1 = 1 = 1^2$$
$$1 + 3 = 4 = 2^2$$
$$1 + 3 + 5 = 9 = 3^2$$
$$1 + 3 + 5 + 7 = 16 = 4^2$$
$$1 + 3 + 5 + 7 + 9 = 25 = 5^2$$

This is shown pictorially in Figure 2.3.

Some numbers are 'perfect' numbers where 'perfect' is a technical term and the number is equal to the sum of its proper divisors. 6 is divisible by 1, 2, and 3.

6 = 1 + 2 + 3, making 6 a perfect number.

There is a sequence of perfect numbers, the next being 28:

$$28 = 1 + 2 + 4 + 7 + 14.$$

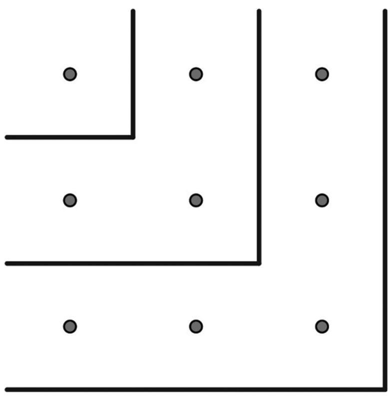

Figure 2.3 Placing counters in the shape of a square illustrates that the sum of the first n odd numbers is n^2.

496, 8128 are the next two and there are currently 48 known perfect numbers. The Pythagoreans were aware of the idea of perfect numbers and had the terms 'under-perfect' and 'over-perfect' for numbers whose factors added up to less or more than the number respectively.

That some Pythagoreans were interested in what we would call numerology is undeniable. They did attribute non-mathematical properties to numbers. So 2 and 3 were associated with male and female, while 5 was associated with marriage and 10 was seen as a divine or special number. There are some important points to make here, though. Numbers do have properties and it would not have been easy for the Pythagoreans, in the context of what was known in ancient Greece, to distinguish what we would consider mathematical and numerological properties. There was a wide spectrum of interest in number among the Pythagoreans. Some would have been what we would see as mathematicians, some what we would see as numerologists and some would have mixed aspects of these two extremes together. Many ancient societies had forms of numerology where there were numerically good days to do things and bad days to do things (superstition about Friday the 13th is a hangover of this sort of thinking).[26] Did the Pythagoreans go beyond this sort of thinking? I would suggest they did, not only in their purely mathematical thinking, but in some of their forms of numerology as well.

What we find in the Pythagoreans, but not in other early cultures, is the attempt to apply what are thought to be certain privileged numbers, the tetraktys of 1, 2, 3, and 4 or the perfect number 10 generated from the tetraktys to draw conclusions about the nature and structure of the heavens or the world. So there are 10 bodies in the heavens because that is the perfect number and their motion is related to the tetraktys because that is related to the celestial music.

Philolaus on music

Philolaus of Croton lived from *c.* 470 to *c.* 385 BCE. Philolaus and Archytas were the most significant contributors to the Pythagorean tradition we know of in the presocratic period.

Philolaus wrote one book, *On Nature*, which if Pythagoras wrote nothing is probably the first book of the Pythagorean tradition, of which a few fragments survive. He worked on astronomy, cosmology, and music theory.

Philolaus did important work on the mathematical theory of music and harmony. He was aware of the basic Pythagorean ideas of a 2:1 ratio for an octave, 4:3 for a musical fourth, and 3:2 for a musical fifth. Philolaus introduced the ratios of 9:8 for a whole note and 256/243 for a semitone. The way this works is that if we take our root note as 1, then the next whole note is 9/8 (1 x 9/8). The following note is then 9/8 x 9/8 = 81/64. So using the key of C major, the Philolaus values for the notes can be generated as in Table 2.1.

Table 2.1 ***Philolaus' scale. The first row gives modern note names, the second row is the ratio between notes, the third row is the note expressed as a ratio relative to the root note.***

C	D	E	F	G	A	B	C
	9/8	9/8	256/243	9/8	9/8	9/8	256/243
1	9/8	81/64	4/3	3/2	27/16	243/128	2

While the 256/243 ratio looks a little complex, each number here is based on a tetraktys number. So 9/8 is 3 × 3/2 × 2 × 2, 81/64 is 9/8 × 9/8, 27/16 is 3 × 3 × 3/2 × 2 × 2 × 2, and even the arbitrary looking numbers as 128, 256, and 243 are powers of 2 and 3. 128 = 2^7, 256 = 2^8, and 243 = 3^5 (= 3 x 81, when 81 = 3 × 27 and 27 = 3^3).

In modern musical theory, we have something called 12-tone equal temperament (12ET), where for an octave there are 12 equally sized semitones. The ratio between all neighbouring semitones in 12ET is $\sqrt[12]{2}$ (the 12th root of two). Pythagorean scales (also known in modern terminology as 'just intonation') do not have this property. Both the ratios between notes and the position of the notes within the octave can be expressed in terms of 'cents'. Ratios are said to be '100 cents' when they match the 12ET ratio, where 1200 cents make up one octave. Positions are said to be 100 cents when they match the 12ET positions. The differences between 12ET and Philolaus are shown in Table 2.2.

Table 2.2 ***Philolaus against the modern scale. The first row gives modern note names, the second row is equal tempered notes expressed in cents, and the third row is Philolaus notes expressed in cents.***

C		D		E	F		G		A		B	C
0	100	200	300	400	500	600	700	800	900	1000	1100	1200
		203.91		407.82	498.04		701.96		905.87		1109.78	1200

There are advantages and disadvantages to modern 12ET. The advantages are that keyboards tuned to 12ET, and instruments with the frets placed according to 12ET (standard modern guitars), can be played in any key equally well without retuning. This allows key changes within one piece of music much more easily and also facilitates ensemble playing. Chords also sound better in 12ET, especially the more complex chords used for jazz.

The disadvantage of 12ET is that the harmonies produced do not sound quite as pure as those of the Pythagorean scale. So Pythagorean harmonies are still sometimes used where tuning the instrument is not an issue, as for example with the human voice, where barber shop quartets can make use of Pythagorean harmonies.

Cosmology

Philolaus gave us a specific model of the heavens shown in Figure 2.4. In many ways this is a remarkable model of the heavens for antiquity. Moving outwards

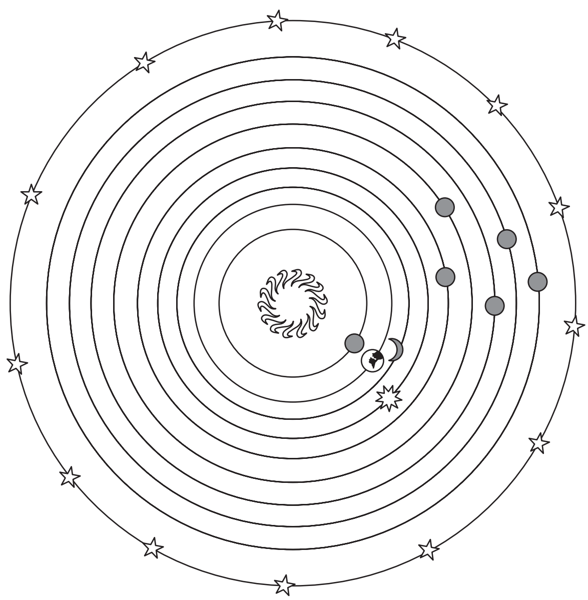

Figure 2.4 Philolaus' model of the heavens. Moving outwards from the middle, the celestial bodies are a central fire, a counter-earth, the earth, the moon, the sun, the five naked-eye planets (Mercury, Venus, Mars, Jupiter, Saturn), and the stars. The counter-earth prevents the central fire being visible from earth.

from the middle, the celestial bodies are a central fire, a counter-earth, the earth, the moon, the sun, the five naked eye planets (Mercury, Venus, Mars, Jupiter, Saturn), and the stars. One reason that this is remarkable is that it is one of the very few ancient models of the heavens which has the earth in motion, rather than immobile and at the centre of the cosmos. It is also remarkable that it is not the sun that is at the centre of the cosmos in place of the earth, but a central fire. No reason has been passed down to us as to why the earth was placed in motion. We cannot see the central fire as the counter-earth is in a synchronous orbit with the earth and always blocks our view of the central fire. How well this model could account for the phenomena is still open to debate. Aristotle has a rather critical view of this cosmology:

> The Italian philosophers known as Pythagoreans take the contrary view. At the centre, they say, is fire, and the earth is one of the stars, creating night and day by its circular motion about the centre. They further construct another earth in opposition to ours to which they give the name counter-earth. In all this they are not seeking for theories and causes to account for observed facts, but rather forcing their observations and trying to accommodate them to certain theories and opinions of their own.[27]

This leads us into another famous Pythagorean idea, the 'harmony of the spheres'.

Music of the spheres

Aristotle gives us several passages on the Pythagoreans and music in the heavens. He tells us in the *Metaphysics*:

> Since, again, they saw that the modifications and the ratios of the musical scales were expressible in numbers;-since, then, all other things seemed in their whole nature to be modelled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number. And all the properties of numbers and scales which they could show to agree with the attributes and parts and the whole arrangement of the heavens, they collected and fitted into their scheme; and if there was a gap anywhere, they readily made additions so as to make their whole theory coherent. E.g. as the number 10 is thought to be perfect and to comprise the whole nature of numbers, they say that the bodies which move through the heavens are ten, but as the visible bodies are only nine, to meet this they invent a tenth — the 'counter-earth'.[28]

We do not know how the Pythagoreans applied their music theory to the heavens, only that they did. When asked the question of why we cannot hear this celestial music, they replied that, just as workers in a smithy do not hear the beating of the hammers, we do not hear the celestial music. It is part of the background of the cosmos and we are simply too accustomed to it to be able to hear it.

Later in antiquity, Plato also adopted the idea of a celestial music in his *Republic*. His *Timaeus* is also interesting in that he uses the ratios of Philolaus' music theory to give the spacing between the celestial bodies. Much later than this, in the late sixteenth and early seventeenth centuries, the great astronomer Johannes Kepler took up this idea, this time with the sun in the centre and the earth and planets orbiting it.

Archytas

Archytas of Tarentum (*c.* 428 BCE–*c.* 347 BCE), another early Pythagorean, was important for his work in mathematics, cosmology, and music theory. Plato clearly treats Archytas as a Pythagorean when he says:

> As our eyes are suited to astronomy, so our ears are suited to harmony, for these are brother disciplines, as the Pythagoreans say and we, Glaucon, agree.[29]

There is a similar grouping of the sciences to Archytas. Archytas does not seem to have been interested in the idea of metempsychosis, nor is there any sense of mysticism or numerology. Archytas was extremely concrete in everything he said and clearly regarded the art of calculation, or logistic, as the key science. So Archytas says:

> It seems to me that those concerned with the science make distinctions well and it is by no means surprising that they understand individual entities as they are. Having made good distinctions concerning wholes they are also able to see well how things are according to their parts. Concerning geometry, arithmetic, and spherics he gave clear distinctions and not least concerning music. These sciences seem to be akin.[30]

Elsewhere he states;

> It seems that logistic is far better than the other crafts in respect of wisdom and deals with its topics more concretely than geometry. In those ways in which geometry is lacking logistic utilises demonstration.[31]

Archytas on cosmology

There is an interesting thought experiment in cosmology that is attributed to Archytas which was much debated and was very influential in antiquity.[32] If someone were to stand close to the edge of a finite cosmos and attempted to thrust a staff beyond the edge of the cosmos, what would happen? If they succeed, then this cannot be the limit of space, and so there must be a new limit further on. This thought experiment though is infinitely replicable. Wherever a new edge is supposed we can imagine someone standing close to it and thrusting a staff beyond it. So space must be infinite. One reply is physical and practical, that it is impossible to stand close enough to the edge of the cosmos in this manner and so the idea of thrusting a staff beyond the edge of the cosmos is impossible too. A more subtle reply is that outside the cosmos neither time nor space exist and it is impossible to thrust the staff where there is no time or space. Our intuition that we thrust the staff beyond the edge of the cosmos is incorrect and so space is finite after all.

Archytas and mathematics

Archytas worked on one of the notorious problems for ancient mathematics, the Delian problem, of doubling the volume of a cube.[33] While initially this looks simple, in fact it is very tricky, especially within the confines of the mathematical techniques then known to the ancients. Archytas developed the work of Hippocrates of Chios. If we suppose that L is the length of a side of the original cube, one can then generate a series of ratios such that L:a:: a:b:: b:2L.[34] One can derive the relation $L:2L = L^3:a^3$. As $L^3:a^3$ is in the ratio of 1:2, a^3 is twice L^3, and the cube can be generated with sides of length a. Archytas' solution is too complex to give here. It involved constructing four similar triangles in the proportions suggested by Hippocrates, employing an imaginary rotation of triangles and then calculating of their points of intersection. Archytas' solution is one of the most remarkable pieces of technical mathematics, visualization and mathematical ingenuity in antiquity.

Archytas also demonstrated a very important property of what are known as superparticular ratios. These ratios are of the type where n + 1:n. These were important for Pythagorean musical theory, which used ratios such as 3:2, 4:3, and 9:8. If x bears the same proportion to y as y does to z, then y is the mean proportional of x and z (if x:y:: y:z). A double octave (4:1) can be split into two octaves with a mean proportional as 4:2 is the same proportion as 2:1. Archytas demonstrated that there is no mean proportional for numbers in superparticular ratios. This means that critical musical ratios, such as 3:2, 4:3, and 9:8 (which all have the form n + 1:n) have no mean proportional and cannot be split in to two equal parts.

Archytas and music theory

Archytas produced a variation on Philolaus' musical scale, using 9:8, 8:7, and 28:27 to generate the notes up to the fourth ($9/8 \times 8/7 \times 28/27 = 4/3$). Archytas worked on two other types of scale, in modern terminology the chromatic and the enharmonic. A chromatic scale includes all twelve semitones. The key ratios for Archytas' chromatic scale are 32:27, 243:224 and 28:27 ($32/27 \times 243/224 \times 28/27 = 4/3$). In the chromatic scale, $A^{\#} = B^{b}$. In an enharmonic scale this is not so, and what we would call $A^{\#}$ differs from B^{b}. The key ratios for Archytas' enharmonic scale are 5:4, 36:35, 28:27 ($5/4 \times 36/35 \times 28/27 = 4/3$).

In contrast to Philolaus, who seemed to be producing an ideal scale, Archytas is now generally agreed to have been describing the scales in use during his time.[35] With Philolaus, certain numbers are privileged by either being part of, or directly derivable from, the tetraktys. This is not the case for several of the ratios which Archytas uses, such as 8:7, 28:27, 32:27, 243:224, 5:4, and 36:35. One might argue that to some extent Philolaus had a numerological approach to music theory, while Archytas did not. Archytas also had a physical theory of pitch.[36] The pitch of a sound in this theory is related to how quickly it travels, a sound travelling more quickly being of higher pitch. In fact the speed of sound is a constant for a given medium and it is frequency that is critical to pitch, how rapidly a string vibrates determining the frequency rather than the speed of the sound.

Plato

Plato (428/427–348/347 BCE) is important both as a philosopher and as someone who was interested in and promoted the study of mathematics. In the philosophy of mathematics the view of 'Realism', or 'Platonic Realism' – the idea that numbers have an independent existence apart from the things they count – dates from Plato. Plato used mathematics as a model for other types of knowledge. It is said that the words 'Let no one ignorant of geometry enter here' were inscribed over the entrance to Plato's academy, the research school which he founded. Plato also gave mathematics and geometry a critical role in the education of the guardians of Plato's ideal state.

Whether Plato was himself a Pythagorean, or whether some of his dialogues should be considered as Pythagorean, has been a matter of some debate. Plato was clearly knowledgeable about both Pythagoras and Pythagorean ideas and it is clear he was to some extent influenced by Pythagorean ideas. It is now widely accepted that it would be misleading to consider Plato an outright Pythagorean, and that while there is an influence, it would also be incorrect to consider any of his dialogues to be simply Pythagorean, either in derivation or in content. In the twentieth century the classicist A.E. Taylor held that Plato's *Timaeus* was derived from Pythagorean sources, but this view is now largely discarded.[37] It is very important in considering the relation between Plato and the Pythagoreans not to use blanket terms like 'number mysticism'. The dominance of positivist and empiricist ideas in the twentieth century, with its rejection of anything which was not either logically true or empirically verifiable, tended to blur the differences between Pythagorean and Platonic approaches to number.

Let us begin with some important differences between Plato and the Pythagoreans. Plato did not accept the notion of the soul as a harmony.[38] In terms of cosmology, the cosmos of Plato's *Timaeus* is finite and bounded where Archytas argued for an infinite, unbounded cosmos.[39] The *Timaeus* also has a very different account of the number, nature, motions, and organization of the celestial bodies from that of Philolaus.[40] There are two other important structural differences between Plato and the Pythagoreans in approaching cosmology, the first of which we have seen something of already.

One can broadly categorize the Pythagorean approach as arithmetical.[41] That is they were interested in the relation of numbers to the world. Plato had a much more geometrical conception of the cosmos. There are the 1, 1, $\sqrt{2}$, and 1, $\sqrt{3}$, 2 triangles from which the cubes of earth and the tetrahedra, octahedra, and ikosahedra of fire, air, and water are formed (see Figure 2.5). It is these shapes that form the basis of Plato's cosmos, not numbers themselves.

A second issue is the generation of the musical scale. Philolaus' scale uses the tetraktys of 1, 2, 3, and 4 to generate its ratios, the justification being that these are part of the tetraktys and that $1 + 2 + 3 + 4 = 10$, the Pythagorean perfect number. While the Pythagoreans have ten heavenly bodies, Plato simply accepts there are seven heavenly bodies (moon, sun, five naked-eye planets) and has seven terms as basic to his musical scale (1, 2, 3, 4, 8, 9, 27),[42] which are the

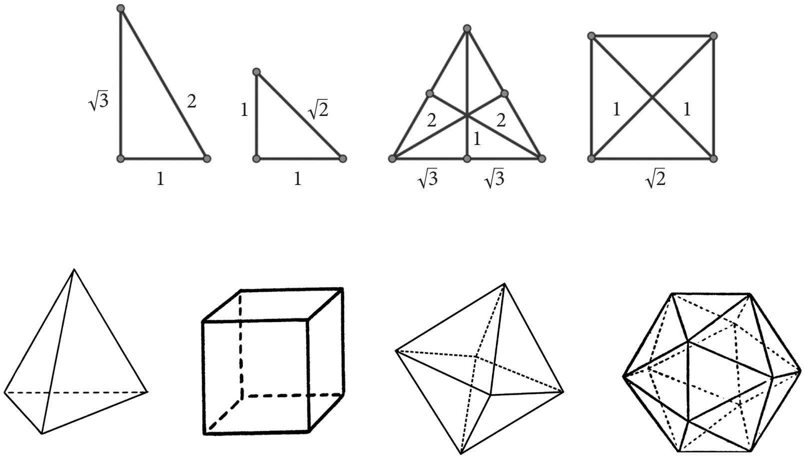

Figure 2.5 (a) Plato's two fundamental triangles, which can be used to construct the equilateral triangle, square, and thence (b) the tetrahedron, cube, octahedron, and icosahedron.

relative lengths of the soul stuff that the demiurge, Plato's geometer/craftsman god, uses to fashion the orbits for these bodies.[43] Plato then generates a tone and semitone scale from these terms.[44] Again, the derivation is geometrical (dividing the soul stuff into circles) rather than purely arithmetical as with the Pythagoreans. So while the Pythagoreans have a numerological derivation of cosmology and of music, Plato has a cosmological derivation of music.

A very important part of Plato's thinking in his *Timaeus* is the idea of the demiurge who organizes the best possible cosmos from a pre-existing chaos. In order to make the cosmos good and comprehensible to humans, this god imposes number and geometrical figure onto the chaos, which is why, for example, the fundamental particles for Plato have geometrical form. One important aspect of this cosmology is that the demiurge needs criteria for all that he does and he finds those criteria in mathematics and geometry. While we might find it strange that one form of triangle should be better than another, Plato did not:

> This we hypothesise as the principle of fire and of the other bodies . . . but the principles of these which are higher are known only to God and whoever is friendly to him. It is necessary to give an account of the nature of the four best bodies, different to each other, with some able to be produced out of the others by dissolution . . . We must be eager then to bring together the best four types of body, and to state that we have adequately grasped the nature of these bodies. Of the two triangles the isosceles has one nature, the scalene an unlimited number. Of this unlimited number we must select the best, if

we intend to begin in the proper manner. If someone has singled out anything better for the construction of these bodies, his victory will be that of a friend rather than an enemy. We shall pass over the many and postulate the best triangles.[45]

If we ask how the heavens should be arranged, what the ratios should be between the orbits of the planets, then Plato's answer is that the demiurge does this using a musical scale. That produces the best arrangement and one that is mathematically comprehensible to humans. This line of thought was both important and long lasting. One can find Johannes Kepler following it in the late fifteenth and early sixteenth centuries. For Kepler, the question is why the orbits of the planets have certain ratios, and why there are this many planets. A second issue is why the elliptical orbits of the planets have specific eccentricities and the planets specific speeds. His answer to the first question is geometrical. The planets have the same ratios as constructions of the five known Platonic solids, and there are this number of planets because there are this number of Platonic solids. His answer to the second question is musical. The planets have these eccentricities and speeds because those values conform to a celestial music. For an explicitly Christian aspect to Kepler's astronomy, see Owen Gingerich's Chapter 4 in this volume.

The early Pythagoreans and cosmic number

The early Pythagoreans are different from Plato and Kepler in that they did not believe in a creator god organizing the cosmos, as far as we can tell from the evidence. So there is no issue for them concerning the criteria by which a creator god did this, even if there was for figures like Plato and Kepler. However, they were interested in the application of number and music theory to the cosmos and how that might make the cosmos good, and make the cosmos comprehensible to humans. Philolaus says that:

Nature in the cosmos was fitted together out of unlimited and limited things, both the whole cosmos and the things in it.[46]

Who or what fitted the cosmos together from unlimited and limited things is not an issue here.[47] The key is that there is a difference between a pre-cosmic state and the cosmos for the Pythagoreans and it relates to harmonization. Philolaus elsewhere states that:

The first thing to be fitted together, the one, is in the middle of the sphere and is called the hearth.[48]

In order for a cosmos to be a proper cosmos, it must have a mathematical structure. That may not be quite so strong or explicit as in Plato here, but nevertheless there is still an important sense of it. I have translated the Greek somewhat conservatively here as 'to fit together', but the Greek word also has a musical sense of to bring into tune, as in to tune an instrument, or even to compose music. It also has a moral sense above simply (as, say with a carpenter) an appropriate or suitable fitting together. Another statement by Philolaus is also important here:

> Concerning nature and harmony they hold this: The being of objects, eternal being and nature itself are susceptible to divine but not human knowledge, although it is not possible for any of the things which are and are known by us to have been generated if the things out of which the cosmos was put together, both the limited and the unlimited things, had not existed beforehand. However, as these origins did exist and were neither alike nor of the same kind, it would not have been possible for them to have been ordered if harmony had not, in some way, been applied to them. On the one hand like things and things of the same kind were not bonded by harmony, while unlike things, things not of the same kind and things not corresponding in order needed to be closed up tightly in harmony if they were to remain held fast in order.[49]

And further:

> All things which are known have number. Without this, it is not possible for anything at all to be understood or known.[50]

So harmony is critical to the structure of the cosmos and number is critical for the possibility of human knowledge of the cosmos.

Let us return to some aspects of numerology that we discussed earlier. The Pythagoreans, and indeed Plato, have been said to employ numerology and 'number mysticism'. In a sense that is true, as both employ numbers in their account of the cosmos in a way that would not be recognized as mathematical physics today. They use privileged numbers and attempt to say how these led to, or constitute a good arrangement of, the world. However, this was not a simple or primitive numerology. They had important philosophical reasons for their application of number. They needed to explain how the cosmos was good, that is in good order and aesthetically good, a standard assumption among the ancients, and how humans could have knowledge of the cosmos. From Plato onwards, there is an assumption of a god organizing the cosmos. For each of the actions of that god there has to be a reason and some of those reasons are supposed to be mathematical or geometrical.[51] This is a far cry from the modern view of an accidental universe where we fit mathematics to what we observe as best we can. It is important to understand that the ancients asked a different question about cosmology, which was how has this all come about for the best, given that the cosmos is so congenial to human beings and appears comprehensible to humans as well?

Plato and a Pythagorean code?

One interesting recent development on the relation between Plato and Pythagoras has been the work of historian of science Jay Kennedy.[52] His claim is that there is a musical structure to each of Plato's works, based on Pythagorean principles, and that this structure effectively contains a code, revealing information about Pythagoras and about Plato's own views. Kennedy claims that Plato organized his work stichometrically, that is he was aware of the number lines in each of his works and that Plato divided each of his works into 12 parts, corresponding to the 12 notes in a musical scale. The claim is then that Plato had means of indicating the transition from one 12th to another, by making a reference to say, divine justice, or a speech by one of his characters may begin at a 12th part of a work. There is also supposed to be a harmonic organization to Plato's works based on a 12-note division of the octave. It is claimed that Plato writes predominantly of ideas he supports at harmonious parts of the scale and predominantly of ideas he does not approve of at dissonant parts. Kennedy claims that Plato's works are 'fundamentally Pythagorean' and that we can find encoded information about Pythagoras in them. Kennedy supports these theses with statistical analysis which at first sight certainly appears impressive in its breadth and claimed results.

If these claims are verified, they will radically change our understanding of both Pythagoras, Plato, and the way that Plato used mathematics. However, this is a big 'if'. There has been considerable debate concerning the methods and assumptions that Kennedy has used. It is not clear that the musical structure Kennedy finds in Plato is an appropriate one for that period. Effectively, Kennedy finds a 12ET structure where one would expect a Pythagorean structure. On the statistical side, there is an important difference between claims for accuracy, which Kennedy makes, and claims for statistical significance. So if I make a similar claim about Shakespeare, that he divided his works into 12 parts and marked the transition from one part to another with the word 'and', this would no doubt be accurate in the sense that 'and' will occur in these places, but not statistically significant. The real test for Kennedy's work would be running proper tests for statistical significance and it is by no means clear that Kennedy's claims would survive such tests.

Conclusion

The idea that Pythagoras himself was an important mathematician, that he discovered or proved Pythagoras' theorem, that he discovered the mathematical ratios underpinning music, that he thought in terms of a world constituted from number or was the originator of the idea of music of the heavenly spheres, has now been rejected. In its place we have a Pythagoras who was primarily interested in the fate of the soul after death, an expert on religious ritual and founder of a religious sect, but who may have been interested in and made some contribution to mathematics as well. Recent changes in how we understand the relation of science and religion and the relation of science and magic can accommodate this new, more interesting picture, one that is more firmly based in the reliable evidence we have about Pythagoras.

The early Pythagoreans, as I have stressed, had a wide diversity of views. Philolaus and Archytas seem much less interested in the fate of the soul and much more interested in mathematical music theory. Philolaus seems to have based his theory firmly in the numbers of the tetraktys, while Archytas seems to have been more interested in what contemporary musicians actually played and how that might be described mathematically. The two broad groups of followers of Pythagoras, the *akousmatikoi* and the *mathematikoi* – the listeners and the learners – also seem to have taken quite different approaches to the teachings of Pythagoras and to the study of mathematics.

Plato is very interesting, both as a figure in the history and philosophy of mathematics in himself and as someone who was influenced by the early Pythagorean tradition. Plato also illuminates some aspects of the Pythagorean use of number in thinking about the structure of the cosmos and how humans can gain knowledge of the cosmos.

It is very important when we look at the early Pythagoreans to place them in their philosophical and scientific context. If we take what they were doing as answers to modern questions, then they come across as interested in an odd number mysticism. If we understand them as looking at ancient questions on the nature of the cosmos, without the benefit of an understanding of modern scientific methods, then they have some interesting and diverse things to say about the application of mathematics to understanding how the cosmos is good, why a god may have organized the cosmos in this way, and how we humans can have some knowledge of that cosmos.

Notes and references

1. The classic discussion here is W. Burkert, *Lore and Science in Ancient Pythagoreanism*, Harvard University Press, 1972, but see also L. Zhmud, *Pythagoras and the Early Pythagoreans*, Oxford University Press, 2012, for the most recent contribution.
2. See C. Huffman, The Pythagorean tradition, in A.A. Long (ed.), *The Cambridge Companion to Early Greek Philosophy*, Cambridge University Press, 1999, 66–87 and also Burkert, *Lore and Science* and Zhmud, *Pythagoras and the Early Pythagoreans*.
3. Burkert, *Lore and Science*.
4. D.W. Graham, *The Texts of Early Greek Philosophy*, Cambridge University Press, 2010 is an excellent recent collection with English translation. On the reports of Plato and Aristotle, and other sources, the classic work is Burkert, *Lore and Science*.
5. Plato, *Republic*, 531a, 600a, *Sophist*, 242c, Aristotle, *Metaphysics* I/v, *De Caelo* II/xiii.
6. Plato, *Republic*, 600b.
7. Diogenes Laertius VIII, 36, Xenophanes Fragment 7. The fragments noted here and elsewhere in this chapter can be found in D.W. Graham, *The Texts of Early Greek Philosophy*, Cambridge University Press, 2010, which is an excellent recent collection with English translation, while G.S. Kirk, J.E. Raven, and M. Schofield, *The Presocratic Philosophers*, Cambridge University Press, 2nd edn, 1983 is the most readily available collection, again with an English translation.
8. Homer, *Odyssey*, XI 489.
9. Cf. Huffman, The Pythagorean tradition, pp. 70–1.
10. S.M. Shirokogoroff, *Psychomental Complex of the Tungus Trubner*, K. Paul, Trench, 1935, p. 269.
11. *Busiris*, 28.
12. See P. Kingsley, 1995, *Ancient Philosophy, Mystery and Magic*, Clarendon Press, 1995.
13. See here Huffman, The Pythagorean tradition, pp. 66–87; Zhmud, *Pythagoras and the Early Pythagoreans*.
14. Pseudo-Plutarch, I, 3, 8.
15. See A.D. Gregory, *Plato's Philosophy of Science*, Duckworth, 2000, pp. 238 ff.
16. K.R. Popper, The nature of philosophical problems and their roots in science, *The British Journal for the Philosophy of Science*, **3**(10), 1952, 124–56, pp. 147–8.

17. See Huffman, The Pythagorean tradition, pp. 66–87, Zhmud, *Pythagoras and the Early Pythagoreans*.
18. Aristotle, *Metaphysics*, I/8, trans. Ross.
19. Aristotle, *Metaphysics*, XIII/6, 1080b16–22.
20. Aristotle Fragment, 191.
21. Aristotle Fragment, 191.
22. P. Kingsley, *Ancient Philosophy*.
23. F.M. Cornford, Mysticism and science in the Pythagorean tradition, *The Classical Quarterly*, **16**(3/4), Jul.–Oct., 1922, 137–50, p. 139.
24. There is no evidence that the Pythagoreans ever used this sort of numerology. See Burkert, *Lore and Science*; A.D. Gregory, *The Presocratics and the Supernatural*, Bloomsbury, 2013.
25. D. Stove, *The Plato Cult and Other Philosophical Follies*, Blackwell, 1991, chapter 7.
26. Concern about Friday 13th seems to be a largely twentieth-century phenomenon and due to an amalgamation of 13 as an unlucky number and Friday as an unlucky day.
27. Aristotle, *On the Heavens*, II/13, trans. Stocks.
28. Aristotle, *Metaphysics*, I/5, trans. Ross, cf. Aristotle, *On the Heavens* II/13 and Aristotle, *On the Heavens*, III/1, trans. Stocks: 'The same consequences follow from composing the heaven of numbers, as some of the Pythagoreans do who make all nature out of numbers.'
29. Plato, *Republic*, 530d.
30. Archytas Fragment 1.
31. Archytas Fragment 4.
32. Simplicius *Physics Commentary*, 467, 26 ff.
33. See T.L. Heath, *A History of Greek Mathematics*, 2 vols, Clarendon Press, 1921; I. Mueller, Greek arithmetic, geometry and harmonics: Thales to Plato, in C.C.W. Taylor (ed.), *Routledge History of Philosophy Vol. I: From the Beginning to Plato*, Routledge, 2001, pp. 271–322; C. Huffman, *Archytas of Tarentum: Pythagorean, Philosopher and Mathematician King*, Cambridge University Press, 2010.
34. L is in proportion to a, as a is to b, as b is to 2L.
35. See A. Barker, *Greek Musical Writings*, Cambridge University Press, 1989, p. 50; C. Huffman, *Archytas of Tarentum*, pp. 63–4 and 419–20.
36. See Barker, *Greek Musical Writings*; Huffman, *Archytas of Tarentum*.
37. A.E. Taylor, *A Commentary on Plato's Timaeus*, Clarendon Press, 1928.
38. Plato, *Phaedo*, 86b ff.
39. See Simplicius, *Commentary on Aristotle's Physics*, 467, 26.
40. See Aristotle *de Caelo*, 293a18ff., Aetius II, 7,7 ff.

41. See K.R. Popper, *The Open Society and its Enemies*, Routledge and Kegan Paul, 1945, p. 248 note 9 and Popper, The nature of philosophical problems, p. 87ff.; also R. Wright, *Cosmology in Antiquity*, Routledge, 1995, p. 54.
42. Plato, *Timaeus*, 35c.
43. Plato, *Timaeus*, 36d.
44. Plato, *Timaeus*, 35d.
45. Plato, *Timaeus*, 53d4–54a6.
46. Philolaus Fragment 1, cf. Fragment 2, esp. 'it is clear that the cosmos and the things in it were fitted together by limited and unlimited things'.
47. Or indeed whether it fitted itself together in the manner of the Milesians.
48. Philolaus, Fragment 7.
49. Philolaus, Fragment 6.
50. Philolaus, Fragment 4, cf. Fragment 3, esp. 'there will not be anything that is going to know if everything is unlimited'.
51. The *locus classicus* here is Plato's *Timaeus*, see A.D. Gregory and R. Waterfield, *Plato: Timaeus and Critias*, Oxford University Press, 2008, and see also Chapter 4 in this volume on Kepler.
52. J. Kennedy, Plato's forms, pythagorean mathematics, and stichometry, *Apeiron* **43**(1), 2010, 1–31, *The Musical Structure of Plato's Dialogues*, Acumen Publishing, 2011; cf. A.D. Gregory, Kennedy and stichometry: some methodological considerations, *Apeiron*, **45**(2), 157–79.