Chapter 1

Syzygies and minimal resolutions

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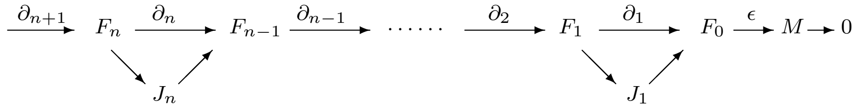
The essence of linear algebra over a field resides in the fact that every vector space is free; that is, has a spanning set of linearly independent vectors. The study of linear algebra over more general rings attempts to approximate this situation by the method of free resolutions. When a module M is not free we make a first approximation to its being free by taking a surjective homomorphism $\epsilon : F_0 \to M$ where F_0 is free to obtain an exact sequence

$$0 \to J_1 \to F_0 \stackrel{\epsilon}{\to} M \to 0.$$

Repeating the construction we approximate J_1 in turn by a free module to obtain an exact sequence $0 \to J_2 \to F_1 \to J_1 \to 0$. Iterating and splicing we obtain a *free resolution of* M in the sense of Hilbert [2]



We study the relationship between the intermediate modules J_n, the so-called *syzygies of* M, and those free resolutions of M which are in some sense minimal.

1. Introduction:

The notion of *free resolution* has its origin in the classical theory of invariants [2] and the study of graded modules over polynomial rings $\mathbf{F}[x_1, \ldots, x_n]$ where $\mathbf{F}$ is a field. In this context there is a well defined notion of *minimal free resolution.* Such minimal resolutions have a strong uniqueness property; not only are they themselves unique up to isomorphism but in addition any other free resolution is a direct sum of the minimal free resolution with a free acyclic complex. In [1], Eilenberg gave an extension of this uniqueness property by essentially formal arguments. However, despite the elegance of Eilenberg's approach, its scope remains relatively narrow.

The main technical limitation of Eilenberg's theory arises from his definition of minimality. This places so strong a restriction on the class of rings to which it may be applied as to render it *a priori* inapplicable to many cases of interest. Consequently we are forced to reformulate matters in a rather more general context.

Our primary notion is that of a *special class* $\mathfrak{S}$ of projectives in an abelian category $\mathfrak{A}$; the precise formulation is given in §6. Suffice to say here that $\mathfrak{S}$ plays a role analogous to that of finitely generated stably free modules over a ring. For an object $M \in \mathfrak{A}$ we consider $\mathfrak{S}$-*resolutions of* M, that is, exact sequences in $\mathfrak{A}$ of the form

$$\mathbf{S} \;=\; (\cdots \stackrel{\partial_{n+1}}{\to} S_n \stackrel{\partial_n}{\to} \cdots \to S_1 \stackrel{\partial_1}{\to} S_0 \to M \to 0)$$

where each $S_r \in \mathfrak{S}$. To such a resolution we may add a $\mathfrak{S}$-resolution of 0

$$\mathbf{T} \;=\; (\cdots \to T_n \to \cdots \to T_1 \to T_0 \to 0)$$

to obtain another $\mathfrak{S}$-resolution $\mathbf{S} \oplus \mathbf{T}$ of M thus

$$\mathbf{S} \oplus \mathbf{T} \;=\; (\cdots \to S_n \oplus T_n \to \cdots \to S_1 \oplus T_1 \to S_0 \oplus T_0 \to M \to 0).$$

$\mathbf{S}$ is said to be *minimal* when, for any $\mathfrak{S}$-resolution $\mathbf{S}'$ there exists a commutative diagram

$$\left(\begin{array}{ccccccccc} \cdots \stackrel{\partial'_{n+1}}{\to} & S'_n & \stackrel{\partial'_n}{\to} & \cdots\cdots \stackrel{\partial'_1}{\to} & S'_0 & \stackrel{\eta}{\to} & M & \to 0 \\ & \varphi_n \downarrow & & & \varphi_0 \downarrow & & \downarrow \mathrm{Id}_M & \\ \cdots \stackrel{\partial_{n+1}}{\to} & S_n & \stackrel{\partial_n}{\to} & \cdots\cdots \stackrel{\partial_1}{\to} & S_0 & \stackrel{\epsilon}{\to} & M & \to 0 \end{array}\right)$$

where each φ_r is epimorphic. When they exist, minimal resolutions are unique in the following sense:

Theorem A : Let $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ be $\mathfrak{S}$-resolutions of M; if $\mathbf{S}$ is minimal then $\widetilde{\mathbf{S}} \cong \mathbf{S} \oplus \mathbf{T}$ for some $\mathfrak{S}$-resolution $\mathbf{T}$ of 0. In particular, if $\widetilde{\mathbf{S}}$ is also minimal then $\widetilde{\mathbf{S}} \cong \mathbf{S}$.

In applications the requirement that M has an $\mathfrak{S}$-resolution is usually a very strong restriction. We may relax it by considering partial $\mathfrak{S}$-resolutions or n-stems. Thus an n-*stem* over M is an exact sequence in $\mathfrak{A}$ of the form

$$\mathbf{S} \;=\; (S_n \stackrel{\partial_n}{\to} \cdots \to S_1 \stackrel{\partial_1}{\to} S_0 \to M \to 0)$$

where each $S_r \in \mathfrak{S}$. The n-stem $\mathbf{S}^{(n)}$ is *minimal* when, for any n-stem $\widetilde{\mathbf{S}}^{(n)}$ there exists a commutative diagram

$$\left(\begin{array}{ccccccc} \widetilde{S}_n \stackrel{\widetilde{\partial_n}}{\to} & \cdots\cdots \stackrel{\widetilde{\partial_1}}{\to} & \widetilde{S}_0 & \stackrel{\widetilde{\eta}}{\to} & M & \to 0 \\ \varphi_n \downarrow & & \varphi_0 \downarrow & & \downarrow \mathrm{Id}_M & \\ S_n \stackrel{\partial_n}{\to} & \cdots\cdots \stackrel{\partial_1}{\to} & S_0 & \stackrel{\epsilon}{\to} & M & \to 0 \end{array}\right)$$

in which each φ_r is epimorphic. For n-stems, Theorem A is modified to:

Theorem B : If $\mathbf{S}^{(n)}$, $\widetilde{\mathbf{S}}^{(n)}$ are n-stems over M and $\mathbf{S}^{(n)}$ is minimal then $\widetilde{\mathbf{S}}^{(n-1)} \cong_{\mathrm{Id}_m} \mathbf{S}^{(n-1)} \oplus \mathbf{T}^{(n-1)}$ for some (n-1)-stem $\mathbf{T}^{(n-1)}$ over 0.

If $\mathbf{S}^{(n)} = (S_n \stackrel{\partial_n}{\to} \cdots \to S_1 \stackrel{\partial_1}{\to} S_0 \to M \to 0)$ is an n-stem its *syzygies* are the intermediate objects $(J_r)_{1 \le r \le n}$ obtained via the canonical decomposition of ∂_r as the composition of a monomorphism i_r and an epimorphism p_r thus:

$$\begin{array}{ccc} S_r & \xrightarrow{\ \ \partial_r\ \ } & S_{r-1} \\ & {\scriptstyle p_r} \searrow \quad \nearrow {\scriptstyle i_r} & \\ & J_r & \end{array}$$

Minimality also implies a relation amongst syzygies. If J, $\widetilde{J} \in \mathfrak{A}$ we say that $\widetilde{J}$ *splits over* J when $\widetilde{J} \cong J \oplus T$ for some $T \in \mathfrak{S}$; we will prove:

Theorem C : Let $\mathbf{S}^{(n)}$ and $\widetilde{\mathbf{S}}^{(n)}$ be n-stems over M having syzygies $(J_r)_{1 \le r \le n}$ $(\widetilde{J}_r)_{1 \le r \le n}$ respectively; if $\mathbf{S}^{(n)}$ is minimal then $\widetilde{J}_r$ splits over J_r for $1 \le r \le n-1$.

2. Some categorical preliminaries :

We assume familiarity with the notions of *category* and *functor* ([5]). We denote by $\mathcal{A}$b the category of abelian groups and homomorphisms. In what follows we shall work with subcategories $\mathfrak{A}$ of $\mathcal{A}$b which satisfy certain tameness conditions. These are defined formally below. However, it is instructive to consider them as they relate to two basic examples; thus suppose that Λ is a ring and consider

$\mathcal{M}\mathrm{od}_\Lambda$: the category of right Λ-modules and Λ-homomorphisms:

By a graded Λ-module we mean a Λ-module M given as a direct sum $M = \bigoplus_{n \ge 0} M_n$ where each M_n is a Λ-submodule. A graded homomorphism $f : M \to N$ between two such graded modules is then a Λ-homomorphism satisfying $f(M_n) \subset N_n$ for each n and we may form

$\mathcal{G}(\Lambda)$: the category of *graded* right Λ-modules and Λ-homomorphisms.

Observe that $\mathcal{M}\mathrm{od}_\Lambda$ may be regarded as the subcategory of $\mathcal{G}(\Lambda)$ consisting of graded modules in which $M_r = 0$ for $r > 0$. In turn, $\mathcal{G}(\Lambda)$ may be regarded as a subcategory of $\mathcal{A}\mathrm{b}$ by forgetting both the grading and the Λ-structure. In the above examples the following notions are well defined;

(i) Zero ; **(ii)** Kernels; **(iii)** Images; **(iv)** Exact sequences;**(v)** Quotients.

In either case the nature of 'zero' should be obvious. Any module has a zero and hence a zero submodule. When $f : M \to N$ is a Λ-homomorphism

$$\mathrm{Ker}(f) = \{\mathbf{x} \in M \mid f(\mathbf{x}) = 0\} \ ; \ \mathrm{Im}(f) = \{f(\mathbf{x}) \mid \mathbf{x} \in M\}.$$

Then $\mathrm{Ker}(f)$ is a submodule of M and $\mathrm{Im}(f)$ a submodule of N. Moreover if f is a graded homomorphism then $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are graded by

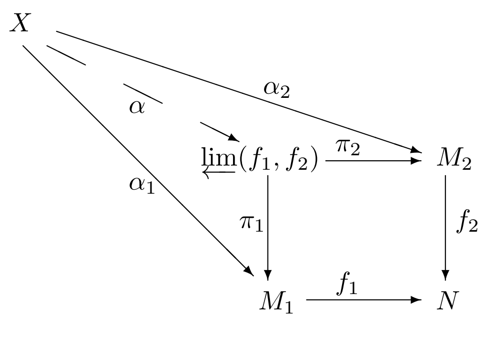
$$\mathrm{Ker}(f)_n = \mathrm{Ker}(f) \cap M_n \ ; \ \mathrm{Im}(f)_n = \mathrm{Im}(f) \cap N_n.$$

A sequence of morphisms $A_1 \stackrel{\alpha_1}{\to} A_2 \stackrel{\alpha_2}{\to} \dots \stackrel{\alpha_{n-1}}{\to} A_n \stackrel{\alpha_n}{\to} A_{n+1}$ is said to be *exact* when $\mathrm{Ker}(\alpha_{r+1}) = \mathrm{Im}(\alpha_r)$ for $1 \le r \le n-1$. If $K \subset M$ is a Λ-submodule the quotient group M/K admits a natural Λ-module structure. Moreover, if K is a graded submodule of the graded module M then M/K is graded by $(M/K)_n = M_n/K_n$. One may also construct

(vi) Pullbacks ; **(vii)** Direct products ; **(viii)** Pushouts ; **(ix)** Direct sums

We first recall the definitions. If $\mathfrak{A}$ is a category and $f_i : M_i \to N$ are morphisms in $\mathfrak{A}$ $(i = 1, 2)$ then by a *pullback* for f_1, f_2 we mean an object $\varprojlim(f_1, f_2)$ in $\mathfrak{A}$ together with morphisms $\pi_i : \varprojlim(f_1, f_2) \to M_i$ such that $f_1 \circ \pi_1 = f_2 \circ \pi_2$ which possess the following *universal property*:

If $\alpha_i : X \to M_i$ are morphisms in $\mathfrak{A}$ such that $f_1 \circ \alpha_1 = f_2 \circ \alpha_2$ then there exists a unique morphism $\alpha : X \to \varprojlim(f_1, f_2)$ making the following diagram commute



When $\varprojlim(f_1, f_2)$ exists the uniqueness condition on α guarantees that $\varprojlim(f_1, f_2)$ is unique up to isomorphism in $\mathfrak{A}$. We say that $\mathfrak{A}$ *has pullbacks* when $\varprojlim(f_1, f_2)$ exists for any pair of morphisms $f_i : M_i \to N$ $(i = 1, 2)$. In $\mathcal{M}\mathrm{od}_\Lambda$ pullbacks are defined by

$$\varprojlim(f_1, f_2) \;=\; \{(m_1, m_2) \in M_1 \times M_2 \mid f_1(m_1) \;=\; f_2(m_2)\}$$

where $\pi_i : \varprojlim(f_1, f_2) \;\to M_i$ is the obvious projection map. Moreover, in the special case where $N \;=\; 0$ the pullback construction simply yields the *direct product* $M_1 \times M_2$ showing that any pullback $\varprojlim(f_1, f_2)$ is a submodule of $M_1 \times M_2$. Note that in $\mathcal{G}(\Lambda)$ a direct product $M \times M'$ of graded modules admits a grading given by $(M \times M')_r \;=\; M_r \times M'_r$ which in turn induces a grading on any pullback contained therein.

Pushout is the dual notion to pullback. Here it is useful to recall that if $\mathfrak{A}$ is a category the *dual category* $\mathfrak{A}^*$ has the same objects and morphisms as $\mathfrak{A}$ but with the direction of all arrows reversed. One says that $\mathfrak{A}$ has *pushouts* when the dual category $\mathfrak{A}^*$ has pullbacks. Thus if $f_r : N \to M_i$ $(r \;=\; 1, 2)$ are morphisms in $\mathfrak{A}$ by a *pushout* for f_1, f_2 we mean an object $\varinjlim(f_1, f_2)$ in $\mathfrak{A}$ together with morphisms $i_r : M_r \to \varinjlim(f_1, f_2)$ such that $i_1 \circ f_1 \;=\; i_2 \circ f_2$ which possess the universal property which is dual to pullback. When $\varinjlim(f_1, f_2)$ exists the uniqueness condition on α again guarantees that $\varinjlim(f_1, f_2)$ is unique up to isomorphism. In the special case where $N \;=\; 0$ the pushout construction yields the *direct sum* $M_1 \oplus M_2$. In both $\mathcal{M}\mathrm{od}_\Lambda$ and $\mathcal{G}(\Lambda)$ the direct sum $M_1 \oplus M_2$ coincides with the direct product $M_1 \times M_2$ with the canonical injections $i_r : M_r \to M_1 \oplus M_2$

$$i_1(\mathbf{x}) \;=\; (\mathbf{x}, 0) \;\; ; \;\; i_2(\mathbf{x}) \;=\; (0, \mathbf{x}).$$

In $\mathcal{M}\mathrm{od}_\Lambda$ $\varinjlim(f_1, f_2) \;=\; (M_1 \oplus M_2)/\mathrm{Im}(f_1 \oplus -f_2)$. Note that this module has a natural grading when f_1, f_2 are graded homomorphisms so that $\mathcal{G}(\Lambda)$ also has pushouts.

In what follows we work with categories $\mathfrak{A}$ in which the above notions **(i) - (ix)** are all present. Recall that a category $\mathfrak{A}$ is said to be *abelian* (cf [4], [5]) when the following properties **(I)**, **(II)**. **(III)** hold†:

(I) $\mathfrak{A}$ has a zero object;

(II) $\mathfrak{A}$ has pullbacks and every monomorphism is a kernel;

(III) $\mathfrak{A}$ has pushouts and every epimorphism is a cokernel.

In any abelian category $\mathfrak{A}$ we define an addition on all $\mathrm{Hom}_{\mathfrak{A}}(A, B)$ thus:

$$+ \;:\; \mathrm{Hom}_{\mathfrak{A}}(A, B) \times \mathrm{Hom}_{\mathfrak{A}}(A, B) \;\to\; \mathrm{Hom}_{\mathfrak{A}}(A, B) \;\; ; \;\; f{+}g \;=\; (f, g){\circ}\delta$$

where $\delta : A \to A \oplus A$ is the diagonal and the morphism $(f, g) : A \oplus A \to B$

†We note (cf [5] Chapter 1) that there are many apparently different, though equivalent, ways of defining the notion of *abelian category*.

is induced from $f : A \to B$ and $g : A \to B$ by regarding $A \oplus A$ as a pushout. When $\mathfrak{A}$ is an abelian category we have the following additivity property whose proof is left as an exercise:

(x) $\mathrm{Hom}_{\mathfrak{A}}(A, B)$ is naturally an abelian group for any $A, B \in \mathfrak{A}$.

Recall that a category $\mathfrak{A}$ is said to be *small* when its objects form a set rather then merely a class. In this context, we note the following theorem of Lubkin ([4], [5])

Theorem 2.1 : If $\mathfrak{A}$ is a small abelian category there is a functor $\iota : \mathfrak{A} \to \mathcal{A}\mathrm{b}$ which preserves addition, exact sequences and for which $\mathrm{Hom}_{\mathfrak{A}}(A, B) \overset{i_*}{\rightarrowtail} \mathrm{Hom}_{\mathrm{Ab}}(\iota(A), \iota(B))$ is injective for all $A, B \in \mathfrak{A}$.

Lubkin's Theorem has the practical consequence that diagrams in any abelian category can be regarded simply as diagrams of additive abelian groups and homomorphisms; we take advantage of this in what follows.

By a *tame category* we mean one which is equivalent to a small subcategory of $\mathcal{A}\mathrm{b}$ and which is *abelian* in the above sense. In consequence we see that every small abelian category is tame. Evidently $\mathcal{M}\mathrm{od}_\Lambda$ and $\mathcal{G}(\Lambda)$ are abelian categories. However, without some size restriction on the objects neither category is tame. One especially convenient restriction is to consider only rings Λ which are countable. We then denote by $\mathcal{M}\mathrm{od}_\Lambda^\infty$ the full subcategory of $\mathcal{M}\mathrm{od}_\Lambda$ consisting of *countably generated* modules. Likewise $\mathcal{G}^\infty(\Lambda)$ will denote the full subcategory of $\mathcal{G}(\Lambda)$ whose underlying modules are countably generated. It follows easily that:

Proposition 2.2: If Λ is a countable ring then $\mathcal{M}\mathrm{od}_\Lambda^\infty$ and $\mathcal{G}^\infty(\Lambda)$ are tame abelian categories.

3. Splitting and projectives :

In what follows, $\mathfrak{A}$ will denote a tame abelian category. We recall the following basic result, the *Five Lemma* which, via Lubkin's Theorem, it suffices to prove in $\mathcal{A}\mathrm{b}$.

(3.1) Suppose given a commutative diagram in $\mathfrak{A}$ with exact rows

$$\begin{array}{ccccccccc} A_1 & \overset{\alpha_1}{\to} & A_2 & \overset{\alpha_2}{\to} & A_3 & \overset{\alpha_3}{\to} & A_4 & \overset{\alpha_4}{\to} & A_5 \\ f_1\downarrow & & f_2\downarrow & & f_3\downarrow & & f_4\downarrow & & f_5\downarrow \\ B_1 & \overset{\beta_1}{\to} & B_2 & \overset{\beta_2}{\to} & B_3 & \overset{\beta_3}{\to} & B_4 & \overset{\beta_4}{\to} & B_5 \end{array}$$

If f_1, f_2, f_4 and f_5 are all isomorphisms then f_3 is also an isomorphism.

Given objects A, $C \in \mathfrak{A}$ there are a canonical morphisms $i_A : A \to A \oplus C$ and $\pi_C : A \oplus C \to C$ allowing the construction of the *trivial exact sequence*

$$\mathcal{T} \;=\; (0 \to \;\; A \;\; \xrightarrow{i_A} \;\; A \oplus C \;\; \xrightarrow{\pi_C} \;\; C \;\; \to 0).$$

An exact sequence $\mathcal{E} \;=\; (0 \to C \xrightarrow{i} B \xrightarrow{p} A \to 0)$ in $\mathfrak{A}$ is said to *split* when it is isomorphic to the trivial exact sequence by means of a commutative diagram as follows:

$$\begin{array}{ccccccc} 0 \to & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \to 0 \\ & \downarrow \mathrm{Id}_A & & \downarrow \psi & & \downarrow \mathrm{Id}_C & \\ 0 \to & A & \xrightarrow{i_A} & A \oplus C & \xrightarrow{\pi_C} & C & \to 0. \end{array}$$

It follows from the Five Lemma that such a splitting ψ is necessarily an isomorphism. We say that $\mathcal{E}$ *splits on the left* when there exists a morphism $r : B \to A$ such that $r \circ i \;=\; \mathrm{Id}_A$. Finally we say that $\mathcal{E}$ *splits on the right* when there exists a morphism $s : A \to B$ such that $p \circ s \;=\; \mathrm{Id}_C$. If ψ is a splitting of $\mathcal{E}$ then $r = \pi_A \circ \psi$ is a left splitting of $\mathcal{E}$. Conversely if $r : B \to C$ is a left splitting of $\mathcal{E}$ then $\psi \;=\; \begin{pmatrix} r \\ p \end{pmatrix} : B \to A \oplus C$ is a splitting. If ψ is a splitting of $\mathcal{E}$ then $s \;= \psi^{-1} \circ i_C : C \to B$ is a right splitting. Conversely if s is a right splitting then by the Five Lemma, $(i, s) : A \oplus C \to B$ is necessarily an isomorphism and $\psi \;=\; (i, s)^{-1}$ is then a splitting. To summarise:

(3.2) $\mathcal{E}$ splits $\iff$ $\mathcal{E}$ splits on the left $\iff$ $\mathcal{E}$ splits on the right.

We say that an object $Q \in \mathfrak{A}$ is *projective* when every exact sequence of the form $0 \to C \xrightarrow{i} B \xrightarrow{p} Q \to 0$ splits. The following is fundamental:

Proposition 3.3: (Schanuel's Lemma) Let $(0 \to D_r \xrightarrow{i_r} P_r \xrightarrow{f_r} M \to 0)$ be exact sequences in $\mathcal{M}\mathrm{od}_\Lambda$ $(r = 1, 2)$; if P_1 and P_2 are projective then $D_1 \oplus P_2 \cong D_2 \oplus P_1$.

Proof : Form the pullback $Q = \varprojlim(f_1, f_2)$ Then there is a short exact sequence $0 \to D_2 \to Q \xrightarrow{\pi_1} P_1 \to 0$ which splits as P_1 is projective. Hence $Q \cong D_2 \oplus P_1$. Likewise the short exact sequence $0 \to D_1 \to Q \xrightarrow{\pi_2} P_2 \to 0$ splits as P_2 is projective. Thus $D_1 \oplus P_2 \cong Q \cong D_2 \oplus P_1$ as claimed. $\square$

4. Some standard diagrams :

Consider the following commutative diagram in a tame abelian category $\mathfrak{A}$ in which it is assumed that all rows and columns are exact.

$$\textbf{(4.1)}\qquad \left\{ \begin{array}{ccccccc}
 & & & & & 0 & \\
 & & & & & \downarrow & \\
 & S_- & \xrightarrow{\widehat{i}} & S & \xrightarrow{\widehat{p}} & S_+ & \\
 & \downarrow j_- & & \downarrow j & & \downarrow j_+ & \\
 & \widetilde{K} & \xrightarrow{\widetilde{i}} & \widetilde{F} & \xrightarrow{\widetilde{p}} & \widetilde{J} & \\
 & \downarrow \varphi_- & & \downarrow \varphi & & \downarrow \varphi_+ & \\
0 \longrightarrow & K & \xrightarrow{i} & F & \xrightarrow{p} & J & \rightarrow 0 \\
 & & & & & \downarrow & \\
 & & & & & 0 &
\end{array} \right.$$

By Lubkin's Theorem we may replace it by an equivalent diagram in $\mathcal{A}\mathbf{b}$. A straightforward diagram chase then shows that, in (4.1):

(4.2) φ and $\widehat{p}$ are both epimorphic $\Longleftrightarrow$ $\widetilde{p}$ and φ_- are both epimorphic.

Consider likewise

$$\textbf{(4.3)}\qquad \left\{ \begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & C_2 & \stackrel{\gamma_2}{\to} & C_1 & \stackrel{\gamma_1}{\to} & C_0 \\
 & & \downarrow j_2 & & \downarrow j_1 & & \downarrow j_0 \\
B_3 & \stackrel{\beta_3}{\to} & B_2 & \stackrel{\beta_2}{\to} & B_1 & \stackrel{\beta_1}{\to} & B_0 \\
\downarrow \varphi_3 & & \downarrow \varphi_2 & & \downarrow \varphi_1 & & \downarrow \varphi_0 \\
A_3 & \stackrel{\alpha_3}{\to} & A_2 & \stackrel{\alpha_2}{\to} & A_1 & \stackrel{\alpha_1}{\to} & A_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array} \right.$$

(4.4) Suppose in (4.3) that the columns are all exact; if the rows $(A_3 \stackrel{\alpha_3}{\to} A_2 \stackrel{\alpha_2}{\to} A_1 \stackrel{\alpha_1}{\to} A_0)$ and $(B_3 \stackrel{\beta_3}{\to} B_2 \stackrel{\beta_2}{\to} B_1 \stackrel{\beta_1}{\to} B_0)$ are exact then $(C_2 \stackrel{\gamma}{\to} C_1 \stackrel{\gamma}{\to} C_0)$ is also exact.

In the following commutative diagram $\mathcal{C}$ over $\mathfrak{A}$ we assume all rows and columns are exact:

(4.5)

$$\mathcal{C} \;=\; \left\{ \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_2 & \xrightarrow{i_C} & C_1 & \xrightarrow{p_C} & C_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j_2} & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_0} & & \\
0 & \longrightarrow & B_2 & \xrightarrow{i_B} & B_1 & \xrightarrow{p_B} & B_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \\
0 & \longrightarrow & A_2 & \xrightarrow{i_A} & A_1 & \xrightarrow{p_A} & A_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \right.$$

We say that the diagram $\mathcal{C}$ of (4.5) *splits completely* when there are morphisms $r_t : B_t \to C_t$ for $t = 0, 1, 2$ such that $r_t \circ j_t = \mathrm{Id}_{C_t}$ and such that the following diagram commutes

$$\begin{array}{ccccc}
C_2 & \xrightarrow{i_C} & C_1 & \xrightarrow{p_C} & C_0 \\
\uparrow{\scriptstyle r_2} & & \uparrow{\scriptstyle r_1} & & \uparrow{\scriptstyle r_0} \\
B_2 & \xrightarrow{i_B} & B_1 & \xrightarrow{p_B} & B_0
\end{array}$$

The triple (r_0, r_1, r_2) is then called a *complete splitting of* $\mathcal{C}$. Evidently r_0 is a (left) splitting of the exact sequence

$$0 \to C_0 \overset{j_0}{\to} B_0 \overset{\varphi_0}{\to} A_0 \to 0$$

and we say the complete splitting (r_0, r_1, r_2) *extends the splitting* r_0.

Theorem 4.6: Assume in (4.5) above that all rows and columns are exact and that A_1 and C_0 are projective; then any (left) splitting r_0 of the right hand column extends to a complete splitting (r_0, r_1, r_2) of $\mathcal{C}$.

5. A comparison theorem for resolutions:

Given an integer $n \geq 0$ and an object $M \in \mathfrak{A}$, by an *n-resolution* we mean an exact sequence in $\mathfrak{A}$ of the form

$$\mathbf{E}^{(n)} \;=\; (E_n \stackrel{\partial_n}{\to} E_{n-1} \stackrel{\partial_{n-1}}{\to} \ldots \cdots \stackrel{\partial_2}{\to} E_1 \stackrel{\partial_1}{\to} E_0 \stackrel{\epsilon}{\to} M \to 0).$$

We allow ourselves to write $E_{-1} = M$ and $\partial_0 = \epsilon$ whenever it is notationally convenient to do so. Whilst later we shall require the resolving objects E_r to be projective of a special type, here we impose no restriction other than exactness. We denote by $\mathfrak{A}(n)$ the category whose objects are such sequences and in which morphisms are commutative ladders

$$\begin{array}{c} \widetilde{\mathbf{E}}^{(n)} \\ \varphi \downarrow \\ \mathbf{E}^{(n)} \end{array} = \left(\begin{array}{ccccccccc} \widetilde{E}_n & \stackrel{\widetilde{\partial}_n}{\to} & \widetilde{E}_{n-1} & \stackrel{\widetilde{\partial}_{n-1}}{\to} \cdots \cdots \stackrel{\widetilde{\partial}_1}{\to} & \widetilde{E}_0 & \stackrel{\widetilde{\epsilon}}{\to} & \widetilde{M} & \to 0 \\ \varphi_n \downarrow & & \varphi_{n-1} \downarrow & & \varphi_0 \downarrow & & \downarrow \varphi_- & \\ E_n & \stackrel{\partial_n}{\to} & E_{n-1} & \stackrel{\partial_{n-1}}{\to} \cdots \cdots \stackrel{\partial_1}{\to} & E_0 & \stackrel{\epsilon}{\to} & M & \to 0 \end{array} \right)$$

We also allow the limiting case $n = \infty$ in the obvious way. If $\varphi_- : \widetilde{M} \to M$ is an epimorphism we say that φ is a *dominating morphism over* φ_- when each φ_r is also an epimorphism. We agree to write $\mathbf{E}^{(n)} \preceq \widetilde{\mathbf{E}}^{(n)}$ in the special case where $\varphi_- = \mathrm{Id}_M : M \to M$.

For the rest of this section we pick a specific dominating morphism $\varphi : \widetilde{\mathbf{E}}^{(n)} \to \mathbf{E}^{(n)}$ over Id_M. Defining $T_r = \mathrm{Ker}(\varphi_r)$, $j_r : T_r \to \widetilde{E}_r$ will denote the 'inclusion' and $\widehat{\partial}_r : T_r \to T_{r-1}$ the 'restriction' giving a commutative diagram:

$$\textbf{(5.1)} \quad \left\{ \begin{array}{ccccccccc} T_n & \stackrel{\widehat{\partial}_n}{\to} & T_{n-1} & \stackrel{\widehat{\partial}_{n-1}}{\to} \cdots \stackrel{\widehat{\partial}_1}{\to} & T_0 & \to & 0 & \to 0 \\ j_n \downarrow & & j_{n-1} \downarrow & & j_0 \downarrow & & \downarrow & \\ \widetilde{E}_n & \stackrel{\widetilde{\partial}_n}{\to} & \widetilde{E}_{n-1} & \stackrel{\widetilde{\partial}_{n-1}}{\to} \cdots \stackrel{\widetilde{\partial}_1}{\to} & \widetilde{E}_0 & \stackrel{\widetilde{\epsilon}}{\to} & M & \to 0 \\ \varphi_n \downarrow & & \varphi_{n-1} \downarrow & & \varphi_0 \downarrow & & \downarrow & \\ E_n & \stackrel{\partial_n}{\to} & E_{n-1} & \stackrel{\partial_{n-1}}{\to} \cdots \stackrel{\partial_1}{\to} & E_0 & \stackrel{\epsilon}{\to} & M & \to 0. \end{array} \right.$$

Although $\widehat{\partial}_{n-1} \circ \widehat{\partial}_n = 0$ it is not, in general, true that $\mathrm{Ker}(\widehat{\partial}_{n-1})$ is the same as $\mathrm{Im}(\widehat{\partial}_n)$. Noting this loss of information at the top left hand corner, it nevertheless follows, by induction from (4.4), that the following portion of (5.1) has exact rows and columns:

$$
\textbf{(5.2)}\left\{
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & T_{n-1} & \overset{\widehat{\partial}_{n-1}}{\rightarrow} & T_{n-2} & \overset{\widehat{\partial}_{n-2}}{\rightarrow} & \cdots & \overset{\widehat{\partial}_1}{\rightarrow} & T_0 & \rightarrow & 0 \rightarrow 0 \\
 & & j_{n-1}\downarrow & & j_{n-2}\downarrow & & & & j_0\downarrow & & \downarrow \\
\widetilde{E}_n & \overset{\widetilde{\partial}_n}{\rightarrow} & \widetilde{E}_{n-1} & \overset{\widetilde{\partial}_{n-1}}{\rightarrow} & \widetilde{E}_{n-2} & \overset{\widetilde{\partial}_{n-2}}{\rightarrow} & \cdots & \overset{\widetilde{\partial}_1}{\rightarrow} & \widetilde{E}_0 & \overset{\widetilde{\epsilon}}{\rightarrow} & M \rightarrow 0 \\
\varphi_n\downarrow & & \varphi_{n-1}\downarrow & & \varphi_{n-2}\downarrow & & & & \varphi_0\downarrow & & \downarrow \\
E_n & \overset{\partial_n}{\rightarrow} & E_{n-1} & \overset{\partial_{n-1}}{\rightarrow} & E_{n-2} & \overset{\partial_{n-2}}{\rightarrow} & \cdots & \overset{\partial_1}{\rightarrow} & E_0 & \overset{\epsilon}{\rightarrow} & M \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & & & 0 & & 0
\end{array}\right.
$$

In the above we define $J_r = \mathrm{Ker}(\partial_{r-1})$ for $1 \le r \le n+1$. When $r \le n$ then it is also true that $J_r = \mathrm{Im}(\partial_r)$ and we then write $\partial_r = i_r \circ p_r$ for the canonical decomposition of ∂_r through its image with i_r monomorphic and p_r epimorphic:

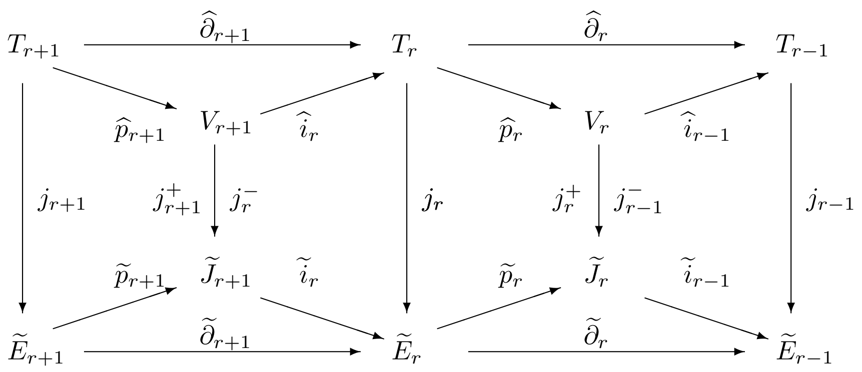
$$
\begin{array}{ccc}
E_r & \xrightarrow{\partial_r} & E_{r-1} \\
 & p_r \searrow \quad \nearrow i_r & \\
 & J_r &
\end{array}
$$

Likewise we consider the corresponding decompositions for the $\widetilde{\partial}_r$ to obtain commutative diagrams as follows:

$$
\textbf{(5.3)}\quad
\begin{array}{ccccccccc}
\widetilde{E}_{r+1} & & \xrightarrow{\widetilde{\partial}_{r+1}} & & \widetilde{E}_r & & \xrightarrow{\widetilde{\partial}_r} & & \widetilde{E}_{r-1} \\
 & \widetilde{p}_{r+1}\searrow & & \nearrow\widetilde{i}_r & & \widetilde{p}_r\searrow & & \nearrow\widetilde{i}_{r-1} & \\
 & & \widetilde{J}_{r+1} & & & & \widetilde{J}_r & & \\
\downarrow\varphi_{r+1} & & \varphi^+_{r+1}\downarrow\varphi^-_r & & \downarrow\varphi_r & & \varphi^+_r\downarrow\varphi^-_{r-1} & & \downarrow\varphi_{r-1} \\
 & & J_{r+1} & & & & J_r & & \\
 & p_{r+1}\nearrow & & \searrow i_r & & p_r\nearrow & & \searrow i_{r-1} & \\
E_{r+1} & & \xrightarrow{\partial_{r+1}} & & E_r & & \xrightarrow{\partial_r} & & E_{r-1}
\end{array}
$$

where, depending on context, both φ^-_r and φ^+_{r-1} denote the restriction $\varphi_{r-1|\widetilde{J}_r} : \widetilde{J}_r \to J_r$. Now taking the corresponding decompositions for the $\widehat{\partial}_r$ we get commutative diagrams

(5.4)



where both j^-_r and j^+_{r-1} both denote the 'inclusion' $V_r \to \widetilde{J}_r$. We assemble (5.3) and (5.4) into commutative diagrams $\mathcal{D}(r)$ for $1 \le r \le n-2$;

$$\mathcal{D}(r) \quad = \quad \left\{ \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 \to & & V_{r+1} & \stackrel{\widehat{i_r}}{\to} & T_r & \stackrel{\widehat{p_r}}{\to} & V_r \longrightarrow 0 \\ & & \downarrow j^+_r & & \downarrow j_r & & \downarrow j^-_r \\ 0 \longrightarrow & & \widetilde{J_{r+1}} & \stackrel{\widetilde{i_r}}{\to} & \widetilde{E}_r & \stackrel{\widetilde{p_r}}{\longrightarrow} & \widetilde{J}_r \longrightarrow 0 \\ & & \downarrow \varphi^+_r & & \downarrow \varphi_r & & \downarrow \varphi^-_r \\ 0 \longrightarrow & & J_{r+1} & \stackrel{i_r}{\to} & E_r & \stackrel{p_r}{\longrightarrow} & J_r \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array} \right.$$

In the special case $r = 0$ we obtain

$$\mathcal{D}(0) \quad = \quad \left\{ \begin{array}{cccccc} & 0 & & 0 & & \\ & \downarrow & & \downarrow & & \\ 0 \to & V_1 & = & T_0 & \to & 0 \\ & \downarrow j^+_0 & & \downarrow j_0 & & \downarrow \\ 0 \to & \widetilde{J_1} & \stackrel{\widetilde{i_0}}{\longrightarrow} & \widetilde{E}_0 & \stackrel{\widetilde{\epsilon}}{\longrightarrow} & M \to 0 \\ & \downarrow \varphi^+_0 & & \downarrow \varphi_0 & & || \;\; \mathrm{Id} \\ 0 \to & J_1 & \stackrel{i_0}{\longrightarrow} & E_0 & \stackrel{\epsilon}{\longrightarrow} & M \to 0 \\ & \downarrow & & \downarrow & & \downarrow \\ & 0 & & 0 & & 0 \end{array} \right.$$

As $\widetilde{\epsilon} \circ j_0 = \epsilon \circ \varphi_0 \circ j_0 = 0$ then the 'inclusion' $j^+_0 : V_1 \to \widetilde{J}_1 = \mathrm{Ker}(\widetilde{\epsilon})$ and 'restriction' $\varphi^+_0 : \varphi_{1|\widetilde{J}_1} : \widetilde{J}_1 \to J_1$ are both well defined. We note:

Proposition 5.5: All the rows and columns of $\mathcal{D}(0)$ are exact.

Proof : Exactness of the rows and of the right hand and middle columns is tautological. Thus it suffices to show that:

(a) φ_0^+ is epimorphic and (b) $\mathrm{Ker}(\varphi_0^+) = \mathrm{Im}(j_0^+)$.

For (a), observe that in the following subdiagram of $\mathcal{D}(0)$ all rows and columns are exact.

$$\begin{array}{ccccccc}
 & & & T_0 & \xrightarrow{\widehat{p_0}} & 0 & \\
 & & & \downarrow j_0 & & \downarrow & \\
 & \widetilde{J_1} & \overset{\widetilde{i_0}}{\hookrightarrow} & \widetilde{E}_0 & \xrightarrow{\widetilde{\epsilon}} & M & \\
 & \downarrow \varphi_0^+ & & \downarrow \varphi_0 & & \| \ \mathrm{Id} & \\
0 \longrightarrow & J_1 & \overset{i_0}{\hookrightarrow} & E_0 & \xrightarrow{\epsilon} & M & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}$$

As both φ_0 and $\widehat{p}_0$ are epimorphic then φ_0^+ is epimorphic by (4.2).
To prove (b) we may again, by Lubkin's Theorem, assume that the diagram consists of abelian groups and homomorphisms in which monomorphisms become inclusions thus:

$$\begin{array}{cccccc}
 & 0 & & 0 & & \\
 & \downarrow & & \downarrow & & \\
0 \to & V_1 & = & T_0 & \to & 0 \\
 & \cap j_0^+ & & \cap j_0 & & \downarrow \\
0 \to & \widetilde{J_1} & \overset{\widetilde{i_0}}{\hookrightarrow} & \widetilde{E}_0 & \xrightarrow{\widetilde{\epsilon}} & M \to 0 \\
 & \downarrow \varphi_0^+ & & \downarrow \varphi_0 & & \| \ \mathrm{Id} \\
0 \to & J_1 & \overset{i_0}{\hookrightarrow} & E_0 & \xrightarrow{\epsilon} & M \to 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
 & 0 & & 0 & & 0
\end{array}$$

The inclusion $\mathrm{Im}(j_0^+) \subset \mathrm{Ker}(\varphi_0^+)$ then follows by restriction from $\varphi_0 \circ j_0 = 0$. Thus suppose $x \in \widetilde{J}_1$ satisfies $\varphi_0^+(x) = 0$; then $\widetilde{i}_0(x) \in \mathrm{Ker}(\varphi_0) = T_0 = V_1$, completing the proof. $\square$

Before proceeding we first note:

(5.6) the rows of each $\mathcal{D}(r)$ are exact ;

(5.7) the middle column of each $\mathcal{D}(r)$ is exact ;

(5.8) the right hand column of $\mathcal{D}(r)$ is identical to the left hand column of $\mathcal{D}(r-1)$ for $1 \leq r \leq n-1$.

We arrive at the following 'weak comparison' theorem:

Theorem 5.9: Let $\varphi : \widetilde{\mathbf{E}}^{(n)} \to \mathbf{E}^{(n)}$ be a dominating morphism over Id_M where $n \geq 2$; then the rows and columns of $\mathcal{D}(r)$ are exact for $0 \leq r \leq n-2$.

Proof : For $n = 2$ this is simply a restatement of (5.5). Thus we may suppose that $n \geq 3$. Let $\mathbf{C}(r)$ be the statement that the rows and columns of $\mathcal{D}(r)$ are exact. As $\mathbf{C}(0)$ is true, again by (5.5), it suffices to show that $\mathbf{C}(r-1) \Longrightarrow \mathbf{C}(r)$ for $1 \leq r \leq n-2$.

Via the Lubkin imbedding it suffices to prove the statement for the corresponding diagram of abelian groups and homomorphisms. By induction the right hand column of $\mathcal{D}(r)$ is exact as it coincides with the left hand column of $\mathcal{D}(r-1)$. As observed in (5.7) the middle column of $\mathcal{D}(r)$ is exact so it suffices to show the left hand column of $\mathcal{D}(r)$ is exact. As j_r^+ is a monomorphism it suffices to show:

(a) φ_r^+ is epimorphic and (b) $\mathrm{Ker}(\varphi_r^+) = \mathrm{Im}(j_r^+)$.

To show (a), note that in the following subdiagram of $\mathcal{D}(r)$ all rows and columns are exact;

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & T_r & \stackrel{\widehat{p_r}}{\to} & V_r & \longrightarrow & 0 \\
 & & \downarrow j_r & & \downarrow j_r^- & & \\
\widetilde{J}_{r+1} & \stackrel{\widetilde{i_r}}{\to} & \widetilde{E}_r & \stackrel{\widetilde{p_r}}{\longrightarrow} & \widetilde{J}_r & \longrightarrow & 0 \\
\downarrow \varphi_r^+ & & \downarrow \varphi_r & & \downarrow \varphi_r^- & & \\
J_{r+1} & \stackrel{i_r}{\to} & E_r & \stackrel{p_r}{\longrightarrow} & J_r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

As φ_r and $\widehat{p}_r$ are epimorphic it follows by (4.2) that φ_r^+ is epimorphic.

To prove (b) suppose $x \in \widetilde{J}_{r+1} = \mathrm{Ker}(\widetilde{\partial}_r)$ satisfies $\varphi_r^+(x) = 0$. We must produce an element $y \in V_{r+1} = \mathrm{Ker}(\widehat{\partial}_r)$ such that $j_r(y) = x$. Consider the following portion of the diagram established in (5.2). Observe that as $r \leq n-2$ this subdiagram is well defined.

$$
\begin{array}{ccccccc}
 & & T_{r+1} & \stackrel{\widehat{\partial}_{r+1}}{\rightarrow} & T_r & \stackrel{\widehat{\partial}_r}{\rightarrow} & T_{r-1} \\
 & & j_{r+1}\downarrow & & j_r\downarrow & & j_{r-1}\downarrow \\
\widetilde{E}_{r+2} & \stackrel{\widetilde{\partial}_{r+2}}{\rightarrow} & \widetilde{E}_{r+1} & \stackrel{\widetilde{\partial}_{r+1}}{\rightarrow} & \widetilde{E}_r & \stackrel{\widetilde{\partial}_r}{\rightarrow} & \widetilde{E}_{r-1} \\
\varphi_{r+2}\downarrow & & \varphi_{r+1}\downarrow & & \varphi_r\downarrow & & \varphi_{r-1}\downarrow \\
E_{r+2} & \stackrel{\partial_{r+2}}{\rightarrow} & E_{r+1} & \stackrel{\partial_{r+1}}{\rightarrow} & E_r & \stackrel{\partial_r}{\rightarrow} & E_{r-1} \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}
$$

The conditions on $x \in \widetilde{E}_r$ are $\widetilde{\partial}_r(x) = 0$ and $\varphi_r(x) = 0$. By exactness of the middle row we may choose $w \in \widetilde{E}_{r+1}$ such that $\widetilde{\partial}_{r+1}(w) = x$. Then $\varphi_r \circ \widetilde{\partial}_{r+1}(w) = 0$ so that $\partial_{r+1} \circ \varphi_{r+1}(w) = 0$. By exactness of the bottom row choose $z \in E_{r+2}$ such that $\partial_{r+2}(z) = \varphi_{r+1}(w)$.

As $\varphi_{r+2} : \widetilde{E}_{r+2} \to E_{r+2}$ is epimorphic, choose $\zeta \in \widetilde{E}_{r+2}$ such that $\varphi_{r+2}(\zeta) = z$; then $\partial_{r+2} \circ \varphi_{r+2}(\zeta) = \varphi_{r+1}(w)$. Put $\mu = w - \widetilde{\partial}_{r+2}(\zeta) \in \widetilde{E}_{r+1}$ so that $\varphi_{r+1}(\mu) = 0$. Choose $\eta \in T_{r+1}$ such that

$$j_{r+1}(\eta) \ = \ \mu \ = \ w - \widetilde{\partial}_{r+2}(\zeta).$$

Then $\widetilde{\partial}_{r+1} \circ j_{r+1}(\eta) = \widetilde{\partial_{r+1}}(w) - \widetilde{\partial_{r+1}}\widetilde{\partial_{r+2}}(\zeta)$. Put $y = \widehat{\partial}_{r+1}(\eta)$. Then $y \in V_{r+1}$ and $j_r(y) = x$. This completes the proof. □

The statement of (5.9) extends to the limiting case $n = \infty$ as follows:

Corollary 5.10: Let $\varphi : \widetilde{\mathbf{E}}^{(\infty)} \to \mathbf{E}^{(\infty)}$ be a dominating morphism over Id_M; then the rows and columns of $\mathcal{D}(r)$ are exact for all $r \geq 0$.

6. Finiteness conditions and stability:

Let $\mathfrak{A}$ be a tame abelian category. By a *special class* in $\mathfrak{A}$ we mean a class of objects $\mathfrak{S} \subset \mathfrak{A}$ satisfying the following properties $\mathfrak{S}(1)$-$\mathfrak{S}(3)$:

$\mathfrak{S}(1)$: Each $S \in \mathfrak{S}$ is projective and $0 \in \mathfrak{S}$;

$\mathfrak{S}(2)$: If $0 \to X \to Y \to S \to 0$ is exact in $\mathfrak{A}$ and $S \in \mathfrak{S}$ then

$$X \in \mathfrak{S} \Longleftrightarrow Y \in \mathfrak{S}.$$

Finally we have a 'finiteness' property. If $S, T \in \mathfrak{S}$ let $\mathbf{e}(S,T)$ denote the set of integers k for which there exists an epimorphism $S \to \underbrace{T \oplus \cdots \oplus T}_{k}$.

$\mathfrak{S}(3)$: If $S, T \in \mathfrak{S}$ and $T \neq 0$ then $\mathbf{e}(S,T)$ is bounded above.

It follows from $\mathfrak{S}(1)$ and $\mathfrak{S}(2)$ that $\mathfrak{S}$ is closed with respect to coproducts;

(6.1). $X \in \mathfrak{S}$ and $Y \in \mathfrak{S} \implies X \oplus Y \in \mathfrak{S}$.

Likewise $\mathfrak{S}$ is closed with respect to isomorphism;

(6.2) $X \in \mathfrak{S}$ and $X \cong_{\mathfrak{A}} Y \implies Y \in \mathfrak{S}$.

Recall that a finitely generated module M over a ring Λ is said to be *stably free* when $M \oplus \Lambda^m \cong \Lambda^{m+n}$ for some integers m, n.

(6.3) The class $\mathcal{SF}$ of finitely generated stably free Λ-modules is a special class in $\mathcal{M}\mathrm{od}_\Lambda$.

Similarly we define a class $\mathcal{GSF}$ of objects in $\mathcal{G}(\Lambda)$ as follows:

$$M \in \mathcal{GSF} \iff \text{each } M_r \text{ is finitely generated stably free over } \Lambda.$$

(6.4) The class $\mathcal{GSF}$ is a special class in $\mathcal{G}(\Lambda)$.

There is a relation, $\mathfrak{S}$-equivalence, defined on the objects of $\mathfrak{A}$ by

$$X \sim X' \iff X \oplus S \cong X' \oplus S' \quad \text{for some } S, S' \in \mathfrak{S}.$$

We define a class $\mathcal{F}(0)$ of objects in $\mathfrak{A}$ as follows; $M \in \mathcal{F}(0)$ when there exists an epimorphism $\eta : S \to M$ for some $S \in \mathfrak{S}$.

Proposition 6.5 : Let $M \in \mathcal{F}(0)$; if $T \in \mathfrak{S}$ is such that $M \oplus T \cong M$ then $T = 0$.

Proof : Let $\varphi : S \to M$ be an epimorphism where $S \in \mathfrak{S}$, and suppose that there is an isomorphism $\psi_1 : M \to M \oplus T$ where $T \in \mathfrak{S}$. Then for each positive integer k we obtain an isomorphism $\psi_k : M \to M \oplus T^{(k)}$ on putting $\psi_k = (\psi_{k-1} \oplus \mathrm{Id}_T) \circ \psi_1$ for $k \geq 2$. Hence for each positive integer k we obtain an epimorphism $\eta_k : S \to T^{(k)}$ on putting $\eta_k = \pi_k \circ \psi_k \circ \varphi$. This contradicts property $\mathfrak{S}(3)$ unless $T = 0$. □

Corollary 6.6 : Let $S \in \mathfrak{S}$; if $\varphi : S \to S$ is an epimorphism then φ is an isomorphism.

Proof : Suppose that $\varphi : S \to S$ is an epimorphism. As S is projective then from the exact sequence

$$0 \to \mathrm{Ker}(\varphi) \to S \xrightarrow{\varphi} S \to 0$$

there is an isomorphism $\psi_1 : S \to S \oplus \mathrm{Ker}(\varphi)$ and $\mathrm{Ker}(\varphi) \in \mathfrak{S}$. By (6.5) $\mathrm{Ker}(\varphi) = 0$, so that φ is monomorphic and hence an isomorphism. □

We first introduce a general definition; if $M_1, M_2 \in \mathcal{F}(0)$ we say that M_2 *splits over* M_1, written $M_1 \dashv M_2$, when there is an isomorphism $M_1 \oplus T \cong M_2$ in which $T \in \mathfrak{S}$. Evidently one has:

Proposition 6.7 : If $M_1 \dashv M_2$ then $M_1 \sim M_2$.

It is straightforward to see that the relation '$\dashv$' is transitive; that is :

(6.8) If $M_1 \dashv M_2$ and $M_2 \dashv M_3$ then $M_1 \dashv M_3$.

More subtly, '$\dashv$' is also anti-symmetric in the sense that, for $M_1, M_2 \in \mathcal{F}(0)$,

Proposition 6.9 : $M_1 \dashv M_2 \ \wedge \ M_2 \dashv M_1 \implies M_1 \cong M_2$.

Proof : The hypothesis allows us to write $M_2 \cong M_1 \oplus T_1$ and $M_1 \cong M_2 \oplus T_2$ for some $T_1, T_2 \in \mathfrak{S}$. Thus $M_1 \cong M_1 \oplus T$ where $T = (T_1 \oplus T_2) \in \mathfrak{S}$. It follows from (6.6) above that $T = 0$. Hence $T_2 = 0$ and $M_1 \cong M_2$. □

Corollary 6.10 : If Ω is an $\mathfrak{S}$-class of type $\mathcal{F}(0)$ then the relation '$\dashv$' induces a partial ordering on the isomorphism types of Ω.

If X is an object in $\mathfrak{A}$ the $\mathcal{S}$-class $[X]$ is defined to be the collection of isomorphism classes of objects Y in $\mathfrak{A}$ which are $\mathfrak{S}$-equivalent to X:

$$[X] = \{Y \in \mathfrak{A} \mid Y \sim X\}/\cong$$

As $\mathfrak{A}$ is equivalent to a small subcategory of $\mathcal{A}$b it follows that

(6.11) $[X]$ is a set for each object $X \in \mathfrak{A}$.

We denote by $\mathfrak{S}(n)$ the full subcategory of $\mathfrak{A}(n)$ consisting of exact sequences of the form

$$\mathbf{S}^{(n)} = (S_n \xrightarrow{\partial_n} S_{n-1} \xrightarrow{\partial_{n-1}} \dots \cdots \xrightarrow{\partial_2} S_1 \xrightarrow{\partial_1} S_0 \xrightarrow{\epsilon} M \to 0)$$

in which $S_0, \dots, S_n \in \mathfrak{S}$. Such a sequence will be called an *n-stem* over M. Moreover $J_{r+1} = \mathrm{Ker}(\partial_r)$ is called the $(r+1)^{st}$ *syzygy* of $\mathbf{S}^{(n)}$. We say that M is of type $\mathcal{F}(n)$ when there exists an n-stem over M. In general this condition is a nontrivial restriction on M.

Theorem 6.12 : Let $M \in \mathfrak{A}$ and $S \in \mathfrak{S}$; then

$$M \in \mathcal{F}(n) \iff M \oplus S \in \mathcal{F}(n).$$

Proof : Let $\mathcal{P}(n)$ be the statement of the theorem; we first prove $\mathcal{P}(0)$. Suppose that $\epsilon : S_0 \to M$ is an epimorphism. Then $\epsilon \oplus \mathrm{Id} : S_0 \oplus S \to M \oplus S$ is also an epimorphism so that if $M \in \mathcal{F}(0)$ then $M \oplus S \in \mathcal{F}(0)$. Conversely suppose that $M \oplus S \in \mathcal{F}(0)$ and let $\eta : S_0 \to M \oplus S$ be an epimorphism.

Taking $\pi_1 : M \oplus S \to M$, $\pi_2 : M \oplus S \to S$ to be the canonical projections, put $S' = \mathrm{Ker}(\pi_2 \circ \eta)$. Applying τ we obtain an exact sequence

$$0 \to \tau(S') \to \tau(S_0) \stackrel{\tau(\pi_1 \circ \eta)}{\longrightarrow} \tau(S) \to 0.$$

It follows that the sequence $0 \to S' \to S_0 \stackrel{\pi_1 \circ \epsilon}{\to} S \to 0$ is also exact so that $S' \in \mathfrak{S}$ by property $\mathfrak{S}(3)$. However, $\tau(\pi_1 \circ \eta) : \tau(S') \to \tau(M)$ is epimorphic so that $\pi_1 \circ \eta : S' \to M$ is epimorphic and hence $M \in \mathcal{F}(0)$. This proves $\mathcal{P}(0)$. Now suppose that $\mathcal{P}(n-1)$ is true for $n \geq 1$, let $M \in \mathcal{F}(n)$ and let

$$S_n \stackrel{\partial_n}{\to} \cdots \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0$$

be an n-stem. Letting $i : S_0 \to S_0 \oplus S$ be the canonical morphism define

$$\delta_r = \begin{cases} i \circ \partial_1 & r = 1 \\ \partial_r & r > 1 \end{cases}$$

We see that $S_n \stackrel{\delta_n}{\to} \cdots \stackrel{\delta_1}{\to} S_0 \oplus S \stackrel{\epsilon \oplus \mathrm{Id}}{\to} M \oplus S \to 0$ is exact. As $S_0 \oplus S \in \mathfrak{S}$ then $M \oplus S \in \mathcal{F}(n)$.

Conversely suppose that $S_n \stackrel{\delta_n}{\to} \cdots \stackrel{\delta_1}{\to} S_0 \stackrel{\eta}{\to} M \oplus S \to 0$ is an n-stem where $M \oplus S \in \mathcal{F}(n)$. We may decompose this into a pair of exact sequences

(*) $$S_n \stackrel{\delta_n}{\to} \cdots \stackrel{\delta_2}{\to} S_1 \stackrel{p}{\to} K \to 0$$

(**) $$0 \to K \stackrel{i}{\to} S_0 \stackrel{\eta}{\to} M \oplus S \to 0$$

where $\delta_1 = i \circ p$. Take $\pi_1 : M \oplus S \to M$, $\pi_2 : M \oplus S \to S$ to be the canonical projections and put $S' = \mathrm{Ker}(\pi_2 \circ \epsilon)$. As in the proof of $\mathcal{P}(0)$, $S' \in \mathfrak{S}$ and $\pi_1 \circ \epsilon : S' \to M$ is epimorphic. Moreover there is an isomorphism of K' with $\mathrm{Ker}(\pi_1 \circ \eta : S' \to M)$ giving an exact sequence

(***) $$0 \to K' \to S' \stackrel{\pi_1 \circ \eta}{\to} M \to 0.$$

Splicing (***) with (*) gives an n-stem $S_n \stackrel{\delta_n}{\to} \cdots \stackrel{\delta_1}{\to} S' \stackrel{\pi_1 \circ \eta}{\to} M \to 0$; hence $M \in \mathcal{F}(n)$. This completes the proof. □

It follows immediately that:

Corollary 6.13 : If $M \sim M'$ then $M \in \mathcal{F}(n) \iff M' \in \mathcal{F}(n)$.

In view of (6.13) we extend the condition $\mathcal{F}(n)$ from objects in $\mathfrak{A}$ to $\mathfrak{S}$-classes by saying that the $\mathfrak{S}$-class $[K]$ satisfies $\mathcal{F}(n)$ when K satisfies $\mathcal{F}(n)$.

7. A strong comparison theorem for syzygies :

Given an n-stem $\mathbf{S}^{(n)} = (S_n \stackrel{\partial_n}{\to} S_{n-1} \stackrel{\partial_{n-1}}{\to} \ldots\cdots \stackrel{\partial_2}{\to} S_1 \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0)$ over M put $J_{r+1} = \mathrm{Ker}(\partial_r)$. Suppose given another n-stem over M

$$\widetilde{\mathbf{S}}^{(n)} = (\widetilde{S}_n \stackrel{\widetilde{\partial}_n}{\to} \widetilde{S}_{n-1} \stackrel{\widetilde{\partial}_{n-1}}{\to} \ldots\cdots \stackrel{\widetilde{\partial}_2}{\to} \widetilde{S}_1 \stackrel{\widetilde{\partial}_1}{\to} \widetilde{S}_0 \stackrel{\widetilde{\epsilon}}{\to} M \to 0)$$

with $\widetilde{J}_{r+1} = \mathrm{Ker}(\widetilde{\partial}_r)$ and suppose that $\varphi : \widetilde{\mathbf{S}}^{(n)} \to \mathbf{S}^{(n)}$ is a dominating homomorphism. From the results of §5, for $0 \le r \le n-2$ we obtain commutative diagrams $\mathcal{D}(r)$ in which all rows and columns are exact

$$\mathcal{D}(r) \quad = \quad \left\{ \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 \to & & V_{r+1} & \stackrel{\widehat{i_r}}{\hookrightarrow} & T_r & \stackrel{\widehat{p_r}}{\to} & V_r \longrightarrow 0 \\ & & \downarrow j_r^- & & \downarrow j_r & & \downarrow j_r^+ \\ 0 \longrightarrow & & \widetilde{J}_{r+1} & \stackrel{\widetilde{i_r}}{\hookrightarrow} & \widetilde{S}_r & \stackrel{\widetilde{p_r}}{\longrightarrow} & \widetilde{J}_r \longrightarrow 0 \\ & & \downarrow \varphi_r^- & & \downarrow \varphi_r & & \downarrow \varphi_r^+ \\ 0 \longrightarrow & & J_{r+1} & \stackrel{i_r}{\hookrightarrow} & S_r & \stackrel{p_r}{\longrightarrow} & J_r \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array} \right.$$

and in which $V_0 = 0$ and $J_0 = \widetilde{J}_0 = M$. As $0 \to T_r \to \widetilde{S}_r \to S_r \to 0$ is exact and $S_r, \widetilde{S}_r \in \mathfrak{S}$ it follows that $\widetilde{S}_r \cong T_r \oplus S_r$ and hence:

(7.1) Each $T_r \in \mathfrak{S}$.

As $V_0 = 0$ then $V_1 = T_0$ so that :

(7.2) $V_1 \in \mathfrak{S}$.

From the exact sequences $0 \to V_{r+1} \stackrel{\widehat{i}_r}{\longrightarrow} T_r \stackrel{\widehat{p}_r}{\longrightarrow} V_r \to 0$ it follows from $\mathfrak{S}(2)$ that

(7.3) $V_r \in \mathfrak{S}$ for $1 \le r \le n-1$.

Consequently;

(7.4) $0 \to V_{r+1} \stackrel{\widehat{i}_r}{\longrightarrow} T_r \stackrel{\widehat{p}_r}{\longrightarrow} V_r \to 0$ splits for $0 \le r \le n-2$.

Hence:

(7.5) $T_r \cong V_{r+1} \oplus V_r$ for $0 \le r \le n-2$.

Theorem 7.6 : Let M be an object in $\mathcal{F}(n)$ and let $\mathbf{S}^{(n)}$, $\widetilde{\mathbf{S}}^{(n)}$ be n-stems over M; if $\mathbf{S}^{(n)} \preceq \widetilde{\mathbf{S}}^{(n)}$ then $\widetilde{J}_r$ splits over J_r for $1 \le r \le n-1$.

Proof : By (5.9) we have diagrams $\mathcal{D}(r)$ with exact rows and columns for $0 \le r \le n-2$. First consider $\mathcal{D}(0)$

$$
\mathcal{D}(0) \quad = \quad \left\{ \begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & V_1 & = & T_0 & \stackrel{\widehat{p_0}}{\longrightarrow} & 0 \longrightarrow 0 \\
 & & \downarrow j_0^- & & \downarrow j_0 & & \downarrow \\
0 \longrightarrow & & \widetilde{J}_1 & \stackrel{\widetilde{i_0}}{\hookrightarrow} & \widetilde{S}_0 & \stackrel{\widetilde{\epsilon}}{\longrightarrow} & M \longrightarrow 0 \\
 & & \downarrow \varphi_0^- & & \downarrow \varphi_0 & & || \ \mathrm{Id} \\
0 \longrightarrow & & J_1 & \stackrel{i_0}{\hookrightarrow} & S_0 & \stackrel{\epsilon}{\longrightarrow} & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array} \right.
$$

By hypothesis we have that $S_0 \in \mathfrak{S}$ so that the middle column splits. If $\rho : \widetilde{S}_0 \to T_0$ is a left splitting of the middle column then $\rho \circ \widetilde{i}_0$ is a left splitting of the left-hand column and $\widetilde{J}_1 \cong J_1 \oplus V_1$. Suppose, inductively, that $0 \to V_r \stackrel{j_r^+}{\to} \widetilde{J}_r \stackrel{\varphi_r^+}{\to} J_r \to 0$ splits for $t \leq r \leq n-2$ and consider

$$
\mathcal{D}(r) \quad = \quad \left\{ \begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & V_{r+1} & \stackrel{\widehat{i_r}}{\hookrightarrow} & T_r & \stackrel{\widehat{p_r}}{\to} & V_r \longrightarrow 0 \\
 & & \downarrow j_r^- & & \downarrow j_r & & \downarrow j_r^+ \\
0 \longrightarrow & & \widetilde{J}_{r+1} & \stackrel{\widetilde{i_r}}{\hookrightarrow} & \widetilde{S}_r & \stackrel{\widetilde{p_r}}{\longrightarrow} & \widetilde{J}_r \longrightarrow 0 \\
 & & \downarrow \varphi_r^- & & \downarrow \varphi_r & & \downarrow \varphi_r^+ \\
0 \longrightarrow & & J_{r+1} & \stackrel{i_r}{\hookrightarrow} & S_r & \stackrel{p_r}{\longrightarrow} & J_r \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array} \right.
$$

As $r \leq n-2$ then we see from (7.3) that $V_r, V_{r+1} \in \mathfrak{S}$. Moreover $S_r \in \mathfrak{S}$ so that both S_r and V_r are projective. It follows from (4.6) that the sequence $0 \to V_{r+1} \to \widetilde{J}_{r+1} \to J_{r+1} \to 0$ splits and so $\widetilde{J}_{r+1} \cong J_{r+1} \oplus V_{r+1}$. As $V_{r+1} \in \mathfrak{S}$ this completes the proof. □

Corollary 7.7 : Let $\mathbf{S}^{(n)}$ and $\widetilde{\mathbf{S}}^{(n)}$ be n-stems over M with syzygies $(J_r)_{1 \leq r \leq n}$ $(\widetilde{J}_r)_{1 \leq r \leq n}$ respectively; if $\mathbf{S}^{(n)}$ is minimal then $\widetilde{J}_r$ splits over J_r for $1 \leq r \leq n-1$.

Thus we have proved Theorem C of the Introduction.

In the limiting case an ∞-stem will be called a *complete* $\mathfrak{S}$*-resolution* of M. The statement of (7.7) is then modified to:

Corollary 7.8 : Let $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ be complete $\mathfrak{S}$-resolutions of M with syzygies $(J_r)_{1\leq r}, (\widetilde{J}_r)_{1\leq r}$; if $\mathbf{S}$ is minimal then $\widetilde{J}_r$ splits over J_r for all r.

8. Uniqueness of minimal resolutions :

Let $M \in \mathcal{F}(n)$ and let $\mathbf{S}^{(n)}$ be an n-stem over M

$$\mathbf{S}^{(n)} \ = \ (S_n \stackrel{\partial_n}{\to} S_{n-1} \stackrel{\partial_{n-1}}{\to} \ldots\cdots \stackrel{\partial_2}{\to} S_1 \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0).$$

We say that $\mathbf{S}^{(n)}$ is a *minimal* n-stem when, given any other n-stem $\widetilde{\mathbf{S}}$ over M, there is a dominating morphism $\varphi : \widetilde{\mathbf{S}}^{(n)} \to \mathbf{S}^{(n)}$ over Id_M thus

$$\begin{matrix} \widetilde{\mathbf{S}}^{(n)} \\ \varphi \downarrow \\ \mathbf{S}^{(n)} \end{matrix} \quad = \quad \begin{pmatrix} \widetilde{S}_n & \stackrel{\widetilde{\partial}_n}{\to} \cdots\ \cdots \stackrel{\widetilde{\partial}_1}{\to} & \widetilde{S}_0 & \stackrel{\widetilde{\epsilon}}{\to} & M & \to 0 \\ \varphi_n \downarrow & & \varphi_0 \downarrow & & \downarrow \mathrm{Id}_M & \\ S_n & \stackrel{\partial_n}{\to} \cdots\ \cdots \stackrel{\partial_1}{\to} & S_0 & \stackrel{\epsilon}{\to} & M & \to 0 \end{pmatrix}$$

In particular, φ_r is epimorphic for each r. A straightforward deduction from (7.7) and (6.6) then shows:

Proposition 8.1 : If $\mathbf{S}^{(n)}$, $\widetilde{\mathbf{S}}^{(n)}$ are both minimal n-stems over M then $\mathbf{S}^{(n)} \ \cong_{\mathrm{Id}_M} \ \widetilde{\mathbf{S}}^{(n)}$.

This is easily strengthened to allow variation of the differentials as follows:

Proposition 8.2 : Suppose given n-stems over M as follows:

$$\mathbf{S}^{(n)} \ = \ (S_n \stackrel{\partial_n}{\to} \ldots\cdots \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0);$$

$$\widehat{\mathbf{S}}^{(n)} \ = \ (S_n \stackrel{\delta_n}{\to} \ldots\cdots \stackrel{\delta_1}{\to} S_0 \stackrel{\eta}{\to} M \to 0);$$

if $\mathbf{S}^{(n)}$ is minimal then so also is $\widehat{\mathbf{S}}^{(n)}$.

Let $M, M' \in \mathcal{F}(n)$ and let $\mathbf{S}^{(n)}$, $\mathbf{T}^{(n)}$ be n-stems over M, M' respectively:

$$\mathbf{S}^{(n)} = (S_n \stackrel{\partial_n}{\to} S_{n-1} \stackrel{\partial_{n-1}}{\to} \ldots\cdots \stackrel{\partial_2}{\to} S_1 \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0);$$

$$\mathbf{T}^{(n)} = (T_n \stackrel{\partial'_n}{\to} T_{n-1} \stackrel{\partial'_{n-1}}{\to} \ldots\cdots \stackrel{\partial'_2}{\to} T_1 \stackrel{\partial'_1}{\to} T_0 \stackrel{\eta}{\to} M' \to 0).$$

We may form an n-stem $\mathbf{S}^{(n)} \oplus \mathbf{T}^{(n)}$ over $M \oplus M'$ thus

$$S_n \oplus T_n \stackrel{\delta_n}{\to} S_{n-1} \oplus T_{n-1} \stackrel{\delta_{n-1}}{\to} \ldots\cdots \stackrel{\delta_2}{\to} S_1 \oplus T_1 \stackrel{\delta_1}{\to} S_0 \oplus T_0 \stackrel{\epsilon\oplus\eta}{\to} M \oplus M' \to 0$$

where $\delta_r \ = \ \begin{pmatrix} \partial_r & 0 \\ 0 & \partial'_r \end{pmatrix}$. An n-stem $\mathbf{T}^{(n)}$ over 0 is simply an exact sequence $\mathbf{T}^{(n)} \ = \ (T_n \to T_{n-1} \to \ldots\cdots \to T_1 \to T_0 \to 0)$ where each

$T_r \in \mathfrak{S}$. Moreover, $\mathbf{S}^{(n)} \oplus \mathbf{T}^{(n)}$ is then an n-stem over $M \oplus 0 \cong M$. We now have the following which is Theorem B of the Introduction:

Theorem 8.3 : If $\mathbf{S}^{(n)}$, $\widetilde{\mathbf{S}}^{(n)}$ are n-stems over M and $\mathbf{S}^{(n)}$ is minimal then $\widetilde{\mathbf{S}}^{(n-1)} \cong_{\mathrm{Id}_M} \mathbf{S}^{(n-1)} \oplus \mathbf{T}^{(n-1)}$ for some (n-1)-stem $\mathbf{T}^{(n-1)}$ over 0.

Proof : Given a dominating homomorphism $\varphi : \widetilde{\mathbf{S}}^{(n)} \to \mathbf{S}^{(n)}$ we construct, as in (5.2), a commutative diagram in which all rows and columns are exact

$$\left\{\begin{array}{cccccccccc}
 & & 0 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & T_{n-1} & \stackrel{\widehat{\partial}_{n-1}}{\to} & T_{n-2} & \stackrel{\widehat{\partial}_{n-2}}{\to} \cdots \stackrel{\widehat{\partial}_1}{\to} & T_0 & \to & 0 & \to 0 \\
 & & j_{n-1}\downarrow & & j_{n-2}\downarrow & & j_0\downarrow & & \downarrow & \\
\widetilde{S}_n & \stackrel{\widetilde{\partial}_n}{\to} & \widetilde{S}_{n-1} & \stackrel{\widetilde{\partial}_{n-1}}{\to} & \widetilde{S}_{n-2} & \stackrel{\widetilde{\partial}_{n-2}}{\to} \cdots \stackrel{\widetilde{\partial}_1}{\to} & \widetilde{S}_0 & \stackrel{\widetilde{\epsilon}}{\to} & M & \to 0 \\
\varphi_n\downarrow & & \varphi_{n-1}\downarrow & & \varphi_{n-2}\downarrow & & \varphi_0\downarrow & & \downarrow & \\
S_n & \stackrel{\partial_n}{\to} & S_{n-1} & \stackrel{\partial_{n-1}}{\to} & S_{n-2} & \stackrel{\partial_{n-2}}{\to} \cdots \stackrel{\partial_1}{\to} & S_0 & \stackrel{\epsilon}{\to} & M & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 & & 0 & & 0 &
\end{array}\right.$$

In particular we have an $(n-1)$-stem over the zero object, namely

$$\mathbf{T} = (T_{n-1} \stackrel{\widehat{\partial}_{n-1}}{\to} T_{n-2} \stackrel{\widehat{\partial}_{n-2}}{\to} \cdots \stackrel{\widehat{\partial}_1}{\to} T_0 \to 0 \to 0).$$

For $0 \leq k \leq n-2$ we obtain commutative diagrams $\mathcal{D}(k)$ as in §5 in which all rows and columns are exact. As the right hand column of $\mathcal{D}(0)$ is trivially split and both $\widetilde{S}_0$ and 0 are projective we may, by (4.6) construct a complete splitting $(r_0^+, r_0, 0)$ of $\mathcal{D}(0)$ as follows:

$$\begin{array}{ccccc}
V_1 & = & T_0 & \longrightarrow & 0 \\
\uparrow r_0^+ & & \uparrow r_0 & & \uparrow \\
\widetilde{J}_1 & \stackrel{\widetilde{i}_0}{\longrightarrow} & \widetilde{S}_0 & \stackrel{\widetilde{\epsilon}}{\longrightarrow} & M
\end{array}$$

Next consider $\mathcal{D}(1)$, recalling that the right hand column of $\mathcal{D}(1)$ is identical with the left hand column of $\mathcal{D}(0)$. Defining $r_1^- = r_0^+$ we see that r_1^- is a (left) splitting of the right hand column of $\mathcal{D}(1)$. As V_1 and S_1 are projective then, by (4.6), r_1^- extends to a complete splitting (r_1^-, r_1, r_1^+) of $\mathcal{D}(1)$.

Suppose inductively that for $t \leq k-1$ we have constructed complete splittings (r_t^+, r_t, r_t^-) of $\mathcal{D}(t)$ in such a way that $r_t^- = r_{t-1}^+$. Defining

$r_k^- = r_{k-1}^+$ gives a splitting of the right hand column of $\mathcal{D}(k)$. Now $S_k \in \mathfrak{S}$ by hypothesis and we have seen in (7.3) that $V_k \in \mathfrak{S}$; thus both S_k and V_k are projective. It again follows from (4.6) that we may extend r_k^- to a complete splitting (r_k^+, r_k, r_k^-) of $\mathcal{D}(k)$.

Inductively, for k in the range $0 \le k \le n-2$, we construct complete splittings (r_k^+, r_k, r_k^-) of $\mathcal{D}(k)$ such that $r_k^- = r_{k-1}^+$ when $k \ge 1$. Finally, applying (4.6) to

$$\mathcal{E} = \begin{cases} 0 \to & T_{n-1} & \stackrel{j_1}{\to} & \widetilde{S}_{n-1} & \stackrel{\varphi_1}{\to} & S_{n-1} & \to 0 \\ & \widehat{p}_{n-1}\downarrow & & \widetilde{p}_{n-1}\downarrow & & p_{n-1}\downarrow & \\ 0 \to & V_{n-1} & \stackrel{j_0}{\to} & \widetilde{J}_{n-1} & \stackrel{\varphi_0}{\to} & J_{n-1} & \to 0 \end{cases}$$

we may construct a left splitting $r_{n-1} : \widetilde{S}_{n-1} \to T_{n-1}$ of the exact sequence

$$0 \to T_{n-1} \stackrel{j_{n-1}}{\to} \widetilde{S}_{n-1} \stackrel{\varphi_{n-1}}{\to} S_{n-1} \to 0$$

making the following diagram commute;

$$\begin{array}{ccc} T_{n-1} & \stackrel{r_1}{\leftarrow} & \widetilde{S}_{n-1} \\ \downarrow \widehat{p}_{n-1} & & \downarrow \widetilde{p}_{n-1} \\ V_{n-1} & \stackrel{r_0}{\leftarrow} & \widetilde{J}_{n-1} \end{array}$$

It follows that we have constructed a morphism of exact sequences

$$\begin{array}{c} \widetilde{\mathbf{S}}^{(n-1)} \\ \mathbf{r}\downarrow \\ \mathbf{T} \end{array} = \begin{cases} \widetilde{S}_{n-1} & \stackrel{\widetilde{\partial}_{n-1}}{\to} & \widetilde{S}_{n-2} & \stackrel{\widetilde{\partial}_{n-2}}{\to} \cdots \stackrel{\widetilde{\partial}_1}{\to} & \widetilde{S}_0 & \stackrel{\widetilde{\epsilon}}{\to} M \to 0 \\ r_{n-1}\downarrow & & r_{n-2}\downarrow & & r_0\downarrow & \downarrow \\ T_{n-1} & \stackrel{\widehat{\partial}_{n-1}}{\to} & T_{n-2} & \stackrel{\widehat{\partial}_{n-2}}{\to} \cdots \stackrel{\widehat{\partial}_1}{\to} & T_0 & \to 0 \to 0 \end{cases}$$

Then $\binom{\varphi}{\mathbf{r}} : \widetilde{\mathbf{S}}^{(n-1)} \to \mathbf{S}^{(n-1)} \oplus \mathbf{T}^{(n-1)}$ is the required isomorphism. $\square$

In the case of complete $\mathfrak{S}$-resolutions we may continue the construction of the complete splittings (r_k^+, r_k, r_k^-) indefinitely to obtain Theorem A of the Introduction, namely:

Theorem 8.4 : Let $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ be complete $\mathfrak{S}$-resolutions of M; if $\mathbf{S}$ is minimal then $\widetilde{\mathbf{S}} \cong \mathbf{S} \oplus \mathbf{T}$ for some complete $\mathfrak{S}$-resolution $\mathbf{T}$ of 0.

9. The structure of the stable syzygies $\Omega_n(M)$:

If $M \in \mathcal{F}(0)$ then there is an exact sequence $0 \to J \to S \to M \to 0$ with $S \in \mathfrak{S}$. We write

$$\Omega_1(M) = [J].$$

$\Omega_1(M)$ is called the *first stable syzygy* of M relative to $\mathfrak{S}$. It is well defined as, by (3.3), the $\mathfrak{S}$-class $[J]$ depends only upon M. Moreover:

(9.2) Let M, $M' \in \mathcal{F}(0)$; if $M \sim M'$ then $\Omega_1(M) = \Omega_1(M')$.

More generally, if $M, M' \in \mathcal{F}(n)$ then there are exact sequences

$$0 \to J \to S_n \to \cdots \to S_0 \to M \to 0$$

$$0 \to J' \to S'_n \to \cdots \to S'_0 \to M' \to 0$$

with $S_i, S'_j \in \mathfrak{S}$; (9.2) now generalizes straightforwardly to give:

(9.3) If $M \sim M'$ then $J \sim J'$.

If $(0 \to J \to S_n \to \cdots \to S_0 \to M \to 0)$ is an n-stem over M we write

$$\Omega_{n+1}(M) = [J].$$

$\Omega_{n+1}(M)$ is the $(n+1)^{st}$-*stable syzygy of* M *relative to* $\mathfrak{S}$; to uniformize notation we shall write the stable class $[M]$ of M as $[M] = \Omega_0(M)$. From (9.3) we now obtain:

(9.4) Let M, $M' \in \mathcal{F}(n)$; if $M \sim M'$ then $\Omega_{n+1}(M) = \Omega_{n+1}(M')$.

One sees easily that:

(9.5) If $M \in \mathcal{F}(n)$ then $\Omega_r(M)$ satisfies $\mathcal{F}(n-r)$ for $1 \le r \le n$.

If M satisfies $\mathcal{F}(n)$ then although $\Omega_{n+1}(M)$ is defined, it need not, in general, satisfy $\mathcal{F}(0)$. In this context, for $M \in \mathcal{F}(n)$ we see that:

(9.6) $\Omega_{n+1}(M)$ satisfies $\mathcal{F}(0) \iff M$ satisfies $\mathcal{F}(n+1)$.

10. Realizing elements of $\Omega_n(M)$ as syzygies :

We say that $M \in \mathfrak{A}$ is 1-*coprojective* when, for any $S \in \mathfrak{S}$, any exact sequence of the form $0 \to S \to X \to M \to 0$ splits; then:

(10.1) If $M \sim M'$ then M is 1-coprojective $\iff$ M' is 1-coprojective.

We have the following 'realization lemma' (cf [3] p. 107) :

(10.2) If M is a 1-coprojective of type $\mathcal{F}(0)$ then any $J \in \Omega_1(M)$ occurs in an exact sequence $0 \to J \to S \to M \to 0$ in which $S \in \mathfrak{S}$.

More generally, we say that $M \in \mathfrak{A}$ is $(n+1)$-*coprojective* when $\Omega_r(M)$ is defined and 1-coprojective for $0 \le r \le n$.

Proposition 10.3 : Suppose that $M \in \mathcal{F}(n)$ is $(n+1)$-coprojective; then for any sequence $(J_r)_{1 \le r \le n+1}$ with $J_r \in \Omega_r(M)$ there exists an n-stem

$$\mathbf{S} \;=\; (S_n \stackrel{\partial_n}{\to} S_{n-1} \stackrel{\partial_{n-1}}{\to} \ldots\cdots \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0)$$

in which $J_r \cong \mathrm{Ker}(\partial_{r-1})$ for $1 \le r \le n+1$.

Proof : By induction on n. Taking $J_1 = J$ and putting $\partial_0 = \epsilon$ then the statement for $n = 0$ is simply (10.2). Thus suppose that $n = 1$ and let $J_1 \in \Omega_1(M)$, $J_2 \in \Omega_2(M)$; by (10.2) there is an object $S_0 \in \mathfrak{S}$ and an exact sequence $0 \to J_1 \stackrel{i_0}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0$. The hypothesis $M \in \mathcal{F}(1)$ implies that $J_1 \in \mathcal{F}(0)$. As $\Omega_2(M) \;=\; \Omega_1(J_1)$ then $J_2 \in \Omega_1(J_1)$ so we may apply (10.2) to obtain an exact sequence $0 \to J_2 \stackrel{i_1}{\to} S_1 \stackrel{\pi_1}{\to} J_1 \to 0$ where $S_1 \in \mathfrak{S}$. Splicing these two sequences together by putting $\partial_1 = i_0 \circ \pi_1$ we obtain an exact sequence $(0 \to J_2 \stackrel{i_1}{\to} S_1 \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0)$ where S_0, $S_1 \in \mathfrak{S}$. Then $(S_1 \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0)$ is a 1-stem with the stated properties.

In general, suppose proved for $n-1$ where $n \ge 2$ and let $(J_r)_{1 \le r \le n+1}$ be a sequence with $J_r \in \Omega_r(M)$. By hypothesis there exists an $n-1$-stem

$$\mathbf{S}' \;=\; (S'_{n-1} \stackrel{\partial_{n-1}}{\to} \ldots\cdots \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0)$$

in which $S_0 \ldots S_{n-1} \in \mathfrak{S}$ and $J_r \cong \mathrm{Ker}(\partial_{r-1})$ for $1 \le r \le n$. We may write this in co-augmented form as

$$\mathbf{S}' \;=\; (0 \to J_n \stackrel{i_{n-1}}{\to} S'_{n-1} \stackrel{\partial_{n-1}}{\to} \ldots\cdots \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0)$$

The hypothesis $M \in \mathcal{F}(n)$ implies that $J_n \in \mathcal{F}(0)$. As $\Omega_{n+1}(M) = \Omega_1(J_n)$ we see that $J_{n+1} \in \Omega_1(J_n)$. Apply (10.2) again to obtain an exact sequence

$$0 \to J_{n+1} \stackrel{i_n}{\to} S_n \stackrel{\pi_n}{\to} J_n \to 0$$

where $S_n \in \mathfrak{S}$. Splicing these last two sequences together gives an n-stem with the stated properties

$$\textbf{(*)} \qquad 0 \to J_{n+1} \stackrel{i_n}{\to} S_n \stackrel{\partial_n}{\to} \ldots\cdots \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0.$$

where $\partial_n = i_{n-1} \circ \pi_n$. □

We say that $\Omega_r(M)$ is *relatively straight* when there exists $N_0 \in \Omega_r(M)$ such that any other $N \in \Omega_r(M)$ may be written in the form $N \cong N_0 \oplus T$ for some $T \in \mathfrak{S}$. We note the following consequence of minimality.

Theorem 10.4 : Suppose that $M \in \mathcal{F}(n+1)$ admits a minimal $(n+1)$-stem $\mathbf{S}^{(n+1)}$. If $\Omega_{n-1}(M)$ is 1-coprojective then $\Omega_n(M)$ is relatively straight.

Proof : Write $\mathbf{S}^{(n+1)} = (S_{n+1} \stackrel{\partial_{n+1}}{\to} S_n \stackrel{\partial_n}{\to} \ldots\cdots \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0)$ and for $1 \le r \le n+1$ put $J_r = \mathrm{Im}(\partial_r)$. Choose $J \in \Omega_n(M)$. We must show there exists $T \in \mathfrak{S}$ such that $J \cong J_n \oplus T$. Write the truncation $\mathbf{S}^{(n-2)}$ in co-augmented form

$$\mathbf{S}^{(n-2)} = (J_{n-1} \stackrel{i_{n-2}}{\to} S_{n-2} \stackrel{\partial_{n-2}}{\to} \ldots\cdots \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0).$$

Clearly $J \in \Omega_1(J_{n-1})$ as $\Omega_n(M) = \Omega_1(J_{n-1})$ and, by hypothesis, J_{n-1} is 1-coprojective. Thus by (10.1), there exists an exact sequence

$$\mathbf{E} = (0 \to J \to E_0 \to J_{n-1} \to 0)$$

where $E_0 \in \mathfrak{S}$. As $M \in \mathcal{F}(n+1)$ and $J \in \Omega_n(M)$ then $J \in \mathcal{F}(1)$ so there exists a 1-stem $\mathbf{F} = (F_1 \to F_0 \to J \to 0)$. Yoneda product $\mathbf{F} \circ \mathbf{E} \circ \mathbf{S}^{(n-2)}$ gives an $n+1$-stem

$$\widetilde{\mathbf{S}}^{(n+1)} = (F_1 \stackrel{\delta_{n+1}}{\to} F_0 \stackrel{\delta_n}{\to} E_0 \stackrel{\delta_{n-1}}{\to} \ldots\cdots \stackrel{\delta_1}{\to} \widetilde{S}_0 \stackrel{\epsilon}{\to} M \to 0)$$

where $\widetilde{S}_r = S_r$ for $r \le n-2$ and where $J = \mathrm{Im}(\delta_n)$. As $\mathbf{S}^{(n+1)}$ is minimal it follows from (7.7) that J splits over $J_n = \mathrm{Im}(\partial_n)$. Thus, as claimed, there exists $T \in \mathfrak{S}$ such that $J \cong J_n \oplus T$. □

11. Minimal epimorphisms :

We define a category $\mathfrak{S}_{(-)}$ in which the objects are pairs (S, ϵ) where $S \in \mathfrak{S}$ and where ϵ is an epimorphism in $\mathfrak{A}$ with domain S and whose codomain is some, as yet unspecified, object in $\mathfrak{A}$. Morphisms in $\mathfrak{S}_{(-)}$ are then commutative squares of morphisms in $\mathfrak{A}$ thus:

$$\begin{array}{ccc} S' & \stackrel{\epsilon'}{\longrightarrow} & M' \\ \varphi \downarrow & & \downarrow \varphi_- \\ S & \stackrel{\epsilon}{\longrightarrow} & M. \end{array}$$

In this case we say that φ is a *morphism over* φ_-. In practice we shall only consider the case where φ_- is an isomorphism and usually, though not always, we shall take φ_- to be the identity morphism. For $M \in \mathfrak{A}$ we define a subcategory $\mathfrak{S}_M$ of $\mathfrak{S}_{(-)}$ by restricting morphisms to be commutative squares of the form

$$\begin{array}{ccc} S' & \stackrel{\epsilon'}{\longrightarrow} & M \\ \varphi \downarrow & & \downarrow \mathrm{Id}_M \\ S & \stackrel{\epsilon}{\longrightarrow} & M. \end{array}$$

If (S, ϵ), (S', ϵ') are objects in $\mathfrak{S}_M$ we write $(S, \epsilon) \preceq (S', \epsilon')$ whenever there exists a morphism $\varphi : (S', \epsilon') \to (S, \epsilon)$ in which $\varphi : S' \to S$ is an epimorphism in $\mathfrak{A}$. It is straightforward to observe that:

(11.1) If $(S, \epsilon) \preceq (S', \epsilon')$ and $(S', \epsilon') \preceq (S'', \epsilon'')$ then $(S, \epsilon) \preceq (S'', \epsilon'')$.

Slightly more subtle is :

(11.2) $(S, \epsilon) \preceq (S', \epsilon') \ \wedge \ (S', \epsilon') \preceq (S, \epsilon) \iff (S, \epsilon) \cong (S', \epsilon')$.

It follows that :

(11.3) The relation '$\preceq$' induces a partial ordering on the isomorphism classes of $\mathfrak{S}_M$.

For $E \in \mathfrak{S}$ we define the *base stabilisation* functor $\beta_E : \mathfrak{S}_M \to \mathfrak{S}_{M \oplus E}$ thus:

$$\beta_E \begin{pmatrix} S' & \xrightarrow{\epsilon'} & M \\ \varphi \downarrow & & \downarrow \mathrm{Id}_M \\ S & \xrightarrow{\epsilon} & M \end{pmatrix} = \begin{pmatrix} S' \oplus E & \xrightarrow{\epsilon' \oplus \mathrm{Id}} & M \oplus E \\ \varphi \oplus \mathrm{Id}_E \downarrow & & \downarrow \mathrm{Id}_M \\ S \oplus E & \xrightarrow{\epsilon \oplus \mathrm{Id}} & M \oplus E \end{pmatrix};$$

that is, β_E acts on objects by $\beta_E(S, \epsilon) = (S \oplus E, \epsilon \oplus \mathrm{Id}_M)$ and on morphisms by $\beta_E(\varphi) = \varphi \oplus \mathrm{Id}_E$. Observe that β_E is order preserving:

(11.4) $(S, \epsilon) \preceq (S', \epsilon') \implies \beta_E(S, \epsilon) \preceq \beta_E(S', \epsilon')$.

Write $(S, \epsilon) \in \mathfrak{S}_{M \oplus E}$ as an exact sequence $0 \to K \hookrightarrow S \xrightarrow{\epsilon} M \oplus E \to 0$ and put $T = \mathrm{Ker}(\pi_E \circ \epsilon)$ where $\pi_E : M \oplus E \to E$ is the canonical projection. We obtain a pair of exact sequences

$$0 \to T \to S \to S/T \to 0 \quad ; \quad 0 \to T/K \to S/K \to S/T \to 0.$$

and a Noether isomorphism $S/T \cong (M \oplus E)/M \cong E$. In particular, $S/T \in \mathfrak{S}$. We may assemble the above into a commutative diagram with exact rows and columns

(*)
$$\left\{ \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 \to & & K & = & K & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & & T & \to & S & \xrightarrow{\tilde{\pi}} & S/T & \longrightarrow 0 \\ & & \downarrow \nu' & & \downarrow \nu & & || \ \mathrm{Id} \\ 0 \longrightarrow & & T/K & \to & S/K & \xrightarrow{\pi} & S/T & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array} \right.$$

in which $\nu, \nu', \tilde{\pi}$ and π are all canonical morphisms. As S and $S/T \cong E$ are both in $\mathfrak{S}$ it follows from the middle row of (*) and property $\mathfrak{S}(2)$ that $T \in \mathfrak{S}$. As $S/T \cong E$ is projective we may choose a morphism

$\widetilde{\sigma} : S/T \to S$ which splits the middle row of (*) on the right. Now define $\sigma = \nu \circ \widetilde{\sigma} : S/T \to S/K$. As $\pi \circ \nu = \widetilde{\pi}$ we see that

(**) $\pi \circ \sigma = \mathrm{Id}_{S/T}$.

That is, σ splits the bottom row of (*) on the right. Taking the corresponding left splittings $\widetilde{\lambda} = \mathrm{Id}_S - \widetilde{\sigma}\widetilde{\pi}\,;\ \lambda = \mathrm{Id}_S - \sigma\pi,$ one verifies easily that $\lambda \circ \nu = \nu' \circ \widetilde{\lambda}$. In addition to (*) we have another commutative diagram with exact rows and columns

$$
(***)\qquad \left\{
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & 0 \leftarrow & K & = & K & \leftarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & 0 \longleftarrow & T & \stackrel{\widetilde{\lambda}}{\longleftarrow} & S & \stackrel{\widetilde{\sigma}}{\longleftarrow} & S/T & \longleftarrow 0 & \\
 & & \downarrow \nu' & & \downarrow \nu & & \|\ \mathrm{Id} & & \\
 & 0 \longleftarrow & T/K & \stackrel{\lambda}{\longleftarrow} & S/K & \stackrel{\sigma}{\longleftarrow} & S/T & \longleftarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\right.
$$

Thus there exists a Noether isomorphism $\natural_1 : (S, \nu) \stackrel{\simeq}{\longrightarrow} (S, \epsilon)$ for some $(S, \nu) \in \mathfrak{S}_{S/K}$. As T and S/T are both in $\mathfrak{S}$ then $(T, \nu') \in \mathfrak{S}_{T/K}$ and $\beta_{S/T}(T, \nu')$ is well defined. Now consider the isomorphisms

$$\widetilde{h} : S \to T \oplus S/T \quad ; \quad h : S/K \to T/K \oplus S/T$$

$$\widetilde{h}(x) = (\widetilde{\lambda}(x), \widetilde{\pi}(x) \quad ; \quad h(x) = (\lambda(x), \pi(x).$$

Then $\widetilde{h}$ defines an isomorphism $\widetilde{h} : (S, \nu) \stackrel{\simeq}{\longrightarrow}_h \beta_{S/T}(T, \nu')$ over h and there is another Noether isomorphism $\natural_2 : \beta_{S/T}(T, \nu') \stackrel{\simeq}{\longrightarrow} \beta_E(T, \eta)$ where $\eta = \epsilon_{|T} : T \to M$. The composition $\natural_2 \circ \widetilde{h} \circ \natural_1^{-1} : (S, \epsilon) \stackrel{\simeq}{\longrightarrow} \beta_E(T, \eta)$ is an isomorphism over $\mathrm{Id}_{M \oplus E}$. We have shown:

Theorem 11.5 : $\beta_E : \mathfrak{S}_M \to \mathfrak{S}_{M \oplus E}$ is surjective on isomorphism classes.

For (S, ϵ), (S', ϵ') in $\mathfrak{S}_M$ consider morphisms $\varphi : \beta_E(S', \epsilon') \to \beta_E(S, \epsilon)$ in $\mathfrak{S}_{M \oplus E}$. Any such morphism is, at least, a morphism $\varphi : S' \oplus E \to S \oplus E$ in $\mathfrak{A}$ and so may be described as a matrix of $\mathfrak{A}$-morphisms

$$\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \text{where} \qquad \begin{cases} A : S' \to S \ ; \ B : E \to S\ ; \\ C : S' \to E \ ; \ D : E \to E. \end{cases}$$

A straightforward calculation shows there is a $1 - 1$ correspondence

$$\mathrm{Hom}_{\mathfrak{S}_{(-)}}(\beta_E(S',\epsilon'),\beta_E(S,\epsilon)) \overset{\simeq}{\longleftrightarrow} \left\{ \begin{pmatrix} A & B \\ & \\ 0 & \mathrm{Id}_E \end{pmatrix} \mid \begin{array}{c} A \in \mathrm{Hom}_{\mathfrak{S}_{(-)}}((S',\epsilon'),(S,\epsilon)) \\ \\ B \in \mathrm{Hom}_{\mathfrak{A}}(E,\mathrm{Ker}(\epsilon)) \end{array} \right\}.$$

Suppose given an $\mathfrak{A}$-morphism $\varphi = \begin{pmatrix} A & B \\ 0 & \mathrm{Id}_E \end{pmatrix} : S' \oplus E \to S \oplus E$. If A is epimorphic then so is φ. Conversely if φ is epimorphic, then since $S \oplus E$ is projective there exists an $\mathfrak{A}$-morphism $\sigma : S \oplus E \to S' \oplus E$ such that $\varphi \circ \sigma = \mathrm{Id}_{S\oplus E}$. Writing $\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ it follows easily that $\sigma_{21} = 0$ and $A \circ \sigma_{11} = \mathrm{Id}_S$; hence A is also epimorphic. From this it follows that

$$\beta_E(S,\epsilon) \preceq \beta_E(S',\epsilon') \implies (S,\epsilon) \preceq (S',\epsilon').$$

As a consequence we see that:

Corollary 11.6 : For any $E \in \mathfrak{S}$, β_E induces an order preserving bijection on isomorphism types $\beta_E : \mathfrak{S}_M \xrightarrow{\simeq} \mathfrak{S}_{M\oplus E}$.

An epimorphism (S,ϵ) in $\mathfrak{S}_M$ is said to be *absolutely minimal* when $(S,\epsilon) \preceq (S',\epsilon')$ for each $(S',\epsilon') \in \mathfrak{S}_M$. We may verify directly that:

(11.7) If (S,ϵ), (S',ϵ') are both absolutely minimal over M then

$$(S,\epsilon) \cong (S',\epsilon').$$

We say that $\mathcal{A}\mathrm{bs}(M)$ holds precisely when there exists an absolutely minimal epimorphism (S,ϵ) in $\mathfrak{S}_M$. By (11.6) satisfaction of this condition depends only upon the $\mathfrak{S}$-class of M; that is:

Corollary 11.8 : If $M \sim M'$ then $\mathcal{A}\mathrm{bs}(M)$ holds $\iff$ $\mathcal{A}\mathrm{bs}(M')$ holds.

This condition may be reformulated to say:

(11.9) $\mathcal{A}\mathrm{bs}(M)$ holds $\iff$ M admits a minimal 0-stem.

12. An existence criterion :

When $M \in \mathfrak{A}$ we define a subcategory $\mathfrak{S}(n)_M$ of $\mathfrak{S}(n)$ by restricting morphisms to be commutative diagrams of the form

$$\begin{array}{c} \widetilde{\mathbf{S}} \\ \varphi\downarrow \\ \mathbf{S} \end{array} = \begin{pmatrix} & \widetilde{S}_n \overset{\widetilde{\partial_n}}{\to} & \cdots\cdots \overset{\widetilde{\partial_1}}{\to} & \widetilde{S}_0 & \overset{\widetilde{\eta}}{\to} & M & \to 0 \\ \varphi_n\downarrow & & & \varphi_0\downarrow & & \downarrow \mathrm{Id}_M & \\ & S_n \overset{\partial_n}{\to} & \cdots\cdots \overset{\partial_1}{\to} & S_0 & \overset{\epsilon}{\to} & M & \to 0 \end{pmatrix}.$$

For $\mathbf{S}$, $\mathbf{S}'$, $\mathbf{S}''$ in $\mathfrak{S}(n)_M$ we may generalize the results of §11 as follows:

(12.1) If $\mathbf{S} \preceq \mathbf{S}'$ and $\mathbf{S}' \preceq \mathbf{S}''$ then $\mathbf{S} \preceq \mathbf{S}''$.

(12.2) If $\mathbf{S} \preceq \mathbf{S}'$ and $\mathbf{S}' \preceq \mathbf{S}$ then $\mathbf{S} \cong \mathbf{S}'$.

(12.3) The relation '$\preceq$' induces a partial ordering on the isomorphism classes of $\mathfrak{S}_M(n)$.

For $E \in \mathfrak{S}$ there is a base stabilisation functor $\beta_E : \mathfrak{S}(n)_M \to \mathfrak{S}(n)_{M\oplus E}$ which transforms

$$\begin{array}{c} \widetilde{\mathbf{S}} \\ \varphi \downarrow \\ \mathbf{S} \end{array} = \left(\begin{array}{cccccc} \widetilde{S}_n \stackrel{\widetilde{\partial_n}}{\to} & \cdots\cdots \stackrel{\widetilde{\partial_1}}{\to} & \widetilde{S}_0 & \stackrel{\widetilde{\eta}}{\to} & M & \to 0 \\ \varphi_n \downarrow & & \varphi_0 \downarrow & & \downarrow \mathrm{Id}_M & \\ S_n \stackrel{\partial_n}{\to} & \cdots\cdots \stackrel{\partial_1}{\to} & S_0 & \stackrel{\epsilon}{\to} & M & \to 0 \end{array}\right)$$

to

$$\begin{array}{c} \beta_E(\widetilde{\mathbf{S}}) \\ \beta_E(\varphi) \downarrow \\ \beta(\mathbf{S}) \end{array} = \left(\begin{array}{cccccc} \widetilde{S}_n \stackrel{\widetilde{\partial_n}}{\to} & \cdots \stackrel{\widetilde{i}\circ\widetilde{\partial_1}}{\to} & \widetilde{S}_0 \oplus E & \stackrel{\widetilde{\eta}\oplus \mathrm{Id}}{\to} & M \oplus E & \to 0 \\ \varphi_n \downarrow & & \varphi_0 \downarrow & & \downarrow \mathrm{Id}_M & \\ S_n \stackrel{\partial_n}{\to} & \cdots \stackrel{i\circ\partial_1}{\to} & S_0 \oplus E & \stackrel{\epsilon\oplus \mathrm{Id}}{\to} & M \oplus E & \to 0 \end{array}\right)$$

where $\widetilde{i} : \widetilde{S}_0 \to \widetilde{S}_0 \oplus E$ and $i : S_0 \to S_0 \oplus E$ are the canonical morphisms.

(12.4) β_E is order preserving; that is, $\mathbf{S} \preceq \mathbf{S}' \implies \beta_E(\mathbf{S}) \preceq \beta_E(\mathbf{S}')$.

(12.5) $\beta_E : \mathfrak{S}(n)_M \to \mathfrak{S}(n)_{M\oplus E}$ is surjective on isomorphism classes.

(12.6) If $\mathbf{S}$, $\widetilde{\mathbf{S}}$ are objects in $\mathfrak{S}(n)_M$ then $\beta_E(\mathbf{S}) \preceq \beta_E(\widetilde{\mathbf{S}}) \implies \mathbf{S} \preceq \widetilde{\mathbf{S}}$.

(12.7) β_E induces an order preserving bijection on isomorphism types

$$\beta_E : \mathfrak{S}(n)_M \xrightarrow{\simeq} \mathfrak{S}(n)_{M\oplus E}.$$

We have the following useful consequence of (12.7):

(12.8) If $\mathbf{S}$ is a minimal n-stem over M then $\beta_E(\mathbf{S})$ is a minimal n-stem over $M \oplus E$.

We say that $\mathcal{M}\mathrm{in}_n(M)$ *holds* when M admits a minimal n-stem. Note that the condition $\mathcal{M}\mathrm{in}_0(M)$ is simply a re-statement of $\mathcal{A}\mathrm{bs}(M)$. Moreover from (12.7) it follows immediately that :

(12.9) If $M \sim M'$ then $\mathcal{M}\mathrm{in}_n(M)$ holds $\iff$ $\mathcal{M}\mathrm{in}_n(M')$ holds.

Thus satisfaction of the condition $\mathcal{M}\mathrm{in}_n(M)$ depends only upon the $\mathfrak{S}$-class $[M]$ of $M \in \mathfrak{A}$. Observe that for $M \in \mathcal{F}(n)$ we have:

Theorem 12.10 : $\mathcal{A}\mathrm{bs}(M) \wedge \mathcal{M}\mathrm{in}_{n-1}(\Omega_1(M)) \implies \mathcal{M}\mathrm{in}_n(M)$.

Proof : Let $\mathbf{S}^{(0)} = (0 \to K \stackrel{i}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0)$ be a minimal 0-stem over M and let $\mathbf{S}' = (S'_{n-1} \stackrel{\delta_{n-1}}{\to} \cdots\cdots \stackrel{\delta_1}{\to} S'_0 \stackrel{\eta}{\to} K \to 0)$ be a minimal $(n-1)$-stem over K. After re-indexing thus $S_{r+1} = S'_r$; $\partial_{r+1} = \delta_r$ we may splice $\mathbf{S}'$ with $\mathbf{S}^{(0)}$ to obtain an n-stem

$$\mathbf{S} = \mathbf{S}' \circ \mathbf{S}^{(0)} = \left(S_n \stackrel{\partial_n}{\to} \cdots\cdots \stackrel{\partial_2}{\to} S_1 \stackrel{\partial_1}{\to} S_0 \stackrel{\epsilon}{\to} M \to 0 \right)$$

where $\partial_1 = i \circ \eta$. We claim that $\mathbf{S}$ is minimal; that is, given an n-stem $\widetilde{\mathbf{S}}$ over M we must produce a dominating morphism $\Psi : \widetilde{\mathbf{S}} \to \mathbf{S}$ over Id_M. Thus write $\widetilde{\mathbf{S}}$ as a Yoneda product $\widetilde{\mathbf{S}} = \widetilde{\mathbf{S}}' \circ \widetilde{\mathbf{S}}^{(0)}$ where

$$\widetilde{\mathbf{S}}' = \left(\widetilde{S}_n \stackrel{\widetilde{\partial_n}}{\to} \cdots\cdots \stackrel{\widetilde{\partial_2}}{\to} \widetilde{S}_1 \stackrel{\widetilde{\eta}}{\to} \widetilde{K} \to 0 \right) \; ; \; \widetilde{\mathbf{S}}^{(0)} = \left(0 \to \widetilde{K} \stackrel{\widetilde{i}}{\to} \widetilde{S}_0 \stackrel{\widetilde{\epsilon}}{\to} M \to 0 \right).$$

Then there is a dominating morphism of 0-stems $\psi_0 : \widetilde{\mathbf{S}}^{(0)} \to \mathbf{S}^{(0)}$

$$(*) \quad = \quad \left\{ \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 \to & & E & \cong & E & \longrightarrow & 0 \\ & & \downarrow j_- & & \downarrow j_0 & & \downarrow \\ 0 \longrightarrow & & \widetilde{K} & \stackrel{\widetilde{i}}{\to} & \widetilde{S}_0 & \stackrel{\widetilde{\epsilon}}{\longrightarrow} & M \longrightarrow 0 \\ & & \downarrow \psi_- & & \downarrow \psi_0 & & \| \ \mathrm{Id} \\ 0 \longrightarrow & & K & \stackrel{i}{\to} & S_0 & \stackrel{\epsilon}{\longrightarrow} & M \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array} \right.$$

Observe that $E \in \mathfrak{S}$ and, in (4.8), (4.17), $\widetilde{K} \cong K \oplus E$. Thus by (12.8) $\beta_E(\mathbf{S}')$ is a minimal $(n-1)$-stem over $\widetilde{K} \cong K \oplus E$. Hence there exists a dominating morphism $\psi' : \widetilde{\mathbf{S}}' \to \beta_E(\mathbf{S}')$. Composition with the canonical morphism $\pi : \beta_E(\mathbf{S}') \to \mathbf{S}'$ then takes the form

$$\pi \circ \psi' = \left(\begin{array}{cccccccc} \widetilde{S}_n \stackrel{\widetilde{\partial_n}}{\to} & \cdots & \widetilde{S}_2 & \stackrel{\widetilde{\partial_2}}{\to} & \widetilde{S}_1 & \stackrel{\widetilde{p_1}}{\to} & K \oplus E & \to 0 \\ \psi'_n \downarrow & & \psi'_2 \downarrow & & \pi \circ \psi'_1 \downarrow & & \pi \downarrow & \\ S_n \stackrel{\partial_n}{\to} & \cdots & S_2 & \stackrel{\partial_2}{\to} & S_1 & \stackrel{p_1}{\to} & K & \to 0 \end{array} \right)$$

Rewriting $\widetilde{K} \cong K \oplus E$ we may splice $\pi \circ \psi'$ with $\psi^{(0)}$ to obtain a morphism over Id_M

$$\Psi = \left(\begin{array}{cccccccc} \widetilde{S}_n \stackrel{\widetilde{\partial_n}}{\to} & \cdots & \widetilde{S}_1 & \stackrel{\widetilde{\partial_1}}{\to} & \widetilde{S}_0 & \stackrel{\widetilde{\epsilon}}{\to} & M & \to 0 \\ \Psi_n \downarrow & & \Psi_1 \downarrow & & \Psi_0 \downarrow & & \mathrm{Id} \downarrow & \\ S_n \stackrel{\partial_n}{\to} & \cdots & S_1 & \stackrel{\partial_1}{\to} & S_0 & \stackrel{\epsilon}{\to} & M & \to 0 \end{array} \right)$$

where $\Psi_r = \pi \circ \psi'_r$ for $r = 1$ and $\Psi_r = \psi'_r$ otherwise. Thus each Ψ_r is epimorphic and Ψ is a dominating morphism. □

From (12.10) we deduce our criterion for the existence of a minimal n-stem:

Theorem 12.11 : Let $M \in \mathcal{F}(n)$ and suppose that $\mathcal{A}\mathrm{bs}(\Omega_r(M))$ holds for $0 \leq r \leq n$; then M admits a minimal n-stem.

In conclusion, we point out that Eilenberg's results from [1] can all be accommodated under the aegis of (12.11). For example, when Λ is a local ring, we take $\mathfrak{A}$ to be the category of finitely generated Λ-modules and $\mathfrak{S} \subset \mathfrak{A}$ to be the subclass of finitely generated free modules. Likewise, when Λ is semisimple, we take $\mathfrak{A}$ to be the category of locally finitely generated graded Λ-modules and $\mathfrak{S} \subset \mathfrak{A}$ to be the subclass of quasi-free modules. In either case, every such module M belongs to $\mathcal{F}(\infty)$ and satisfies $\mathrm{Abs}(M)$. Hence every such module has a complete minimal resolution. However, as we shall show elsewhere, there are many more examples of minimal resolutions which are excluded *a priori* from Eilenberg's framework.

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