Permutation graphs and unique games

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Abstract

We study the value of unique games as a graph-theoretic parameter. This is obtained by labeling edges with permutations. We describe the classical value of a game as well as give a necessary and sufficient condition for the existence of an optimal assignment based on a generalisation of permutation graphs and graph bundles. In considering some special cases, we relate XOR games to EDGE BIPARTIZATION, and define an edge-labeling with permutations from Latin squares.

1   Introduction

We shall work with undirected graphs $G = (V, E)$ without self-loops or multiple edges. An edge between vertices u and v is denoted by $\{u, v\}$ or uv. A *vertex-labeling*, $l : V(G) \mapsto S$, is an assignment of elements of a given set S to the vertices of a graph G. An *edge-labeling*, $L : E(G) \mapsto S'$, is an assignment of elements of a given set S' to the edges of G. A *permutation* of a set $S = \{x_1, ..., x_n\}$ is a bijective function $\pi : S \mapsto S$. As usual, a permutation π can be defined by a set of ordered pairs $\{(x_1, \pi(x_1)), ..., (x_n, \pi(x_n))\}$. A *fixed point* of a permutation π is an element $x \in S$ such that $\pi(x) = x$. By $(x_1 x_2 ... x_n)$, we denote a permutation π such that $\pi(x_1) = x_2, ..., \pi(x_{n-1}) = x_n, \pi(x_n) = x_1$. The set of all permutations on $[n] = \{0, ..., n - 1\}$ is denoted by S_n. We consider edge-labelings with permutations. For undirected graphs, we generally consider labelings with transpositions only, *i.e.*, permutations π such that $\pi^{-1} = \pi$. For directed graphs we may use any subset of S_n.

Let G be a graph and $K : E(G) \mapsto S_n$ an edge-labeling assigning a permutation $\pi_i : [n] \mapsto [n]$ to every edge $e_i \in E(G)$. A *contradiction* in a vertex-labeling $k : V(G) \mapsto [n]$ is an edge uv such that $\pi(k(u)) \neq k(v)$, where π is the permutation assigned to uv. For a given graph G and an edge-labeling $K : E(G) \mapsto S_n$, we say that a vertex-labeling $k : V \mapsto [n]$ is *consistent* if it gives no contradictions. In such a case, the edge-labeling K is said to be *vertex-proper*.

The *contradiction number*, $\beta_C(G, K)$, is the minimum number of contradictions for a given graph G and an edge-labeling $K : E \mapsto S_n$, where the minimum is taken over all possible vertex-labelings:

$$\beta_C(G, K) = \min_k |\{e_i = uv \in E(G) : \pi_i(k(u)) \neq k(v), \pi_i = K(e_i)\}| .$$

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For a given labeled graph $(G, K : E \mapsto S_n)$, we say that a vertex-labeling $K : V \mapsto [n]$ is *optimal* if it gives exactly $\beta_C(G, K)$ contradictions. It is clear that every consistent vertex-labeling is optimal. For a given graph G and an edge-labeling $K : E \mapsto S_n$, the *assignment number*, $\beta'_C(G, K)$, is the number of consistent vertex-labelings possible for G and K.

In [8], labeled graphs are used in the study of *contextuality*. When investigating contextuality, we attempt to quantify how much the outcome of a measurement of a physical observable depends on the context in which it is measured. By a *context*, we understand a set of observables which can be measured at the same time. By a *contextual game*, we mean a scenario in which two players are asked to assign values a and b to certain randomly chosen variables $x \in X$ and $y \in Y$, respectively, under some requirements. Neither of the players knows which variable was chosen for the other one and they are not allowed to communicate with each other during the game. For each pair, x, y, of variables which may be chosen, we want the values a and b to satisfy certain constraints. If a and b satisfy the constraints, both players win, otherwise they both lose. The maximum probability of winning a given game with a classical strategy is called the *classical value* of the game and denoted by ω. Similarly, the *quantum value* of the game, denoted by ω^*, is the highest probability of winning the game using quantum resources, commonly entanglement shared between the players.

A *unique game* (see, for example, [6]) is a contextual game in which the variables can take values from the set $[n]$ and the constraints are in the form $\pi_{xy}(a) = b$, where a and b are values assigned to the variables x and y, respectively, and π is a permutation of $[n]$.

Classically, the optimal strategy for any unique game is to deterministically assign values to all of the variables so as to satisfy as many of the constraints as possible. Every constraint is satisfied with probability either 0 or 1. Quantum theory allows for strategies in which each constraint is satisfied with some probability $0 \leq p_{xy} \leq 1$. In some cases, the quantum value of a game may be higher than its classical value, because the players share entanglement.

A unique game with n possible answers can be defined in terms of a graph G with an edge-labeling $K \colon E(G) \mapsto S_n$. The set of variables corresponds to $V(G)$. Two vertices x and y of G are adjacent if and only if we may ask about the values of both at the same time. The edge-labeling K represents the constraints. If the answers for a given pair x, y should satisfy $\pi_{xy}(a) = b$, then $K(xy) = \pi_{xy}$. The most common scenario is one where the sets X and Y of questions each player may be asked are disjoint. These games are represented by bipartite labeled graphs. However, single-player games on non-bipartite graphs are also considered. Thus, the classical value of a game can be written as follows:

$$\omega(G, K) = \max_{P \in C} \frac{1}{|E(G)|} \sum_{\{x,y\} \in E} V(a, b|x, y) P(a, b|x, y)$$

where $V(a, b|x, y)$ is the indicator function, *i.e.* $V(a, b|x, y) = 1$ if $\pi_{xy}(a) = b$ and $V(a, b|x, y) = 0$, otherwise. Hence, the quantum value of a game, denoted by $\omega^*(G, K)$, is defined in exactly the same way, but with the maximum taken over all strategies involving shared entanglement. For the purpose of this paper, we do need to go beyond this deliberately vague definition.

It is easy to see that a classical strategy, or deterministic assignment of values to all variables in the game, is the same as a vertex-assignment of the labeled graph representing the game. Thus, the classical value of the game can be given in terms of the contradiction number of the labeled graph representing the game:

$$\omega(G, K) = 1 - \frac{\beta_C(G, K)}{|E(G)|}.$$

The simplest and best-known class of unique games are *XOR games*, in which the variables can only take the values 0 and 1. The constraints in those games are given in the form $a + b \equiv i \pmod 2$, where $i \in \{0, 1\}$. XOR games can be described in terms of signed graphs and thus the classical value of an XOR game is connected to the notion of balance in a signed graph.

Labeled graph equivalence, defined in section 4, allows us to introduce an equivalence for unique games. Two unique games are considered equivalent if they are associated to equivalent labeled graphs. Since equivalence preserves the contradiction number of a labeled graph, it is clear that equivalent games have the same classical value. The quantum value is also preserved, as graph isomorphisms and switches represent the relabeling of the inputs x, y and outputs a, b, respectively. Thus, equivalent games have the same classical and quantum values.

The permutation graph $KG = \overline{K_n} \bowtie^K G$, which we describe in more detail in Section 2, was first introduced with the purpose of studying the contradiction number of labeled graphs (G, K) and classical value of the corresponding games. However, the structure and components of KG also provide information about optimal quantum strategies. Let $(G, K : E(G) \mapsto S_n)$ be a connected labeled graph such that KG is not a connected graph. If $K' : E(G) \mapsto S_d$, for $d < n$, is an edge-labeling of G, such that K' is a component of G, then any strategy for the game defined by (G, K') is also a valid strategy for (G, K). It follows that $\omega^*(G, K) \geq \omega^*(G, K')$. However, in many cases, $\omega^*(G, K)$ is equal to the maximum value $\omega^*(G, K')$ over all labelings K' such that $K'G$ is a component of KG. The structure and components of the permutation graph KG will most likely play an important role in future studies of unique games.

In this paper, we focus on the contradiction and assignment numbers of labeled graphs in general as well as graphs whose labels are given by specific classes of permutations. In Section 2, we introduce the permutation graph KG obtained from a labeled graph (G, K) and show how it can be used to study consistent labelings. In Section 3, we study the effects of some operations, such as deleting an edge or identifying two vertices, on the contradiction and assignment numbers of a labeled graph. In Section 4, we define an equivalence relation for labeled graphs as a generalized version of signed graph equivalence and show that if two labeled graphs are equivalent, then their contradiction and assignment numbers are equal. In sections 5 and 6 we focus on labelings with specific classes of permutations. While this has not immediate practical interest, it is a playground to introduce new mathematical notions and gain a better understanding of special cases. In particular, we consider, the set $L_n = \{\pi_i \in S_n : \pi_i(x) \equiv i - x \pmod n$ for $x, i \in [n]\}$, $x + y \equiv i \pmod 2$, which corresponds to the class of *GXOR games*, studied in [8], and S_2, which we compare to signed graphs.

We begin with some observations on labeled trees and cycles that arguably give simplest cases.

Observation 1. *For any tree all edge-labelings are vertex-proper.*

Proof. Let T be a tree, and let $K : E(T) \mapsto S_n$ be an edge-labeling. Given the tree property, any two vertices $u, v \in V(T)$ are connected by exactly one path. For any path $P_t = (v_1, ..., v_t)$, we can define a permutation $\pi_{P_t} = K(v_{t-1}v_t)...K(v_1v_2)$. Let v be any vertex of T. Assign the value $k(v)$ to the vertex v. Then, for every $u \in V(T) - \{v\}$, let $k(u) = \pi_{P_t}(k(v))$, where P_t is the path connecting u to v. This assignment is consistent. $\square$

It is clear that a graph with only one cycle (often called unicyclic) can have at most one contradiction. Whether or not there is a contradicion in the cycle can be determined through the composition of all permutations assigned to the edges of the cycle. For a given cycle

$C_t = (v_1, ..., v_t)$, we define a permutation $\pi_{C_t} = K(e_1)K(e_2)...K(e_t)$, where $e_i \cap e_{i+1} = \{v_i\}$, for all i and $v_t = v_0$.

Theorem 1. *A cycle C_t has a consistent vertex-labeling for a given edge-labeling K if and only if π_{C_t} has at least one fixed point.*

Proof. Suppose $k : V(C) \mapsto [n]$ is a consistent vertex-assignment of C. For any $v_i \in C$, $k(v_{i+1}) = K(e_{i+1})(k(v_i)) = K(e_1)K(e_2)...K(e_{i+1})(k(v_0))$. Since $v_n = v_0$, and therefore $k(v_t) = k(v_0)$, we have $\pi_c(k(v_0)) = k(v_0)$. $\qquad\square$

Corollary 1. *The number of fixed points of π_{C_t} is equal to the number of consistent vertex-labelings of C_t.*

It follows that the number of contradictions in a given graph is at most the number of cycles. It may, however, be greater than the number of cycles containing contradictions.

2 Permutation graphs

To study the contradiction and assignment numbers in a given graph G with edge-labeling $K : E(G) \mapsto \Pi \subset S_n$, we define the graph $KG = \overline{K_n} \bowtie^K G$. The *permutation graph* $G \bowtie^\Pi H$ of two graphs G and H is as follows.

Definition 1. *Let G and H be two graphs and let $\Pi \colon E(H) \mapsto S_n$ (where $n = |V(G)|$) be a function assigning a permutation π_{ij} of $V(G)$ to each edge $v_i v_j$ of H.*

1. *Let $G_1, ..., G_n$ be copies of G. For any vertex $v \in V(G)$, we denote the copy of $u \in V(G)$ in $V_i = V(G_i)$ as u_i and for any set of vertices $S \subseteq V(G)$, the copy of S in G_i is denoted by S_i.*

2. *Two vertices $u_i \in V_i$ and $w_j \in V_j$ (i.e. copies of u in V_i and w in V_j, $i < j$) are adjacent in $G \bowtie^\Pi H$ if and only if $ij \in E(H)$ and $\pi_{ij}(u) = w$, where $\pi_{ij} = \Pi(v_i v_j)$.*

In [2], this type of graph is referred to as a *permutation graph over a graph H*. Every edge-labeling $K : E \mapsto S_n$ of a graph G gives rise to a graph $KG = \overline{K_n} \bowtie^K G$, which can also be defined as follows.

Definition 2. *Let $(G, K \colon E(G) \mapsto S_n)$ be a labeled graph.*

1. *For every vertex $v_i \in V(G)$ take an independent set $V_i = \{v_{i0}, ...v_{in-1}\}$.*

2. *Two vertices $v_{ij} \in V_j, v_{st} \in V_s$ are adjacent in KG if and only if $v_i v_s \in E(G)$ and $\pi_{is}(j) = t$, where $\pi_{is} = K(v_i v_s)$.*

This object is strictly linked to the notion of a permutation voltage graph originally introduced in [1], and it can be seen as a generalization of permutation graphs and graph bundles [5].

Lemma 2. *Let K' be an edge-labeling of the graph KG such that all edges in $E(KG)$ corresponding to a certain $e \in E(G)$ are labeled with the same permutation as e. The edge-labeling K' is vertex-proper.*

Proof. For every vertex $v_i \in V(G)$, let $v_{i0}, ..., v_{in-1}$ denote the corresponding vertices in KG. Let K' be an edge-labeling of KG assigning the permutation $K(e)$ to all edges corresponding to e. The vertex-assignment $k' : V(KG) \mapsto [n]$, in which $k'(v_{ij}) = j$, is consistent. □
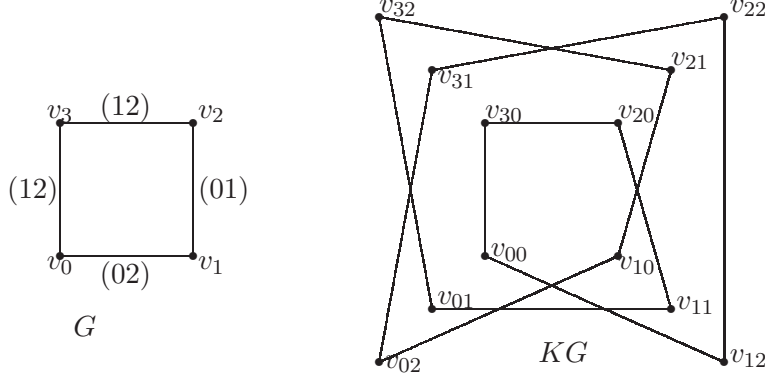

Figure 1: The graphs G and KG. The vertex-assignment $k : V(KG) \mapsto [3]$, where $k(v_{ij}) = j$ is consistent for (KG, K'). There is no consistent vertex-assignment for (G, K).

Theorem 3. *The graph G has a consistent vertex-labeling for a given edge-labeling K if and only if KG has a component isomorphic to G. If G is connected then the number of consistent vertex-labelings is equal to the number of such components.*

Proof. Let $V_i = \{v_{i0}, ..., v_{in-1}\}$ denote the set of vertices in KG corresponding to $v_i \in V(G)$. For any V_i, V_j and vertex $u \in V_i$, $|N_{V_j}(u)| = 1$ if $v_j \in N_G(v_i)$ and $|N_{V_j}(u)| = 0$ if $v_j \notin N_G(v_i)$. Therefore, for any component H of KG, if $|V_i \cap V(H)| = c$, then $|V_j \cap V(H)| = c$, for all j such that $v_j \in N_G(v_i)$. If G is a connected graph, then for any component H of KG $|V_i \cap V(H)| = |V_j \cap V(H)|$ for any i and j. Thus, a component isomorphic to G must contain exactly one vertex corresponding to each $v \in V(G)$ and one edge corresponding to each $e \in E(G)$. By Lemma 1, this component has a consistent vertex-assignment. Now, let $k : V(G) \mapsto [n]$ be a consistent vertex-assignment of G. For every $v \in V(G)$, $k(v)$ determines $k(u)$ for all $u \in N_G(v)$. For any $v_i \in V(G)$, there exists a vertex $v_{ik(v_i)} \in V_i$. For this vertex $k'(v_{ik(v_i)}) = k(v_i)$ and for all $u_{jh} \in N_{KG}(v_{ik(v_i)})$, $k'(u_{jh}) = k(u_j)$. Thus, each consistent assignment of G defines a component of KG isomorphic to G. □

Corollary 2. *If G is a connected graph then any component of KG with the same number of vertices as G is isomorphic to G.*

Proof. Let G be a connected graph and let $V_i = \{v_{i0}, ..., v_{in-1}\}$ denote the set of vertices in KG corresponding to $v_i \in V(G)$. For any V_i, V_j and vertex $u \in V_i$, $|N_{V_j}(u)| = 1$ if $v_j \in N_G(v_i)$ and $|N_{V_j}(u)| = 0$ if $v_j \notin N_G(v_i)$. Therefore, for any pair V_i, V_j and any component H of KG,

$|H \cap V_i| = |H \cap V_j|$. Thus, any component with $|V(G)|$ vertices contains exactly one vertex from each V_i and it is isomorphic to G. $\qquad\square$

Let us consider the case where $K : E \mapsto S_3$.

Theorem 4. *For any edge-labeling $K : E(G) \mapsto S_3$, any connected graph G has either $3, 1$, or 0 consistent vertex-labelings.*

Proof. For any component H of KG, $|V_i \cap V(H)| = |V_j \cap V(H)|$, for all $V_i, V_j \subset V(KG)$. It follows that KG can have either $3, 1$, or 0 components isomorphic to G. $\qquad\square$

For a given $K : E \mapsto S_n$, we say that a graph (or its subgraph) is *good* if it has n consistent vertex-labelings. If it has no consistent vertex-labeling, we say that it is *bad*. Otherwise, the graph is *ugly*.

3 Operations on labeled graphs

Obviously, for any graph G and edge-labeling $K : E(G) \mapsto S_n$ if $\beta'_C(G, K) > 0$ then $\beta_C(G, K) = 0$. The assignment and contradiction numbers of various subgraphs of a given graph may also be useful in calculating $\beta_C(G, K)$ and $\beta'_C(G, K)$.

3.1 Subgraphs of labeled graphs

For a labeled graph (H, K_H) we consider a subgraph G of H with an edge-labeling $K_G : E(G) \mapsto S_n$, where $K_G(e) = K_H(e)$ for all $e \in E(G)$. We compare the contradiction and assignment numbers of (G, K_G) to those of (H, K_H). This will help us study the effect of various graph operations on these parameters.

Observation 2. *Let H be a graph and K_H an edge-labeling of H with a set of permutations. Let G be a subgraph of H and let the edge-labeling K_G of G be defined as $K_G(e) = K_H(e)$ for all $e \in E(G)$. Then $\beta_C(G, K_G) \leq \beta_C(H, K_H)$.*

Proof. Let $k : V(H) \mapsto [n]$ be any vertex-assignment of H and $k_G : V(G) \mapsto [n]$ a vertex assignment of G such that $k_G(v) = k(v)$ for all $v \in V(G)$. It is clear that k must have at least as many contradictions as k_G. Thus, $\beta_C(G, K_G) \leq \beta_C(H, K_H)$. $\qquad\square$

Observation 3. *Let H be a connected graph and K_H an edge-labeling of H with a set of permutations. Let G be a subgraph of H and let the edge-labeling K_G of G be defined as $K_G(e) = K_H(e)$ for all $e \in E(G)$. Then $\beta'_C(G, K_G) \geq \beta'_C(H, K_H)$.*

Proof. Let $k : V(H) \mapsto [n]$ be any consistent vertex-assignment of H. The assignment $k_G : V(G) \mapsto [n]$ of G such that $k_G(v) = k(v)$ for all $v \in V(G)$ must also be consistent. In a consistent vertex-assignment of a connected graph the value assigned to one vertex determines the values in all other vertices. Thus, for any consistent k_G there is at most one k such that $k_G(v) = k(v)$. $\qquad\square$

Theorem 5. *For any graph G, edge-labeling $K : E(G) \mapsto S_n$ and edge $e \in E(G)$, we have*

$$\beta_C(G, K) - 1 \leq \beta_C(G - e, K) \leq \beta_C(G, K).$$

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Proof. If the edge e is a contradiction in some optimal vertex-assignment $k : V(G) \mapsto [n]$, then in $G - e$ the assignment k is still optimal but has $\beta_C(G, K) - 1$ contradictions. Otherwise every optimal vertex-assignment of G is also an optimal assignment of $G - e$ and $\beta_C(G, K) = \beta_C(G - e, K)$. $\qquad\square$

Observation 4. *For any graph G, edge-labeling $K : E(G) \mapsto S_n$ and edge $e \in E(G)$, we have*

$$\beta'_C(G - e, K) \geq \beta'_C(G, K).$$

Proof. It follows from the above theorem that any consistent vertex-assignment of G remains consistent for $G - e$. $\qquad\square$

3.1.1   Identifying two vertices of a labeled graph

Let (G, K_G) be a labeled graph and let v_1 and v_2 be two vertices of G. By $H = G_{v_1 = v_2}$ we denote the graph obtained by identifying v_1 and v_2. The vertices v_1 and v_2 are replaced by a new vertex v, that is, $V(H) = (V(G) - \{v_1, v_2\}) \cup \{v\}$. The edge set of H is defined as follows.

1. $N_H(v) = (N_G(v_1) \cup N_G(v_2)) - \{v_1, v_2\}$;

2. $N_H(u) = N_G(u) \cup \{v\} - \{u, v\}$ for $u \in N_H(v)$;

3. $N_H(u) = N_G(u)$ for $u \notin N_H(v)$.

Let $K_H \colon E(H) \mapsto S_n$ be an inherited edge-labeling, i.e., an edge-labeling of H in which:

1. $K_H(u_1 u_2) = K_G(u_1 u_2)$ for $u_1, u_2 \in V(H) - \{v\}$;

2. If $u \in N_H(v)$, then $K_H(uv) = K_G(uv_i)$, where $uv_i \in E(G)$.

Theorem 6. *For any labeled graph (G, K_G) and pair of vertices $v_1, v_2 \in V(G)$, the contradiction number of (H, K_H), where $H = G_{v_1 = v_2}$ and K_H is the inherited edge-labeling, satisfies the following inequalities:*

$$\beta_C(G, K_G) - 1 \leq \beta_C(H, K_H) \leq \beta_C(G, K_G) + \min\{\deg_G v_1, \deg_G v_2\}.$$

Proof. Assume without loss of generality that $\deg_G(v_1) \geq \deg_G(v_2)$. For a given vertex-assignment $k \colon V(G) \mapsto [n]$, we define a vertex-assignment $k' \colon V(H) \mapsto [n]$, where $k'(v) = k(v_1)$ and $k'(u) = k(u)$ for $u \in V(H) - \{v\}$.

If there exists an optimal vertex-assignment k of (G, K) such that $k(v_1) = k(v_2)$, then k' is an optimal assignment in which $k'(v) = k(v_1) = k(v_2)$. Consequently, if $v_1 v_2$ is a contradiction in (G, K_G), then $\beta_C(H, K_H) = \beta_C(G, K_G) - 1$ and, if $v_1 v_2$ is not a contradiction, then $\beta_C(H, K_H) = \beta_C(G, K_G)$.

If k is an optimal vertex-assignment of (G, K_G) in which $k(v_1) \neq k(v_2)$, then k' is a vertex-assignment of (H, K_H) with no more than $\beta_C(G, K_G) - 1$ and at most $\beta_C(G, K_G) + \deg_G(v_2)$ contradictions. It follows that $\beta_C(G, K_G) - 1 \leq \beta_C(H, K_H) \leq \beta_C(G, K_G) + \min\{\deg_G v_1, \deg_G v_2\}$. $\qquad\square$

The arguments used in the proof of Theorem 6 also imply that if $\beta'_C(G, K_G) > 0$, then the assignment number of (H, K_H) is equal to the number of consistent vertex-assignments k of (G, K_G) such that $k(v_1) = k(v_2)$.

We now consider labeled graphs (G, K_G) which are not connected. Let G_1 and G_2 denote two subgraphs of G such that $E(V(G_1), V(G_2)) = \emptyset$ and let $K_1 \colon E(G_1) \mapsto S_n$ and $K_2 \colon E(G_2) \mapsto S_n$ be edge labelings of G_1 and G_2, respectively, in which $K_i(e) = K_G(e)$ for all edges e of G_i.

Theorem 7. *Let (G, K_G) be a disconnected labeled graph and let $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ be two vertices of G not connected by a path. If $H = G_{v_1=v_2}$ and K_H is the inherited edge-labeling of H, then the following inequalities hold:*

$$\beta'_C(G_1, K_1) + \beta'_C(G_2, K_2) - n \leq \beta'_C(H, K_H) \leq \min\{\beta'_C(G_1, K_1), \beta'_C(G_2, K_2)\}.$$

Proof. Observation 3 implies that $\beta'_C(H, K_H) \leq \beta'_C(G_1, K_1)$ and $\beta'_C(H, K_H) \leq \beta'_C(G_2, K_2)$. It follows that $\beta'_C(H, K_H) \leq \min\{\beta'_C(G_1, K_1), \beta'_C(G_2, K_2)\}$.

Let us note that for a given vertex u and $t \in [n]$ a labeled graph has at most one consistent vertex-assignment k such that $k(u) = t$.

Now let A_i for $i \in \{1, 2\}$ denote the set of values $t \in [n]$ for which consistent vertex-assignments $k_i^t \colon V(G_i) \mapsto [n]$ such that $k_i^t(v) = t$. The assignment $k^t \colon V(H) \mapsto [n]$ such that $k^t(v) = t$, for $t \in [n]$, exists if and only if $t \in A_1 \cap A_2$. It follows that, $\beta'_C(H, K_H) = |A_1 \cap A_2|$.

It is clear that $|A_1 \cup A_2| \leq n$ and that $|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2|$. Thus, $\beta'_C(H, K_H) = \beta'_C(G_1, K_1) + \beta'_C(G_2, K_2) - |A_1 \cup A_2|$, and therefore

$$\beta'_C(H, K_H) \geq \beta'_C(G_1, K_1) + \beta'_C(G_2, K_2) - n$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Since $\beta'_C(G, K) > 0$ implies $\beta_C(G, K) = 0$, we have the following corollary.

Corollary 3. *Let G be a disconnected graph with components G_1 and G_2 and, let K_1 and K_2 be edge-labelings of G_1 and G_2, respectively, $H = G_{v_1=v_2}$ for $v_1 \in V(G_1), v_2 \in V(G_2)$ and let K_H be the inherited edge-labeling of G. If $\beta'_C(G_1, K_1) + \beta'_C(G_2, K_2) > n$, then $\beta_C(H, K_H) = 0$.*

Proof. If $\beta'_C(G_1, K_1) + \beta'_C(G_2, K_2) > n$, then $0 < \beta'_C(G_1, K_1) + \beta'_C(G_2, K_2) - n \leq \beta'_C(H, K_H)$ and therefore $\beta_C(H, K_H) = 0$. $\qquad\square$

4 Labeled graph equivalence

A signed graph Σ consists of a graph G and a labeling $\hat{K} \colon E(G) \mapsto \{+, -\}$. We say that a cycle of Σ is *balanced* if it contains an even number of negative (" $-$ ") edges. A signed graph is *in balance* if all of its cycles are balanced. It is easy to see that this corresponds directly to an edge-labeling $K \colon E(G) \mapsto S_2$, where $K(e) = id$ if $\hat{K}(e) = +$ and $K(e) = (01)$ otherwise. A signed graph $\Sigma = (G, \hat{K})$ is in balance if and only if $\beta_C(G, K) = 0$. Two signed graphs Σ_1 and Σ_2 are said to be *equivalent* if they have the same underlying graph G and the same set of balanced cycles. The *switching operation* is defined by changing the signs $\hat{K}(e)$ of all edges incident to a certain vertex $v \in V(G)$. Two signed graphs are equivalent if and only if one

can be obtained from the other by some isomorphism of the underlying graphs and a sequence of switches. We can define a similar equivalence for labeled graphs. Since different labelings of the same graph G, such that β_C and β'_C of each cycle are the same, do not necessarily need to have the same β_C or β'_C for the whole graph, an equivalence based on that would be meaningless. Instead, we consider two labeled graphs (G_1, K_1) and (G_2, K_2) to be equivalent if one can be obtained from the other one by an isomorphism between G_1 and G_2, replacing a directed edge uv labeled with π with an edge vu labeled with π^{-1}, and switching operations $s(v, \sigma)$ defined as follows. For any graph G and edge-labeling $K : E(G) \mapsto S_n$, let $v \in V(G)$ be any vertex of G and let σ be any permutation of $[n]$. For every vertex $u \in N_G(v)$:

1. if $uv \in E$, we replace $K(uv) = \pi$ with $K'(uv) = \sigma\pi$,

2. if $vu \in E$, we replace $K(vu) = \pi$ with $K'(vu) = \pi\sigma^{-1}$.

Here, $N_G(v)$ is the open neighborhood of v in G.

Theorem 8. *For any given $G_1, G_2, K_1 : E(G_1) \mapsto S_n, K_2 : E(G_2) \mapsto S_n$ the labeled graphs (G_1, K_1) and (G_2, K_2) are equivalent if and only if there exists an isomorphism $F : V(K_1G_1) \mapsto V(K_2G_2)$ such that for every $v_i \in V(G_1)$ there exists a $u_i \in V(G_2)$ such that $F(\{v_{i0}, ..., v_{in-1}\}) = \{u_{i0}, ..., u_{in-1}\}$*

Proof. First note that if $\pi(x) = y$, then $\pi^{-1}(y) = x$. It folows that changing a directed edge $uv \in E(G_1)$ labeled with a permutation $K_1(uv) = \pi$ to vu labeled with π^{-1} results in (G_2, K_2) identical to (G_1, K_1).

Suppose (G_2, K_2) can be obtained from (G_1, K_1) by some isomorphism $f : V(G_1) \mapsto V(G_2)$ and a series of switches $s_1, ..., s_t$ where $s_k = s(v_k, \sigma_k)$. We can define functions $\hat{f}, \hat{s}_1, ..., \hat{s}_t : V(K_1G_1) \mapsto V(K_2G_2)$ as follows: $\hat{f}(v_{ij}) = v_{f(i)j}$ and

$$\hat{s}_k(v_{ij}) = \begin{cases} v_{ij}, & i \neq k; \\ v_{k\sigma_k(j)}, & i = k. \end{cases}$$

It is easy to see that the function $F = \hat{f} \circ \hat{s}_1 \circ ... \circ \hat{s}_t$ is an isomorphism from K_1G_1 to K_2G_2 and that for every $v_i \in V(G_1)$ there exists a $u_i \in V(G_2)$ such that $F(\{v_{i0}, ..., v_{in-1}\}) = \{u_{i0}, ..., u_{in-1}\}$

Now, let $F : V(K_1G_1) \mapsto V(K_2G_2)$ be an isomorphism such that $F(V_i) = U_j$, where $V_i = \{v_{i0}, ..., v_{in-1}\} \subset V(K_1G_1)$ and $U_i = \{u_{i0}, ..., u_{in-1}\} \subset V(K_2G_2)$. This isomorphism can be presented as $F = \hat{f} \circ \hat{s}_1 \circ ... \circ \hat{s}_t$ such that $\hat{f}(v_{ij}) = v_{i'j}$ and

$$\hat{s}_k(v_{ij}) = \begin{cases} v_{ij}, & i \neq k; \\ v_{kj'}, & i = k; \end{cases}$$

where $i' = f(i)$ for some isomorphism $f : G_1 \mapsto G_2$ and $j' = \sigma_k(j)$ for some permutation $\sigma_k \in S_n$. $\qquad\square$

It follows that equivalent labeled graphs must have the same contradiction and assignment numbers. All equivalent labelings of a given graph G must also have an optimal vertex-labeling with the same set of contradictions.

5   Labeling with S_2

Labeling with S_2 is equivalent to the problem of balance in a signed graph. In [3] Harary proved the following result.

Theorem 9 (Harary [3]). *A signed graph is balanced if and only if its set of vertices can be partitioned into two disjoint subsets in such a way that each positive edge joins two vertices in the same subset while each negative edge joins two vertices from different subsets.*

It follows that a signed graph is *in balance* if and only if the corresponding labeled graph has a consistent vertex-assignment.

Edge-labelings with S_2 correspond to a class of unique games known as XOR games, in which the value of each variable can be either 0 or 1 and the constraints are given in the form of the XOR of those values, *i.e.* the constraint is satisfied if and only if $x + y \equiv i \pmod{2}$, where i is either 0 (*i.e.* $\pi = id$) or 1 ($\pi = (01)$). It is easy to see that for the edge-labeling $K : E \mapsto S_n$ of any graph G such that $K(e) = id$ for every $e \in E$, $\beta_C(G, K) = 0$, $\beta'_C(G, K) = n$ and the only consistent vertex-assignments are those which assign the same value to every vertex. Let us consider the edge-labeling $K : E \mapsto S_2$ in which $K(e) = (01)$ for all $e \in E$.

Proposition 10. *The edge-labeling $K : E \mapsto \{(01)\} \subset S_2$ (i.e. a labeling using only the permutation $\pi = (01)$) of a given graph $G \neq \overline{K_n}$ is vertex-proper if and only if G is bipartite.*

Proof. Let $k : V \mapsto [2]$ be a consistent vertex assignment. Let $A = \{v \in V : c(v) = 0\}$ and $B = \{v \in V : k(v) = 1\}$. The two sets are independent and $A \cup B = V$. Thus G is a bipartite graph. Let G be a bipartite graph. The vertex set of G can be paritioned into two independent subsets A and B. The vertex assignments k and k' such that $k(v) = 0$ if $v \in A$ and $k(v) = 1$ if $v \in B$ and $k'(v) = 0$ if $v \in B$ and $k'(v) = 1$ if $v \in A$ are the only consistent vertex-assignments. $\qquad\square$

The next fact follows directly from Theorem 8.

Proposition 11. *Let $K : E \mapsto \{(ab)\} \subset S_n$ for $n \geq 3$ be an edge labeling of a non-bipartite graph $G \neq \overline{K_n}$ in which $K(e) = (ab)$ for all $e \in E$. The only consistent vertex-labelings are $k : V \mapsto \{c\} \subset [n]$, where c is a fixed point of the permutation (ab). If G is bipartite, a consistent vertex-labeling using a and b exists.*

What can we say about $\beta_C(G, (01))$, for a generic graph G? Let us assume that G is non-bipartite. Then, $\beta_C(G, (01))$ is the minimum number of edges that we need to delete from G to obtain a bipartite graph. Computing $\beta_C(G, (01))$ is at least as hard as asking whether a bipartite graph can be obtained by deleting at most k edges from G. This problem is called EDGE BIPARTIZATION (or, equivalently, (unweighted) MINIMUM UNCUT). The problem is a way to measure how close G is to being bipartite. It is known to be MaxSNP-hard and can be approximated to a factor of $O(\sqrt{\log n})$ in polynomial time, where n is the total number of vertices (see [4]). Because of this point, we can simplify the notation: the *edge bipartization number* of a graph G, $\beta_C^2(G)$, is the minimum number of edges that we need to delete in order to obtain a bipartite graph. By the above definitions, $\beta_C(G, (01)) = \beta_C^2(G)$.

6   Edge-labelings with Latin squares

A *Latin square* L of order n is an $n \times n$ array $\{l_{ij}\}_{n \times n}$ such that $l_{ij} \neq l_{is}$ and $l_{ij} \neq l_{tj}$ for $s \neq j$ and $t \neq i$. A an $n \times n$ Latin square defines a set of n permutations $\pi_i : [n] \mapsto [n]$. For each row in the array π_i is defined by the set of pairs $\{(0, l_{i0}), ..., (n-1, l_{in-1})\}$. Let us consider a particular set $L_n = \{\pi_0, ..., \pi_{n-1}\}$ of permutations where, for every $i \in [n]$ we have $\pi_i(x) \equiv i - x \pmod{n}$. Note that L_n corresponds to the Latin square in which $l_{ij} \equiv i - j \pmod{n}$, for all $i, j \in [n]$. In any set of permutations defined by a Latin square, we have $\pi_i(x) \neq \pi_j(x)$ if $i \neq j$. Since every element $i \in [n]$ occurs exactly once in each column of a given Latin square L, clearly every $i \in [n]$ is a fixed point of exactly one permutation in the set defined by L. If the set of permutations contains the identity, no other permutation may have a fixed point. Furthermore, if n is odd, then each permutation in L_n has exactly one fixed point.

Observation 5. *For any edge-labeling $K : E \mapsto L_n$, a cycle of even length can have either 0 or n consistent vertex-labelings.*

Proof. Let $L'_n = \{\sigma_0, ..., \sigma_{n-1}\}$ be a set of permutations such that $\sigma_i(x) \equiv i + x \pmod{n}$. It is clear that for any two permutations $\pi_i, \pi_j \in L_n$, $\pi_j \pi_i \in L'_n$, since $\pi_j \pi_i(x) = j - (i - x) \equiv (j - i) + x \pmod{n}$. Now, for any $\sigma_i, \sigma_j \in L'_n$, we have $\sigma_j \sigma_i \in L'_n$, since $\sigma_j \sigma_i(x) = j + (i + x) \equiv (i + j) + x \pmod{n}$. It follows that $\sigma_k ... \sigma_1 \in L'_n$ for any finite k. Now let C_{2k} be any even cycle and let $K : E(C) \mapsto L_n$ be an edge-labeling of C_{2k}. Let $\pi_C = \pi_{2k} ... \pi_1$, where π_i is the permutation assigned to the i-th edge of C_{2k}. Obviously, $\pi_C = \sigma_k ... \sigma_1$, where $\sigma_i = \pi_{2i} \pi_{2i-1}$. Since $\sigma_i \in L'_n$ for $i = 1, 2, ..., k$, this implies that $\pi_C \in L'_n$. The only permutation in L'_n which has a fixed point is $\sigma_0 = id$. Thus, either $\pi_C = id$ and C_{2k} has n consistent vertex-assignments or π_C has no fixed point and C_{2k} has no consistent assignments. □

Observation 6. *For any edge-labeling $K : E \mapsto L_n$, for $n \geq 3$ a cycle of odd length has at most 2 consistent vertex-labelings.*

Proof. Let $L'_n = \{\sigma_0, ..., \sigma_{n-1}\}$ be a set of permutations such that $\sigma_i(x) \equiv i + x \pmod{n}$. Now, let C_{2k+1} be a cycle of odd length. and let $K : E(C) \mapsto L_n$ be an edge-labeling of C_{2k+1}. Let $\pi_C = \pi_{2k+1} ... \pi_1$, where π_i is the permutation assigned to the i-th edge of C_{2k+1}. Then $\pi_C = \pi_{2k+1} \sigma$, where $\sigma = \pi_{2k} ... \pi_1$. By the above observation, $\sigma \in L'_n$. For all $x \in [n]$ we have $\pi_{2k+i}(x) \equiv i - x \pmod{n}$ and $\sigma(x) \equiv j + x \pmod{n}$ for some $i, j \in [n]$. Thus $\pi(\sigma(x)) = i - (j + x) \equiv (i - j) + x \pmod{n}$ and therefore $\pi_C \in L_n$. It follows that C_{2k+1} has exactly one vertex-assignment if n is odd and either zero or two if n is even. □

An interesting set of permutations is L'_n, in which for every $i \in [n]$ we have $\sigma(x)_i = i + x$. Here $\sigma^{-1} \neq \sigma$ except for $\sigma_0 = id$, so we can only consider those labelings on a directed graph. A contradiction in a given vertex-labeling k is now a directed edge uv such that $\sigma(k(u)) \neq k(v)$, where σ is the permutation assigned to the edge uv.

Observation 7. *Let G be a directed graph and let $K : E(G) \mapsto L'_n$ be an edge-labeling of G. The graph G can have either 0 or n consistent vertex-labelings.*

Proof. A graph G can only be ugly if it contains ugly cycles. Thus, let us consider a cycle $C_t = \{v_1, ..., v_t, v_{t+1} = v_1\} \in G$ and let σ_i denote the permutation assigned to the i-th edge of C_n. Let $\pi_{C_t} = \pi'_t ... \pi'_1$, where $\pi'_i = \pi_i$ if $e_i = v_i v_{i+1}$ and $\pi'_i = \pi_i^{-1}$ if $e_i = v_{i+1} v_i$. The assignment

number of the C_t is equal to the number of fixed points of π_{C_t}. For any $\sigma_1, \sigma_2 \in L'_n$ we have $\sigma_2\sigma_1, \sigma_1^{-1} \in L'_n$, so $\pi_{C_t} \in L'_n$. The only permutation is L'_n which has a fixed point is the identity. Thus, either $\pi_{C_t} = id$ and C_t is good or π_{C_t} has no fixed point and C_t is bad. $\square$

6.1 Bipartite graphs

A bipartite graph represents a two-player game. The referee asks each of the players, who are not allowed to communicate, to assign a value from $[n]$ to a variable. The players win if their answers satisfy the constraint $\pi(a) = b$. The value of such a game is the maximum probability of winning.

Theorem 12. *For any edge-labeling $K : E \mapsto L_n$ a bipartite graph G is bad if and only if it contains a bad chordless cycle.*

Proof. Let G be a bipartite graph. Obviously, if G has no cycles, it has to be good. By Observation 6, the graph G contains no ugly cycles. Thus, G is bad if and only if it contains a bad cycle. We will now prove that if G contains a bad cycle, it must contain a bad chordless cycle. Now, let $C_n = v_1...v_n$ for some $n \geq 6$ be a bad cycle in G. Let $e_i = v_i v_{i+1}$ for $i \in \{1, ..., 2k-1\}$, $e_{2k} = v_{2k}v_1$ and $K(e_i) = \pi_i$. Assume that an edge $e_{n+1} = v_1 v_k$ exists, for some $k \in \{4, 6, ..., n-2\}$. Let $\pi_{n+1} = K(e_{n+1})$. Suppose both $v_1 v_2...v_k$ and $v_k v_{k+1}...v_1$ are good cycles. Then $\pi_{n+1}\pi_k\pi_{k-1}...\pi_1 = id$ and $\pi_{n+1}\pi_n...\pi_{k+1} = id$. It follows that $\pi_{n+1} = \pi_n...\pi_{k+1}$ and therefore $\pi_n...\pi_1 = id$, a contradiction. $\square$

In the complete bipartite graph $K_{s,t}$ all chordless cycles have length 4 and hence we have the following:

Corollary 4. *For any edge-labeling $K : E \mapsto L_n$, a complete bipartite graph $K_{s,t}$ is bad if and only if it contains a bad 4-cycle.*

Proof. It follows from Observation 6 that bipartite graphs contain no ugly cycles. Thus $K_{s,t}$ is bad if and only if it contains at least one bad cycle. First, we will prove that if $K_{s,t}$ contains a bad 6-cycle, it must also contain a bad 4-cycle. Let $v_1 v_2 v_3 v_4 v_5 v_6$ be a cycle of length 6, where $e_i = v_i v_i + 1$ for $i \in \{1, ..., 5\}$, $e_6 = v_6 v_1$ and $K(e_i) = \pi_i$. Since $K_{s,t}$ is a complete bipartite graph, an edge $e_7 = v_1 v_4$ exists. Let $K(e_7) = \pi_7$. Suppose all 4-cycles in $K_{s,t}$ are good. Then $\pi_7\pi_3\pi_2\pi_1 = id$ and $\pi_7^{-1}\pi_6\pi_5\pi_4 = id$. It follows that $\pi_6\pi_5\pi_4 = \pi_7$. Therefore $\pi_6\pi_5\pi_4\pi_3\pi_2\pi_1 = id$ and the 6-cycle $v_1 v_2 v_3 v_4 v_5 v_6$ is good. Thus a bad 6-cycle can only exist if there is a bad 4-cycle. Now, let $v_1...v_{2k}$ be a cycle, where $k > 3$. Let $e_i = v_i v_{i+1}$ for $i \in \{1, ..., 2k-1\}$, $e_{2k} = v_{2k}v_1$ and $K(e_i) = \pi_i$. Assume that a bad $2k-2$-cycle can only exist if a bad 4-cycle exists. Let $e_{2k+1} = v_1 v_4$ and $K(e_{2k+1}) = \pi_{2k+1}$. If all 4-cycles in the graph are good, then $\pi_{2k+1}\pi_3\pi_2\pi_1 = id$ and $\pi_{2k+1}^{-1}\pi_{2k}\pi_{2k-1}...\pi_4 = id$. It follows that $\pi_{2k}\pi_{2k-1}...\pi_4 = \pi_{2k+1}$ and therefore $\pi_{2k}...\pi_1 = id$. Thus, the $2k$-cycle must be good. $\square$

6.2 Other graphs

Every non-bipartite graph contains at least one odd cycle. It follows, that for any edge-labeling $K : E \mapsto L_n$ where $n \geq 3$ a non-bipartite graph cannot be good.

Observation 8. *Let G be a non-bipartite graph and let $K : E \mapsto L_n$ for some $n \geq 3$ be an edge-labeling of G. If n is odd then $\beta'_C(G, K) \leq 1$. If n is even then $\beta'_C(G, K) \leq 2$.*

Proof. If H is a subgraph of G and K' is an edge-labeling of H such that $K'(e) = K(e)$, then $\beta'_C(G, K) \leq \beta'_C(H, K')$. By Observation 5 the assignment number of an odd cycle is at most 2 for an even n and 1 for an odd n. Thus, the assignment number of any graph containing an odd cycle must be at most 2 for even n and at most 1 for odd n. $\square$

It follows that the assignment number of a graph labeled with L_n has to be either $0, 1, 2$ or n.

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