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Inference about Non-Identified SVARs*

Raffaella Giacomini[†] and Toru Kitagawa[‡]

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Abstract

We propose a method for conducting inference on impulse responses in structural vector autoregressions (SVARs) when the impulse response is not point identified because the number of equality restrictions one can credibly impose is not sufficient for point identification and/or one imposes sign restrictions. We proceed in three steps. We first define the object of interest as the identified set for a given impulse response at a given horizon and discuss how inference is simple when the identified set is convex, as one can limit attention to the set's upper and lower bounds. We then provide easily verifiable conditions on the type of equality and sign restrictions that guarantee convexity. These cover most cases of practical interest, with exceptions including sign restrictions on multiple shocks and equality restrictions that make the impulse response locally, but not globally, identified. Second, we show how to conduct inference on the identified set. We adopt a robust Bayes approach that considers the class of all possible priors for the non-identified aspects of the model and delivers a class of associated posteriors. We summarize the posterior class by reporting the "posterior mean bounds", which can be interpreted as an estimator of the identified set. We also consider a "robustified credible region" which is a measure of the posterior uncertainty about the identified set. The two intervals can be obtained using a computationally convenient numerical procedure. Third, we show that the posterior bounds converge asymptotically to the identified set if the set is convex. If the identified set is not convex, our posterior bounds can be interpreted as an estimator of the convex hull of the identified set. Finally, a useful diagnostic tool delivered by our procedure is the posterior belief about the plausibility of the imposed identifying restrictions.

Keywords: Partial causal ordering, Ambiguous beliefs, Posterior bounds, Credible region

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1 Introduction

Structural vector autoregressions (SVARs) (Sims (1980)) are one of the main tools in macroeconomic policy analysis and their use is widespread in empirical work. The common practice in SVAR analysis is to assume a sufficient number of identifying restrictions that guarantee that knowledge of the reduced-form parameters can uniquely pin down the underlying structural parameters. The estimation results and the policy implications of the model crucially rely on these identifying restrictions but the literature is far from having reached a consensus on which restrictions are credible or justifiable on economic grounds. For example, for the common cases of causal ordering restrictions (Sims (1980) and Bernanke (1986)) or long-run neutrality restrictions (Blanchard and Quah (1993) among others), economic theory can be invoked to justify only a small number of the restrictions that are typically needed to achieve point identification. For this reason, one often ends up conducting policy analysis in the context of small models and a tradeoff emerges between the empirical fit of the model and the plausibility of the imposed identifying restrictions. This paper offers a way to make the choice of the variables to include in the model independent of the number of credible identifying restrictions that are available.

We propose a method for conducting inference on an impulse response when imposing a subset of the equality restrictions that are needed for point identification and/or when imposing sign restrictions on the model's parameters or on the impulse responses (as in Canova and Nicolo (2002), Faust (1998), and Uhlig (2005)). The method gives the researcher the flexibility to only impose the set of identifying restrictions that she deems credible, for example by assuming a partial causal ordering for the variables in the model.

We proceed in three steps. We first define the object of interest of our analysis as the identified set for a given impulse response at a given horizon, which is a subset of the real line. We characterize the identified set implied by imposing a collection of under-identifying equality restrictions and/or sign restrictions. Our first insight is that conducting inference about the identified set is facilitated by restricting attention to convex sets, in which case one only needs to focus on the set's upper and lower bounds. We then provide sufficient conditions that can be used to verify if a collection of equality and/or sign restrictions gives rise to a convex set for the impulse response of interest. The conditions are easy to verify and cover most cases of practical interest, with exceptions including sign restrictions on multiple shocks and equality restrictions that make the impulse response locally, but not globally, identified. It is worth emphasizing that the method that we propose can be implemented regardless of whether the convexity conditions hold. When convexity is violated, our procedure can be interpreted as providing an estimator of the convex hull of the impulse response identified set.

To fix ideas and gain intuition about the convexity result, consider a simplified model for the $n-$dimensional vector y_t, $A_0 y_t = \varepsilon_t$, so that the impulse response of interest is one element of A_0^{-1}. In the absence of identifying restrictions, knowledge of the reduced-form parameter $E\left(u_t u_t'\right) = \Sigma$ from the model $y_t = u_t$ is not enough to pin down a unique A_0 but, rather, it gives a set of observationally equivalent $A_0 : \left\{A_0 = Q'\Sigma_{tr}^{-1},\ Q \in \mathcal{O}(n)\right\}$ (with Σ_{tr} the lower triangular matrix from the Cholesky decomposition of Σ), for all rotation matrices Q ranging over the space of orthonormal matrices $\mathcal{O}(n)$. This in turns defines the identified set for the impulse response r of the $i-th$ variable to the $j-th$ shock as $\{r = e_i'\Sigma_{tr}Qe_j = e_i'\Sigma_{tr}q_j,\ q_j \in \mathcal{S}\}$ (with e_i the $i-th$ column vector of the identity matrix I_n), for the $j-th$ column vector q_j of Q ranging over the n-dimensional unit sphere $\mathcal{S}$. Imposing identifying restrictions can be viewed as progressively restricting the space of feasible $q_j's$ until the impulse response becomes point identified. Our first result is to provide sufficient conditions on the type of equality and inequality restrictions that ensure that the identified set for r is convex. Intuitively, in the case of equality restrictions, convexity is guaranteed when the subspace of $\mathcal{S}$ obtained by imposing the restrictions has dimension greater than one since in this case any two elements q_j and $q_{j'}$ are path-connected, which in turn yields a convex identified set for r because r is a continuous function of q_j. When the subspace of $\mathcal{S}$ has dimension one non-convexity can occur because, for example, the impulse response identified set consists of two disconnected points - meaning that the impulse response is locally, but not globally identified.

The second step of our analysis considers a method for conducting inference on the impulse response identified set. Frequentist inference on the impulse response identified set has been considered by Moon et al. (2013) in special cases of sign and zero restrictions, but the validity of their method is not known for our general class of equality and sign restrictions. Moreover, a drawback of their approach is its high computational cost as the construction of confidence intervals involves a grid-search of non-rejection regions of hypothesis tests using resampling methods. In ongoing research, Gafarov and Montiel-Olea (2014) show that it is possible to develop a computationally convenient approach to frequentist inference when the number of restrictions is smaller than $n-2$. The majority of the literature on SVARs with sign restrictions has adopted a Bayesian approach to inference. Bayesian inference has some appealing features, such as the ability to make inferential statements that are conditional on the data and do not rely on asymptotic approximations, as well as the possibility to incorporate prior beliefs about the reduced-form parameters. It is also computationally convenient. However, Bayesian interval estimators do not converge asymptotically to the true identified set (Moon and Schorfheide, 2012) but lead to informative inference due to the effect of the choice of prior, an effect that does not disappear asymptotically as in the point-identified case (Kadane (1974), Poirier (1998), and Moon and Schorfheide (2012). See also the recent discussion in Baumeister and Hamilton (2013)). This can be perceived as a drawback as it is not clear how to choose such a prior in the context of partially identified SVARs.

In this paper, we consider a robust Bayes approach to inference that overcomes the sensitivity of standard Bayesian inference to the choice of prior. To gain intuition about the robust Bayes approach, note that a single prior for the reduced-form parameters cannot yield a single posterior for the structural parameters if the parameters are not identified, but one must also choose a prior for the non-identified components of the model (in our case the rotation matrix Q). This is a difficult task if the only available prior knowledge about Q is exhausted by the set of identifying restrictions. Choosing an uninformative prior for Q does not necessarily provide a solution as it can lead to unintentionally informative priors for the impulse responses, which in turn may influence the posterior analysis no matter how large the sample size is. The robust Bayes approach overcomes this drawback by not choosing a single prior for Q, but considering instead the class of all possible priors for Q that satisfy the identifying restrictions. Combining the multiple priors for Q with a single prior for the reduced-form parameters and applying Bayes rule yields a class of posteriors, which can be interpreted as summarizing the posterior distribution of the impulse response identified set. In practice, we suggest reporting two intervals for a given impulse response: a posterior mean bounds interval, which can be interpreted as an estimator of the identified set, and an associated robustified credible region, which is a measure of the posterior uncertainty about the identified set. The two intervals are computationally convenient to obtain by applying a numerical procedure that uses draws for only the upper and lower bounds of the identified set.

The multiple prior approach that characterizes robust Bayes inference can be motivated in our context by assuming that one has "ambiguous beliefs" about the aspects of the model for which the data are not informative, in the sense of not being able to judge which priors are more credible than others within a given class. In a SVAR, the data are only informative about the reduced form parameters and not about the rotation matrices, so our proposal can be interpreted as reporting the posterior information for the impulse response based only on the well-updated part of the beliefs, or, alternatively, based only on the shape of the likelihood if the prior for the reduced form parameters is uninformative. Robust Bayes inference has been considered in the statistics literature (Berger and Berliner (1986), DeRobertis and Hartigan (1981), and Wasserman (1990)) and in econometrics (Chamberlain and Leamer (1976) and Leamer (1982)) for the linear regression model, but in both cases only for point identified models. Kitagawa (2012) focuses on the construction of robustified credible regions in a general partially identified model and provides conditions such that the robustified credible region attains correct frequentist coverage for the identified set. This paper builds on Kitagawa (2012) and extends the approach to the more complex setting of SVAR models, where the conditions in Kitagawa (2012) are violated. In addition to considering the robustified credible sets analyzed by Kitagawa (2012), the paper broadens the analysis in several directions: we characterize the topology of the identified set and provide conditions for its convexity; we introduce the posterior mean bounds and interpret them as an estimator of the identified set; we investigate the asymptotic frequentist properties of both posterior mean bounds and robustified credible sets.

The third step of our analysis investigates the asymptotic properties of our procedure and shows that the posterior mean bounds and robustified credible region asymptotically converge to the identified set, when the set is convex. This implies that in large samples one can expect our posterior mean bounds estimator to be similar to the estimator of the identified set obtained in the context of the frequentist approach.

Finally, a useful diagnostic tool that is a by-product of our analysis is the ability to separately report the posterior belief for the plausibility of the imposed identifying restrictions and the posterior belief for the impulse responses, conditional on the imposed assumptions being plausible (in the sense of not contradicting the observed data). Note that if one were to adopt a frequentist approach it would generally be difficult to separate these two types of sample information, as discussed by Sims and Zha (1998).

The remainder of the paper is organized as follows. Section 2 introduces the notation and the general analytical framework of SVARs with equality and/or sign restrictions. Section 3 characterizes the impulse response identified set and provides conditions for its convexity. Section 4 proposes a robust Bayes approach to conducting inference on the identified set. Section 5 presents a numerical procedure for computing an estimator of the identified set and its associated robustified credible region. Section 6 shows that the proposed estimator converges asymptotically to the true identified set. An empirical example is contained in Section 7. The proofs are collected in the Appendix.

2 The Econometric Framework

Consider a SVAR(p) model

$$A_0 y_t = a + \sum_{j=1}^{p} A_j y_{t-j} + \epsilon_t \quad \text{for } t = 1, \ldots, T,$$

where y_t is an $n \times 1$ vector, ϵ_t an $n \times 1$ vector white noise process, normally distributed with mean zero and variance-covariance matrix I_n, the $n \times n$ identity matrix. Note that we assume the structural shocks to be uncorrelated, as is common in the SVAR literature. The initial conditions $y_1, \ldots, y_p$ are given.

The reduced form VAR representation of the model is

$$y_t = b + \sum_{j=1}^{p} B_j y_{t-j} + u_t, \tag{2.1}$$

where $b = A_0^{-1} a$, $B_j = A_0^{-1} A_j$, $u_t = A_0^{-1} \epsilon_t$, and $E(u_t u_t') \equiv \Sigma = A_0^{-1} \left(A_0^{-1}\right)'$. We denote the reduced form parameters by $\phi = (B, \Sigma) \in \Phi \subset \mathcal{R}^{n+n^2p} \times \Omega$, where $B = [b, B_1, \ldots, B_p]$ and Ω is the space of positive-semidefinite matrices. We restrict the domain Φ to the set of $\phi' s$ such that the reduced form VAR(p) model can be inverted into a VMA(∞) model.

We denote the h-th horizon impulse response matrix by the $n \times n$ matrix IR^h, $h = 0, 1, 2, \ldots$, where the (i, j)-element of IR^h represents the effect on the i-th variable in y_{t+h} of a unit shock to the j-th element of ϵ_t. Assuming the reduced form lag polynomial $\left(I_n - \sum_{j=1}^{p} B_j L^p\right)$ is invertible, the VMA(∞) representation of the reduced form model (2.1) is

$$\begin{aligned} y_t &= c + \sum_{j=0}^{\infty} C_j (B) u_{t-j} \\ &= c + \sum_{j=0}^{\infty} C_j (B) A_0^{-1} \epsilon_{t-j}, \end{aligned} \tag{2.2}$$

where $C_j(B)$ is the j-th coefficient matrix of the inverted lag polynomial $\left(I_n - \sum_{j=1}^{p} B_j L^j\right)^{-1}$, which depends only on B. The impulse response IR^h is then given by

$$IR^h = C_h (B) A_0^{-1}. \tag{2.3}$$

The long-run impulse response matrix is defined as

$$IR^{\infty} = \lim_{h \to \infty} IR^h = \left(I_n - \sum_{j=1}^{p} B_j \right)^{-1} A_0^{-1} \tag{2.4}$$

and the long-run cumulative impulse response matrix is defined as

$$CIR^{\infty} = \sum_{h=0}^{\infty} IR^h = \left(\sum_{h=0}^{\infty} C_h (B) \right) A_0^{-1}. \tag{2.5}$$

In the absence of any identifying restrictions, knowledge of the reduced form parameters ϕ does not pin down a unique A_0. We can express the set of observationally equivalent A_0's given Σ using an orthonormal matrix $Q \in \mathcal{O}(n)$, where $\mathcal{O}(n)$ is the set of $n \times n$ orthonormal matrices. The individual column vectors in Q are denoted by $[q_1, q_2, \ldots, q_n]$. Denote the Cholesky decomposition of Σ by $\Sigma = \Sigma_{tr} \Sigma_{tr}'$, where Σ_{tr} is the unique lower-triangular Cholesky factor with nonnegative diagonal elements. Since any A_0 of the form $A_0 = Q' \Sigma_{tr}^{-1}$ satisfies $\Sigma = (A_0' A_0)^{-1}$, in the absence of any identifying restrictions $\left\{ A_0 = Q' \Sigma_{tr}^{-1} : Q \in \mathcal{O}(n) \right\}$ forms the set of A_0's that are consistent with the reduced-form variance-covariance matrix Σ (Uhlig (2005) Proposition A.1). Since the likelihood function only depends on the reduced form parameters ϕ, the data do not contain any information about Q, which leads to ambiguity in decomposing Σ. If the imposed identifying restrictions fail to identify A_0, it means that for a given Σ there are multiple Q's yielding the structural parameter matrix A_0 which satisfies the imposed restrictions.

In the absence of any identifying restrictions on A_0, the only restrictions to be imposed on Q are the sign normalization restrictions for the structural shocks. Following the identification analysis in Christiano, Eichenbaum, and Evans (1999), we impose the sign normalization restrictions on A_0, such that the diagonal elements of A_0 are all nonnegative. This means that a unit positive change in a structural shock is interpreted as a one standard-deviation positive shock to the corresponding endogenous variable.

Once the sign normalization restrictions on A_0 are imposed, the set of observationally equivalent A_0's corresponding to Σ can be expressed as

$$\left\{A_0 = Q'\Sigma_{tr}^{-1} : Q \in \mathcal{O}(n), \quad diag\left(Q'\Sigma_{tr}^{-1}\right) \geq 0\right\}, \tag{2.6}$$

where the inequality restriction $diag\left(Q'\Sigma_{tr}^{-1}\right) \geq 0$ means that all diagonal elements of $A_0 = Q'\Sigma_{tr}^{-1}$ are nonnegative. By denoting the column vectors of Σ_{tr}^{-1} as $\left[\sigma^1, \sigma^2, \ldots, \sigma^n\right]$, the sign normalization restriction can be written as a collection of linear inequalities

$$\left(\sigma^i\right)' q_i \geq 0 \quad \text{for all } i = 1, \ldots, n.$$

Suppose one is interested in a specific impulse response, say the (i, j^*)-th element of IR^h,

$$r_{ij^*}^h \equiv e_i' C_h(B) \Sigma_{tr} Q e_{j^*} \equiv c_{ih}'(\phi) q_{j^*},$$

where e_i is the i-th column vector of I_n and $c_{ih}'(\phi)$ is the i-th row vector of $C_h(B)\Sigma_{tr}$. For simplicity, we sometimes make i, j^*, and h implicit in our notation unless confusions arises, and use $r \in \mathcal{R}$ to denote the impulse response of interest, i.e., $r \equiv r_{ij^*}^h$. When we want to emphasize the dependence of r on the reduced form parameters ϕ and the rotation matrix Q, we express r as $r(\phi, Q)$. Note that the analysis developed below for the impulse responses can be easily extended to the structural parameters A_0 and $[A_1, \ldots, A_p]$, since each structural parameter can be expressed by the inner product of a vector depending on ϕ and a column vector of Q, e.g., the (i, j)-th element of A_l can be obtained as $e_j'\left(\Sigma_{tr}^{-1} B_l\right)' q_i$.

3 The Impulse Response Identified Set

In this section we characterize the impulse response identified set obtained by imposing a collection of under-identifying equality restrictions and/or sign restrictions and provide conditions that guarantee that the set is convex.

3.1 Under-identifying Equality Restrictions

We begin by considering a set of equality restrictions that are not sufficient for point identification of the impulse response. The equality restrictions could for example be zero restrictions on some off-diagonal elements of A_0^{-1}, e.g., a subset of the restrictions imposed by the common Sims-Bernanke recursive identification strategy that sets the upper-triangular components of A_0^{-1} to zero. This amounts to assuming only a partial causal ordering for the variables in the model while allowing arbitrary contemporaneous relationships among the remaining variables (see Example 3.1 below). More in general, if some structural equation in the system represents the behavioral response of a sector or an economic agent to the macroeconomic variables, zero restrictions can be placed on the elements of A_0 by invoking economic theory or available institutional knowledge. Note that, since the contemporaneous impulse response matrix is $IR^0 = A_0^{-1}$, zero restrictions that set some of the contemporaneous impulse responses to zero can be seen as zero restrictions on the corresponding elements of A_0^{-1} and hence can be treated as part of the causal ordering restrictions. Our framework also accommodates zero restrictions on the lagged coefficients $\{A_l : l = 1, \dots p\}$ as well as restrictions on the long-run impulse responses, which are zero restrictions on some elements of the long-run impulse response $IR^{\infty} = \left(I - \sum_{j=1}^{p} B_j\right)^{-1} \Sigma_{tr} Q$ or the long-run cumulative impulse response, $CIR^{\infty} = \sum_{h=0}^{\infty} C_h(B)\Sigma_{tr} Q$.

Since A_0^{-1}, A_0, $\{A_l : l = 1, \dots, p\}$, and $\left\{IR^h : h = 1, 2 \dots, \infty\right\}$ are products of Q and a matrix that depends only on the reduced-form parameters, all the zero restrictions listed above can be viewed as imposing linear constraints on the columns of Q, with coefficients depending on the reduced-form parameters $\phi = (\Sigma, B)$. For example:

$$\begin{aligned}
\left((i,j)\text{-th element of } A_0^{-1}\right) &= 0 \iff \left(e_i'\Sigma_{tr}\right) q_j = 0, \\
\left((i,j)\text{-th element of } A_0\right) &= 0 \iff \left(\Sigma_{tr}^{-1} e_j\right)' q_i = 0, \\
\left((i,j)\text{-th element of } A_l\right) &= 0 \iff \left(\Sigma_{tr}^{-1} B_l e_j\right)' q_i = 0, \\
\left((i,j)\text{-th element of } CIR^{\infty}\right) &= 0 \iff \left[e_i' \sum_{h=0}^{\infty} C_h(B)\Sigma_{tr}\right] q_j = 0.
\end{aligned} \tag{3.1}$$

We can thus represent a collection of zero restrictions in the following general form:

$$F(\phi, Q) \equiv \begin{pmatrix} F_1(\phi) q_1 \\ F_2(\phi) q_2 \\ \vdots \\ F_n(\phi) q_n \end{pmatrix} = \mathbf{0}, \tag{3.2}$$

where $F_i(\phi)$ is an $f_i \times n$ matrix that depends only on the reduced form parameters $\phi = (B, \Sigma)$. Each row vector in $F_i(\phi)$ corresponds to the coefficient vector of a zero restriction that constrains q_i as in (3.1), and $F_i(\phi)$ stacks all the coefficient vectors that multiply q_i into a matrix. Hence, f_i is the number of zero restrictions constraining q_i. If the set of zero restrictions does not constrain q_i, $F_i(\phi)$ does not exist and thus $f_i = 0$.

Without loss of generality, we make the following assumption on the ordering of the variables in the model.

Notation 3.1 *(Ordering of variables) The variables in the SVAR are ordered so that the number of equality restrictions f_i imposed on the $i-th$ column of Q (i.e., the rows of $F_i(\phi)$ in (3.2)) satisfy $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$. In case of ties, if the impulse response of interest is that to the j^*-th structural shock, let the $j^* - th$ variable be ordered first. That is, set $j^* = 1$ when no other* *column vector has a larger number of restrictions than q_{j^*}. If $j^* \geq 2$, then order the variables so that $f_{j^*-1} > f_{j^*}$.*

Note that our assumption for the ordering of the variables pins down a unique j^*, while it does not necessarily yield a unique ordering for the other variables if some of them admit the same number of constraints. However, the condition for the convexity of the identified set for the impulse responses to the j^*-th structural shock that we provide in Lemma 3.1 is not affected by the ordering chosen for the other variables as long as the f_i's are in decreasing order.

Rubio-Ramirez et. al. (2010) focus on point identification in SVARs subject to equality restrictions of the form (3.2) and their conditions for point identification provide a starting point for our analysis. Rubio-Ramirez et. al. (2010) define the parameters to be exactly identified if, for almost every $\phi \in \Phi$, there exist unique $(A_0, A_1, \ldots, A_p)$ satisfying the identifying restrictions, which can be equivalently stated as saying that there is a unique Q satisfying $F(\phi, Q) = \mathbf{0}$ and the sign normalizations. They then show that, under regularity assumptions, a necessary and sufficient condition for point identification is that $f_i = n - i$ for all $i = 1, \ldots, n$. Here we consider restrictions that make the SVAR partially identified because

$$f_i \leq n - i \text{ for all } i = 1, \ldots, n, \tag{3.3}$$

with strict inequality for at least one $i \in \{1, \ldots, n\}$. This means that there are multiple Q's satisfying $F(\phi, Q) = \mathbf{0}$ and the sign normalizations at almost every value of ϕ. Denote by $\mathcal{Q}(\phi|F)$ the set of Q's that satisfy the restrictions (3.2) and the sign normalization given ϕ,

$$\mathcal{Q}(\phi|F) = \left\{Q \in \mathcal{O}(n) : F(\phi, Q) = \mathbf{0},\, diag\left(Q'\Sigma_{tr}^{-1}\right) \geq 0\right\}.$$

The identified set for the impulse response is a set-valued map from ϕ to a subset in $\mathcal{R}$ that gives the range of $r(\phi, Q)$ when Q varies over its domain $\mathcal{Q}(\phi|F)$,

$$IS_r(\phi|F) = \{r(\phi, Q) : Q \in \mathcal{Q}(\phi|F)\}.$$

The class of under-identified models that we consider here does not exhaust the universe of all possible non-identified SVARs, since there exist models that do not satisfy (3.3), but for which the structural parameters are not globally identified for some values of the reduced form parameters with a positive measure. For instance, the example given in Section 4.4 of Rubio-Ramirez, Waggoner, and Zha (2010) provides an example with $n = 3$ and $f_1 = f_2 = f_3 = 1$, where the structural parameters are locally identified but their global identification fails. Such locally-, but not globally-identified models would give rise to non-convex identified sets so this case will be ruled out by the conditions of Lemma 3.1 below. For another example, the zero restrictions given in page 77 of Christiano, Eichenbaum, and Evans (1999) correspond to a case with $n = 3$ and $f_1 = f_2 = f_3 = 1$, where even local identification fails. This case will also be ruled out by the conditions of Lemma 3.1.

The next lemma shows sufficient conditions such that the impulse response identified set $IS_r(\phi|F)$ is ϕ-a.s. convex.

Lemma 3.1 *(Convexity of the impulse response identified set under equality restrictions) Let* $\{r = c'_{ih}(\phi) q_{j^*} : i = 1, \ldots, n, \ h = 0, 1, 2, \ldots\}$ *be the impulse responses to the* j^**-th structural shock. Consider a collection of zero restrictions of the form given by (3.2), where the order of the variables is such that* $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$ *and* $f_{j^*-1} > f_{j^*}$ *if* $j^* \geq 2$. *Assume* $f_i \leq n - i$ *holds for all* $i = 1, 2, \ldots, n$. *Then, the identified set for* $r = c'_{ih}(\phi) q_{j^*}$ *is non-empty and bounded for every* $i \in \{1, \ldots, n\}$ *and* $h = 0, 1, 2, \ldots$, ϕ*-a.s. In addition, the identified set is convex for every* $i \in \{1, \ldots, n\}$ *and* $h = 0, 1, 2, \ldots$, ϕ*-a.s., if any of the following mutually exclusive conditions holds:*

(i) $j^* = 1$ and $f_1 < n - 1$.

(ii) $j^* \geq 2$, and $f_i < n - i$ for all $i = 1, \ldots, (j^* - 1)$.

(iii) $j^* \geq 2$ and there exists $1 \leq i^* \leq (j^* - 1)$ such that $[q_1, \ldots, q_{i^*}]$ is exactly identified (as in Definition 3.1) and $f_i < n - i$ for all $i = i^* + 1, \ldots, j^*$.

Proof. See Appendix A. ■

Below we define exact identification for a subset of the column vectors of Q.

Definition 3.1 *(Exact identification of column vectors of* Q*) Consider a collection of zero restrictions of the form given by (3.2), where the order of the variables is consistent with* $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$. *We say that the first* j*-th column vectors of* Q, $[q_1, \ldots, q_j]$ *are exactly identified if, for almost every* $\phi \in \Phi$, $\left(\sum_{i=1}^{j} f_j\right)$*-number of constraints*

$$\begin{pmatrix} F_1(\phi) q_1 \\ F_2(\phi) q_2 \\ \vdots \\ F_j(\phi) q_j \end{pmatrix} = \mathbf{0}$$

and the sign-normalizations $\left(\sigma^i\right)' q_i \geq 0$, $i = 1, \ldots, j$, *pin down a unique* $[q_1, \ldots, q_j]$.

If $rank(F_i(\phi)) = f_i$ for all $i = 1, \ldots, j$, ϕ-a.s., a *necessary* condition for exact identification of $[q_1, \ldots, q_j]$ is that $f_i = n - i$ for all $i = 1, 2, \ldots, j$. One can check if the condition is also sufficient by assessing if the following algorithm developed in Rubio-Ramirez, Waggoner, and Zha (2010) yields a unique set of orthonormal vectors $[q_1, \ldots, q_j]$ for every ϕ randomly drawn from a prior supporting the whole Φ (e.g., a normal-Wishart prior for (B, Σ)).

Algorithm 3.2 *(Successive construction of orthonormal vectors, Algorithm 1 in Rubio-Ramirez, Waggoner, and Zha (2010)) Consider a collection of zero restrictions of the form given by (3.2), where the order of the variables is consistent with* $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$. *Assume* $f_i = n - i$ *for* *all* $i = 1, \dots, j$*, and* $rank(F_i(\phi)) = f_i$ *for all* $i = 1, \dots, j$*,* ϕ*-a.s. Let* q_1 *be a unit length vector satisfying* $F_1(\phi) q_1 = 0$*, which is unique up to sign since* $rank(F_1(\phi)) = n - 1$ *by assumption. Given* q_1*, find orthonormal vectors* $q_2, \dots, q_j$*, by solving*

$$\begin{pmatrix} F_i(\phi) \\ q_1' \\ \vdots \\ q_{i-1}' \end{pmatrix} q_i = 0,$$

successively for $i = 2, 3, \dots, j$*. If*

$$rank \begin{pmatrix} F_i(\phi) \\ q_1' \\ \vdots \\ q_{i-1}' \end{pmatrix} = n - 1 \text{ for } i = 2, \dots, j, \tag{3.4}$$

and q_i*,* $i = 1, \dots, j$*, obtained by this algorithm satisfies* $(\sigma^i)' q_i \neq 0$ *for almost all* $\phi \in \Phi$*, i.e., the sign normalization restrictions determine a unique sign for the* q_i'*s, then* $[q_1, \dots, q_j]$ *is exactly identified.*[1]

Lemma 3.1 shows that, when a set of zero restrictions satisfies $f_i \leq n - i$ for all $i = 1, 2, \dots, n$, the identified set for the impulse response is never empty for all variables and horizons, so any of the zero restrictions cannot be refuted by data. Furthermore, convexity of the identified set is guaranteed under additional restrictions as summarized by conditions (i) - (iii) of the lemma.

We now provide several examples to illustrate how to order the variables and how to verify the conditions for convexity of the impulse response identified set using Lemma 3.1.

Example 3.1 *(Partial causal ordering) Consider a SVAR with quarterly observations of* $(\pi_t, \Delta gdp_t, m_t, i_t)'$*, where* π_t *is inflation,* Δgdp_t *real GDP growth,* m_t *the monetary aggregate and* i_t *the nominal interest rate. Consider the under-identifying restrictions imposed on* A_0^{-1}*,*

$$\begin{pmatrix} u_t^{\pi} \\ u_t^{\Delta gdp} \\ u_t^{m} \\ u_t^{i} \end{pmatrix} = \begin{pmatrix} a^{11} & a^{12} & 0 & 0 \\ a^{21} & a^{22} & 0 & 0 \\ a^{31} & a^{32} & a^{33} & a^{34} \\ a^{41} & a^{42} & a^{43} & a^{44} \end{pmatrix} \begin{pmatrix} \epsilon_t^{\pi} \\ \epsilon_t^{\Delta gdp} \\ \epsilon_t^{m} \\ \epsilon_t^{i} \end{pmatrix}. \tag{3.5}$$

As in Example 1 in Kilian (2013), we interpret the first two equations as an aggregate supply and an aggregate demand equation, the third equation as a money demand equation, and the last equation

[1] A special situation where the rank conditions of (3.4) are guaranteed at almost every ϕ is when σ^i is linearly independent of the row vectors in $F_i(\phi)$ for all $i = 1, \dots, n$, and the row vectors of $F_i(\phi)$ are spanned by the row vectors of $F_{i-1}(\phi)$ for all $i = 2, \dots, j$. This condition holds in the recursive identification scheme, where we impose a triangularity restriction on A_0^{-1}. See Example 3.2.

as a monetary policy reaction function of the central bank. The zero restrictions imply that the private sectors in the economy do not react to the contemporaneous money demand and interest rate, which may be a credible assumption if the private sectors cannot react within the quarter. If in addition we set $a^{12} = a^{34} = 0$, we would have the classical recursive identification restrictions which guarantee point identification. These additional restrictions are however often difficult to justify, as $a^{12} = 0$ implies a horizontal supply curve and $a^{34} = 0$ implies a money demand that is inelastic to the nominal interest rate.

Suppose the object of interest are the impulse responses to the monetary policy shock ϵ_t^i. Let $\left[q^{\pi}, q^{\Delta gdp}, q^m, q^i\right]$ be a 4×4 orthogonal matrix with the order of columns same as in (3.5). By (3.1), the imposed restrictions imply two restrictions on q^m and two restrictions on q^i. Hence, an ordering of the variables that is consistent with Notation 3.1 is $y_t = (i_t, m_t, \pi_t, \Delta gdp_t)'$, and the corresponding numbers of restrictions are $(f_1, f_2, f_3, f_4) = (2, 2, 0, 0)$. Since $j^* = 1$ and $f_1 = 2$, condition (i) of Lemma 3.1 guarantees that the impulse response identified sets are ϕ-a.s. convex.

Suppose the object of interest are the impulse responses to a demand shock $\epsilon^{\Delta gdp}$. In this case, we order the variables as $y_t = (i_t, m_t, \Delta gdp_t, \pi_t)$, and $j^* = 3$. Note that none of the conditions Lemma 3.1 apply in this case, so Lemma 3.1 does not guarantee convexity of the impulse response identified sets.

Example 3.2 *Consider adding to the case in Example 3.1 a long-run money neutrality restriction, which sets the long-run impulse response of output gdp to monetary policy shock ϵ^i to zero. This results in adding one more restriction to q^i, as we are adding a zero restriction on the $(2, 4)$-th element of the long-run cumulative impulse response matrix CIR^{∞}. As a result, we can again order the variables as $y_t = (i_t, m_t, \pi_t, \Delta gdp_t)'$ and we have $(f_1, f_2, f_3, f_4) = (3, 2, 0, 0)$. It can be shown that in this case the first two columns $[q_1, q_2]$ are exactly identified,[2] implying that the impulse responses to ϵ^i and ϵ^m are each point-identified. The impulse responses to $\epsilon^{\Delta gdp}$ are instead partially identified and their identified sets are convex, as condition (iii) of Lemma 3.1 applies to $y_t = (i_t, m_t, \Delta gdp_t, \pi_t)'$ with $j^* = 3$.*

As an alternative to the long-run money neutrality restriction, let us assume $a^{12} = 0$. Then, an ordering of the variables when the objects of interest are the impulse responses to ϵ^i is given by $y_t = (i_t, m_t, \Delta gdp_t, \pi_t)'$ with $j^* = 1$ and $(f_1, f_2, f_3, f_4) = (2, 2, 1, 0)$. In comparison to the case of Example 3.1, additionally imposing $a^{12} = 0$ does not change j^*. A close inspection of the proof of Lemma 3.1 shows that, if adding restrictions does not change the order of the variables and the number of zero restrictions up to the j^*-th variable, the identified set for the impulse responses to the j^*-th shock does not change for every $\phi \in \Phi$. Hence, adding $a^{12} = 0$ does not bring any additional identifying information for the impulse responses to the monetary policy shock. We can

[2] In the current case $F_2(\phi)$ is a submatrix of $F_1(\phi)$, implying that the vector space spanned by the rows of $F_1(\phi)$ contains the vector space spanned by the rows of $F_2(\phi)$ for every $\phi \in \Phi$. Hence, the rank condition for exact identification (3.4) holds.

generalize this observation, as stated in the next corollary (see Appendix A for a proof).

Corollary 3.1 *Let a set of zero restrictions, an ordering of variables* $(1, \ldots, j^*, \ldots, n)$*, and the corresponding number of zero restrictions* $(f_1, \ldots, f_n)$ *satisfy* $f_i \leq n - i$ *for all* i*,* $f_1 \geq \cdots \geq f_n \geq 0$*, and* $f_{j^*-1} > f_{j^*}$*, as in Notation 3.1. Consider imposing additional zero restrictions. Let* $\pi(\cdot) : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ *be a permutation that reorders the variables to be consistent with Notation 3.1 after adding the new restrictions, and let* $\left(\tilde{f}_{\pi(1)}, \ldots, \tilde{f}_{\pi(n)}\right)$ *be the new number of restrictions. If* $\tilde{f}_{\pi(i)} \leq n - \pi(i)$ *for all* $i = 1, \ldots, n$*,* $(\pi(1), \ldots, \pi(j^*)) = (1, \ldots, j^*)$*, and* $(f_1, \ldots, f_{j^*}) = \left(\tilde{f}_1, \ldots, \tilde{f}_{j^*}\right)$*, i.e., adding the zero restrictions does not change the order of the variables and the number of restrictions for the first* j^* *variables, then the additional restrictions do not tighten the identified sets for the impulse response to the* j^**-th shock for every* $\phi \in \Phi$*.*

Example 3.3 *Consider relaxing one of the zero restrictions in (3.5),*

$$\begin{pmatrix} u_t^{\pi} \\ u_t^{\Delta gdp} \\ u_t^{m} \\ u_t^{i} \end{pmatrix} = \begin{pmatrix} a^{11} & a^{12} & 0 & 0 \\ a^{21} & a^{22} & 0 & a^{24} \\ a^{31} & a^{32} & a^{33} & a^{34} \\ a^{41} & a^{42} & a^{43} & a^{44} \end{pmatrix} \begin{pmatrix} \epsilon_t^{\pi} \\ \epsilon_t^{\Delta gdp} \\ \epsilon_t^{m} \\ \epsilon_t^{i} \end{pmatrix},$$

where the $(2, 4)$*-th element of* A_0^{-1} *is now unconstrained, i.e., the aggregate demand equation is allowed to respond contemporaneously to the monetary policy shock. If our interest is in the impulse responses to monetary policy shock* ϵ_t^i*, an ordering of the variables can be given by* $y_t = (m_t, i_t, \pi_t, \Delta gdp_t)'$ *with* $j^* = 2$*. Condition (ii) of Lemma 3.1 is satisfied, so that the impulse response identified sets are convex. In fact, Lemma A.1 in the appendix implies that, in situations where condition (ii) of Lemma 3.1 applies, the zero restrictions imposed on the preceding shocks to the* j^**-th structural shocks do not tighten the identified sets for the* j^**-th shock impulse responses compared to the case where no zero restrictions are imposed on them. In the current context, this means that dropping the two zero restrictions on* q_m *does not change the identified sets for the impulse responses to* ϵ_t^i*. The next corollary shows invariance of the identified sets when relaxing the zero restrictions, which partially overlaps with the implications of Corollary 3.1.*

Corollary 3.2 *Let a set of zero restrictions, an ordering of variables* $(1, \ldots, j^*, \ldots, n)$*, and the corresponding number of zero restrictions* $(f_1, \ldots, f_n)$ *satisfy* $f_i \leq n - i$ *for all* i*,* $f_1 \geq \cdots \geq f_n \geq 0$*, and* $f_{j^*-1} > f_{j^*}$*, as in Notation 3.1. Under any of the conditions (i) - (iii) of Lemma 3.1, the identified set for the impulse responses to the* j^**-th structural shock does not change when relaxing any or all of the zero restrictions on* $q_{j^*+1}, \ldots, q_{n-1}$*. Furthermore, in the case where condition (ii) of Lemma 3.1 is satisfied, the identified set for the impulse responses to the* j^**-th structural shock does not change when relaxing any or all of the zero restrictions on* $q_1, \ldots, q_{j^*-1}$*. In the case where condition (iii) of Lemma 3.1 is satisfied, the identified set for the impulse responses* *to the j^*-th structural shock does not change when relaxing any or all of the zero restrictions on* $q_{i^*+1}, \ldots, q_{j^*-1}$.

3.2 Sign Restrictions

Sign restrictions on the impulse responses could be considered alone or could be added to the zero restrictions as a way to tighten the impulse response identified set. It is straightforward to incorporate sign restrictions on the impulse responses into the current framework. Given the zero restrictions $F(\phi, Q) = \mathbf{0}$, we maintain the order of the variables as specified in the previous section. When only imposing sign restrictions, the order of the variables can be arbitrary, while we let the variable whose structural shock is of interest appear first, $j^* = 1$. For a vector $x = (x_1, \ldots, x_m)'$, $x \geq \mathbf{0}$ means $x_i \geq 0$ for all $i = 1, \ldots, m$, and $x > \mathbf{0}$ means $x_i \geq 0$ for all $i = 1, \ldots, m$ and $x_i > 0$ for some $i \in \{1, \ldots, m\}$.

Suppose that sign restrictions are placed on the responses to the j-th structural shock and let $s_{h,j}$ be the number of sign restrictions placed on the h-th horizon impulse responses. Since the impulse response vector to the j-th structural shock is given by the j-th column vector of $IR^h = C_h(B)\Sigma_{tr}Q$, we can write the sign restrictions on the h-th horizon response vector as

$$S_{h,j}(\phi)\, q_j \geq \mathbf{0},$$

where the inequality stands for component-wise weak inequalities, $S_{h,j}(\phi) \equiv D_{h,j} C_h(B)\Sigma_{tr}$ is a $s_{h,j} \times n$ matrix, and $D_{h,j}$ is the $s_{h,j} \times n$ selection matrix that selects the sign restricted responses from the $n \times 1$ response vector $C_h(B)\Sigma_{tr} q_j$. The nonzero elements of $D_{h,j}$ equal 1 or -1 depending on whether the corresponding impulse responses are restricted to be positive or negative. By stacking the coefficient matrices $S_{h,j}(\phi)$ over multiple horizons, we express the whole set of sign restrictions on the responses to the j-th shock as

$$S_j(\phi)\, q_j \geq \mathbf{0}, \tag{3.6}$$

where $S_j(\phi)$ is a $\left(\sum_{h=0}^{\bar{h}_j} s_{h,j}\right) \times n$ matrix defined by $S_j(\phi) = \left[S_{0,j}(\phi)', \ldots, S_{\bar{h}_j,j}(\phi)\right]'$. If no sign restrictions are placed on the $\tilde{h}$-th horizon responses, $0 \leq \tilde{h} \leq \bar{h}$, we set $s_{\tilde{h},j} = 0$ and interpret $S_{\tilde{h},j}(\phi)$ as not present in the construction of $S_j(\phi)$.

Note that the sign restrictions considered here do not have to be limited to the impulse responses. Since $A_0' = \Sigma_{tr}^{-1\prime} Q$ and $A_l' = B_l'\left(\Sigma_{tr}^{-1}\right)' Q$, $l = 1, \ldots, p$, any sign restrictions on structural parameters appearing in the j-th row of A_0 or A_l take the form of linear inequalities for q_j, so these sign restrictions could be appended to $S_j(\phi)$ in (3.6).

Let $\mathcal{I}_S \subset \{1, 2, \ldots, n\}$ be the set of indices such that $j \in \mathcal{I}_S$ if some of the impulse responses to the j-th structural shock are sign-constrained. The set of all the sign constraints can be accordingly expressed by

$$S_j(\phi)\, q_j \geq \mathbf{0} \quad \text{for } j \in \mathcal{I}_S. \tag{3.7}$$

As a shorthand notation, we represent the entire set of sign restrictions by $S(\phi, Q) \geq \mathbf{0}$.

Given $\phi \in \Phi$, let $\mathcal{Q}(\phi|F,S)$ be the set of Q's that jointly satisfy the sign restrictions (3.7), zero restrictions (3.2), and the sign normalizations,

$$\mathcal{Q}(\phi|F,S) = \left\{Q \in \mathcal{O}(n) : S(\phi, Q) \geq \mathbf{0},\ F(\phi, Q) = \mathbf{0},\ diag\left(Q'\Sigma_{tr}^{-1}\right) \geq \mathbf{0}\right\}. \tag{3.8}$$

Unlike in the case with only under-identifying zero restrictions, $\mathcal{Q}(\phi|F,S)$ can be an empty set, depending on ϕ and the imposed sign restrictions. If $\mathcal{Q}(\phi|F,S)$ is nonempty, the identified set for r, denoted by $IS_r(\phi|F,S)$, is given by the range of r with the domain of Q given by $\mathcal{Q}(\phi|F,S)$. If $\mathcal{Q}(\phi|F,S)$ is empty, the identified set of r is defined as an empty set.

The next lemma extends Lemma 3.1 to the case with sign restrictions.

Lemma 3.2 *(Convexity of the impulse response identified set under equality and sign restrictions) Let $\{r = c'_{ih}(\phi)\, q_{j^*} : i = 1, \ldots, n,\ h = 0, 1, 2, \ldots\}$ be the impulse responses of interest. Assume $\mathcal{I}_S = \{j^*\}$, i.e., the sign restrictions are placed only on the impulse responses to the j^*-th structural shock.*

(i) Suppose that the zero restrictions $F(\phi, Q) = \mathbf{0}$ satisfy one of the conditions (i) and (ii) of Lemma 3.1. If there exists a unit length vector $q \in \mathcal{R}^n$ such that

$$F_{j^*}(\phi)\, q = 0 \text{ and } \begin{pmatrix} S_{j^*}(\phi) \\ \left(\sigma^{j^*}\right)' \end{pmatrix} q > \mathbf{0}, \tag{3.9}$$

then $IS_r(\phi|F,S)$ is nonempty and convex for every $i \in \{1, \ldots, n\}$ and $h = 0, 1, 2, \ldots$.

(ii) Suppose that the zero restrictions $F(\phi, Q) = \mathbf{0}$ satisfy condition (iii) of Lemma 3.1. Accordingly, let $[q_1(\phi), \ldots, q_{i^*}(\phi)]$ be the first i^*-th orthonormal vectors that are exactly identified. If there exists a unit length vector $q \in \mathcal{R}^n$ such that

$$\begin{pmatrix} F_{j^*}(\phi) \\ q'_1(\phi) \\ \vdots \\ q'_{i^*}(\phi) \end{pmatrix} q = 0 \text{ and } \begin{pmatrix} S_{j^*}(\phi) \\ \left(\sigma^{j^*}\right)' \end{pmatrix} q > \mathbf{0}, \tag{3.10}$$

then $IS_r(\phi|F,S)$ is nonempty and convex for every $i \in \{1, \ldots, n\}$ and $h = 0, 1, 2, \ldots$.

Proof. See Appendix A. ■

Lemma B.1 of Moon, Schorfheide, and Granziera (2013) shows convexity of the impulse response identified set for the special case where $\mathcal{I}_S = \{j^*\}$ and zero restrictions are imposed only on q_{j^*}, i.e., $j^* = 1$ and $f_i = 0$ for all $i = 2, \ldots, n$ in our notation. Lemma 3.2 extends their result to the case where zero restrictions are placed on the column vectors of Q other than q_{j^*}. Assumptions (3.9) or (3.10) of Lemma 3.2 imply that the set of feasible q's subject to the zero and sign restrictions is not degenerate in the sense that it does not collapse to a one-dimensional subspace in $\mathcal{R}^n$. If the set of feasible q's becomes degenerate, a non-convex identified set arises since the intersection of a one-dimensional subspace in $\mathcal{R}^n$ with the unit sphere consists of two disconnected points only. If the set of ϕ's that leads to such degeneracy has measure zero in Φ, then, as a corollary of Lemma 3.2, we can claim that the impulse response identified set is convex, ϕ-a.s., conditional on it being nonempty.

If sign restrictions are imposed on impulse responses to some structural shock other than the j^*-th shock, i.e., $\mathcal{I}_S$ contains an index other than j^*, the identified set for an impulse response can become non-convex, as we show in the next simple example.[3]

Example 3.4 *Consider a SVAR(0) model,*

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = A_0^{-1} \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}.$$

Let $\Sigma_{tr} = \begin{pmatrix} \sigma_{11} & 0 \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$, *where* $\sigma_{11} \geq 0$ *and* $\sigma_{22} \geq 0$. *Positive semidefiniteness of* $\Sigma = \Sigma_{tr}\Sigma_{tr}'$ *requires* $\sigma_{22} \geq 1$, *while* σ_{21} *is left unconstrained. Denoting an orthonormal matrix by* $Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$, *we can express the contemporaneous impulse response matrix as*

$$IR^0 = \begin{pmatrix} \sigma_{11}q_{11} & \sigma_{11}q_{12} \\ \sigma_{21}q_{11} + \sigma_{22}q_{21} & \sigma_{21}q_{12} + \sigma_{22}q_{22} \end{pmatrix}.$$

Consider restricting the sign of the $(1,2)$*-th element of* IR^0 *to being positive,* $\sigma_{11}q_{12} \geq 0$. *Since* $\Sigma_{tr}^{-1} = (\sigma_{11}\sigma_{22})^{-1} \begin{pmatrix} \sigma_{22} & 0 \\ -\sigma_{21} & \sigma_{11} \end{pmatrix}$, *the sign normalization restrictions give* $\sigma_{22}q_{11} - \sigma_{21}q_{21} \geq 0$ *and* $\sigma_{11}q_{22} \geq 0$. *We now show that the identified set for the* $(1,1)$*-th element of* IR^0 *is non-convex for a set of* Σ *with a positive measure. Note first that the second column vector of* Q *is constrained to* $\{q_{12} \geq 0, q_{22} \geq 0\}$, *so that the set of* $(q_{11}, q_{21})'$ *orthogonal to* $(q_{12}, q_{22})'$ *is constrained to*

$$\{q_{11} \geq 0, q_{21} \leq 0\} \cup \{q_{11} \leq 0, q_{21} \geq 0\}.$$

When $\sigma_{21} < 0$, *intersecting this union set with the half-space defined by the first sign normalization restriction* $\{\sigma_{22}q_{11} - \sigma_{21}q_{21} \geq 0\}$ *yields two disconnected arcs,*

$$\left\{ \begin{pmatrix} q_{11} \\ q_{21} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} : \theta \in \left(\left[\frac{1}{2}\pi, \frac{1}{2}\pi + \psi\right] \cup \left[\frac{3}{2}\pi + \psi, 2\pi\right] \right) \right\},$$

[3] Consider the example given in Section 4.4 of Rubio-Ramirez (2010), where $n = 3$ and zero restrictions satisfying $f_1 = f_2 = f_3 = 1$. Their paper shows that the identified set for an impulse response consists of two distinct points. If we interpret the zero restrictions on the second and third variables as pairs of linear inequality restrictions for q_2 and q_3 with opposite signs, convexity of $IS_r(\phi|F,S)$ fails. In this counterexample, the assumption of $\mathcal{I}_S = \{j\}$ fails.

where $\psi = \arccos\left(\frac{\sigma_{22}}{\sqrt{\sigma_{22}^2+\sigma_{21}^2}}\right) \in \left[0, \frac{1}{2}\pi\right]$. *Accordingly, the identified set for* $r = \sigma_{11}q_{11}$ *is given by the union of two disconnected intervals*

$$\left[\sigma_{11}\cos\left(\frac{1}{2}\pi+\psi\right), 0\right] \cup \left[\sigma_{11}\cos\left(\frac{3}{2}\pi+\psi\right), \sigma_{11}\right].$$

Since $\{\sigma_{21} < 0\}$ *has a positive measure in the space of* Σ*, the identified set is non-convex with a positive measure.*

4 Inference on the Identified Set: a Robust Bayes Approach

In this section we consider a robust Bayes approach to conducting inference on the impulse response identified set. The procedure delivers two intervals: a posterior mean bounds interval, interpreted as an estimator of the identified set, and a robustified credible region, interpreted as a measure of the posterior uncertainty about the identified set. It is important to remark here that the robust Bayes interpretation of these intervals is valid regardless of whether the identified set is convex or not.

4.1 Multiple Priors and Posterior Bounds

Let $\tilde{\pi}_\phi$ be a probability measure on the reduced form parameter space Φ. To construct a prior distribution for ϕ consistent with the zero restrictions $F(\phi, Q) = \mathbf{0}$ and the sign restrictions $S(\phi, Q) \geq \mathbf{0}$, we trim the support of $\tilde{\pi}_\phi$ as follows:

$$\pi_\phi \equiv \tilde{\pi}_{\phi|\Phi_{F,S}} \equiv \frac{\tilde{\pi}_\phi 1\left\{\mathcal{Q}(\phi|F,S) \neq \emptyset\right\}}{\tilde{\pi}_\phi\left(\left\{\mathcal{Q}(\phi|F,S) \neq \emptyset\right\}\right)}, \tag{4.1}$$

where the conditioning event $\Phi_{F,S}$ in the notation of $\tilde{\pi}_{\phi|\Phi_{F,S}}$ is the set of reduced form parameter values that are consistent with the imposed restrictions, $\Phi_{F,S} = \{\phi \in \Phi : \mathcal{Q}(\phi|F,S) \neq \emptyset\}$. By construction, the prior π_ϕ assigns probability one to the distribution of data that is consistent with the identifying restrictions, i.e., $\pi_\phi\left(\left\{\mathcal{Q}(\phi|F,S) \neq \emptyset\right\}\right) = 1$. A joint prior for $(\phi, Q) \in \Phi \times \mathcal{O}(n)$ that has ϕ-marginal π_ϕ can be expressed as $\pi_{\phi,Q} = \pi_{Q|\phi}\pi_\phi$, where $\pi_{Q|\phi}$ is supported only on $\mathcal{Q}(\phi|F,S) \subset \mathcal{O}(n)$. Since the structural parameters $(A_0, A_1, \ldots, A_p)$ and the impulse responses are functions of (ϕ, Q), $\pi_{\phi,Q}$ induces a unique prior distribution for the structural parameters and the impulse responses. Conversely, a prior distribution for $(A_0, A_1, \ldots, A_p)$ that incorporates the sign normalizations induces a prior for $\pi_{\phi Q}$. If one conducts SVAR analysis with a prior distribution for $(A_0, A_1, \ldots, A_p)$, the prior for ϕ induced by the prior for $(A_0, A_1, \ldots, A_p)$ is well-updated by the data, while the conditional prior $\pi_{Q|\phi}$, which is implicitly induced by the prior for $(A_0, A_1, \ldots, A_p)$, remains unchanged.

In the exact identification case where the imposed restrictions and the sign normalizations can pin down a unique Q (i.e., $\mathcal{Q}(\phi|F,S)$ is a singleton), $\pi_{Q|\phi}$ is degenerate and gives a point mass at such Q. As a result, specifying π_ϕ suffices to induce a single posterior distribution for the structural coefficients and the impulse responses. In contrast, in the partially identified case where $\mathcal{Q}(\phi|F,S)$ is non-singleton for ϕ's with a positive measure, specifying only π_ϕ cannot yield a unique posterior distribution for the impulse responses. To obtain a posterior distribution for the impulse responses, as desired in the standard Bayesian approach, we need to specify $\pi_{Q|\phi}$, which is supported only on $\mathcal{Q}(\phi|F,S) \subset \mathcal{O}(n)$ at each $\phi \in \Phi$. In empirical practice, however, it is a challenging task for a researcher to come up with a "reasonable" specification for $\pi_{Q|\phi}$ especially when the prior knowledge that she considers credible is exhausted by the zero restrictions and the sign restrictions. Even when it is feasible to specify $\pi_{Q|\phi}$, the fact that $\pi_{Q|\phi}$ is never updated by the data makes the posterior distribution for the impulse response sensitive to the choice of $\pi_{Q|\phi}$ even asymptotically, so that a limited confidence in the choice of $\pi_{Q|\phi}$ leads to an equally limited credibility in the posterior inference. Since (ϕ, Q) and the structural parameters $(A_0, A_1, \ldots, A_p)$ are one-to-one (under the sign normalizations), the difficulty of specifying a prior for $\pi_{Q|\phi}$ can be equivalently stated as the difficulty of specifying a joint prior for all structural parameters with fixing the prior for ϕ at π_ϕ.

The robust Bayes procedure considered in this paper aims to make the posterior inference free from the choice of $\pi_{Q|\phi}$. More specifically, we specify a single prior for the reduced form parameters ϕ, which the data are always informative about, whereas we introduce a set of priors (ambiguous belief) for $\pi_{Q|\phi}$. Let $\Pi_{Q|\phi}$ denote a collection of conditional priors $\pi_{Q|\phi}$. Given a single prior for ϕ, π_ϕ, let $\pi_{\phi|Y}$ be the posterior distribution for ϕ obtained by the Bayesian reduced-form VAR, where Y stands for a sample. The class of conditional priors that impose no restrictions other than the zero restrictions and/or the sign restrictions is defined as

$$\Pi_{Q|\phi} = \left\{ \pi_{Q|\phi} : \pi_{Q|\phi}\left(\mathcal{Q}\left(\phi|F,S\right)\right) = 1,\ \pi_\phi\text{-almost surely} \right\}. \tag{4.2}$$

In words, it consists of arbitrary $\pi_{Q|\phi}$'s as far as they assign probability one to the set of Q's that satisfy the imposed restrictions.

The posterior for ϕ combined with the prior class $\Pi_{Q|\phi}$ generates the class of joint posteriors for (ϕ, Q),

$$\Pi_{\phi Q|Y} = \left\{ \pi_{\phi Q|Y} = \pi_{Q|\phi}\pi_{\phi|Y} : \pi_{Q|\phi} \in \Pi_{Q|\phi} \right\},$$

which coincides with the class of posteriors obtained by applying Bayes rule to each prior in the class $\left\{ \pi_{\phi,Q} = \pi_\phi \pi_{Q|\phi} : \pi_{Q|\phi} \in \Pi_{Q|\phi} \right\}$. This class of posteriors for (ϕ, Q) induces the class of posteriors for impulse response, $r = r(\phi, Q)$,

$$\Pi_{r|Y} \equiv \left\{ \pi_{r|Y}(\cdot) = \pi_{\phi,Q|Y}\left(r\left(\phi,Q\right) \in \cdot\right) : \pi_{\phi Q|Y} \in \Pi_{\phi Q|Y} \right\}. \tag{4.3}$$

We summarize the posterior class for r by constructing the bounds of the posterior means of r and the posterior probabilities.

Proposition 4.1 *Let a prior for ϕ, π_ϕ, be given, and assume $\pi_\phi\left(\{\phi : \mathcal{Q}(\phi|F,S) \neq \emptyset\}\right) = 1$. Let a prior class for $\pi_{Q|\phi}$ be given by (4.2).*

(i) The bounds of the posterior probabilities for an event $\{r \in G\}$, where G is a measurable subset in $\mathcal{R}$, are given by $\left[\pi_{r|Y*}(G), \pi^*_{r|Y}(G)\right]$, where

$$\begin{aligned}
\pi_{r|Y*}(G) &\equiv \inf\left\{\pi_{r|Y}(G) : \pi_{\phi Q|Y} \in \Pi_{\phi Q|Y}\right\} \\
&= \pi_{\phi|Y}\left(IS_r(\phi|S,F) \subset G\right), \\
\pi^*_{r|Y}(G) &\equiv \sup\left\{\pi_{r|Y}(G) : \pi_{\phi Q|Y} \in \Pi_{\phi Q|Y}\right\} \\
&= \pi_{\phi|Y}\left(IS_r(\phi|S,F) \cap G \neq \emptyset\right), \\
&= 1 - \pi_{r|Y*}(G^c).
\end{aligned}$$

(ii) The range of the posterior means $E(r|Y)$ with the posterior class $\Pi_{r|Y}$ given in (4.3) is

$$\left[\int_\Phi \ell(\phi) d\pi_{\phi|Y}, \int_\Phi u(\phi)\, d\pi_{\phi|Y}\right], \tag{4.4}$$

where $\ell(\phi)$ is the lower bound of $IS_r(\phi|F,S)$, $\ell(\phi) = \inf\{r(\phi,Q) : Q \in \mathcal{Q}(\phi|F,S)\}$, and $u(\phi)$ is the upper bound of $IS_r(\phi|F,S)$, $u(\phi) = \sup\{r(\phi,Q) : Q \in \mathcal{Q}(\phi|F,S)\}$.

Proof. The first claim is a corollary of Theorem 3.1 in Kitagawa (2012). For a proof of the second claim, see Appendix A. ■

Note that the construction of these bounds is valid irrespective of whether $IS_r(\phi|S,F)$ is a convex interval or not, so the formulas of the posterior probability bounds and the mean bounds apply to any set-identified SVARs. The presented posterior probability bounds are convex in the sense that every value in $\left[\pi_{r|Y*}(G), \pi^*_{r|Y}(G)\right]$ is attained by some posterior in $\Pi_{\phi,Q|Y}$ (see Lemma B.1 of Kitagawa (2012) for a proof of this statement). As the expressions for $\pi_{r|Y*}(G)$ and $\pi^*_{r|Y}(G)$ suggest, the bounds of the posterior probabilities can be computed by the posterior probability that G contains and intersects with the identified set of r, respectively. If the impulse response is point-identified in the sense of $IS_r(\phi|S,F)$ being $\pi_{\phi|Y}$-almost surely a singleton, the posterior probability bounds collapse to a point for every G, leading to a single posterior.

As the analytical expressions of the posterior bounds show, we can approximate these posterior probability bounds if we can compute $IS_r(\phi|S,F)$ at values of ϕ randomly drawn from its posterior $\pi_{\phi|Y}$. Computation of $IS_r(\phi|S,F)$ can be greatly simplified if $IS_r(\phi|S,F)$ is guaranteed to be convex, e.g., the cases where Lemma 3.1 and Lemma 3.2 apply, since obtaining convex $IS_r(\phi|S,F)$ is reduced to computing $\ell(\phi)$ and $u(\phi)$.

The posterior mean bounds (4.4) are given by the mean of the lower and upper bounds of $IS_r(\phi|S,F)$ taken with respect to the posterior of ϕ. The range of posterior means is convex irrespective of whether the identified set of r is convex or not.

Building on Proposition 4.1, our robust Bayes inference proposes to report the posterior mean bounds of (4.4). As a robustified credible region, we consider reporting an interval satisfying

$$\pi_{r|Y*}(C_\alpha) \geq \alpha. \tag{4.5}$$

C_α is interpreted as an interval estimate for the impulse response r, such that the posterior probability put on C_α is greater than or equal to α uniformly over the posteriors in the class (4.3). There are multiple ways to construct C_α satisfying (4.5). One proposal is to consider the interval that has shortest width (Kitagawa (2012)) and satisfies (4.5) with equality. We hereafter refer to it as the *robustified credible region with lower credibility* α. We can also define C_α by mapping the highest posterior density region of ϕ to the real line via the set-valued map $IS_r(\cdot|S,F)$ (Moon and Schorfheide (2011)), which can be conservative in the sense that (4.5) can hold with inequality See also Kline and Tamer (2013) and Liao and Simoni (2013) for alternative proposals to constructing C_α.

5 Computing Posterior Bounds

This subsection presents an algorithm to numerically approximate the posterior mean bounds and the robustified credible region discussed in Proposition 4.1, using random draws of ϕ from its posterior. The algorithm assumes that $IS_r(\phi|F,S)$ is ϕ-a.s. convex. Therefore, the conditions derived in Lemmas 3.1 and 3.2 should be checked prior to implementation.

Algorithm 5.1 *Let $F(\phi,Q)=\mathbf{0}$ and $S(\phi,Q)\geq \mathbf{0}$ be given, and let $r = c'_{ih}(\phi)\, q_{j^*}$ be the impulse response of interest.*

(Step 1) Specify $\tilde{\pi}_\phi$, a prior for the reduced form parameters ϕ. The proposed $\tilde{\pi}_\phi$ need not satisfy $\tilde{\pi}_\phi(\{\phi : \mathcal{Q}(\phi|F,S)\neq\emptyset\}) = 1$. Run a Bayesian reduced form VAR to obtain the posterior $\tilde{\pi}_{\phi|Y}$.

(Step 2) Draw a reduced form parameter vector ϕ from $\tilde{\pi}_{\phi|Y}$. Given the draw of ϕ, examine if $\mathcal{Q}(\phi|F,S)$ is empty or not by following the subroutine (Step 2.1) - (Step 2.3) below.[4]

(Step 2.1) Let $z_1 \sim \mathcal{N}(0,I_n)$ be a draw of an n-variate standard normal random variable. Let $\mathcal{M}_1 z_1$ be the $n\times 1$ residual vector in the linear projection of z_1 onto a $n\times f_1$ regressor matrix $F_1(\phi)'$. Set $\tilde{q}_1 = \mathcal{M}_1 z_1$. For $i=2,3,\ldots,n$, run the following procedure sequentially; draw $z_i\sim\mathcal{N}(0,I_n)$, and compute $\tilde{q}_i = \mathcal{M}_i z_i$, where $\mathcal{M}_i z_i$ is the residual vector in the linear projection of z_i onto the $n\times(f_i+i-1)$ regressor matrix, $\left[F_i(\phi)', \tilde{q}_1,\ldots,\tilde{q}_{i-1}\right]$.

[4] Instead of our algorithm for drawing Q from $\mathcal{Q}(\phi|F,S)$ (Step 2.1 - 2.3), one could alternatively use an algorithm that Arias, Rubio-Ramirez, and Waggoner (2013) developed in their Theorem 4, which draws Q from the Haar measure condtional on the zero restrictions.

(Step 2.2) *Given* $\tilde{q}_1, \ldots, \tilde{q}_n$ *obtained in the previous step, define*

$$Q = \left[sign\left(\left(\sigma^1\right)' \tilde{q}_1 \right) \frac{\tilde{q}_1}{\|\tilde{q}_1\|}, \ldots, sign\left((\sigma^n)' \tilde{q}_n \right) \frac{\tilde{q}_n}{\|\tilde{q}_n\|} \right],$$

where $\|\cdot\|$ *is the Euclidian metric in* $\mathcal{R}^n$. Q *can be seen as a draw of an orthogonal matrix from* $\mathcal{Q}(\phi|F)$.[5]

(Step 2.3) [6]*If* Q *obtained in (Step 2.2) satisfies the sign restrictions* $S(\phi, Q) \geq \mathbf{0}$, *retain this* Q *and proceed to (Step 3). Otherwise, repeat (Step 2.1) and (Step 2.2) at most* L *times (e.g.,* $L = 10000$*), until obtaining* Q *satisfying* $S(\phi, Q) \geq \mathbf{0}$. *If none of* L *number of draws of* Q *satisfies* $S(\phi, Q) \geq \mathbf{0}$, *approximate* $\mathcal{Q}(\phi|F, S)$ *to be empty, and go back to Step 2 to obtain a new draw of* ϕ.

(Step 3) *Given* ϕ *and* Q *obtained in (Step 2) and (Step 2.3), compute the lower and upper bounds of* $IS_r(\phi|S, F)$ *by solving the following nonlinear optimization with equality and inequality constraints,*[7]

$$\begin{aligned} & \ell(\phi) = \arg\min_{Q} c'_{ih}(\phi)\, q_{j^*}, \\ s.t. \quad & Q'Q = I_n, \quad F(\phi, Q) = \mathbf{0}, \\ & diag(Q'\Sigma_{tr}^{-1}) \geq 0, \text{ and } S(\phi, Q) \geq \mathbf{0}, \end{aligned}$$

and $u(\phi) = \arg\max_Q c'_{ih}(\phi)\, q_{j^*}$ *under the same set of constraints.*

(Step 4) *Repeat (Step 2) - (Step 3)* M *times, and obtain* M *draws of the intervals,* $[\ell(\phi_m), u(\phi_m)]$, $m = 1, \ldots, M$. *Approximate the posterior mean bounds of Proposition 4.1 by the sample averages of* $(\ell(\phi_m) : m = 1, \ldots, M)$ *and* $(u(\phi_m) : m = 1, \ldots, M)$.

(Step 5) *To obtain an approximation of the robustified credible region with credibility* $\alpha \in (0, 1)$, *define* $d(r, \phi) = \max\{|r - \ell(\phi)|, |r - u(\phi)|\}$, *and let* $\hat{z}_\alpha(r)$ *be the sample* α*-th quantile of* $(d(r, \phi_m) : m = 1, \ldots, M)$. *An approximated robustified credible region for* r *is obtained as an interval centered at* $\arg\min_r \hat{z}_\alpha(r)$ *with radius* $\min_r \hat{z}_\alpha(r)$ *(Proposition 5.1 of Kitagawa (2012)).*

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[5]If $\left(\sigma^i\right)' \tilde{q}_i$ is zero for some i, we can set $sign\left(\left(\sigma^i\right)' \tilde{q}_i\right)$ at 1 or -1 randomly.

[6]Skip this step is there are no sign restrictions.

[7]In the empirical application in section 7, we used the "auglag" function available in an R package "alabama", which implements the augumented Lagrangean multiplier method for a nonlinear optimization with equality and inequality constraints. At each ϕ, we used Q obtained in (Step 2.3) as an initial value for the nonlinear optimization. For all the models considered, the optimization algorithm converged under the default convergence criterion at every draw of ϕ.

In the above algorithm, the non-linear optimization part of (Step 3) can be computationally unstable and time-consuming, especially when the number of variables and constraints are large and convergence to the optimum is slow. If one encounters such computational challenges in a given application, a more computationally stable algorithm can be used, in which (Step 3) above is replaced with (Step 3') below. A downside of this alternative algorithm is that the approximated identified set is smaller than $IS_r(\phi|F,S)$ at every draw of ϕ, resulting in approximate posterior bounds that are shorter than the actual ones. Nonetheless, these alternative bounds still provide a consistent estimator of the identified set, as the number of draws of Q's goes to infinity.

(Step 3') Iterate (Step 2.1) - (Step 2.3) K times and let $\left(Q_l : l = 1, \ldots, \tilde{K}\right)$ be the draws that satisfy the sign restrictions. (If none of the draws satisfy the sign restrictions, we draw a new ϕ and iterate (Step 2.1) - (Step 2.3) again). Let $q_{j^*,k}$, $k = 1, \ldots, \tilde{K}$, be the j^*-th column vector of Q_k. We then approximate $[\ell(\phi), u(\phi)]$ by $[\min_k c'_{ih}(\phi)\, q_{j^*,k}, \max_k c'_{ih}(\phi)\, q_{j^*,k}]$.

In a situation where the zero and sign restrictions satisifes their parsimony condition in Gafarov and Montiel-Olea (2014), closed form expressions for the optimum in (Step 3) obtained in Gafarov and Montiel-Olea (2014) can be used and they resolve the aforementioned potential computational issues at (Step 3).

5.1 Assessing the Plausibility of the Identifying Restrictions

By calculating the proportion of drawn ϕ's passing (Step 2.3) of Algorithm 5.1, we can obtain an approximation of the posterior probability (corresponding to the non-trimmed prior $\tilde{\pi}_\phi$) of having a nonempty identified set, $\tilde{\pi}_{\phi|Y}\left(\{\phi : \mathcal{Q}(\phi|F,S) \neq \emptyset\}\right)$. With only zero restrictions satisfying $f_i \leq n-i$ for all $i = 1, \ldots, n$, the set of admissible Q's, $\mathcal{Q}(\phi|F)$, is never empty as shown in Lemma 3.1, so the data are never able to detect violation of the imposed assumptions irrespective of the choice of $\tilde{\pi}_\phi$. In contrast, with the sign restrictions $\mathcal{Q}(\phi|F,S)$ can become empty for some ϕ, so that, if we specify $\tilde{\pi}_\phi$ that supports the entire Φ (e.g., the normal -Wishart prior for $\phi = (B, \Sigma)$), the data allow us to update the belief about the *plausibility* of the imposed assumptions (i.e., the posterior probability of having a non-empty identified set). As is also discussed in Kline and Tamer (2013), we consider the posterior *plausibility* of the imposed assumptions as an important quantity to report in empirical applications, since it can convey the *upper bound* of the credibility (most optimistic belief) of the imposed assumptions after observing data.[8] In fact, the posterior *plausibility* of the

[8] An alternative quantity that is informative for assessing the plausibility of the imposed restrictions is the prior-posterior odds of the nonemptiness of the identified set,

$$O_{F,S} = \frac{\tilde{\pi}_{\phi|Y}\left(\{\phi : \mathcal{Q}(\phi|F,S) \neq \emptyset\}\right)}{\tilde{\pi}_{\phi}\left(\{\phi : \mathcal{Q}(\phi|F,S) \neq \emptyset\}\right)}.$$

$O_{F,S}$ exceeding one indicates that the data are in favor of "plausibility of the imposed assumptions."

imposed assumptions is not specific to the current robust Bayes proposal, but in principle it can be computed as a by-product of the MCMC algorithm in the Bayesian structural VAR analysis with the sign restricted impulse responses, although it has been rarely reported in the literature.

In the frequentist approach Moon, Schorfheide, and Granziera (2013), it is instead not straightforward to separate out the inferential statement about the plausibility of the assumptions from the confidence statement about the identified set. It is possible to observe non-empty frequentist confidence intervals for an empty identified set as a result to sampling variation in the estimators of the set's upper and lower bounds. In contrast, the robust Bayes approach proposed above enables us to separately quantify these two types of information on the basis of the posterior distribution of the reduced form parameters, by reporting both the posterior probability of having an empty identified set and the posterior bounds conditional on ϕ yielding a nonempty identified set.

6 Asymptotic Properties

This section shows consistency of the posterior bounds for the identified set. Let $\phi_0 \in \Phi$ be the true value of the reduced form parameters, and let $Y^T = (y_1, \ldots, y_T)$ denote a sample of size T generated from the probability distribution of the data, $p(Y^T|\phi_0)$. We assume posterior consistency for the reduced form parameters, meaning $\lim_{T\to\infty} \pi_{\phi|Y^T}(G) = 1$ for every G open neighborhood of ϕ_0, $p(Y^T|\phi_0)$-a.s.

The next proposition shows posterior consistency of the impulse response identified set as well as consistency of the posterior mean bounds and robustified credible region for the true identified set.

Proposition 6.1 *Suppose that $IS_r(\phi|F,S)$ is a non-empty and continuous correspondence at $\phi = \phi_0$, and let $[\ell(\phi_0), u(\phi_0)]$ be the convex hull of $IS_r(\phi_0|F,S)$.*

(i) $\lim_{T\to\infty} \pi_{\phi|Y^T}(\{\phi : d_H(IS_r(\phi|F,S), IS_r(\phi_0|F,S)) > \epsilon\}) = 0$, $p(Y^T|\phi_0)$-a.s., where $d_H(\cdot,\cdot)$ is the Hausdorff distance.

(ii) If $\ell(\phi)$ and $u(\phi)$, $\phi \sim \pi_{\phi|Y^T}$, are uniformly integrable, $p(Y^T|\phi_0)$-a.s.,[9], the range of the posterior means converges to $[\ell(\phi_0), u(\phi_0)]$ as $T \to \infty$, $p(Y^T|\phi_0)$-a.s., i.e.,

$$\int_\Phi \ell(\phi)\, d\pi_{\phi|Y^T} \to \ell(\phi_0) \quad \text{and}$$
$$\int_\Phi u(\phi)\, d\pi_{\phi|Y^T} \to u(\phi_0), \quad \text{as } T \to \infty,\ p(Y^T|\phi_0)\text{-a.s.},$$

[9] The uniform integrability of $\ell(\phi)$ and $u(\phi)$, $p(Y^T|\phi)$-a.s. means

$$\sup_T \int_{|\ell(\phi)|>c} |\ell(\phi)|\, d\pi_{\phi|Y^T} \to 0, \text{ and}$$
$$\sup_T \int_{|u(\phi)|>c} |u(\phi)|\, d\pi_{\phi|Y^T} \to 0,$$

as $c \to \infty$, $p(Y^T|\phi)$-a.s.

and the shortest-width robustified credible region with credibility $\alpha \in (0,1)$ *converges to* $[\ell(\phi_0), u(\phi_0)]$, $p(Y^T|\phi_0)$*-a.s.*

Proof. See Appendix A. ■

The first claim of this proposition shows that the identified set $IS_r(\phi|F,S)$ viewed as a random set induced by the posterior of ϕ converges to the true identified set in the Hausdorff metric. This claim only relies on continuity of the identified set correspondence and does not rely on convexity of $IS_r(\phi_0|F,S)$. If $IS_r(\phi_0|F,S)$ is convex, as is implied under the conditions of Lemma 3.1 or 3.2, Proposition 6.1 (ii) shows that the posterior mean bounds and the robustified credible region constructed in (Step 5) of Algorithm 5.1 converge to the true convex identified set. On the other hand, if the true identified set is non-convex, then, the posterior mean bounds and the robustified credible regions are consistent for the convex hull of the true identified set.

The continuity of $IS_r(\phi|F,S)$ at $\phi = \phi_0$ assumed in this proposition is crucial for guaranteeing consistency of the posterior bounds. The continuity of $IS_r(\phi|F,S)$ can be ensured by imposing a set of more primitive conditions involving a rank condition for the coefficient matrices of the zero and sign restrictions. We clarify them in the next proposition. In the statement of the proposition, for $x = (x_1, \ldots, x_m) \in \mathcal{R}^m$, $x >> 0$ means $x_i > 0$ for all $i = 1, \ldots, m$.

Proposition 6.2 *Consider the set-up of Lemma 3.2, where* $\mathcal{I}_S = \{j^*\}$*, i.e., the sign restrictions are placed only on the* j^**-th structural shock. Let* $\{r = c'_{ih}(\phi)q_{j^*} : i = 1, \ldots, n, h = 0, 1, 2, \ldots\}$ *be the impulse responses of interest.*

(i) Suppose that the zero restrictions $F(\phi, Q) = \mathbf{0}$ *satisfy one of the conditions (i) and (ii) of Lemma 3.1. If there exists* $G \subset \Phi$ *an open neighborhood of* ϕ_0 *such that* $rank(F_{j^*}(\phi)) = f_{j^*}$ *for all* $\phi \in G$*, and if there exists a unit length vector* $q \in \mathcal{R}^n$ *such that*

$$F_{j^*}(\phi_0)\, q = 0 \text{ and } \begin{pmatrix} S_{j^*}(\phi_0) \\ \left(\sigma^{j^*}(\phi_0)\right)' \end{pmatrix} q >> \mathbf{0},$$

then the identified set correspondence $IS_r(\phi)$ *is continuous at* $\phi = \phi_0$ *for every* $i = 1, \ldots, n$ *and* $h = 0, 1, 2, \ldots$.

(ii) Suppose that the zero restrictions $F(\phi, Q) = \mathbf{0}$ *satisfy condition (iii) of Lemma 3.1, and let* $[q_1(\phi), \ldots, q_{i^*}(\phi)]$ *be the first* i^**-th column vectors of* Q *that are exactly identified. If there exists* $G \subset \Phi$ *an open neighborhood of* ϕ_0 *such that* $\begin{pmatrix} F_{j^*}(\phi) \\ q'_1(\phi) \\ \vdots \\ q'_{i^*}(\phi) \end{pmatrix}$ *is a full row-rank matrix for all* $\phi \in G$,

and if there exists a unit length vector $q \in \mathcal{R}^n$ such that

$$\begin{pmatrix} F_{j^*}(\phi_0) \\ q_1'(\phi_0) \\ \vdots \\ q_{i^*}'(\phi_0) \end{pmatrix} q = 0 \text{ and } \begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} q >> \mathbf{0},$$

then the identified set correspondence $IS_r(\phi)$ is continuous at $\phi = \phi_0$ for every $i = 1, \ldots, n$ and $h = 0, 1, 2, \ldots$.

Proof. See Appendix A. ∎

7 An Empirical Example

We illustrate the use of the posterior bound analysis developed above in a four-variable SVAR, where the vector of observables consists of the nominal interest rate i_t, real GDP growth Δy_t, inflation rate π_t, and real money balances m_t. The data set we use is from Aruoba and Schorfheide (2011), and it is used in the empirical illustration of Moon et al (2013). The data are quarterly observations for the period 1965:I to 2005:I from the FRED2 database of the Federal Reserve Bank of St. Louis. For the details of the construction of the variables, see Aruoba and Schorfheide (2011).

We specify the four variable structural VAR as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} i_t \\ \Delta y_t \\ \pi_t \\ m_t \end{pmatrix} = c + \sum_{j=1}^{2} A_j \begin{pmatrix} i_{t-j} \\ \Delta y_{t-j} \\ \pi_{t-j} \\ m_{t-j} \end{pmatrix} + \begin{pmatrix} \epsilon_{i,t} \\ \epsilon_{y,t} \\ \epsilon_{\pi,t} \\ \epsilon_{m,t} \end{pmatrix},$$

where we specify the lag to be $p = 2$. The output variable is transformed to first differences (GDP growth) to ensure invertibility of the VAR. The order of variables presented here is chosen in such way that the set of zero restrictions introduced below are compatible with the condition $f_1 \geq f_2 \geq f_3 \geq f_4$. Suppose that the impulse response of interest is the output response to a monetary policy shock, $\frac{\partial y_{t+h}}{\partial \epsilon_{i,t}}$, i.e., $j^* = 1$. For the sign normalizations, recall that the diagonal elements of the coefficient matrix in the left-hand side are restricted to being nonnegative, so the output response is estimated with respect to a unit standard deviation contractionary monetary policy shock.

We consider several specifications of the SVARs that use different combinations of the following set of zero and sign restrictions.

Restrictions:

(i) The monetary authority does not respond to contemporaneous GDP growth, $a_{12} = 0$.

(ii) The instantaneous impulse response of the real GDP to a monetary policy shock is zero, $IR^0(\Delta y, i) = 0$.

(iii) The long-run impulse response of the real GDP level to a monetary policy shock is zero, $CIR^\infty(\Delta y, i) \approx \sum_{h=1}^{H} \frac{\partial \Delta y_{t+h}}{\delta \epsilon_{i,t}} = 0$, with $H = 80$.

(iv) The inflation response to a contractionary monetary policy shock is nonpositive for one quarter, $\frac{\partial \pi_{t+h}}{\partial \epsilon_{i,t}} \leq 0$ for $h = 0, 1$, the interest rate response is nonnegative for one quarter, $\frac{\partial i_{t+h}}{\partial \epsilon_{i,t}} \geq 0$ for $h = 0, 1$, and the response of the real money balances is nonpositive for one quarter, $\frac{\partial m_{t+h}}{\partial \epsilon_{i,t}} \geq 0$, for $h = 0, 1$.

To compare the identifying power and informativeness of each restriction, we conduct the posterior bound analysis for seven different combinations of the restrictions. Table 1 summarizes the combinations of restrictions that we consider. The restrictions (i) through (iii) are zero restrictions that constrain the first column vector of Q, so $f_1 = 1$ if only one assumption out of (i) - (iii) are imposed (Model II and IV), and $f_1 = 2$ if two out of (i) - (iii) are imposed. On the other hand, no zero restrictions are placed on the other columns of Q, so $f_2 = f_3 = f_4 = 0$ hold for all the models. The sign restrictions on the impulse responses are given by (iv), which are identical to those considered in Moon, Schorfheide, and Granziera (2013). We impose the sign restrictions on all the specifications. Note that, in all the models considered, the impulse response of interest is partially identified, so the posterior bounds are not expected to collapse to a point. For all specifications, condition (i) of Lemma 3.2 guarantees convexity of the impulse response identified set.

Table 1: Definition of Models and Posterior Plausibility

Restriction \ Model	I	II	III	IV	V	VI	VII
(i) $a_{12} = 0$	-	O	-	-	O	O	-
(ii) $IR^0(\Delta y, i) = 0$	-	-	O	-	O	-	O
(iii) $CIR^\infty(\Delta y, i) = 0$	-	-	-	O	-	O	O
(iv) sign restrictions	O	O	O	O	O	O	O
$\Pr(IS_r(\phi \vert F, S) \neq \emptyset \vert data)$	1.00	1.00	1.00	1.00	0.99	0.93	0.98

Note: "O" indicates the restriction is imposed

The prior for the reduced form parameters ($\tilde{\pi}_\phi$, as defined in Section 4) is common to all the models and it is specified to be improper $d\tilde{\pi}_\phi(B, \Sigma) \propto |\Sigma|^{-\frac{4+1}{2}}$. This prior for ϕ corresponds to the Jeffreys' prior for the reduced form Gaussian VAR, and the posterior for ϕ is nearly identical to the likelihood with the current sample size. The bottom row of Table 1 reports the posterior probabilities for the plausibility of the imposed restrictions (nonemptiness of the identified set). In all the specifications considered, these probabilities are approximately one or nearly one for the current sample.

In addition to the posterior bound analysis, we further consider standard Bayesian inference based on a single prior, for the purpose of assessing how much extra information is added to the posterior inference by the non-updated part of the prior. We introduce a prior for Q that builds on the agnostic prior of Uhlig (2005). Specifically, we obtain the approximated posterior for the impulse responses based on the MCMC draws of the impulse responses. The draws for the impulse responses are obtained by iterating Step (2.1) - (2.3) of Algorithm 5.1, and retaining the draws of Q that satisfy the sign restrictions.

Figures 1 and 2 present the posterior distributions and the posterior bounds for the impulse responses.[10] In implementing Algorithm 5.1, we draw ϕ's until we obtain 1000 realizations of the nonempty identified set $IS(\phi|S, F)$. In all the models considered, we employ the non-linear optimization step of Algorithm 5.1 (Step 3). Since we use the same prior for ϕ in every model and the posterior probabilities of having nonempty identified sets are close to one for all the models, the posterior bounds differ across the models mainly due to the different identifying restrictions. Model I in Figure 3 shows that the posterior of ϕ combined with only the sign restrictions does not lead to informative inference for output responses; their posterior distributions vary over a wide range and we can observe that the posterior mean bounds are as wide as the posterior credible region of the single prior standard Bayesian procedure. Note that the posterior mean bounds and the robustified credible regions are as wide as the point estimator of the identified sets and the frequentist confidence intervals reported in Moon, Schorfheide, and Granziera (2013). This similarity with frequentist inference for impulse response identified sets is not surprising given the consistency property of the posterior bounds (Proposition 6.1). When one zero restriction is additionally imposed (Model II - IV), the posterior mean bounds and the robustified credible regions get substantially tighter, although an informative conclusion is hard to draw (except the negative impact in the short horizon in Model III). With two additional zero restrictions (Model V - VII), the posterior mean bounds become informative for the sign of the output response for short to middle-range horizons. Specifically, when the imposed zero restrictions include restriction (ii) (Model V and VII), the range of posterior means of output responses is negative for $h = 0$ up to $h = 10$. On the other hand, if restrictions (i) and (iii) are jointly imposed, the range of posterior means is positive for short horizons, and we obtain the opposite conclusion to Models V and VII. These results on relatively more informative posterior bounds show that, despite the lack of point-identification, the posterior is less sensitive to the choice of prior for Q once any of the two zero restrictions is imposed.

It is worth noting that both the posterior mean bounds and the lower robustified credible region become tighter as more restrictions are added. As far as the posterior probabilities of the nonemptiness of the identified set are one, this monotonic gain in informativeness of the posterior

[10] These figures summarize the marginal distribution of the impulse response at each horizon, and do not capture the dependence of the responses across different horizons.

bounds holds *irrespective of the realized values of the observations*, since adding zero equality or sign restrictions monotonically reduces the size of the prior class without changing the posterior of ϕ.[11] This property of "more restrictions, more informative inference" does not necessarily hold if we report frequentist confidence intervals for the true identified set.

8 Conclusion

We develop a robust Bayes inference procedure for non-identified structural vector autoregressions subject to under-identifying zero and/or sign restrictions. The proposed procedure reports the range of posterior means and posterior probabilities for a given impulse response, when the prior varies over the class that consists of any priors for the non-identified components of the model that satisfy the restrictions. When the identified set is convex, the posterior bounds are easy to compute even for a large number of restrictions and we provide easy-to-check conditions for convexity that are verified for a large class of non-identified SVARs with zero and/or sign restrictions. The range of posterior quantities we derived can be interpreted as conducting Bayesian inference about the identified set, and the posterior mean bounds and the robustified credible region converge asymptotically to the true identified set. Our procedure can be a useful tool for separating the information about the impulse responses contained in the data from any prior input that is not updated by the data. Given that the shape of the likelihood is an object of interest for both Bayesians and frequentists, we believe that both may find the proposed analysis useful in summarizing and visualizing the information about the impulse responses contained in the observed likelihood, as is also advocated in Sims and Zha (1998) in the point-identified context.

Note that the robustified credible region we provide are for a specific impulse response at a given horizon. If one wanted to provide inferential statements about multiple impulse responses, it would be in principle possible to define the range of posterior probabilities, but this presents challenges both in terms of visualization and computation. This is true in the point identified case (see the discussion in Inoue and Kilian (2013)), and it appears even more challenging in the set identified case. We thus leave this endeavour for future research.

A final priority for future research is showing the asymptotic frequentist coverage properties of our robustified credible sets.

[11] Manski (2003) calls the general principle of the trade-off between the strength of assumptions and informativenss of the conclusion "the law of decreasing credibility." Manski defines this concept in terms of the true identified set, while our posterior bounds analysis respects this principle in the posterior inferential statement at every possible realization of data.

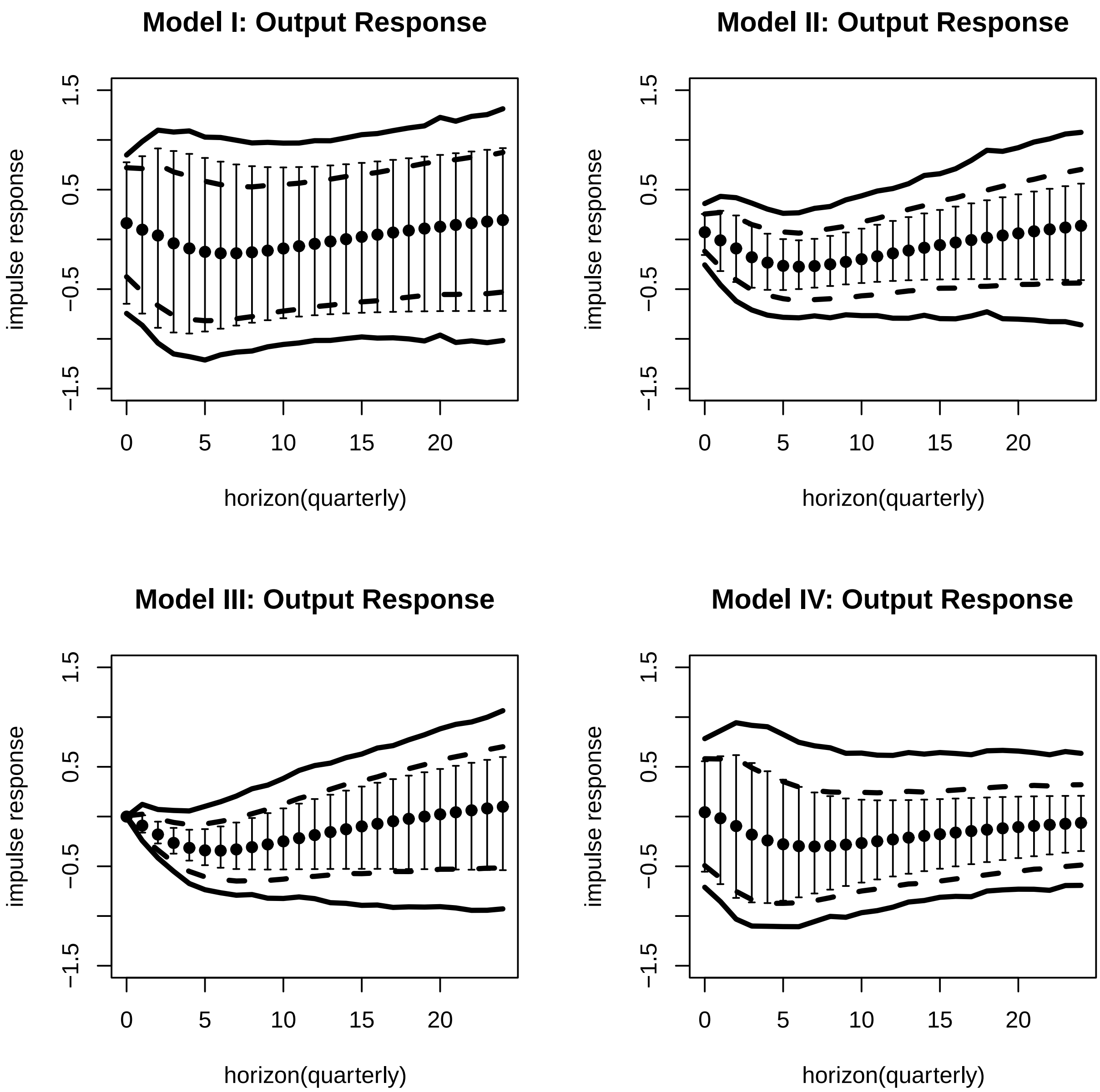

Figure 1: **Plots of Output Impulse Responses.** See Table 1 for the definition of each model. In each figure, the points plot the posterior means with the single prior for Q, the vertical bars show the posterior mean bounds with the multiple priors for Q, the dashed curves connect the upper/lower bounds of the highest posterior density regions with credibility 90% with the single prior for Q, and the solid curves connect the upper/lower bounds of our posterior robustified credible regions with credibility 90%.

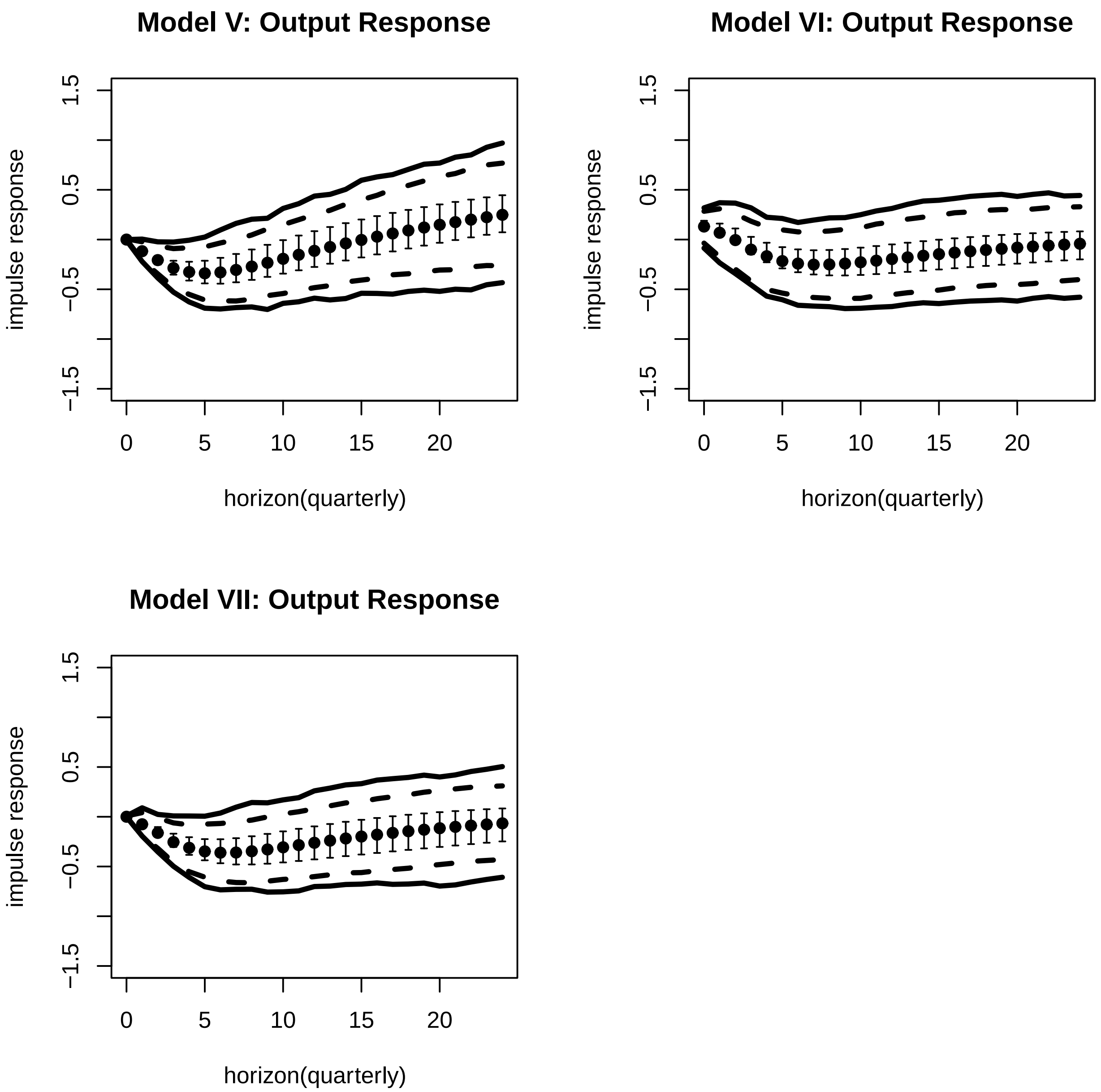

Figure 2: **Plots of Output Impulse Responses for Model IV - VII.** See Table 1 for the definition of each model. See the caption of Figure 3 for remarks.

Appendix

A Proofs

The proofs given below use the following notation. For given $\phi \in \Phi$, and $i = 1, \ldots, n$, let $\tilde{f}_i(\phi) \equiv rank(F_i(\phi))$. Let $\mathcal{F}_i^{\perp}(\phi)$ be the linear subspace of $\mathcal{R}^n$ that is orthogonal to the row vectors of $F_i(\phi)$. If no zero restrictions are placed on q_i, we interpret $\mathcal{F}_i^{\perp}(\phi)$ to be $\mathcal{R}^n$. Note that, the dimension of $\mathcal{F}_i^{\perp}(\phi)$ is equal to $n - \tilde{f}_i(\phi)$. We let $\mathcal{H}_i(\phi)$ be the half-space in $\mathcal{R}^n$ defined by the sign normalization restriction $\left\{z \in \mathcal{R}^n : \left(\sigma^i\right)' z \geq 0\right\}$, where σ^i is the i-th column vector of Σ_{tr}^{-1}. The unit sphere in $\mathcal{R}^n$ is denoted by $\mathcal{S}^{n-1}$. Given linearly independent vectors, $A = [a_1, \ldots, a_j] \in \mathcal{R}^{n \times j}$, denote the linear subspace in $\mathcal{R}^n$ that is orthogonal to the column vectors of A by $\mathcal{P}(A)$. Note that the dimension of $\mathcal{P}(A)$ is $n - j$.

Proof of Lemma 3.1. Fix $\phi \in \Phi$, and let $Q_{1:i} = [q_1, \ldots, q_i]$ be an $n \times i$ matrix of orthogonal vectors in $\mathcal{R}^n$. The set of feasible Q's satisfying the zero restrictions and the sign normalizations, $\mathcal{Q}(\phi|F)$, can be written in the following recursive manner,

$$\begin{aligned}
Q &= [q_1, \ldots, q_n] \in \mathcal{Q}(\phi|F) \\
&\quad \text{if and only if } Q = [q_1, \ldots, q_n] \text{ satisfies} \\
q_1 &\in D_1(\phi) \equiv \mathcal{F}_1^{\perp}(\phi) \cap \mathcal{H}_1(\phi) \cap \mathcal{S}^{n-1}, \\
q_2 &\in D_2(\phi, q_1) \equiv \mathcal{F}_2^{\perp}(\phi) \cap \mathcal{H}_2(\phi) \cap \mathcal{P}(q_1) \cap \mathcal{S}^{n-1}, \\
q_3 &\in D_3(\phi, Q_{1:2}) \equiv \mathcal{F}_3^{\perp}(\phi) \cap \mathcal{H}_3(\phi) \cap \mathcal{P}(Q_{1:2}) \cap \mathcal{S}^{n-1}, \\
&\quad \vdots \\
q_j &\in D_j\left(\phi, Q_{1:(j-1)}\right) \equiv \mathcal{F}_j^{\perp}(\phi) \cap \mathcal{H}_j(\phi) \cap \mathcal{P}(Q_{1:(j-1)}) \cap \mathcal{S}^{n-1}, \qquad (A.1) \\
&\quad \vdots \\
q_n &\in D_n\left(\phi, Q_{1:(n-1)}\right) \equiv \mathcal{F}_n^{\perp}(\phi) \cap \mathcal{H}_n(\phi) \cap \mathcal{P}(Q_{1:(n-1)}) \cap \mathcal{S}^{n-1}.
\end{aligned}$$

where $D_i\left(\phi, Q_{1:(i-1)}\right) \subset \mathcal{R}^n$ is the set of feasible q_i's given $Q_{1:(i-1)} = [q_1, \ldots, q_{i-1}]$, the set of $(i-1)$ orthonormal vectors in $\mathcal{R}^n$ preceding i. Nonemptiness of the identified set for $r_{ij}^h = c_{ih}(\phi)\, q_j$ follows if the feasible domain of the orthogonal vector, $D_i\left(\phi, Q_{1:(i-1)}\right)$ is nonempty at every $i = 1, \ldots, n$.

Note that, by the assumption $f_1 \leq n-1$, $\mathcal{F}_1^{\perp}(\phi) \cap \mathcal{H}_1(\phi)$ is the half-space of the linear subspace of $\mathcal{R}^n$ with dimension $n - \tilde{f}_1(\phi) \geq n - f_1 \geq 1$. Hence, $D_1(\phi)$ is nonempty for every $\phi \in \Phi$. For $i = 2, \ldots, n$, $\mathcal{F}_i^{\perp}(\phi) \cap \mathcal{H}_i(\phi) \cap \mathcal{P}(Q_{1:(i-1)})$ is the half-space of the linear subspace of $\mathcal{R}^n$ with dimension at least

$$\begin{aligned}
n - \tilde{f}_i(\phi) - \dim(\mathcal{P}(Q_{1:(n-1)})) &\geq n - f_i - (i-1) \\
&\geq 1,
\end{aligned}$$

where the last inequality follows by the assumption $f_i \leq n - i$. Hence, $D_i\left(\phi, Q_{1:(i-1)}\right)$ is non-empty for every $\phi \in \Phi$. We thus conclude that $\mathcal{Q}(\phi|F)$ is nonempty, and this implies nonemptiness of the impulse response identified sets for every $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n\}$, and $h = 0, 1, 2, \ldots$. The boundedness of the identified sets follows since $\left| r_{ij}^h \right| \leq \| c_{ih}(\phi) \| < \infty$ for any $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n\}$, and $h = 0, 1, 2, \ldots$, where the boundedness of $\| c_{ih}(\phi) \|$ is ensured by the restriction on ϕ such that the reduced form VAR is invertible to VMA(∞).

Next, we show convexity of the identified set of the impulse response to the j^*-th shock under each one of conditions (i) - (iii). Suppose $j^* = 1$ and $f_1 < n-1$ (condition (i)). Since $\tilde{f}_1(\phi) < n-1$ for all $\phi \in \Phi$, $D_1(\phi)$ is a path-connected set because it is an intersection of the half-space with dimension at least 2 and the unit sphere. Since the impulse response is a continuous function of q_1, the identified set of $r_{i1}^h = c_{ih}(\phi) q_1$ is an interval, as the range of a continuous function with a path-connected domain is always an interval (see, e.g., Propositions 12.11 and 12.23 in Sutherland (2009)).

Suppose $j^* \geq 2$ and assume condition (ii) holds. Denote the set of feasible q_{j^*}'s by $\mathcal{E}_{j^*}(\phi) \equiv \left\{ q_{j^*} \in \mathcal{S}^{n-1} : Q \in \mathcal{Q}(\phi|F) \right\}$. The next lemma provides a specific expression of $\mathcal{E}_{j^*}(\phi)$. We defer its proof to a later part of this appendix.

Lemma A.1 *Suppose $j^* \geq 2$ and assume condition (ii) of Lemma 3.1 holds. Then $\mathcal{E}_{j^*}(\phi) = \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{S}^{n-1}$.*

This lemma shows that $\mathcal{E}_{j^*}(\phi)$ is an intersection of a half-space of a linear subspace with dimension $n - f_{j^*} \geq j^* \geq 2$ with the unit sphere, so $\mathcal{E}_{j^*}(\phi)$ is a path-connected set on $\mathcal{S}^{n-1}$. Hence, convexity of $IS_r(\phi|F)$ holds.

Next, suppose condition (iii) holds. Let $Q_{1:i^*}(\phi) \equiv [q_1(\phi), \ldots, q_{i^*}(\phi)]$ be the first i^* columns of feasible $Q \in \mathcal{Q}(\phi|F)$, that are common for all $Q \in \mathcal{Q}(\phi|F)$, ϕ-a.s., by exact identification of the first i^* columns. In this case, the set of feasible q_{j^*}'s can be expressed as in the next lemma (see a later part of this appendix section for its proof).

Lemma A.2 *Suppose $j^* \geq 2$ and assume condition (iii) of Lemma 3.1 holds. Then, whenever $Q_{1:i^*}(\phi) = (q_1(\phi), \ldots, q_{i^*}(\phi))$ is uniquely determined as a function of ϕ (this is the case ϕ-a.s. by the assumption of exact identification), $\mathcal{E}_{j^*}(\phi) = \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{P}(Q_{1:i^*}(\phi)) \cap \mathcal{S}^{n-1}$.*

This lemma shows that $\mathcal{E}_{j^*}(\phi)$ is an intersection of a half-space of a linear subspace with dimension $n - f_{j^*} - i^* \geq j^* + 1 - i^* \geq 2$ with the unit sphere, so $\mathcal{E}_{j^*}(\phi)$ is a path-connected set on $\mathcal{S}^{n-1}$. Hence, convexity of $IS_r(\phi|F)$ holds.

For the cases under condition (i) or (ii), since $\phi \in \Phi$ is arbitrary, the convexity of the impulse response identified set holds for every $\phi \in \Phi$. As for the case of condition (iii), the exact identification of $[q_1(\phi), \ldots, q_{i^*}(\phi)]$ assumes its unique determination for only ϕ-a.s., so convexity of the identified set holds ϕ-a.s. ■

Proof of Lemma 3.2. Suppose $j^* = 1$ and $f_1 < n-1$ (condition (i) of Lemma 3.1). Using the notation introduced in (A.1), the set of feasible q_1's can be denoted by $D_1(\phi) \cap \{x \in \mathcal{R}^n : S_1(\phi) x \geq 0\}$.

Let $\tilde{q}_1 \in D_1(\phi)$ be a unit length vector that satisfies $\begin{pmatrix} S_1(\phi) \\ (\sigma^1)' \end{pmatrix} \tilde{q}_1 > \mathbf{0}$. Such $\tilde{q}_1$ is guaranteed to exist by the assumption. Let $q_1 \in D_1(\phi) \cap \{x \in \mathcal{R}^n : S_1(\phi)x \geq 0\}$ be arbitrary. Note that $q_1 \neq -\tilde{q}_1$ must hold, since otherwise some of the sign restrictions are violated. Consider

$$q_1(\lambda) = \frac{\lambda q_1 + (1-\lambda)\tilde{q}_1}{\|\lambda q_1 + (1-\lambda)\tilde{q}_1\|}, \quad \lambda \in [0,1],$$

which is a connected path in $D_1(\phi) \cap \{x \in \mathcal{R}^n : S_1(\phi)x \geq \mathbf{0}\}$ since the denominator is nonzero for all $\lambda \in [0,1]$ by the fact that $q_1 \neq -\tilde{q}_1$. Since q_1 is arbitrary, we can connect any points in $D_1(\phi) \cap \{x \in \mathcal{R}^n : S_1(\phi)x \geq \mathbf{0}\}$ by connected-paths via $\tilde{q}_1$. Hence, $D_1(\phi) \cap \{x \in \mathcal{R}^n : S_1(\phi)x \geq \mathbf{0}\}$ is path-connected, and convexity of the impulse response identified set follows.

Suppose $j^* \geq 2$ and assume that the imposed zero restrictions satisfy condition (ii) of Lemma 3.1. Let $\mathcal{E}_{j^*}(\phi) \equiv \{q_{j^*} \in \mathcal{S}^{n-1} : Q \in \mathcal{Q}(\phi|F,S)\}$, and, by the assumption, let $\tilde{q}_{j^*} \in \mathcal{E}_{j^*}(\phi)$ be chosen so as to satisfy $\begin{pmatrix} S_{j^*}(\phi) \\ [\sigma^{j^*}(\phi)]' \end{pmatrix} \tilde{q}_{j^*} > \mathbf{0}$. For any $q_{j^*} \in \mathcal{E}_{j^*}(\phi)$, $q_{j^*} \neq -\tilde{q}_{j^*}$ must be true, since otherwise q_{j^*} would violate some of the imposed sign restrictions. Consider constructing a path between q_{j^*} and $\tilde{q}_{j^*}$ as follows. For $\lambda \in [0,1]$, let

$$q_{j^*}(\lambda) = \frac{\lambda \tilde{q}_{j^*} + (1-\lambda) q_{j^*}}{\|\lambda \tilde{q}_{j^*} + (1-\lambda) q_{j^*}\|}, \tag{A.2}$$

which is a continuous path on the unit sphere since the denominator is nonzero for all $\lambda \in [0,1]$ by the construction of $\tilde{q}_{j^*}$. Along this path, $F_{j^*}(\phi) q_{j^*}(\lambda) = \mathbf{0}$ and the sign restrictions hold. Hence, for every $\lambda \in [0,1]$, if there exists $Q(\lambda) \equiv [q_1(\lambda), \ldots, q_{j^*}(\lambda), \ldots, q_n(\lambda)] \in \mathcal{Q}(\phi|F,S)$, where the j^*-th column is set to $q_{j^*}(\lambda)$, then the path-connectedness of $\mathcal{E}_{j^*}(\phi)$ follows. The recursive construction of Algorithm 3.2 can be used to construct such $Q(\lambda) \in \mathcal{Q}(\phi|F,S)$. For $i = 1, \ldots, (j^*-1)$, we recursively obtain $q_i(\lambda)$ that solves

$$\begin{pmatrix} F_i(\phi) \\ q_1'(\lambda) \\ \vdots \\ q_{i-1}'(\lambda) \\ q_{j^*}'(\lambda) \end{pmatrix} q_i(\lambda) = \mathbf{0}, \tag{A.3}$$

and satisfies $[\sigma^i(\phi)]' q_i(\lambda) \geq 0$. Such a $q_i(\lambda)$ always exists since the rank of the matrix multiplied to $q_i(\lambda)$ is at most $f_i + i$, which is less than n under condition (ii). For $i = (j^*+1), \ldots, n$, a direct application of Algorithm 3.2 yields a feasible $q_i(\lambda)$. Thus, existence of $Q(\lambda) \in \mathcal{Q}(\phi|F,S)$, $\lambda \in [0,1]$, is established. We therefore conclude that $\mathcal{E}_{j^*}(\phi)$ is path-connected under condition (ii), and the convexity of impulse response identified sets holds for every variable and every horizon. This completes the proof of (i) in the lemma.

Now, suppose that the imposed zero restrictions satisfy condition (iii) of Lemma 3.1. Let $[q_1(\phi), \ldots, q_{i^*}(\phi)]$ be the first i^*-th columns of feasible $Q's$, that are common for all $Q \in \mathcal{Q}(\phi|F,S)$, ϕ-a.s., by exact identification of the first i^*-columns. Let $\tilde{q}_{j^*} \in \mathcal{E}_{j^*}(\phi)$ be chosen so as to satisfy $\begin{pmatrix} S_{j^*}(\phi) \\ (\sigma^{j^*})' \end{pmatrix} \tilde{q}_{j^*} > \mathbf{0}$, and $q_{j^*} \in \mathcal{E}_{j^*}(\phi)$ be arbitrary. Consider $q_{j^*}(\lambda)$ in (A.2) and construct $Q(\lambda) \in \mathcal{Q}(\phi|F,S)$ as follows. The first i^*-th column of $Q(\lambda)$ must be $[q_1(\phi), \ldots, q_{i^*}(\phi)]$, ϕ-a.s., by the assumption of exact identification. For $i = (i^*+1), \ldots, (j^*-1)$, we can recursively obtain $q_i(\lambda)$ that solves

$$\begin{pmatrix} F_i(\phi) \\ q_1'(\phi) \\ \vdots \\ q_{i^*}'(\phi) \\ q_{i^*+1}'(\lambda) \\ \vdots \\ q_{i-1}'(\lambda) \\ q_{j^*}'(\lambda) \end{pmatrix} q_i(\lambda) = \mathbf{0} \tag{A.4}$$

and satisfies $\left[\sigma^i(\phi)\right]' q_i(\lambda) \geq 0$. There always exist such $q_i(\lambda)$ because $f_i < n - i$ for all $i = (i^*+1), \ldots, (j^*-1)$. The rest of column vectors $q_i(\lambda)$, $i = j^*+1, \ldots, n$, of $Q(\lambda)$ are obtained successively by applying Algorithm 3.2. Having shown a feasible construction of $Q(\lambda) \in \mathcal{Q}(\phi|F,S)$ for $\lambda \in [0,1]$, we conclude that $\mathcal{E}_{j^*}(\phi)$ is path-connected, and convexity of the impulse response identified sets follows for every variable and every horizon. ■

In what follows, we provide proofs for the lemmas used in the proof of Lemma 3.1.

Proof of Lemma A.1. Given zero restrictions $F(\phi, Q) = \mathbf{0}$ and the set of feasible orthogonal matrices $\mathcal{Q}(\phi|F)$, define the projection of $\mathcal{Q}(\phi|F)$ with respect to the first i-th column vectors,

$$\mathcal{Q}_{1:i}(\phi|F) \equiv \{[q_1, \ldots, q_i] : Q \in \mathcal{Q}(\phi|F)\}.$$

Following the recursive representation of (A.1), $\mathcal{E}_{j^*}(\phi) \equiv \left\{q_{j^*} \in \mathcal{S}^{n-1} : Q \in \mathcal{Q}(\phi|F)\right\}$ can be written as

$$\begin{aligned} \mathcal{E}_{j^*}(\phi) &= \bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \left[\mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{P}(Q_{1:(j^*-1)}) \cap \mathcal{S}^{n-1}\right] \\ &= \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \left[\bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \mathcal{P}(Q_{1:(j^*-1)})\right] \cap \mathcal{S}^{n-1}. \end{aligned}$$

Hence, the conclusion follows if we can show $\bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \mathcal{P}(Q_{1:(j^*-1)}) = \mathcal{S}^{n-1}$. To show this claim, let $q \in \mathcal{S}^{n-1}$ be arbitrary, and we construct $Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)$ such that $q \in$

$\mathcal{P}(Q_{1:(j^*-1)})$ holds. Specifically, construct q_i, $i = 1, \ldots, (j^* - 1)$, successively, by solving

$$\begin{pmatrix} F_i(\phi) \\ q_1' \\ \vdots \\ q_{i-1}' \\ q' \end{pmatrix} q_i = \mathbf{0},$$

and choose the sign of q_i to satisfy its sign normalization. Under condition (ii) of Lemma 3.1, $q_i \in \mathcal{S}^{n-1}$ solving these equalities exist since the rank of the coefficient matrix is at most $f_i + i < n$. Thus-obtained $Q_{1:(j^*-1)} = [q_1, \ldots, q_{j^*-1}]$ belongs to $\mathcal{Q}_{1:(j^*-1)}(\phi|F)$ by construction, and it is orthogonal to q. Hence, $q \in \mathcal{P}(Q_{1:(j^*-1)})$. Since q is arbitrary, we obtain $\bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \mathcal{P}(Q_{1:(j^*-1)}) = \mathcal{S}^{n-1}$. ■

Proof of Lemma A.2. Let $Q_{1:i^*}(\phi) \equiv [q_1(\phi), \ldots, q_{i^*}(\phi)]$ be the first i^*-th columns of feasible $Q \in \mathcal{Q}(\phi|F)$, that are common for all $Q \in \mathcal{Q}(\phi|F)$, ϕ-a.s., by exact identification of the first i^*-columns. As in the proof of Lemma A.1, $\mathcal{E}_{j^*}(\phi)$ can be written as

$$\begin{aligned} \mathcal{E}_{j^*}(\phi) &= \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \left[\bigcup_{Q_{1:(j^*-1)} \in \mathcal{Q}_{1:(j^*-1)}(\phi|F)} \mathcal{P}(Q_{1:(j^*-1)}) \right] \cap \mathcal{S}^{n-1} \\ &= \mathcal{F}_{j^*}^{\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{P}(Q_{1:i^*}(\phi)) \cap \bigcup_{Q_{(i^*+1):(j^*-1)} \in \mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)} \mathcal{P}(Q_{(i^*+1):(j^*-1)}) \cap \mathcal{S}^{n-1}, \end{aligned}$$

where $\mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F) = \left\{ Q_{(i^*+1):(j^*-1)} = [q_{i^*+1}, \ldots, q_{j^*-1}] : Q \in \mathcal{Q}(\phi|F) \right\}$ is the projection of $\mathcal{Q}(\phi|F)$ with respect to the (i^*+1)-th to (j^*-1)-th columns of Q. We now show that, under condition (iii) of Lemma 3.1, $\bigcup_{Q_{(i^*+1):(j^*-1)} \in \mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)} \mathcal{P}(Q_{(i^*+1):(j^*-1)}) = \mathcal{S}^{n-1}$ holds. Let $q \in \mathcal{S}^{n-1}$ be arbitrary, and we consider constructing $Q_{(i^*+1):(j^*-1)} \in \mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)$ such that $q \in \mathcal{P}(Q_{(i^*+1):(j^*-1)})$ holds. For $i = (i^*+1), \ldots, (j^*-1)$, we recursively obtain q_i by solving

$$\begin{pmatrix} F_i(\phi) \\ q_1'(\phi) \\ \vdots \\ q_{i^*}'(\phi) \\ q_{i^*+1}' \\ \vdots \\ q_{i-1}' \\ q' \end{pmatrix} q_i = \mathbf{0},$$

and choose the sign of q_i to be consistent with the sign normalization. Under condition (iii) of Lemma 3.1, $q_i \in \mathcal{S}^{n-1}$ solving these equalities exist since the rank of the coefficient matrix is at most

$f_i + i < n$ for all $i = (i^* + 1), \ldots, (j^* - 1)$. Thus-obtained $Q_{(i^*+1):(j^*-1)} = [q_{i^*+1}, \ldots, q_{j^*-1}]$ belongs to $\mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)$ by construction, and it is orthogonal to q. Hence, $q \in \mathcal{P}(Q_{(i^*+1):(j^*-1)})$. Since q is arbitrary, $\bigcup_{Q_{(i^*+1):(j^*-1)} \in \mathcal{Q}_{(i^*+1):(j^*-1)}(\phi|F)} \mathcal{P}(Q_{(i^*+1):(j^*-1)}) = \mathcal{S}^{n-1}$ is shown. ■

Proof of Corollaries 3.1 and 3.2. As for a proof of Corollary of 3.1, the successive construction of the feasible column vectors q_i, $i = 1, \ldots, n$, show that the additional zero restrictions that do not change the order of variables nor the zero restrictions for those preceding j^* does not constrain the set of feasible q_{j^*}'s.

As for Corollary 3.2, dropping the zero restrictions imposed for those following the j^*-th variable does not change the order of variables nor the construction of the set of feasible q_{j^*}'s. Under condition (ii) of Lemma 3.1, Lemma A.1 above shows that the set of feasible q_{j^*}'s does not depend on any of $F_i(\phi)$, $i = 1, \ldots, (j^* - 1)$. Hence, removing or altering them (as far as condition (ii) of Lemma 3.1 holds) does not affect the set of feasible q_{j^*}'s. Under condition (iii) of Lemma 3.1, Lemma A.2 shows that the set of feasible q_{j^*}'s does not depend on any of $F_i(\phi)$, $i = (i^* + 1), \ldots, (j^* - 1)$. Hence, relaxing the zero restrictions constraining $[q_{i^*+1}, \ldots, q_{j^*-1}]$ does not affect the set of feasible q_{j^*}'s. ■

Proof of Proposition 4.1 (ii). The proof proceeds by applying the proof of Proposition 4.1 of Kitagawa (2012). Let $r(\phi, Q) = c'_{ih}(\phi) q_j$ be the impulse response of interest. By Lemma A.4 of Kitagawa (2012) and Proposition 10.3 of Denneberg (1994), the upper bound of the posterior means of $r(\phi, Q)$ satisfies the following equality,

$$\sup_{\pi_{\phi Q|Y} \in \Pi_{\phi Q|Y}} \int r(\phi, Q) d\pi_{\phi Q|Y} = \int r(\phi, Q) d\pi^*_{\phi Q|Y},$$

where the integral with respect to the upper probability $\int r(\phi, Q) d\pi^*_{\phi Q|Y}$ stands for the generalized Choquet integral (Denneberg (1994), pp62),

$$\int r(\phi, Q) d\pi^*_{\phi Q|Y} = \int_{-\infty}^{0} \left[\pi^*_{\phi Q|Y} \left(\{ r(\phi, Q) \geq \tilde{r} \} \right) - 1 \right] d\tilde{r} + \int_{0}^{\infty} \pi^*_{\phi Q|Y} \left(\{ r(\phi, Q) \geq \tilde{r} \} \right) d\tilde{r}.$$

By the current Proposition 4.1 (i), we have that

$$\begin{aligned} \pi^*_{\phi Q|Y} \left(\{ r(\phi, Q) \geq \tilde{r} \} \right) &= \pi^*_{r|Y} \left(\{ r \geq \tilde{r} \} \right) \\ &= \pi_{\phi|Y} \left(IS(\phi|F, S) \cap \{ r \geq \tilde{r} \} \neq \emptyset \right). \end{aligned}$$

Note that $IS(\phi|F, S) \cap \{ r \geq \tilde{r} \} \neq \emptyset$ is true if and only if $\{ u(\phi) \geq \tilde{r} \}$. Hence, we have

$$\begin{aligned} \int r(\phi, Q) d\pi^*_{\phi Q|Y} &= \int_{-\infty}^{0} \left[\pi_{\phi|Y} \left(u(\phi) \geq \tilde{r} \right) - 1 \right] d\tilde{r} + \int_{0}^{\infty} \pi_{\phi|Y} \left(u(\phi) \geq \tilde{r} \right) d\tilde{r} \\ &= -\int_{-\infty}^{0} \pi_{\phi|Y} \left(u(\phi) < \tilde{r} \right) d\tilde{r} + \int_{0}^{\infty} \pi_{\phi|Y} \left(u(\phi) \geq \tilde{r} \right) d\tilde{r} \\ &= E_{\phi|Y}(u(\phi)), \end{aligned}$$

where the last line follows by the identity $E(X) = -\int_{-\infty}^{0} \Pr(X < x)dx + \int_{0}^{\infty} \Pr(X \geq x)dx$ that holds for any integrable random variable X. The lower bound of the posterior means can be obtained similarly by replacing $r(\phi, Q)$ above with $-r(\phi, Q)$. Any posterior means between the lower and upper bounds can be obtained by a mixture of the priors putting probability masses at the lower and upper bounds, so the range of the posterior means is convex. ■

Proof of Proposition 6.1. (i) Let $\epsilon > 0$ be arbitrary, and denote the identified set of an impulse response by $IS(\phi)$ for short. Recall that $IS(\cdot)$ is a compact-valued correspondence as implied from Lemma 4.1. Accordingly, the assumption of continuity of the identified set correspondence at ϕ_0 is equivalent to Hausdorff continuity of $IS(\cdot)$ at ϕ_0 (see, e.g., Proposition 5 in Chapter E of Ok (2007)), implying that there exists an open neighborhood G of ϕ_0 such that $d_H(IS(\phi), IS(\phi_0)) < \epsilon$ holds for every $\phi \in G$. Consider

$$\begin{aligned} \pi_{\phi|Y^T}(\{\phi : d_H(IS(\phi), IS(\phi_0)) > \epsilon\}) &= \pi_{\phi|Y^T}(\{\phi : d_H(IS(\phi), IS(\phi_0)) > \epsilon\} \cap G) \\ &\quad + \pi_{\phi|Y^T}(\{\phi : d_H(IS(\phi), IS(\phi_0)) > \epsilon\} \cap G^c) \\ &\leq \pi_{\phi|Y^T}(G^c), \end{aligned}$$

where the last line follows because $\{\phi : d_H(IS(\phi), IS(\phi_0)) > \epsilon\} \cap G = \emptyset$ by the construction of G. The posterior consistency of ϕ yields $\lim_{T\to\infty} \pi_{\phi|Y^T}(G^c) = 0$, $p(Y^T|\phi_0)$-a.s.

(ii) The posterior consistency of ϕ implies that ϕ converges in probability (in terms of $\pi_{\phi|Y^T}$) to ϕ_0 as $T \to \infty$. Since continuity of the identified set correspondence implies that $\ell(\phi)$ and $u(\phi)$ are continuous at ϕ_0, $\ell(\phi)$ and $u(\phi)$ converge in probability to $\ell(\phi_0)$ and $u(\phi_0)$ as $T \to \infty$, respectively. Combined with the assumption of the uniform integrability of $\ell(\phi)$ and $u(\phi)$, the convergences in probability of $\ell(\phi)$ and $u(\phi)$ imply their convergences in mean (see, e.g., Proposition 4.12 in Kallenberg (2001)).

To show the convergence of robustified credible regions, recall the notation introduced in (Step 5) of Algorithm 5.1, $d(r, \phi) = \max\{|r - \ell(\phi)|, |r - u(\phi)|\}$, and let $z_\alpha(r)$ be the α-th quantile of the posterior distribution of $d(r, \phi)$. By Proposition 5.1 of Kitagawa (2012), the shortest width robustified credible region can be written as

$$C_\alpha^{shortest} = [r^* - z_\alpha(r^*), r^* + z_\alpha(r^*)],$$

where $r^* \in \arg\min_r z_\alpha(r)$. Note that the convex hull of $IS_r(\phi_0|F, S)$, $[\ell(\phi_0), u(\phi_0)]$, can be written as

$$\left[\arg\min_r d(r, \phi_0) - \min_r d(r, \phi_0), \arg\min_r d(r, \phi_0) + \min_r d(r, \phi_0)\right].$$

Note also that $d(r, \phi_0)$ is continuous in r and has a unique minimum. Hence, $r^* \pm z_\alpha(r^*)$ converges to $\arg\min_r d(r, \phi_0) \pm \min_r d(r, \phi_0)$, if $z_\alpha(r)$ converges to $d(r, \phi_0)$ uniformly over r, $p(Y^T|\phi_0)$-a.s.

To show this uniform convergence, consider

$$\begin{aligned} |z_\alpha(r) - d(r,\phi_0)| &\leq |z_\alpha(r) - d(r,\phi)| + |d(r,\phi) - d(r,\phi_0)| \\ &\leq \frac{1}{\alpha \wedge (1-\alpha)} \rho_\alpha(d(r,\phi) - z_\alpha(r)) + |d(r,\phi) - d(r,\phi_0)|, \end{aligned} \tag{A.5}$$

where $\rho_\alpha(\cdot)$ is the check loss function, $\rho_\alpha(u) = \alpha u 1\{u \geq 0\} - (1-\alpha)u 1\{u < 0\}$, and the second line uses $|u| \leq \frac{1}{\alpha\wedge(1-\alpha)}\rho_\alpha(u)$. Let $\Delta(\phi) = \max\{|\ell(\phi) - \ell(\phi_0)|, |u(\phi) - u(\phi_0)|\}$. Since $|d(r,\phi) - d(r,\phi_0)| \leq \Delta(\phi)$, taking the posterior expectation on (A.5) leads to

$$\begin{aligned} |z_\alpha(r) - d(r,\phi_0)| &\leq \frac{1}{\alpha \wedge (1-\alpha)} E_{\phi|Y^T}[\rho_\alpha(d(r,\phi) - z_\alpha(r))] + E_{\phi|Y^T}(\Delta(\phi)) \\ &\leq \frac{1}{\alpha \wedge (1-\alpha)} E_{\phi|Y^T}[\rho_\alpha(d(r,\phi) - d(r,\phi_0))] + E_{\phi|Y^T}(\Delta(\phi)) \\ &\leq \frac{1}{\alpha \wedge (1-\alpha)} E_{\phi|Y^T}[\rho_\alpha(\Delta(\phi)) + \rho_\alpha(-\Delta(\phi))] + E_{\phi|Y^T}(\Delta(\phi)) \\ &= \left[\left\{\left[1 \vee \left(\frac{\alpha}{1-\alpha}\right)\right] + \left[1 \vee \left(\frac{1-\alpha}{\alpha}\right)\right]\right\} + 1\right] E_{\phi|Y^T}(\Delta(\phi)), \end{aligned}$$

where the second line follows since posterior α-th quantile $z_\alpha(r)$ minimizes $E_{\phi|Y^T}[\rho_\alpha(d(r,\phi) - z)]$ in z. Since the left-hand side of this inequality does not depend on r, the uniform convergence of $|z_\alpha(r) - d(r,\phi_0)|$ follows if $E_{\phi|Y^T}(\Delta(\phi)) \to 0$ as $T \to \infty$. This holds true, because

$$\begin{aligned} E_{\phi|Y^T}(\Delta(\phi)) &\leq E_{\phi|Y^T}(|\ell(\phi) - \ell(\phi_0)|) + E_{\phi|Y^T}(|u(\phi) - u(\phi_0)|) \\ &\to 0, \text{ as } T \to \infty, \end{aligned}$$

where the last line follows from the convergences in probability of $\ell(\phi)$ and $u(\phi)$ and uniform integrability of $\ell(\phi)$ and $u(\phi)$. ■

Proof of Proposition 6.2. (i) Following the notation introduced in the proofs of Lemmas 3.1 and 3.2, the upper and lower bounds of the impulse response identified set for $r = c'_{ih}(\phi)q_{j^*}$ are written as

$$\begin{aligned} u(\phi)/\ell(\phi) &= \max/\min_{q_{j^*}} c'_{ih}(\phi)q_{j^*}, \\ \text{s.t., } q_{j^*} &\in \mathcal{E}_{j^*}(\phi) \text{ and } S_{j^*}(\phi)q_{j^*} \geq \mathbf{0}. \end{aligned} \tag{A.6}$$

When $j^* = 1$ (condition (i) of Lemma 3.1), $\mathcal{E}_1(\phi)$ is given by $D_1(\phi)$ defined in (A.1). On the other hand, when $j^* \geq 2$ and condition (ii) of Lemma 3.1 holds, Lemma A.1 given in the proof of Lemma 3.1 provides a specific expression for $\mathcal{E}_{j^*}(\phi)$. Accordingly, in either case, the constrained set of q_{j^*} in (A.6) can be expressed as

$$\tilde{\mathcal{E}}_{j^*}(\phi) \equiv \left\{ q \in \mathcal{S}^{n-1} : F_{j^*}(\phi)q = \mathbf{0}, \begin{pmatrix} S_{j^*}(\phi) \\ (\sigma^{j^*}(\phi))' \end{pmatrix} q \geq \mathbf{0} \right\}.$$

The objective function of (A.6) is continuous in q_{j^*}, so, by the theorem of Maximum (see, e.g., Theorem 9.14 of Sundaram (1996)), the continuity of $u(\phi)$ and $\ell(\phi)$ is obtained if $\tilde{\mathcal{E}}_{j^*}(\phi)$ is shown to be a continuous correspondence at $\phi = \phi_0$.

To show continuity of $\tilde{\mathcal{E}}_{j^*}(\phi)$, note first that $\tilde{\mathcal{E}}_{j^*}(\phi)$ is a closed and bounded correspondence, so upper-semicontinuity and lower-semicontinuity of $\tilde{\mathcal{E}}_{j^*}(\phi)$ can be defined in terms of sequences (see, e.g., Propositions 21 of Border (2013)),

- $\tilde{\mathcal{E}}_{j^*}(\phi)$ is upper-semicontinuous (usc) at $\phi = \phi_0$ if and only if, for any sequence $\phi^v \to \phi_0$, $v = 1, 2, \ldots$, and any $q_{j^*}^v \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$, there is a subsequence of $q_{j^*}^v$ with limit in $\tilde{\mathcal{E}}_{j^*}(\phi_0)$.
- $\tilde{\mathcal{E}}_{j^*}(\phi)$ is lower-semicontinuous (lsc) at $\phi = \phi_0$ if and only if, $\phi^v \to \phi_0$, $v = 1, 2, \ldots$, and $q_{j^*}^0 \in \tilde{\mathcal{E}}_{j^*}(\phi_0)$ imply that there is a sequence $q_{j^*}^v \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$ with $q_{j^*}^v \to q_{j^*}^0$.

In the proofs given below, we use the same index v to denote a subsequence, just to compress notation.

Usc: Since $q_{j^*}^v$ is a sequence on the unit-sphere, it has a convergent subsequence $q_{j^*}^v \to q_{j^*}$. Since $q_{j^*}^v \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$, $F_{j^*}(\phi^v) q_{j^*}^v = \mathbf{0}$ and $\begin{pmatrix} S_{j^*}(\phi^v) \\ (\sigma^{j^*}(\phi^v))' \end{pmatrix} q_{j^*}^v \geq \mathbf{0}$ hold for all v. Since $F_{j^*}(\cdot)$ and $\begin{pmatrix} S_{j^*}(\cdot) \\ (\sigma^{j^*}(\cdot))' \end{pmatrix}$ are continuous in ϕ, these equality and sign restrictions hold at the limit as well. Hence, $q_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi_0)$.

Lsc: Our proof of lsc proceeds similarly to the proof of Lemma 3 in Moon et al (2013). Let $\phi^v \to \phi_0$ be arbitrary. Let $q_{j^*}^0 \in \tilde{\mathcal{E}}_{j^*}(\phi_0)$, and define $\mathbf{P}^0 = F_{j^*}(\phi_0)' \left[F_{j^*}(\phi_0) F_{j^*}(\phi_0)'\right]^{-1} F_{j^*}(\phi_0)$ be the projection matrix onto the space spanned by the row vectors of $F_{j^*}(\phi_0)$. By the assumption, $F_{j^*}(\phi)$ has full row-rank in the open neighborhood of ϕ_0, so $\mathbf{P}^0$ and $\mathbf{P}^v = F_{j^*}(\phi^v)' \left[F_{j^*}(\phi^v) F_{j^*}(\phi^v)'\right]^{-1} F_{j^*}(\phi^v)$ are well-defined for all large v. Let $\boldsymbol{\xi}^* \in \mathcal{R}^n$ be a vector satisfying $\begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} \left[I - \mathbf{P}^0\right] \boldsymbol{\xi}^* >> \mathbf{0}$, which exists by the assumption. Let

$$\eta = \min \left\{ \begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} \left[I - \mathbf{P}^0\right] \boldsymbol{\xi}^* \right\} > 0,$$

and define

$$\begin{aligned} \boldsymbol{\xi} &= \frac{2}{\eta} \boldsymbol{\xi}^*, \\ \epsilon^v &= \left\| \begin{pmatrix} S_{j^*}(\phi^v) \\ (\sigma^{j^*}(\phi^v))' \end{pmatrix} \left[I - \mathbf{P}^v\right] - \begin{pmatrix} S_{j^*}(\phi_0) \\ (\sigma^{j^*}(\phi_0))' \end{pmatrix} \left[I - \mathbf{P}^0\right] \right\|, \\ q_{j^*}^v &= \frac{\left[I - \mathbf{P}^v\right] \left[q_{j^*}^0 + \epsilon^v \boldsymbol{\xi}\right]}{\left\| \left[I - \mathbf{P}^v\right] \left[q_{j^*}^0 + \epsilon^v \boldsymbol{\xi}\right] \right\|}. \end{aligned}$$

Since $\mathbf{P}^v$ converges to $\mathbf{P}^0$, $\epsilon^v \to 0$. Furthermore, $\left[I - \mathbf{P}^0\right] q^0_{j^*} = q^0_{j^*}$ implies that $q^v_{j^*}$ converges to $q^0_{j^*}$ as $v \to \infty$. Note that $q^v_{j^*}$ is orthogonal to $F_{j^*}(\phi^v)$ by construction. Furthermore, note that

$$\begin{aligned}
&\begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} q^v_{j^*} \\
= \;& \frac{1}{\left\|\left[I - \mathbf{P}^v\right]\left[q^0_{j^*} + \epsilon^v \boldsymbol{\xi}\right]\right\|} \begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} \left[\left[I - \mathbf{P}^v\right]\left[q^0_{j^*} + \epsilon^v \boldsymbol{\xi}\right]\right] \\
\geq \;& \frac{1}{\left\|\left[I - \mathbf{P}^v\right]\left[q^0_{j^*} + \epsilon^v \boldsymbol{\xi}\right]\right\|} \begin{pmatrix} \left(\begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} \left[I - \mathbf{P}^v\right] - \begin{pmatrix} S_{j^*}(\phi_0) \\ \left(\sigma^{j^*}(\phi_0)\right)' \end{pmatrix} \left[I - \mathbf{P}^0\right] \right) q^0_{j^*} \\ + \epsilon^v \begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} \left[I - \mathbf{P}^v\right] \boldsymbol{\xi} \end{pmatrix} \\
\geq \;& \frac{1}{\left\|\left[I - \mathbf{P}^v\right]\left[q^0_{j^*} + \epsilon^v \boldsymbol{\xi}\right]\right\|} \left(-\epsilon^v \left\|q^0_{j^*}\right\| \mathbf{1} + \epsilon^v \begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} \left[I - \mathbf{P}^v\right] \boldsymbol{\xi} \right) \\
= \;& \frac{\epsilon^v}{\left\|\left[I - \mathbf{P}^v\right]\left[q^0_{j^*} + \epsilon^v \boldsymbol{\xi}\right]\right\|} \left(\frac{2}{\eta} \begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} \left[I - \mathbf{P}^v\right] \boldsymbol{\xi}^* - \mathbf{1} \right),
\end{aligned}$$

where the third line follows by $\begin{pmatrix} S_{j^*}(\phi_0) \\ \left(\sigma^{j^*}(\phi_0)\right)' \end{pmatrix} \left[I - \mathbf{P}^0\right] q^0_{j^*} = \begin{pmatrix} S_{j^*}(\phi_0) \\ \left(\sigma^{j^*}(\phi_0)\right)' \end{pmatrix} q^0_{j^*} \geq \mathbf{0}$. By the construction of $\boldsymbol{\xi}^*$ and η, $\frac{2}{\eta} \begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} \left[I - \mathbf{P}^v\right] \boldsymbol{\xi}^* > \mathbf{1}$ holds for all large v. This implies that $\begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} q^v_{j^*} \geq 0$ holds for all large v, implying that $q^v_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$ for all large v. Hence, $\tilde{\mathcal{E}}_{j^*}(\phi)$ is lsc at $\phi = \phi_0$.

(ii) *Usc*: Under condition (iii) of Lemma 3.1, Lemma A.2 implies that the constraint set of q_{j^*} in (A.6) can be expressed as

$$\tilde{\mathcal{E}}_{j^*}(\phi) \equiv \left\{ q \in \mathcal{S}^{n-1} : \begin{pmatrix} F_{j^*}(\phi) \\ q_1'(\phi) \\ \vdots \\ q_{i^*}'(\phi) \end{pmatrix} q = \mathbf{0}, \; \begin{pmatrix} S_{j^*}(\phi) \\ \left(\sigma^{j^*}(\phi)\right)' \end{pmatrix} q \geq \mathbf{0} \right\}.$$

Let $q^v_{j^*}$, $v = 1, 2, \ldots$, be a sequence on the unit-sphere, such that $q^v_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi^v)$ holds for all v. This has a convergent subsequence $q^v_{j^*} \to q_{j^*}$. Since $F_i(\phi)$ are continuous in ϕ for all $i = 1, \ldots, i^*$, $q_i(\phi)$, $i = 1, \ldots, i^*$, are continuous in ϕ as well, implying that the equality restrictions and the sign restrictions, $\begin{pmatrix} F_{j^*}(\phi^v) \\ q_1'(\phi^v) \\ \vdots \\ q_{i^*}'(\phi^v) \end{pmatrix} q^v_{j^*} = \mathbf{0}$ and $\begin{pmatrix} S_{j^*}(\phi^v) \\ \left(\sigma^{j^*}(\phi^v)\right)' \end{pmatrix} q^v_{j^*} \geq \mathbf{0}$ must hold at the limit $v \to \infty$. Hence,

$q_{j^*} \in \tilde{\mathcal{E}}_{j^*}(\phi_0)$.

Lsc: Define $\mathbf{P}^0$ and $\mathbf{P}^v$ as the projection matrices onto the row vectors of $\begin{pmatrix} F_{j^*}(\phi_0) \\ q_1'(\phi_0) \\ \vdots \\ q_{i^*}'(\phi_0) \end{pmatrix}$ and $\begin{pmatrix} F_{j^*}(\phi^v) \\ q_1'(\phi^v) \\ \vdots \\ q_{i^*}'(\phi^v) \end{pmatrix}$, respectively. The imposed assumptions imply that $\mathbf{P}^v$ and $\mathbf{P}^0$ are well-defined for all large v, and $\mathbf{P}^v \to \mathbf{P}^0$. With the current definition of $\mathbf{P}^v$ and $\mathbf{P}^0$, lower-semicontinuity of $\tilde{\mathcal{E}}_{j^*}(\phi)$ can be shown by repeating the same argument as in the proof of part (i) of the current proposition. We omit details for brevity. ■

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