

ROBUST FLUX ERROR ESTIMATION OF AN UNFITTED NITSCHKE METHOD FOR HIGH-CONTRAST INTERFACE PROBLEMS

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ABSTRACT. We prove an optimal error estimate for the flux variable for a stabilized unfitted Nitsche finite element method applied to an elliptic interface problem with discontinuous constant coefficients. Our result shows explicitly that this error estimate is totally independent of the diffusion coefficients. Interface problems; high-contrast; unfitted Nitsche method; finite elements.

1. INTRODUCTION

In this paper we study the error estimation of an unfitted Nitsche finite element method for the following elliptic interface problem with discontinuous constant coefficients: Let $\Omega \subset \mathbb{R}^2$ be an open polygonal domain with an immersed smooth interface Γ , such that $\bar{\Omega} = \bar{\Omega}^- \cup \bar{\Omega}^+$, and Γ encloses either Ω^- or Ω^+ . Consider the problem

$$\begin{aligned} (1.1a) \quad & -\nabla \cdot (\rho^\pm \nabla u^\pm) = f^\pm && \text{in } \Omega^\pm, \\ (1.1b) \quad & u^\pm = 0 && \text{on } \partial\Omega^\pm \setminus \Gamma, \\ (1.1c) \quad & [u] = 0 && \text{on } \Gamma, \\ (1.1d) \quad & [\rho \nabla u \cdot \mathbf{n}] = 0 && \text{on } \Gamma. \end{aligned}$$

The jumps on the interface Γ are defined as

$$(1.2) \quad [\rho \nabla u \cdot \mathbf{n}] = \rho^- \nabla u^- \cdot \mathbf{n}^- + \rho^+ \nabla u^+ \cdot \mathbf{n}^+ \quad \text{and} \quad [u] = u^+ - u^-,$$

where $u^\pm = u|_{\Omega^\pm}$ and \mathbf{n}^\pm is the unit outward pointing normal to Ω^\pm . We furthermore assume that the diffusion coefficients $\rho^+ \geq \rho^- > 0$ are constant.

There have been several numerical methods for problem (1.1). See for example [2], [22], [8], [11], [16], [14], [4], [9], [5], [18], [17], [23], [7], [3], [21], [1], [19], [15]. The method we will consider below uses meshes that are not necessarily aligned with the mesh (i.e. unfitted meshes). There are several papers dealing with methods (see [1], [3], [4], [5], [9], [14], [15], [17], [19], [23]) using unfitted meshes. One of the advantages of using unfitted meshes is the fact that re-meshing is not required for problems where the interface is moving. Nevertheless, the majority of the unfitted methods do not address the analysis of high contrast problems. Some exceptions are found [9], [6],[23], [7], [15]. In particular in [9], energy error estimates

independent of the contrast of the coefficients (i.e. ρ^+/ρ^-) were proved. However, the estimates were not completely independent of the coefficients, a factor of $1/\sqrt{\rho^-}$ was present in the right-hand side. More recently, in [15] an interface finite element method was designed and certain error estimates independent of coefficients contrast were proved on problems with smooth interfaces. Specifically, the error estimate achieved in [15] for the energy error was of the form

$$(1.3) \quad \|\sqrt{\rho}\nabla(u - u_h)\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{\rho^-}} h \|f\|_{L^2(\Omega)},$$

where we also observe a factor of $1/\sqrt{\rho^-}$ in the estimate. One of the key ingredients in [15] was to add a stabilization term that penalized the jump of the gradients across edges of the triangulation. This idea was borrowed from the stabilized Nitsche's methods developed by Burman and co-authors; see for example [7]. Here in this paper we analyze a variant of the method introduced in [7] and prove the following error estimate totally independent of contrast

$$(1.4) \quad \|\rho\nabla(u - u_h)\|_{L^2(\Omega)} \leq C h \|f\|_{L^2(\Omega)},$$

where constant C is independent of ρ^\pm . It is important to note that this estimate is for the flux error $\rho\nabla(u - u_h)$. The previous analysis in [15, 9, 7] used energy arguments to establish error estimates for the energy error $\sqrt{\rho}\nabla(u - u_h)$, resulting in the dependence of $1/\rho^-$. Notice that a simple application of estimate (1.3) will give

$$\|\rho^+\nabla(u - u_h)^+\|_{L^2(\Omega^+)} \leq \frac{\sqrt{\rho^+}}{\sqrt{\rho^-}} h \|f\|_{L^2(\Omega)},$$

Hence, we see that our result (1.4) here is much sharper for this quantity. The main ingredient of the analysis is the use of a discrete extension result from Ω^+ to all of Ω . We note that this technique can be extended to conforming finite element discretizations and it opens the possibility to establish sharper results also for other discretizations as well (in particular the method studied in [15]).

The paper is organized as follows. In the next section we describe the Nitsche's finite element method. In Section 3 we provide an error estimate based on an energy argument. In Section 4 we improve the main result, obtaining an error estimate independent of the contrast for the diffusion coefficients for the flux. In the following section we discuss extension of the method and results for: interface problem with non homogeneous jumps, and the three dimensional problem. In Section 6 we present numerical results that validate the theoretical results. We conclude with an appendix that contains proofs of some crucial lemmas.

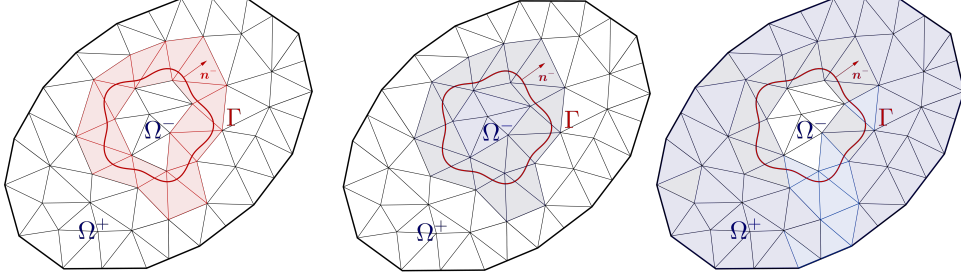


Figure 1. Illustration of the definition of set \mathcal{T}_h^Γ and discrete sub-domains Ω_h^\pm . Left figure: elements in \mathcal{T}_h^Γ (red transparent). Center figure: sub-domain Ω_h^- (blue transparent). Right figure: sub-domain Ω_h^+ (blue transparent).

2. FINITE ELEMENT METHOD

2.1. Preliminaries. Let $\{\mathcal{T}_h\}_{h>0}$ be an admissible family of triangulations of Ω . We adopt the convention that elements T and element edges e are open sets. We use over-line symbol to refer to their closure. For each triangular element $T \in \mathcal{T}_h$, let h_T denotes its diameter and define the global parameter of the triangulation by $h = \max_T h_T$. We assume that \mathcal{T}_h is shape regular, i.e. there exists $\kappa > 0$ such that for every $T \in \mathcal{T}_h$ the radius ρ_T of its inscribed circle satisfies

$$(2.1) \quad \rho_T > h_T/\kappa.$$

The set of elements cutting the interface Γ , and restricted to Ω^+ and Ω^- are also of interest. They are defined by:

$$\begin{aligned} \mathcal{T}_h^\pm &:= \{T \in \mathcal{T}_h : T \cap \Omega^\pm \neq \emptyset\}, \\ \mathcal{T}_h^\Gamma &:= \{T \in \mathcal{T}_h^\pm : \bar{T} \cap \Gamma \neq \emptyset\}. \end{aligned}$$

In particular for $T \in \mathcal{T}_h^\Gamma$ we denote $T_\Gamma = \bar{T} \cap \Gamma$. Observe that the definition of \mathcal{T}_h^Γ guarantees that $\sum_{T \in \mathcal{T}_h^\Gamma} |T_\Gamma| = |\Gamma|$. Under these definitions we define the discrete domains

$$\Omega_h^\pm := \text{Int} \left(\bigcup_{T \in \mathcal{T}_h^\pm} \bar{T} \right).$$

See Figure 1 for an illustration of these definitions. The set of all the edges of \mathcal{T}_h^Γ restricted to the interior of Ω_h^+ and Ω_h^- is also considered

$$\mathcal{E}_h^{\Gamma, \pm} := \{e = \text{Int}(\partial T_1 \cap \partial T_2) : T_1, T_2 \in \mathcal{T}_h^\pm, \text{ and } T_1 \cap \Gamma \neq \emptyset \text{ or } T_2 \cap \Gamma \neq \emptyset\}.$$

Standard finite element notation for the normal jumps is introduced as follows: for a piecewise smooth function v with support in \mathcal{T}_h , the jump of its gradient across an interior edge $e = \text{Int}(\partial T_1 \cap \partial T_2)$ is defined by

$$[[\nabla v]] = \nabla v|_{T_1} \cdot \mathbf{n}_1 + \nabla v|_{T_2} \cdot \mathbf{n}_2,$$

where \mathbf{n}_1 and \mathbf{n}_2 are the unit normal vectors to e , pointing outwards to T_1 and T_2 , respectively.

2.2. Stabilized unfitted Nitsche method. In this section we introduce a slightly simplified version of the Nitsche method for high-contrast interface problems by [7], Section 3.3. We begin by denoting the standard finite element space of continuous piecewise linear polynomials with support in Ω_h^\pm by:

$$V_h^\pm = \{v \in \mathcal{C}(\Omega_h^\pm) : v|_T \in \mathbb{P}^1(T), \forall T \in \mathcal{T}_h^\pm, \text{ and } v|_{\partial\Omega^\pm \setminus \Gamma} \equiv 0\}.$$

The finite element space is defined by means of the restrictions of piecewise linear functions to Ω_h^+ and Ω_h^- , i.e.

$$V_h = V_h^- \times V_h^+.$$

The jumps across the interface of a function $v \in V_h$ are defined as in (1.2).

We now consider a finite element method based on: the weak formulation of problem (1.1), penalty terms of the jump across the interface, and stabilization terms on edges in $\mathcal{E}_h^{\Gamma, \pm}$. Find $u_h = (u_h^-, u_h^+) \in V_h$, such that:

$$(2.2) \quad a_h(u_h, v) = (f^+, v^+)_{\Omega^+} + (f^-, v^-)_{\Omega^-}, \quad \text{for all } v \in V_h,$$

where $(\cdot, \cdot)_{\Omega^\pm}$ denotes the L^2 product in Ω^\pm and $a_h(\cdot, \cdot)$ is a bilinear form defined by

$$(2.3) \quad \begin{aligned} a_h(u_h, v) &= \int_{\Omega^+} \rho^+ \nabla u_h^+ \cdot \nabla v^+ dx + \int_{\Omega^-} \rho^- \nabla u_h^- \cdot \nabla v^- dx \\ &+ \int_{\Gamma} (\rho^- \nabla v^- \cdot \mathbf{n}^- [u_h] + \rho^- \nabla u_h^- \cdot \mathbf{n}^- [v]) ds + \sum_{T \in \mathcal{T}_h^\Gamma} \frac{\gamma}{h_T} \rho^- \int_{T_\Gamma} [u_h][v] ds \\ &+ \gamma_g^- \sum_{e \in \mathcal{E}_h^{\Gamma, -}} |e| \int_e \rho^- \llbracket \nabla v^- \rrbracket \llbracket \nabla u_h^- \rrbracket ds + \gamma_g^+ \sum_{e \in \mathcal{E}_h^{\Gamma, +}} |e| \int_e \rho^+ \llbracket \nabla v^+ \rrbracket \llbracket \nabla u_h^+ \rrbracket ds, \end{aligned}$$

where γ, γ_g^- , and γ_g^+ are positive parameters to be chosen, and $|e|$ denotes the diameter of e , i.e. the size of the edge in two dimensions. Note that although we consider the case where $\Gamma \cap \partial\Omega = \emptyset$, the method is well defined when the interface crosses the boundary of the domain. However, we only analyze the embedded case. If the interface crosses the boundary of the domain the solution will not necessarily be in $H^2(\Omega^+ \cup \Omega^-)$. In addition, a technical tool that we utilized in the proof of our main result is the existence of a stable extension (see Lemma 4.1). It is not clear that this extension will exist in some cases where the interface crosses the boundary of the domain.

Remark 1. We point out that the terms in (2.3) involving integration on Γ can be generalized to

$$\int_{\Gamma} (\{\rho \nabla v\}_w [u_h] + \{\rho \nabla u_h\}_w [v]) ds + \sum_{T \in \mathcal{T}_h^\Gamma} \frac{\gamma}{h_T} \tilde{\rho} \int_{T_\Gamma} [u_h][v] ds.$$

The weighted average

$$\{\rho \nabla v \cdot \mathbf{n}^-\}_w := (w_- \rho^- \nabla v_- + w_+ \rho^+ \nabla v_+) \cdot \mathbf{n}^-,$$

where the weights $w_-(x) \in [0, 1]$ and $w_+(x) = 1 - w_-(x)$ and $\tilde{\rho}$ are chosen properly, see [8]. The case $w_-(x) = 1$ and $\tilde{\rho}(x) = \rho^-(x)$ reduces to the one in (2.3). Another choice considered in the literature (see [11, 6, 12]) is the harmonic average given by $w_- = \frac{\rho^+}{\rho^+ + \rho^-}$ and $\tilde{\rho} = \frac{2\rho^+\rho^-}{\rho^+ + \rho^-}$. In this case we obtain

$$\{\rho \nabla v \cdot \mathbf{n}^-\}_w = \frac{\tilde{\rho}}{2} (\nabla v_- + \nabla v_+) \cdot \mathbf{n}^-.$$

In this paper we concentrate in the analysis of the choice (2.3), however, since $\rho^- \leq \tilde{\rho} \leq 2\rho^-$, the analysis for the harmonic average case follows straightforwardly.

Let us define the broken Sobolev spaces

$$H_h^2(\Omega_h^\pm) = \{v \in H^1(\Omega_h^\pm) : v|_{T^\pm} \in H^2(T^\pm), \text{ for all } T \in \mathcal{T}_h^\pm\}.$$

The energy norm $\|\cdot\|_V$, induced by the bilinear form a_h , is defined for $v = (v^-, v^+) \in H_h^2(\Omega_h^-) \times H_h^2(\Omega_h^+)$ by

$$\begin{aligned} \|v\|_V^2 &= \|\sqrt{\rho} \nabla v\|_{L^2(\Omega)}^2 + \sum_{T \in \mathcal{T}_h^\Gamma} \frac{1}{h_T} \|\sqrt{\rho^-}[v]\|_{L^2(T_\Gamma)}^2 \\ &\quad + \sum_{e \in \mathcal{E}_h^{\Gamma,-}} |e| \|\sqrt{\rho^-}[\nabla v^-]\|_{L^2(e)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \|\sqrt{\rho^+}[\nabla v^+]\|_{L^2(e)}^2. \end{aligned}$$

Note that in this definition we use the following notation:

$$\|\sqrt{\rho} \nabla v\|_{L^2(\Omega)}^2 = \|\sqrt{\rho^-} \nabla v^-\|_{L^2(\Omega^-)}^2 + \|\sqrt{\rho^+} \nabla v^+\|_{L^2(\Omega^+)}^2.$$

3. STANDARD A PRIORI ERROR ANALYSIS

3.1. Stability and best approximation results. We will need the following technical proposition for the proof of coercivity; proof can be found in Appendix A.

Proposition 1. *Consider a node z of the triangulation \mathcal{T}_h such that $z \in \overline{\Omega^-}$. Let Δ_z be the patch of elements associated to z , i.e. $\Delta_z = \text{Int}(\cup\{\overline{T} : T \in \mathcal{T}_h \text{ and } z \in \partial T\})$. Then for h small enough, there exists an element $T_z \in \Delta_z$ such that:*

$$(3.1) \quad |T_z \cap \Omega^-| \geq Ch_{T_z}^2,$$

where $C > 0$ is a constant independent of h_{T_z} .

Coercivity of the bilinear form a_h is proved below.

Lemma 1. *There exists a constant $c > 0$ such that*

$$(3.2) \quad c\|v\|_V^2 \leq a_h(v, v), \quad \text{for all } v \in V_h.$$

Proof. Let $v \in V_h$. Observe that the bilinear form a_h is symmetric, then it follows

$$\begin{aligned} a_h(v, v) = & \|\sqrt{\rho^-} \nabla v\|_{L^2(\Omega)}^2 + 2 \int_{\Gamma} \rho^- \nabla v^- \cdot \mathbf{n}^- [v] ds + \sum_{T \in \mathcal{T}_h^\Gamma} \frac{\gamma}{h_T} \|\sqrt{\rho^-} [v]\|_{L^2(T_\Gamma)}^2 \\ & + \gamma_g^- \sum_{e \in \mathcal{E}_h^{\Gamma, -}} |e| \rho^- \|[\![\nabla v^-]\!] \|_{L^2(e^-)}^2 + \gamma_g^+ \sum_{e \in \mathcal{E}_h^{\Gamma, +}} |e| \rho^+ \|[\![\nabla v^+]\!] \|_{L^2(e)}^2. \end{aligned}$$

In order to prove (3.2) it is enough to bound the non positive term (second term). Let $T \in \mathcal{T}_h^\Gamma$. Applying Cauchy-Schwarz inequality we obtain

$$\left| \int_{T_\Gamma} \rho^- \nabla v^- \cdot \mathbf{n}^- [v] ds \right| \leq \left(\sqrt{\rho^- h_T} \|\nabla v^- \cdot \mathbf{n}^- \|_{L^2(T_\Gamma)} \right) \left(\sqrt{\frac{\rho^-}{h_T}} \| [v] \|_{L^2(T_\Gamma)} \right).$$

Summing over $T \in \mathcal{T}_h^\Gamma$ and applying arithmetic-geometric inequality give

$$\left| \int_{\Gamma} \rho^- \nabla v^- \cdot \mathbf{n}^- [v] ds \right| \leq \sum_{T \in \mathcal{T}_h^\Gamma} \left(\varepsilon \rho^- h_T \|\nabla v^- \cdot \mathbf{n}^- \|_{L^2(T_\Gamma)}^2 + \frac{\rho^-}{\varepsilon h_T} \| [v] \|_{L^2(T_\Gamma)}^2 \right).$$

Let $T \in \mathcal{T}_h^\Gamma$ and let $z \in \overline{\Omega}^-$ be a node of T . By Proposition 1, there exists a triangle T_z satisfying (3.1). Now, consider the shortest sequence of edges $E(T) = \{e_1, e_2, \dots, e_N\}$ such that

$$\begin{cases} z \in \overline{e_j}, & j = 1, \dots, N, \\ e_1 \subset \partial T \text{ and } e_N \subset \partial T_z, \\ e_j, e_{j+1} \subset \partial T_j, T_j \in \mathcal{T}_h, & j = 1, \dots, N-1. \end{cases}$$

Note that by its definition $E(T) \subset \mathcal{E}_h^{\Gamma, -}$. Then, observing that the tangential jump of ∇v^- is zero along edges, we have

$$\begin{aligned} h_T \rho^- \|\nabla v^- \cdot \mathbf{n}^- \|_{L^2(T_\Gamma)}^2 & \leq h_T \rho^- \frac{|T_\Gamma|}{|e_1|} \|\nabla v^- \|_{L^2(e_1)}^2 \leq \kappa^2 \rho^- |e_1| \|\nabla v^- \|_{L^2(e_1)}^2 \\ & \leq \kappa^2 \rho^- |e_1| \left(\|[\![\nabla v^-]\!] \|_{L^2(e_1)}^2 + \|\nabla v^- |_{T_1} \|_{L^2(e_1)}^2 \right) \\ & \leq \kappa^2 \rho^- |e_1| \|[\![\nabla v^-]\!] \|_{L^2(e_1)}^2 + \kappa^4 \rho^- |e_2| \|\nabla v^- |_{T_1} \|_{L^2(e_2)}^2 \\ & \quad \vdots \\ & \leq c(\kappa) \rho^- \sum_{e \in E(T)} |e| \|[\![\nabla v^-]\!] \|_{L^2(e)}^2 + C(\kappa) \rho^- |e_N| \|\nabla v^- |_{T_z} \|_{L^2(e_N)}^2 \\ & \leq c(\kappa) \rho^- \sum_{e \in E(T)} |e| \|[\![\nabla v^-]\!] \|_{L^2(e)}^2 + \tilde{C}(\kappa) \|\sqrt{\rho^-} \nabla v^- \|_{L^2(T_z)}^2, \end{aligned}$$

where κ is the shape regularity constant defined in (2.1). Observe that by property (3.1) in Proposition 1 and since the test function is piecewise linear, we can estimate the last term above as

$$\|\sqrt{\rho^-} \nabla v^- \|_{L^2(T_z)}^2 \leq C \|\sqrt{\rho^-} \nabla v^- \|_{L^2(T_z \cap \Omega^-)}^2.$$

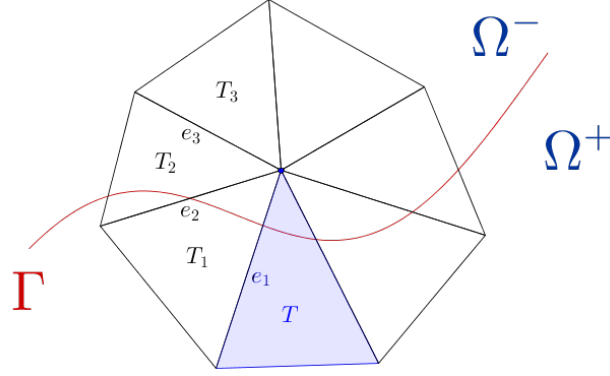


Figure 2. Illustration of set $E(T)$.

Hence, considering that each nodal patch contains finite number of elements implies that, after summing over $T \in \mathcal{T}_h^\Gamma$, the terms above are repeated at most finitely many times. Then, it follows

$$\sum_{T \in \mathcal{T}_h^\Gamma} \rho^- h_T \|\nabla v^- \cdot \mathbf{n}^-\|_{L^2(T_\Gamma)}^2 \leq \tilde{C}(\kappa) \left(\|\sqrt{\rho^-} \nabla v\|_{L^2(\Omega^-)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,-}} |e| \rho^- \|\llbracket \nabla v^- \rrbracket\|_{L^2(e)}^2 \right).$$

Therefore, coercivity follows by choosing γ large enough in terms of $C(\kappa)$ and ε . \square

With the aim of proving continuity of the bilinear form a_h , we define the following augmented norm:

$$(3.3) \quad \|v\|_{V_A}^2 = \|v\|_V^2 + \sum_{T \in \mathcal{T}_h^\Gamma} h_T \|\sqrt{\rho^-} \nabla v^- \cdot \mathbf{n}^-\|_{L^2(T_\Gamma)}^2.$$

Continuity of the bilinear form follows from its definition and Cauchy-Schwarz inequality. The result is stated as follows:

Lemma 2. (Continuity) *Suppose that $v^\pm, w^\pm \in H_h^2(\Omega^\pm)$. Then, there exists a constant $C > 0$, independent of v and w , such that*

$$a_h(w, v) \leq C \|w\|_{V_A} \|v\|_{V_A}.$$

Additionally, if $w^\pm \in H_h^2(\Omega^\pm)$ and $v \in V_h$ we have

$$(3.4) \quad a_h(w, v) \leq C \|w\|_{V_A} \|v\|_V.$$

In order to discuss the Galerkin orthogonality of method (2.2)-(2.3) we need to define extension operators. Consider the following well known extension result (see [13]).

Lemma 3. *Assume that $u^\pm := u|_{\Omega^\pm} \in H^2(\Omega^\pm)$. Then, there exist extensions $u_E^\pm \in H^2(\Omega)$, such that $u_E^\pm|_{\Omega^\pm} = u^\pm$ and*

$$\|u_E^\pm\|_{H^i(\Omega)} \leq C \|u^\pm\|_{H^i(\Omega^\pm)}, \quad \text{for } i = 0, 1, 2,$$

for a constant $C > 0$ depending only on Ω^\pm .

A particular case is proved in Appendix B by means of an even extension. From now on we simply denote u_E^\pm by u^\pm .

Lemma 4. (*Galerkin orthogonality*) *Suppose that u solves the (1.1) and suppose that $u|_{\Omega^\pm} \in H^2(\Omega^\pm)$. Then, we have that*

$$a_h(u, v) = (f, v), \quad \forall v \in V_h,$$

where we use the notation $u = (u^+, u^-)$. Hence, Galerkin orthogonality holds:

$$a_h(u - u_h, v) = 0, \quad \forall v \in V_h.$$

The main result of this section, the best approximation result, is stated below. It follows easily from coercivity, continuity and Galerkin orthogonality.

Theorem 1. (*Best approximation*) *Let $\Omega \subseteq \mathbb{R}^2$ be an open polygonal domain. Let u be a solution of problem (1.1) and assume that $u|_{\Omega^\pm} \in H^2(\Omega^\pm)$. Let u_h be solution of the discrete problem (2.2). Then, there exists a constant $C > 0$ independent of h , such that*

$$\|u - u_h\|_V \leq C \inf_{v \in V_h} \|u - v\|_{V_A}.$$

3.2. Energy error estimates. In order to prove an error estimate in terms of the V -norm we introduce the following interpolation operator: Define $I_h : H^1(\Omega_h^+) \times H^1(\Omega_h^-) \rightarrow V_h$ such that

$$(I_h u)^\pm = J_h u^\pm, \quad \text{for } u = (u^+, u^-) \in H^1(\Omega_h^+) \times H^1(\Omega_h^-),$$

where J_h is the interpolant onto the standard continuous piecewise linear polynomials introduced in [20]. Consequently, the following estimate for the interpolation error follows from the properties of the Scott-Zhang interpolation operator and the extension result: Lemma 3.

Lemma 5. *Consider the definition of the interpolation operator I_h given above. Then, there exists a constant $C > 0$, independent of h , such that:*

$$(3.5) \quad \|u - I_h u\|_{V_A} \leq Ch(\sqrt{\rho^+} \|D^2 u\|_{L^2(\Omega^+)} + \sqrt{\rho^-} \|D^2 u\|_{L^2(\Omega^-)}).$$

In addition, we assume that the following elliptic regularity bound holds,

$$(3.6) \quad \rho^+ \|D^2 u\|_{L^2(\Omega^+)} + \rho^- \|D^2 u\|_{L^2(\Omega^-)} \leq C \|f\|_{L^2(\Omega)}.$$

For instance, this bound is satisfied in the two-dimensional case (see [9]) for Ω convex and polyhedral.

An energy error estimate follows from Theorem 1, Lemma 5 and elliptic regularity (3.6). We state it in the corollary below.

Corollary 1. (Standard energy error estimate) Let u be a solution of problem (1.1), and let u_h be the solution of the discrete problem (2.2). Suppose that Ω is convex. Then, there exists $C > 0$ independent of ρ^\pm , such that

$$\|u - u_h\|_V \leq \frac{Ch}{\sqrt{\rho^-}} \|f\|_{L^2(\Omega)}.$$

4. ERROR ESTIMATE FOR THE FLUX $\rho\nabla(u - u_h)$

In order to prove the main result of this paper, the error estimate for the flux, we need a discrete extension result.

Lemma 6. (*Discrete extension*) Assume that the triangulation \mathcal{T}_h is quasi-uniform. Let $v_h \in V_h^+$. Then, there exists a function $Ev_h \in V_h^c = \{v \in C_0(\Omega) : v|_T \in \mathbb{P}^1(T), \forall T \in \mathcal{T}_h\}$, such that $Ev_h = v_h$ in Ω_h^+ and

$$\|Ev_h\|_{H^1(\Omega)} \leq C \|v_h\|_{H^1(\Omega_h^+)},$$

with $C > 0$ independent of h .

Proof. See Appendix C. □

Considering the discrete extension lemma, we state a bound for the H^1 -norm in Ω_h^+ . The proof follows easily using Proposition 1 (with roles of Ω^+ and Ω^- reversed), which allows us to control the terms in Ω_h^+ by terms in Ω^+ . Similar statements have appeared before in [5] and [19].

Lemma 7. Let $v \in V_h^+$. Then, we have

$$\|v\|_{H^1(\Omega_h^+)}^2 \leq C \left(\|v\|_{H^1(\Omega^+)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \|[\![\nabla v]\!] \|_{L^2(e)}^2 \right).$$

Proof. We first bound the H^1 semi-norm. By the argument given in proof of Lemma 1 which uses Proposition 1 it follows that

$$\|\nabla v\|_{L^2(\Omega_h^+)}^2 \leq C \left(\|\nabla v\|_{L^2(\Omega^+)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \|[\![\nabla v]\!] \|_{L^2(e)}^2 \right)$$

where C depends on the shape regularity constant κ . Now, in order to bound the L^2 norm let T be an element Ω_h^+ not totally contained in Ω^+ . Let z be a node of T such that $z \in \Omega^+$. Then, by Proposition 1, there exists an element T_z in the patch of z such that :

$$|T_z \cap \Omega^+| \geq Ch_{T_z}^2.$$

Then, using the fact that v is a linear function on T we have that

$$|v(x)| \leq |v(z)| + h_T |\nabla v|_T, \quad \text{for all } x \in T,$$

and then we have

$$\|v\|_{L^2(T)} \leq Ch_T |v(z)| + h_T \|\nabla v\|_{L^2(T)}.$$

Using an inverse estimate we get

$$|v(z)| \leq Ch_{T_z}^{-1} \|v\|_{L^2(T_z)}.$$

It is not difficult to see, using the fact that $|T_z \cap \Omega^+| \geq Ch_{T_z}^2$ and that v is linear on T_z that

$$\|v\|_{L^2(T_z)} \leq C \|v\|_{L^2(T_z \cap \Omega^+)}.$$

Therefore, we obtain

$$\|v\|_{L^2(T)} \leq C \|v\|_{L^2(T_z \cap \Omega^+)} + h_T \|\nabla v\|_{L^2(T)}.$$

Using this inequality repeatedly we obtain that

$$\|v\|_{L^2(\Omega_h^+)} \leq C \left(\|v\|_{L^2(\Omega^+)} + h \|\nabla v\|_{L^2(\Omega_h^+)} \right),$$

which proves the result after applying the estimate derived for the H^1 semi-norm. \square

Now we are in position to state and prove the main result of this paper. Observe that if we have a conforming piecewise linear discretization with an interface Γ aligning the mesh \mathcal{T}_h , then the proof would be short. Precisely, the fact that the mesh does not align the interface creates some extra technicalities on the proof.

Theorem 2. (Main result) *Let u be a solution of problem (1.1) and let u_h be solution of the discrete problem (2.2). Assume the triangulation is quasi-uniform and that $(\rho^+ \|D^2 u\|_{L^2(\Omega^+)} + \rho^- \|D^2 u\|_{L^2(\Omega^-)})$ is bounded. Then, there exists a constant $C > 0$, independent of h and ρ^\pm , such that*

$$\|\rho \nabla(u - u_h)\|_{L^2(\Omega)} \leq Ch(\rho^+ \|D^2 u\|_{L^2(\Omega^+)} + \rho^- \|D^2 u\|_{L^2(\Omega^-)}).$$

We remind the reader that

$$\|\rho \nabla(u - u_h)\|_{L^2(\Omega)}^2 = \|\rho^- \nabla(u - u_h)^-\|_{L^2(\Omega^-)}^2 + \|\rho^+ \nabla(u - u_h)^+\|_{L^2(\Omega^+)}^2.$$

Proof. Observe that the bound in Ω^- is given by Theorem 1 and (3.5)

$$\|\rho^- \nabla(u - u_h)^-\|_{L^2(\Omega^-)} \leq Ch(\rho^+ \|D^2 u\|_{L^2(\Omega^+)} + \rho^- \|D^2 u\|_{L^2(\Omega^-)}).$$

Thus, it remains to prove

$$(4.1) \quad \|\rho^+ \nabla(u - u_h)^+\|_{L^2(\Omega^+)} \leq Ch(\rho^+ \|D^2 u\|_{L^2(\Omega^+)} + \rho^- \|D^2 u\|_{L^2(\Omega^-)}).$$

Consider the function $v_h = (I_h u - u_h)^+ \in V_h^+$. If Ω^+ is the inclusion, i.e., $\partial\Omega^+ \cap \partial\Omega = \emptyset$, we redefine v_h so that it has average zero on Ω^+ , i.e., $v_h = v_h - \frac{1}{|\Omega^+|} \int_{\Omega^+} v_h$. Either case Poincaré's inequality holds:

$$\|v_h\|_{H^1(\Omega^+)} \leq C \|\nabla v_h\|_{L^2(\Omega^+)} = C \|\nabla(I_h u - u_h)^+\|_{L^2(\Omega^+)},$$

for a constant $C > 0$. Applying Lemma 6 it follows that there exists an extension $Ev_h \in V_h^c$ such that $Ev_h = v_h$ in Ω_h^+ , and $\|Ev_h\|_{H^1(\Omega)} \leq C\|v_h\|_{H^1(\Omega_h^+)}$. Combining this estimate with Lemma 7, we have

$$(4.2) \quad \|Ev_h\|_{H^1(\Omega)} \leq C \left(\|\nabla(I_h u - u_h)^+\|_{L^2(\Omega^+)} + \left(\sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \|\llbracket \nabla(I_h u - u_h)^+ \rrbracket\|_{L^2(e)}^2 \right)^{1/2} \right).$$

Consequently, using $((Ev_h)^-, (Ev_h)^+) \in V_h$ as test function (where $(Ev_h)^\pm := Ev_h|_{\Omega_h^\pm}$), and the consistency of the method (2.2)-(2.3), we obtain

$$\begin{aligned} 0 &= a_h(u - u_h, Ev_h) \\ &= \int_{\Omega^+} \rho^+ \nabla(u - u_h)^+ \cdot \nabla(Ev_h)^+ + \int_{\Omega^-} \rho^- \nabla(u - u_h)^- \cdot \nabla(Ev_h)^- \\ &\quad + \int_{\Gamma} \rho^- \nabla Ev_h^- \cdot \mathbf{n}^- [u - u_h] ds \\ &\quad + \sum_{e \in \mathcal{E}_h^{\Gamma,-}} |e| \int_e \rho^- \llbracket \nabla(Ev_h)^- \rrbracket \llbracket \nabla(u - u_h)^- \rrbracket ds \\ &\quad + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \int_e \rho^+ \llbracket \nabla(Ev_h)^+ \rrbracket \llbracket \nabla(u - u_h)^+ \rrbracket ds. \end{aligned}$$

Here we used that Ev_h is continuous across the interface Γ . Multiplying by ρ^+ and adding and subtracting $I_h u^+$ in the first and last term give

$$\begin{aligned} &\|\rho^+ \nabla(I_h u - u_h)^+\|_{L^2(\Omega^+)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \|\rho^+ \llbracket \nabla(I_h u - u_h)^+ \rrbracket\|_{L^2(e)}^2 = \\ &\quad - \rho^+ \int_{\Omega^+} \rho^+ \nabla(u - I_h u)^+ \cdot \nabla(Ev_h)^+ - \rho^+ \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \int_e \rho^+ \llbracket \nabla(Ev_h)^+ \rrbracket \llbracket \nabla(u - I_h u)^+ \rrbracket ds \\ &\quad - \rho^+ \int_{\Omega^-} \rho^- \nabla(u - u_h)^- \cdot \nabla(Ev_h)^- - \rho^+ \int_{\Gamma} \rho^- \nabla(Ev_h)^- \cdot \mathbf{n}^- [u - u_h] ds \\ &\quad - \rho^+ \sum_{e \in \mathcal{E}_h^{\Gamma,-}} |e| \int_e \rho^- \llbracket \nabla(Ev_h)^- \rrbracket \llbracket \nabla(u - u_h)^- \rrbracket ds =: I_1 + I_2, \end{aligned}$$

where I_1 denotes the first two terms and I_2 the last three terms. First we bound I_1 . Applying Cauchy-Schwarz and arithmetic geometric inequalities,

and using that $\nabla(Ev_h)^+ = \nabla(I_h u - u_h)^+$ in Ω_h^+ yield

$$\begin{aligned} 2|I_1| &\leq \|\rho^+ \nabla(I_h u - u)^+\|_{L^2(\Omega^+)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \|\rho^+ \llbracket \nabla(I_h u - u)^+ \rrbracket\|_{L^2(e)}^2 \\ &\quad + \|\rho^+ \nabla(I_h u - u_h)^+\|_{L^2(\Omega^+)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \|\rho^+ \llbracket \nabla(I_h u - u_h)^+ \rrbracket\|_{L^2(e)}^2 \\ &\leq \|\rho^+ \nabla(I_h u - u)^+\|_{L^2(\Omega^+)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \|\rho^+ \llbracket \nabla(I_h u - u)^+ \rrbracket\|_{L^2(e)}^2 + |I_1| + |I_2|, \end{aligned}$$

which implies that

$$(4.3) \quad |I_1| \leq \|\rho^+ \nabla(I_h u - u)^+\|_{L^2(\Omega^+)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \|\rho^+ \llbracket \nabla(I_h u - u)^+ \rrbracket\|_{L^2(e)}^2 + |I_2|.$$

Similarly, we apply Cauchy-Schwarz inequality for the three terms in I_2 after taking as common factor $\rho^+ \sqrt{\rho^-}$, i.e.

$$\begin{aligned} \left| \sqrt{\rho^-} \int_{\Omega^-} \nabla(u - u_h)^- \cdot \nabla(Ev_h)^- dx \right| &\leq \|\sqrt{\rho^-} \nabla(u - u_h)^-\|_{L^2(\Omega^-)} \|\nabla(Ev_h)^-\|_{L^2(\Omega^-)}, \\ \left| \sqrt{\rho^-} \int_{\Gamma} \nabla(Ev_h)^- \cdot \mathbf{n}^- [u - u_h] ds \right| &\leq \sum_{T \in \mathcal{T}_h^\Gamma} \frac{1}{\sqrt{h_T}} \|\sqrt{\rho^-} [u - u_h]\|_{L^2(T_\Gamma)} \sqrt{h_T} \|\nabla(Ev_h)^- \cdot \mathbf{n}^-\|_{L^2(T_\Gamma)}, \\ \left| \sqrt{\rho^-} \sum_{e \in \mathcal{E}_h^{\Gamma,-}} |e| \int_e \llbracket \nabla(Ev_h)^- \rrbracket \llbracket \nabla(u - u_h)^- \rrbracket ds \right| &\leq \sum_{e \in \mathcal{E}_h^{\Gamma,-}} |e| \|\sqrt{\rho^-} \llbracket \nabla(u - u_h)^- \rrbracket\|_{L^2(e)} \|\llbracket \nabla(Ev_h)^- \rrbracket\|_{L^2(e)}. \end{aligned}$$

Observe that terms involving $u - u_h$ are all bounded by $\|u - u_h\|_V$. Hence,

$$\begin{aligned} |I_2| &\leq C \rho^+ \sqrt{\rho^-} \|u - u_h\|_V \left(\|\nabla(Ev_h)^-\|_{L^2(\Omega^-)} \right. \\ &\quad \left. + \left(\sum_{T \in \mathcal{T}_h^\Gamma} h_T \|\nabla(Ev_h)^- \cdot \mathbf{n}^-\|_{L^2(T_\Gamma)}^2 \right)^{1/2} + \left(\sum_{e \in \mathcal{E}_h^{\Gamma,-}} |e| \|\llbracket \nabla(Ev_h)^- \rrbracket\|_{L^2(e)}^2 \right)^{1/2} \right). \end{aligned}$$

For the terms involving Ev_h we use the argument in the proof of Proposition 1 to bound term in T_Γ by terms in a region of size h_T^2 . In addition, applying inverse inequalities we obtain

$$\left(\sum_{T \in \mathcal{T}_h^\Gamma} h_T \|\nabla(Ev_h)^- \cdot \mathbf{n}^-\|_{L^2(T_\Gamma)}^2 \right)^{1/2} + \left(\sum_{e \in \mathcal{E}_h^{\Gamma,-}} |e| \|\llbracket \nabla(Ev_h)^- \rrbracket\|_{L^2(e)}^2 \right)^{1/2} \leq C \|\nabla Ev_h\|_{L^2(\Omega)}.$$

Thus, by estimate (4.3) and the estimate above for I_2 , it follows that

$$\begin{aligned} |I_1| + |I_2| &\leq \|\rho^+ \nabla(I_h u - u)^+\|_{L^2(\Omega^+)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \|\rho^+ \llbracket \nabla(I_h u - u)^+ \rrbracket\|_{L^2(e)}^2 \\ &\quad + C \sqrt{\rho^-} \|u - u_h\|_V \|\rho^+ \nabla Ev_h\|_{L^2(\Omega)}. \end{aligned}$$

By approximation properties of the Scott-Zhang interpolant we have

$$(4.4) \quad \begin{aligned} \|\rho^+ \nabla(I_h u - u)^+\|_{L^2(\Omega^+)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \|\rho^+ \llbracket \nabla(I_h u - u)^+ \rrbracket\|_{L^2(e)}^2 \\ \leq Ch^2(\rho^+)^2 \|D^2 u^+\|_{L^2(\Omega^+)}^2. \end{aligned}$$

Using the error estimate Theorem 1 and (3.5) and definitions of I_1 and I_2 we conclude

$$\begin{aligned} \|\rho^+ \nabla(I_h u - u_h)^+\|_{L^2(\Omega^+)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \|\rho^+ \llbracket \nabla(I_h u - u_h)^+ \rrbracket\|_{L^2(e)}^2 \\ \leq C(h \|\rho^+ \nabla E v_h\|_{L^2(\Omega)} (\rho^+ \|D^2 u\|_{L^2(\Omega^+)} + \rho^- \|D^2 u\|_{L^2(\Omega^-)}) \\ + h^2(\rho^+)^2 \|D^2 u^+\|_{L^2(\Omega^+)}^2). \end{aligned}$$

Therefore, applying (4.2) we have

$$\|\rho^+ \nabla(I_h u - u_h)\|_{L^2(\Omega^+)}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \|\rho^+ \llbracket \nabla(I_h u - u_h)^+ \rrbracket\|_{L^2(e)}^2 \leq Ch^2(\rho^+ \|D^2 u\|_{L^2(\Omega^+)} + \rho^- \|D^2 u\|_{L^2(\Omega^-)})^2.$$

Finally, applying triangle inequality, the estimate for the interpolation error (4.4), and last inequality we obtain the desired bound (4.1). \square

We can now use the elliptic regularity result to prove the following corollary.

Corollary 2. Assuming the hypothesis of Theorem 2 and in addition assuming that our domain is such that (3.6) holds, then

$$\|\rho \nabla(u - u_h)\|_{L^2(\Omega)} \leq Ch \|f\|_{L^2(\Omega)}.$$

We conclude this section by stating an L^2 -error estimate. The proof follows easily from a duality argument (see [15]). We omit the details.

Lemma 8. (L^2 error estimate) Assuming the hypothesis of previous corollary we have

$$\|(u - u_h)^-\|_{L^2(\Omega^-)} + \|(u - u_h)^+\|_{L^2(\Omega^+)} \leq \frac{C}{\rho^-} h^2 \|f\|_{L^2(\Omega)}.$$

5. EXTENSIONS OF THE METHOD

In this section we discuss and state some straightforward extensions of method (2.2)-(2.3).

5.1. Non homogeneous jump conditions. We consider problem (1.1) with a non homogeneous jump conditions in equations (1.1c) and (1.1d), i.e.

$$\begin{aligned}
(5.1a) \quad & -\nabla \cdot (\rho^\pm \nabla u^\pm) = f^\pm && \text{in } \Omega^\pm, \\
(5.1b) \quad & u^\pm = 0 && \text{on } \partial\Omega^\pm \setminus \Gamma, \\
(5.1c) \quad & [u] = \alpha && \text{on } \Gamma, \\
(5.1d) \quad & [\rho \nabla u \cdot \mathbf{n}] = \beta && \text{on } \Gamma,
\end{aligned}$$

where α, β are a smooth functions given on the interface. The method in this case follows from a standard derivation of the variational formulation:

$$\begin{aligned}
(5.2) \quad a_h(u_h, v) &= (f^+, v^+)_{\Omega^+} + (f^-, v^-)_{\Omega^-} + \int_{\Gamma} (\beta v^+ + \rho^- \nabla v^- \cdot \mathbf{n}^- \alpha) ds \\
&+ \sum_{T \in \mathcal{T}_h^\Gamma} \frac{\gamma}{h_T} \rho^- \int_{T_\Gamma} [v] \alpha ds,
\end{aligned}$$

for all $v \in V_h$.

All the proofs would generalize easily. In particular, the main result Theorem 2 holds. However, if one wants a result in terms of the data of the problem (e.g. f , β and α) then one would need a regularity result like the one in [9] for this more complicated problem which does not seem to appear in the literature.

5.2. Three dimensional problem. To extend the method in three dimensions is straightforward. In order to prove the same result one needs the regularity results from [9] to hold in three dimensions which we have not found in the literature. Moreover, we need to be able to prove the geometric result and extension results in three dimensions: Proposition 1 and Lemma 6. We believe that these results should hold.

6. NUMERICAL EXAMPLES

This section illustrates numerically the a priori error estimates proved in Section 3 and Section 4. We consider a two dimensional example with a non trivial immersed and closed interface. In particular, the example supports the optimal order of convergence for the error of the flux $\rho \nabla(u - u_h)$. We summarize our experimental results in tables, displaying the following errors and experimental orders of convergence (eoc):

$$\begin{aligned}
e_h^0 &:= \|u - u_h\|_{L^2(\Omega)}, & e_h^\infty &:= \|u - u_h\|_{L^\infty(\Omega)}, \\
e_{h,\rho}^1 &:= \|\rho(\nabla u - \nabla u_h)\|_{L^2(\Omega)}, & e_{h,\rho}^{1,\infty} &:= \|\rho(\nabla u - \nabla u_h)\|_{L^\infty(\Omega)},
\end{aligned}$$

$$\text{eoc}(e) := \frac{\log(e_{h_{l+1}}/e_{h_l})}{\log(h_{l+1}/h_l)}.$$

Our theoretical results predict optimal convergence of: error e_h^0 (second order), and error e_h^1 (first order). We also test the convergence of the errors e_h^∞ and $e_h^{1,\infty}$.

The finite element approximation by scheme (2.2)-(2.3) is computed with a sequence of uniform triangulations non matching the interface. The parameter of the triangulation is given by $h = 2^{-(l+3/2)}$, for $l = 1, \dots, 7$. Computations were performed in MATLAB including the solution of the linear system by means of command "\".

- (1) Consider problem (1.1) in a square domain $\Omega = (-1, 1)^2$ and an immersed interface $\Gamma = \{x \in \Omega : x_1^2 + x_2^2 = (1/3)^2\}$. We will test both cases: Ω^- as inclusion, and Ω^+ as inclusion. Consider the following exact solution:

$$(6.1) \quad u(x) = \begin{cases} \frac{r^\alpha}{\rho^-} & , \text{ if } x \in \Omega^-, \\ \frac{r^\alpha}{\rho^+} + (1/3)^\alpha \left(\frac{1}{\rho^-} - \frac{1}{\rho^+} \right) & , \text{ if } x \in \Omega^+, \end{cases}$$

where $r = \sqrt{x_1^2 + x_2^2}$ and $\alpha = 2$. We test with values of the diffusion coefficients $\rho^+ = 10^4$ and $\rho^- = 1$. Tables 1 and 2 summarize the results obtained by method (2.2)-(2.3) with stabilization parameters $\gamma = 10$ and $\gamma_g^\pm = 10$, with Ω^- and Ω^+ as inclusion, respectively.

l	e_h^0	eoc	e_h^∞	eoc	$e_{h,\rho}^1$	eoc	$e_{h,\rho}^{1,\infty}$	eoc
1	1.9e-02		5.5e-02		3.7e-01		5.9e-01	
2	6.2e-03	1.64	1.5e-02	1.86	1.4e-01	1.38	2.7e-01	1.11
3	1.1e-03	2.46	2.8e-03	2.45	5.6e-02	1.31	1.2e-01	1.17
4	1.7e-04	2.69	5.0e-04	2.48	2.6e-02	1.11	5.4e-02	1.17
5	2.8e-05	2.66	9.8e-05	2.34	1.3e-02	1.03	2.4e-02	1.14
6	4.6e-06	2.59	1.9e-05	2.39	6.4e-03	1.01	1.2e-02	1.03
7	8.4e-07	2.45	4.2e-06	2.16	3.2e-03	1.00	6.0e-03	1.01

Table 1. Example (1): errors and experimental orders of convergence (eoc) for problem (1.1) and solution (6.1) with $\Omega^- = \{x \in \Omega : x_1^2 + x_2^2 < (1/3)^2\}$ and $\Omega^+ = \Omega \setminus (\Omega^- \cup \Gamma)$.

Optimal convergence of the errors, second order for e_h^0 and e_h^∞ and first order for $e_{h,\rho}^1$ and $e_{h,\rho}^\infty$, for Example (1) is observed in Tables 1 and 2 for both cases Ω^- and Ω^+ as inclusion.

In addition, in order to test the independence of the coefficient ρ^\pm of our estimate in Theorem 2, we consider the exact solution (6.1) with a fix mesh corresponding to $h = 2^{-(5+3/2)}$. We compute errors e_h^0 , $e_{h,\rho}^1$ and $e_{h,\sqrt{\rho}}^1 = \|\sqrt{\rho}(u - u_h)\|_{L^2(\Omega)}$ for decreasing values of ρ^- and increasing values of ρ^+ . We summarize the results in Tables 3 and 4, corresponding to Ω^- as inclusion and Ω^+ as inclusion, respectively.

l	e_h^0	eoc	e_h^∞	eoc	$e_{h,\rho}^1$	eoc	$e_{h,\rho}^{1,\infty}$	eoc
1	3.7e-02		5.9e-02		3.6e-01		6.0e-01	
2	8.3e-03	2.15	1.3e-02	2.23	1.4e-01	1.41	2.7e-01	1.15
3	1.5e-03	2.45	2.9e-03	2.09	5.6e-02	1.29	1.2e-01	1.17
4	2.7e-04	2.50	5.8e-04	2.35	2.6e-02	1.09	5.3e-02	1.18
5	5.2e-05	2.35	9.8e-05	2.56	1.3e-02	1.02	2.4e-02	1.14
6	1.2e-05	2.16	2.1e-05	2.25	6.4e-03	1.01	1.2e-02	1.01
7	2.8e-06	2.06	4.2e-06	2.29	3.2e-03	1.00	5.8e-03	1.05

Table 2. Example (1): errors and experimental orders of convergence (eoc) for problem (1.1) and solution (6.1) with $\Omega^+ = \{x \in \Omega : x_1^2 + x_2^2 < (1/3)^2\}$ and $\Omega^- = \Omega \setminus (\Omega^+ \cup \Gamma)$.

ρ^-	ρ^+	e_h^0	$e_{h,\rho}^1$	$e_{h,\sqrt{\rho}}^1$
1e+00	1e+01	3.0e-05	1.3e-02	5.5e-03
1e-01	1e+02	2.8e-04	1.3e-02	1.3e-02
1e-02	1e+03	2.8e-03	1.3e-02	4.0e-02
1e-03	1e+04	2.8e-02	1.3e-02	1.3e-01
1e-04	1e+05	2.8e-01	1.3e-02	4.0e-01

Table 3. Example (1): errors for mesh parameter $h = 2^{-(5+3/2)}$. $\Omega^- = \{x \in \Omega : x_1^2 + x_2^2 < (1/3)^2\}$ and $\Omega^+ = \Omega \setminus (\Omega^- \cup \Gamma)$.

ρ^-	ρ^+	e_h^0	$e_{h,\rho}^1$	$e_{h,\sqrt{\rho}}^1$
1e+00	1e+01	5.4e-05	1.3e-02	1.2e-02
1e-01	1e+02	5.2e-04	1.3e-02	3.9e-02
1e-02	1e+03	5.2e-03	1.3e-02	1.2e-01
1e-03	1e+04	5.3e-02	1.3e-02	3.9e-01
1e-04	1e+05	3.9e-01	1.3e-02	1.2e+00

Table 4. Example (1): errors for mesh parameter $h = 2^{-(5+3/2)}$. $\Omega^+ = \{x \in \Omega : x_1^2 + x_2^2 < (1/3)^2\}$ and $\Omega^- = \Omega \setminus (\Omega^+ \cup \Gamma)$.

Tables 3 and 4 show that error $e_{h,\rho}^1$ is practically invariant, corroborating that in the main result of our paper Theorem 2, the estimate is totally independent of the diffusion coefficients ρ^\pm . Errors e_h^0 and $e_{h,\sqrt{\rho}}^1$ seems to be dependent of the coefficients as our estimates in Section 3 show.

- (2) Consider the two dimensional domain $\Omega = (-1, 1)^2$ with the immersed interface Γ defined by $\Gamma = \{x = (x_1, x_2) \in \Omega : \|x\|_2 = r = 1/18 + 0.2 \sin(5s), s \in [0, 2\pi)\}$. We define Ω^- as the interior domain,

i.e. $\partial\Omega^- = \Gamma$. We further consider the following exact solution

$$(6.2) \quad u(x) = \begin{cases} \frac{1}{\rho^-}(x_1^2 + x_2^2)^2 & , \text{ if } (x, y) \in \Omega^-, \\ \frac{1}{\rho^+}x_2\sqrt{x_1^2 + x_2^2} & , \text{ if } (x, y) \in \Omega^+, \end{cases}$$

We set the stabilization parameters to be: $\gamma = 10$, $\gamma_g^+ = 10$ and $\gamma_g^- = 10$. Note that in this case the jump of the solution and the jump of the flux are nonzero. Table 5 shows the errors and experimental orders of convergence obtained by method (5.2)-(2.3). As in the previous example, we test the independence of the coefficient ρ^\pm of our estimate in Theorem 2. We consider the exact solution (6.2) with a fix mesh corresponding to $h = 2^{-(5+3/2)}$. We compute errors e_h^0 , $e_{h,\rho}^1$ and $e_{h,\sqrt{\rho}}^1 = \|\sqrt{\rho}(u - u_h)\|_{L^2(\Omega)}$ for decreasing values of ρ^- and increasing values of ρ^+ . We summarize these results in Table 6.

We observe optimal convergence of the errors, second order for e_h^0 and e_h^∞ and first order for $e_{h,\rho}^1$ and $e_{h,\rho}^\infty$, for Example (2) in Table 5. Table 6 shows that error $e_{h,\rho}^1$ is practically invariant, supporting our claim that the estimate for the error of the flux is totally independent of the diffusion coefficients ρ^\pm for the case of non-homogeneous jumps.

l	e_h^0	eoc	e_h^∞	eoc	$e_{h,\rho}^1$	eoc	$e_{h,\rho}^{1,\infty}$	eoc
1	2.4e-2	—	7.1e-2	—	5.1e-1	—	1.0e+0	—
2	1.0e-2	1.23	2.6e-2	1.46	2.2e-1	1.18	7.8e-1	0.39
3	3.2e-3	1.68	1.4e-2	0.91	9.8e-2	1.19	5.1e-1	0.62
4	8.6e-4	1.91	4.4e-3	1.63	3.7e-2	1.41	2.9e-1	0.84
5	1.7e-4	2.35	1.0e-3	2.13	1.4e-2	1.43	1.4e-1	1.07
6	2.8e-5	2.60	1.9e-4	2.42	5.9e-3	1.23	6.2e-2	1.14

Table 5. Example (2): errors for mesh parameter $h = 2^{-(5+3/2)}$, using method (5.2)-(2.3).

APPENDIX A. PROOF PROPOSITION 1

Consider a node z of the triangulation \mathcal{T}_h and the patch of elements Δ_z (defined in Proposition A) associated to z . Since we are assuming that h is small enough and the interface is smooth we have that: the interface intersects each edge of triangulation at most once, or the interface coincides with an edge. Therefore, if $z \in \bar{\Omega}^-$, there exists at least one node $z' \in \bar{\Delta}_z$ with $z' \in \bar{\Omega}^-$, and the edge \bar{e} connecting z and z' is completely contained in $\bar{\Omega}^-$. If there exists more than one node satisfying this property then it would follow that an element of the patch of z is fully contained in Ω^- .

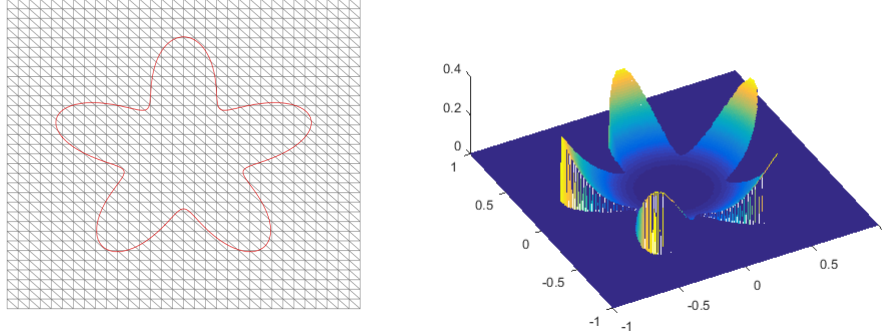


Figure 3. Example (2): Left figure: a non-fitted triangulation of the interface. Right figure: approximate solution by method (5.2)-(2.3).

ρ^-	ρ^+	e_h^0	$e_{h,\rho}^1$	$e_{h,\sqrt{\rho}}^1$
1e+00	1e+01	1.6e-04	1.4e-02	1.1e-02
1e-01	1e+02	1.7e-03	1.4e-02	3.4e-02
1e-02	1e+03	1.7e-02	1.4e-02	1.1e-01
1e-03	1e+04	1.7e-01	1.4e-02	3.4e-01
1e-04	1e+05	1.7e-00	1.4e-02	1.1e+00

Table 6. Example (2): errors and experimental orders of convergence (eoc) with $\rho^- = 1$, $\rho^+ = 10^5$, using method (5.2)-(2.3).

We then prove the remaining case. See Figure 4 for an illustration of these definitions.

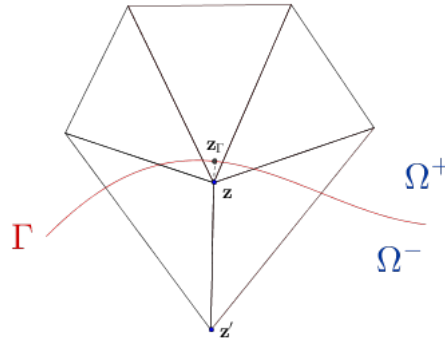


Figure 4. Appendix A: Illustration of definitions.

Let z_Γ the point on Γ resulting of extending the segment e connecting z and z' . Denote by A_1 and A_2 the regions separated by the segment $e_\Gamma =$

$\text{Int}(\overline{z'/z_\Gamma})$, such that $A_1 \cup A_2 \cup e_\Gamma = \Delta_z \cap \Omega^-$. Then, by Lemma 4 in [15], we have

$$|e_\Gamma|^2 \leq C \max\{|A_1|, |A_2|\}.$$

Thus, the result follows from $|e| \leq |e^\Gamma|$ and shape regularity.

APPENDIX B. PROOF OF EVEN EXTENSION

In this section we give an explicit construction of an even extension from Ω^+ to Ω . We apply this to continuous functions that are in $H^1(\Omega^+)$. The extension result (B.1) is a well-known result in partial differential equations. We sketch the proof with the aim to obtain the explicit construction of the extension, which will be used in the proof of Lemma 6.

For simplicity, assume that Ω^+ is the inclusion (i.e. $\partial\Omega^+ = \Gamma$). For $\epsilon > 0$ define the tubular neighborhood

$$R_\epsilon = \{x \in \Omega : d(x) := \text{dist}(x, \partial\Omega^+) \leq \epsilon\}.$$

Since Γ is smooth, and for ϵ small enough, for each $x \in R_\epsilon$ there exists a unique point $x_{\partial\Omega^+} \in \partial\Omega^+$ such that

$$|x - x_{\partial\Omega^+}| = \text{dist}(x, \partial\Omega^+).$$

Moreover, define the unit normal vector to $\partial\Omega^+$ pointing towards x as: $n(x) = (x - x_{\partial\Omega^+})/|x - x_{\partial\Omega^+}|$. Hence, for each $x \in R_\epsilon$ we define its ‘‘reflection’’ $\tilde{x} = \tilde{x}(x)$ by

$$\tilde{x} = x_{\partial\Omega^+} - d(x)n(x).$$

Now, given $v \in C(\overline{\Omega}^+)$ we define $\tilde{G}v \in C(\Omega^+ \cup R_\epsilon)$ such that:

$$\tilde{G}v(x) = \begin{cases} v(x), & \text{if } x \in \overline{\Omega}^+ \\ v(\tilde{x}), & \text{if } x \in R_\epsilon \setminus \Omega^+. \end{cases}$$

Our extension is complete by considering a cutoff function $\eta \in C_c(\Omega^+ \cup R_\epsilon)$, such that $\eta \equiv 1$ on $\Omega^+ \cup R_{\epsilon/2}$. Of course, we extend η to all of Ω by zero. Thus, the extension $Gv \in C_c(\Omega)$ is defined as follows

$$Gv = \eta \tilde{G}v.$$

The estimate is a well-known result.

$$(B.1) \quad \|Gv\|_{H^1(\Omega)} \leq C \|v\|_{H^1(\Omega^+)}.$$

APPENDIX C. PROOF OF DISCRETE EXTENSION LEMMA 6

We first introduce notation. For each node x of the triangulation \mathcal{T}_h we consider the patch associated to x , denoted by Δ_x . Moreover, we consider Δ_x^- as the restrictions to Ω^- , i.e.,

$$\Delta_x^- = \Delta_x \cap \Omega^-.$$

Considering definitions introduced in Appendix B, we define the reflection of Δ_x^-

$$\tilde{\Delta}_x^- = \{\tilde{y}(y) : y \in \Delta_x^-\} \subset \overline{\Omega}^+.$$

Let $v \in V_h^+$, then, by (B.1) there exists $Gv \in H_0^1(\Omega)$ such that $Gv = v$ in Ω^+ , and

$$\|Gv\|_{H^1(\Omega)} \leq C\|v\|_{H^1(\Omega^+)}.$$

We proceed constructing a stable interpolation operator of the extended function Gv onto V_h^c , invariant on V_h^+ . Define $P_1(Gv)$ for any node x of the triangulation \mathcal{T}_h by

$$P_1(Gv)(x) = \begin{cases} v(x), & \text{if } x \in \overline{\Omega}^+, \\ \frac{1}{|\Delta_x^-|} \int_{\Delta_x^-} Gv(y) dy, & \text{if } x \in \Omega^-, \end{cases}$$

i.e., P_1 preserves v in $\overline{\Omega}^+$ and is the Clément interpolant (see [10]) of $G(v)$ on Ω^- . Then, as a consequence of the definition of P_1 we have that

$$(C.1) \quad \|P_1(Gv)\|_{H^1(\Omega)} \leq C\|Gv\|_{H^1(\Omega)}.$$

Observe that this definition does not guarantee that $P_1(Gv)$ coincides with v for a node in $\Omega_h^+ \setminus \Omega^+$. Then we need to correct the definition of the interpolant on these nodes. The needed extension is defined as follows

$$Ev := P_2(Gv)(x) = \begin{cases} v(x), & \text{if } x \in \overline{\Omega}_h^+, \\ \frac{1}{|\Delta_x^-|} \int_{\Delta_x^-} Gv(y) dy, & \text{if } x \in \Omega \setminus \overline{\Omega}_h^+. \end{cases}$$

Notice that $P_2(Gv)$ and $P_1(Gv)$ agree at every node except the nodes x in the following set

$$S_h = \{x \in \Omega_h^+ \setminus \Omega^+ : x \text{ is a node of some triangle } T \in \mathcal{T}_h^F\}.$$

Define also the collection of triangles that have a node in S .

$$\mathcal{M}_h = \{T \in \mathcal{T}_h : \text{at least one of three vertices of } T \text{ belongs to } S_h\}.$$

If we define $e_h = P_2(Gv) - P_1(Gv)$ we see that

$$\|\nabla e_h\|_{L^2(\Omega)}^2 = \sum_{T \in \mathcal{M}_h} \|\nabla e_h\|_{L^2(T)}^2.$$

For each $T \in \mathcal{M}_h$, let $x_T \in S$ be such that $|e_h(x_T)| = \max_{y \in T} |e_h(y)|$. Then it is simple to show, using inverse estimates, that

$$\|\nabla e_h\|_{L^2(T)} \leq C |e_h(x_T)|.$$

By definition we have

$$|e_h(x_T)| = \left| v(x_T) - \frac{1}{|\Delta_{x_T}^-|} \int_{\Delta_{x_T}^-} Gv(y) dy \right| = \left| v(x_T) - \frac{1}{|\Delta_{x_T}^-|} \int_{\tilde{\Delta}_{x_T}^-} Gv(\tilde{y}) J(\tilde{y}) d\tilde{y} \right|,$$

where $J(\tilde{y})$ is the Jacobian of the map $\tilde{y}(y) \rightarrow y$. Applying the definition of Gv from the previous section, using that $v(\tilde{x}_T) = v(x_T)$, and the fact that $\int_{\tilde{\Delta}_{x_T}^-} J(\tilde{y}) d\tilde{y} = |\Delta_{x_T}^-|$ we get

$$|e_h(x_T)| = \left| \frac{1}{|\Delta_{x_T}^-|} \int_{\tilde{\Delta}_{x_T}^-} (v(\tilde{x}_T) - v(\tilde{y})) J(\tilde{y}) d\tilde{y} \right| \leq C \text{diameter}(\tilde{\Delta}_{x_T}^-) \|\nabla v\|_{L^\infty(\tilde{\Delta}_{x_T}^-)}.$$

We see that $\tilde{\Delta}_{x_T}^- \subset B_{\frac{d}{2}h}(T) = \{y \in \Omega_h^+ : \text{dist}(y, T) < \frac{d}{2}h\}$ for a d large enough but independent of h and T .

Using an inverse estimate we have

$$\text{diameter}(\tilde{\Delta}_{x_T}^-) \|\nabla v\|_{L^\infty(\tilde{\Delta}_{x_T}^-)} \leq C \|\nabla v\|_{L^2(B_{dh}(T))}.$$

Hence, we get

$$\|\nabla e_h\|_{L^2(\Omega)}^2 \leq C \sum_{T \in \mathcal{M}_h} \|\nabla v\|_{L^2(B_{dh}(T))}^2 \leq C \|\nabla v\|_{L^2(\Omega_h^+)}^2.$$

Finally, using the triangle inequality we get

$$\|\nabla E v\|_{L^2(\Omega)} \leq \|\nabla P_1(Gv)\|_{L^2(\Omega)} + C \|\nabla v\|_{L^2(\Omega_h^+)}.$$

The result now follows from using (C.1) and Poincaré's inequality.

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