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Response to Reviewers:	<p>Dear Editor</p> <p>Please find herewith the further-revised version of our paper. This takes into account the recent very helpful remarks from yourself, after the reviewers had (we understand) accepted the earlier version.</p> <p>Best regards,</p> <p>Samire Balta and Frank Smith.</p>

Inviscid and low-viscosity flows in multi-branching and reconnecting networks

Samire Balta and Frank Smith *

Abstract. We study unsteady flow problems of inviscid and low-viscosity fluids in multi-branching networks, reconnecting networks and large networks. The outer end pressures are assumed. Systematic solutions for fast flow are based on overall features of vessel shapes, lengths and end-pressure distribution. This is for rigid or elastic vessels. Favourable end-pressure gradients combined with constricting vessels lead to steady states or forced oscillations. Other conditions however can produce high internal transient pressures and flow surges even when the end conditions remain mild. For a small or large network a single nonlinear evolution equation is derived under the additional assumption of nearly equal vessel flows; this admits the influence of all the end pressures as well as the overall features of the vessels and determines the flow through the whole network.

Keywords: networks, flow, surges, end-pressures

MSC codes: 35Q35, 74F10, 76D09, 76M45

1. Introduction

Our concern is with problems of multiple branching and reconnecting of fluid-dynamical networks in engineering, industrial, geological and biomedical applications [1]-[10]. These occur in applications such as hydraulic fracturing or fracking in oil and gas recovery, while understanding of angiogenesis has prompted much work on arterial and similar networks and their evolution. Plumbing, waste disposal and other household-like matters are other applications.

An underlying issue is pressure control in simple or complex networks: we take the outer end pressures as prescribed and the unsteady flow velocities as unknown. A systematic approach is sought for rigid vessels first and then, in an appendix, elastic vessels. Other issues arise on the effects of vessel shape in natural or designed configurations and on the magnitudes of pressures produced inside a controlled network. These internal pressures if extreme or sustained or of high frequency can cause severe damage to the containing vessels, as can the corresponding wall shear stresses. A further issue is the evolving behaviour within a network rather than supposed steady-state properties. This is partly due to the wish to understand how quickly or slowly changes in the

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pressure control affect the network. In many applications a range of approximately 10-30% contraction in cross-sectional area occurs along the typical containing vessel and through the typical branch junction. It is clear also that nonlinear effects and global responses rather than detailed local responses can be important. Numerous computational studies have been performed as in the references above combining pioneering research, close relations with real network data and painstaking numerical work. There is much interest in comparatively large Reynolds numbers as well as in low flow rates [11]. Overall flow-through in physiological networks tends to be restricted by viscous resistance but that is not necessarily the case in engineering, industrial and geological networks.

The complementary approach taken here tackles the evolution in a network when end-pressures change substantially and predicts conditions under which flow surges of unexpectedly high velocity can occur even when the end conditions are of mild form. The paper also allows for overall features of constriction, dilation, branching and networks rather than smaller-scale features. Constrictions and branch junctions are important sites medically for flow blockage and disease initiation [12, 13]. They significantly influence cardiovascular performance in the human torso and cerebrovascular effects in the head. The latter effects include stroke and specifically arteriovenous malformations [14, 15], where one upstream (mother) vessel supplies incident blood flow to many downstream (daughter) vessels instead of just the conventional two. Additionally reconnections can occur whereby the many daughter vessels merge further downstream into a few grand-daughter vessels. Geological fractures can also lead to reconnections.

The problems addressed have unsteady nonlinear behaviour of inviscid or low-viscosity flow. See figure 1. The fluid is incompressible and Newtonian and its motion is driven by the known end-pressures. The variables used are non-dimensionalised on a typical flow velocity u_D , vessel length l_D , fluid density ρ_D and kinematic viscosity ν_D where the subscripts D refer to dimensional quantities. The representative time is l_D/U_D and the typical dimensional pressure value is $\rho_D u_D^2$. The characteristic Reynolds number Re is then $U_D l_D / \nu_D$ which is supposed large. See comparisons between analysis and simulations in [12, 16, 17]. The velocity components in the Cartesian directions x, y are written u, v respectively, the vector $\mathbf{u}$ is (u, v) , t denotes the time and p is the pressure. The flow response is two-dimensional (2D) and the small diffusive effects can be taken as laminar or turbulent. With the operator D/Dt denoting $\partial/\partial t + \mathbf{u} \cdot \nabla$ we work mostly on the 2D unsteady Euler equations

Figure 1. Schematic diagram. (a) Large network, comprising bifurcations and multi-branchings; $M = 4$ generations are shown in this case and the number of furthestmost downstream ends where pressure p is prescribed is $N = 13$. The upstream pressure $\hat{p}_0$ in the mother vessel is also prescribed. (b) Basic 1-to-2 branching with prescribed pressures $\hat{p}_n$ at the outermost ends of the mother and two daughter vessels for $n = 0, 1, 2$; the vessel widths $\hat{f}_n, f_n$ are also known but the velocities u_n close to the branching zone are unknown.

$$\nabla \cdot \mathbf{u} = 0, \quad \frac{D\mathbf{u}}{Dt} = -\nabla p \quad (1.1a, b)$$

of mass conservation and momentum balance; the vorticity which is necessary for solution of (1.1) is assumed given by the incoming flow upstream. Later we re-consider the Navier-Stokes equations which in the laminar regime have

$$\frac{D\mathbf{u}}{Dt} = -\nabla p + Re^{-1} \nabla^2 \mathbf{u} \quad (1.2)$$

for a viscous fluid. Another interesting approach in terms of viscous effects [7] is the widely used engineering model in which effects in slender vessels along the x axis are represented by

$$\frac{\partial q}{\partial t} + \frac{\partial(\alpha q^2/A)}{\partial x} + A \frac{\partial p}{\partial x} + K_R \frac{q}{A} = 0, \quad \frac{\partial A}{\partial t} + \frac{\partial q}{\partial x} = 0. \quad (1.3a, b)$$

This gives an estimate for the flux $q(x, t)$, with $p(x, t)$ being independent of the cross-sectional coordinates and with the end pressures being prescribed. Here A is the cross-sectional area in 3D or the vessel width in the present 2D problems, while the constant α is a momentum correction coefficient of order unity usually and K_R denotes a viscous resistance parameter proportional to ν_D which is small in general. The

system (1.3a,b) coupled with modelling for wall flexibility (relating A to p , for example) is particularly useful for predicting features due to biomedical networks of vessels, and the corresponding system will also be addressed below.

§2 is on one-to-two branching with prescribed end pressures. A single vessel is considered first, to highlight the fast-flow solutions which apply in all the problems below, and then solutions for a specific bifurcation are examined as the vessel lengths and shapes are varied. Steady inviscid flow in a single contracting vessel with prescribed end pressures would give a quadratic for the flow rate, rendering the solution non-existent or non-unique, apart from a marginal case which will be encountered below. The present study considers instead the unsteady problem for branching networks of vessels. The 10-30% range of area variation described earlier is used as guidance for the analysis. §3, §4, §5 are on problems of one-to-many branching, reconnection and large networks successively. Final discussion is presented in §6, while viscous effects and wall flexibility are described in the appendices.

2. One-to-two branching

The branchings have long vessels (with typical length scale $|x| \gg 1$) compared with the local branch-zones (junctions) which have $O(1)$ length scales. We start with the unsteady motion in a single vessel before joining three vessels together into a bifurcation and, in later sections, joining more together.

2.1. A SINGLE VESSEL

In a single long vessel inviscid boundary-layer-like arguments apply [12] to (1.1). The streamwise momentum balance in (1.1b) applies in full whereas the lateral momentum balance gives the pressure $p(x, t)$ being independent of y and the vorticity is predominantly $-u_y$ (since v_x is relatively small). In the major case of zero incoming vorticity u is thus $u(x, t)$ independently of y . So then the streamwise momentum balance yields (2.1b) below. Also continuity in (1.1a) then gives v as $-y u_x + a(x, t)$; the unknown function $a(x, t)$ here can be evaluated however from the two kinematic boundary conditions holding on v at the upper and lower walls and this leads to (2.1a) below. Hence we have the unsteady-flow equations

$$f_t + (uf)_x = 0, \quad (2.1a)$$

$$u_t + u u_x = -p_x. \quad (2.1b)$$

Here f is the internal width of the vessel. End-pressure values $p = p_a, p_b$ of $O(1)$ are imposed at the upstream and downstream ends $x = a, b$. We seek an asymptotic description

$$u = E^{-1/2}U_0(T) + E^{1/2}U_1(x, T) + \dots, \quad (2.2a)$$

$p = O(1)$, with $x \sim 1$, $t = E^{-1/2}T$ for the almost-spatially-uniform width

$$f = H_0(T) + EH_1(x, T). \quad (2.2b)$$

A key point is that for steady flow with a contraction, if the pressure drop is $O(1)$ then the flow rate increases as the typical degree E of contraction decreases and is expected to be $O(E^{-1/2})$ if the contraction is $O(E)$ when $E \ll 1$. This supports the form (2.2a,b). The shape (2.2b) comprises a spatially uniform value H_0 and a small deviation $O(E)$ with H_1 being of order unity. The contributions (2.2a,b) indicate high velocities, $O(1)$ pressures due to the end conditions, an $O(1)$ length of vessel and a pressure-driven time scale in turn. From the dominant balances in (2.1a,b), we have $H'_0 + U_0H_{1x} + H_0U_{1x} = 0$, $U'_0 + U_0U_{1x} = -p_x$ respectively, allowing U_{1x} to be eliminated, and then integration with respect to x shows that the leading-order velocity $U_0(T)$ along the vessel satisfies

$$L \frac{d}{dT} \left(\frac{U_0}{H_0} \right) + \{H_1(a, T) - H_1(b, T)\} \left(\frac{U_0}{H_0} \right)^2 = \frac{\{p_a(T) - p_b(T)\}}{H_0}. \quad (2.3)$$

Here $L = b - a$ is the vessel length. The thickness deviations H_1 , end-pressures $p = p_a, p_b$, end stations a, b and length L can be time-dependent. The ratio U_0/H_0 represents the unknown velocity per unit width.

The equation (2.3) is a Riccati equation [18, 19] for U_0/H_0 . The signs of the coefficients play an important part. The case of positive $H_1(a, T) - H_1(b, T)$ and positive $p_a(T) - p_b(T)$ is referred to herein as “the regular case”: the vessel width decreases in the supposed direction of flow from station a to station b the pressure gradient is favourable in that direction. A favourable gradient corresponds to decreasing pressure and an adverse one to increasing pressure. The two special situations of steady flow and of a parallel-sided channel admit forward flow solutions then since each counteracts the positive pressure difference on the right-hand side of (2.3) in any regular case by a positive effect on the left, namely inertial or accelerating respectively with forward flow. The reduction to the single nonlinear ordinary differential equation (2.3) for properties along the entire vessel includes the overall influences of vessel shape, vessel length and pressure differences.

The outcome, i.e. ultimate behaviour of the solution, from any initial conditions with non-negative U_0 is one of the following as T increases:

A nonlinear steady state /forced oscillation with $U_0 \geq 0$ at large T ;
(2.4a)

A finite-time blow-up of U_0 ;
(2.4b)

A finite-time reversal of U_0 .
(2.4c)

These outcomes come from the balances in (2.3), from numerical solutions or from the Riccati transformation. Outcome (2.4a) has finite amplitudes and includes nonlinear oscillations due to periodic forcing, for instance. Outcome (2.4b) has $U_0 \rightarrow \infty$ within a finite time say T_B with $U_0 \propto 1/(T_B - T)$. (2.4c) has $U_0 \rightarrow 0+$ within a finite time. The third outcome includes cases where U_0 would tend to $-\infty$ at a later finite time but the reversal of the flow before that stage, or indeed during oscillations, is generally inconsistent. This is due to viscous effects in (1.2) as described in §6. Regular cases lead to the outcome (2.4a); irregular cases can produce (2.4b,c). The outcomes (2.4a-c) give moderately large, enormous or considerably reduced flow rates respectively.

2.2. ONE MOTHER AND TWO DAUGHTER VESSELS

The one-to-two branching problem (figure 1) has the mother and two daughter vessels being long whereas the branching zone is of $O(1)$ axial length and $O(1)$ width. The latter zone is controlled by

$$(u, v, p) = (u^*, v^*, p^*) + \dots, \quad (2.5)$$

with (x, y) of $O(1)$, $t = L^*T$, where $L^* \gg 1$ is the ratio of the typical vessel length over the typical branching-zone distance. The components u^*, v^*, p^* are of $O(1)$. The governing equations are the quasi-steady version of (1.1),

$$\nabla \cdot \mathbf{u}^* = 0, (\mathbf{u}^* \cdot \nabla) \mathbf{u}^* = -\nabla p^* \quad (2.6)$$

with $\mathbf{u}^* = (u^*, v^*)$. These are subject to matching to pressure and velocity in the uni-directional motions at large distances in terms of x, y ; the pressures there are unknown in advance, in contrast with the pressures at the outer ends of the mother and daughter vessels (at large $|X|$ below). The time scale is the convective time in the long vessels. Inside the mother and daughters the coordinate y is normal to the walls, the characteristic length is large and the flow solution expands as

$$(u, v, p) = (u^{**}, \frac{v^{**}}{L^*}, p^{**}) [1 + o(1)], \quad (2.7)$$

with $x = L^*X$, $y = O(1)$, $t = L^*T$. Here X , T , u^{**} , v^{**} , p^{**} are $O(1)$. The vessel extends from zero X to a finite positive value of X downstream. From (1.1) the dominant balances are as in (2.1a, b) with the main pressure $p^{**}(X, T)$ being independent of y , provided that the vorticity is zero. (An analogous derivation can be based on (1.2) provided the reduced Reynolds number is high.) In the case of steady wall shapes $f(X)$ with which this paper is mostly but not entirely concerned an integration of (2.1a) yields $u^{**} = Q(T)f^{-1}(X)$. Here Q is the scaled mass flux. Then (2.1b) along the length $L = b - a$ of the vessel gives

$$I \frac{dQ}{dT} + \frac{1}{2} Q^2 (f^{-2}(b) - f^{-2}(a)) = p_a(T) - p_b(T), \quad (2.8a)$$

$$\text{where } I = \int_a^b f^{-1} dX. \quad (2.8b)$$

Here (2.8a) is the Riccati equation for the unknown flux Q . It is consistent with (2.3) when f takes the form (2.2b) with E small. We bear in mind that (2.8a,b) is an exact form from (2.1) whereas (2.3) is an approximation, albeit a fairly accurate one. The accuracy can be assessed as in figure 2 and indicates that (2.3), (2.8a,b) agree well numerically over much of the parameter range of interest. Concerning the example in figure 2, $p_a - p_b$ is set as 1, while $a = 0$, $b = 1$, and (2.8a, b) has $f = (1 - EX)^2$ for two cases $E = 0.1$ and $E = 0.4$; so (2.3) takes $H_0 = 1, H_1 = -2X$. (The velocities in (2.8a,b) are comparable with

Figure 2. Mass flux Q vs scaled time T according to (a) (2.3), (b) (2.8a,b), for 10% and 40% shape reductions in a single vessel based on end differences of $1/f^2$. The near-equality for small times is as expected. The results also indicate use of (a) or (b) as alternatives in the typical range of interest for contractions and dilations.

the square-root of the pressure differences but the velocities in (2.3) are significantly larger by virtue of the expansion (2.2a)). The match with the branching zone requires $Q(T)f^{-1}(0+) = u^*(\infty)$. The conditions at the vessel ends are on pressure as in figure 1: p^{**} is prescribed as the value $\hat{p}_0$ at the upstream end of the mother tube and $\hat{p}_n$ at the downstream ends of the two daughters for $n = 1, 2$.

As mentioned previously the solution depends on the vorticity, which is determined by the incident flow. When the incident vorticity is zero (2.6) leads to the jump relations

$$p_n + \frac{1}{2} Q_n^2 f_n^{-2}(a_n) = p_0 + \frac{1}{2} Q_0^2 f_0^{-2}(b_0), \text{ for } n = 1, 2, \quad (2.9a)$$

$$Q_0 = \sum_{n=1}^2 Q_n, \quad (2.9b)$$

over the short scale [7, 12, 15]. Over the longer scale (2.8a, b) lead to

$$I_n \frac{dQ_n}{dT} + Q_n^2 (f_n^{-2}(b_n) - f_n^{-2}(a_n))/2 = p_{a_n}(T) - p_{b_n}(T), \quad (2.10a)$$

$$\text{where } I_n = \int_{a_n}^{b_n} f_n^{-1}(X) dX, \quad (2.10b)$$

in the three vessels with respective subscripts $n = 0, 1, 2$. See figure 1. The one-to-two branching has internal end-pressures p_{b0}, p_{a1}, p_{a2} which are unknown but the external end-pressures $\hat{p}_0, \hat{p}_1, \hat{p}_2$, i.e. p_{a0}, p_{b1}, p_{b2} respectively, are prescribed. It follows that we have six equations for the three fluxes Q_n for $n = 0, 1, 2$ and three internal pressures p_{b0}, p_{a1}, p_{a2} . Eliminating the three internal pressures produces a system of three equations for the unknown velocities u_0, u_1, u_2 at the branching zone or node:

$$\frac{1}{2}(\sigma_n u_n^2 - \sigma_0 u_0^2) = \bar{p}_n - \bar{L}_n \frac{du_n}{dT} - \bar{L}_0 \frac{du_0}{dT}, \text{ for } n = 1, 2, \quad (2.11a, b)$$

$$f_0 u_0 = \sum_{n=1}^2 f_n u_n. \quad (2.11c)$$

Here $\sigma_n = f_n^2/\hat{f}_n^2$ is the square of the vessel area ratio measured from the node end to the other end; $\bar{p}_n = \hat{p}_0 - \hat{p}_n$ denotes the prescribed pressure drops. Each $Q_n = f_n u_n$ where f_n is evaluated at the node and $u_n = u_n^{**}(a_n, T)$ for $n = 1, 2$, $u_0 = u_0^{**}(b_0, T)$. Also $\bar{L}_n (= I_n f_n)$ which is from the integral I in (2.8b) acts as an effective length or longitudinal area. The $f_n, \hat{f}_n$ are vessel thicknesses at locations a_n, b_n respectively for $n = 1, 2$ but at b_0, a_0 respectively for $n = 0$. In regular

cases the pressure drops are positive and the vessels constrict such that $\sigma_1 > 1, \sigma_2 > 1$ whereas $\sigma_0 < 1$. The coefficients are assumed given at this stage, cf. the influence of flexibility in Appendix B. The full dynamics for the three-vessel network here is exactly as would be predicted by the small-deviation system of (2.3). The complete network dynamics is also equivalent to a Bernoulli-like balance for unsteady flow. The next step is to investigate the evolution of the system.

2.3. SOLUTIONS

Concerning numerical solutions, figure 3 is for the case of $f_0 = 1, f_n = 0.4, \sigma_n = 1$. The pressure differences $\bar{p}_n$ depend on scaled time T as $nT/(1+T)$ for $n = 1, 2$, which constitute a regular case starting from rest; this is an arbitrary choice to illustrate the flow properties. Solutions are shown for the mother- and daughter-vessel velocities over a range of $\bar{L}_0$ values from zero to 5. All lead to steady states at large times but increasing any $\bar{L}_n$ value for $n = 0, 1$ or 2 is seen to delay the trend towards that state, in line with an analysis where time T scales on the maximum $\bar{L}_n$. We note that the nominally zero value of $\bar{L}_0$ in one case above should be considered as $0+$, since the length must remain larger than the junction length scale, and also that for convenience only one pair of u_1, u_2 curves is shown but three u_0 curves. Figure 4 has the same f_0, f_1 as above but the daughter-width factor f_2 takes the values 0.1 and 0.5 while $\sigma_n, \bar{L}_n$ are maintained as unity. The velocities decrease as f_2 decreases due to the blockage effect with u_0 being particularly suppressed but u_1 also falls significantly. The internal pressures produced at the branching zone are also presented along with the corresponding end pressures for comparison. Little adverse pressure gradient is induced internally, in vessel 1, for a short time before the approach to a steady state emerges. The regular case in figure 5 corresponds to varying $\bar{L}_2$, which takes the values 0.5 and 3 , provoking the internal and end pressures as shown. The adverse pressure gradient is quite substantial at earlier times in both vessels 1, 2. Figure 6 then has the $\bar{L}_n$ all equal to unity and the f_n as originally but with the square ratio σ_2 varying; $\sigma_0 = \sigma_1 = 1$ is maintained. This produces a mixture of regular and irregular settings. The velocities show a marked increase in response as σ_2 is decreased and indicate blow-up in the most adverse settings. The internal pressure gradients become markedly adverse for smaller σ_2 values.

Steady states are found in all regular cases (figures 3-5) as $T \rightarrow \infty$. Such states do not usually arise in irregular cases however.

Linearised properties are considered next. These apply when the imposed pressure amplitudes $\bar{p}_n(T)$ are small or when the motion starts

Figure 3. Induced velocities u_n , $n = 0-2$, plotted against time T for length factors $\bar{L}_0$ (written L_0) of 0, 1, 5; the value 0 here should be regarded as $0+$. Increased length of the mother vessel delays the approach to a steady state.

Figure 4. Velocities u_n and pressures p_n, P_n vs T , where shape effect f_2 is 0.1 or 0.5. The solid curves show u_n, p_n , the dotted curves show P_n , for $n = 0, 1, 2$ respectively. (The P_n are the imposed end pressure differences $\bar{p}_n$.) This shows the effects of alterations in the width of one daughter vessel (that for $n = 2$).

from rest, as the nonlinear terms become suppressed. This is seen in parts of figures 3-6. The solution can then be obtained explicitly,

$$u_0 = S^{-1} \sum_{n=1}^2 f_n g_n \bar{L}_n^{-1}, \quad u_n = (g_n - \bar{L}_0 u_0) \bar{L}_n^{-1} \quad (2.12a, b)$$

Figure 5. Velocities u_n and internal pressures $p_n, n = 0-2$, vs T , as length effect $\bar{L}_2$ takes values 0.5 or 3. Imposed end pressures are shown as P_n . (The solid curves show u_n, p_n , the dotted curves show $P_n = \bar{p}_n$, while the values 0.5, 3 refer to length.) This indicates significant influence from the daughter ($n = 2$) length.

Figure 6. As in figure 5 but the shape factors σ_2 are varied, reducing from 1.5 to zero. Symbols 0, 0.5, 1, 1.5 refer to the σ_2 values on the nearest curve. Solid curves show u_0, p_0 ; dashed u_1, p_1 ; dotted u_2, p_2 . Blow-up (flow surge) within a finite time is suggested for a range of shape factors.

for $n = 1, 2$ where

$$g_n = \int_0^T \bar{p}_n(T) dT + \bar{L}_n u_n(0) + \bar{L}_0 u_0(0) \quad (n = 1, 2) \quad (2.12c)$$

and $S = \bar{L}_0 \sum_{n=0}^2 f_n / \bar{L}_n$. The induced velocities are positive when the pressure gradients are favourable. This response persists in principle as long as the pressure integrals above remain small but if the imposed pressures become steady at sufficiently large times the integrals increase linearly with T at large T . So the velocities must do likewise. Over a longer time scale the nonlinear effects then come back into the reckoning. The response (2.12a-c) holds for small times in almost any setting where the motion starts from rest. The contributions g_n in (2.12a-c) then become linear in T at small T , specifically $\bar{p}_n(0)T$, determining the small-time behaviour.

Large pressures are of interest. When the imposed nondimensional pressure differences are of amplitudes that are large of order Λ , where $\Lambda \gg 1$ is a nondimensional parameter representing typical amplitudes, then the induced velocities can be large also (of order $\Lambda^{1/2}$) or the response is of high frequency. For comparatively large velocities over an extended period, the behaviour is quasi-steady. This period lasts for all T in regular cases and it also lasts while the pressure gradient is favourable in cases of constricting geometry. When the pressure gradient changes linearly at some $O(1)$ time T_c say from favourable to adverse the quasi-steady state ceases to exist due to a square-root format; see (2.9a,b). The change is seen most easily for a single vessel as in (2.3) or equivalently (2.8a, b) or (2.10a, b), and here we take the case (2.10a, b). The change thus occurs over a short time scale of $O(\Lambda^{-1/3})$ around $T = T_c$ within which

$$Q_n = \Lambda^{1/3} r(T^*) + \dots, \quad (2.13a)$$

$$p^C = -\Lambda^{2/3} p_C T^* + \dots \quad (2.13b)$$

where p^C is the imposed pressure difference $(p_{a_n} - p_{b_n})(T)$, p_C is an $O(1)$ positive constant and $f_n^{-2}(b_n) - f_n^{-2}(a_n)$ is $O(1)$ and positive. This leads from (2.10a,b) to the normalised nonlinear equation

$$\frac{dr}{dT^*} + r^2 = -T^*. \quad (2.13c)$$

The local $O(1)$ time variable is T^* . Matching to the earlier large amplitudes the unknown function $r(T^*)$ is subject to the initial condition $r \sim (-T^*)^{1/2}$ as T^* tends to $-\infty$. The appropriate solution of (2.13c) is hence the reversed Airy-function (Ai) ratio

$$r = \varphi^{-1} \frac{d\varphi}{dT^*} \quad \text{with} \quad \varphi = Ai(-T^*), \quad (2.13d)$$

to within a multiplicative constant. The square-root behaviour of r is retrieved for $T^* \rightarrow -\infty$ but as time increases φ reaches a maximum at $T^* = T_C^*$ ($= 1.01879\dots$, see [20]). The zero of the slope $Ai'(-T_C^*)$ there indicates that the effective velocity response

$$u = 0 \text{ (} r = 0 \text{) at finite time } T^* = T_C^* \quad (2.13e)$$

is encountered. This onset of flow reversal means that the overall solution breaks down then. At a subsequent finite time the Airy representation above would also show φ reaching a zero, implying that r would tend to ∞ as in negative blow-up, but the first response (2.13e) encountered is the dominant one as in (2.4c). A similar account applies to the full system (2.11a-c). Results (not presented here) for fluctuating differences in end pressure have shown such breakdown; see also figure 6. The critical time in (2.13e) is the latest at which the imposed pressure difference could be switched off, i.e. p^C or $\bar{p}_n$ set to zero, without provoking flow breakdown. Thus the system can withstand only a limited amount of adverse pressure gradient (between T^* of zero and T_C^*) without breaking down.

Finite-time blow-up is also of interest here. This may occur in irregular settings according to the computational results. The orders of magnitude in (2.11a,b) suggest each u_n can grow large like $1/(T_B - T)$ at some finite time $T = T_B -$. All the velocities blow up together in view of the connectivity of the system, giving the pattern

$$u_n = \mu_n (T_B - T)^{-1} + \dots \text{ as } T \rightarrow T_B - \quad (2.14)$$

for all n . The constant coefficients μ_n are positive to correspond to forward blow-up and they are in line with (2.4b) whereas negative μ_n would yield backward flow; the latter is taken to be irrelevant since it would be preceded by zero flow or small flow reversal which leads to a different type of structural breakdown as addressed in §6. Substitution into (2.11a-c) yields the nonlinear algebraic system

$$\frac{1}{2}(\sigma_n \mu_n^2 - \sigma_0 \mu_0^2) = -\bar{L}_n \mu_n - \bar{L}_0 \mu_0, \text{ for } n = 1, 2, \quad (2.15a,b)$$

$$f_0 \mu_0 = \sum_{n=1}^2 f_n \mu_n, \quad (2.15c)$$

controlling the constants μ_n . The imposed pressures have no effect in the blow-up system. The results presented in figure 7 show the left- and right-hand sides of (2.15c) plotted against positive μ_0 , with (2.15a,b) used to determine each μ_n subject to a positive rather than negative root. Figure 7 has just one vessel being irregular, namely the mother

one as $\sigma_0 > 1$, but this is still sufficient to produce positive roots for μ_0 over the parameter range and hence blow-up. A single-vessel irregularity inside the system does not necessarily produce blow-up. The networking can have a calming effect since the single irregular vessel on its own would lead to blow-up. Regular cases avoid blow-up according to computations and to (2.15c) since the right-hand side is always less than the left-hand side in regular cases.

3. One-to-many branching

The extension for the version with N daughter vessels stems directly from the previous section. It gives

$$\frac{1}{2}(\sigma_n u_n^2 - \sigma_0 u_0^2) = \bar{p}_n - \bar{L}_n \frac{du_n}{dT} - \bar{L}_0 \frac{du_0}{dT}, \text{ for } n = 1, \dots, N, \quad (3.1a)$$

$$f_0 u_0 = \sum_{n=1}^N f_n u_n, \quad (3.1b)$$

for the unknown velocities u_0, u_n at the branching zone, with n from 1 to N . The pressure differences are imposed as functions of scaled time T , and likewise the coefficients $\sigma_n, \bar{L}_n, f_n$ which have definitions analogous with those in the previous section are prescribed. The nonlinear system is an N th- or $(N+1)$ th-order ordinary differential one in view of the coupling between the many first-order differential equations.

Numerical solutions are addressed first. Figure 8 is for $N = 11$ daughter vessels in a regular setting. All the $f_n, \bar{L}_n, \sigma_n$ parameters are unity throughout. The unsteady pressure differences $\bar{p}_n$ are imposed as $(1 + n/10)T/(1 + T)$ and thus are skewed towards larger n values. As for figure 3, the choice of $\bar{p}_n$ is made in order to demonstrate the main solution features. The velocities u_n exhibit totally forward flow, increase monotonically with n but not necessarily with time T and they eventually approach a steady state at large times. These and related results display very similar evolutions for the different velocity components and suggest flows representing near-equality of input and output essentially. Likewise the properties of large N values become relevant.

Near-equality is investigated now. Here the upstream and downstream total areas are nearly identical. The induced velocities are also nearly identical: with a small parameter E as in §2.1 the expressions

$$f_n = \kappa_n + E f_n^* \text{ for } n = 0 \text{ to } N, \text{ with } \sum_{n=1}^N \kappa_n = \kappa_0, \sigma_n = 1 + E \sigma_n^*, \quad (3.2a)$$

$$\hat{f}_n = \kappa_n + E \hat{f}_n^* \text{ for } n = 0 \text{ to } N, \quad (3.2b)$$

$$u_n = E^{-1/2} \bar{U}_n + E^{1/2} \tilde{U}_n + \dots \text{ with } \bar{U}_0 = \bar{U}_n \text{ for all } n, \quad (3.2c)$$

apply, where $\sigma_n^* = 2(f_n^* - \hat{f}_n^*)/\kappa_n$. The results in figure 8 suggest the above expressions. The effective lengths and imposed pressures are assumed to be $O(1)$, while the time scale is large such that $T = E^{-1/2} \hat{T}$. We recall that the $\hat{f}_n$ are the values at the upstream and downstream ends of the vessels rather than at the branching zone. From (3.1a, b) we obtain the balance

Figure 7. Right- and left-hand sides of (2.15c), respectively a , b , plotted against μ_0 for varying σ_0 from 5 to 30 with $\sigma_1 = 1, \sigma_2 = 2$. Each location at which the dotted curves cross the solid curve indicates a root and hence blow-up (flow surge). Such blow-up occurs over the entire parameter range examined here.

Figure 8. Multi-branching with $N = 11$: induced velocities vs time T . All cases approach a steady state as T increases.

$$\hat{L}\bar{U}'_0(\hat{T}) + \hat{\Gamma}\bar{U}_0^2(\hat{T}) = p_{eff}(\hat{T}), \quad (3.3a)$$

where the coefficients are defined by

$$\hat{L} = \kappa_0 \bar{L}_0 + \sum_{n=1}^N \kappa_n \bar{L}_n, \quad (3.3b)$$

Figure 9. For continuum limit of multi-branching, showing (first, as the thinner curves) representative velocities vs T and (second, as the thicker curves) the influence of a sudden stopping of the applied end-pressure differences. In the first case the analogy with the results of figure 8 seems clear. In the second case there is algebraic decay of the velocities after the imposed pressures cease.

$$\hat{\Gamma} = \hat{f}_0^* - \sum_{n=1}^N \hat{f}_n^*, \quad (3.3c)$$

$$p_{eff} = \sum_{n=1}^N \kappa_n \bar{p}_n. \quad (3.3d)$$

The Riccati form (3.3a) has coefficients (3.3b-d) representing an effective length, shape effect and imposed pressure difference respectively but acting for the whole one-to-many multi-branching system rather than for only a single vessel. This somewhat surprising result is extended to a larger nonlinear network in §6. In regular cases the coefficients (3.3b-d) are found to be positive. The properties for the entire system are therefore as described by (2.4a). For irregular cases the summations in (3.3b-d) determine how much irregularity can be tolerated without causing blow-up or choking. For instance, adverse pressure gradients acting on certain of the daughter vessels can be withstood provided the whole summation in (3.3d) remains favourable. Finally here, the central example of equal daughter widths $\kappa_n = 1/N$ for $n = 1$ to N with $\kappa_0 = 1$ should be examined. It shows the equivalent length factor $\hat{L}$ to be $\bar{L}_0 + (\bar{L}_1 + \dots + \bar{L}_N)/N$, the equivalent pressure drop to be $\hat{p}_0 - (\hat{p}_1 + \dots + \hat{p}_N)/N$ over the whole system and the equivalent cross-sectional area drop to be that from the upstream mother to the total from all the daughters downstream. This enables the mother

contributions to be gauged against the mean effects of the daughter contributions to the overall behaviour.

Large N values provide additional insight. The large- N behaviour takes the system from the discrete form in (4.1a,b) to a continuum form, namely

$$\frac{1}{2}(\sigma u^2 - \sigma_0 u_0^2) = \bar{p} - \bar{L} \frac{\partial u}{\partial T} - \bar{L}_0 \frac{\partial u_0}{\partial T}, \text{ for the unknowns } u(y, T), u_0(T), \quad (3.4a)$$

$$f_0 u_0 = \int f u \, dy. \quad (3.4b)$$

The scaled y acts normally across the vessels from 0 to 1. The pressure difference, length and shape are prescribed as functions of y , while the integral shown extends from zero to unity and constrains the system. Solutions are presented in figure 9 for regular settings. First, the quasi-density function f is taken as 0.5 for all y , and $\sigma, \bar{L}, \sigma_0, \bar{L}_0$ are kept as unity while the unsteady end-pressure difference function is imposed as $yT/(1+T)$, a normal pressure gradient in effect. The velocity response starting from rest is found to increase in y because of the enhanced longitudinal pressure gradient at increased y . As T increases the flow solution gradually tends towards a steady state. Second, the effects of a sudden stopping of the applied pressure difference at a scaled time T of 2 are included. Here the velocities decay like $1/T$ at large times. A nearly-uniform scenario is again found to produce a Riccati-like response as in (3.3a) for the mother velocity $u_0(T)$ for the entire system.

4. Reconnections

A basic reconnection involves one mother vessel followed by two daughters which in turn are followed by a single final vessel to produce a $1 \rightarrow 2 \rightarrow 1$ formation. With more daughters present this becomes a $1 \rightarrow \text{many} \rightarrow 1$ formation. In either case additional consideration is necessary in the modelling. For the $1 \rightarrow 2 \rightarrow 1$ formation we assume that the unknown pressures at the downstream ends of the two daughters must be equal as in a Kutta condition. We treat the ensuing motion beyond as two zero-vorticity flows proceeding beside each other inside the single grand-daughter vessel continuing on from the two daughter flows. The width of the grand-daughter vessel is taken to be equal to the sum of the two daughter-vessel widths at reconnection as a first model.

The consequent governing equations for this reconnection are

$$\frac{1}{2}(\sigma_n u_n^2 - \sigma_0 u_0^2) = \hat{p}_0 - \hat{p}_n - \bar{L}_n \frac{du_n}{dT} - \bar{L}_0 \frac{du_0}{dT}, \text{ for } n = 1, 2, \quad (4.1a)$$

$$f_0 u_0 = \sum_{n=1}^2 f_n u_n, \quad (4.1b)$$

for the mother-daughters contribution as in the previous section but with $N = 2$ here, and

$$-(f_1 + f_2)\Gamma_3 = (f_1 u_1^{-2} + f_2 u_2^{-2})(\hat{p}_1 - \hat{p}_3) - \bar{L}_3(f_1 u_1^{-2} \frac{du_1}{dT} + f_2 u_2^{-2} \frac{du_2}{dT}), \quad (4.1c)$$

$$\hat{p}_1 = \hat{p}_2. \quad (4.1d)$$

Here (4.1c,d) are respectively for the single grand-daughter motion and the equi-pressure condition at the reconnection. The scaled end pressures $\hat{p}_0, \hat{p}_3$ are prescribed. So there are five equations above for the five unknown functions of time $u_0, u_1, u_2, \hat{p}_1, \hat{p}_2$ representing the internal induced velocities and pressures. Eliminating $\hat{p}_1, \hat{p}_2$ leads to three equations for the three velocities which can be solved as before computationally and analytically depending on the parameter ranges.

Numerical solutions are presented in figure 10. These are primarily for σ_n equal to unity for $n = 0, 1, 2$, while $\Gamma_3 = -0.2$, the f_n values are 1, 0.4, 0.3 for $n = 0, 1, 2$ respectively, the $\bar{L}_n$ effective lengths are 1, 1, 4, 1 in turn and a pressure difference $\hat{p}_0 - \hat{p}_3$ is imposed that increases monotonically from zero to a steady form at large T . The velocities produced from rest initially evolve towards a steady state as T increases with the components u_1, u_2 approaching equality then. This agrees with the effect of unequal daughter lengths diminishing as the time derivatives reduce significantly. The additional solutions are for the same regular case except that σ_1 is set equal to 2, which causes a reduction in all the internal velocities due to the increased resistance. The two daughter velocities approach different values at large T , with $u_1/u_2 \rightarrow (\sigma_1/\sigma_2)^{1/2}$, which gives the value $2^{-1/2}$ as the trend in the computations indicates. Near-equality analysis is similar to that in §3.

Figure 10. Reconnecting vessels: velocities produced for $\sigma_0 = 1$ (thinner curves) and $\sigma_0 = 2$ (thicker curves). The approach to a steady state is clear as T increases.

5. Larger networks

We show results here for three generations before enlarging this to any number of generations.

Figure 11. In a $1 \rightarrow 2 \rightarrow 4$ network: evolving solutions for applied end-pressure differences of $1 + nT/(1 + T)$ with $n = 1-4$. The mother-vessel velocity is u_0 , the two daughter velocities have superscripts (2) and the four grand-daughter velocities have superscript (3). This case is close to being regular but the system nevertheless blows up within a finite time.

Concerning numerical solutions, for three generations $1 \rightarrow 2 \rightarrow 4$ the configuration (2.11a-c) occurring three times needs to account for the fact that the only pressures known in advance are at the upstream

end of the mother, $\hat{p}_0$, and the downstream ends of the four grand-daughters, say $\hat{p}_n^{(3)}$ for $n = 1$ to 4. Results are shown in figure 11 for a case which is marginal, i.e. close to being regular. The imposed end-pressure differences are $1 + nT/(1 + T)$ from the mother to the n th grand-daughter vessel for $n = 1$ to 4. The effective vessel lengths are unity and areas (widths) are 0.4 except for the mother value of unity but the σ_n values are unity. The evolving velocities starting from a rest state are presented for each of the seven vessels. The response remains quite benign at early times but leads eventually to a blow-up at a finite time.

On the matter of any number of generations, we consider any number M of generations in a bifurcating tree as in $1 \rightarrow 2 \rightarrow 4 \rightarrow \dots \rightarrow \hat{M}$ where $\hat{M} = 2^{M-1}$; see also figure 1. This is based on (2.11a-c) being repeated at each bifurcation, on avoiding overlaps in counting, on eliminating the internal pressures from the interactive equations and on allowing for all the prescribed end pressures. Large matrix inversion or considerable iteration seems unavoidable in the numerical methodology then in general. Similar considerations apply to the continuum limit. More bountiful directly here is a treatment based on near-equality which as shown below allows a description of the whole network in terms of a single Riccati equation.

Turning to near-equality in large networks, we see that the solution expands as

$$f_n^{(m)} = \kappa_n^{(m)} + E f_n^{*(m)}, \quad (5.1a)$$

$$\hat{f}_n^{(m)} = \kappa_n^{(m)} + E \hat{f}_n^{*(m)}, \quad (5.1b)$$

$$\sigma_n^{(m)} = 1 + O(E), \quad (5.1c)$$

$$\sum_{j=2n-1}^{2n} \kappa_j^{(m+1)} = \kappa_n^{(m)}, \quad (5.1d)$$

$$u_n^{(m)} = E^{-1/2} \bar{U}_n^{(m)} + E^{1/2} \tilde{U}_n^{(m)} + \dots \quad (5.1e)$$

$$\text{with } \bar{U}_0 = \bar{U}_n^{(m)} \text{ for all } m, n. \quad (5.1f)$$

The property in (5.1a-d) of nearly identical total cross-sectional area and shape at each bifurcation continues on from the mother vessel (the first generation) in which

$$f_0 = \kappa_0 + E f_0^*, \quad (5.1g)$$

$$\hat{f}_0 = \kappa_0 + E \hat{f}_0^*, \quad (5.1h)$$

$$\sigma_0 = 1 + O(E), \quad (5.1i)$$

$$\text{and } \sum_{j=1}^2 \kappa_j^{(2)} = \kappa_0, \quad (5.1j)$$

to every subsequent generation $m = 2, \dots, M$ with bifurcation ($n = 1$ to $N(m)$ where $N(m)$ is 2^{m-1}) at each stage. Likewise the response (5.1c) of large but nearly equal flow velocities everywhere is repeated through the generations present. The notation is as in §3 including the time scale $T = E^{-1/2} \hat{T}$. The governing equation for the unknown mother-flow velocity $\bar{U}_0(\hat{T})$ is

$$\hat{L} \frac{d\bar{U}_0}{d\hat{T}} + \hat{\Gamma} \bar{U}_0^2 = p_{eff}(\hat{T}). \quad (5.2a)$$

This also governs the velocity in every vessel at leading order. The coefficients which act as effective lengths, shape effects and imposed pressure differences for the entire system are now given by

$$\hat{L} = \sum_{m=1}^M \sum_{n=1}^{N(m)} \kappa_n^{(m)} \bar{L}_n^{(m)}, \quad (5.2b)$$

$$\hat{\Gamma} = \hat{f}_1^{*(1)} - \sum_{n=1}^{N(M)} \hat{f}_n^{*(M)}, \quad (5.2c)$$

$$p_{eff} = \sum_{n=1}^{N(M)} \kappa_n^{(M)} \bar{p}_n^{(M)}. \quad (5.2d)$$

Here $\bar{L}_1^{(1)} \equiv \bar{L}_0$, $\kappa_1^{(1)} \equiv \kappa_0$, $\hat{f}_1^{*(1)} \equiv \hat{f}_0^*$. Hence a nonlinear Riccati-like description namely (5.2a) covers the flow evolution in the whole of this larger network as well as the much smaller networks (or even single vessels) addressed before. In (5.2b-d) the quantity $\bar{p}_n^{(M)}$, to emphasize, is the pressure difference between the mother (upstream end) and the final descendant M (downstream end). The quantity $\hat{f}_n^{*(M)}$ is the effective area deviation at the downstream end of that final descendant and $\hat{f}_1^{*(1)}$ is the corresponding deviation furthestmost upstream at the very start of the first generation (the mother vessel). The effects from all other constrictions and dilations inside the network cancel out within this regime. In contrast the effects from the lengths of the vessels inside the network are leading-order ones as is evident in the double summation in (5.2b). Overall regular and irregular cases respond as for the earlier problems. The specific example of equal vessel widths throughout any given generation yields $\kappa_1^{(2)} = \kappa_2^{(2)} = \kappa_1^{(1)}/2, \dots, \kappa_n^{(m)} = \kappa_1^{(1)}/m$ for all n . Hence if $\kappa_1^{(1)}$ is unity the active length $\hat{L}$ is $\bar{L}_1^{(1)} + (\bar{L}_1^{(2)} + \bar{L}_2^{(2)})/2 + \dots +$

$(\bar{L}_1^{(M)} + \dots + \bar{L}_N^{(M)})/N$, the active pressure difference is $\bar{p}_1^{(M)} + \dots + \bar{p}_N^{(M)}$ and the active shape influence remains the relative area factor (5.2c).

6. Further discussion

The present work in unsteady flow branchings and reconnections has tackled problems in networks of various sizes by means of an unusual broad-features analysis. The solutions have shown the combined effects of vessel shapes, the production of internal pressures, how end-pressures affect the network, and flow surges (blow-up) with high velocity. The same approach holds for the pressure-shape and velocity-shape interactions addressed in Appendix B. It is found that adding extra vessels in a network tends to take energy away from blow-up. Constriction also increases the overall regular flow-through but details are significant; some dilation is tolerable without inducing flow surges.

Given that fluid-flow networks are of wide concern, further systematic investigation would be desirable. The influences of three-dimensionality, vorticity, viscous effects (discussed in Appendix A with regard to (1.3)) and wall flexibility (Appendix B) are of great interest. Concerning viscous effects further the thin wall layer for (2.7) has thickness $O(Re^{-1/2} L_n^{1/2} u_n^{-1/2})$, is quasi-steady and of Blasius form for positive motion. If the core motion slows sufficiently then the wall layer expands, breaks up and so induces vorticity in any subsequent reversed flow: see also (2.4c).

Most fruitful is the near-equality work, which applies across all present cases and which reduces to a single nonlinear evolution equation for the entire network as in (5.2a). Averaging or weighted averaging of effects from imposed end-pressures, vessel lengths and end-shapes based on rational analysis and asymptotics (see (5.2b-d)) determines the large-network behaviour. It indicates that flow surges and correspondingly high internal pressures inside a network can be avoided by adjustment of the end-values in the mean, specifically reducing the mean outlet cross-sectional area sufficiently or reducing the mean outlet pressure level sufficiently or both.

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Appendix

A.

Engineering model and viscous effects

In the engineering model in (1.3a,b), for each vessel the flux q and effective area A are expanded

$$A = H_0(T) + \Delta H_1(x, T), \quad (\text{A.1})$$

$$q = \Delta^{-1/2} Q_0(T) + \Delta^{1/2} Q_1(x, T) + \dots, \quad (\text{A.2})$$

for a slow time scale $t = \Delta^{-1/2} T$: compare (2.2a,b). This is with a

Figure 12. With viscosity present, mass flux q vs time t from (a) (A1)-(A6) in appendix A, shown as dashed curves, (b) (1.3a, b), solid curves, for a single vessel. Area $A = (1 - \Delta x)^2$ with parameter values $\Delta = 0.1, \alpha = 1, L(\text{length}) = H_0 = P$ (pressure drop) = 1. Here $K_R = \Delta^{1/2} \bar{K}_R$ and the K_R value is shown next to each curve. The responses at larger t following the closeness to each other (and to the inviscid solution) of results at K_R values of 0.025, 0.1, 0.4 for $0 < t < 1$ approximately attests to the two time scales holding for small K_R .

typical axial distance x and pressure scale p of $O(1)$. The parameter Δ is small, α is an $O(1)$ constant and K_R is small. Substitution into (1.3a) shows that viscous effects first have significant impact when K_R is $0(\Delta^{1/2})$. The governing equations are

$$\frac{dQ_0}{dT} + \alpha(2 H_0^{-1} Q_0 \frac{\partial Q_1}{\partial x} - H_0^{-2} Q_0^2 \frac{\partial H_1}{\partial x}) + H_0 \frac{\partial p}{\partial x} + \bar{K}_R H_0^{-1} Q_0 = 0, \quad (\text{A.3})$$

$$\frac{dH_0}{dT} + \frac{\partial Q_1}{\partial x} = 0, \quad (\text{A.4})$$

where $K_R = \Delta^{1/2} \bar{K}_R$ with $\bar{K}_R$ of $O(1)$. The second equation here stems from (1.3b). Elimination of Q_{1x} leads to a single equation relating Q_0, p . Following that, the substitution $Q_0 = RS$ results in the equation

$$S \frac{dR}{dT} + (\varphi_2 S^2) R^2 = -H_0 \frac{\partial p}{\partial x} \quad (\text{A.5})$$

relating $R(T), p$. Here the functions $\varphi_2 = -\alpha H_0^{-2} H_{1x}$ and

$$S(T) = H_0^{2\alpha} \exp(-\bar{K}_R \int_0^T H_0^{-1} dT) \quad (\text{A.6})$$

are known if the wall shape is prescribed: see also appendix B. An integration of (A5) with respect to x from inlet to outlet then yields a Riccati equation for the ratio function $R(T)$ consistent with the form (2.3) in the absence of viscous effects together with α of unity.

Solutions of this Riccati equation for various $\bar{K}_R$ values given in figure 12 show two time scales becoming apparent when $\bar{K}_R$ is small. Over the $O(1)$ scale of T for $\bar{K}_R \ll 1$ the factor S is simply $H_0^{2\alpha}(1 - \bar{K}_R H_0^{-1} T + \dots)$ for the example of uniform basic width H_0 , making S constant and so R evolves significantly because of (A5), but over the longer $O(1/\bar{K}_R)$ scale S becomes nontrivial as $H_0^{2\alpha} \exp(-H_0^{-1} \bar{T})$ when $T = \bar{T}/\bar{K}_R$ and new viscous-inviscid evolution is expected. Eventually the nonlinear contribution in (A5) can then peter out, connecting with the more fully developed stage where unsteadiness and pressure gradient dominate. Figure 12 is for the case $\Delta = 0.1$ and also includes a comparison between solutions of (1.3a, b) and (A1)-(A6), which are seen to yield reasonable agreement.

B. Wall flexibility

Interaction between wall shape and fluid pressure is present for multiple branching if the mother vessel wall for example is flexible. We then have the governing equations

$$p_n + \frac{1}{2} u_n^2 = p_0 + \frac{1}{2} u_0^2, \text{ for } n = 1 \text{ to } N, \quad (\text{B.1})$$

$$h_0 u_0 = \sum_{n=1}^N h_n u_n, \quad (\text{B.2})$$

$$L_n \frac{du_n}{dT} + \Gamma_n^* u_n^2 = p_n - \hat{p}_n \text{ for } n = 1 \text{ to } N, \quad (\text{B.3})$$

$$L_0 \frac{du_0}{dT} + \Gamma_0^* u_0^2 = \hat{p}_0 - p_0. \quad (\text{B.4})$$

The pressures $\hat{p}_n$ are prescribed end values for all n and the factors Γ_n^* are known for $n \geq 1$. Also $\Gamma_0^* = (H_0(0, T) - H_0(L_0, T))/h_0$ is now unknown and interacts with the unknown mother pressure in the sense that

$$p_0 - c_0 = -\kappa^2 \Gamma_0^*. \quad (\text{B.5})$$

The constant κ^2 is given and c_0 is a known pressure level. The incident perturbation thickness $H_0(0, T)$ is prescribed whereas the thickness $H_0(L_0, T)$ at the junction responds to local pressure, leading to the relation (B5). A cut-off would be added to the relation at increased pressure values.

If the daughter vessels are uniform spatially then the $\Gamma_n^* = 0$ and manipulation yields a system of $N + 2$ equations for π_0 and all the u_n . This comprises the mass-flux balance in (B1), (B2) combined with

$$L_n \frac{du_n}{dT} = p_0 - \hat{p}_n + \frac{1}{2}(u_0^2 - u_n^2) \text{ for } n = 1 \text{ to } N, \quad (\text{B.6})$$

$$L_0 \frac{du_0}{dT} - (p_0 - c_0)u_0^2/\kappa^2 = \hat{p}_0 - p_0. \quad (\text{B.7})$$

A steady state solution is $u_n = \lambda\kappa$ where $\lambda = (p_0/(p_0 - c_0))^{1/2}$, with $\hat{p}_0 = 0, p_0 = \hat{p}_n$ for $n = 1$ to $N, h_0 = \sum_{n=1}^N h_n$. The linear stability of this may be examined through a small disturbance proportional to $\exp(\beta T)$ and gives rise to the two roots

$$\beta = \frac{(\hat{p}_n - c_0)}{\kappa L_0}, \text{ or } \beta = -\frac{\kappa}{L_0}, \quad (\text{B.8})$$

for β when $N = 2$ with all L_n equal and $h_n = 1/N$. In regular cases $\hat{p}_n < 0, \hat{p}_n < c_0$ and so both roots correspond to stability. In irregular cases the inequalities reverse and give instability. We might expect the same conclusion for general N and for larger networks. These results along with (B5) support the view that regular cases lead to steady states or oscillations and irregular ones to choking or blow-up.

Interaction between wall shape and local shear stress can be modelled by supposing the wall shear stress to be dependent on the local velocity alone, compare the Blasius wall layer of §6 and see [10]. With a flexible-walled mother vessel akin to that earlier in this appendix the interaction takes the form

$$\Gamma_0^* = \hat{\kappa}^{-2} u_0^s. \quad (\text{B.9})$$

The power s is positive, with values $\frac{1}{2}$ and 1 being of particular note. The positive coefficient $\hat{\kappa}^{-2}$ is associated with the perturbed width of the vessel at the branching being enhanced when the wall shear stress increases subject to an inherent cut-off value. Further research would be interesting on the implications of coupling (B9) with the multi-branching system such as (B1)-(B4).

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