THE ALGEBRA OF FUNCTIONS WITH ANTIDOMAIN AND RANGE

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Abstract. We give complete, finite quasiequational axiomatisations for algebras of unary partial functions under the operations of composition, domain, antidomain, range and intersection. This completes the extensive programme of classifying algebras of unary partial functions under combinations of these operations. We also look at the complexity of the equational theories and provide a nondeterministic polynomial upper bound. Finally we look at the problem of finite representability and show that finite algebras can be represented as a collection of functions over a finite base set provided that intersection is not in the signature.

1. Introduction

The abstract algebraic study of partial maps goes back at least to Menger [35] and the subsequent work of Schweizer and Sklar [42, 43, 44, 45]. A large body of work has followed, some of it specifically building on the work of Schweizer and Sklar (such as Schein [38]) but numerous other contributions with independent motivation starting from semigroup theory (where there is a close relationship to ample and weakly ample semigroups; see Hollings [21] for a survey), category theory (where there is a very close connection with restriction categories [7, 8]) and constructions in computer science [26, 27]. Moreover there is a close connection to the more heavily developed algebraic theory of binary relations; see Maddux [31] or Hirsch and Hodkinson [17] in general and articles such as Hollenberg [20] (which also delves into equational properties of partial maps) and Desharnais, Möller and Struth [9], where the development is closer in nature to the theme of applications of the algebra of partial maps. Of course, often the motivation has been across several of these fronts at once, with much of the category-theoretic development focussed toward computer science motivation, and articles such as Jackson and Stokes [22, 27] and Manes [32] attempting in part to provide new links between the various perspectives. We make no attempt at a full survey here. Some further references are given below, but other discussion and history can be found in Schein's early (but already substantial) survey article [39] or [40] and in articles such as [8, 21, 27].

In each of the above approaches, the fundamental operation of composition of partial maps is accompanied by additional operations capturing facets of what it means to be a partial map. Operations modelling the domain of a partial map are particularly ubiquitous in the literature but other frequently occurring operations

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are those modelling range, intersection, fixset, domain-complement or antidomain, as well as programming-specific constructions such as if-then-else and looping.

Perhaps the foremost goal in the development of an abstract approach to partial maps is (finite) axiomatisability: the construction of (finite) systems of axioms that can be proved sound and complete over fragments of the first order theory of systems of partial maps closed under the given operations. In this context, the present article completes a long programme of investigation by giving sound and complete (and finite) axiomatisations for arguably the last two remaining natural families of signatures.

- Theorem 4.1 shows finite axiomatisation for the algebras of partial maps with composition, antidomain and range. Corollary 4.2 extends this to signatures including preferential union and intersection.

This answers the final open problem stated in [27, §5]. As is explained in the introductory sections of [27], the signature consisting of composition, antidomain, range, intersection and preferential union is, in an informal sense at least, the richest natural case: all previously studied operations and relations on semigroups of partial maps can be expressed using terms in this signature.[1] We feel that the the existence of a finite complete axiomatisation for this "master signature" is significant. It is in stark contrast with the world of algebras of binary relations, where the most obvious "master signature" is the Tarski signature. There, the representable algebras admit no finite axiomatisation (Monk [36]), and the finite representable algebras are not even recursive (Hirsch and Hodkinson [16]). Moreover, these properties emerge for even quite weak reducts; see [18] and [37].

By Theorem 4.1 we have that abstract algebras satisfying the axioms can be represented as algebras of functions. A specific problem is whether finite algebras can be represented on finite bases. To this end we revisit Schein's elegant method of representation for the signature consisting of composition, domain and range in Section 5.

- In Theorem 5.1 we show how to adjust Schein's representation so that finite algebras representable in Schein's signature have representations over a finite domain, and in Theorem 5.3 we extend this result to algebras including the operation of antidomain as well.

This gives a positive answer to the first of the two open questions stated in [25] (see §10 there). The corresponding problems for signatures containing intersection are left unanswered here, but Brett McLean has a forthcoming article [33] that will extend our results on finite representability to signatures including intersection too.

A related issue to finite axiomatisability is the complexity of the equational theory.

- Theorem 6.1 shows that the equational theory of these systems (and hence for reducts of these signatures) is in the class co-NP, and is co-NP-complete for those reducts containing composition and antidomain.

We also briefly consider operations modelling looping. In this case we cannot claim completeness. Moreover this is not possible, as for sufficiently rich signatures,

[1]We wish to stress that we do not claim that all possible operations of partial functions can be expressed in terms of composition, antidomain, range, intersection and preferential union, possibly with looping. We only claim that all previously considered operations can be expressed in this way.

it is shown in Goldblatt and Jackson [14] that there is no recursive system of axioms that will be sound and complete for even just the equational properties of partial maps with looping.

The rest of the paper is organized as follows. Section 2 presents motivations for and basic definitions of operations on partial maps. Section 3 introduces various laws that hold true for the algebras of partial maps and surveys some characterisations of representability for various relevant signatures. Aside from soundness of these laws (which are all completely routine and can be found in the cited literature), the only result here that is called on for use in our proofs is Schein's representation for semigroups with domain and range [38] (and its extension to intersection in [25]). We give a brief presentation of this in Section 5, making the present article essentially self-contained. The main result and its proof are given in Section 4. In Section 5 we show that a modification of Schein's representation gives finite representations for finite algebras even for signatures including antidomain (provided that intersection is not included). Our main result then gives a finite representation for finite semigroups with antidomain and range. Finally, in Section 6 we give our results on the complexity of the equational theory of various signatures in the algebra of partial maps.

2. Preliminaries: operations and representability

By a (unary) *function*[2] on a set X, we mean a partial map $f\colon X \to X$. We use $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ to denote the domain and range of f respectively, and often write $(x, y) \in f$ for $f(x) = y$. Domain and range may be recorded as functions by way of unary operations: we define $\mathsf{d}(f)$, the *domain* of f (as a function), as the identity relation restricted to $\mathrm{dom}(f)$

$$\mathsf{d}(f) := \{(x, x) \mid \exists y\ (x, y) \in f\},$$

and $\mathsf{r}(f)$, the *range* of f, as the identity relation restricted to $\mathrm{ran}(f)$

$$\mathsf{r}(f) := \{(y, y) \mid \exists x\ (x, y) \in f\}.$$

Given two functions f and g on X, their *composition* $f\,;g$ is defined as

$$f\,;g := \{(x, y) \mid \exists z((x, z) \in f \ \&\ (z, y) \in g)\},$$

that is, $(f\,;g)(x) = g(f(x))$, while $f \cdot g$ denotes their *intersection*

$$f \cdot g := \{(x, y) \mid (x, y) \in f \ \&\ (x, y) \in g\}.$$

Observe that if f and g are functions on X then so are $\mathsf{d}(f), \mathsf{r}(f), f\,;g$ and $f \cdot g$. These definitions also apply to general binary relations on X.

Many of the motivations mentioned in the Introduction give rise to other operations of importance. For example, the domain operation d models the modal possibility operator of dynamic logic: if r is any binary relation on the set X, and id_S denotes the identity on some subset $S \subseteq X$ then the modal possibility relation $\langle r\rangle$ over r maps S to the set $\{x \in X \mid (\exists y \in S)(x, y) \in r\}$, which is the set of points fixed by $\mathsf{d}(r\,;\mathrm{id}_S)$. Obviously, $\mathsf{d}(r)$ itself coincides with the restriction of the identity to $\langle r\rangle\mathtt{true}$. The domain operation alone cannot express negation or

[2]We prefer the briefer "function" here to "partial function", a convention with long tradition in the abstract treatment of partial maps [39, 42], and which is also consistent with other reasonably standard conventions concerning real functions of real variables (such as `sqrt`, whose domain is a proper subset of the reals). A function defined on all of X is a *total* function.

modal necessity, but this can be achieved using "antidomain". The *antidomain* of $f\colon X \to X$ is the function

$$\mathsf{a}(f) := \{(x,x) \mid x \in X \ \&\ \forall y\ (x,y) \notin f\}.$$

(Again, this definition works for any binaray relation, not just for functions.) Note that X occurs as a parameter in the definition of antidomain just like the top element occurs in the definition of complement in Boolean set algebras. Observe that $\mathsf{d}(f)$ can be defined using a as $\mathsf{d}(f) = \mathsf{a}(\mathsf{a}(f))$, while the empty function 0 arises as $0 = \mathsf{a}(x) \,;\, x$ (any x) and the identity relation

$$1' := \{(x,x) \mid x \in X\}$$

arises as $1' = \mathsf{a}(0)$. When combined with composition, a enables a full modelling of modal logics over a semigroup of relations. Indeed, for any subset $S \subseteq X$, the modal necessity operator $[r]$ over r maps S to the set $\{x \in X \mid (\forall y \in X)(x,y) \in r \Rightarrow y \in S\}$,which coincides with the set of points fixed by $\mathsf{a}(r \,;\, \mathsf{a}(\mathrm{id}_S))$, while complementation is modelled because $X \backslash S$ is the set of points fixed by $\mathsf{a}(\mathrm{id}_S)$.Conversely, the value of $\mathsf{a}(r)$ itself arises as the identity on $[r]\mathtt{false}$. Thus algebras of binary relations with composition and a coincide exactly to the basic compositional fragment of propositional dynamic logic, with the functional case corresponding to the deterministic fragment of this. The reader is referred to a text such as [3] or [15] for a more detailed treatment of dynamic logic, or to articles such as Hollenberg [20] (where antidomain is called dynamic negation), Desharnais, Möller and Struth [9], Desharnais, Jipsen and Struth [10] or Jackson and Stokes [27] for algebraic modelling of dynamic logics via antidomain.

As is explained in [27], many natural operations can be written in terms of the signature $\{;, \cdot, \mathsf{a}, \mathsf{r}\}$. For example, the *fixset* operation

$$\mathsf{fix}(f) := \{(x,x) \mid (x,x) \in f\}$$

can also be defined using $\cdot$ and d as $\mathsf{fix}(f) = \mathsf{d}(f) \cdot f$, or using $\cdot$ and ; by $\mathsf{fix}(f) = f \cdot (f \,;\, f)$. The fixset operation is considered in Goldblatt and Jackson [14] for example, where it is identified as a trigger for an explosion in the computational complexity of fragments of dynamic logic. However, two additional operations that cannot be expressed by $\{;, \cdot, \mathsf{a}, \mathsf{r}\}$ are the *preferential union* operation $\sqcup$ and the *maximum iterate* operation $^{\uparrow}$. The preferential union $f \sqcup g$ of f with g is defined as

$$(f \sqcup g)(x) := \begin{cases} f(x) & \text{if } f(x) \text{ is defined,} \\ g(x) & \text{otherwise.} \end{cases}$$

In other words it is $f(x) \cup (\mathsf{a}(f) \,;\, g(x))$, a union which always returns a function on functional arguments. The signature $\{;, \mathsf{a}, \sqcup\}$ also arises in the work of Cockett and Manes, as $\mathsf{Fun}(;, \mathsf{a}, \sqcup)$ coincides with the one-object classical restriction categories of [8]; see the introduction and Section 2.3.2 of [27] for discussion of this connection. Preferential union can express *if-then-else* statements: $\mathtt{if}\ f\ \mathtt{then}\ g\ \mathtt{else}\ \ h = \mathsf{d}(f) \,;\, g \ \sqcup\ \mathsf{a}(f) \,;\, h$ and in fact $f \sqcup g$ coincides with $\mathtt{if}\ \mathsf{d}(f)\ \mathtt{then}\ f\ \mathtt{else}\ g$, so is identical to the override operation of Berendsen et al. [2]. Similarly, the update operation of [2] has $f\ \mathtt{update}\ g$ given by $\mathtt{if}\ \mathsf{d}(f) \,;\, \mathsf{d}(g)\ \mathtt{then}\ g\ \mathtt{else}\ f$, so that update is a derived term, given $;, \sqcup, \mathsf{a}$. Other variants of union are discussed in [27, §2.3.2].

The semantics of the *maximum iterate* operation are given by

$$f^{\uparrow} = \bigsqcup_{n<\omega} f^n \,;\, \mathsf{a}(f).$$

This operation can express while statements: $\mathtt{while}(d)p = (\mathsf{d}(d)\,;p)^{\uparrow}\,;\mathsf{a}(d)$. See [27] for more on this.

Definition 2.1. *Let* $\tau \subseteq \{;, \cdot, \mathsf{d}, \mathsf{r}, \mathsf{a}, \mathsf{fix}, \sqcup, ^{\uparrow}, 0, 1'\}$ *be a similarity type. A* τ-algebra of functions *is a family of functions on some set* X*, the* base *of the algebra, augmented with the operations in* τ *as defined above* (*using* X *as the parameter*)*. We will denote by* $\mathsf{Fun}(\tau)$ *the class of* τ*-algebras of functions. When an abstract* τ*-algebra* $\mathcal{S}$ *is isomorphic to an element of* $\mathsf{Fun}(\tau)$*, we say that* $\mathcal{S}$ *is* (functionally) representable*.*

We will consider only signatures containing ; and usually omit explicit mention of d if a is present, and fix if $\cdot$ is present.

Functional representability has been considered for many subsignatures τ of $\{;, \cdot, \mathsf{d}, \mathsf{r}, \mathsf{a}, \mathsf{fix}, \sqcup, ^{\uparrow}, 0, 1'\}$. The signature $\{;, \mathsf{d}, \mathsf{r}\}$ is one of the most obvious signatures and not surprisingly was one of the earliest signatures to receive serious attention through a series of articles by Schweizer and Sklar [42, 43, 44, 45]. No complete axiomatisation for representability in this signature was found until the work of Schein [38], who gave a complete, finite quasiequational axiomatisation and a proof that the class is not a variety. When intersection is not present in τ, the article [28] provides a table describing finite axiomatisability of $\mathsf{Fun}(\tau)$ and describing whether the class is a variety or a quasivariety. Although the class of functionally representable algebras for the signature $\{;, \mathsf{d}\}$ is a variety [46], all other functional representation classes for signatures without intersection form proper quasivarieties. For the relatively weak signatures τ with $\{;\} \subsetneq \tau \subseteq \{;, \mathsf{r}, \mathsf{fix}\}$, no finite axiomatisation is possible [28], but all remaining cases have known complete finite axiomatisations except for the strongest of the signatures, $\{;, \mathsf{a}, \mathsf{r}\}$. In the present article we will give a complete finite quasiequational axiomatisation for the signature $\{;, \mathsf{a}, \mathsf{r}\}$.

Signatures involving $\cdot$ are discussed in depth in the introductory sections of [27]. Again, aside from the somewhat artificial case of $\{;, \cdot, \mathsf{r}\}$, all cases have known axiomatisations except for $\{;, \cdot, \mathsf{a}, \mathsf{r}\}$. In the present article we give a finite equational axiomatisation characterising functional representability for the signature $\{;, \cdot, \mathsf{a}, \mathsf{r}\}$. This result and the characterisation of representability in the signature $\{;, \mathsf{a}, \mathsf{r}\}$ solve problems posed in the final subsection of [27].

Schein's elegant representation gives only infinite representations for finite $\{;, \mathsf{d}, \mathsf{r}\}$-algebras. In [25] it was shown that this representation also preserves $\cdot$ and fix when they satisfy appropriate (sound) axioms. Schein's method of representation is invoked at one stage of our own representation, and we use a modification to show that for many signatures $\{;, \mathsf{d}, \mathsf{r}\} \subseteq \tau \subseteq \{;, \mathsf{d}, \mathsf{a}, \mathsf{r}, \mathsf{fix}, \sqcup, ^{\uparrow}, 0, 1'\}$ there are finite representations for finite τ-algebras, see Theorem 5.3. This solves the first question in [25, §10]. Cases where $\{;, \cdot, \mathsf{a}, \mathsf{r}\} \subseteq \tau$ will be covered in a forthcoming paper by Brett McLean [33].

Our results for the signatures $\{;, \mathsf{a}, \mathsf{r}\}$ and $\{;, \cdot, \mathsf{a}, \mathsf{r}\}$ are extended to include preferential union, thus subsuming the axiomatisation in [2] (the axioms for the weaker signatures considered in [27] also subsume those of [2]). Our axiomatisability results extend to signatures including $^{\uparrow}$ if we restrict ourselves to finite algebras.

We also look at the computational complexity of the equational theories and prove that the validity problem is in co-NP in all cases when $\tau \subseteq \{;, \cdot, \sqcup, \mathsf{d}, \mathsf{a}, \mathsf{r}, \mathsf{fix}, 0, 1'\}$, and it is co-NP-complete provided $\{;, \mathsf{a}\} \subseteq \tau$. It follows from [14] that the validity problem is Π^1_1-hard when $\{;, \mathsf{a}, \mathsf{fix}, \uparrow\} \subseteq \tau$.

3. Known axiomatisations

In this section we recall from the literature known axiomatizations for classes $\mathsf{Fun}(\tau)$ where $\tau \subseteq \{;, \cdot, \mathsf{d}, \mathsf{r}, \mathsf{a}, \mathsf{fix}, \sqcup, {}^\uparrow, 0, 1'\}$. Reference to these existing results simplifies the statement of our main results, however the only unobvious facts that we call on directly in our proof is the known complete axiomatisations for $\tau = \{;, \mathsf{d}, \mathsf{r}\}$ and $\tau = \{;, \cdot, \mathsf{d}, \mathsf{r}\}$. Even for these cases, we give an overview of the proof method in Section 5, where we identify a new refinement for the case $\tau = \{;, \mathsf{d}, \mathsf{r}\}$.

We introduce some notations and conventions. We will assume throughout that $; \in \tau$ and that d is available as well by either $\mathsf{d} \in \tau$ or $\mathsf{a} \in \tau$ (in which case we define $\mathsf{d}(x) := \mathsf{a}(\mathsf{a}(x))$). Given a (not necessarily representable) τ-algebra $\mathcal{S} = (S, \tau)$, we define the set $D(S)$ of *domain elements* as

$$D(S) := \{s \in S \mid \mathsf{d}(s) = s\}.$$

Lower case Greek letters $\alpha, \beta, \delta, \gamma, \dots$ will denote domain elements.

Next we list some (quasi)equations that are known to be valid in representable algebras. Associativity of ; is assumed throughout, though the reader will be reminded of this in some technical equational deductions. We reserve labelled Roman numerals for axioms, and use standard Arabic numbering for general laws that are consequences of the axioms.

3.1. **Domain and meet.** We start with the *domain axioms*:

$$\mathsf{d}(x) \,;\, x = x, \tag{d.I}$$

$$\mathsf{d}(x) \,;\, \mathsf{d}(y) = \mathsf{d}(y) \,;\, \mathsf{d}(x), \tag{d.II}$$

$$\mathsf{d}(\mathsf{d}(x)) = \mathsf{d}(x), \tag{d.III}$$

$$x \,;\, \mathsf{d}(y) = \mathsf{d}(x \,;\, y) \,;\, x, \tag{d.IV}$$

$$\mathsf{d}(x) \,;\, \mathsf{d}(x \,;\, y) = \mathsf{d}(x \,;\, y), \tag{d.V}$$

and some of their consequences:

$$\mathsf{d}(x) \,;\, \mathsf{d}(x) = \mathsf{d}(x), \tag{3.1}$$

$$\mathsf{d}(x) \,;\, \mathsf{d}(y) = \mathsf{d}(\mathsf{d}(x) \,;\, y), \tag{3.2}$$

$$\mathsf{d}(x \,;\, \mathsf{d}(y)) = \mathsf{d}(x \,;\, y). \tag{3.3}$$

Associative $\{;, \mathsf{d}\}$-algebras obeying axioms (d.I)–(d.V) have been given many names [1, 7, 12, 22, 32, 46] but the name *restriction semigroup* (sometimes, one-sided restriction semigroup) has emerged as the modern standard. We mention that restriction semigroups also have a number of other known axiomatisations. We refer the reader to Desharnais, Jipsen and Struth [10] for discussion of this; a number of other consequences of the given laws are recalled later in the present article.

Besides the semilattice axioms for $\cdot$ we will need the following *meet axioms*:

$$x \mathbin{;} (y \cdot z) = (x \mathbin{;} y) \cdot (x \mathbin{;} z), \tag{m.I}$$

$$x \cdot y = \mathsf{d}(x \cdot y) \mathbin{;} x. \tag{m.II}$$

Theorem 3.1. (1) (*Trokhimenko* [46]; *see also* [22] *or* [32] *for Cayley representation.*) *The class* $\mathsf{Fun}(;, \mathsf{d})$ *is the class of restriction semigroups, that is, it is finitely axiomatised by associativity and* (d.I)–(d.V).

(2) (*Dudek and Trokhimenko* [11], *Jackson and Stokes* [23].) *The class* $\mathsf{Fun}(;, \cdot, \mathsf{d})$ *is finitely axiomatised by associativity,* (d.I)–(d.V)*, the semilattice axioms for* $\cdot$ *and* (m.I)*,* (m.II).

3.2. **Antidomain.** We define 0 as $\mathsf{a}(x) \mathbin{;} x$ (any x), see (a.I) below, and $1'$ by $\mathsf{a}(0)$. We have the following *antidomain axioms*:

$$\mathsf{a}(x) \mathbin{;} x = \mathsf{a}(y) \mathbin{;} y, \tag{a.I}$$

$$x \mathbin{;} \mathsf{a}(y) = \mathsf{a}(x \mathbin{;} y) \mathbin{;} x, \tag{a.II}$$

$$\alpha \mathbin{;} x = \alpha \mathbin{;} y \ \ \& \ \ \mathsf{a}(\alpha) \mathbin{;} x = \mathsf{a}(\alpha) \mathbin{;} y \ \Rightarrow \ x = y. \tag{a.III}$$

$$1' \mathbin{;} y = y, \tag{a.IV}$$

$$0 \mathbin{;} y = 0, \tag{a.V}$$

An associative $\{;, \mathsf{a}\}$-algebra satisfying (a.I)–(a.V) is called a *modal restriction semigroup* [27]. (Note that the law $x;0 = 0$ assumed in [27] follows from $x;0 = x;\mathsf{a}(x);x = \mathsf{a}(x \mathbin{;} x) \mathbin{;} x \mathbin{;} x = 0$, and then $x \mathbin{;} 1' = x \mathbin{;} \mathsf{a}(0) = \mathsf{a}(x \mathbin{;} 0) \mathbin{;} x = \mathsf{a}(0) \mathbin{;} x = x$.) Some consequences deduced in [27] include:

$$\mathsf{a}(x) \mathbin{;} \mathsf{a}(y) = \mathsf{a}(y) \mathbin{;} \mathsf{a}(x), \tag{3.4}$$

$$\mathsf{a}(x) \mathbin{;} \mathsf{a}(y) = \mathsf{a}(x) \mathbin{;} \mathsf{a}(\mathsf{a}(x) \mathbin{;} y), \tag{3.5}$$

$$\alpha \mathbin{;} x = \alpha \mathbin{;} y \ \ \& \ \ \beta \mathbin{;} x = \beta \mathbin{;} y \ \Rightarrow \ (\alpha \vee \beta) \mathbin{;} x = (\alpha \vee \beta) \mathbin{;} y, \tag{3.6}$$

$$\mathsf{a}(\alpha \mathbin{;} x) \mathbin{;} \mathsf{a}(\beta \mathbin{;} x) = \mathsf{a}((\alpha \vee \beta) \mathbin{;} x), \tag{3.7}$$

where $\alpha \vee \beta := \mathsf{a}(\mathsf{a}(\alpha) \mathbin{;} \mathsf{a}(\beta))$ for domain elements α and β.

Theorem 3.2. (*Jackson and Stokes* [27].)

(1) *The class* $\mathsf{Fun}(;, \mathsf{a})$ *is the class of modal restriction semigroups, that is, it is finitely axiomatised by associativity and* (a.I)–(a.V).

(2) *The class* $\mathsf{Fun}(;, \cdot, \mathsf{a})$ *is finitely axiomatised by associativity, the semilattice axioms for* $\cdot$, (m.I) *and* (m.II) (*where* $\mathsf{d} := \mathsf{aa}$)*, and the antidomain axioms* (a.I)–(a.V).

Proof. The proof is essentially covered in [27] but the signatures used there are slightly different, so some translation is needed. For each part, the axioms are easily verified in $\mathsf{Fun}(;, \mathsf{a})$ and $\mathsf{Fun}(;, \cdot, \mathsf{a})$, respectively. Conversely, for the first part, let $(S, ;, \mathsf{a})$ be any associative algebra satisfying (a.I)–(a.V). Axioms (a.I)–(a.V) prove that $0, 1'$ have the usual multiplicative properties and so $(S, ;, \mathsf{a}, 0, 1')$ satisfies the conditions of [27, Definition 3], hence by [27, Theorem 4] it is isomorphic to a member of $\mathsf{Fun}(;, \mathsf{a})$, as required. For the second part, let $(S, ;, \cdot, \mathsf{a})$ satisfy the stated axioms. Define a binary operation $\bowtie$ by $x \bowtie y = \mathsf{d}(x \cdot y) \vee (\mathsf{a}(x) \cdot \mathsf{a}(y))$. The intended meaning of $x \bowtie y$ is the identity function over the points where x and y do not disagree. It is not hard to check that $(S, ;, \bowtie, 0)$ satisfies [27, Definition 19], hence by [27, Theorem 20] it is isomorphic to an algebra of functions where $f \bowtie g$ is the identity restricted to the points where f and g either agree or are both undefined. From this we can recover a representation of $(S, ;, \cdot, \mathsf{a})$ using $f \cdot g = (f \bowtie g) ; f$ and $\mathsf{a}(f) = 0 \bowtie f$. □

The axioms in [27, Definition 19] are equational (albeit expressed in terms of $\bowtie$), so that [27, Theorem 20] shows that $\mathsf{Fun}(;, \cdot, \mathsf{a})$ is a finitely based variety. In Lemma 3.7 below we give a simple law in the signature $\{;, \cdot, \mathsf{a}\}$ that can replace the one implicational law (a.III) in the axioms stated in Theorem 3.2(2).

It is shown in [27] that if we let d denote aa then each of the domain axioms (d.I)–(d.V) hold, and that the set of elements fixed by d includes 0 and $1'$ and forms a Boolean algebra, where the meet of $\mathsf{d}(x)$ and $\mathsf{d}(y)$ is given by $\mathsf{d}(x) ; \mathsf{d}(y)$ and the complement of $\mathsf{d}(x)$ is $\mathsf{a}(x)$ (or equivalently, $\mathsf{a}(\mathsf{d}(x))$, as the law $\mathsf{a}(\mathsf{a}(\mathsf{a}(x))) = \mathsf{a}(x)$ also follows).

3.3. **Range.** We list the following *range axioms*:

(r.I) $\mathsf{d}(\mathsf{r}(x)) = \mathsf{r}(x)$,

(r.II) $x ; \mathsf{r}(x) = x$,

(r.III) $x ; y = x ; z \;\Rightarrow\; \mathsf{r}(x) ; y = \mathsf{r}(x) ; z$,

(r.IV) $\mathsf{r}(x) ; \mathsf{r}(y) = \mathsf{r}(y) ; \mathsf{r}(x)$,

(r.V) $\mathsf{r}(\mathsf{r}(x)) = \mathsf{r}(x)$,

(r.VI) $\mathsf{r}(x ; y) ; \mathsf{r}(y) = \mathsf{r}(x ; y)$,

(r.VII) $\mathsf{r}(\mathsf{r}(x) ; y) = \mathsf{r}(x ; y)$,

(r.VIII) $\mathsf{r}(\mathsf{d}(x)) = \mathsf{d}(x)$.

Theorem 3.3. (1) (*Schein* [38].) *The class* $\mathsf{Fun}(;, \mathsf{d}, \mathsf{r})$ *is finitely axiomatised by associativity,* (d.I)–(d.V) *and* (r.I)–(r.VIII).

(2) (*Jackson and Stokes* [25].) *The class* $\mathsf{Fun}(;, \cdot, \mathsf{d}, \mathsf{r})$ *is finitely axiomatised by associativity,* (d.I)–(d.V)*, the semilattice axioms for* $\cdot$ *and* (m.I) (m.II)*, the range axioms* (r.I)–(r.VIII).

If 0 *is added to the signature then the laws* $0 ; x = x ; 0 = \mathsf{d}(0) = 0$ *ensure that* 0 *can be correctly represented as well.*

Law (r.III) can be omitted from Theorem 3.3(2) because it is shown in [25, Lemma 9.8] to be redundant in the presence of the other axioms for the signature $\{;, \cdot, \mathsf{d}, \mathsf{r}\}$. In particular, $\mathsf{Fun}(;, \cdot, \mathsf{d}, \mathsf{r})$ is a variety.

We note that Theorem 3.3 has also been developed in a category-theoretic setting by Cockett, Guo and Hofstra [5, 6].

In the next section we will give finite axiomatisations of the classes $\mathsf{Fun}(;, \mathsf{a}, \mathsf{r})$ and $\mathsf{Fun}(;, \cdot, \mathsf{a}, \mathsf{r})$ using suitable selections from the above axioms.

3.4. **Preferential union and maximal iterate.** Finally we consider axioms for preferential union and maximal iterate. We have the following two laws

($\sqcup$.I) $\mathsf{d}(x) ; (x \sqcup y) = x$,

($\sqcup$.II) $\mathsf{a}(x) ; (x \sqcup y) = \mathsf{a}(x) ; y$.

As mentioned in the introduction, iteration is far more elusive. We list some axioms and Theorem 3.4 will state the extent to which they are known to be complete:

$$(\uparrow.\mathrm{I}) \qquad \mathsf{d}(x) \mathbin{;} x^{\uparrow} = x \mathbin{;} x^{\uparrow} \quad \text{and} \quad \mathsf{a}(x) \mathbin{;} x^{\uparrow} = \mathsf{a}(x),$$

$$(\uparrow.\mathrm{II}) \qquad \mathsf{d}(x) \mathbin{;} y = \mathsf{d}(x) \mathbin{;} y \mathbin{;} \mathsf{d}(x) \;\Rightarrow\; \mathsf{d}(x) \mathbin{;} y^{\uparrow} = \mathsf{d}(x) \mathbin{;} y^{\uparrow} \mathbin{;} \mathsf{d}(x).$$

Theorem 3.4.
1. *(Jackson and Stokes [27, §3.3].) The class* $\mathsf{Fun}(;, \mathsf{a}, \sqcup)$ *is characterised by the laws characterising the class* $\mathsf{Fun}(;, \mathsf{a})$ *together with laws* ($\sqcup$.I) *and* ($\sqcup$.II)*. Moreover, when laws* ($\sqcup$.I) *and* ($\sqcup$.II) *hold, every representation in the reduct signature* $\{;, \mathsf{a}\}$ *correctly represents* $\sqcup$.
2. *(Jackson and Stokes [27].) The finite members of* $\mathsf{Fun}(;, \mathsf{a}, \sqcup, {}^{\uparrow})$ *are characterised by the set of laws characterising the signature* $\{;, \mathsf{a}, \sqcup\}$ *together with laws* ($\uparrow$.I) *and* ($\uparrow$.II)*. Moreover, when laws* ($\uparrow$.I) *and* ($\uparrow$.II) *hold, every representation in the reduct signature* $\{;, \mathsf{a}, \sqcup\}$ *correctly represents* ${}^{\uparrow}$ *for finite algebras.*

More is true: in part (1), the implication (a.III) used in the characterisation of $\mathsf{Fun}(;, \mathsf{a})$ becomes redundant [27, Proposition 15], while in part (2), the implication ($\uparrow$.II) can be replaced by the less intuitive equational law

$$\mathsf{d}(x) \mathbin{;} \mathsf{a}(y \mathbin{;} \mathsf{a}(x)) \mathbin{;} (y \mathbin{;} \mathsf{a}(\mathsf{d}(x) \mathbin{;} y \mathbin{;} \mathsf{a}(x))^{\uparrow} \mathbin{;} \mathsf{a}(x) = 0,$$

see [27, Proposition 29]. Thus in both cases one can obtain purely equational axioms, although for the latter case it does not follow (and is not true) that $\mathsf{Fun}(;, \mathsf{a}, \sqcup, {}^{\uparrow})$ is a variety, since the equational characterisation applies only to finite algebras.

3.5. **Dependencies in the axiomatisations.** In some of the axiomatisations that we recalled above there are some redundancies. Now we make some efforts at presenting suitable axiom systems without redundancies, though we do not consider this to be an important feature of our presentation and the reader may skip the remainder of this section. Justifying the completeness of small systems of axioms requires us to call on a number of existing results in the literature and in some cases the automated theorem prover Prover9 [34].

Starting with the signature $\{;, \mathsf{d}, \cdot\}$, we note that law (d.I) is in fact redundant in the presence of idempotence of $\cdot$ and (m.II): $\mathsf{d}(x) \mathbin{;} x = \mathsf{d}(x \cdot x) \mathbin{;} x \overset{(\mathsf{m}.\mathrm{II})}{=} x \cdot x = x$. Perhaps surprisingly, Prover9 finds that associativity of $\cdot$ is also redundant in Theorem 3.1(2); we omit the proof, though it is quite easily followed by a human. Mace4 finds that the remaining laws (associativity of ;, idempotence and commutativity of $\cdot$, and laws (d.II)–(d.V), (m.I), (m.II)) have no further redundancies. An equivalent irredundant axiomatisation with one fewer axiom can be obtained by replacing (d.II)–(d.V), with (d.II), (d.IV) and (3.2).

Considering antidomain we have the following dependencies, which in particular show that (a.V) is redundant.

Lemma 3.5. *Let* $(S, ;, \mathsf{a})$ *be a semigroup with respect to* ; *and satisfy* (a.I) *and* (a.II)*, with* 0 *denoting* $\mathsf{a}(a) ; a$ *(for any* $a \in A$*), and* $1'$ *denoting* $\mathsf{a}(0)$*. Then law* (3.8) *holds, and if* (a.IV) *also holds then* (3.9) *holds*:

$$0 \mathbin{;} x = 0 = x \mathbin{;} 0, \tag{3.8}$$

$$1' \mathbin{;} x = x = x \mathbin{;} 1'. \tag{3.9}$$

Proof. First note that $x\,;0 \overset{(\mathsf{a}.\mathrm{I})}{=} x\,;\mathsf{a}(x)\,;x \overset{(\mathsf{a}.\mathrm{II})}{=} \mathsf{a}(x\,;x)\,;x\,;x \overset{(\mathsf{a}.\mathrm{I})}{=} 0$. Then the derived law $y\,;0=0$ gives $0\,;x=\mathsf{a}(0\,;x)\,;0\,;x \overset{(\mathsf{a}.\mathrm{I})}{=} 0$, with the first equality using a right-to-left application of the property $x\,;0=0$. So (3.8) is verified. Next, $x\,;1'=x\,;\mathsf{a}(0) \overset{(\mathsf{a}.\mathrm{II})}{=} \mathsf{a}(x\,;0)\,;0 \overset{(3.8)}{=} \mathsf{a}(0)\,;x \overset{(\mathsf{a}.\mathrm{IV})}{=} x$. □

The irredundancy of laws (a.I)–(a.IV) is established by hand in [27], but is also easily established using Mace4. In the $\{;,\cdot,\mathsf{a}\}$ signature, it turns out that axiom (a.IV) and (a.I) are equivalent in the presence of associativity of ;, the usual semilattice laws for $\cdot$ (idempotence, commutativity and associativity) and the remaining axioms from (a.I)–(a.IV), (m.I) and (m.II). Both deductions admit relatively straightforward human proofs, but we omit the details, as we show in Lemma 4.10 below that the law (a.IV) is already redundant once range information is included. Prover9 also finds that associativity of $\cdot$ is a consequence of associativity of ;, idempotence and commutativity of $\cdot$ along with laws (a.I)–(a.III), (m.I) and (m.II). This proof does not seem to be short and we did not attempt to find a human proof. Mace4 shows that the remaining eight laws have no further redundancies.

Considering range we have the following dependencies.

Lemma 3.6. *Laws* (r.IV)–(r.VIII) *are a consequence of* (d.I)–(d.IV) *and* (r.I)–(r.III) (*and associativity of* ;).

Proof. Once we show law (r.VIII) and (r.VII) we have the axiomatisation given in [39] and [10], from which all the other laws are well-established consequences (even without (r.III)—see [45] for example). To prove (r.VIII) and (r.VII) we will need (r.IV) and the following two laws

$$\mathsf{r}(x)\,;\mathsf{r}(x)=\mathsf{r}(x). \tag{3.10}$$

$$u\,;v=u \Rightarrow \mathsf{r}(u)\,;v=\mathsf{r}(u) \tag{3.11}$$

Law (r.IV) follows almost immediately from (r.I) and (d.II). Law (3.10) similarly follows almost immediately from (r.I) and (3.1) (a known and easy consequence of (d.I)–(d.IV)). Now for law (3.11). If $u=u\,;v$ then $u\,;\mathsf{r}(u)=u\,;v$ by (r.II), which by (r.III) then implies $\mathsf{r}(u)\,;\mathsf{r}(u)=\mathsf{r}(u)\,;v$. But $\mathsf{r}(u)\,;\mathsf{r}(u) \overset{(3.10)}{=} \mathsf{r}(u)$, as concluded in (3.11).

Now for law (r.VIII). We have $\mathsf{d}(x) \overset{(\mathrm{r.II})}{=} \mathsf{d}(x)\,;\mathsf{r}(\mathsf{d}(x))$, which after applications of (r.I) and (d.II) gives

$$\mathsf{d}(x)=\mathsf{r}(\mathsf{d}(x))\,;\mathsf{d}(x). \tag{3.12}$$

Next, $\mathsf{d}(x)\,;\mathsf{d}(x) \overset{(3.1)}{=} \mathsf{d}(x)$, and applying (3.11) to this gives $\mathsf{r}(\mathsf{d}(x))\,;\mathsf{d}(x)=\mathsf{r}(\mathsf{d}(x))$. Combined with (3.12) we have law (r.VIII).

Now for (r.VII). We have $x;y;\mathsf{r}(\mathsf{r}(x);y) \overset{(\mathrm{r.II})}{=} x;\mathsf{r}(x);y;\mathsf{r}(\mathsf{r}(x);y) \overset{(\mathrm{r.II})}{=} x;\mathsf{r}(x);y \overset{(\mathrm{r.II})}{=} x\,;y$. Applying (3.11) to the equality $x\,;y\,;\mathsf{r}(\mathsf{r}(x)\,;y)=x\,;y$ gives

$$\mathsf{r}(x\,;y)\,;\mathsf{r}(\mathsf{r}(x)\,;y)=\mathsf{r}(x\,;y). \tag{3.13}$$

Next, $x;y \overset{(\mathrm{r.II})}{=} x;y;\mathsf{r}(x;y)$, so an application of (r.III) gives $\mathsf{r}(x);y=\mathsf{r}(x);y;\mathsf{r}(x;y)$. Then applying (3.11) to this equality gives

$$\mathsf{r}(\mathsf{r}(x)\,;y)=\mathsf{r}(\mathsf{r}(x)\,;y)\,;\mathsf{r}(x\,;y). \tag{3.14}$$

Equations (3.13) and (3.14) give the desired law (r.VII) after one application of (r.IV). ☐

The laws (d.I)–(d.IV), (r.I)–(r.III) (and associativity of ;) are still not quite without redundancy. Law (d.I) follows quickly from (d.IV), (r.I) and (r.II) as follows: $\mathsf{d}(x) ; x \overset{(\mathsf{r}.\mathrm{II})}{=} \mathsf{d}(x ; \mathsf{r}(x)) ; x \overset{(\mathsf{d}.\mathrm{IV})}{=} x ; \mathsf{d}(\mathsf{r}(x)) \overset{(\mathsf{r}.\mathrm{I})}{=} x ; \mathsf{r}(x) \overset{(\mathsf{r}.\mathrm{II})}{=} x$. Prover9 finds that Law (d.III) turns out to be a further redundancy, though the proof is omitted as it is a little longer. Mace4 easily finds counterexamples to demonstrate that all proper subsets of the laws (d.II), (d.V), (d.IV), (r.I)–(r.III) are strictly weaker, so that this set is an irredundant axiomatisation of $\mathsf{Fun}(;, \mathsf{d}, \mathsf{r})$. With Prover9 we found that an equivalent axiomatisation can be obtained by replacing (d.V) by (3.2).

Finally, we make the following observation, which gives a relatively self contained proof that the class $\mathsf{Fun}(;, \cdot, \mathsf{a})$ is a variety.

Lemma 3.7. *The following equational law can replace the implicational axiom* (a.III) *in Theorem 3.2(2) to form an equivalent equational system of axioms*:

$$\text{(a.VI)} \qquad \mathsf{a}(x) = \mathsf{a}(\mathsf{a}(\alpha) ; x) ; \mathsf{a}(\alpha ; x)$$

Proof. The law (a.VI) is obviously sound (it is an easy consequence of (3.7) for example), so by Theorem 3.2(2) is a consequence of the laws in Theorem 3.2(2). Now we need to show that (a.III) is a consequence of these axioms, but with (a.VI) replacing (a.III).

Now assume that $\alpha ; x = \alpha ; y$ and $\mathsf{a}(\alpha) ; x = \mathsf{a}(\alpha) ; y$ (the premise of (a.III)). So by applying idempotence of $\cdot$ we have $\alpha ; x = (\alpha ; x) \cdot (\alpha ; y) \overset{(\mathsf{m}.\mathrm{I})}{=} \alpha ; (x \cdot y)$ and $\mathsf{a}(\alpha) ; x = (\mathsf{a}(\alpha) ; x) \cdot (\mathsf{a}(\alpha) ; y) \overset{(\mathsf{m}.\mathrm{I})}{=} \mathsf{a}(\alpha) ; (x \cdot y)$. Now $0 \overset{(\mathsf{a}.\mathrm{I})}{=} \mathsf{a}(x) ; x \overset{(\mathsf{a}.\mathrm{VI})}{=} \mathsf{a}(\mathsf{a}(\alpha) ; x) ; \mathsf{a}(\alpha ; x) ; x$, and using $\alpha ; x = \alpha ; (x \cdot y)$ and $\mathsf{a}(\alpha) ; x = \mathsf{a}(\alpha) ; (x \cdot y)$ we obtain $0 = \mathsf{a}(\mathsf{a}(\alpha) ; (x \cdot y)) ; \mathsf{a}(\alpha ; (x \cdot y)) ; x \overset{(\mathsf{a}.\mathrm{VI})}{=} \mathsf{a}(x \cdot y) ; x$.

Next,

$$\begin{aligned}\mathsf{a}(x \cdot y) &\overset{(\mathsf{m}.\mathrm{II})}{=} \mathsf{a}(\mathsf{d}(x \cdot y) ; x) \overset{(3.9)}{=} \mathsf{a}(\mathsf{d}(x \cdot y) ; x) ; \mathsf{a}(0) \\ &= \mathsf{a}(\mathsf{d}(x \cdot y) ; x) ; \mathsf{a}(\mathsf{a}(x \cdot y) ; x) \overset{(\mathsf{a}.\mathrm{VI})}{=} \mathsf{a}(x).\end{aligned}$$

So $\mathsf{d}(x \cdot y) = \mathsf{d}(x)$ and using this fact we have $x \cdot y \overset{(\mathsf{m}.\mathrm{II})}{=} \mathsf{d}(x \cdot y) \cdot x = \mathsf{d}(x) ; x = \mathsf{d}(x \cdot x) ; x \overset{(\mathsf{m}.\mathrm{II})}{=} x \cdot x = x$. By symmetry we have $x \cdot y = y$ also, giving $x = y$ as required. ☐

The law (3.7) can also be used in place of (a.VI), however our proof was longer than the one given here.

4. Characterisation of semigroups of functions with range and antidomain

We are ready to state our main results. An overview of the structure of the proof is given in Subsection 4.1. The proof itself is given in Subsection 4.2.

Theorem 4.1. *The classes* $\mathsf{Fun}(;, \mathsf{a}, \mathsf{r})$ *and* $\mathsf{Fun}(;, \cdot, \mathsf{a}, \mathsf{r})$ *are finitely axiomatisable.*

1. *The proper quasivariety* $\mathsf{Fun}(;, \mathsf{a}, \mathsf{r})$ *is axiomatised by the associativity of* ;, *axioms* (a.I)–(a.V) *and* (r.I)–(r.VIII).

(2) *The variety* $\mathsf{Fun}(;, \cdot, \mathsf{a}, \mathsf{r})$ *is finitely axiomatised over* $\mathsf{Fun}(;, \mathsf{a}, \mathsf{r})$ *by the semilattice axioms for* $\cdot$*, and the axioms* (m.I) *and* (m.II)*.*

Then Theorem 3.4 gives us the following corollary.

Corollary 4.2. *With the additional axioms* ($\sqcup$.I) *and* ($\sqcup$.II)*, both items* (1) *and* (2) *of Theorem 4.1 extend to a complete axiomatisation for functionally representable algebras with* $\sqcup$ *in the signature.*

With axioms ($\uparrow$.I) *and* ($\uparrow$.II) *adjoined these characterisations extend to include maximal iterate in the case of finite algebras.*

4.1. **Overview of proof of Theorem 4.1.** The proof of Theorem 4.1 takes the following form. For each case, the stated axioms are routinely seen to be valid, and indeed correspond to known axiomatisations for representability in various fragments of the signatures, see Section 3. It remains to demonstrate a faithful representation for any algebra satisfying the stated axioms.

The construction of our representation is essentially the same for both items (1) and (2) of the theorem. The approach is reminiscent of Stone's representation for Boolean algebras, where each element b of the Boolean algebra $\mathcal{B}$ is identified with a subset of ultrafilters of $\mathcal{B}$: specifically, b is identified with the set of ultrafilters containing it.

In an algebra $\mathcal{S} = (S, ;, \mathsf{a}, \mathsf{r})$ (or $\mathcal{S} = (S, ;, \cdot, \mathsf{a}, \mathsf{r})$) satisfying the stated axioms, the set of domain elements (elements fixed by $\mathsf{d} = \mathsf{aa}$) forms a Boolean algebra with meet given by ;, and complementation given by a. Ultrafilters in this Boolean algebra of domain elements are used to motivate a related notion of *ultrasubset* in $\mathcal{S}$. These are introduced in Definition 4.4.

After a series of lemmas developing properties of ultrasubsets, we are able to define an algebra on the family of all ultrasubsets of $\mathcal{S}$, which we denote by $\mathcal{S}^\flat$. The signature of $\mathcal{S}^\flat$ is $\{;, \mathsf{d}, \mathsf{r}\}$ (or $\{;, \cdot, \mathsf{d}, \mathsf{r}\}$ if $\cdot$ is the signature of $\mathcal{S}$), and in Lemma 4.8 we show that it satisfies the known axioms for the class $\mathsf{Fun}(;, \mathsf{d}, \mathsf{r})$ (or $\mathsf{Fun}(;, \cdot, \mathsf{d}, \mathsf{r})$ if $\cdot$ is present). In particular then, $\mathcal{S}^\flat$ can be represented as an algebra of functions. For any representation ϕ of $\mathcal{S}^\flat$ over some set X, we show in Lemma 4.9 how to define a related representation $\phi^\sharp$ of the original algebra $\mathcal{S}$ over X (now with respect to original signature including antidomain). This representation $\phi^\sharp$ represents each $s \in S$ with the union of the representations U^ϕ of the ultrasubsets U containing s. Of course, typically a union of functions is not a function, however we show that distinct ultrasubsets containing an element s must always have disjoint domains (under any representation ϕ).

4.2. **Proof of Theorem 4.1.** First we note that the class $\mathsf{Fun}(;, \mathsf{a}, \mathsf{r})$ is a proper quasivariety, as is explained in [28, p. 232]. In the presence of $\cdot$, the implicational law (a.III) can be replaced by an equational law, so that $\mathsf{Fun}(;, \cdot, \mathsf{a}, \mathsf{r})$ is in fact a variety, see the discussion of Theorem 3.2.

Throughout, we consider a fixed algebra $\mathcal{S} = (S, ;, \mathsf{a}, \mathsf{r})$ (or $(S, ;, \cdot, \mathsf{a}, \mathsf{r})$) that satisfies the stated axioms. Recall that we define $\mathsf{d}(x) = \mathsf{a}(\mathsf{a}(x))$. It follows by Theorem 3.3 and Theorem 3.2 that the $\{;, \mathsf{d}, \mathsf{r}\}$-reduct $(S, ;, \mathsf{d}, \mathsf{r})$ (or $\{;, \cdot, \mathsf{d}, \mathsf{r}\}$-reduct $(S, ;, \cdot, \mathsf{d}, \mathsf{r})$), respectively) and $\{;, \mathsf{a}\}$-reduct $(S, ;, \mathsf{a})$ are representable. In particular then, $(S, ;, \mathsf{d})$ is a restriction semigroup, and so satisfies (d.I)–(d.IV).

Recall the definition $\alpha \vee \beta := \mathsf{a}(\mathsf{a}(\alpha) \mathbin{;} \mathsf{a}(\beta))$ for domain elements α, β. Our arguments will make use of the following law, which states the distribution of range over union:

$$\mathsf{r}(\alpha\,;x)\vee\mathsf{r}(\beta\,;x)=\mathsf{r}((\alpha\vee\beta)\,;x). \tag{4.1}$$

Prover9 shows that this law is a consequence of associativity for ; and the laws (a.I)–(a.III) and (r.I)–(r.III). However the proof takes several minutes by computer. The reader who is not troubled by redundancies among the axioms may include this as an axiom.

Any restriction semigroup $(S, ;, \mathsf{d})$ carries a natural order relation defined by $x \leq y$ iff $x = \mathsf{d}(x)\,;y$, or equivalently, iff $\exists z\ x = \mathsf{d}(z)\,;y$. This natural order coincides with the relation

$$\{(x\,;\alpha\,;y,\ x\,;y) \mid \alpha=\mathsf{d}(z),\ x,y,z\in S\},$$

is stable under left and right multiplication and is represented as $\subseteq$ in $\{;, \mathsf{d}\}$-representations. These properties are completely routine syntactic consequences of axioms (d.I)–(d.IV) (see [22] or [32] for example), but as we already know that $\mathcal{S}$ is representable in the signature $\{;, \mathsf{d}, \mathsf{r}\}$ (and hence in particular, in the signature $\{;, \mathsf{d}\}$), the observed properties follow directly from the transparent fact that for functions f, g we have $f \subseteq g$ if and only if $f = \mathsf{d}(f)\,;g$. We use this natural order in the construction of the algebra $\mathcal{S}^\flat$.

Lemma 4.3. (1) *$(D(S), ;, \mathsf{a}, 0, 1')$ forms a Boolean algebra where $0, 1'$ are the bottom and top elements, respectively, ; is Boolean meet and a is Boolean complement.*
(2) *$\leq$ is an order relation (reflexive, transitive, antisymmetric). For $s, t, u \in S$, if $s \leq t$ then $s\,;u \leq t\,;u$, $u\,;s \leq u\,;t$ and $\mathsf{d}(s) \leq \mathsf{d}(t)$.*
(3) *For $s, t \in S$, if $s \leq t$ and $\mathsf{d}(s) = \mathsf{d}(t)$ then $s = t$.*
(4) *For $s, t \in S$ and $\alpha, \beta \in D(S)$, if $s \geq \alpha\,;t$ and $s \geq \beta\,;t$ then $s \geq (\alpha \vee \beta)\,;t$.*
(5) *For $s \in S$ and $\alpha, \beta \in D(S)$, $\alpha\,;s \leq (\alpha\vee\beta)\,;s$ and $\beta\,;s \leq (\alpha\vee\beta)\,;s$.*

Proof. The first part may be proved directly from the antidomain axioms, or alternatively it follows from the first part of Theorem 3.2.

Properties (2)–(3) are discussed at the introduction of the natural order and are basic properties of restriction semigroups (see [22] for example), and also follow from Theorem 3.1.

For the fourth part, suppose $s \geq \alpha;t$ and $s \geq \beta;t$. Then $\alpha;\mathsf{d}(t);s \geq \alpha;t$ and if we apply the domain operation to both sides we get $\alpha;\mathsf{d}(t);\mathsf{d}(s) \geq \alpha;\mathsf{d}(t) \geq \alpha;\mathsf{d}(t);\mathsf{d}(s)$ and the two terms are equal. Hence, by the third part, $\alpha;\mathsf{d}(t);s = \alpha;t$, and similarly $\beta;\mathsf{d}(t);s = \beta;t$. By (3.6), $(\alpha\vee\beta);\mathsf{d}(t);s = (\alpha\vee\beta);t$, so $s \geq (\alpha\vee\beta);\mathsf{d}(t);s = (\alpha\vee\beta);t$, as required.

The fifth part follows from part two, since $\alpha, \beta \leq \alpha \vee \beta$. □

Since $D(S)$ forms a Boolean algebra, there are ultrafilters in $D(S)$. We will denote these by capital Greek letters. Also note that the natural order $\leq$, when restricted to elements in $D(S)$, agrees with the Boolean ordering: $\alpha \leq \beta$ if and only if $\alpha\,;\beta = \alpha$ (this follows from Theorem 3.1). For a subset $X \subseteq S$ of elements, we define

$$\uparrow X := \{s \in S \mid (\exists x \in X) x \leq s\},$$

the *upset* of X in S. At times we will consider the restriction to domain elements of the upset of X, namely $\uparrow X \cap D(S)$. An upset $X = \uparrow X$ is *down directed* if, for every $x, y \in X$, there exists $z \in X$ with $z \leq x$ and $z \leq y$. In the following, we extend the operations on S pointwise to subsets of S, and treat elements of S as singletons when convenient. So for example, for any $s \in S$ and $X \subseteq S$, we let $s\,;X := \{s\,;x \mid x \in X\}$, while $\mathsf{d}(X)\,;X = \{\mathsf{d}(x)\,;y \mid x, y \in X\}$.

Definition 4.4. *Let Δ be an ultrafilter of $D(S)$ and $s \in S$. If $0 \notin \Delta\,;s$ then we say that the upset $\uparrow(\Delta\,;s)$ of S is an* ultrasubset *S.*

Note that the upset (in S) of any ultrafilter Δ of $D(S)$ is an ultrasubset, because $\uparrow(\Delta\,;1') = \uparrow\Delta$.

Lemma 4.5. (i) *If Δ is an ultrafilter of $D(S)$ and $a \in S$ has $0 \notin \Delta\,;a$ then $\Delta = \uparrow\mathsf{d}(\Delta\,;a) \cap \mathsf{d}(S)$, that is, Δ is the upset of $\mathsf{d}(\Delta\,;a)$ in $D(S)$ and so it is the unique ultrafilter of $D(S)$ extending $\mathsf{d}(\Delta\,;a)$.*

(ii) *The following are equivalent for a subset $U \subseteq S$:*
- *U is an ultrasubset of S,*
- *U is a down-directed upset of S and $\uparrow\mathsf{d}(U) \cap D(S)$ is an ultrafilter of $D(S)$,*
- *U is a maximal proper down-directed upset of S.*

Moreover, for any ultrasubset U and $a \in U$, we have $U = \uparrow(\mathsf{d}(U)\,;a)$.

(iii) *If $\uparrow(\Delta_1\,;s_1)$ and $\uparrow(\Delta_2\,;s_2)$ are ultrasubsets of S with $\uparrow(\Delta_1\,;s_1) \subseteq \uparrow(\Delta_2\,;s_2)$ then $\Delta_1 = \Delta_2$ and $\uparrow(\Delta_1\,;s_1) = \uparrow(\Delta_2\,;s_2)$.*

(iv) *If $\uparrow(\Delta\,;s) \cap \uparrow(\Delta\,;t) \neq \varnothing$ then $\uparrow(\Delta\,;s) = \uparrow(\Delta\,;t)$.*

(v) *If $0 \notin \Delta\,;s$ then the upset of $\mathsf{r}(\Delta\,;s)$ in $D(S)$ is an ultrafilter of $D(S)$.*

Proof. (i) This follows because for every $\delta \in \Delta$, the element $\mathsf{d}(\delta\,;a)$ is in $\mathsf{d}(\uparrow(\Delta\,;a))$ and $\mathsf{d}(\delta\,;a) \leq \delta$.

(ii) First assume that $U = \uparrow(\Delta;a)$ is an ultrasubset of $D(S)$. As Δ (an ultrafilter of $D(S)$) is down directed, it follows that so is U, because $(\gamma\,;\delta)\,;a$ is a lower bound of both $\gamma\,;a$ and $\delta\,;a$. Also, $\uparrow\mathsf{d}(U) \cap D(S) = \Delta$ from part (i).

Next, assume that U is a down-directed filter with $\mathsf{d}(U)$ generating an ultrafilter of $D(S)$. We show that U is a maximal down-directed filter. Assume V is a down-directed filter with $U \subseteq V$ but $0 \notin V$, we show that $U = V$. Consider any $a \in V$. Then for any $b \in U$, as $U \subseteq V$, we have a lower bound $c \in V$ for $\{a, b\}$. Now $\mathsf{d}(c) \in \uparrow\mathsf{d}(U)$, because otherwise $\mathsf{a}(c) \in \uparrow\mathsf{d}(U)$ (as $\uparrow\mathsf{d}(U) \cap D(S)$ is an ultrafilter of $D(S)$), which would give an element $d \in U$ with $\mathsf{d}(d) \leq \mathsf{a}(c)$. But then $\mathsf{d}(d)\,;\mathsf{d}(c) = 0$, so there could be no lower bound for d and c in V. So $\mathsf{d}(c) \in \uparrow\mathsf{d}(U)$. Thus there is $u \in U$ with $\mathsf{d}(u) \leq \mathsf{d}(c)$. Let $u' \in U$ be a lower bound of u and b. Then $\mathsf{d}(u')\,;b = u' \in U$, whence $\mathsf{d}(u')\,;b \leq \mathsf{d}(u)\,;b \leq \mathsf{d}(c)\,;b \in U$. But $c = \mathsf{d}(c)\,;b$ and U is upward closed, showing that $a \in U$. Thus $U = V$.

Now assume that U is a maximal down-directed filter with respect to $\leq$. Consider any $a \in U$. We claim that $\uparrow(\mathsf{d}(U)\,;a) = U$. For the inclusion $\uparrow(\mathsf{d}(U)\,;a) \subseteq U$, consider $\mathsf{d}(x)\,;a$ for some $x \in U$. Let $z \in U$ have $z \leq x$ and $z \leq a$, and note that $\mathsf{d}(x)\,;z = z$. Then $\mathsf{d}(x)\,;a \geq \mathsf{d}(x)\,;z = z$, so $\mathsf{d}(x)\,;a \in U$ as claimed. For the reverse inclusion, consider $x \in U$. Then there is $z \in U$ with $z \leq x$ and $z \leq a$. So $\mathsf{d}(z) \in \mathsf{d}(U)$ and $\mathsf{d}(z)\,;a = z = \mathsf{d}(z)\,;x$. Then $x \geq \mathsf{d}(z)\,;x = \mathsf{d}(z)\,;a$, an element of $\mathsf{d}(U)\,;a$. So $x \in \uparrow(\mathsf{d}(U)\,;a)$ giving $\uparrow(\mathsf{d}(U)\,;a) = U$. Note that this also shows that U can be written as $\uparrow(\mathsf{d}(U)\,;a)$ for any element $a \in U$, the final claim of part (ii).

(iii) Part (i) gives $\Delta_1 = \Delta_2$ and the final statement of part (ii) gives $\uparrow(\Delta_1\,;s_1) = \uparrow(\Delta_2\,;s_2)$.

(iv) This also follows from the final statement of (ii). Once the two ultrasubsets have a common element c then both can be written as $\uparrow(\Delta\,;c)$.

(v) To show that $\uparrow\mathsf{r}(\Delta\,;s)$ is a filter, it suffices to show that if $\delta_1, \delta_2 \in \Delta$ then $\mathsf{r}(\delta_1\,;s)\,;\mathsf{r}(\delta_2\,;s)$ is in $\uparrow\mathsf{r}(\Delta\,;s)$. This follows because $\delta := \delta_1\,;\delta_2$ is in Δ and $\mathsf{r}(\delta\,;s)$ is a lower bound of both $\mathsf{r}(\delta_1\,;s)$ and $\mathsf{r}(\delta_2\,;s)$ in $\mathsf{r}(\Delta\,;s)$.

Now we show that it is an ultrafilter. Assume that $\alpha \notin \uparrow\mathsf{r}(\Delta\,;s)$, so $s\,;\alpha \notin \Delta\,;s$. Hence $\mathsf{d}(s;\alpha) \notin \uparrow\mathsf{d}(\Delta;s) = \uparrow\Delta$. So $\mathsf{a}(s;\alpha) \in \Delta$ showing that $s;\mathsf{a}(\alpha) \stackrel{(\mathsf{a}.\mathrm{II})}{=} \mathsf{a}(s;\alpha);s \in \Delta\,;s$. Then $\mathsf{a}(\alpha) \geq \mathsf{r}(\mathsf{a}(s\,;\alpha)\,;s) \in \mathsf{r}(\Delta\,;s)$, giving $\mathsf{a}(\alpha) \in \uparrow\mathsf{r}(\Delta\,;s)$. So the upset of $\mathsf{r}(\Delta\,;s)$ in $D(S)$ is an ultrafilter. □

Lemma 4.6. (i) $\uparrow(\uparrow(\Delta_1\,;s_1)\,;\uparrow(\Delta_2\,;s_2)) = \uparrow(\Delta_1\,;s_1\,;\Delta_2\,;s_2)$.

(ii) *If* $0 \notin \uparrow(\Delta_1\,;s_1\,;\Delta_2\,;s_2)$ *then* $\uparrow(\Delta_1\,;s_1\,;\Delta_2\,;s_2) = \uparrow(\Delta_1\,;(s_1\,;s_2))$.

(iii) $\uparrow(\uparrow(\Delta_1\,;s_1)\cdot\uparrow(\Delta_2\,;s_2)) = \uparrow((\Delta_1\,;s_1)\cdot(\Delta_2\,;s_2))$.

(iv) *If* $0 \notin (\Delta_1\,;s_1)\cdot(\Delta_2\,;s_2)$ *then* $\Delta_1 = \Delta_2$ *and* $\uparrow(\Delta_1\,;s_1) = \uparrow(\Delta_2\,;s_2)$ *and* $\uparrow((\Delta_1\,;s_1)\cdot(\Delta_2\,;s_2)) = \uparrow(\Delta_1\,;(s_1\cdot s_2))$.

Proof. Part (i) is trivial, as is part (iii). For part (ii) note that $\uparrow(\Delta_1\,;s_1\,;\Delta_2\,;s_2) \supseteq \uparrow(\Delta_1\,;s_1\,;s_2)$ and that $\uparrow(\Delta_1\,;s_1\,;s_2)$ is maximal by Lemma 4.5 part (ii). For part (iv), observe that $\uparrow((\Delta_1\,;s_1)\cdot(\Delta_2\,;s_2)) = \uparrow((\Delta_1\cdot\Delta_2)\,;(s_1\cdot s_2))$ is certainly a down-directed filter, moreover, one that contains both $\Delta_1\,;s_1$ and $\Delta_2\,;s_2$. The statement now follows from Lemma 4.5 parts (ii) and (iii). □

Definition 4.7 (Definition of $\mathcal{S}^\flat$)**.** *Let the set* $S^\flat$ *consist of* S *along with the set of ultrasubsets of* S*. To obtain the algebra* $\mathcal{S}^\flat$*, define operations* D, R, $*$ *and* $\wedge$ *on* $S^\flat$ *corresponding to* d, r, ; *and* $\cdot$ (*if present*) *as follows.*

(1) $\mathsf{D}(\uparrow(\Delta;s)) := \uparrow\mathsf{d}(\Delta;s) = \uparrow(\Delta;\mathsf{d}(s))$ $(= \uparrow\Delta$ *when* $\mathsf{d}(s) \in \Delta$*, by Lemma 4.5*(i)),
(2) $\mathsf{R}(\uparrow(\Delta\,;s)) := \uparrow\mathsf{r}(\Delta\,;s)$,
(3) $\uparrow(\Delta_1\,;s_1) * \uparrow(\Delta_2\,;s_2) := \uparrow((\Delta_1\,;s_1)\,;(\Delta_2\,;s_2))$,
(4) (*if* $\cdot$ *is present*) $\uparrow(\Delta_1\,;s_1) \wedge \uparrow(\Delta_2\,;s_2) := \uparrow((\Delta_1\,;s_1)\cdot(\Delta_2\,;s_2))$,

where $\Delta, \Delta_1, \Delta_2$ *are ultrafilters of* $D(S)$ *and* $s, s_1, s_2 \in S$.

Note that, while S is not an ultrasubset by definition (as $0 \in S$), it is covered by Definition 4.7 because $S = \uparrow(\Delta\,;0)$. It is easy to see from Definition 4.7 that S acts as an absorbing zero element with respect to both $*$ and $\wedge$, and that it is fixed by both R and D (simply because 0 is fixed by d and r and is contained in S). We now observe that each of (1)–(4) in Definition 4.7 correctly defines an operation on $S^\flat$. This is trivial in part (1). In part (2), Lemma 4.5(v) shows that the right hand side is either S or can be written in the form $\uparrow(\Gamma\,;\mathsf{r}(s))$ (where Γ is the upset of $\mathsf{r}(\Delta\,;s)$ in $D(S)$), and Lemma 4.6 shows that the same alternatives apply to parts (3) and (4). Hence $S^\flat$ is indeed closed under the above operations.

Lemma 4.8. $\mathcal{S}^\flat = (S^\flat, *, \mathsf{D}, \mathsf{R})$ (*or* $\mathcal{S}^\flat = (S^\flat, *, \wedge, \mathsf{D}, \mathsf{R})$, *respectively*) *is representable. Furthermore, we can assume that the constant element* $S \in S^\flat$ *is represented as the empty function.*

Proof. It suffices to show that $\mathcal{S}^\flat$ satisfies the axioms for representability (d.II)–(d.IV), (r.I)–(r.III) (possibly with (m.I), (m.II) and the semilattice laws for $\cdot$) so that we can apply Theorem 3.3. Checking the axioms is almost immediate because the operations are determined elementwise. For example, we verify (d.IV). Consider ultrasubsets U, V. Now $x \in U * \mathsf{D}(V)$ if and only if $x \geq a\,;\mathsf{d}(b)$ for some $a \in U$ and $b \in V$. But $a\,;\mathsf{d}(b) = \mathsf{d}(a\,;b)\,;a$ which is an element of $\mathsf{d}(U\,;V)\,;U$, a subset of $\mathsf{D}(U * V) * U$. For the reverse inclusion, let $x \in \mathsf{D}(U * V) * U$. There are $a, c \in U$

and $b \in V$ such that $x \geq \mathsf{d}(a \,;\, b) \,;\, c$. Let $e \in U$ be a lower bound of a, c. Then $x \geq \mathsf{d}(e \,;\, b) \,;\, e = e \,;\, \mathsf{d}(b) \in U * \mathsf{D}(V)$, as required. □

Now we provide the crucial lifting of a $\{;, \mathsf{d}, \mathsf{r}\}$ representation of $\mathcal{S}^\flat$ to $\{;, \mathsf{a}, \mathsf{r}\}$ representation of $\mathcal{S}$.

Lemma 4.9. *Let ϕ be faithful representation of $\mathcal{S}^\flat$ as an algebra of functions on a set X in the signature $\{;, \mathsf{d}, \mathsf{r}, 0\}$ (and $\cdot$ if it is present), where 0 is interpreted on $\mathcal{S}^\flat$ as the element S. Without loss of generality, assume that X is the union of domains of elements in the image of $S^\flat$ under ϕ.*

Define $\phi^\sharp$ on S by

$$\phi^\sharp(s) := \bigcup\{\phi(\uparrow(\Delta \,;\, s)) \mid 0 \notin \Delta \,;\, s\}. \tag{4.2}$$

Then $\phi^\sharp$ represents $\mathcal{S}$ as functions on X in the signature $\{;, \mathsf{a}, \mathsf{r}\}$ (and $\cdot$ if it is present).

Proof. First we show that $\phi^\sharp(s)$ is in fact a function on X for every $s \in S$. For this it suffices to show that if $\Delta_1 \neq \Delta_2$ and $0 \notin \Delta_1 \,;\, s,\ \Delta_2 \,;\, s$ then $\phi(\uparrow(\Delta_1 \,;\, s))$ has disjoint domain from $\phi(\uparrow(\Delta_2 \,;\, s))$. Observe that $\Delta_1 \,;\, \Delta_2$ contains 0, since there is $\alpha \in \mathsf{d}(S)$ such that α and $\mathsf{a}(\alpha)$ are in the symmetric difference of Δ_1 and Δ_2 and $0 = \alpha \,;\, \mathsf{a}(\alpha)$. Then, as composition of restrictions of the identity coincides with intersection in representable algebras,

$$\begin{aligned}
\mathsf{d}(\phi(\uparrow(\Delta_1 \,;\, s))) \cap \mathsf{d}(\phi(\uparrow(\Delta_2 \,;\, s))) &= \mathsf{d}(\phi(\uparrow(\Delta_1 \,;\, s))) \,;\, \mathsf{d}(\phi(\uparrow(\Delta_2 \,;\, s))) \\
&= \phi(\mathsf{D}(\uparrow(\Delta_1 \,;\, s))) \,;\, \phi(\mathsf{D}(\uparrow(\Delta_2 \,;\, s))) \\
&= \phi(\mathsf{D}(\uparrow(\Delta_1 \,;\, s)) * \mathsf{D}(\uparrow(\Delta_2 \,;\, s))) \\
&= \phi(\uparrow(\Delta_1 \,;\, \Delta_2)) \\
&= \phi(S) \\
&= \varnothing
\end{aligned}$$

as desired. (Note that this holds even when meet is not in the signature.)

Next we show that if $s \neq t$ in S then $\phi^\sharp(s) \neq \phi^\sharp(t)$. Without loss of generality, assume that $s \not\leq t$. We show that there is an ultrafilter Δ of $D(S)$ such that $\uparrow(\Delta ; s)$ is an ultrasubset and $\uparrow(\Delta \,;\, s) \neq \uparrow(\Delta \,;\, t)$. As ϕ is a faithful representation, we then have $\phi(\uparrow(\Delta \,;\, s)) \neq \phi(\uparrow(\Delta \,;\, t))$.

We claim that the set $I = \{\alpha \in D(S) \mid \alpha \,;\, s \leq \alpha \,;\, t\}$ is an ideal in the Boolean algebra of domain elements (Lemma 4.3). For downward closure, suppose $\alpha \in I$ and $\alpha_0 \leq \alpha$. Then $\alpha_0 ; s \leq \alpha ; s \leq \alpha ; t$, so $\alpha_0 ; s \leq \alpha_0 ; \alpha ; t = \alpha_0 ; t$ by Lemma 4.3(2), so $\alpha_0 \in I$. Now suppose $\alpha, \beta \in I$. We have $\alpha \,;\, s \leq \alpha \,;\, t$ and $\beta \,;\, s \leq \beta \,;\, t$, hence $\alpha \,;\, s,\ \beta \,;\, s \leq (\alpha \vee \beta) \,;\, t$ by Lemma 4.3(5). By Lemma 4.3(4) it follows that $(\alpha \vee \beta) \,;\, s \leq (\alpha \vee \beta) \,;\, t$ so $\alpha \vee \beta \in I$, as required. So I is an ideal. Clearly also, it avoids $\mathsf{d}(s)$. Thus we may extend the principal filter of $\mathsf{d}(s)$ in $D(S)$ to an ultrafilter Δ disjoint from I. Now $0 \notin \Delta \,;\, s$ because if $\alpha \,;\, s = 0$, then trivially $\alpha \,;\, s \leq \alpha \,;\, t$ so that α would be in I. Thus $\uparrow(\Delta \,;\, s)$ is an ultrasubset. Moreover, it is clear that $t \notin \uparrow(\Delta ; s)$ (as then we would have $\beta \in \Delta$ with $\beta ; s \leq \beta ; t$, contradicting the choice of Δ). So by Lemma 4.5, it follows that either $\uparrow(\Delta ; t)$ is equal to S or it is disjoint from $\uparrow(\Delta \,;\, s)$. In either case there are $x, y \in X$ with (x, y) related by $\phi(\uparrow(\Delta \,;\, s))$ but not by $\phi(\uparrow(\Delta ; t))$. Since we showed that for $\Gamma \neq \Delta$, $\phi(\uparrow(\Delta ; s))$ and $\phi(\uparrow(\Gamma ; t))$ have disjoint domains, it follows that (x, y) is related by $\phi^\sharp(s)$ but not by $\phi^\sharp(t)$ as required.

We show that $\phi^\sharp$ is a homomorphism.

Preservation of ;: For all $s, t \in S$,

$$\begin{aligned}
\phi^\sharp(s) \,;\, \phi^\sharp(t) &= \left(\bigcup_{0 \notin \Delta ; s} \phi\left(\uparrow(\Delta \,;\, s)\right) \right) ; \left(\bigcup_{0 \notin \Gamma ; t} \phi\left(\uparrow(\Gamma \,;\, t)\right) \right) \\
&= \bigcup_{0 \notin \Delta ; s} \bigcup_{0 \notin \Gamma ; t} \phi\left(\uparrow(\Delta \,;\, s)\right) ; \phi\left(\uparrow(\Gamma \,;\, t)\right) \\
&= \bigcup_{0 \notin \Delta ; s,\ 0 \notin \Gamma ; t} \phi\left(\uparrow(\Delta \,;\, s) * \uparrow(\Gamma \,;\, t)\right) \\
&= \bigcup_{0 \notin \Delta ; s,\ 0 \notin \Gamma ; t} \phi\left(\uparrow((\Delta \,;\, s) \,;\, (\Gamma \,;\, t))\right) \\
&= \bigcup_{0 \notin \Delta ; s ; t} \phi\left(\uparrow(\Delta \,;\, s \,;\, t)\right) \\
&= \phi^\sharp(s \,;\, t).
\end{aligned}$$

Note that the penultimate equality uses the fact that $\uparrow(\Delta \,;\, s \,;\, \Gamma \,;\, t) = \uparrow(\Delta \,;\, s \,;\, t)$ when $0 \notin \Delta \,;\, s \,;\, \Gamma \,;\, t$ (by Lemma 4.6), and $\phi(\uparrow(\Delta \,;\, s \,;\, \Gamma \,;\, t)) = \phi(S) = \varnothing$ when $0 \in \Delta \,;\, s \,;\, \Gamma \,;\, t$.

Preservation of d: Even though the operation d is only a derived operation in $(S, ;, \mathsf{a}, \mathsf{r})$, it is convenient to verify that it is preserved before showing preservation of a. We have

$$\begin{aligned}
\phi^\sharp(\mathsf{d}(s)) &= \bigcup_{0 \notin \Delta ; \mathsf{d}(s)} \phi\left(\uparrow(\Delta \,;\, \mathsf{d}(s))\right) \\
&= \bigcup_{0 \notin \Delta ; s} \phi(\uparrow \Delta) \\
&= \bigcup_{0 \notin \Delta ; s} \phi(\mathsf{D}(\uparrow(\Delta ; s))) \\
&= \mathsf{d}\left(\bigcup_{0 \notin \Delta ; s} \phi(\uparrow(\Delta \,;\, s)) \right) \\
&= \mathsf{d}(\phi^\sharp(s))
\end{aligned}$$

as required. The second equality uses the fact that $0 \notin \Delta \,;\, \mathsf{d}(s)$ if and only if $0 \notin \Delta \,;\, s$.

Preservation of a: As $\mathsf{a}(s) \,;\, s = 0$, $\phi^\sharp(0) = \phi(S) = \varnothing$, $;$ is preserved and $\phi^\sharp(\mathsf{a}(s))$ is contained in the identity, we must have $\mathrm{dom}(\phi^\sharp(\mathsf{a}(s))) \subseteq X \setminus \mathrm{dom}(\phi^\sharp(s))$. Let $x \in X \setminus \mathrm{dom}(\phi^\sharp(s))$, that is, $(x, x) \notin \mathsf{d}(\phi^\sharp(s))$. Now as X is a union of the domains of elements of $\phi(S^\flat)$, it follows that there is an ultrafilter Δ of $\mathsf{d}(S)$ such that $x \in \mathrm{dom}(\phi(\uparrow \Delta))$, that is, $(x, x) \in \mathsf{d}(\phi(\uparrow \Delta))$. As $(x, x) \notin \mathsf{d}(\phi^\sharp(s)) = \bigcup\{\phi(\uparrow \Gamma) \mid 0 \notin \uparrow(\Gamma \,;\, s)\}$, we must have $\uparrow(\Delta \,;\, s) = S$ showing that $\mathsf{a}(s) \in \Delta$. Then $(x, x) \in \bigcup_{\mathsf{a}(s) \in \Delta} \phi(\uparrow \Delta) = \phi^\sharp(\mathsf{a}(s))$ as required.

Preservation of r: First use the fact that $s \,;\, \mathsf{r}(s) = s$ and the fact that $;$ is preserved and $\mathsf{r}(s) = \mathsf{d}(\mathsf{r}(s))$ to deduce that $\phi^\sharp(\mathsf{r}(s))$ is a restriction of the identity element whose domain contains the range of $\phi^\sharp(s)$. So $\mathsf{r}(\phi^\sharp(s)) \subseteq \phi^\sharp(\mathsf{r}(s))$.

For the other direction assume that $(y,y) \in \phi^\sharp(\mathsf{r}(s))$. We show that there is $x \in X$ with $(x,y) \in \phi^\sharp(s)$. Now, as $(y,y) \in \phi^\sharp(\mathsf{r}(s))$, there is Δ with $(y,y) \in \phi(\uparrow(\Delta\,;\mathsf{r}(s)))$. Hence $\mathsf{r}(s) \in \Delta$ and $\uparrow\Delta = \uparrow(\Delta\,;\mathsf{r}(s))$. Consider the filter in $D(S)$ generated by $F := \{\mathsf{a}(\alpha) \mid \mathsf{r}(\alpha\,;s) \notin \Delta\}$. To show this is a proper filter, observe that $\mathsf{r}(\alpha\,;s) \vee \mathsf{r}(\beta\,;s) \overset{(4.1)}{=} \mathsf{r}((\alpha \vee \beta)\,;s)$. So if $\mathsf{a}(\alpha)$ and $\mathsf{a}(\beta)$ are in F then (as Δ is a prime filter) we have $\mathsf{r}(\alpha\,;s) \vee \mathsf{r}(\beta\,;s)$ not in Δ, whence $\mathsf{a}(\alpha)\cdot\mathsf{a}(\beta) = \mathsf{a}(\alpha \vee \beta) \in F$. Let Γ be any ultrafilter of $D(S)$ extending F. If $0 \in \Gamma\,;s$ then there is $\alpha \in \Gamma$ with $0 = \alpha\,;s$ so that $\mathsf{r}(\alpha\,;s) = 0$, which would give $\mathsf{a}(\alpha) \in F \subseteq \Gamma$, a contradiction. So $\uparrow(\Gamma\,;s)$ is an ultrasubset. Now we show that $\mathsf{r}(\uparrow(\Gamma\,;s)) \subseteq \Delta$. Consider any $\alpha \in \Gamma$. Then as $\mathsf{a}(\alpha) \notin F$, it follows that $\mathsf{r}(\alpha\,;s) \in \Delta$. Thus $\mathsf{R}(\uparrow(\Gamma\,;s)) = \uparrow\Delta$. Now recall that $\uparrow\Delta = \uparrow(\Delta\,;\mathsf{r}(s))$. Since ϕ preserves R as range and $(y,y) \in \phi(\uparrow(\Delta\,;\mathsf{r}(s))) = \phi(\uparrow\Delta) = \phi(\mathsf{R}(\uparrow(\Gamma\,;s)))$, there must be $x \in X$ with $(x,y) \in \phi(\uparrow(\Gamma\,;s))$. Then $(x,y) \in \phi^\sharp(s)$ as required.

We have not used $\cdot$ to establish the preservation of $;, \mathsf{a}, \mathsf{r}$, so if $\cdot$ is not present then the proof of Lemma 4.9 is complete.

Preservation of $\cdot$ when it is present: Using that meet is interpreted as intersection in representable algebras and Lemma 4.6 part (iii),

$$\begin{aligned}\phi^\sharp(s)\cdot\phi^\sharp(t) &= \phi^\sharp(s) \cap \phi^\sharp(t)\\ &= \left(\bigcup_{0\notin\Delta;s} \phi(\uparrow(\Delta\,;s))\right) \cap \left(\bigcup_{0\notin\Gamma;t} \phi(\uparrow(\Gamma\,;t))\right)\\ &= \bigcup_{0\notin\Delta;s,\ 0\notin\Gamma;t} \phi(\uparrow(\Delta\,;s)) \cap \phi(\uparrow(\Gamma\,;t))\\ &= \bigcup_{0\notin\Delta;s,\ 0\notin\Gamma;t} \phi\left(\uparrow((\Delta\,;s)\cdot(\Gamma\,;t))\right).\end{aligned}$$

Using Lemma 4.6 part (iv),

$$0 \notin (\Delta\,;s)\cdot(\Gamma\,;t) \text{ if and only if } \Delta = \Gamma \text{ and } 0 \notin \Delta\,;(s\cdot t)$$

whence

$$\bigcup_{0\notin\Delta;s,\ \Gamma;t} \phi\left(\uparrow((\Delta\,;s)\cdot(\Gamma\,;t))\right) = \bigcup_{0\notin\Delta;(s\cdot t)} \phi(\uparrow(\Delta\,;(s\cdot t))) = \phi^\sharp(s\cdot t)$$

as desired. □

Proof of Theorem 4.1. Let $\mathcal{S}$ be an algebra satisfying the stated axioms. We defined $\mathcal{S}^\flat$ in Definition 4.7, and showed in Lemma 4.8 that $\mathcal{S}^\flat$ is representable in the reduct signature $\{;, \mathsf{d}, \mathsf{r}, 0\}$ (with meet, when required). Let ϕ be any such representation. Then Lemma 4.9 shows that $\phi^\sharp$ is the desired (faithful) representation of $\mathcal{S}$ in the full signature including antidomain. □

The axiomatization given in Theorem 4.1 contains redundancy. We already noted that instead of axioms (a.I)–(a.V) and (r.I)–(r.VIII) it suffices to take axioms (a.I)–(a.IV) and (r.I)–(r.III). In addition, once the irredundant set of axioms (a.I)–(a.IV) are combined with (r.I)–(r.III) (with d denoting aa), the law (a.IV) becomes redundant.

Lemma 4.10. *Law* (a.IV) *is a consequence of associativity along with* (a.I)–(a.III) *and* (r.I)–(r.III).

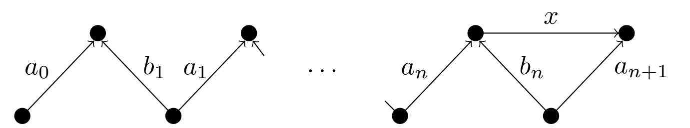

FIGURE 1. A permissible sequence $(a_0, b_1, a_1, \dots, a_n, b_n, a_{n+1})$ where $b_n \,;\, x = a_{n+1}$. This sequence can be reduced to $(a_0, b_1, a_1, \dots, a_n \,;\, x)$.

Proof. By Lemma 3.5 we may use law (3.8). Observe (using associativity implicitly) that $\mathsf{a}(0)\,;x \overset{(3.8)}{=} \mathsf{a}(x\,;0)\,;x \overset{(\mathsf{a}.\mathrm{I})}{=} \mathsf{a}(x\,;\mathsf{a}(\mathsf{r}(x))\,;\mathsf{r}(x))\,;x \overset{(\mathsf{a}.\mathrm{II})}{=} \mathsf{a}(\mathsf{a}(x\,;\mathsf{r}(x))\,;x\,;\mathsf{r}(x))\,;x \overset{(\mathsf{r}.\mathrm{II})}{=} \mathsf{a}(\mathsf{a}(x\,;\mathsf{r}(x))\,;x)\,;x \overset{(\mathsf{a}.\mathrm{II})}{=} \mathsf{a}(x\,;\mathsf{a}(\mathsf{r}(x)))\,;x \overset{(\mathsf{a}.\mathrm{II})}{=} x\,;\mathsf{a}(\mathsf{a}(\mathsf{r}(x))) = x\,;\mathsf{d}(\mathsf{r}(x)) \overset{(\mathsf{r}.\mathrm{I})}{=} x$. □

5. FINITE REPRESENTATION FOR DOMAIN, RANGE AND COMPOSITION

Following Lemma 4.9, to find a finite representation for finite algebras in signatures such as antidomain and range, it suffices to find finite representations for finite algebras in signatures involving just domain and range. We now revisit Schein's representation for the signature $\{;, \mathsf{d}, \mathsf{r}\}$ and present an identification that yields a finite representation in the case of finite algebras. Using the results from previous sections of this article, we then obtain a finite representation for finite representable algebras in various meet-free signatures extending $\{;, \mathsf{d}, \mathsf{r}\}$, see Theorem 5.3 below.

Schein's original argument is in Russian [38]. The notes [40] have an English presentation of a similar representation, while an English presentation of the original argument (but also including $\cdot$) is given in [25], and also in a category-theoretic setting in [5]. Because we need some details for our refinement, we present a concise presentation of the method here (in the signature $\tau = \{;, \mathsf{d}, \mathsf{r}\}$). With only a little checking of details, the reader should have enough to reprove Theorem 3.3, however the reader can consult one of the references just listed for a more detailed treatment.

Let $\mathcal{S} = (S, ;, \mathsf{d}, \mathsf{r})$, possibly with 0, be an associative algebra satisfying the domain and range axioms, (d.II)–(d.IV) and (r.I)–(r.III). The base of Schein's representation consists of certain finite sequences of elements from $S \backslash \{0\}$.

For $n \geq -1$, a sequence $(a_0, b_0, \dots, a_n, b_n, a_{n+1}) \in S^{2n+3}$ is *permissible* if $\mathsf{r}(a_i) = \mathsf{r}(b_i)$ and $\mathsf{d}(b_i) = \mathsf{d}(a_{i+1})$, for all $i \leq n$. A permissible sequence $(a_0, b_0, \dots, a_n, b_n, a_{n+1})$ *reduces* to a sequence $(a_0, b_0, \dots, a_n \,;\, x)$ if $b_n \,;\, x = a_{n+1}$. See Figure 1.

The sequence $(a_0, b_0, \dots, b_{n-1}, a_n ; x)$ is itself permissible because of the following. Notice that $\mathsf{d}(a_n\,;x) \leq \mathsf{d}(a_n) = \mathsf{d}(b_{n-1})$ by (d.V). Also, $b_n\,;\mathsf{d}(x) \overset{(\mathsf{d}.\mathrm{IV})}{=} \mathsf{d}(b_n\,;x)\,;b_n = \mathsf{d}(a_{n+1})\,;b_n = \mathsf{d}(b_n)\,;b_n = b_n$. Thus $\mathsf{d}(x) \geq \mathsf{r}(b_n) = \mathsf{r}(a_n)$. So $\mathsf{d}(a_n\,;x) \overset{(3.3)}{=} \mathsf{d}(a_n\,;\mathsf{d}(x)) = \mathsf{d}(a_n) = \mathsf{d}(b_{n-1})$, showing that $(a_0, b_0, \dots, a_n \,;\, x)$ is permissible. Reduction is easily shown to be unique using the implication (r.III): $b\,;x = b\,;y \Rightarrow \mathsf{r}(b)\,;x = \mathsf{r}(b)\,;y$.

A permissible sequence is *reduced* if no reductions are possible. Given a sequence $\bar{a}$, we denote the unique reduced sequence obtained by reducing $\bar{a}$ by $\mathrm{nf}(\bar{a})$, the *normal form* of $\bar{a}$ and let NF be the set of reduced permissible sequences. Schein defines a representation θ on NF as follows. For an element a of S, the representation

a^θ of a will be a function with the following domain

$$\mathrm{dom}(a^\theta) = \{(a_0, b_0, \ldots, a_n, b_n, a_{n+1}) \in \mathsf{NF} \mid \mathsf{d}(a) \geq \mathsf{r}(a_{n+1})\}$$

and then

$$a^\theta(a_0, b_0, \ldots, a_n, b_n, a_{n+1}) = \mathrm{nf}(a_0, b_0, \ldots, a_n, b_n, a_{n+1}\ ; a).$$

It can be shown, with the aid of Figure 1, for any element a and permissible sequence $(a_0, b_0, \ldots, a_{n+1})$, that $a_{n+1};\mathsf{d}(a) = a_{n+1}$ if and only if $\mathrm{nf}(a_0, b_0, \ldots, a_{n+1})$ is in the domain of a^θ, and that in this case, $\mathrm{nf}(a_0, b_0, \ldots, a_{n+1};a) = a^\theta(\mathrm{nf}(a_0, b_0, \ldots, a_{n+1}))$, (for example, see [25, Lemmas 4.7, 4.8] for full details). Hence

$$a^\theta(\mathrm{nf}(a_0, b_0, \ldots, a_{n+1})) = \mathrm{nf}(a_0, b_0, \ldots, b_n, a_{n+1}\ ; a) \tag{5.1}$$

for any permissible sequence $(a_0, \ldots, a_{n+1})$ and $a \in S$ such that $\mathsf{d}(a) \geq \mathsf{r}(a_{n+1})$.

This representation preserves ;, d and r (Schein [38]) as well as 0 as $\varnothing$ (if $0 = \mathsf{d}(0)$ is present). For example, to check r, the reduced permissible sequence $(a_0, \ldots, a_{n+1})$ is in the domain of the function $(\mathsf{r}(a))^\theta$ if and only if $\mathsf{r}(a_{n+1}) \leq \mathsf{d}(\mathsf{r}(a)) = \mathsf{r}(a)$ and in that case $(\mathsf{r}(a))^\theta(a_0, \ldots, a_{n+1}) = (a_0, \ldots, a_{n+1})$. Also a^θ is defined on $\mathrm{nf}(a_0, \ldots, a_{n+1},\ a\ ; \mathsf{r}(a_{n+1}),\ \mathsf{d}(a\ ; \mathsf{r}(a_{n+1})))$ and

$$\begin{aligned} & a^\theta(\mathrm{nf}(a_0, \ldots, a_{n+1},\ a\ ; \mathsf{r}(a_{n+1}),\ \mathsf{d}(a\ ; \mathsf{r}(a_{n+1})))) \\ &= \ \mathrm{nf}(a_0, \ldots, a_{n+1},\ a\ ; \mathsf{r}(a_{n+1}),\ \mathsf{d}(a\ ; \mathsf{r}(a_{n+1}))\ ; a) \\ &= \ (a_0, \ldots, a_{n+1}) \end{aligned}$$

using a reduction with $x = \mathsf{r}(a_{n+1})$, since $a\ ; \mathsf{r}(a_{n+1})\ ; \mathsf{r}(a_{n+1}) = \mathsf{d}(a\ ; \mathsf{r}(a_{n+1}))\ ; a$.

For the other direction assume that $\mathrm{nf}(a_0, \ldots, b_n, a_{n+1}\ ; a)$ is in the range of a^θ. Since $(a_0, \ldots, b_n, a_{n+1}\ ; a) = (a_0, \ldots, b_n, a_{n+1}\ ; a\ ; \mathsf{r}(a))$, applying $(\mathsf{r}(a))^\theta$ fixes $\mathrm{nf}(a_0, \ldots, b_n, a_{n+1}\ ; a)$ (use (5.1)). Therefore $\mathsf{r}(a)^\theta$ is equal to the identity restricted to the range of the function a^θ, as required. Similarly, d is represented correctly by θ.

This representation also preserves $\cdot$ if it is present and the appropriate axioms are satisfied (Jackson and Stokes [25]). Except in trivial cases there are infinitely many reduced permissible words over S, and then this representation is over an infinite domain, even when S is finite. We now observe a further identification that for finite S will produce a faithful representation over a finite domain for $\{;, \mathsf{d}, \mathsf{r}, 0\}$, though not in general for $\cdot$. Note that the proof method is illustrated over a single basic example in Example 5.2; the reader might find this useful while reading the proof.

Theorem 5.1. *Let $\mathcal{S}$ be an associative $\{;, \mathsf{d}, \mathsf{r}\}$-algebra satisfying the domain and range axioms* (d.II)–(d.IV) *and* (r.I)–(r.III). *Then $\mathcal{S}$ has a representation on a base of size at most $|S|^{1+|S|}$ and at most $(|S|-1)^{|S|}$ if there is a 0 element with $\mathsf{d}(0) = 0$.*

Proof. Let us say that the *address* of a sequence $(a_0, b_0, \ldots, a_n, b_n, a_{n+1})$ is a_{n+1}, and denote the address of sequence $\bar{a}$ by $\mathrm{add}(\bar{a})$. The *view* of a reduced permissible sequence $\bar{a}$ is the set

$$\begin{aligned} \mathrm{view}(\bar{a}) &= \{(x, y) \in S \times S \mid \bar{a} \in \mathrm{dom}(x^\theta) \text{ and } \mathrm{add}(x^\theta(\bar{a})) = y\} \\ &= \{(x, \mathrm{add}(x^\theta(\bar{a}))) \mid \mathrm{add}(\bar{a})\ ; \mathsf{d}(x) = \mathrm{add}(\bar{a})\}. \end{aligned}$$

A view is a partial function from S to S, hence when S is finite, the number of views is at most $|S|^{1+|S|}$. If there is $0 = \mathsf{d}(0)$ then we may replace S by $S \backslash \{0\}$, giving at most $(|S| - 1)^{|S|}$ views.

Let $\bar{a} = \mathrm{nf}(\bar{a}) = (a_0, b_0, \ldots, a_n, b_n, a_{n+1})$ be a reduced sequence. Then

$$\begin{aligned}(\mathsf{r}(\mathrm{add}(\bar{a})))^\theta(\bar{a}) &= (\mathsf{r}(a_{n+1}))^\theta(a_0, b_0, \ldots, a_n, b_n, a_{n+1}) \\ &= \mathrm{nf}(a_0, b_0, \ldots, a_n, b_n, a_{n+1}\,;\mathsf{r}(a_{n+1})) \\ &= \mathrm{nf}(a_0, b_0, \ldots, a_n, b_n, a_{n+1}) \\ &= \bar{a}.\end{aligned}$$

Hence $(\mathsf{r}(\mathrm{add}(\bar{a})), \mathrm{add}(\bar{a})) \in \mathrm{view}(\bar{a})$. In particular, for a sequence (c) of length 1, we have $(\mathsf{r}(c), c) \in \mathrm{view}(c)$ and $\mathrm{view}(c) = \{(x, c\,;x) \mid \mathsf{r}(c) \le \mathsf{d}(x)\}$.

Define an equivalence relation $\equiv$ on reduced permissible sequences by

$$\bar{a} \equiv \bar{b} \text{ if } \mathrm{view}(\bar{a}) = \mathrm{view}(\bar{b}). \tag{5.2}$$

Since $(\mathsf{r}(\mathrm{add}(\bar{a})), \mathrm{add}(\bar{a})) \in \mathrm{view}(\bar{a})$, we have that

$$\mathrm{view}(\bar{a}) = \mathrm{view}(\bar{b}) \text{ implies } \mathrm{add}(\bar{a}) = \mathrm{add}(\bar{b}).$$

In particular this shows that distinct permissible sequences of length 1 lie in distinct equivalence classes modulo $\equiv$. Also, if S is a finite set then there are only finitely many possible views, so that $\equiv$ has finitely many blocks. We now show that the functions x^θ preserve this equivalence relation, and that domains of functions x^θ are unions of blocks of the equivalence relation. Thus if X denotes the set of reduced permissible words, then $\mathcal{S}$ is also represented on $X/{\equiv}$ by the map

$$s^\Theta = (s^\theta/{\equiv}). \tag{5.3}$$

Fix $x \in S$ and assume that $\bar{a} \equiv \bar{b}$. By the definition of view we have that $\bar{a} \in \mathrm{dom}(x^\theta)$ if and only if $\bar{b} \in \mathrm{dom}(x^\theta)$. This shows that the domain of x^θ is a union of blocks.

Next we show that $x^\theta(\bar{a})$ is equivalent modulo $\equiv$ to $x^\theta(\bar{b})$. Let (z, c) be in the view of $x^\theta(\bar{a})$. So $x^\theta(\bar{a}) \in \mathrm{dom}(z^\theta)$ and $\mathrm{add}(z^\theta(x^\theta(\bar{a}))) = c$. That is, $\bar{a} \in \mathrm{dom}((x;z)^\theta)$ and $\mathrm{add}((x;z)^\theta(\bar{a})) = c$. Hence $((x;z), c) = ((x;z), \mathrm{add}((x;z)^\theta(\bar{a}))) \in \mathrm{view}(\bar{a}) = \mathrm{view}(\bar{b})$ as $\bar{a}$ and $\bar{b}$ have the same view. So $\bar{b} \in \mathrm{dom}((x\,;z)^\theta)$ and $\mathrm{add}((x\,;z)^\theta(\bar{b})) = c$. Thus $x^\theta(\bar{b}) \in \mathrm{dom}(z^\theta)$ and $\mathrm{add}(z^\theta(x^\theta(\bar{b}))) = c$, that is, $(z, c) \in \mathrm{view}(x^\theta(\bar{b}))$, as required.

The faithfulness of the representation Θ of $\mathcal{S}$ on $X/{\equiv}$ follows from the faithfulness of the representation θ of $\mathcal{S}$ on X, since this is witnessed over sequences of length 1 and distinct length 1 sequences are never equivalent. $\square$

The following basic example may aid the reader.

Example 5.2. *Consider the 5-element algebra in the signature $\{0, ;, \cdot, \mathsf{d}, \mathsf{r}\}$ consisting of elements $\{0, a, b, d, r\}$ with 0, d and r domain elements and with $\mathsf{d}(a) = \mathsf{d}(b) = d$ and $\mathsf{r}(a) = \mathsf{r}(b) = r$. All elements are disjoint (meeting to 0 under $\cdot$) and the only nonzero products with respect to ; are those forced by the usual properties of d and r, for example $d\,;a\,;r = a$. Note that*

$$\mathsf{d}(x) = \begin{cases} r & \text{if } x = r \\ d & x \ne r \end{cases} \quad \text{and } \mathsf{r}(x) = \begin{cases} d & \text{if } x = d \\ r & x \ne d \end{cases}$$

for non-zero x. So, a sequence $(x_0, x_1, \ldots, x_{2n})$ (where $n \ge 0$) of non-zero elements is permissible if (i) $(x_{2i}, x_{2i+1}) \in \{(d, d)\} \cup \{(w, z) \mid d \notin \{w, z\}\}$ *and* (ii) $(x_{2i+1}, x_{2i+2}) \in \{(r, r)\} \cup \{(w, z) \mid r \notin \{w, z\}\}$, *for $i < n$. Their equivalence relation $\equiv$ has six blocks, namely, the singleton $\{(r)\}$, the singleton $\{(d)\}$, and for $s = (a), (b), (a, d), (b, d)$, the set of all permissible sequences ending with the string s.*

We now extend this finite representation result to larger signatures including antidomain.

Theorem 5.3. *Let τ be a signature with $\{;, \mathsf{d}, \mathsf{r}\} \subseteq \tau \subseteq \{;, \mathsf{d}, \mathsf{a}, \mathsf{r}, \mathsf{fix}, 0, 1'\}$ or with $\{;, \mathsf{a}, \mathsf{r}, \sqcup\} \subseteq \tau \subseteq \{;, \mathsf{d}, \mathsf{a}, \mathsf{r}, \mathsf{fix}, \sqcup, {}^{\uparrow}, 0, 1'\}$. Every finite, representable τ-algebra $\mathcal{S} = (S, \tau)$ is representable over a base of size at most $|S|^{|S|+1}$.*

Proof. First assume that $\mathsf{a} \notin \tau$. So we are considering $\{;, \mathsf{d}, \mathsf{r}\} \subseteq \tau \subseteq \{;, \mathsf{d}, \mathsf{r}, \mathsf{fix}, 0, 1'\}$. The case $\tau = \{;, \mathsf{d}, \mathsf{r}\}$ is covered by Theorem 5.1, and it is clear that the representation Θ defined in (5.3) correctly represents 0 and $1'$ if one or both of these are present. We now show that Θ also already represents the operation fix correctly if $\mathsf{fix} \in \tau$. In [25] it was shown that, because the laws $\mathsf{fix}(x) ; x = \mathsf{fix}(x)$, $\mathsf{d}(\mathsf{fix}(x)) = \mathsf{fix}(x)$ and $x ; y = x \Rightarrow x ; \mathsf{fix}(y) = x$ are satisfied, Schein's representation will correctly represent fix. We now observe that fix is still correctly represented after applying the identification $\equiv$.

Consider an element x and a reduced sequence $\bar{a}$ in the domain of x^θ such that x^Θ fixes the $\equiv$-class $[\bar{a}]$ of $\bar{a}$, that is, $\bar{a} \equiv x^\theta(\bar{a})$. Thus the view of $x^\theta(\bar{a})$ is identical to that of $\bar{a}$. In particular, $x^\theta(\bar{a})$ has the same address as $\bar{a}$, and is either strictly shorter than $\bar{a}$ or is identical to $\bar{a}$. Let a denote the address of $\bar{a}$, and let $\bar{a}'$ denote $x^\theta(\bar{a})$ and $\bar{a}''$ denote $x^\theta(\bar{a}')$ and so on. Each element of $\bar{a}, \bar{a}', \bar{a}'', \ldots$ has the same view as $\bar{a}$ (so in particular, the same address, a). The sequence $\bar{a}, \bar{a}', \bar{a}'', \ldots$ is eventually constant, so eventually we arrive at some $\bar{b} \equiv \bar{a}$ that is fixed by x^θ. Because the address of $\bar{b}$ is a, we then have $a ; x = a$, which gives $a ; \mathsf{fix}(x) = a$. So, for any reduced permissible sequence $\bar{c} \equiv \bar{a}$, we have $x^\theta(\bar{c}) = \bar{c}$, since $\mathrm{add}(\bar{c}) = a$. Hence

$$\begin{aligned}
([\bar{a}], [\bar{a}]) \in \mathrm{fix}(x^\Theta) \quad &\iff \quad x^\Theta([\bar{a}]) = [\bar{a}] \\
&\iff \quad x^\theta(\bar{c}) = \bar{c} \ \ (\text{all } \bar{c} \equiv \bar{a}) \\
&\iff \quad (\bar{c}, \bar{c}) \in \mathrm{fix}(x^\theta) \ \ (\text{all } \bar{c} \equiv \bar{a}) \\
&\iff \quad (\bar{c}, \bar{c}) \in (\mathrm{fix}(x))^\theta \ \ (\text{all } \bar{c} \equiv \bar{a}) \\
&\iff \quad ([\bar{a}], [\bar{a}]) \in (\mathrm{fix}(x))^\Theta.
\end{aligned}$$

This completes the proof for cases where $\mathsf{a} \notin \tau$.

Now assume $\mathsf{a} \in \tau$ and let $\mathcal{S} \in \mathsf{Fun}(\tau)$ be finite and representable. We will temporarily ignore $\sqcup$ and ${}^{\uparrow}$ if they are present. Since $\{;, \mathsf{a}\} \subseteq \tau$ the set $D(S)$ of domain elements of $\mathcal{S}$ forms a Boolean algebra and since $\mathcal{S}$ is finite, this Boolean algebra is atomic. Consider the set $S^{at} = \{s \in \mathcal{S} : \mathsf{d}(s) \text{ is an atom of } D(S)\} \cup \{0\}$. It is clear that S^{at} is closed under all the operations of τ except a, so let $\mathcal{S}^{at}$ be the algebra with universe S^{at} and operations in $\tau \cap \{;, \mathsf{d}, \mathsf{r}, \mathsf{fix}, 0, 1'\}$ inherited from $\mathcal{S}$ (in fact it is isomorphic to the algebra $\mathcal{S}^\flat$ of Definition 4.7). By the previous part, Θ is a representation of $\mathcal{S}^{at}$ with respect to the signature $\tau \cap \{;, \mathsf{d}, \mathsf{r}, \mathsf{fix}, 0, 1'\}$ over a set of size at most $(|S^{at}| - 1)^{|S^{at}|} \leq (|S| - 1)^{|S|}$. By Lemma 4.9, Θ extends to a representation ϕ of $\mathcal{S}$ over the same base by letting

$$s^\phi = \bigcup \{(d ; s)^\Theta \mid d \in At(D(S))\}$$

see (4.2). Thus $\mathcal{S}$ is representable over a set of size at most $(|S| - 1)^{|S|}$.

All of this ignored $\sqcup$ and ${}^{\uparrow}$ if they were present in τ (with $\mathsf{a} \in \tau$). For these operations observe that if $\sqcup \in \tau$ then as $\mathcal{S} \in \mathsf{Fun}(\tau)$, then laws ($\sqcup$.I) and ($\sqcup$.II) will hold, so that the "moreover" statement of Theorem 3.4 part (1) ensures the correct representation of $\sqcup$. If both $\sqcup$ and $^\uparrow$ are in τ, then laws ($\sqcup$.I), ($\sqcup$.II) and ($\uparrow$.I), ($\uparrow$.II) hold and the "moreover" statement of Theorem 3.4 part (2) shows that both $\sqcup$ and $^\uparrow$ are correctly represented provided that ; and a have been correctly represented (which we showed could be achieved on a set of size at most $(|S|-1)^{|S|}$). □

Of course if S is infinite, an application of the downward Löwenheim–Skolem Theorem yields a representation on a base set of size at most $|S|$.

In general, intersection is not preserved by the representation method in Theorem 5.1. If $\bar{a}$ is in the domain of x^θ and y^θ with $x \cdot y = 0$ then it is still possible that the view of $x^\theta(\bar{a})$ coincides with the view of $y^\theta(\bar{a})$. This occurs in Example 5.2 when $x = a$, $y = b$ and $\bar{a} = (a, b, d)$ for example.

We now turn to signatures excluding range; finite representability can be extracted from [27], but the approach we take here gives an alternative perspective. We first make the following definition, which we use again later.

Definition 5.4. *Let $\mathcal{S}$ be a concrete algebra of functions on a base X, in some signature $\tau \subseteq \{;, \cdot, \mathsf{d}, \mathsf{a}, \mathsf{fix}, \sqcup, ^\uparrow, 0, 1'\}$. For a subset $Y \subseteq X$ and function $f \in \mathcal{S}$, we let the* relativization *$f{\restriction_Y}$ of f to Y be $f \cap (Y \times Y)$. Operations in τ may also be relativised, with only antidomain and identity requiring amended definitions: we let $\mathsf{a}{\restriction_Y}(f) = \mathsf{a}(f) \cap (Y \times Y)$ and $1'{\restriction_Y} = 1' \cap (Y \times Y)$. We let $\mathcal{S}{\restriction_Y}$ denote the set $\{f{\restriction_Y} \mid f \in \mathcal{S}\}$ and $\mathcal{S}_Y$ denote the τ-algebra generated by $\mathcal{S}{\restriction_Y}$ under the relativised operations.*

We mention that in general, all three of $\mathcal{S}$, $\mathcal{S}{\restriction_Y}$ and $\mathcal{S}_Y$ can be algebraically different. Indeed $\mathcal{S}{\restriction_Y}$ need not even be closed under composition: even if $f ; g = h$ in $\mathcal{S}$ it is possible that $f{\restriction_Y} ; g{\restriction_Y}$ is different to $h{\restriction_Y}$. So $\mathcal{S}_Y$ can have "extra" elements not obtained directly by relativising from $\mathcal{S}$. Also, distinct functions $f \neq g$ in $\mathcal{S}$ might act identically on Y, in which case $f{\restriction_Y} = g{\restriction_Y}$. In our use of relativisation, the set Y is chosen carefully so as to keep certain properties of the original $\mathcal{S}$ intact. In the proof of the next theorem for example we choose Y so that $\mathcal{S}_Y$ is in fact closed under composition and find $\mathcal{S}_Y \cong \mathcal{S}$.

Theorem 5.5. *Let $; \in \tau \subseteq \{;, \cdot, \mathsf{d}, \mathsf{a}, \mathsf{fix}, \sqcup, ^\uparrow, 0, 1'\}$ (but no range operation). If $\mathcal{S} \in \mathsf{Fun}(\tau)$ is finite then it has a representation on a base of size at most $|S|^3$.*

Proof. Let $\mathcal{S} \in \mathsf{Fun}(\tau)$ be a finite algebra of functions over a base X. For each pair (s, t) where $s, t \in \mathcal{S}$ and $s \neq t$, let $x_{s,t}$ be an arbitrary element of X such that s disagrees with t at $x_{s,t}$, so either both s, t are defined at $x_{s,t}$ and $s(x_{s,t}) \neq t(x_{s,t})$ or just one of them is defined (such a point must exist, since s, t are distinct functions). Let

$$Y = \{x_{s,t}, u(x_{s,t}) \mid s, t, u \in \mathcal{S},\ s \neq t,\ u \text{ defined at } x_{s,t}\}.$$

Clearly $|Y| \leq |\mathcal{S}|^3$. Observe that

$$y \in Y,\ v \in \mathcal{S},\ v(y) \text{ defined } \Rightarrow v(y) \in Y \tag{5.4}$$

because if $y = u(x_{s,t})$ then $v(y) = v(u(x_{s,t})) = (u ; v)(x_{s,t}) \in Y$. We now show that with this choice of Y, the relativisation $\mathcal{S}{\restriction_Y}$ of $\mathcal{S}$ to Y is isomorphic to $\mathcal{S}$.

We claim that the map $\theta \colon s \mapsto s{\restriction_Y}$ is an isomorphism from $\mathcal{S}$ to $\mathcal{S}{\restriction_Y}$. For any $s \neq t \in \mathcal{S}$, $s{\restriction_Y}$ disagrees with $t{\restriction_Y}$ at $x_{s,t}$, so $\theta(s) \neq \theta(t)$ whence θ is injective. Clearly $0{\restriction_Y} = 0$, $1'{\restriction_Y} = 1'_Y$ by definition, and $(s{\restriction_Y}) \cdot (t{\restriction_Y}) = (s \cdot t){\restriction_Y}$, so θ respects $0, 1'$ and $\cdot$. Equation (5.4) shows that θ also respects $;, \mathsf{d}, \mathsf{fix}, \sqcup, ^\uparrow$. Take composition, for example.

$$\begin{aligned}
(x,y)\in\theta(u;v) &\iff x,y\in Y \ \&\ (x,y)\in(u;v) \\
&\iff x\in Y \ \&\ y=(u;v)(x)=v(u(x)) && \text{by (5.4)}\\
&\iff (x,u(x))\in\theta(u) \ \&\ (u(x),y)\in\theta(v) && \text{by (5.4)}\\
&\iff (x,y)\in\theta(u)\,;\theta(v)
\end{aligned}$$

Similarly, for maximum iterate, we have the following.

$$\begin{aligned}
(x,y)\in\theta(u^{\uparrow}) &\iff x,y\in Y \ \&\ (x,y)\in u^{\uparrow}\\
&\iff x,y\in Y \ \&\ (\exists k\geq 0)\exists x_0,\dots,x_k \bigwedge_{i<k}(x_i,x_{i+1})\in u\\
&\qquad\qquad \&\ x=x_0 \ \&\ y=x_k\notin \mathrm{dom}(u)\\
&\iff (\exists x_0,\dots,x_k\in Y)\bigwedge_{i<k}(x_i,x_{i+1})\in\theta(u)\\
&\qquad\qquad \&\ x=x_0 \ \&\ y=x_k\notin\mathrm{dom}(\theta(u))\\
&\iff (x,y)\in(\theta(u))^{\uparrow}
\end{aligned}$$

Checking the preservation of the other operations is similar. Hence θ is an isomorphism. □

Similarly, the finite representation property is easy to establish for signatures that cannot express d. This leaves one group of cases.

Problem 5.6. *Let* $\{;,\cdot,\mathsf{d},\mathsf{r}\}\subseteq\tau\subseteq\{;,\cdot,\mathsf{d},\mathsf{a},\mathsf{r},\mathsf{fix},\sqcup,{}^{\uparrow},0,1'\}$. *Is it the case that every finite member of* $\mathsf{Fun}(\tau)$ *has a representation on a finite base? In particular, does the finite representation property hold for the signature* $\tau=\{;,\cdot,\mathsf{a},\mathsf{r},0\}$ *and the signature* $\{;,\cdot,\mathsf{d},\mathsf{r},0\}$*?*

Remark 5.7. *Since the submission of this manuscript, Brett McLean* [33] *has succeeded in implementing a careful inductive argument to establish the finite representation property for the signatures involving* $\{;,\cdot,\mathsf{a},\mathsf{r}\}$. *The case* $\{;,\cdot,\mathsf{d},\mathsf{r},0\}$ *can also be covered by extending McLean's proof.*

6. Equational Theory

Let $\tau\subseteq\{;,\cdot,\mathsf{d},\mathsf{r},\mathsf{a},\mathsf{fix},\sqcup,0,1'\}$. A *term* is either a single variable symbol, a constant 0 or $1'$, or recursively $(t_1\,;t_2),(t_1\cdot t_2),(t_1\sqcup t_2),\mathsf{d}(t),\mathsf{r}(t),\mathsf{a}(t)$ or $\mathsf{fix}(t)$ (if the relevant operations are in τ), where t_1,t_2,t are terms. We write $t(\bar{x})$ for a term using only variables in the n-tuple of variables $\bar{x}$, but not necessarily all of them. For $\mathcal{S}\in\mathsf{Fun}(\tau)$ and a n-tuple $\bar{a}$ of elements (functions) in $\mathcal{S}$, we interpret $t(\bar{a})$ as a partial function over the base of $\mathcal{S}$ as we explained in the Preliminaries section. An *equation* has the form $t(\bar{x})=s(\bar{x})$, where $\bar{x}$ is an n-tuple of variables and $t(\bar{x}),s(\bar{x})$ are terms. It is valid if for every $\mathcal{S}\in\mathsf{Fun}(\tau)$ and every n-tuple $\bar{a}$ of elements of $\mathcal{S}$, the partial functions $t(\bar{a})$ and $s(\bar{a})$ are identical. (For the sake of simplicity, we will call $t(\bar{a})$ etc. terms as well.)

Theorem 6.1. *Let* $\tau\subseteq\{;,\cdot,\mathsf{d},\mathsf{r},\mathsf{a},\mathsf{fix},\sqcup,0,1'\}$ *and* Σ_τ *be the set of equations valid over* $\mathsf{Fun}(\tau)$.

1. Σ_τ *is co-NP.*
2. *If* $\{;,\mathsf{a}\}\subseteq\tau$ *then* Σ_τ *is co-NP-complete.*

Proof. For the first part, let $\bar{x}$ be an n-tuple of variables and suppose the equation $u(\bar{x}) = v(\bar{x})$ is not valid over $\mathsf{Fun}(\tau)$. Then there is a concrete algebra of functions $\mathcal{S} \in \mathsf{Fun}(\tau)$ and $u(\bar{a}) \neq v(\bar{a})$, for some n-tuple $\bar{a} = (a_0, a_1, \ldots, a_{n-1})$ of elements in $\mathcal{S}$. Let X denote the set on which the elements of $\mathcal{S}$ are functions. The two distinct functions $u(\bar{a}), v(\bar{a})$ must disagree on at least one point of X (disagreement includes the possibility that one partial function is defined at this point but not the other). Recall the notion of relativisation from Definition 5.4. The plan is to find a finite subset $Y \subseteq X$ of size no more than the sum of the lengths of the terms $u(\bar{a}), v(\bar{a})$ such that the equation already fails in the relativised algebra $\mathcal{S}_Y$ of functions on the set Y. Some observations here are useful:

- We do not require that $\mathcal{S}_Y$ be isomorphic to $\mathcal{S}$, only that it continues to witness failure of the law $u(\bar{x}) = v(\bar{x})$.
- The algebra $\mathcal{S}_Y$ on Y failing $u(\bar{x}) = v(\bar{x})$ might possibly be exponentially larger than $|Y|$, however the base of $\mathcal{S}_Y$ is Y, so a non-deterministic machine can, in quadratic time, choose an evaluation of $\bar{x}$ to a tuple $\bar{b}$ of functions over Y, choose a point $y \in Y$ and verify that $u(\bar{b})$ does not agree with $v(\bar{b})$ at y.

Our task is to construct Y in such a way that $u(\bar{x}) = v(\bar{x})$ still fails in $\mathcal{S}_Y$. We consider terms $t(\bar{a})$ constructed from $a_0, a_1, \ldots, a_{n-1}$ using operations in τ. Such a term is directly evaluated in $\mathcal{S}$ using the set-theoretically defined operations of $\mathcal{S}$. We write $t(\bar{a})\restriction_Y$ for that function in $\mathcal{S}\restriction_Y$ obtained from the tuple of elements $(a_0 \cap (Y \times Y), a_1 \cap (Y \times Y), \ldots, a_{n-1} \cap (Y \times Y))$, using the (relativised) constants and operations of $\mathcal{S}\restriction_Y$. Warning: it is not in general true that $t(\bar{a})\restriction_Y = t(\bar{a}) \cap (Y \times Y)$, for example let a be the function $\{(y, z)\}$ (some $y \neq z \in X$) and let $Y = \{y\}$, then $a\restriction_Y$ is empty and hence $(\mathsf{d}(a))\restriction_Y = \varnothing$, whereas $\mathsf{d}(a) \cap (Y \times Y) = \{(y, y)\}$.

For fixed $x \in X$ and term $t(\bar{a})$ using only the elements $a_0, \ldots, a_{n-1}$ of $\mathcal{S}$, we construct a finite subset $\Sigma(t(\bar{a}), x)$ of X. When applied to the terms $u(\bar{a})$ and $v(\bar{a})$ selected above (and for which $u(\bar{a})$ and $v(\bar{a})$ disagree at some point), this will provide a small finite subset of X in which $u(\bar{x}) \neq v(\bar{x})$ remains witnessed. The construction is by induction on the complexity of subterms of $t(\bar{a})$.

$$
\begin{aligned}
\Sigma(0, x) &= \{x\} \\
\Sigma(1', x) &= \{x\} \\
\Sigma(a_i, x) &= \begin{cases} \{x, a_i(x)\} & \text{if } a_i \text{ is defined on } x \\ \{x\} & \text{otherwise} \end{cases} \quad (i < n) \\
\Sigma(\mathsf{fix}(s(\bar{a})), x) &= \{x\} \\
\Sigma(\mathsf{d}(s(\bar{a})), x) &= \Sigma(\mathsf{a}(s(\bar{a})), x) = \Sigma(s(\bar{a}), x) \\
\Sigma(\mathsf{r}(s(\bar{a})), x) &= \begin{cases} \Sigma(s(\bar{a}), y) & \text{some arbitrary } y \text{ with } s(\bar{a})(y) = x \\ \{x\} & \text{if no such } y \text{ exists} \end{cases} \\
\Sigma(s_1(\bar{a}) \,;\, s_2(\bar{a}), x) &= \begin{cases} \Sigma(s_1(\bar{a}), x) \cup \Sigma(s_2(\bar{a}), s_1(\bar{a})(x)) & \text{if } (s_1(\bar{a})(x)) \text{ is defined} \\ \Sigma(s_1(\bar{a}), x) & \text{otherwise} \end{cases} \\
\Sigma(s_1(\bar{a}) \cdot s_2(\bar{a}), x) &= \Sigma(s_1(\bar{a}), x) \cup \Sigma(s_2(\bar{a}), x) \\
\Sigma(s_1(\bar{a}) \sqcup s_2(\bar{a}), x) &= \Sigma(s_1(\bar{a}), x) \cup \Sigma(s_2(\bar{a}), x)
\end{aligned}
$$

Clearly, the size of $\Sigma(t(\bar{a}), x)$ is no bigger than twice the length of the term $t(\bar{a})$. We may drop the $\bar{a}$ and refer to a term simply as t. Observe that $x \in \Sigma(t, x)$ and that $t(x) \in \Sigma(t, x)$ whenever t is defined on x.

We claim that, for any Y with $\Sigma(t,x) \subseteq Y \subseteq X$, the following holds:

$$t(x) = t{\restriction_Y}(x) \tag{6.1}$$

including that $t(x)$ is defined iff $t{\restriction_Y}(x)$ is defined.

For the base cases, observe that 0 is the empty function in both $\mathcal{S}$ and $\mathcal{S}{\restriction_Y}$, $1'(x) = 1'_Y(x) = x$, since $x \in \Sigma(1', x) \subseteq Y$, and for $t = a_i$ (some $i < n$), a_i is defined at x iff $a_i{\restriction_Y} = a \cap (Y \times Y)$ is defined at x (since $\{x, a_i(x)\} \subseteq \Sigma(a_i, x) \subseteq Y$) and if defined they are equal.

Next suppose $t = r \cdot s$ (some terms r, s). Then $r \cdot s$ is defined at x iff $r(x) = s(x)$ ($= y$, say) iff $r{\restriction_Y}(x) = s{\restriction_Y}(x) = y$ (inductively) iff $(r \cdot s){\restriction_Y}(x) = y$. Similarly $r \sqcup s$ is defined at x (say $(r \sqcup s)(x) = y$) iff either $r(x) = y$ or r is not defined at x but $s(x) = y$. This holds if and only if $r{\restriction_Y}(x) = y$ or $r{\restriction_Y}$ is not defined at x but $s{\restriction_Y}(x) = y$ (inductively) iff $(r \sqcup s){\restriction_Y}(x) = y$. The case $t = \mathsf{fix}(s)$ (some term s) is also similar (after all $\mathsf{fix}(v) = 1' \cdot v$).

Consider the case $t = \mathsf{a}(s)$ (some term s). We have: $(\mathsf{a}(s))(x)$ is defined and equal to x iff $s(x)$ is not defined iff (by induction) $s{\restriction_Y}(x)$ is not defined iff $(\mathsf{a}s){\restriction_Y}(x)$ is defined and equal to x, since $\Sigma(s, x) = \Sigma(\mathsf{a}(s), x) \subseteq Y \subseteq X$. The case $t = \mathsf{d}(s)$ is similar.

Now let $t = r \,;\, s$ (some terms r, s). If $r \,;\, s$ is defined at x then $(r \,;\, s)(x) = s(r(x))$ and $\{x, r(x), s(r(x))\} \subseteq \Sigma(r \,;\, s, x) \subseteq Y$. So $r{\restriction_Y}(x) = r(x)$ is defined and $s{\restriction_Y}(r(x)) = s(r(x))$ is also defined, hence $(r \,;\, s){\restriction_Y}(x) = s(r(x))$ is defined. Conversely, if $(r \,;\, s){\restriction_Y}(x)$ is defined then $r(x) = r{\restriction_Y}(x)$ is defined and $s{\restriction_Y}(r(x)) = s(r(x))$ is also defined, so $(r \,;\, s)(x) = (r \,;\, s){\restriction_Y}(x)$ is also defined.

Finally consider the case $t = \mathsf{r}(s)$ (some term s). If $\mathsf{r}(s)$ is defined at x then by definition of $\Sigma(\mathsf{r}(s), x)$ there is $y \in \Sigma(\mathsf{r}(s), x)$ with $s(y) = x$ and $\Sigma(\mathsf{r}(s), x) = \Sigma(s, y)$. By induction, $s{\restriction_Y}(y) = x$, so $(\mathsf{r}(s)){\restriction_Y}(x) = x$. Conversely, if $(\mathsf{r}(s)){\restriction_Y}(x)$ is defined then there is $y \in Y$ with $s{\restriction_Y}(y) = x$, hence $s(y) = x$ and $(\mathsf{r}(s))(x) = x$, as required. This proves the claim (6.1).

Now recall that $u(\bar{x}) = v(\bar{x})$ was an equation failing in $\mathcal{S}$ under some assignment mapping the tuple of variables $\bar{x}$ to the tuple of elements $\bar{a}$ of $\mathcal{S}$ such that the two functions $u(\bar{a}), v(\bar{a})$ are distinct. That is, $u(\bar{a})$ and $v(\bar{a})$ disagree at some point, say $u(\bar{a})$ disagrees with $v(\bar{a})$ at $x \in X$. Let $Y = \Sigma(u(\bar{a}), x) \cup \Sigma(v(\bar{a}), x)$. By the claim, $u(\bar{a}){\restriction_Y}(x)$ agrees with $u(\bar{a})(x)$ (both defined and equal or neither defined) and $v(\bar{a})(x)$ agrees with $v(\bar{a}){\restriction_Y}(x)$, hence $u(\bar{a}){\restriction_Y}$ disagrees with $v(\bar{a}){\restriction_Y}$ at x. So the equation $u(\bar{x}) = v(\bar{x})$ fails in $\mathcal{S}_Y$ under the assignment mapping x_i to $a_i \cap (Y \times Y)$, for $i < n$.

Thus an equation $u(\bar{x}) = v(\bar{x})$ fails to be valid over $\mathsf{Fun}(\tau)$ if and only if it fails in some concrete algebra $\mathcal{S}$ of functions on a set Y, with the size $|Y|$ being only linear in terms of the equation. It follows that we can test the failure of equations by nondeterministically generating a labelled directed graph of this size and verifying if the equation fails.

For the second part, we reduce the validity problem for propositional formulas (see [13, §A9] for example) to membership of Σ_τ. We may assume that our propositional language includes only the connectives $\neg, \wedge$. Take a propositional formula ϕ and replace each proposition p by $\mathsf{d}(f_p)$ for some function symbol f_p unique to p and replace $\neg$ and $\wedge$ by a and ;, respectively, to obtain a term ϕ^*. The required reduction maps ϕ to the equation $\phi^* = 1'$. This reduction is correct by Lemma 4.3. $\square$

As mentioned in the introduction, one cannot hope for such a result when maximal iterate is included in the signature, as the equational theory is known to be Π^1_1-hard [14], at least in signatures containing $\{;, \mathsf{fix}, \mathsf{a}, {}^\uparrow\}$. For signatures avoiding antidomain and maximal iterate, one might expect that co-NP is only a rough upper bound of the complexity. An efficient algorithm for deciding equalities in the language $\{;, \cdot, \mathsf{d}\}$ can be found in [24]: once routine denesting of domain terms has occurred, this process is trivially seen to be polynomial time decision procedure. A comprehensive understanding in the signature $\{;, \cdot, \mathsf{d}, \mathsf{r}\}$ would be very interesting. We mention here the work of Kambites [29] and Kambites and Kazda [30], where there is a transparent description of term reduction in a very closely related class of semigroup-related structures (the *adequate semigroups*), which is shown to be polynomial time solvable.

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