CONVEXITY OF THE CARRYING SIMPLEX FOR DISCRETE-TIME PLANAR COMPETITIVE KOLMOGOROV SYSTEMS

STEPHEN BAIGENT, DEPT. MATH, UCL, GOWER STREET, LONDON WC1E 6BT.

ABSTRACT. We consider the geometry of carrying simplices of discrete-time competitive Kolmogorov systems. An existence theorem for the carrying simplex based upon the Hadamard graph transform is developed, and conditions for when the transform yields a sequence of convex or concave graphs are determined. As an application it is shown that the planar Leslie-Gower model has a carrying simplex that is convex or concave.
Keywords: Carrying simplex, convexity and concavity, globally attracting invariant manifold.

1. INTRODUCTION

Many planar discrete-time competitive ecological models with a repelling origin have a carrying simplex; that is, a one-dimensional invariant manifold linking two axial fixed points that attracts all points except the origin [6, 14, 15, 19, 23, 24, 14]. Examples of the carrying simplex are shown in Figure 1 for the Leslie-Gower model discussed below.

Here we study the geometry of the carrying simplex of the well-known planar, discrete-time Leslie-Gower model from ecology. This model is competitive and its convergent dynamics are well-understood through application of known results for planar competitive systems [21, 5] (see results summary below).

Rather than global stability of fixed points *per se*, we are principally concerned with the geometry of the carrying simplex. It is known that the Leslie-Gower model has a carrying simplex for a large range of relevant parameter values [12, 18]. The aim of this paper is to obtain existence of the carrying simplex through a dynamic approach and to determine the shape of the carrying simplex. In particular, we seek conditions for the carrying simplex to be convex or concave.

Our method is similar to that developed for the continuous-time planar competitive Lotka-Volterra equations [1]. We use the Hadamard graph transform method: Start with the straight line joining the invariant axial fixed points and evolve this line forward under the discrete-time flow [9, 20]. By global attraction, this line evolves under the flow to the unique carrying simplex. The well-studied monotonicity properties of planar competitive systems that are usually applied pointwise (e.g. [6, 20]) to establish ordering between points, in the graph transform context lead to preservation of the sign of the gradient of the evolving curve. This fact, plus injectivity of the map on suitable domains, is exploited to obtain existence of a unique invariant curve joining the axial fixed points which is identified as the carrying simplex. Our conditions for the carrying simplex to exist are slightly different
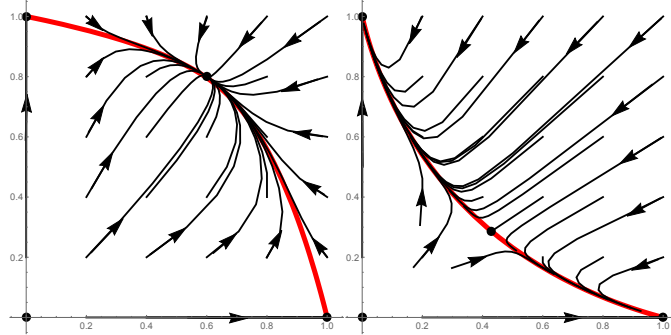
FIGURE 1.   Examples of the carrying simplex (the red line joining the axial fixed points) for the Leslie-Gower Model (see equation (9)). (a) shows a convex carrying simplex Σ with a globally stable interior fixed point ($\alpha = 2, \beta = 2, a = 1/2, b = 1/3$). (b) shows a concave carrying simplex Σ with an interior fixed point that repels in Σ ($\alpha = 2, \beta = 2, a = 2, b = 5/3$). Note that the accepted terminology for carrying simplicies states that Σ is convex if the line segment joining two points on Σ lie on or below Σ. When a convex Σ is the graph of a function, that function is then concave.

from those recently used by other authors (e.g. [12, 18]) and are motivated in part by requiring phase space volume to be decreasing with time.

To investigate the carrying simplex geometry, the first step is to show that the curvature at every point of the evolving curve is either positive or negative after one iteration. The second step is to show that the sign of curvature cannot change. When these two steps can be achieved, standard results from convex analysis then show that the carrying simplex, which is the limit of the evolving curves, is either convex or concave. For the Leslie-Gower model we are able to show that the carrying simplex is only ever convex or concave with a single inequality on the parameters to distinguish between the two possibilities. This is completely analogous to the planar Lotka-Volterra model in continuous time [22, 1].

In the continuous-time planar competitive Lotka-Volterra model, it turns out that stability of interior equilibria can be related to the geometry of the carrying simplex. When the carrying simplex is convex a unique interior fixed point is globally stable (attracts all interior orbits), whereas it is unstable when the carrying simplex is concave. As shown in [25] this convexity-global stability relation extends to $n-$dimensional competitive Lotka-Volterra systems. In a forthcoming paper [2], we clarify the convexity-global stability relationship for discrete-time competitive Kolmogorov systems. As might be expected, for the Leslie-Gower model considered here, a unique fixed point is globally stable when the carrying simplex is convex (see Figure 1 (a)) and unstable when it is concave (Figure 1 (b)).

2. Carrying simplices for discrete-time planar competitive systems

The planar discrete-time systems that we study are of the form

(1)
$$\begin{aligned} x_{t+1} &= F(x_t, y_t) = x_t f(x_t, y_t) \\ y_{t+1} &= G(x_t, y_t) = y_t g(x_t, y_t) \end{aligned} \qquad t \in \mathbb{N} := \{0, 1, 2, \ldots\}.$$

We will set $T(x, y) = (F(x, y), G(x, y))$ and assume that F, G are at least twice differentiable on an open set in $\mathbb{R}^2$ containing $\mathbb{R}^2_+$ (here $\mathbb{R}_+ = [0, \infty)$). In all the models we consider, phase space is $\mathbb{R}^2_+$ and $f > 0, g > 0$ on $\mathbb{R}^2_+$. We use D to denote the derivative, and DT means the Jacobian matrix for T.

We will say that a map $T : A \to \mathbb{R}^2$ ($A \subset \mathbb{R}^2_+$) is *competitive* when $x < y$ whenever $T(x) < T(y)$ and $x, y \in A$, and strongly competitive when $x \ll y$ whenever $T(x) < T(y)$ and $x, y \in A$. Ruiz-Herrera [18] calls a map *retrotone* when it has the similar property: $x_i < y_i$ whenever $T(x) < T(y)$ and $x_i \neq 0$, $x, y \in A$.

Introduce the assumptions

(C1) $f_x, f_y, g_x, g_y < 0$ on $\mathbb{R}^2_+$;

(C2) Equation (1) has a unique and globally attracting fixed point when restricted to each positive axis; denote them by $e_1 = (A, 0)$ and $e_2 = (0, B)$ where $A, B > 0$;

(C3) $\rho(M(x, y)) < 1 \;\; \forall (x, y) \in \Omega = [0, A] \times [0, B]$ where ρ denotes spectral radius and

(2)
$$M(x, y) = - \begin{pmatrix} \frac{x}{f(x,y)} f_x(x, y) & \frac{x}{f(x,y)} f_y(x, y) \\ \frac{y}{g(x,y)} g_x(x, y) & \frac{y}{g(x,y)} g_y(x, y) \end{pmatrix}.$$

(C4) There is a $\lambda > 1$ such that $[0, \lambda A] \times [0, \lambda B]$ is an attractor for $\mathbb{R}^2_+$.

Remark 1. *The conditions above are stated for $\mathbb{R}^2_+$, but are a special case of the $n-$dimensional problem as treated in [12, 18] (under less restrictive conditions on differentiability of F, G). Other related conditions for the $n-$dimensional problem appear in [19, 15, 7] and the recent paper [16].*

Remark 2. *Since $\rho(M(x, y)) < 1$, $I - M(x, y)$ is invertible and $(DT)^{-1} = (I - M)^{-1} diag(1/f, 1/g) = (\sum_{k=0}^{\infty} M^k) diag(1/f, 1/g) > 0$ by virtue of C1. Hence C1 and C3 imply that $(DT)^{-1} \gg 0$ on Ω [12, 18]. In proposition 4.1 of [18], the author uses this property of the inverse on $\mathbb{R}^N_+$ and proposition 2.1 of [14] to conclude that T is retrotone and one-to-one on $\mathbb{R}^N_+$. An alternative approach worth mentioning is to note that since Ω is rectangular, results for maps whose derivative are $P-$matrices may be used. In particular, theorem 5 in [8] applies to yield $T(x) \leq T(y) \Rightarrow x \leq y$ whenever $x, y \in \Omega$.*

C2 shows that each axial fixed point is globally attracting on its respective positive coordinate axis and implies that $f(0, 0) > 1, g(0, 0) > 1$. Accordingly $DT(0, 0) = diag(f(0, 0), g(0, 0))$ has eigenvalues exceeding unity and hence the origin $O = (0, 0)$ is a repeller in $\mathbb{R}^2_+$. It follows from Remark 2 and Theorem 4.1 of [24], that T is injective on Ω. Condition C3, which appears in [12, 18], also plays an important role in ensuring that the carrying simplex is unique. The last condition (C4) ensures that all points in $\mathbb{R}^2_+$ eventually enter $[0, \lambda A] \times [0, \lambda B]$ and remain there.

Under conditions C1 - C4 there exists a unique carrying simplex; that is, a Lipschitz invariant manifold passing through all non-trivial fixed points (i.e. those other than O) of (1) and that attracts $\mathbb{R}^2_+ \setminus O$ [12, 18, 7].

Many of the techniques utilised to study competitive discrete-time maps owe their origins to P. de Mottoni and Shiaffino's study of the attractors and repellers of the competitive Poincaré map of the periodic competitive Lotka-Volterra equations [6]. In particular Theorem 5 of [6] describes the carrying simplex Σ (although not named as such there) as the relative boundary of the basin of repulsion of O for the Poincaré map, and shows that Σ is the graph of a decreasing function connecting the axial equilibria. In [17] Theorem 5, the carrying simplex appears as the unstable manifold of a saddle fixed point p, and is a decreasing curve that joins fixed points, but the case where p is an interior stable fixed point is not covered. For a similar description of Σ in terms of stable and unstable manifolds, see also Theorem 6.9 of [10].

3. Alternative conditions for existence of the planar carrying simplex

In some applications the map T is not injective on the rectangle Ω and some refinements are necessary. We introduce an approach that also leads naturally to the investigation of the curvature of Σ. We use the graph transform method introduced by Hadamard [9], originally for differential equations, and widely used in the study of stable, unstable and centre manifolds of the fixed points of maps (see, for example, [11]).

We recall that $\Lambda \subset U(\text{open})$ is absorbing (for T in U) if for each bounded $B \subset U$ there exists a $\tau(B)$ such that $T^t(B) \subset \Lambda$ for all $t \geq \tau(B)$.

Theorem 1. *Assume*

(C1') *There exists a rectangle $R = [0, a] \times [0, b]$ with $a > A, b > B$ (with A, B as in C2) where $F_x > 0, G_y > 0, F_y \leq 0, G_x \leq 0$ and $\int_\gamma F_y \, ds < 0$ for any line segment $\gamma \subset R$ of positive length.*

(C2) *Equation (1) has a unique and globally attracting fixed point when restricted to each positive axis: $e_1 = (A, 0), e_2 = (0, B)$ where $A, B > 0$;*

(C3') *There exists a compact and connected absorbing set $\Lambda \subseteq R$ for T such that $\det DT > 0$ in Λ, and $\left(\frac{1}{fg} \det DT \right) < 1$ on $\Lambda \setminus O$.*

Then there exists a carrying simplex Σ for (1); that is a 1-dimensional invariant and Lipschitz manifold that attracts $\mathbb{R}^2_+ \setminus O$.

Conditions C1' and C3' imply that the restriction $T : \Lambda \to T\Lambda$ is injective. To see this note that T is a proper map (using the closed mapping theorem and that Λ is compact) and locally invertible, since $\det DT > 0$ in Λ. Thus by Lemma 2.3.4 in [3] (since we have assumed that Λ is connected) the number of elements in $T^{-1}(y)$ is a constant r for each $y \in T\Lambda$. To show that $r = 1$ we just recall that we assume throughout that $f, g > 0$ on $\mathbb{R}^2_+$ so that $T^{-1}(0) = \{0\}$.

The following lemma shows that when $M(x, y) \geq 0$ (as under C1), $C3'$ can be linked to $C3$.

Lemma 1. *When $M(x, y) \geq 0$, for some $(x, y) \in \mathbb{R}^2_+$ the condition $\rho(M(x, y)) < 1$ is equivalent to $0 < \frac{\det DT(x,y)}{f(x,y)g(x,y)} < 1$.*

Proof. Note that $\frac{1}{fg} \det DT = \det(I - M) = (1 - \lambda_1)(1 - \lambda_2)$ where λ_1, λ_2 are the eigenvalues of M. Since $M \geq 0$, both eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\max\{\lambda_1, \lambda_2\} \geq 0$. If $\rho(M) < 1$ then $|\lambda_1| < 1, |\lambda_2| < 1$ and $0 < (1 - \lambda_1)(1 - \lambda_2) < 1$ so that $0 < \frac{1}{fg} \det DT < 1$. □

We will split the proof of Theorem 1 into 2 parts. The first part will show the existence of an invariant curve that connects the 2 axial fixed points. The second part proves that this curve attracts all points in $\mathbb{R}_+^2 \setminus O$. We use $\mathsf{graph}(\phi)$ to denote the graph of a function ϕ and for each set S, $\phi|_S$ denotes the restriction of ϕ to S.

We will use the Schauder fixed point theorem that says that a continuous operator Φ from a convex compact subset K of a Banach space B into K has a fixed point, i.e. an $x^* \in K$ such that $\Phi(x^*) = x^*$ (e.g. [4]). Since we are concerned with invariant manifolds of a discrete-time flow, our approach is to find the carrying simplex Σ as the fixed point of a sequence of manifolds $\{M_t\}_{t=0}^{\infty}$ in a suitable convex space K of manifolds. We choose K to be the set bounded Lipschitz manifolds of Lipschitz constant at most one 1, since the invariant limits are always objects with no holes on which a reduced system of one dimension less can be studied. At some steps in the construction, however it is easier to work with functions ϕ_t whose graphs $\mathsf{graph}(\phi_t) = M_t$, since the Lipschitz property of each graph M_t comes (in our setting) from the fact that each function ϕ_t is a decreasing function (in the coordinates of (1)).

In the following we use a simple coordinate change (a rotation by $\pi/4$) to relate decreasing functions to bounded rank-1 Lipschitz manifolds.

Proof. (of Theorem 1)

3.0.1. *Existence of an invariant manifold.* Existence uses the assumptions C1' and C2 only plus the Schauder fixed point theorem. Here we find it more convenient to work with a new rotated coordinate system.

The flow of (1) in new coordinates $(u, v) = (\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}})$ is given by $u_{t+1} = \frac{1}{\sqrt{2}} \left(F(\frac{u+v}{\sqrt{2}}, \frac{v-u}{\sqrt{2}}) - G(\frac{u+v}{\sqrt{2}}, \frac{v-u}{\sqrt{2}}) \right) = \mathsf{F}(u, v)$ and $v_{t+1} = \frac{1}{\sqrt{2}} \left(F(\frac{u+v}{\sqrt{2}}, \frac{v-u}{\sqrt{2}}) + G(\frac{u+v}{\sqrt{2}}, \frac{v-u}{\sqrt{2}}) \right) = \mathsf{G}(u, v)$. Let $L_1(A, B)$ denote the set of Lipschitz functions $\phi^* : I_0 := [-\frac{B}{\sqrt{2}}, \frac{A}{\sqrt{2}}] \to \mathbb{R}_+$ with Lipschitz constant at most 1 that satisfy $\phi^*(\frac{A}{\sqrt{2}}) = \frac{A}{\sqrt{2}}$ and $\phi^*(\frac{-B}{\sqrt{2}}) = \frac{B}{\sqrt{2}}$. The map T induces a map $\mathcal{T}^* : L_1(A, B) \to L_1(A, B)$. To see this, consider distinct $u, v \in I_0$ and let us suppose that $\phi^* \in L_1(A, B)$. If $\phi^* \mapsto \mathcal{T}^*\phi^*$ under the flow T, then $\mathcal{T}^*\phi^*$ has Lipschitz constant at most 1 on I_0 if

$$|\mathsf{G}(u, \phi^*(u)) - \mathsf{G}(v, \phi^*(v))| \leq |\mathsf{F}(u, \phi^*(u)) - \mathsf{F}(v, \phi^*(v))|, \quad \forall u, v \in I_0.$$

But for Γ the line segment joining the points with parameter values u, v,

$$\mathsf{G}(u, \phi^*(u)) - \mathsf{G}(v, \phi^*(v)) = \int_\Gamma \mathsf{G}_u \, ds(u - v) + \int_\Gamma \mathsf{G}_v \, ds(\phi^*(u) - \phi^*(v))$$

$$= (u - v) \left(\int_\Gamma \mathsf{G}_u \, ds + \int_\Gamma \mathsf{G}_v \, ds \left(\frac{\phi^*(u) - \phi^*(v)}{u - v} \right) \right)$$

and similarly

$$\mathsf{F}(u, \phi^*(u)) - \mathsf{F}(v, \phi^*(v)) = (u - v) \left(\int_\Gamma \mathsf{F}_u \, ds + \int_\Gamma \mathsf{F}_v \, ds \left(\frac{\phi^*(u) - \phi^*(v)}{u - v} \right) \right).$$

Now $\mathsf{F}_u = \frac{1}{2}(F_x - G_x - F_y + G_y)$, $\mathsf{F}_v = \frac{1}{2}(F_x - G_x + F_y - G_y)$, $\mathsf{G}_u = \frac{1}{2}(F_x + G_x - F_y - G_y)$, $\mathsf{G}_v = \frac{1}{2}(F_x + G_x + F_y + G_y)$. Let $\mathsf{a} = \int_\Gamma F_x \, ds$, $\mathsf{b} = \int_\Gamma F_y \, ds$, $\mathsf{c} = \int_\Gamma G_x \, ds$, $\mathsf{d} = \int_\Gamma G_y \, ds$. Then we obtain

$$
\frac{|\mathsf{G}(u, \phi^*(u)) - \mathsf{G}(v, \phi^*(v))|}{|\mathsf{F}(u, \phi^*(u)) - \mathsf{F}(v, \phi^*(v))|} = \frac{\left| \int_\Gamma \mathsf{G}_u \, ds + \int_\Gamma \mathsf{G}_v \, ds \left(\frac{\phi^*(u) - \phi^*(v)}{u - v} \right) \right|}{\left| \int_\Gamma \mathsf{F}_u \, ds + \int_\Gamma \mathsf{F}_v \, ds \left(\frac{\phi^*(u) - \phi^*(v)}{u - v} \right) \right|}
$$

$$
(3) \qquad\qquad = \frac{|\mathsf{a} + \mathsf{c} - \mathsf{b} - \mathsf{d} + (\mathsf{a} + \mathsf{b} + \mathsf{c} + \mathsf{d})\theta|}{|\mathsf{a} - \mathsf{c} - \mathsf{b} + \mathsf{d} + (\mathsf{a} - \mathsf{c} + \mathsf{b} - \mathsf{d})\theta|} = \zeta(\theta)
$$

where $\theta = \frac{\phi^*(u) - \phi^*(v)}{u - v}$.

We will now use that $\mathsf{a} > 0, \mathsf{d} > 0$ and $\mathsf{b} < 0$ (see condition C1') and $\mathsf{c} \leq 0$ together with $|\theta| \leq 1$ to show that (i) the denominator of ζ cannot vanish, and (ii) $\zeta(\theta) \leq 1$ for $\theta \in [-1, 1]$. For (i), the denominator vanishes if and only if $\theta = \frac{\mathsf{a} - \mathsf{c} - \mathsf{b} + \mathsf{d}}{\mathsf{d} + \mathsf{c} - \mathsf{a} - \mathsf{b}}$. But $\mathsf{a} - \mathsf{c} > 0$ and $\mathsf{d} - \mathsf{b} > 0$ so that $\left| \frac{\mathsf{a} - \mathsf{c} - \mathsf{b} + \mathsf{d}}{\mathsf{d} + \mathsf{c} - \mathsf{a} - \mathsf{b}} \right| = \left| \frac{\frac{\mathsf{a} - \mathsf{c}}{\mathsf{d} - \mathsf{b}} + 1}{\frac{\mathsf{a} - \mathsf{c}}{\mathsf{d} - \mathsf{b}} - 1} \right| > 1$. This shows (i). For (ii), first note that the expression $\zeta(\theta)$ in (3) is monotonic in θ. When $\theta = -1$ we have $\zeta(-1) = \left| \frac{-\mathsf{b} - \mathsf{d}}{\mathsf{d} - \mathsf{b}} \right| < 1$ (since $\mathsf{b} < 0$), and when $\theta = +1$ we have $\zeta(+1) = \left| \frac{\mathsf{a} + \mathsf{c}}{\mathsf{a} - \mathsf{c}} \right| \leq 1$. Since ζ is continuous and monotonic on $[-1, 1]$ and $\zeta(-1) \leq 1$, $\zeta(+1) \leq 1$ we have $\zeta(\theta) \leq 1$ for all $\theta \in [-1, 1]$. The conclusion is that $\mathcal{T}^* \phi^* \in L_1(A, B)$ if $\phi^* \in L_1(A, B)$.

$\mathcal{T}^* : L_1(A, B) \to L_1(A, B)$ is continuous and satisfies $(\mathcal{T}^* \phi^*)(\mathsf{F}(u, \phi^*(u))) = \mathsf{G}(u, \phi^*(u))$, $u \in I_0$ for each $\phi^* \in L_1(A, B)$. By C2, $f(0, 0) > 1, g(0, 0) > 1$ which implies there exists an $\nu > 0$ such that $f(x, y) > 1$ and $g(x, y) > 1$ for $(x, y) \in \mathbb{R}_+^2 \setminus E_\nu$, where $E_\nu := \{(x, y) \in \mathbb{R}_+^2 : x + y \geq \nu\}$. E_ν is forward invariant under T. Now define the set $L_{1,\nu}(A, B) = \{\max\{\phi^*, \nu\} : \phi^* \in L_1(A, B)\}$. Then $L_{1,\nu}(A, B)$ is mapped into itself under $\mathcal{T}^*$, and is a compact convex subset of the Banach space of continuous functions on I_0. Hence by the Schauder fixed point theorem $\mathcal{T}^*$ has a fixed point in $L_{1,\nu}(A, B)$ which we denote by $\hat{\phi}^*$. By construction, $\Sigma = \mathsf{graph}(\hat{\phi}^*)$ is an invariant manifold of Lipschitz rank 1 with $(A, 0), (0, B) \in \Sigma$ and $O \notin \Sigma$.

We can say more about Σ, namely that it may be also represented as the graph (in x, y coordinates) of a locally Lipschitz and decreasing function $\hat{\phi}$.

For suppose that $\hat{\phi}^*$ does not become a locally Lipschitz and decreasing function $\hat{\phi}$ under the change of coordinates from (u, v) to (x, y). Then there are distinct points $u, v \in I_0$ such that $\hat{\phi}^*(u) - \hat{\phi}^*(v) = v - u$. Since $\mathsf{graph}(\hat{\phi}^*)$ is invariant $\mathcal{T}^* \hat{\phi}^* = \hat{\phi}^*$ and $\left| \frac{\mathcal{T}^* \hat{\phi}^*(u) - \mathcal{T}^* \hat{\phi}^*(v)}{u - v} \right| < 1$ by setting $\theta = -1$ in (3) and using that $\mathsf{d} > 0$, and $\mathsf{b} < 0$. This contradicts that $\hat{\phi}^*(u) - \hat{\phi}^*(v) = v - u$. Hence $\hat{\phi}$ is a locally Lipschitz and decreasing function.

3.0.2. *Global attraction.* To obtain global attraction of $\mathbb{R}_+^2 \setminus O$ to Σ we will use injectivity and contraction of phase space by using a suitable measure. Since the absorbing set $\Lambda \subset R$, it is sufficient to show that Σ attracts $R \setminus O$. Here we find it more convenient to work with x, y coordinates, and nonincreasing functions.

Let $L_{d,loc}([0, a])$ denote the set of locally Lipschitz functions $\phi : [0, a] \to [0, b]$ that are strictly decreasing on their support. Take $\phi_0 \in L_{d,loc}([0, a])$ with nonempty support $[0, \chi_0) \subset [0, a]$. It is straight-forward to show (formally) that under the

mapping T the function ϕ_0 evolves to a new function ϕ_1 given implicitly by

$$(4) \qquad \phi_1(F(x, \phi_0(x)) = G(x, \phi_0(x)), \ x \in [0, a].$$

To properly define ϕ_1, we need that $F(\cdot, \phi_0(\cdot))$ is invertible on $[0, \chi_1]$ where $\chi_1 = T_1(\chi_0, 0)$. We have for $0 \le x' \le x \le \chi_1$ that

$$F(x, \phi_0(x)) - F(x', \phi_0(x')) \ = \ F(x, \phi_0(x)) - F(x', \phi_0(x)) + F(x', \phi_0(x)) - F(x', \phi_0(x')).$$

Now $F(x, \phi_0(x)) > F(x', \phi_0(x))$ since $F_x > 0$, and the combination that ϕ_0 is decreasing and $F_y \le 0$ implies that $F(x', \phi_0(x)) \ge F(x', \phi_0(x'))$. Hence $F(\cdot, \phi_0(\cdot))$ is strictly increasing and hence invertible on $[0, \chi_1]$. Moreover,

$$\begin{aligned}
\phi_1(F(x, \phi_0(x)) - \phi_1(F(x', \phi_0(x')) \ &= \ G(x, \phi_0(x)) - G(x', \phi_0(x')) \\
&= \ \{G(x, \phi_0(x)) - G(x', \phi_0(x))\} \\
&\quad + \{G(x', \phi_0(x)) - G(x', \phi_0(x'))\} < 0
\end{aligned}$$

by virtue of (C1'), so that since $F(\cdot, \phi_0(\cdot))$ is invertible, ϕ_1 is decreasing for $x \in [0, \chi_1]$. Hence ϕ_1 is a decreasing function on $[0, \chi_1]$ and we extend ϕ_1 to $[0, a]$ by defining $\phi_1(x) = 0$ for $x \in [\chi_1, a]$. Hence functions in $L_{d,loc}([0, a])$ are mapped into functions in $L_{d,loc}([0, a])$ by the map T.

Choose any $(x_0, y_0) \in R := [0, a] \times [0, b]$ and let ϕ_0 be any smooth decreasing function with $\mathsf{graph}(\phi_0) \subset R$, $y_0 = \phi_0(x_0)$ and $\phi_0(\chi) = 0$ for some $\chi \in (0, a]$. Construct the sequence of decreasing functions $\{\phi_t\}_{t \in \mathbb{N}}$ with each $\phi_t \in L_{d,loc}([0, a])$ using the above construction so that

$$(5) \qquad \phi_{t+1}(F(x, \phi_t(x)) = G(x, \phi_t(x)), \ x \in [0, a].$$

For t large enough, say $t \ge \tau$, $\mathsf{graph}(\phi_t) \subset \Lambda \cap E_\nu$.

Set $Q_t = \{(x, y) \in R : \min\{\phi_t(x), \hat{\phi}(x)\} \le y \le \max\{\phi_t(x), \hat{\phi}(x)\}\}$. For $t \ge \tau$, $Q_t \subset \Lambda \cap E_\nu$ and T is injective on Λ, T is invertible on Q_t. For each $\epsilon > 0$ introduce the area form $dA = (x + \epsilon)^{-1}(y + \epsilon)^{-1}dx\, dy$. Then for $t \ge \tau + 1$, and σ satisfying $\sup_{\Lambda \cap E_\nu} \frac{1}{fg} \det DT < \sigma < 1$,

$$\begin{aligned}
\int_{Q_t} dA \ &= \ \int_{T^{-1}Q_t} \frac{1}{(T_1(x, y) + \epsilon)(T_2(x, y) + \epsilon)} \det DT(x, y)\, dx\, dy \\
&= \ \int_{Q_{t-1}} \frac{1}{(x + \epsilon/f(x,y))(y + \epsilon/g(x,y))} \left\{ \frac{1}{f(x,y)g(x,y)} \det DT(x, y) \right\} dx\, dy \\
&< \ \sigma \int_{Q_{t-1}} \frac{1}{(x + \epsilon/f(x,y))(y + \epsilon/g(x,y))}\, dx\, dy \\
&= \ \int_{Q_{t-1}} \frac{\sigma(x + \epsilon)(y + \epsilon)}{(x + \epsilon/f(x,y))(y + \epsilon/g(x,y))}\, dA < \sigma \int_{Q_{t-1}} dA,
\end{aligned}$$

using that $Q_t \in \Lambda \cap E_\nu$ for $t \ge \tau + 1$, and by choosing $\epsilon > 0$ sufficiently small. To summarise, we have shown that the dA–measure of the set of points between the graph of the decreasing continuous curve ϕ_t and the Lipschitz invariant manifold Σ decreases with iterations. Moreover, we have $\int_{Q_t} dA \to 0$ as $t \to \infty$. Since $dA = \frac{1}{(x+\epsilon)(y+\epsilon)} dx\, dy$, the (Lebesgue) area of $Q_t \to 0$ as $t \to \infty$.

Now we show that as a consequence $\phi_t \to \hat{\phi}$ pointwise. We work with the scaled $\psi_t(v) := \frac{1}{\xi_t} \phi_t(\chi_t v)$, and $\hat{\psi}(v) = \frac{1}{B} \hat{\phi}(vA)$, $v \in [0, 1]$ where $\chi_t = T_1^t(\chi_0, 0)$, $\xi_t = T_2^t(0, \xi_0)$ and $\chi_0 = \chi$, $\xi_0 = \phi_0(0)$. Suppose that there exists an $s \in (0, 1)$ such that $\psi_t(s) \nrightarrow \hat{\psi}(s)$ as $t \to \infty$. Then there is an $\epsilon > 0$ such that for all $N \in \mathbb{N}$

there exists $t(N) \geq N$ for which $|\psi_{t(N)}(s) - \hat{\psi}(s)| \geq \epsilon$. If $\psi_{t(N)}(s) \geq \hat{\psi}(s) + \epsilon$ then let $s' := (\hat{\psi})^{-1}(\hat{\psi}(s) + \epsilon) < s$ and $J = [s', s]$. Then the area $S_N := \int_J \psi_{t(N)}(z) - \hat{\psi}(z)\, dz > \int_J \hat{\psi}(s) + \epsilon - \hat{\psi}(z)\, dz > 0$, since $\psi_{t(N)}$ is a strictly decreasing function. If $\psi_{t(N)}(s) \leq \hat{\psi}(s) - \epsilon$ then let $s'' := (\hat{\psi})^{-1}(\hat{\psi}(s) - \epsilon) > s$ and $J' = [s, s'']$. Then the area $T_N := \int_{J'} \hat{\psi}(z) - \psi_{t(N)}(z)\, dz > \int_{J'} \hat{\psi}(z) + \epsilon - \hat{\psi}(s)\, dz > 0$. We may choose a subsequence $\{t(N_k)\}_{k \in \mathbb{N}}$ of $\{t(N)\}_{N \in \mathbb{N}}$ such that either $\psi_{t(N_k)}(s) \geq \hat{\psi}(s) + \epsilon$ for all $k \in \mathbb{N}$ or $\psi_{t(N_k)}(s) \leq \hat{\psi}(s) - \epsilon$ for $k \in \mathbb{N}$. For simplicity, let us assume that former case holds; the latter case is similar. Then $S_{N_k} \geq \mu$ for some $\mu > 0$ for all $k \in \mathbb{N}$. On the other hand, as the area of Q_N tends to zero as $N \to \infty$, the area $S_{N_k} \to 0$ as $k \to \infty$, and so we obtain a contradiction. Hence $\psi_t(s) \to \hat{\psi}(s)$ as $t \to \infty$ for each $s \in (0, 1)$. $\square$

4. CURVATURE OF THE CARRYING SIMPLEX

Consider the convergent sequence $\{\phi_t\}_{t \in \mathbb{N}}$ in $L_{d,loc}([0, a])$ described immediately above, with the initial curve $\phi_0(x) = B - x/A$. Since ϕ_t inherits the smoothness of ϕ_0 we have on $[0, A]$

(6) $$\phi'_{t+1}(F(x, \phi_t(x)) = \frac{G_x(x, \phi_t(x)) + G_y(x, \phi_t(x))\phi'_t(x)}{F_x(x, \phi_t(x)) + F_y(x, \phi_t(x))\phi'_t(x)}.$$

By differentiation of (6), we find that

$$(F_x + F_y\hat{\phi}'_t)\phi''_{t+1} \circ F + \frac{G_x + G_y\phi'_t}{(F_x + F_y\phi'_t)^2}(F_{xx} + 2F_{xy}\phi'_t + F_{yy}\phi'^2_t + F_y\phi''_t) =$$

$$\frac{G_{xx} + 2G_{xy}\phi'_t + G_{yy}\phi'^2_t + G_y\phi''_t}{F_x + F_y\phi'_t}.$$

It is possible to write this last expression in a more concise form by setting $\xi = (1, \phi'_t)^T$:
(7)
$$\phi''_{t+1} \circ F = \frac{1}{(DF^T\xi)^3}\left((DF^T\xi)\xi^T D^2 G\xi - (DG^T\xi)\xi^T D^2 F\xi\right) + \left(\frac{F_x G_y - F_y G_x}{(DF^T\xi)^3}\right)\phi''_t.$$

We now note that taking into account C1' and C3', the coefficient of ϕ''_t in (7) is always positive. Let us set

(8) $$\Delta_t = (DF^T\xi)\xi^T D^2 G\xi - (DG^T\xi)\xi^T D^2 F\xi,$$

where the righthand side is evaluated at $(x, \phi_t(x))$.

We always start by taking ϕ_0 to be the linear function whose graph joins the two axial fixed points; the principal reason being that then $\phi''_0(x) = 0$ for $x \in [0, A]$ and $\chi_t = A$, $\xi_t = B$ for all $t \in \mathbb{N}$. Arguing by induction, suppose that ϕ_t is convex and $\Delta_t(x) > 0$ for all $x \in [0, A]$ and $t \in \mathbb{N}$. Then we see that, for each $t \in \mathbb{N}$, $\phi''_{t+1} \circ F > 0$ for all $x \in [0, A]$, so by inverting $F(\cdot, \phi_t(\cdot))$ as above we see that $\phi''_{t+1}(x) > 0$ for $x \in [0, A]$. Thus $\{\phi_t\}_{t \in \mathbb{N}}$ is a sequence of bounded convex functions. Similarly, for each $t \in \mathbb{N}$, if $\Delta_t(x) < 0$ for all $x \in [0, A]$ then $\phi''_{t+1} < 0$ and $\{\phi_t\}_{t \in \mathbb{N}}$ is then a sequence of bounded concave functions. Thus in the case where each $\phi''_t > 0$ for $t = 1, 2, \ldots$, Σ is the graph of a convex function, and in the case where $\phi''_t < 0$ for $t = 1, 2, \ldots$, Σ is the graph of a concave function.

Definition 1 ((Convexity/Concavity of Σ).)**.** *Let the carrying simplex Σ be the graph of a function ϕ on $[0, A]$. Then we say that Σ is convex(convex) when the function ϕ is concave(convex).*

Theorem 2. *Consider the system (1) subject to conditions C1, C2 and C3'. Then for $\Delta_t, t \in \mathbb{N}$ defined by (8):*

 (1) *If $\Delta_t > 0$ for all $t \in \mathbb{N}$ then the carrying simplex is concave.*
 (2) *If $\Delta_t < 0$ for all $t \in \mathbb{N}$ then the carrying simplex is convex.*

5. APPLICATION: LESLIE-GOWER MODEL

For this model

(9) $$F(x, y) = \frac{\alpha x}{1 + x + ay}, \quad G(x, y) = \frac{\beta y}{1 + y + bx}.$$

The fixed points are $e_0 = (0, 0)$, $e_1 = (\alpha - 1, 0)$, $e_2 = (0, \beta - 1)$ and $p = (\frac{a(1-\beta)+\alpha-1}{1-ab}, \frac{\beta-1+b(1-\alpha)}{1-ab})$ which only exists when either (a) $a < \frac{\alpha-1}{\beta-1} < \frac{1}{b}$ or (b) $b < \frac{\alpha-1}{\beta-1} < \frac{1}{a}$. Cushing et al. [5] show that if (a) $\alpha, \beta < 1$ then e_0 is globally asymptotically stable on $\mathbb{R}_+^2$, (b) $\alpha > 1, \beta < 1$ then e_1 is globally asymptotically stable on $\mathrm{int}\mathbb{R}_+^2$, (c) $\alpha < 1, \beta > 1$ then e_2 is globally asymptotically stable on $\mathrm{int}\mathbb{R}_+^2$. When $\alpha > 1, \beta > 1$, e_0 is a repeller and there are 4 distinct cases: When (a) $b(\alpha - 1) > \beta - 1, \alpha - 1 > a(\beta - 1)$ then e_1 is asymptotically stable on $\mathrm{int}\mathbb{R}_+^2$ and e_2 is a saddle, (b) $b(\alpha-1) < \beta-1, \alpha-1 < a(\beta-1)$ e_2 is asymptotically stable on $\mathrm{int}\mathbb{R}_+^2$ and e_1 is a saddle, (c) when $b(\alpha - 1) < \beta - 1, \alpha - 1 > a(\beta - 1)$ then the interior fixed point p is globally asymptotically stable on $\mathrm{int}\mathbb{R}_+^2$ and (d) when $b(\alpha - 1) > \beta - 1, \alpha - 1 < a(\beta - 1)$ then the interior fixed point p is a saddle.

Here we are concerned with the case $\alpha, \beta > 1$.

First we must show existence and uniqueness of the carrying simplex.

Lemma 2. *The Leslie-Gower model (9) has a unique carrying simplex for all $a, b > 0$ and $\alpha, \beta > 1$.*

Proof. Conditions C1' and C2 are easy to verify with $R = \mathbb{R}_+^2$ and $A = \alpha - 1$, $B = \beta - 1$. Now consider C3':

$$DT(x, y) = \begin{pmatrix} \frac{\alpha(1+ay)}{(1+x+ay)^2} & \frac{-\alpha a x}{(1+x+ay)^2} \\ \frac{-\beta b y}{(1+bx+y)^2} & \frac{\beta(1+bx)}{(1+bx+y)^2} \end{pmatrix}, \quad M(x, y) = \begin{pmatrix} \frac{x}{1+x+ay} & \frac{ax}{1+x+ay} \\ \frac{by}{1+bx+y} & \frac{y}{1+bx+y} \end{pmatrix}.$$

Thus $\det DT(x, y) = \frac{\alpha\beta(1+ay+bx)}{(1+x+ay)^2(1+bx+y)^2} > 0$ on $\mathbb{R}_+^2$. T is strongly competitive on $\mathrm{int}\mathbb{R}_+^2$. T is competitive and a local homeomorphism on $\mathbb{R}_+^2$, so that by Theorem 4.1 of [24], T is injective on the whole of $\mathbb{R}_+^2$. Moreover, $0 < \frac{1}{fg}\det DT = \frac{1+ay+bx}{(1+x+ay)(1+y+bx)} < 1$ in $\mathbb{R}_+^2 \setminus O$ and hence condition C3' is satisfied. By Theorem 1 Σ exists. For the absorbing set Λ we may choose $\Lambda = [0, \alpha] \times [0, \beta]$. $\square$

We now reveal the geometry of Σ:

Theorem 3. *For the map defined by (9) with $\alpha, \beta > 1$ and $a, b > 0$ there is a unique carrying simplex Σ which is convex, concave, or a straight line. Set $\delta = (1 + a(\beta - 1))(1 + b(\alpha - 1)) - \alpha\beta$. Then if (i) $\delta = 0$, Σ is the straight line segment joining $e_1 = (\alpha - 1, 0)$ and $e_2 = (0, \beta - 1)$, (ii) $\delta > 0$ then Σ is concave, and if (iii) $\delta < 0$ then Σ is convex.*

(Thus, as a reminder $\hat{\phi}$ is a convex function when $\delta > 0$, but Σ is referred to as concave).

Proof. Let us first find conditions for the line $e_1 e_2$ to be invariant. The straight line through e_1 and e_2 is given by $\phi(x) = \beta - 1 + \left(\frac{1-\beta}{\alpha-1}\right)x$. By invariance we require from $\phi(F(x, \phi(x))) = G(x, \phi(x))$ the identity

$$\frac{\beta\phi(x)}{1 + \phi(x) + bx} = \phi\left(\frac{\alpha x}{1 + x + a\phi(x)}\right).$$

Hence we compute

$$1 - \beta - \frac{x\alpha(1-\beta)}{(\alpha-1)\left(a\left(\frac{x(1-\beta)}{\alpha-1} + \beta - 1\right) + x + 1\right)} + \frac{\beta\left(\frac{x(1-\beta)}{\alpha-1} + \beta - 1\right)}{bx + \frac{x(1-\beta)}{\alpha-1} + \beta} = 0.$$

This simplifies to

$$\frac{x(\beta-1)(x - \alpha + 1)(a(\beta-1)(b(\alpha-1)+1) + b(\alpha-1) - \alpha\beta + 1)}{((x+1)(\alpha-1) - a(\beta-1)(x-\alpha+1))(x(b(\alpha-1) - \beta + 1) + (\alpha-1)\beta)} = 0.$$

For this to hold for all $x \in [0, \alpha-1]$ we need

$$a(\beta-1)(b(\alpha-1)+1) + b(\alpha-1) - \alpha\beta + 1 = 0,$$

which tidies to

$$\delta = (1 + a(\beta-1))(1 + b(\alpha-1)) - \alpha\beta = 0.$$

This proves part (i) of the theorem.

Now we start with $\phi_0(x) = (1-\beta)(\frac{x}{\alpha-1} - 1)$. It is easily shown from (8) that

$$\Delta_t = \frac{2\alpha\beta\left(a\phi_t(x) + 1 - ax\phi_t'(x)\right)\left(b\phi_t(x) - (bx+1)\phi_t'(x)\right)}{(a\phi_t(x) + x + 1)^3(bx + \phi_t(x) + 1)^3}$$
$$(10) \qquad \times \left((1 - a + (1-ab)x)\phi_t'(x) + (ab-1)\phi_t(x) + b - 1\right).$$

We note that since each $\phi_t' < 0$, $\Delta_t > 0(< 0)$ when

$$(11) \qquad h_t(x) = ((1-ab)x + 1 - a)\phi_t'(x) + (ab-1)\phi_t(x) + b - 1 > 0(< 0).$$

A simple computation shows that when $t = 0$, $h_0(x) = \frac{\delta}{\alpha-1}$ for all $x \in [0, \alpha-1]$. In fact, as we now show that when $\delta > 0$, $h_t(0) \geq \frac{\delta}{\alpha} > 0$ for all $t \in \mathbb{N}$. We have $h_t(0) = (1-a)\phi_t'(0) + (ab-1)(\beta-1) + b - 1$. Arguing by induction, we have $h_0(0) > 0$ if $\delta > 0$. If $h_t(0) \geq 0$ then $(1-a)\phi_t'(0) \geq 1 - b + (1-ab)(\beta-1)$, and using (11)

$$
\begin{aligned}
h_{t+1}(0) &= (1-a)\phi_{t+1}'(0) + (ab-1)(\beta-1) + b - 1 \\
&= (1-a)\left(\frac{-b(\beta-1) + \phi_t'(0)}{\alpha\beta/(1 + a(\beta-1))}\right) + (ab-1)(\beta-1) + b - 1 \\
&= (1 + a(\beta-1))\left(\frac{-b(\beta-1)(1-a) + (1-a)\phi_t'(0)}{\alpha\beta}\right) + (ab-1)(\beta-1) + b - 1 \\
&\geq (1 + a(\beta-1))\left(\frac{-b(\beta-1)(1-a) + 1 - b + (1-ab)(\beta-1)}{\alpha\beta}\right) \\
&\quad + (ab-1)(\beta-1) + b - 1 \\
&= \frac{1}{\alpha}(1 + b(\alpha-1) + a(1 + b(\alpha-1))(\beta-1) - \alpha\beta) = \frac{\delta}{\alpha}.
\end{aligned}
$$

Thus when $\delta > 0$, $h_t(0) > 0$ for all $t \in \mathbb{N}$. When $\delta < 0$, $h_0(0) < 0$, so now $(1 - a)\phi_t'(0) \leq 1 - b + (1 - ab)(\beta - 1)$. Now apply induction in the same way to obtain $h_{t+1} \leq \frac{\delta}{\alpha} < 0$. The conclusion is that when $\delta > 0$, $h_t(0) > 0$ for all $t \in \mathbb{N}$ and when $\delta < 0$, $h_t(0) < 0$ for all $t \in \mathbb{N}$.

At the other end of the interval, $x = \alpha - 1$, we have the recursion relation for the derivative: $w_{t+1} = \frac{\frac{\alpha\beta}{1+b(\alpha-1)} w_t}{1 - a(\alpha-1)w_t}$, where $w_t = \phi_t'(\alpha - 1)$. As usual we consider the sequence w_t with $w_0 = \frac{\beta-1}{1-\alpha}$. The gradient of the function $\vartheta(w) = \frac{\frac{\alpha\beta}{1+b(\alpha-1)} w}{1 - a(\alpha-1)w}$ at $w = 0$ is $\xi = \frac{\alpha\beta}{1+b(\alpha-1)} > 0$. When $\xi > 1$ there are two fixed points, namely $w_1^* = 0$ and $w_2^* = \frac{1+b(\alpha-1)-\alpha\beta}{a(\alpha-1)(1+b(\alpha-1))} < 0$. When $\xi < 1$, $w^* = 0$ is the unique fixed point. Also we have $\vartheta(\frac{\beta-1}{1-\alpha}) = \frac{\alpha\beta}{(1+b(\alpha-1))(1+a(\beta-1))} \frac{\beta-1}{\alpha-1}$. Thus if $\delta > 0$, $\vartheta(w_0) > w_0$ and if $\delta < 0$ then $\vartheta(w_0) < w_0$. Since ϑ is increasing, when $\delta > 0$, the sequence w_t with $w_0 = \frac{\beta-1}{1-\alpha}$ satisfies $w_{t+1} = \vartheta^{t+1}(w_0) = \vartheta^t(\vartheta(w_0)) > \vartheta^t(w_0) = w_t$, so that w_t is then an increasing sequence. On the other hand, if $\delta < 0$ then w_t with the same $w_0 = \frac{\beta-1}{1-\alpha}$ is a decreasing sequence.

When $\xi < 1$, then $\delta > 0$, and so w_t increases from $w_0 < 0$ to the next fixed point which is $w^* = 0$. When $\xi > 1$ there are two fixed points $0, w_2^*$ and $w_0 - w_2^* = \frac{-\delta}{a(\alpha-1)(1+b(\alpha-1))}$. When $\delta > 0$ we have $w_0 < w_2^*$ and w_t increases to the next fixed point which is w_2^*, and when $\delta < 0$ we have $w_0 > w_2^*$ and w_t decreases to the next fixed point which is also w_2^*.

We recall that $h_0(\alpha - 1) = \frac{\delta}{\alpha-1}$. Using information of the derivatives obtained above, we will now show that when $\delta > 0$ then $h_t(\alpha-1) > 0$ for all $t \in \mathbb{N}$ and when $\delta < 0$ then $h_t(\alpha-1) < 0$ for all $t \in \mathbb{N}$. We set $v(s) = ((1-ab)(\alpha-1)+1-a)s+b-1$. Since v is affine in s, when $I = [c, d]$ is an interval of $\mathbb{R}$, we have $v(s) > 0$ for $s \in [c, d]$ if and only if $v(c) > 0$ and $v(d) > 0$.

Suppose that $\xi < 1$, which implies that $\delta > 0$. Then also we have from $\xi = \frac{\alpha\beta}{1+b(\alpha-1)} < 1$, upon rearrangement that $b > \frac{\alpha\beta-1}{\alpha-1} > 1$ since $\alpha, \beta > 1$. We established above that when $\xi < 1$, $w_t \in [\frac{1-\beta}{\alpha-1}, 0)$. But $v(\frac{1-\beta}{\alpha-1}) = \frac{\delta}{\alpha-1} > 0$ and $v(0) = b-1 > 0$. Hence we see that $h_t(\alpha-1) = v(w_t) > 0$ for all $t \in \mathbb{N}$ when $\xi < 1$.

When $\xi > 1$ we may have either $\delta > 0$ or $\delta < 0$, but in both cases monotone convergence to w_2^*: When $\delta > 0, w_t \uparrow w_2^*$ so $w_t \in [\frac{1-\beta}{\alpha-1}, w_2^*)$, and when $\delta < 0, w_t \downarrow w_2^*$ so $w_t \in (w_2^*, \frac{1-\beta}{\alpha-1}]$. Now, after simplification, we find that $v(w_2^*) = \frac{\alpha\delta}{a(\alpha-1)(1+b(\alpha-1))}$. Thus when $\delta > 0$, v is positive at both $s = \frac{1-\beta}{\alpha-1}$ and $s = w_2^*$, so that we conclude that when $\xi > 1$ and $\delta > 0$ then $h_t(\alpha-1) > 0$ for all $t \in \mathbb{N}$. Lastly, when $\xi > 1$ and $\delta < 0$, v is negative at both $s = w_2^*$ and $s = \frac{1-\beta}{\alpha-1}$, so that we conclude that when $\xi > 1$ and $\delta < 0$ then $h_t(\alpha-1) < 0$ for all $t \in \mathbb{N}$.

To summarise the above result: If $\delta > 0$ then $h_t(0) > 0, h_t(\alpha-1) > 0$ for all $t \in \mathbb{N}$ and if $\delta < 0$ then $h_t(0) < 0, h_t(\alpha-1) < 0$ for all $t \in \mathbb{N}$.

Having studied h_t at the endpoints $x = 0, \alpha - 1$, we now consider $h_t(x)$ where $x \in (0, \alpha - 1)$.

Differentiating equation (11) we obtain

$$h_t'(x) = ((1 - ab)x + (1 - a))\phi_t''(x).$$

Thus $h_t'(x) = 0$ for some $x^* \in (0, \alpha - 1)$ if and only if at least one of the factors $(1 - ab)x^* + 1 - a$ and $\phi_t''(x^*)$ vanish.

If t is the first time that ϕ''_{t+1} changes sign on $(0, \alpha - 1)$, then since $\phi''_t(x) \geq 0$ for $x \in (0, \alpha - 1)$, the only point in $(0, \alpha - 1)$ where h'_t can change sign is $x^* = \frac{a-1}{1-ab} \in (0, \alpha - 1)$ and this requires either (a) $1 < a < 1/b$ and $a - 1 < (\alpha - 1)(1 - ab)$ or (b) $1 > a > 1/b$ and $a - 1 > (\alpha - 1)(1 - ab)$. In the case where neither of (a) or (b) hold, h_t is either nonincreasing or nondecreasing on $[0, \alpha - 1]$ and hence $h_t(x) > 0$ for all $x \in [0, \alpha - 1]$ when $\delta > 0$ and $h_t(x) > 0$ for all $x \in [0, \alpha - 1]$ when $\delta < 0$. This leaves cases (a) and (b).

In case (a), we note that

$$
\begin{aligned}
\delta &= (1-a)(1-b) + b\alpha(1-a) + a\beta(1-b) + (ab-1)\alpha\beta \\
&= (1-a)(1-b) + b\alpha(1-a) + \beta(a(1-b) + (ab-1)\alpha) \\
&= (1-a)(1-b) + b\alpha(1-a) + \beta(a-1 + (ab-1)(\alpha-1)) < 0.
\end{aligned}
$$

The derivative $h'_t(x) \geq 0$ for $x \in (0, x^*)$ and $h'_t(x) \leq 0$ for $x \in (x^*, 1-\alpha)$. Hence h_t is nondecreasing to a maximum $h_t(x^*)$ at $x = x^*$ and nonincreasing for $x \in (x^*, \alpha-1)$. Since $h_t(x^*) = (ab-1)\phi_t(x^*) + (b-1)$ and in case (a) we have $a > 1, ab < 1$ so that $b < 1$, $h_t(x^*) < 0$. Since this is the maximum value of h_t in $(0, \alpha - 1)$ and we have already established above that $h_t(0) < 0$, $h_t(\alpha - 1) < 0$ when $\delta < 0$ we conclude that $h_t(x) < 0$ for all $x \in [0, \alpha - 1]$. The case (b) is similar, with $\delta > 0$ and h_t reaching a minimum in $[0, \alpha - 1]$ of $h_t(x^*) > 0$, so that when $\delta > 0$ we have $h_t(x) > 0$ for all $x \in [0, \alpha - 1]$.

Putting all this together we find that when $\delta > 0 (< 0)$, $h_t(x) > 0 (< 0)$ for all $x \in [0, \alpha - 1]$ and $t \in \mathbb{N}$. From (10) the same is true for Δ_t and hence by Theorem 2 we see that Σ is concave when $\delta > 0$ and convex when $\delta < 0$. $\square$

6. Discussion

We have reviewed conditions for the existence of a carrying simplex in planar discrete-time competitive Kolmogorov systems, and introduced a condition that implies that a measure of phase space area is decreasing with iterations. Using this condition we prove the existence and global attraction of the carrying simplex which is the graph of a decreasing function that joins the two axial fixed points. Our method is based on the graph transform of Hadamard and also provides us with a method for determining the sign of the curvature of the carrying simplex. When applied to the well-known planar Leslie-Gower model of population dynamics, a model which is recognised to display similar properties to the continuous time planar Lotka-Volterra competition model, we find that, as in the continuous time case, the carrying simplex can be only convex or concave. In the continuous time Lotka-Volterra competition model, through the use of the Split Lyapunov method [25, 13], we know that the geometry of the carrying simplex determines the stability of interior fixed points: when the carrying simplex is convex(concave) the interior fixed point is stable(unstable). In the discrete Leslie-Gower model, it is known that for weak(strong) competition the interior fixed point is stable(unstable), and also that for weak(strong) competition the carrying simplex is convex(concave). However, there is a similar link between the geometry of the carrying simplex and stability of interior fixed points in that an interior fixed point is stable(unstable) when competition is weak(strong) and in these particular cases the carrying simplex is convex(concave). In future work [2] we will look at the direct link between the geometry of the carrying simplex and stability of fixed points.

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