On the Sectorial Property of the Caputo Derivative Operator

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Abstract

In this note, we establish the sectorial property of the Caputo fractional derivative operator of order $\alpha \in (1,2)$ with a zero Dirichlet boundary condition.

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1. Introduction

We consider the following Sturm-Liouville problem with a left-sided Caputo fractional derivative in the leading term: find u and $\lambda \in \mathbb{C}$ such that

$$\begin{aligned} -{}_0^CD_x^\alpha u + qu &= \lambda u \quad \text{in } D \equiv (0,1), \\ u(0) = u(1) &= 0, \end{aligned} \tag{1.1}$$

where $\alpha \in (1,2)$ is the order of the derivative, $q \in L^\infty(D)$ and ${}_0^CD_x^\alpha u$ is the left-sided Caputo derivative of order α defined in (2.1) below. In case of $\alpha = 2$, the fractional derivative ${}_0^CD_x^\alpha u$ coincides with the usual second-order derivative u'', and the model (1.1) recovers the classical Sturm-Liouville problem, which has been extensively studied [1].

The interest in the model (1.1) mainly stems from modeling superdiffusion processes, in which the mean squares variance grows faster than that in a Gaussian process. It arises in e.g., subsurface flow and magnetized plasma [2, 3]; see also [4, Chapter 11] for an application in harmonic analysis.

Analytically, very little is known about the Sturm-Liouville problem (1.1). In the case of $q \equiv 0$, the eigenvalues $\{\lambda_j\} \subset \mathbb{C}$ are zeros of the Mittag-Leffler function $E_{\alpha,2}(-\lambda)$, where the two parameter

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Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by (with $\Gamma(\cdot)$ being the Gamma function)

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)},$$

and the corresponding eigenfunctions are given by $xE_{\alpha,2}(-\lambda_j x^\alpha)$ [5]. We refer to [6] for detailed discussions on the zeros of Mittag-Leffler functions. The numerical experiments in [5] indicate that the eigenvalues to problem (1.1) lie in a sector in the complex plane, i.e., the Caputo fractional derivative is a sectorial operator; see also [7] for a Galerkin finite element method for computing the eigenvalues. In this work we shall rigorously establish the sectorial property of the Caputo derivative operator. This property plays an important role in spectral analysis, and underlies the use of semigroup theory for studying the related time-dependent space fractional diffusion equation [8].

The rest of the paper is organized as follows. In Section 2, we describe preliminaries of fractional calculus. In Section 3, we establish the sectorial property of the Caputo derivative operator.

2. Preliminaries on fractional calculus

Now we recall preliminaries on fractional calculus. For any positive non-integer real number α with $n-1<\alpha<n$, $n\in\mathbb{N}$, the (formal) left-sided Caputo fractional derivative of order α is defined by (see, e.g., [9, p. 92])

$${}_0^C D_x^\alpha \phi = {}_0I_x^{n-\alpha}\left(\frac{d^n\phi}{dx^n}\right), \tag{2.1}$$

and the (formal) left-sided Riemann-Liouville fractional derivative of order α is defined by [9, p. 70]:

$${}_0^R D_x^\alpha \phi = \frac{d^n}{dx^n}\left({}_0I_x^{n-\alpha}\phi\right). \tag{2.2}$$

Here ${}_0I_x^\gamma$ for $\gamma>0$ is the left-sided Riemann-Liouville integral operator of order γ defined by

$$({}_0I_x^\gamma\phi)(x) = \frac{1}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1}\phi(t)dt.$$

The right-sided versions of fractional-order integrals and derivatives are defined analogously by

$$({}_xI_1^\gamma\phi)(x) = \frac{1}{\Gamma(\gamma)}\int_x^1 (t-x)^{\gamma-1}\phi(t)\,dt,$$

and

$${}_x^C D_1^\alpha \phi = (-1)^n\,{}_xI_1^{n-\alpha}\left(\frac{d^n\phi}{dx^n}\right), \qquad {}_x^R D_1^\alpha \phi = (-1)^n\frac{d^n}{dx^n}\left({}_xI_1^{n-\alpha}\phi\right).$$

The integral operator ${}_0I_x^\gamma$ satisfies a semigroup property, i.e., for $\gamma,\delta>0$, there holds [9, Lemma 2.3, p. 73]

$${}_0I_x^{\gamma+\delta}\phi = {}_0I_x^\gamma\,{}_0I_x^\delta\phi \quad \forall\phi\in L^2(D), \tag{2.3}$$

and it holds also for the right-sided Riemann-Liouville integral operator ${}_xI_1^\gamma$. We also recall a useful change of integration order formula [9, Lemma 2.7, p. 76]:

$$({}_0I_x^\beta\phi,\psi) = (\phi,\,{}_xI_1^\beta\psi) \quad \forall\phi,\psi\in L^2(D). \tag{2.4}$$

3. Sectorial property

Now we establish the sectorial property of the Caputo derivative operator $\mathcal{A} = -{}_0^C D_x^\alpha + q$, i.e.,

$$\mathcal{A}u(x) = -\frac{1}{\Gamma(2-\alpha)} \int_0^x (x-t)^{1-\alpha} u''(t)dt + q(x)u(x),$$

with a zero Dirichlet boundary condition. First we recall the concept of sectorial operators [8, Definition 3.8, p. 97]. For $\theta \in (0, \pi/2]$ and $\kappa \in \mathbb{R}$, we define a sector $\Sigma_{\theta,\kappa}$ by $\Sigma_{\theta,\kappa} = \{z \in \mathbb{C} : z \neq \kappa, \ |\arg(z-\kappa)| < \theta + \pi/2\}$. Then a densely defined operator A on a Banach space X is called a sectorial operator on X if there exists numbers $\theta \in (0, \pi/2]$ and $\kappa \in \mathbb{R}$ such that (i) the sector $\Sigma_{\theta,\kappa} \subset \rho(A)$, the resolvent set of A, and (ii) for any $\epsilon \in (0, \theta)$ there exists a constant $\gamma = \gamma(\epsilon) > 0$ with

$$\|(A + zI)^{-1}\|_{L(X)} \leq \frac{\gamma(\epsilon)}{|z-\kappa|}, \quad \forall z \in \bar{\Sigma}_{\epsilon,\kappa}, \ z \neq \kappa.$$

By Theorem 3.6 of [8], the sectorial property of an operator follows from the V-coercivity of the associated bilinear form, i.e., it is continuous and satisfies a Gärding type inequality. We shall use this result to prove the desired sectorial property.

We first consider the special case $q \equiv 0$, and denote the Caputo derivative operator with $q \equiv 0$ by $\mathcal{A}_0$, i.e.,

$$\mathcal{A}_0 u(x) = -\frac{1}{\Gamma(2-\alpha)} \int_0^x (x-t)^{1-\alpha} u''(t)dt.$$

We appeal to the adjoint technique, since the spectrum of the adjoint operator $\mathcal{A}_0^*$ (with respect to $L^2(D)$) is conjugate to that of $\mathcal{A}_0$. The domain $\mathrm{dom}(\mathcal{A}_0)$ of the derivative operator $\mathcal{A}_0$ is given by

$$\mathrm{dom}(\mathcal{A}_0) = \left\{ u \in L^2(D) : \ u'' \in \mathrm{dom}({}_0I_x^\alpha) \text{ with } u(0) = u(1) = 0 \right\}.$$

Next we derive a representation of the adjoint $\mathcal{A}_0^*$. Using (2.4) and integration by parts, we deduce

$$\begin{aligned}
-(\mathcal{A}_0 u, \phi) &= ({}_0I_x^{2-\alpha} u'', \phi) = (u'', {}_xI_1^{2-\alpha}\phi) \\
&= (u' {}_xI_1^{2-\alpha}\phi)|_{x=0}^1 - (u', ({}_xI_1^{2-\alpha}\phi)') \\
&= -u'(0)({}_xI_1^{2-\alpha}\phi)(0) - (u', {}_xI_1^{2-\alpha}\phi') \\
&= -u'(0)({}_xI_1^{2-\alpha}\phi)(0) + (u, ({}_xI_1^{2-\alpha}\phi')').
\end{aligned}$$

Here the third line follows from the fact that ${}_x^R D_1^{\alpha-1}\phi = {}_x^C D_1^{\alpha-1}\phi$ for $\phi \in H^1(D)$ with $\phi(1) = 0$ [7, Lemma 4.1]. Therefore, the adjoint operator $\mathcal{A}_0^*$ of $\mathcal{A}_0$ is given by

$$\mathcal{A}_0^*\phi(x) = -\frac{d}{dx} \int_x^1 \frac{(t-x)^{1-\alpha}}{\Gamma(2-\alpha)} \phi'(t)dt$$

with its domain $\mathrm{dom}(\mathcal{A}_0^*)$ given by

$$\mathrm{dom}(\mathcal{A}_0^*) = \left\{ \phi \in H^1(D) : \phi(1) = 0, \ ({}_xI_1^{2-\alpha}\phi)(0) = 0, \ {}_xI_1^{2-\alpha}\phi' \in H^1(D) \right\}.$$

It is easy to see that the nonlocal condition $({}_xI_1^{2-\alpha}\phi)(0) = 0$ is explicitly given by $(x^{1-\alpha}, \phi) = 0$. Since $\mathrm{dom}(\mathcal{A}_0) \neq \mathrm{dom}(\mathcal{A}_0^*)$, the operator $\mathcal{A}_0$ is not self-adjoint. The operator $\mathcal{A}_0^*$ involves a nonlocal constraint in its domain $\mathrm{dom}(\mathcal{A}_0^*)$, and hence it is inconvenient for direct treatment. To this end, we introduce an operator $\tilde{\mathcal{A}}_0$ defined by

$$\tilde{\mathcal{A}}_0\tilde{\phi} = -\frac{d}{dx}\int_x^1 \frac{(s-x)^{1-\alpha}}{\Gamma(2-\alpha)}\tilde{\phi}'(s)\,ds$$

with its domain

$$\mathrm{dom}(\tilde{\mathcal{A}}_0) = \left\{\tilde{\phi} \in L^2(D) : {}_xI_1^{2-\alpha}\tilde{\phi}' \in H^1(D) \text{ with } \tilde{\phi}(0) = \tilde{\phi}(1) = 0\right\}.$$

The operator $\tilde{\mathcal{A}}_0$ is the right-sided Riemann-Liouville fractional derivative, since in view of [7, Lemma 4.1], there holds ${}_xI_1^{2-\alpha}\tilde{\phi}' = ({}_xI_1^{2-\alpha}\tilde{\phi})'$. Hence it is coercive and continuous in the space $H_0^{\alpha/2}(D)$ [7], and hence it is sectorial (see [8, Theorem 3.6] or [10, Lemma 2.1]). Next we construct the perturbation term. Given any $\phi \in \mathrm{dom}(\mathcal{A}_0^*)$, we set $\tilde{\phi}(x) = \phi(x) - \phi(0)(1-x) \in \mathrm{dom}(\tilde{\mathcal{A}}_0)$, i.e., $\phi(x) = \tilde{\phi}(x) + \phi(0)(1-x) \in \mathrm{dom}(\mathcal{A}_0^*)$. The nonlocal condition $({}_xI_1^{2-\alpha}\phi)(0) = 0$ in $\mathrm{dom}(\mathcal{A}_0^*)$ implies $\int_0^1 x^{1-\alpha}((1-x)\phi(0) + \tilde{\phi})dx = 0$, i.e., the constant $\phi(0)$ can be uniquely constructed from $\tilde{\phi}$ by $\phi(0) = -c_\alpha \int_0^1 \tilde{\phi}(x)x^{1-\alpha}dx$, with $c_\alpha = \Gamma(4-\alpha)/\Gamma(2-\alpha)$. Hence, the operator $\mathcal{A}_0^*$ can be expressed using $\tilde{\mathcal{A}}_0$ as

$$\begin{aligned}\mathcal{A}_0^*\phi &= -\frac{d}{dx}\int_x^1 \frac{(s-x)^{1-\alpha}}{\Gamma(2-\alpha)}\phi'(s)\,ds = -\frac{d}{dx}\int_x^1 \frac{(s-x)^{1-\alpha}}{\Gamma(2-\alpha)}(\tilde{\phi}'(s) - \phi(0))\,ds \\ &= \tilde{\mathcal{A}}_0\tilde{\phi}(x) + c'_\alpha(1-x)^{1-\alpha}\int_0^1 \tilde{\phi}(t)t^{1-\alpha}dt,\end{aligned}$$

with $c'_\alpha = c_\alpha/\Gamma(2-\alpha)$. Namely, the perturbation term is given by $c'_\alpha(1-x)^{1-\alpha}\int_0^1 \tilde{\phi}(t)t^{1-\alpha}dt$, which is a rank-one operator. By the standard Sobolev embedding theorem, there holds

$$\begin{aligned}|((1-x)^{1-\alpha}\int_0^1 \tilde{\phi}(t)t^{1-\alpha}dt, \tilde{\phi})| &= |\int_0^1 \tilde{\phi}(t)t^{1-\alpha}dt\int_0^1 (1-x)^{1-\alpha}\tilde{\phi}(x)dx| \\ &\leq c\|\tilde{\phi}\|_{L^p(D)}^2 \leq c\|\tilde{\phi}\|_{L^2(D)}^{2-\gamma}\|\tilde{\phi}\|_{H^{\alpha/2}(D)}^{\gamma},\end{aligned} \tag{3.1}$$

where $p > 1/(2-\alpha)$ and some $\gamma \in (0,2)$. This together with Young's inequality indicates that the associated bilinear form for $\mathcal{A}_0^*$ satisfies a Gärding type coercivity inequality, and clearly also the continuity on the space $H_0^{\alpha/2}(D)$. This and [8, Theorem 3.6] imply that the operator $\mathcal{A}_0^*$ has the sectorial property and so does the operator $\mathcal{A}_0$.

In the general case $q \neq 0$, we proceed like before to deduce

$$\begin{aligned}\mathcal{A}^*\phi &= -\frac{d}{dx}\int_x^1 \frac{(s-x)^{1-\alpha}}{\Gamma(2-\alpha)}\phi'(s)\,ds + q\phi \\ &= -\frac{d}{dx}\int_x^1 \frac{(s-x)^{1-\alpha}}{\Gamma(2-\alpha)}(\tilde{\phi}'(s) - \phi(0))\,ds + q(\tilde{\phi} - c_\alpha(x^{1-\alpha}, \tilde{\phi})(1-x)) \\ &= \tilde{\mathcal{A}}\tilde{\phi}(x) + c'_\alpha(1-x)^{1-\alpha}(x^{1-\alpha}, \tilde{\phi}) - c_\alpha q(1-x)(x^{1-\alpha}, \tilde{\phi}),\end{aligned}$$

where the domain $\mathrm{dom}(\tilde{\mathcal{A}}) = \mathrm{dom}(\tilde{\mathcal{A}}_0)$, the domain of the operator $\tilde{\mathcal{A}}_0$. From this relation and the estimate (3.1), we deduce that the associated bilinear form satisfies a Gärding type coercivity inequality, and thus the operator $\mathcal{A}$ has the sectorial property. By summarizing the preceding discussions, we arrive at the main result of the note.

Theorem 3.1. *For any $q \in L^\infty(D)$, the operator $\mathcal{A} = -{}_0^C D_x^\alpha + q$ is sectorial.*

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