On the syzygies of Poincaré duality algebras

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Abstract

We investigate the structure of the syzygies of Poincaré duality algebras.

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§0 : Introduction :

An algebra Λ augmented over a commutative ring R is said to be a *Poincaré duality algebra* of dimension n (abbreviated to PD^n-algebra) when R admits a free resolution of finite type over Λ

$$0 \to \Lambda \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} F_{n-2} \xrightarrow{\partial_{n-2}} \dots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} \Lambda \xrightarrow{\epsilon} R \to 0.$$

for which
$$\mathrm{Ext}^r_\Lambda(R,\Lambda) \cong \begin{cases} R & r = n \\ 0 & r \neq n. \end{cases}$$

These arise in a variety of different contexts, notably, though not exclusively, as group rings $\Lambda = R[G]$ where G is an orientable Poincaré Duality group [9]. The r^{th}*-syzygy* $\Omega_r(R)$ (cf [7]) is the stable isomorphism class of $\mathrm{Ker}(\partial_{r-1})$ and is independent of the particular free resolution considered. Our aim is to investigate the internal structure of the stable modules $\Omega_r(R)$.

We consider the category $\mathcal{SF}_M$ whose objects are pairs (S,ϵ) where S is a finitely generated stably free Λ-module and where $\epsilon : S \to M$ is a surjective Λ-homomorphism. Morphisms in $\mathcal{SF}_M$ are then commutative squares

$$\begin{array}{ccc} S' & \xrightarrow{\epsilon'} & M \\ \varphi \downarrow & & \downarrow \mathrm{Id}_M \\ S & \xrightarrow{\epsilon} & M. \end{array}$$

By the universal property of projectives, constructing morphisms in this sense is not problematic. What is problematic is to construct morphisms $\varphi : (S',\epsilon') \to (S,\epsilon)$ in which φ is *surjective*. When such a surjective morphism exists we shall write $(S,\epsilon) \preceq (S',\epsilon')$. We shall prove:

(*) '$\preceq$' is a partial ordering on the set of isomorphism types of $\mathcal{SF}_M$.

There is a functor, *base stabilisation*, $\beta : \mathcal{SF}_M \to \mathcal{SF}_{M\oplus\Lambda}$ which gives the following stability theorem:

(**) β is an order preserving bijection $\mathcal{SF}_M \xrightarrow{\simeq} \mathcal{SF}_{M\oplus\Lambda}$ on isomorphism types.

We recall the stability relation '$\sim$' on Λ-modules M, M' (cf Chapter 1 of [7]);

$$M' \sim M \iff M' \oplus \Lambda^{n_1} \cong M \oplus \Lambda^{n_2}$$

for some integers $n_1, n_2 \geq 0$. Then ' $\sim$ ' is an equivalence on isomorphism classes of Λ-modules. For any Λ-module M, we denote by $[M]$ the corresponding *stable module*; that is, the set of isomorphism classes of modules M' such that $M' \sim M$. Given a finitely presented Λ-module M and a surjective homomorphism $\epsilon : \Lambda^m \to M$, the syzygy $\Omega_1(M)$ is defined to be the stable module $[\mathrm{Ker}(\epsilon)]$ consisting of all modules M' such that $M' \sim \mathrm{Ker}(\epsilon)$. By Schanuel's Lemma, this stable module is independent of the particular epimorphism ϵ chosen. More generally we may take a surjective homomorphism $\epsilon : S \to M$ where S is a finitely generated stably free module. We shall parametrize the stable module $\Omega_1(M)$ by means of the category $\mathcal{SF}_M$. There is a partial ordering relation '$\dashv$' on the isomorphism classes $J \in \Omega_1(M)$ defined by writing $J \dashv J'$ when there is an isomorphism $J' \cong J \oplus T$ where T is finitely generated and stably free. We define a mapping $\kappa : \mathcal{SF}_M \to \Omega_1(M)$ by $\kappa(S, \epsilon) = \mathrm{Ker}(\epsilon)$; then κ is order preserving in the sense that

$$(S, \epsilon) \;\preceq\; (S', \epsilon') \;\Longrightarrow\; \kappa(S, \epsilon) \;\dashv\; \kappa(S', \epsilon').$$

For any Λ-module M, the ring $\mathrm{End}_\Lambda(M)$ is also an algebra over R via the homomorphism $\lambda : R \to \mathrm{End}_\Lambda(M)$; $\alpha \mapsto \lambda_\alpha$, where $\lambda_\alpha(x) = \alpha \cdot x$. There is a canonical ring homomorphism $[\,] : \mathrm{End}_\Lambda(M) \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(M)$ where $\mathcal{D}$er is the 'derived module category' of Λ (cf [7]). Composing we get a ring homomorphism, denoted by the same symbol, $\lambda : R \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(M)$.

Theorem A : If $\mathrm{Ext}^1(M, \Lambda) = 0$ and $\lambda : R \xrightarrow{\simeq} \mathrm{End}_{\mathcal{D}\mathrm{er}}(M)$ is an isomorphism then $\kappa : \mathcal{SF}_M \to \Omega_1(M)$ is an order preserving bijection on isomorphism classes.

When Λ is a Poincaré duality algebra of dimension n the hypotheses of Theorem A are satisfied by $M = \Omega_{r-1}(R)$ for $1 \leq r \leq n-1$; then we obtain :

Theorem B : Let Λ be an n-dimensional Poincaré duality algebra augmented over the commutative ring R ; then $\kappa : \mathcal{SF}_{\Omega_{r-1}(R)} \to \Omega_r(R)$ is an order preserving bijection on isomorphism classes for $1 \leq r \leq n-1$.

Whilst the above results are proved by general homological methods they can nevertheless be interpreted in numerical terms. Under mild conditions on Λ, $\Omega_r(R)$ has the structure of a graph endowed with a 'height function' $h : \Omega_r(R) \to \mathbf{N}$. A 'counting function' c_r then counts the number (possibly infinite) of distinct isomorphism classes of modules in $\Omega_r(R)$ at each height n. A precise determination of c_r reveals the extent to which cancellation holds or fails within $\Omega_r(R)$.

To any finitely generated module M we may associate a rather different counting function χ_M. Approximately stated, $\chi_M(n)$ counts the number of distinct isomorphism classes of surjective Λ-homomorphisms $\Lambda^{m+n} \twoheadrightarrow M$ where m is the minimal number of generators of M; the precise definition is given in §4. We first show that χ_M depends only upon the stable class of M. In our context, we may therefore associate a counting function χ_r with $\Omega_r(R)$. It then follows from Theorem B, under the same hypotheses, that for $n \geq 2$:

(I) $c_r \equiv \chi_{r-1}$ for $1 \leq r \leq n-1$.

The standard examples of PD^n-algebras come with an extra feature, namely an algebra anti-involution which allows the identification of left and right modules. Whilst we do not require this for (I) above, nevertheless the existence of such an anti-involution gives rise to the following duality between counting functions:

(II) $c_r \equiv c_{n+1-r}$ for $2 \leq r \leq n-1$.

Clearly (II) requires that $n \geq 3$. In that case, combining (I) and (II) gives the following :

(III) $\chi_r \equiv \chi_{n-r-1}$ for $1 \leq r \leq n-2$.

This recalls the formula $\theta_r = \theta_{n-r-1}$ for the torsion subgroup θ_r of $H_r(X, \mathbf{Z})$ when X is a closed orientable n-manifold. We note that the duality relation (III) neglects the extreme case $r = 0$. We say that the ring Λ has stably free cancellation property SFC when every stably free Λ-module is free. In that case a weak form of the duality relation continues to hold, namely :

(IV) $\chi_0 = \chi_{n-1} \equiv 1$ provided Λ has SFC and R is a principal ideal domain.

In the final section we apply these results to establish the following existence criterion for minimal resolutions of R;

Theorem C: Let Λ is an involuted PD^n-algebra with $n \geq 2$, augmented over a principal ideal domain R; then the following conditions are equivalent:

(i) R has a minimal free resolution over Λ;

(i) Λ satisfies SFC and $\mathrm{Abs}(\Omega_k(R))$ holds for $1 \leq k \leq [\frac{n+1}{2}] - 1$;

(ii) Λ satisfies SFC and $\Omega_k(R)$ is straight for $2 \leq k \leq [\frac{n+1}{2}]$.

§1 : Stable modules and their graphical representations:

Throughout we shall impose restrictions progessively on the rings Λ under consideration. However, without further mention we will assume is that Λ is *weakly finite* in the sense of Cohn ([2], [11]); that is:

(WF) If n is a positive integer and $\varphi : \Lambda^n \to \Lambda^n$ is a surjective Λ-homomorphism then φ is bijective.

This condition is satisfied in many familiar cases; in particular, it is satisfied (cf [12]) when Λ is the integral group ring $\mathbf{Z}[G]$ of any group G.

We begin by recalling some basic notions of stable module theory. For a fuller discussion we refer the reader to Chapter 1 of [7]. It follows from weak finiteness that we also have the following *surjective rank property.* [2]

(1.1) If m, n are positive integers and $\varphi : \Lambda^n \to \Lambda^m$ is a surjective Λ-homomorphism then $m \leq n$.

This has a useful consequence:

(1.2) Let M be a finitely generated Λ-module; then $M \oplus \Lambda^a \cong M \implies a = 0$,

As in the Introduction above, '$\sim$' will denote the stability relation on Λ modules;

$$M' \sim M \iff M' \oplus \Lambda^{n_1} \cong M \oplus \Lambda^{n_2}$$

for some integers $n_1, n_2 \geq 0$. If M is a Λ-module then $[M]$ will denote the corresponding *stable module*. Evidently if $M \sim M'$ then $\mathrm{Ext}^r(M, N) \cong \mathrm{Ext}^r(M', N)$ for any Λ-module N and $r \geq 1$. Thus for a stable module Ω one may define $\mathrm{Ext}^r(\Omega, N) = \mathrm{Ext}^r(M, N)$ for any $M \in \Omega$. Clearly also if $M \sim M'$ then M is finitely generated $\iff$ M' is finitely generated. We say the stable module Ω is finitely generated when any $M \in \Omega$ is finitely generated.

For the rest of the discussion fix a finitely generated stable module Ω. We define a function $g : \Omega \times \Omega \to \mathbf{Z}$, the 'gap function' as follows

$$g(N_1, N_2) = g \iff N_1 \oplus \Lambda^{a+g} \cong N_2 \oplus \Lambda^a$$

where both a and $a + g$ are positive integers. It is a consequence of weak finiteness for Λ that g is well defined. We say that a module $M_0 \in \Omega$ is a *root module* for Ω when $0 \leq g(M_0, L)$ for all $L \in \Omega$. It is an easy consequence of (1.2) that :

(1.3) Any finitely generated stable module Ω-module contains a root module.

From a root module M_0 one derives the 'height function' $h : \Omega \to \mathbf{N}$

$$h(L) = g(M_0, L);$$

h is independent of the particular choice of root module M_0 and so is intrinsic to Ω. One now defines the *counting function* $c^M : \mathbf{N} \to \mathbf{Z}_+ \cup \{\infty\}$ by

$$c^M(n) = |h^{-1}(n)|;$$

in words, $c^M(n)$ is the number (possibly infinite) of distinct isomorphism types of modules $M' \in [M]$ such that $h(M') = n$.

We denote by $\mathcal{S}$ the stable class of the zero module; a module S belongs to $\mathcal{S}$ precisely when it is *stably free*; that is, when $S \oplus \Lambda^a \cong \Lambda^b$ for some integers $a, b \geq 1$; the *rank*, $\mathrm{rk}(S)$ of S, is then defined by

$$\mathrm{rk}(S) = b - a.$$

It is again a consequence of weak finiteness that $\mathrm{rk}(S)$ is well defined and coincides with the height $h(S)$ defined above; in particular $0 < \mathrm{rk}(S)$ when S is nonzero. We note another consequence of weak finiteness:

(1.4) Let $\varphi : S_1 \to S_2$ be a surjective Λ-homomorphism between finitely generated stably free modules S_1, S_2. If $\mathrm{rk}(S_1) = \mathrm{rk}(S_2)$ then φ is an isomorphism.

There is also an obvious generalization of (1.2)

(1.5) Let M, T be finitely generated Λ-modules and suppose that T is stably free ; then $M \oplus T \cong M \implies T = 0$.

The existence of stably free modules which are not free complicates the study of stable modules. However, it involves only finitely generated modules as by the theorem of Gabel [5], [7] any infinitely generated stably free module must be free. We say that Λ has the *stably free cancellation property* ($=$ *SFC*) when this complication does not occur; that is when

$$S \oplus \Lambda^a \;\cong\; \Lambda^b \implies S \;\cong\; \Lambda^{b-a}.$$

One may impose upon a finitely generated stable module Ω the structure of a directed graph in which the vertices are the isomorphism classes of modules $N \in \Omega$. This may be done in a two ways. In the first method, we draw an edge $N_1 \to N_2$ whenever $N_2 \cong N_1 \oplus \Lambda$. The existence of the height function h shows that Ω is a tree, *the Dyer-Sieradski tree* [3], whose roots do not extend infinitely downwards.

For our present purposes this graphical structure requires some refinement. We first introduce a general definition; if M_1, M_2 are finitely generated Λ-modules we say that M_2 *splits over* M_1, written $M_1 \dashv M_2$, when there is an isomorphism $M_1 \oplus T \cong M_2$ in which T is a finitely generated stably free module. Evidently one has :

(1.6) If $M_1 \dashv M_2$ then $M_1 \sim M_2$.

It is straightforward to see that the relation '$\dashv$' is transitive; that is :

(1.7) If $M_1 \dashv M_2$ and $M_2 \dashv M_3$ then $M_1 \dashv M_3$.

Suppose $M_2 \cong M_1 \oplus T_1$ and $M_1 \cong M_2 \oplus T_2$ for some finitely generated stably free modules T_1, T_2. Thus $M_1 \cong M_1 \oplus T$ where $T = (T_1 \oplus T_2)$ is finitely generated stably free. It follows from (1.5) above that $T = 0$. Hence $T_2 = 0$ and $M_1 \cong M_2$. It follows that '$\dashv$' is also anti-symmetric in the sense that:

$$M_1 \dashv M_2 \;\wedge\; M_2 \dashv M_1 \implies M_1 \cong M_2.$$

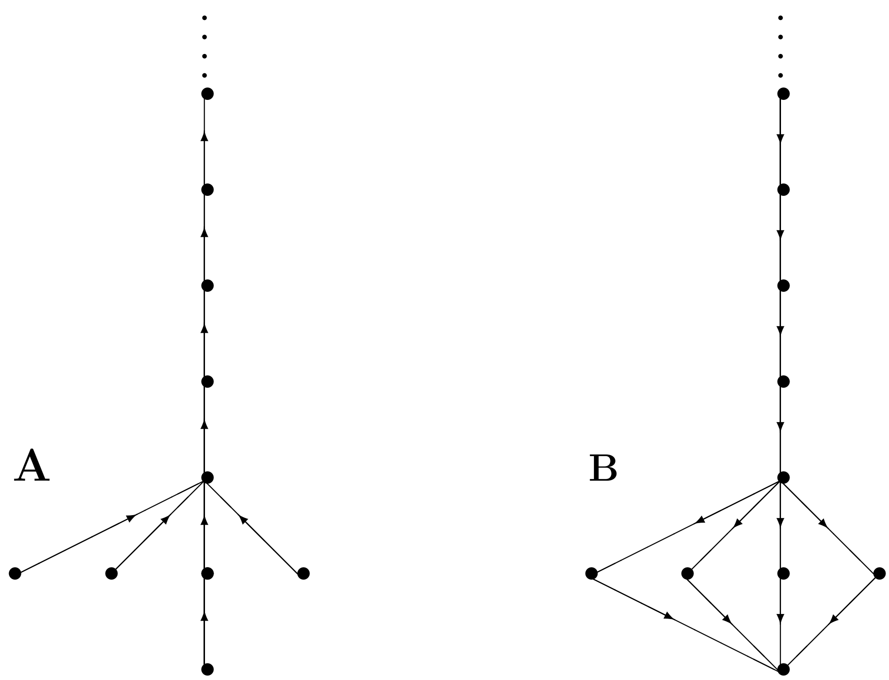
Corollary 1.8 : If Ω is a finitely generated stable module then the relation '$\dashv$' induces a partial ordering on the isomorphism types of Ω.

We may now give an alternative to the Dyer-Sieradski representation of Ω as follows; say that a nonzero stably free module T is *indecomposably stably free* when T cannot be expressed as a direct sum of two nonzero stably free modules. Note that, by Gabel's Theorem, T must then be finitely generated. We introduce a new graphical structure on Ω by drawing an arrow $M_1 \twoheadleftarrow M_2$ whenever $M_2 \cong M_1 \oplus T$ and T is indecomposably stably free.

Figure (1.9) below shows, according to the calculations of Swan [13], two depictions of the class $\mathcal{S}$ of stably free modules over the group ring $\mathbf{Z}[Q_{36}]$ of the finite quaternionic group

$$Q_{36} = \langle x, y \mid x^9 = y^2 ; xyx = y \rangle .$$

(1.9)



On the left, (A) shows the conventional Dyer-Sieradski tree; on the right (B) gives the representation based on the relation '$\dashv$'. The bottom-most node in each graph represents the zero module. Note the reversal of the arrows.

As can be seen, this new structure on Ω is not, in general, a tree; Nevertheless, under the general assumption that Ω is finitely generated we do have :

(1.10) The dual† Dyer-Sieradski tree is a maximal tree in Ω.

† That is, with arrows reversed.

The most elementary form a stable module Ω may take is when the counting function is the constant function $c \equiv 1$; in this case we say that Ω is *straight*.

§2 : The category $\mathcal{SF}_M$:

We denote by $\mathcal{SF}$ the category whose objects are pairs (S, ϵ) where S is finitely generated stably free Λ-module and where ϵ is a surjective Λ-homomorphism with domain S and whose codomain is some, as yet unpecified, Λ-module. Morphisms in $\mathcal{SF}$ are then commutative squares

$$\begin{array}{ccc} S' & \stackrel{\epsilon'}{\longrightarrow} & M \\ \varphi\downarrow & & \downarrow\varphi_0 \\ S & \stackrel{\epsilon}{\longrightarrow} & M \end{array}$$

and φ is then said be a *morphism over* φ_0. When M is a finitely generated Λ-module $\mathcal{SF}_M$ will denote the full subcategory of $\mathcal{SF}$ whose morphisms are defined over Id_M. If (S, ϵ), (S', ϵ') are objects in $\mathcal{SF}_M$ we write $(S, \epsilon) \preceq (S', \epsilon')$ when there exists a morphism $\varphi : (S', \epsilon') \to (S, \epsilon)$ in which $\varphi : S' \to S$ is a surjective Λ-homomorphism. It is straightforward to observe that if $(S', \epsilon') \preceq (S'', \epsilon'')$ and $(S', \epsilon') \preceq (S'', \epsilon'')$ then $(S, \epsilon) \preceq (S'', \epsilon'')$. Slightly more subtle is :

Proposition 2.1 : $(S, \epsilon) \preceq (S', \epsilon') \;\wedge\; (S', \epsilon') \preceq (S, \epsilon) \iff (S, \epsilon) \cong (S', \epsilon')$.

Proof : Suppose that $\varphi : (S', \epsilon') \to (S, \epsilon)$ and $\psi : (S, \epsilon) \to (S', \epsilon')$ are morphisms in $\mathcal{SF}_M$ and that $\varphi : S' \to S$, $\psi : S \to S'$ are both surjective. Then $\psi \circ \varphi : S' \to S'$ is a surjective Λ-homomorphism. As S' is finitely generated stably free it follows from (1.4) that $\psi \circ \varphi$ is an isomorphism. Hence φ is injective and so $\varphi : (S', \epsilon') \to (S, \epsilon)$ is an isomorphism in $\mathcal{SF}_M$, proving ($\Longrightarrow$). The proof of ($\Longleftarrow$) is straightforward. $\square$

It follows that :

(2.2) The relation '$\preceq$' induces a partial ordering on the isomorphism classes of $\mathcal{SF}_M$.

Now suppose that E is a finitely generated stably free Λ-module; we define a functor $\beta_E : \mathcal{SF}_M \to \mathcal{SF}_{M \oplus E}$ (*base stabilisation by* E) by its action on commutative squares thus:

$$\beta_E \begin{pmatrix} S' & \stackrel{\epsilon'}{\longrightarrow} & M \\ \varphi\downarrow & & \downarrow \mathrm{Id}_M \\ S & \stackrel{\epsilon}{\longrightarrow} & M \end{pmatrix} = \begin{pmatrix} S' \oplus E & \stackrel{\epsilon' \oplus \mathrm{Id}}{\longrightarrow} & M \oplus E \\ \varphi \oplus \mathrm{Id}_E \downarrow & & \downarrow \mathrm{Id}_M \\ S \oplus E & \stackrel{\epsilon \oplus \mathrm{Id}}{\longrightarrow} & M \oplus E \end{pmatrix};$$

that is, β_E acts on objects by $\beta_E(S, \epsilon) = (S \oplus E, \epsilon \oplus \mathrm{Id}_M)$ and on morphisms by $\beta_E(\varphi) = \varphi \oplus \mathrm{Id}_E$. Observe that β_E is order preserving in the sense that :

(2.3) If $(S, \epsilon) \preceq (S', \epsilon')$ then $\beta_E(S, \epsilon) \preceq \beta_E(S', \epsilon')$.

We next establish :

Theorem 2.4 : $\beta_E : \mathcal{SF}_M \to \mathcal{SF}_{M\oplus E}$ is surjective on isomorphism classes.

Proof : Let $(S, \epsilon) \in \mathcal{SF}_{M\oplus E}$ and respresent it as an exact sequence

$$0 \to K \hookrightarrow S \xrightarrow{\epsilon} M \oplus E \to 0.$$

Put $T = \epsilon^{-1}(M)$ and $\eta = \epsilon_{|T} : T \to M$. It will suffice to establish the following two statements **(*)** and **(**)**:

(*) $(T, \eta) \in \mathcal{SF}_M$

(**) $(S, \epsilon) \cong_{\mathrm{Id}_{M\oplus E}} \beta_E(T, \eta)$.

First observe that the filtration $K \subset T \subset S$ gives rise to a pair of exact sequences

$$0 \to T \hookrightarrow S \to S/T \to 0$$

$$0 \to T/K \hookrightarrow S/K \to S/T \to 0.$$

which can be assembled into a commutative diagram with exact rows and columns:

$$\textbf{(I)} \qquad \left\{ \begin{array}{ccccccc} & 0 & & 0 & & & \\ & \downarrow & & \downarrow & & & \\ 0 \to & K & = & K & \to & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \longrightarrow & T & \hookrightarrow & S & \xrightarrow{\widetilde{\pi}} & S/T & \longrightarrow 0 \\ & \downarrow \nu' & & \downarrow \nu & & \| \ \mathrm{Id} & \\ 0 \longrightarrow & T/K & \hookrightarrow & S/K & \xrightarrow{\pi} & S/T & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array} \right.$$

in which $\nu, \nu', \widetilde{\pi}$ and π are all natural mappings. As ϵ is surjective there are Noether isomorphisms $S/T \cong (M \oplus E)/M \cong E$. Hence

(II) S/T is finitely generated stably free.

In particular, S/T is projective so we may choose a homomorphism $\widetilde{\sigma} : S/T \to S$ which splits the middle row of (I) on the right ; that is;

(III) $\widetilde{\pi} \circ \widetilde{\sigma} = \mathrm{Id}_{S/T}$.

Hence $S \cong T \oplus S/T$ from which we see also that

(IV) T is finitely generated stably free.

As $T = \epsilon^{-1}(M)$ then $\eta = \epsilon_{|T} : T \to M$ is evidently surjective and so $(T, \eta) \in \mathcal{SF}_M$. This establishes **(*)**.

Now define $\sigma = \nu \circ \widetilde{\sigma} : S/T \to S/K$. As $\pi \circ \nu = \widetilde{\pi}$ we see that

(V) $\pi \circ \sigma = \mathrm{Id}_{S/T}$.

That is, σ splits the bottom row of (I) on the right; there are corresponding left splittings:

$$\widetilde{\lambda} : S \to T \; ; \; \widetilde{\lambda} \; = \; \mathrm{Id}_S \, - \, \widetilde{\sigma}\widetilde{\pi} \quad ; \quad \lambda : S/K \to T/K \; ; \; \lambda \; = \; \mathrm{Id}_S \, - \, \sigma\pi.$$

One verifies easily that $\lambda \circ \nu \;\; = \;\; \nu' \circ \widetilde{\lambda}$. Evidently $(S, \nu) \in \mathcal{SF}_{S/K}$ and there is a Noether isomorphism $\natural_1 : (S, \nu) \stackrel{\simeq}{\longrightarrow} (S, \epsilon)$. As T and S/T are finitely generated stably free then $(T, \nu') \in \mathcal{SF}_{T/K}$ and $\beta_{S/T}(T, \nu')$ is well defined. Now consider the isomorphisms

$$\begin{array}{rclcrcl} \widetilde{h} : S & \to & T \oplus S/T & ; & h : S/K & \to & T/K \oplus S/T \\ \widetilde{h}(x) & = & (\widetilde{\lambda}(x), \widetilde{\pi}(x) & ; & h(x) & = & (\lambda(x), \pi(x). \end{array}$$

Then $\widetilde{h}$ defines an isomorphism $\widetilde{h} : (S, \nu) \stackrel{\simeq}{\longrightarrow}_h \beta_{S/T}(T, \nu')$ over h and there is another Noether isomorphism $\natural_2 : \beta_{S/T}(T, \nu') \stackrel{\simeq}{\longrightarrow} \beta_E(T, \eta)$ where $\eta \; = \; \epsilon_{|T} : T \to M$. The composition $\natural_2 \circ \widetilde{h} \circ \natural_1^{-1} : (S, \epsilon) \stackrel{\simeq}{\longrightarrow} \beta_E(T, \eta)$ is an isomorphism over $\mathrm{Id}_{M \oplus E}$. This establishes **(**)** and completes the proof. □

We next consider morphisms $\varphi : \beta_E(S', \epsilon') \to \beta_E(S, \epsilon)$ in $\mathcal{SF}_{M \oplus E}$. Any such morphism is, at least, a Λ-homomorphism $\varphi : S' \oplus E \to S \oplus E$ and so may be described as a matrix of Λ-homomorphisms

$$\varphi \;=\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \text{where} \qquad \begin{array}{ll} A : S' \to S & B : E \to S \\ C : S' \to E & D : E \to E. \end{array}$$

The condition that φ should describe a morphism $\varphi : \beta_E(S', \epsilon') \to \beta_E(S, \epsilon)$ in $\mathcal{SF}_{M \oplus E}$ is that $(\epsilon \oplus \mathrm{Id}_E) \circ \varphi = (\epsilon' \oplus \mathrm{Id}_E)$ which in matrix terms then becomes

$$\begin{pmatrix} \epsilon & 0 \\ 0 & \mathrm{Id}_E \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;=\; \begin{pmatrix} \epsilon A & \epsilon B \\ C & D \end{pmatrix} \;=\; \begin{pmatrix} \epsilon' & 0 \\ 0 & \mathrm{Id}_E \end{pmatrix}.$$

Hence we require that $\epsilon A = \epsilon'$; $\epsilon B = 0$; $C = 0$; $D = \mathrm{Id}_E$. To summarize, there is a $1 - 1$ correspondence

(2.5) $$\mathrm{Hom}_{\mathcal{SF}}(\beta_E(S', \epsilon'), \beta_E(S, \epsilon)) \stackrel{\simeq}{\longleftrightarrow} \left\{ \begin{pmatrix} A & B \\ 0 & \mathrm{Id}_E \end{pmatrix} \;\middle|\; \begin{array}{c} A \in \mathrm{Hom}_{\mathcal{SF}}((S', \epsilon'), (S, \epsilon)) \\ B \in \mathrm{Hom}_\Lambda(E, \mathrm{Ker}(\epsilon)) \end{array} \right\}.$$

Observe that, purely as Λ-homomorphisms

(2.6) $$\begin{pmatrix} A & B \\ 0 & \mathrm{Id}_E \end{pmatrix} : S \oplus E \to S' \oplus E \text{ is surjective} \iff A : S \to S' \text{ is surjective.}$$

Proposition 2.7 : Let (S, ϵ), (S', ϵ') be objects in $\mathcal{SF}_M$; then

$$\beta_E(S,\epsilon) \preceq \beta_E(S',\epsilon') \iff (S,\epsilon) \preceq (S',\epsilon')$$

Proof : Firstly suppose that $\beta_E(S,\epsilon) \preceq \beta_E(S',\epsilon')$ and that

$$\varphi = \begin{pmatrix} A & B \\ 0 & \mathrm{Id}_E \end{pmatrix} : S' \oplus E \to S \oplus E$$

defines a surjective $\mathcal{SF}_{M\oplus E}$-morphism $\beta_E(S',\epsilon') \to \beta_E(S,\epsilon)$. Then $\epsilon A = \epsilon'$ by above so that $A : (S',\epsilon') \to (S,\epsilon)$ is a morphism in $\mathcal{SF}_M$ and is surjective by (2.6). Thus $(S,\epsilon) \preceq (S',\epsilon')$.

Conversely, suppose that $(S,\epsilon) \preceq (S',\epsilon')$ and that $A : (S',\epsilon') \to (S,\epsilon)$ is a surjective morphism in $\mathcal{SF}_M$; then we have a surjective morphism

$$\begin{pmatrix} A & 0 \\ 0 & \mathrm{Id}_E \end{pmatrix} : S' \oplus E \to S \oplus E$$

in $\mathcal{SF}_{M\oplus E}$ and so $\beta_E(S,\epsilon) \preceq \beta_E(S',\epsilon')$. □

As a consequence we obtain :

Corollary 2.8 : For any finitely generated stably free module E, β_E induces an order preserving bijection on isomorphism types $\beta_E : \mathcal{SF}_M \xrightarrow{\simeq} \mathcal{SF}_{M\oplus E}$.

Proof : By (2.3), (2.4) it suffices to show that β_E is injective on isomorphism types. Suppose (S,ϵ), (S',ϵ') are objects in $\mathcal{SF}_M$ with the property that $\beta_E(S,\epsilon) \cong \beta_E(S',\epsilon')$. Then $\beta_E(S,\epsilon) \preceq \beta_E(S',\epsilon') \preceq \beta_E(S,\epsilon)$ so that $(S,\epsilon) \preceq (S',\epsilon') \preceq (S,\epsilon)$ by (2.7). By (2.1) it now follows that $(S,\epsilon) \cong (S',\epsilon')$. □

Writing $\beta = \beta_\Lambda$ we obtain the statement (**) of the Introduction.

Corollary 2.9 : $\beta : \mathcal{SF}_M \xrightarrow{\simeq} \mathcal{SF}_{M\oplus\Lambda}$ induces an order preserving bijection on isomorphism types.

§3 : The first syzygy and the mapping $\kappa : \mathcal{SF}_M \to \Omega_1(M)$:

Let M be a finitely generated Λ-module; given surjective homomorphisms $\epsilon_i : \Lambda^{m_i} \to M$ $(i = 1,2)$ we obtain exact sequences

$$0 \to J_1 \hookrightarrow \Lambda^{m_1} \xrightarrow{\epsilon_1} M \to 0 \ ; \qquad 0 \to J_2 \hookrightarrow \Lambda^{m_2} \xrightarrow{\epsilon_1} M \to 0.$$

Schanuel's Lemma then assures us that $J_1 \oplus \Lambda^{m_2} \cong J_2 \oplus \Lambda^{m_1}$. In particular $J_1 \sim J_2$. We define the first syzygy $\Omega_1(M)$ to be the stable module determined by any module J which occurs in an exact sequence of the form

(3.1) $$0 \to J \hookrightarrow \Lambda^m \xrightarrow{\epsilon} M \to 0$$

where m is an nonnegative integer. Given such an exact sequence we may stabilize M to $M \oplus \Lambda^n$ to obtain another exact sequence $0 \to J \hookrightarrow \Lambda^{m+n} \to M \oplus \Lambda^n \to 0$, from which we see that :

(3.2) If $M \sim M'$ then $\Omega_1(M) \equiv \Omega_1(M')$.

Observe that, in the definition of $\Omega_1(M)$, one may, more generally, replace the middle term Λ^m by a finitely generated stably free module thus:

(3.3) Let $0 \to J \hookrightarrow S \stackrel{\epsilon}{\to} M \to 0$ be an exact sequence where S is a finitely generated stably free module; then $J \in \Omega_1(M)$.

It follows from (3.3) that we obtain a mapping $\kappa : \mathcal{SF}_M \to \Omega_1(M)$ on defining

$$\kappa(S, \epsilon) = \mathrm{Ker}(\epsilon).$$

We show that κ is order-preserving in the following sense :

Proposition 3.4: If $(S, \epsilon) \preceq (\widetilde{S}, \widetilde{\epsilon})$ then $\kappa(S, \epsilon) \dashv \kappa(\widetilde{S}, \widetilde{\epsilon})$.

Proof : Suppose that $\varphi : (\widetilde{S}, \widetilde{\epsilon}) \to (S, \epsilon)$ is a dominating morphism in $\mathcal{SF}_M$. On putting $T = \mathrm{Ker}(\varphi)$ we have an exact sequence

$$0 \to T \stackrel{j}{\hookrightarrow} \widetilde{S} \stackrel{\varphi}{\to} S \to 0.$$

As S is stably free it is *a fortiori* projective so that the sequence splits to give

$$\widetilde{S} \cong S \oplus T.$$

As both S and $\widetilde{S}$ are finitely generated stably free we see also that :

(I) T is finitely generated stably free.

Defining $J = \mathrm{Ker}(\epsilon)$, $\widetilde{J} = \mathrm{Ker}(\widetilde{\epsilon})$, we now construct the commutative diagram

$$\mathcal{D} \quad = \quad \left\{ \begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & T & = & T & \stackrel{\widehat{p}}{\longrightarrow} & 0 \longrightarrow 0 & \\ & \downarrow j^- & & \downarrow j & & \downarrow & \\ 0 \longrightarrow & \widetilde{J} & \stackrel{\widetilde{i}}{\hookrightarrow} & \widetilde{S} & \stackrel{\widetilde{\epsilon}}{\longrightarrow} & M \longrightarrow 0 & \\ & \downarrow \varphi^- & & \downarrow \varphi & & \| \ \mathrm{Id} & \\ 0 \longrightarrow & J & \stackrel{i}{\hookrightarrow} & S & \stackrel{\epsilon}{\longrightarrow} & M \longrightarrow 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array} \right.$$

in which j^- denotes the inclusion $T \hookrightarrow \widetilde{J}$ and φ^- denotes the restriction of φ to $\widetilde{J}$. The rows of $\mathcal{D}$ are obviously exact. As both φ and $\widehat{p}$ are surjective a straightforward diagram chase shows that φ^- is surjective. Moreover:

(II) $\mathrm{Ker}(\varphi^-) = \mathrm{Im}(j^-)$

We also have :

(III) φ^- is surjective.

To see (III) observe that the inclusion $\mathrm{Im}(j^-) \subset \mathrm{Ker}(\varphi^-)$ follows by restriction from $\varphi \circ j = 0$. Thus suppose $x \in \widetilde{J}$ satisfies $\varphi^-(x) = 0$; then $x \in \mathrm{Ker}(\varphi) = T$.

It follows that :

(IV) The sequence $0 \to T \xrightarrow{j^-} \widetilde{J} \xrightarrow{\varphi^-} J \to 0$ is exact.

Finally, consider the exact sequence defined by the middle column of $\mathcal{D}$

$$0 \to T \overset{j}{\to} \widetilde{S} \overset{\varphi}{\to} S \to 0.$$

As S is projective this sequence splits. We may choose to split it on the left by means of a Λ-homomorphism $r : \widetilde{S} \to T$ such that $r \circ j = \mathrm{Id}_T$. Define $\rho : \widetilde{J} \to T$ by

$$\rho = r \circ \widetilde{i}.$$

However, $j = \widetilde{i} \circ j^-$ so that $\rho \circ j^- = \mathrm{Id}_T$ and ρ is a left splitting of the exact sequence

$$0 \to T \xrightarrow{j^-} \widetilde{J} \xrightarrow{\varphi^-} J \to 0.$$

Thus $J \oplus T \cong \widetilde{J}$. As T is finitely generated stably free then $J \dashv \widetilde{J}$ as required. □

The definition of $\Omega_1(M)$ conceals a subtlety which we must now make explicit. As we have defined it, $J' \in \Omega_1(M)$ when $J' \oplus \Lambda^a \cong J \oplus \Lambda^b$ where J which occurs in an exact sequence of the form described in (3.3). Although we would like to conclude that J' also occurs in an exact sequence $0 \to J' \hookrightarrow S \overset{\epsilon}{\to} M \to 0$ where S is stably free this is false without an extra hypothesis on M. We say that M is *coprojective* when $\mathrm{Ext}^1(M, \Lambda) = 0$. Coprojectivity implies the following 'de-stabilization' result (cf [6] Proposition 2.5).

Proposition 3.5 : Let M be a finitely generated Λ-module and let $J' \in \Omega_1(M)$; if M is coprojective then there exists an exact sequence $0 \to J' \hookrightarrow S \overset{\epsilon}{\to} M \to 0$ in which S is finitely generated stably free.

As an immediate consequence we have:

Corollary 3.6 : Let M be a finitely generated Λ-module such that $\mathrm{Ext}^1(M, \Lambda) = 0$; then $\kappa : \mathcal{SF}_M \to \Omega_1(M)$ is an order preserving surjection on isomorphism classes.

Although in the above argument we require M to be finitely generated we have not imposed this hypothesis on $\Omega_1(M)$. In order to iterate the argument to higher syzygies we require extra finiteness conditions. We say that the finitely generated module M satisfies condition $FT(n)$ when $\Omega_r(M)$ is finitely generated for $1 \le r \le n$.

§4 : The counting function for epimorphisms :

In §1 we defined the counting function c_Ω associated with a stable module Ω. Likewise, for any finitely generated Λ-module M there is a corresponding counting function associated with $\mathcal{SF}_M$. We first define

$$\mu(M) \; = \; \min\{ \, \mathrm{rk}(S) \; | \; (S, \epsilon) \in \mathcal{SF}_M \, \}.$$

Clearly $\mu(M)$ is an integer ≥ 0 and $\mu(M) = 0 \iff M = 0$. We claim that :

Proposition 4.1 : $\mu(M \oplus T) \; = \; \mu(M) \; + \; \mathrm{rk}(T)$ for any finitely generated stably free module T.

Proof : Suppose that $\mu(M) = m$ and that $\epsilon : S \twoheadrightarrow M$ is a surjective Λ homomorphism with S stably free and $\mathrm{rk}(S) = m$. If T is stably free then $\epsilon \oplus \mathrm{Id} : S \oplus T \twoheadrightarrow M \oplus T$ is surjective and $S \oplus T$ is stably free with $\mathrm{rk}(S \oplus T) = m + \mathrm{rk}(T)$; it follows that

$$\mu(M \oplus T) \; \leq \; \mu(M) + \mathrm{rk}(T).$$

We claim that we actually have equality. Thus suppose not and that $\eta : \widetilde{S} \twoheadrightarrow M \oplus T$ is a surjection where $\widetilde{S}$ stably free with $\mathrm{rk}(\widetilde{S}) < \mu(M) + \mathrm{rk}(T)$. Put $S' = \eta^{-1}(M)$ so that $\eta : S' \twoheadrightarrow M$ is surjective. We claim that S' is stably free. To see this, let $\pi : M \oplus T \to T$ denote the projection; then there is an exact sequence

$$0 \to S' \to \widetilde{S} \xrightarrow{\pi \circ \eta} T \to 0$$

which necessarily splits as T is projective. Hence $\widetilde{S} \cong S' \oplus T$ and S' is stably free as claimed. However, $\mathrm{rk}(S') \; = \; \mathrm{rk}(\widetilde{S}) - \mathrm{rk}(T) < \mu(M)$. As $\eta : S' \twoheadrightarrow M$ is surjective this contradicts the definition of $\mu(M)$ and completes the proof. □

In particular, we see that:

(4.2) $\mu(M \oplus \Lambda^n) \; = \; \mu(M) \; + \; n$.

We say that a finitely generated Λ-module M is *generic* when it has the property that if S a stably free module and $S \twoheadrightarrow M$ is a surjective Λ-homomorphism then S is free; note that:

(4.3) When M is generic $\mu(M)$ is the cardinal of a minimal generating set for M.

Put $\mathcal{SF}_M(n) \; = \; \{(S, \epsilon) \in \mathcal{SF}_M \mid \mathrm{rk}(S) \; = \; \mu(M) + n\}$ and define

$$\chi_M(n) \; = \; |\mathcal{SF}_M(n)|.$$

Then χ_M gives a counting function $\chi_M \; : \; \mathbf{N} \to \mathbf{Z}_+ \cup \{+\infty\}$. We note that :

Proposition 4.4 : If $M \sim M'$ then $\chi_M \; = \; \chi_{M'}$.

Proof: By (2.9) β induces a bijection on isomorphism classes $\beta : \mathcal{SF}_M \xrightarrow{\simeq} \mathcal{SF}_{M \oplus \Lambda}$ so it suffices to show that $\beta(\mathcal{SF}_M(n)) \subset \mathcal{SF}_{M \oplus \Lambda}(n)$. If $(S, \epsilon) \in \mathcal{SF}_M(n)$ then

$\mathrm{rk}(S) = \mu(M) + n$ whilst $\beta(S, \epsilon) = (S \oplus \Lambda, \epsilon \oplus \mathrm{Id})$ and $\mathrm{rk}(S \oplus \Lambda) = \mu(M) + n + 1$. However, it follows from (4.1) that $\mu(M) + 1 = \mu(M \oplus \Lambda)$ and so $\mathrm{rk}(S \oplus \Lambda) = \mu(M \oplus \Lambda) + n$. Hence $\beta(S, \epsilon) \in \mathcal{SF}_{M \oplus \Lambda}(n)$ as required. □

(4.5) When M is generic $\chi_M(n)$ is the number of isomorphism types of Λ-epimorphisms $\Lambda^{\mu(M)+n} \twoheadrightarrow M$.

If Λ has a 'stable range' property whereby every stably free Λ-module of rank $\geq N$ is free then any module of the form $M \cong M_0 \oplus \Lambda^N$ is generic. In this case every stable module contains a generic isomorphism class. Observe that (3.6) can be expressed in terms of counting functions thus;

(4.6) If M is finitely presented and coprojective then $c_{\Omega_1(M)}(n) \leq \chi_M(n)$ for all n.

§5 : Absolutely minimal epimorphisms:

Let M be a finitely generated Λ-module; an element (S, ϵ) of $\mathcal{SF}_M(0)$ is said to be a *minimal epimorphism over* M. Such a minimal epimorphism (S, ϵ) is then said to be *absolutely minimal* when $(S, \epsilon) \preceq (S', \epsilon')$ for each $(S', \epsilon') \in \mathcal{SF}_M$. By (2.1) an absolutely minimal epimorphism over M is unique up to isomorphism; that is :

(5.1) If (S, ϵ), (S', ϵ') are both absolutely minimal over M then $(S, \epsilon) \cong (S', \epsilon')$.

We say that M satisfies the condition $\mathcal{A}\mathrm{bs}_\Lambda(M)$ when $\mathcal{SF}_M$ contains an absolutely minimal epimorphism. We also note, from (2.9), that satisfaction of this condition depends only upon the stable isomorphism class of M.

Proposition 5.2 : Let M, M' be finitely generated Λ-modules such that $M \sim M'$; then

$$\mathcal{A}\mathrm{bs}_\Lambda(M) \text{ holds} \iff \mathcal{A}\mathrm{bs}_\Lambda(M') \text{ holds.}$$

Beyond such stability considerations however, the question of whether the condition $\mathcal{A}\mathrm{bs}_\Lambda(M)$ holds is both nontrivial and highly contingent.†.

We wish to relate the condition $\mathcal{A}\mathrm{bs}_\Lambda(M)$ to the counting function χ_M. Clearly the uniqueness property (5.1) may be re-stated thus:

(5.3) If $\mathcal{A}\mathrm{bs}_\Lambda(M)$ holds then $\chi_M(0) = 1$.

The simplest behaviour that we can expect from χ_M is that it takes the constant value 1; otherwise expressed, $\chi_M \equiv 1$. We have :

Proposition 5.4 : If $\chi_M \equiv 1$ then $\mathcal{A}\mathrm{bs}_\Lambda(M)$ holds.

Proof : Let $n \in \mathbf{N}$ and let $(S_n, \epsilon_n) \in \mathcal{SF}_M(n)$. As $\chi_M(n) = 1$ then (S_n, ϵ_n) represents the unique isomorphism class in $\mathcal{SF}_M(n)$. We show that (S_0, ϵ_0) is absolutely minimal.

† Eilenberg [4] previously gave a definition of *minimal epimorphism* although in a much more specialized context than is considered here. The essential property he requires is absolute minimality as defined above. However, under the specialized hypotheses of [4] a minimal epimorphism as defined above is necessarily absolutely minimal. We consider this point at greater length in [8].

As every (S', ϵ') is isomorphic to some (S_n, ϵ_n) it suffices to show $(S_0, \epsilon_0) \preceq (S_n, \epsilon_n)$. However $(S_0 \oplus \Lambda^n, \epsilon_0 \circ \pi) \in \mathcal{SF}_M(n)$ where $\pi : S_0 \oplus \Lambda^n \to S_0$ is the projection. Moreover, one clearly has $(S_0, \epsilon_0) \preceq (S_0 \oplus \Lambda^n, \epsilon_0 \circ \pi)$ via the projection π. As (S_n, ϵ_n) represents the unique isomorphism class in $\mathcal{SF}_M(n)$ then $(S_n, \epsilon_n) \cong (S_0 \oplus \Lambda^n, \epsilon_0 \circ \pi)$ so that $(S_0, \epsilon_0) \preceq (S_n, \epsilon_n)$ as required. □

We wish to consider the extent to which the converse holds. Let T be a finitely generated stably free Λ-module ; for any $(S, \epsilon) \in \mathcal{SF}_M$ let $\pi : S \oplus T \to S$ denote the projection and define an object $\rho_T(S, \epsilon) \in \mathcal{SF}_M$ by $\rho_T(S, \epsilon) = (S \oplus T, \epsilon \circ \pi)$ Evidently if $(S, \epsilon) \in \mathcal{SF}_M(0)$ then $\rho_T(S, \epsilon) \in \mathcal{SF}_M(\mathrm{rk}(T))$.

Proposition 5.5 : Let M be a finitely generated Λ-module for which $\mathcal{A}\mathrm{bs}_\Lambda(M)$ holds and let $(S, \epsilon) \in \mathcal{SF}_M(0)$; then for any $(S', \epsilon') \in \mathcal{SF}_M$ there exists a finitely generated stably free module T such that $(S', \epsilon') \cong \rho_T(S, \epsilon)$.

Proof : If $(S', \epsilon') \in \mathcal{SF}_M$ then by absolute minimality of (S, ϵ) there exists a surjective Λ-homomorphism $\varphi : S' \to S$ such that $\epsilon' = \epsilon \circ \varphi$. Put $T = \mathrm{Ker}(\varphi)$ so that we have an exact sequence

$$0 \to T \xrightarrow{j} S' \xrightarrow{\varphi} S \to 0.$$

As S is projective the sequence splits so that there exists a homomorphism $r : S' \to T$ such that $r \circ j = \mathrm{Id}_T$. Then $\natural = (\varphi, r) : S' \to S \oplus T$ gives the required $\mathcal{SF}_M$-isomorphism $\natural : (S', \epsilon') \xrightarrow{\simeq} \rho_T(S, \epsilon)$. □

We denote by $\nu_\Lambda(n)$ the number (possibly $+\infty$) of distinct isomorphism classes of stably free Λ-modules of rank n. As a consequence of (5.5) we have:

Corollary 5.6 : If $\mathcal{A}\mathrm{bs}_\Lambda(M)$ holds then $\chi_M(n) \le \nu_\Lambda(n)$ for all $n \in \mathbf{N}$.

Λ has property SFC if and only if $\nu_\Lambda \equiv 1$. Thus from (5.4) and (5.6) we see that :

Corollary 5.7 : If Λ has property SFC then $\mathcal{A}\mathrm{bs}_\Lambda(M)$ holds $\Longleftrightarrow$ $\chi_M \equiv 1$.

Likewise from (3.6) and (5.6) we obtain:

Corollary 5.8: Let M be a finitely generated coprojective Λ-module; if $\mathcal{A}\mathrm{bs}_\Lambda(M)$ holds then for all n we have $c_{\Omega_1(M)}(n) \le \nu_\Lambda(n)$.

Corollary 5.9: Let M be a finitely generated coprojective Λ-module ; if $\mathcal{A}\mathrm{bs}_\Lambda(M)$ holds and Λ has property SFC then $\Omega_1(M)$ is straight.

By dimension shifting from (5.8) we get the following consequence of (4.6) and (5.6):

Corollary 5.10: Let M be a module of type $FT(k-1)$ such that $\mathrm{Ext}^k_\Lambda(M, \Lambda) = 0$. If $\mathcal{A}\mathrm{bs}_\Lambda(\Omega_{k-1}(M))$ holds then for all n we have $c_{\Omega_k(M)}(n) \le \nu_\Lambda(n)$.

Similarly we obtain a sufficient condition for the k^{th}-syzygy to be straight.

Corollary 5.11: Let M be a module of type $FT(k-1)$ such that $\mathrm{Ext}^k_\Lambda(M,\Lambda) = 0$. If $\mathcal{A}\mathrm{bs}_\Lambda(\Omega_{k-1}(M))$ holds and Λ has property SFC then $\Omega_k(M)$ is straight.

§6 : An injectivity criterion :

We recall briefly the notion of the derived module category $\mathcal{D}\mathrm{er}(\Lambda)$ of the ring Λ (cf [7] Chapter 5). If $f : M \to N$ is a homomorphism of Λ-modules we write '$f \approx 0$', when f can be written as a composite $f = \xi \circ \eta$ thus

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ & {\scriptstyle\eta}\searrow \quad \nearrow{\scriptstyle\xi} & \\ & P & \end{array}$$

where P is a projective Λ-module and $\eta : M \to P$ and $\xi : P \to N$ are Λ - homomorphisms. The derived module category $\mathcal{D}\mathrm{er}(\Lambda)$ is then the quotient of the category of Λ-modules by the relation '$\approx$'; that is, the objects in $\mathcal{D}\mathrm{er}$ are Λ-modules with

$$\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(M,N) = \mathrm{Hom}_\Lambda(M,N)/\langle M,N\rangle.$$

Note that, as $\langle M,N\rangle$ is a subgroup of $\mathrm{Hom}_\Lambda(M,N)$ then $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(M,N)$ has the natural structure of an abelian group. Moreover, $\mathrm{End}_{\mathcal{D}\mathrm{er}}(M) = \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(M,M)$ has the natural structure of a ring.

So far our results apply equally well to any weakly finite ring Λ. In this section we suppose also that Λ is an algebra augmented over a commutative ring R. In particular, there are ring homomorphisms $R \xrightarrow{i} \Lambda \xrightarrow{\epsilon} R$ such that $\epsilon \circ i = \mathrm{Id}_R$ and such that $i(R)$ is contained in the centre of Λ. Thereby R acquires the structure of a Λ-module. We note the trivial point that although ϵ is Λ-homomorphism, in general i is not. Observe that for any Λ-module J there is a ring homomorphism $\widetilde{\lambda} : R \to \mathrm{End}_\Lambda(J)$ given by $\alpha \mapsto \widetilde{\lambda}_\alpha$ where $\widetilde{\lambda}_\alpha(x) = i(\alpha)\cdot x$ for $x \in J$. Composing with the natural map $[\,] : \mathrm{End}_\Lambda(J) \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$ gives a ring homomorphism $\lambda : R \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$.

Until further notice, M will denote a finitely generated coprojective Λ-module and J a module in $\Omega_1(M)$. We will, appealing to (3.5), likewise fix an exact sequence $\mathcal{E} = (0 \to J \xrightarrow{j} S \xrightarrow{p} M \to 0)$ in which S is finitely generated stably free. We recall the basics of corepresentability ([7] Chap.5) of $\mathrm{Ext}^1(M,-)$. Given an extension $\mathcal{X} = (0 \to J \xrightarrow{\iota} X \xrightarrow{\pi} M \to 0)$, its congruence class in $\mathrm{Ext}^1(M,J)$ will be denoted by $[\mathcal{X}]$. For any Λ-homomorphism $\alpha : J \to J$ there is an extension

$$\alpha_*(\chi) \;=\; (\;0 \to \;\; J \xrightarrow{\iota} \;\; \lim_{\to}(\alpha,\iota) \xrightarrow{\pi} \;\; M \to \;\; 0\;).$$

As M is coprojective, then applying $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(-,J)$ to $\mathcal{E}$ yields an exact sequence

$$\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(S,J) \xrightarrow{j^*} \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(J,J) \xrightarrow{\delta_*} \mathrm{Ext}^1(M,J) \xrightarrow{p^*} \mathrm{Ext}^1(S,J)$$

where $\delta_*([\alpha]) = \alpha_*(\mathcal{E})$. As S is projective then $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(S,J) = \mathrm{Ext}^1(S,J) = 0$ so that δ_* induces an isomorphism $\delta_* : \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(J,J) \to \mathrm{Ext}^1(M,J)$. However the natural map $\mathrm{End}_\Lambda(J) \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$ is surjective so that:

(6.1) $[\mathcal{X}] = [\alpha_*(\mathcal{E})]$ for some $\alpha \in \mathrm{End}_\Lambda(J)$;

(6.2) $[\alpha_*(\mathcal{E})] = [\beta_*(\mathcal{E})] \iff [\alpha] = [\beta] \in \mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$.

An exact sequence $\mathcal{X} = (0 \to J \xrightarrow{\iota} X \xrightarrow{\pi} M \to 0)$ is said to be a *projective extension* when X is projective. We note that (cf [7] (7.21), p. 139).

(6.3) $\alpha_*(\mathcal{E})$ is a projective extension $\iff [\alpha] \in \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J)$.

Given another such extension $\mathcal{X}' = (0 \to J \xrightarrow{\iota'} X' \xrightarrow{\pi'} M \to 0)$ we write $\mathcal{X}' \cong_M \mathcal{X}$ when there is a commutative diagram of Λ-homomorphisms

$$\begin{matrix} \mathcal{X} \\ \downarrow \\ \mathcal{X}' \end{matrix} = \begin{pmatrix} 0 \to & J & \xrightarrow{\iota} & X & \xrightarrow{\pi} & M & \to 0 \\ & \varphi \downarrow & & \widetilde{\varphi} \downarrow & & \| \ \mathrm{Id}_M & \\ 0 \to & J & \xrightarrow{\iota'} & X' & \xrightarrow{\pi'} & M & \to 0 \end{pmatrix}$$

in which φ and hence $\widetilde{\varphi}$ are isomorphisms. Also note:

(6.4) $\mathcal{X}' \cong_M \mathcal{X} \iff [\mathcal{X}'] = [\alpha_*(\mathcal{X})]$ for some $\alpha \in \mathrm{Aut}_\Lambda(J)$.

Finally we will add the assumption that $\lambda_J : R \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$ is an isomorphism:

Proposition 6.5: Suppose that $\lambda_J : R \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$ is an isomorphism; then for any projective extension $\mathcal{X} = (0 \to J \xrightarrow{\iota} X \xrightarrow{\pi} M \to 0)$ there is an isomorphism of extensions $\varphi : \mathcal{E} \xrightarrow{\simeq} \mathcal{X}$ over Id_M.

Proof : By (6.1) we may write $[\mathcal{X}] = [\alpha_*(\mathcal{E})]$ for some $\alpha \in \mathrm{End}_\Lambda(J)$. However as $\mathcal{X}$ is projective then $[\alpha]$ is a unit in $\mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$ by (6.3).

By assumption, $\lambda_J : R \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$ is a ring isomorphism, so that, for some unit $u \in R^*$, $\lambda_J(u) = [\alpha]$. Let $\widehat{u} : J \to J$ be the mapping $\widehat{u}(x) = i(u) \cdot x$. As $u \in R^*$ and $i(R)$ is central in Λ then $\widehat{u}$ is an isomorphism over Λ. Thus $\widehat{u}_*(\mathcal{E}) \cong_M \mathcal{E}$. In particular, there is an isomorphism of extensions $\overline{u} : \mathcal{E} \to \widehat{u}_*(\mathcal{E})$ described by the following diagram.

$$\begin{matrix} \mathcal{E} \\ \overline{u} \downarrow \\ \widehat{u}_*(\mathcal{E}) \end{matrix} = \begin{pmatrix} 0 \to & J & \xrightarrow{i} & S & \xrightarrow{\epsilon} & M & \to 0 \\ & \widehat{u} \downarrow & & \natural \downarrow & & \| \ \mathrm{Id}_M & \\ 0 \to & J & \xrightarrow{\iota} & \lim(\widehat{u}, j) & \xrightarrow{\pi} & M & \to 0 \end{pmatrix}.$$

However, by construction $[u] = [\alpha] = \lambda_J(u) \in \mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$ so that, by (6.2),

$$[\widehat{u}_*(\mathcal{E})] = [\alpha_*(\mathcal{E})] = [\mathcal{X}].$$

Let $c : \widehat{u}_*(\mathcal{E}) \to \mathcal{X}$ be a congruence; taking the composition, $\varphi = c \circ \overline{u} : \mathcal{E} \xrightarrow{\simeq} \mathcal{X}$ is an isomorphism over Id_M as required. □

In (6.5) the hypothesis on λ_J can be transferred to M as follows:

Proposition 6.6 : Let Λ be an algebra augmented over the commutative ring R, and let M, J be Λ-modules such that M is coprojective and $J \in \Omega_1(M)$; then

$$\lambda_M : R \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(M) \text{ is an isomorphism } \iff \lambda_J : R \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(J) \text{ is an isomorphism.}$$

Proof : If $f : M \to M$ is a Λ-homomorphism it is a consequence of the universal property of projective modules that there exists a morphism $\widetilde{f}$ over f

$$\begin{array}{c} \alpha \\ \downarrow \widetilde{f} \\ \beta \end{array} = \left(\begin{array}{ccccc} 0 \to & J \stackrel{j}{\to} & S \stackrel{p}{\to} & M \to & 0 \\ & \downarrow f_- & \downarrow \widetilde{f} & \downarrow f & \\ 0 \to & J \stackrel{j}{\to} & S \stackrel{p}{\to} & M \to & 0 \end{array} \right).$$

Although the homomorphism $f_- : J \to J$ need not be unique, it becomes unique if we work instead in the category $\mathcal{D}\mathrm{er}(\Lambda)$ and the correspondence $f \mapsto [f_-]$ determines a ring homomorphism $\rho_{\mathcal{E}} : \mathrm{End}_{\mathcal{D}\mathrm{er}}(M) \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$. Moreover, as M is coprojective then $\rho_{\mathcal{E}}$ is a ring isomorphism (cf [7] p. 133). However the following diagram commutes;

$$\begin{array}{ccc} & R & \\ {}^{\lambda_M}\swarrow & & \searrow^{\lambda_J} \\ \mathrm{End}_{\mathcal{D}\mathrm{er}}(M) & \xrightarrow{\rho_{\mathcal{E}}} & \mathrm{End}_{\mathcal{D}\mathrm{er}}(J) \end{array}$$

Thus λ_M is an isomorphism if and only if λ_J is also an isomorphism. □

Finally we arrive at our injectivity criterion which is Theorem A of the Introduction:

Theorem 6.7 : Let Λ be an algebra augmented over the commutative ring R, and let M be a finitely presented coprojective Λ-module such that $\lambda_M : R \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(M)$ is an isomorphism ; then $\kappa : \mathcal{SF}_M \to \Omega_1(M)$ is injective on isomorphism classes.

Proof : Suppose that $J \in \Omega_1(M)$ and that (S, ϵ), (S', ϵ') are objects in $\mathcal{SF}_M$ such that $\kappa(S, \epsilon) \cong \kappa(S', \epsilon') \cong J$. By hypothesis M is coprojective and λ_M is an isomorphism so that, by (6.6) λ_J is also an isomorphism. Thus the hypotheses of (6.5) are all realised. Now construct the extensions

$$\mathcal{E} = (0 \to J \stackrel{j}{\to} S \stackrel{\epsilon}{\to} M \to 0) \;;\; \mathcal{E}' = (0 \to J \stackrel{j'}{\to} S' \stackrel{\epsilon'}{\to} M \to 0).$$

As S' is stably free then $\mathcal{E}'$ is certainly a projective extension. Thus by (6.5) there is an isomorphism of extensions $\varphi : \mathcal{E} \stackrel{\simeq}{\longrightarrow}_M \mathcal{E}'$ over Id_M thus

$$\begin{array}{c} \mathcal{E} \\ \varphi \downarrow \\ \mathcal{E}' \end{array} = \left(\begin{array}{ccccccc} 0 \to & J & \stackrel{i}{\to} & S & \stackrel{\epsilon}{\to} & M & \to 0 \\ & \varphi_- \downarrow & & \varphi \downarrow & & \| \; \mathrm{Id}_M & \\ 0 \to & J & \stackrel{j'}{\to} & S' & \stackrel{\epsilon'}{\to} & M & \to 0 \end{array} \right).$$

Thereby φ defines an isomorphism $\varphi : (S, \epsilon) \stackrel{\simeq}{\longrightarrow} (S', \epsilon')$ in $\mathcal{SF}_M$. Thus $\kappa(S, \epsilon) \cong \kappa(S', \epsilon')$ implies that $(S, \epsilon) \cong (S', \epsilon')$ and κ is injective on isomorphism classes as claimed. □

In order to apply (6.7) we need to guarantee the hypotheses that M be coprojective and that $\lambda_M : R \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(M)$ is an isomorphism are both realized. Thus we impose the condition that Λ is a PD^n-algebra; that is, R admits a free resolution of finite type over Λ and $0 \to \Lambda \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} F_{n-2} \xrightarrow{\partial_{n-2}} \dots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} \Lambda \xrightarrow{\epsilon} R \to 0$ and

$$\mathrm{Ext}^r_\Lambda(R, \Lambda) \cong \begin{cases} R & r = n \\ 0 & r \neq n. \end{cases}$$

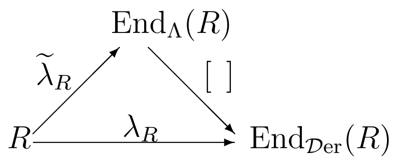
Here $\mathrm{Ext}^0_\Lambda(R, \Lambda) = \mathrm{Hom}_\Lambda(R, \Lambda)$. In particular, all syzygies $\Omega_k(R)$ are finitely generated whilst, by dimension shifting we have :

(6.8) $\Omega_k(R)$ is coprojective for $0 \leq k \leq n-2$.

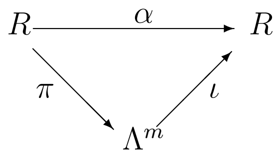
With this hypothesis we have:

Proposition 6.9 : $\lambda_R : R \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(R)$ is an isomorphism provided $n \geq 1$.

Proof : There is a commutative diagram of ring homomorphisms



where $\widetilde{\lambda} : R \to \mathrm{End}_\Lambda(J)$ is given by $\widetilde{\lambda}_\alpha(x) = i(\alpha) \cdot x$ for $x \in J$ and [ ] is the canonical surjection. Now $[\alpha] = 0$ if and only if α factors through a free module thus :



As $\mathrm{Hom}_\Lambda(R, \Lambda) = \mathrm{Ext}^0_\Lambda(R, \Lambda) = 0$ then in the above factorization, π is necessarily zero. Thus $[\alpha] = 0$ implies that $\alpha = 0$; that is, $[\,] : \mathrm{End}_\Lambda(J) \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$ is injective. As [ ] is trivially surjective then $[\,] : \mathrm{End}_\Lambda(J) \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(J)$ is an isomorphism. However, $\widetilde{\lambda}$ is trivially an isomorphism so that λ_R is also an isomorphism. □

Proposition 6.10 : Let Λ be a PD^n-algebra augmented over the commutative ring R ; then $\lambda_{\Omega_r(R)} : R \to \mathrm{End}_{\mathcal{D}\mathrm{er}}(\Omega_r(R))$ is an isomorphism provided $0 \leq r \leq n-1$.

Proof : For $r = 0$ this is simply (6.10). Using the fact that $\Omega_{r-1}(R)$ is coprojective it follows inductively from (6.6) that $\lambda_{\Omega_r(R)}$ is an isomorphism for $1 \leq r \leq n-1$. □

From (3.6), (6.7) and (6.10) we obtain Theorem B of the Introduction thus:

Theorem 6.11 : Let Λ be a PD^n-algebra augmented over the commutative ring R ; then $\kappa : \mathcal{SF}_{\Omega_{r-1}(R)} \to \Omega_r(R)$ is an order-preserving bijection on isomorphism classes for $1 \leq r \leq n-1$.

With the same hypotheses we can interpret (6.12) in terms of counting functions:

(6.12) $\chi_{\Omega_{k-1}(R)} \equiv c_{\Omega_k(R)} \qquad (1 \leq k \leq n-1).$

§7 : The duality relations :

Suppose given a PD^n-algebra Λ augmented over the commutative ring R. In particular, we are given a free resolution of finite type

(7.1) $\mathcal{F} \;=\; (0 \to \Lambda \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} F_{n-2} \xrightarrow{\partial_{n-2}} \dots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} \Lambda \xrightarrow{\epsilon} R \to 0).$

Applying $\mathrm{Hom}_\Lambda(-,\Lambda)$ to the above sequence and systematically employing the requirement that $\mathrm{Ext}^r_\Lambda(R,\Lambda) = 0$ for $r \neq n$ one obtains a dual exact sequence

(7.2) $\mathcal{F}^* \;=\; (0 \to \Lambda^* \xrightarrow{\partial_1^*} F_1^* \xrightarrow{\partial_2^*} F_2^* \xrightarrow{\partial_3^*} \dots \xrightarrow{\partial_{n-1}^*} F_{n-1}^* \xrightarrow{\partial_n^*} \Lambda^* \xrightarrow{\widetilde{\epsilon}} \mathrm{Ext}^n_\Lambda(R,\Lambda) \to 0.$

However laterality is now reversed; if $\mathcal{F}$ is a sequence of right modules then $\mathcal{F}^*$ is naturally a sequence of left modules. To circumvent this difficulty we impose the extra hypothesis that the algebra Λ be *involuted*; that is, we assume there is an R-algebra isomorphism $\tau : \Lambda \to \Lambda^{opp}$ to the *opposite* R-algebra which satisfies $\tau^2 = \mathrm{Id}$. In particular, τ must be R-linear, should satisfy $\tau(1) = 1$ and, crucially,

$$\tau(xy) \;=\; \tau(y)\tau(x).$$

Under this hypothesis one converts a left Λ-module M to a right Λ-module by writing $\mathbf{x} \bullet \alpha \;=\; \tau(\alpha) * \mathbf{x}$ where $\mathbf{x} \in M$ and $\alpha \in \Lambda$. The existence of such an involution is, of course, highly contingent. For general rings Λ no such involution exists and the categories of left and right Λ-modules need not be equivalent. However, in the most familiar examples of augmented algebras such involutions do exist; for example, when Λ is commutative ; then we may take τ to be the identity; when $\Lambda = R[G]$ is a group algebra we may take τ to be the canonical involution induced from $g \mapsto g^{-1}$; that is , $\tau(\sum_{g\in G} a_g g) \;=\; \sum_{g \in G} a_g g^{-1}$. Assuming that Λ admits such an involution one may then legitimately regard $\mathcal{F}^*$ as a free resolution of right Λ-modules. It is convenient to re-index formally and write

(7.3) $E_r \;=\; F^*_{n-r} \quad ; \quad d_r \;=\; \partial^*_{n+1-r}.$

when, observing that $\mathrm{Ext}^n_\Lambda(R,\Lambda) \;\cong\; R$, $\mathcal{F}^*$ assumes the form

(7.4) $\mathcal{F}^* \;=\; (0 \to \Lambda^* \xrightarrow{d_n} E_{n-1} \xrightarrow{d_{n-1}} E_{n-2} \xrightarrow{d_{n-2}} \dots \xrightarrow{d_2} E_1 \xrightarrow{d_1} \Lambda^* \xrightarrow{\widetilde{\epsilon}} R \to 0)$

Comparing the positions of the respective syzygies in $\mathcal{F}$ and $\mathcal{F}^*$ then writing $\delta(J) = \mathrm{Hom}_\Lambda(J,\Lambda)$, we obtain mapping $\delta : \Omega_r(R) \;\to\; \Omega_{n+1-r}(R)$ for $1 \leq r \leq n-1$. We observe when $2 \leq r \leq n-1$ it is also true that $2 \leq n+1-r \leq n-1$. Then applying δ again we see that $\delta \circ \delta$ is the identity on $\Omega_r(R)$. Expressed formally :

Theorem 7.5 : Let Λ be an involuted PD^n algebra augmented over the commutative ring R; for $2 \leq r \leq n-1$, the duality map $J \mapsto J^* = \mathrm{Hom}_\Lambda(J, \Lambda)$ induces a bijection on isomorphism classes $\Omega_r(R) \overset{\simeq}{\longleftrightarrow} \Omega_{n+1-r}(R)$ and satisfies $\delta \circ \delta = \mathrm{Id}$.

We can again interpret (7.5) in terms of counting functions. Writing $c_k = c_{\Omega_k(R)}$, then with the same hypotheses as (7.5) we clearly have :

(7.6) $c_r \equiv c_{n+1-r}$ for $2 \leq r \leq n-1$.

Likewise, writing $\chi_k = \chi_{\Omega_k(R)}$ then it follows from (6.13) and (7.6) that :

(7.7) $\chi_r \equiv \chi_{n-r-1}$ for $1 \leq r \leq n-2$.

The duality of (7.5) breaks down at the extremes of the range, namely when $r = 1$ or, equivalently expressed, when $n + 1 - r = n$. Consider first the exact sequence defined by the augmentation homomorphism

$$0 \to I \overset{\iota}{\hookrightarrow} \Lambda \overset{\epsilon}{\to} R \to 0$$

where $I = \mathrm{Ker}(\epsilon)$ is the augmentation ideal. When $n = 1$ there is nothing to consider as the definition of PD^1-algebra forces an isomorphism $I \cong \Lambda$. However, when $n \geq 2$, dualisation of the augmentation sequence gives an exact sequence in cohomology

$$\mathrm{Hom}_\Lambda(R, \Lambda) \overset{\epsilon^*}{\to} \Lambda^* \overset{\iota^*}{\to} \mathrm{Hom}_\Lambda(I, \Lambda) \overset{\partial}{\to} \mathrm{Ext}^1(R, \Lambda)$$

After making the identification $\mathrm{Ext}^0(R, \Lambda) = \mathrm{Hom}_\Lambda(R, \Lambda)$ the hypothesis that Λ be a PD^n-algebra with $n \geq 2$ requires the two end terms to vanish giving an isomorphism $I^* \cong \Lambda^*$. The augmentation ideal I is a representative $\Omega_1(R)$ whilst $\Omega_n(R)$ is the stable class of $\Lambda^* \cong \Lambda$. Thus duality still gives a mapping $\delta : \Omega_1(R) \to \Omega_n(R)$ which we proceed to investigate.

In the general context of this paper the syzygy $\Omega_n(R)$ is better described as $\mathcal{S}$, the class of finitely generated stably free modules. We denote by $\mathcal{S}_+$ the subclass of *nonzero* finitely generated stably free modules. It is straightforward to see that:

(7.8) The duality $\delta : \mathcal{S}_+ \overset{\simeq}{\longrightarrow} \mathcal{S}_+$ is bijective on isomorphism types.

There is a 'forgetful mapping' $\pi : \mathcal{SF}_R \to \mathcal{S}_+$ given by $\pi(S, \epsilon) = S$. In the case where R is a principal ideal domain we have :

Proposition 7.9 : If Λ is an algebra augmented over a commutative principal ideal domain R then $\pi : \mathcal{SF}_R \to \mathcal{S}_+$ is surjective on isomorphism types.

Proof : Let S be a nonzero finitely generated stably free Λ-module; we must show that there exists a surjective Λ-homomorphism $\eta : S \to R$. Then for some integers $a \geq 0$ $S \oplus \Lambda^a \cong \Lambda^{n+a}$ where $n = \mathrm{rk}(S) > 0$ and hence

$$\begin{aligned} \mathrm{Hom}_\Lambda(S \oplus \Lambda^a, R) &\cong \mathrm{Hom}_\Lambda(S, R) \oplus \mathrm{Hom}_\Lambda(\Lambda^a, R) \\ &\cong \mathrm{Hom}_\Lambda(S, R) \oplus R^a. \end{aligned}$$

However $\mathrm{Hom}_\Lambda(S \oplus \Lambda^a, R) \cong \mathrm{Hom}_\Lambda(\Lambda^{n+a}, R) \cong R^{n+a}$. From the classification of modules over the principal ideal domain R it follows that $\mathrm{Hom}_\Lambda(S, R) \cong R^n$.

As $n > 0$ there is a nonzero homomorphism $\widetilde{\eta} : S \to R$. As R is a principal ideal domain it follows that $\mathrm{Im}(\widetilde{\eta}) = cR$ for some $c \neq 0$. Putting $\eta = \frac{1}{c}\widetilde{\eta}$ then $\eta : S \to R$ is a surjective Λ-homomorphism as required. □

As a consequence we have:

Theorem 7.10 : Let Λ be an involuted PD^n algebra augmented over a commutative principal ideal domain R; if $n \geq 2$ then the duality map $J \mapsto J^* = \mathrm{Hom}_\Lambda(J, \Lambda)$ induces a surjection on isomorphism classes $\delta : \Omega_1(R) \xrightarrow{\simeq} \mathcal{S}_+$.

Proof : Immediate from (6.11), (7.8) and (7.9). □

§8 : The SFC property and the conditions $\mathcal{A}\mathrm{bs}_\Lambda(R)$, $\mathcal{A}\mathrm{bs}_\Lambda(\Omega_{n-1}(R))$:

We maintain our assumption that Λ is an involuted PD^n-algebra augmented over a commutative ring R. Moreover, as at the end of §7 we add the extra hypothesis:

(*) R is a principal ideal domain.

The syzygy $\Omega_{n-1}(R)$ is exceptional in not being coprojective; it has a representative $K = \mathrm{Ker}(\partial_{n-1})$ which satisfies $\mathrm{Ext}^1(K, \Lambda) \cong R \neq 0$. Thus our previous arguments do not apply in this case. Nevertheless, it is possible to say something. We note that:

Proposition 8.1: If $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in R^n$ and c is a generator of the ideal $\langle v_1, v_2, \ldots, v_n\rangle \lhd R$. then there exists $A \in GL_n(R)$ such that

$$(v_1, v_2, \ldots, v_n) \cdot A = (c, 0, \ldots, 0) \in R^n.$$

Proof : As c is a generator of the ideal $\langle v_1, v_2, \ldots, v_n\rangle$ we may write $v_i = cw_i$ and put $\mathbf{w} = (w_1, w_2, \ldots, w_n)$. Then $\langle w_1, w_2, \ldots, w_n\rangle = R$ and there exist $\xi_1, \xi_2, \ldots \xi_n \in R$ such that $\sum_{i=1}^n w_i\xi_i = 1$. Thus $\mathbf{w}$ is a unimodular row. As R is a principle ideal domain then every stably free R module is free. A well known argument (cf [10] p.118) implies the existence of $B \in \mathrm{GL}_n(R)$ with first row $\mathbf{w}$. As $\mathbf{v} = c\mathbf{w}$ then taking $A = B^{-1}$ we see that $(v_1, v_2, \ldots, v_n) \cdot A = (c, 0, \ldots, 0) \in R^n$. □

Proposition 8.2 : If $\mathcal{A}\mathrm{bs}_\Lambda(R)$ holds then Λ has property SFC.

Proof : Suppose that Λ fails to have property SFC and let $\widetilde{S}$ be a non-free stably free Λ-module of minimum possible rank. Note that $\widetilde{S}$ must then be indecomposably stably free. Put $m = \mathrm{rk}(\widetilde{S})$. Then $\widetilde{S} \oplus \Lambda^a \cong \Lambda^{m+a}$ for some $a \geq 1$ and hence

$$\begin{aligned}\mathrm{Hom}_\Lambda(\widetilde{S} \oplus \Lambda^a, R) &\cong \mathrm{Hom}_\Lambda(\widetilde{S}, R) \oplus \mathrm{Hom}_\Lambda(\Lambda^a, R)\\ &\cong \mathrm{Hom}_\Lambda(\widetilde{S}, R) \oplus R^a.\end{aligned}$$

However $\mathrm{Hom}_\Lambda(\widetilde{S} \oplus \Lambda^a, R) \cong \mathrm{Hom}_\Lambda(\Lambda^{m+a}, R) \cong R^{m+a}$. From the classification of modules over the principal ideal domain R we see that :

(*) $\quad \mathrm{Hom}_\Lambda(\widetilde{S}, R) \;\cong\; R^m$

As $m \geq 1$ it follows that there is a nonzero homomorphism $\widetilde{\eta} : \widetilde{S} \to R$. As R is a principal ideal domain it follows that $\mathrm{Im}(\widetilde{\eta}) = cR$ for some $c \neq 0$. Putting $\widetilde{\epsilon} = \frac{1}{c}\widetilde{\eta}$ we see that $\widetilde{\epsilon} : \widetilde{S} \to R$ is a surjective Λ-homomorphism and hence defines an object in $\mathcal{SF}_R$. Under the assumption that $\mathcal{A}\mathrm{bs}_\Lambda(R)$ holds it follows from (8.2) that (Λ, ϵ) is an absolutely minimal object in $\mathcal{SF}_R$. Comparing $(\widetilde{S}, \widetilde{\epsilon})$ with (Λ, ϵ) we see there exists a surjective Λ-homomorphism $\psi : \widetilde{S} \to S$ making the following commute.

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\widetilde{\epsilon}} & R \\ \psi \downarrow & & || \\ \Lambda & \xrightarrow{\epsilon} & R. \end{array}$$

Clearly the exact sequence $\;0 \to \mathrm{Ker}(\psi) \to \widetilde{S} \xrightarrow{\psi} \Lambda \to 0\;$ splits showing that $\widetilde{S} \cong \Lambda \oplus \mathrm{Ker}(\psi)$. As $\widetilde{S}$ is indecomposably stably free then $\mathrm{Ker}(\psi) = 0$. Hence $\psi : \widetilde{S} \xrightarrow{\simeq} \Lambda$ is an isomorphism, contradicting the assumption that $\widetilde{S}$ is not free. Hence Λ has property SFC. $\square$

It is straightforward to see that $\mathrm{Hom}_\Lambda(\Lambda, R) \cong R$ and that the augmentation map $\epsilon : \Lambda \to R$ is an R generator. It follows immediately that :

Proposition 8.3: Each Λ-homomorphism $\eta : \Lambda^N \to R$ has the form $\eta = (a_1\epsilon, \dots, a_N\epsilon)$ where $a_i \in R$; moreover $(a_1\epsilon, \dots, a_N\epsilon) : \Lambda^N \to R$ is surjective $\Leftrightarrow$ $(a_1, \dots, a_N) = R$.

We now see that the converse to (8.2) also holds :

Theorem 8.4 : Λ has property $SFC \iff \mathcal{A}\mathrm{bs}_\Lambda(R)$ holds.

Proof : By (8.2) it suffices to prove ($\Longrightarrow$). Thus suppose that Λ satisfies SFC. We will show that (Λ, ϵ) is an absolutely minimal element in $\mathcal{SF}_R$.

As Λ has property SFC the elements of $\mathcal{SF}_R$ are surjective homomorphisms $\widetilde{\epsilon} : \Lambda^N \to R$. Given such a surjective homomorphism $\widetilde{\epsilon}$ we can, by (8.3), express it in the form $\widetilde{\epsilon} = (a_1\epsilon, \dots, a_N\epsilon)$ where $(a_1, \dots, a_N) = R$. Choose $(b_1, \dots, b_N) \in R$ such that $\sum_{r=1}^N a_r b_r = 1$ and define Λ-linear maps $\varphi : \Lambda^N \to \Lambda$ and $s : \Lambda \to \Lambda^N$ by means of the matrices

$$\varphi = (i(a_1), \dots, i(a_N)) \quad ; \quad s = \begin{pmatrix} i(b_1) \\ \vdots \\ i(b_N) \end{pmatrix}$$

It is straightforward to check that the following diagram commutes:

$$\begin{array}{ccc} \Lambda^N & \xrightarrow{\widetilde{\epsilon}} & R \\ \varphi \downarrow & & || \\ \Lambda & \xrightarrow{\epsilon} & R. \end{array}$$

Moreover, as $\sum_{r=1}^{N} i(a_r)i(b_r) = i(1_R) = 1_\Lambda$ then $\varphi \circ s = \mathrm{Id}_\Lambda$, φ is surjective and (Λ, ϵ) is absolutely minimal. $\square$

We turn now to consider the exceptional syzygy $\Omega_{n-1}(R)$. Suppose given an extension of Λ-modules $\mathcal{X} = (0 \to \Lambda \xrightarrow{i} X \xrightarrow{p} M \to 0)$ and an element $\alpha \in \Lambda$; then we may form the extension $\alpha_*(\chi)$ defined by the bottom row of the following diagram

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \alpha_*(\mathcal{X}) \end{array} = \left(\begin{array}{cccccc} 0 \to & \Lambda & \xrightarrow{i} & X & \xrightarrow{p} & M & \to 0 \\ & \lambda_\alpha \downarrow & & \natural \downarrow & & \| \ \mathrm{Id}_M & \\ 0 \to & \Lambda & \xrightarrow{\iota} & \varinjlim(\lambda_\alpha, i) & \xrightarrow{\pi} & M & \to 0 \end{array} \right)$$

where $\lambda_\alpha(y) = \alpha y$. Thereby $\mathrm{Ext}^1(M, \Lambda)$ acquires the structure of a left Λ-module under the action

$$\begin{array}{rcl} \bullet \ : \ \Lambda \times \mathrm{Ext}^1(M, \Lambda) & \to & \mathrm{Ext}^1(M, \Lambda) \\ \\ (\alpha, [\beta]) & \longmapsto & [\alpha_*(\beta)]. \end{array}$$

Until further notice we will assume that M is some module satisfying the following condition $\mathcal{R}(\mathbf{I})$ and $\mathcal{R}(\mathbf{II})$:

$\mathcal{R}(\mathbf{I})$ there is an extension $\mathcal{E} = (0 \to \Lambda \xrightarrow{i} \Lambda^m \xrightarrow{p} M \to 0)$ for some $m \geq 1$;

$\mathcal{R}(\mathbf{II})$ there is an isomorphism of Λ-modules $\mathrm{Ext}^1(M, \Lambda) \xrightarrow{\simeq} R$ under which $[\mathcal{E}] \mapsto 1$.

On that understanding we have:

Proposition 8.5 : Let the extension $\mathcal{F} = (0 \to \Lambda \xrightarrow{j} F \xrightarrow{q} M \to 0)$ be classified by $[\mathcal{F}] \in \mathrm{Ext}^1(M, \Lambda)$; if F is free then $[\mathcal{F}] = u[\mathcal{E}]$ for some unit $u \in R^*$.

Proof : As $[\mathcal{E}]$ generates $\mathrm{Ext}^1(M, \Lambda)$ over R we may write $[\mathcal{F}] = u\,[\mathcal{E}]$ for some $u \in R$. We claim that u is a unit. As p is surjective and F is free there exists a commutative diagram of Λ-homomorphisms as follows:

$$\begin{array}{ccc} & & F \\ & \stackrel{\widetilde{q}}{\swarrow} & \downarrow q \\ \Lambda^m & \xrightarrow{p} & M \end{array}$$

which may be incorporated in the following commutative diagram :

$$\begin{array}{c} \mathcal{F} \\ \downarrow \\ \alpha_*(\mathcal{F}) \\ \downarrow \\ \mathcal{E} \end{array} = \left(\begin{array}{cccccc} 0 \to & \Lambda & \xrightarrow{j} & F & \xrightarrow{q} & M & \to 0 \\ & \lambda_\alpha \downarrow & & \natural \downarrow & & \| \ \mathrm{Id}_M & \\ 0 \to & \Lambda & \xrightarrow{\iota} & \varinjlim(\lambda_\alpha, j) & \xrightarrow{\pi} & M & \to 0 \\ & \mathrm{Id}_\Lambda \ \| & & \bar{q} \downarrow & & \| \ \mathrm{Id}_M & \\ 0 \to & \Lambda & \xrightarrow{i} & \Lambda^m & \xrightarrow{p} & M & \to 0 \end{array} \right)$$

where we write the restriction of $\widehat{q}$ to Λ as $\widehat{q} = \lambda_\alpha$ for some $\alpha \in \Lambda$ and where $\bar{q}$ is the canonical map induced on the pushout by $\widetilde{q}$. The bottom two rows then define a congruence between $\alpha_*(\mathcal{F})$ and $\mathcal{E}$ so that

$$[\mathcal{E}] = [\alpha_*(\mathcal{F})].$$

Hence $[\mathcal{E}] = \epsilon(\alpha)[\mathcal{F}]$. Now $[\mathcal{F}] = u\,[\mathcal{E}]$ so that $[\mathcal{E}] = \epsilon(\alpha)u\,[\mathcal{E}]$. However, under the isomorphism $\mathrm{Ext}^1(M,\Lambda) \xrightarrow{\simeq} R$, $[\mathcal{E}]$ corresponds to $1_R \in R$. Hence $\epsilon(\alpha)u = 1_R$ and u is a unit as claimed. □

Proposition 8.6 : $p : \Lambda^m \to M$ is a minimal epimorphism.

Proof : If $p : \Lambda^m \to M$ is not minimal then there exists a surjective homomorphism $\theta : S \to M$ where S is a stably free module with $\mathrm{rk}(S) < m$. Comparing the exact sequence $\mathcal{E}' = (0 \to T \xrightarrow{\iota} S \xrightarrow{\theta} M \to 0)$ with $\mathcal{E}$, Schanuel's Lemma assures us that

$$\Lambda^m \oplus T \cong S \oplus \Lambda.$$

Thus T is stably free and $\mathrm{rk}(T) = \mathrm{rk}(S) + 1 - m \leq 0$. Hence $T = 0$ and we have an isomorphism $\theta : S \xrightarrow{\simeq} M$. Thereby M is forced to be stably free, contradicting the assumption that $\mathrm{Ext}^1(M,\Lambda) \cong R \neq 0$. Hence p is minimal as claimed. □

Theorem 8.7 : If Λ has property SFC then $\mathcal{A}\mathrm{bs}_\Lambda(M)$ holds.

Proof : By (5.4) it suffices to show that $\chi_M \equiv 1$ Thus let n be an integer ≥ 1 and let $\Sigma^n(\mathcal{E})$ denote the extension

$$\Sigma^n(\mathcal{E}) = (0 \to \Lambda \oplus \Lambda^n \xrightarrow{j \oplus \mathrm{Id}} \Lambda^m \oplus \Lambda^n \xrightarrow{p_n} M \to 0)$$

where $p_n = p \circ \pi_n$ and $\pi_n : \Lambda^m \oplus \Lambda^n \to \Lambda^m$ is the projection. It follows from (9.2) that $(\Lambda^m \oplus \Lambda^n, p_n) \in \mathcal{SF}_M(n)$. As Λ has property SFC then up to isomorphism any object in $\mathcal{SF}_M(n)$ has the form (Λ^{m+n}, q). To show that $\chi_M(n) = 1$ we must produce an isomorphism $(\Lambda^{m+n}, q) \xrightarrow{\simeq} (\Lambda^m \oplus \Lambda^n, p_n)$. Taking $(\Lambda^{m+n}, q) \in \mathcal{SF}_M(n)$, consider the corresponding extension $\mathcal{F} = (0 \to K \to \Lambda^{m+n} \xrightarrow{q} M \to 0)$. Comparing $\mathcal{F}$ with $\mathcal{E}$ we see that $\Lambda^m \oplus K \cong \Lambda^{m+n} \oplus \Lambda$. As Λ has SFC then $K \cong \Lambda^{n+1}$ so, without loss of generality, we may write $\mathcal{F}$ in the form

$$\mathcal{F} = (0 \to \Lambda^{n+1} \xrightarrow{j} \Lambda^{m+n} \xrightarrow{q} M \to 0).$$

Then $\mathcal{F}$ is classified by an element $[\mathcal{F}] \in \mathrm{Ext}^1(M, \Lambda^{n+1}) \cong R^{n+1}$. Write

$$[\mathcal{F}]^t = (v_1, v_2, \ldots, v_{n+1})$$

where each $v_i \in R$. Let $c \in R$ be a generator of the ideal $\langle v_1, v_2, \ldots, v_{n+1}\rangle \lhd R$. By (9.1) there exists $A \in GL_{n+1}(R)$ such that

$$[\mathcal{F}]^t A^t = (c, 0, \ldots, 0).$$

Put $\widetilde{A} = i(A) \in GL_{n+1}(\Lambda)$. Then there is an isomorphism of extensions

$$\begin{matrix} \mathcal{F} \\ A_* \downarrow \\ A_*(\mathcal{F}) \end{matrix} \quad = \quad \begin{pmatrix} 0 \to & \Lambda^{n+1} & \stackrel{j}{\to} & \Lambda^{m+n} & \stackrel{q}{\to} & M & \to 0 \\ & \widetilde{A} \downarrow & & \natural \downarrow & & \| \ \mathrm{Id}_M & \\ 0 \to & \Lambda^{n+1} & \stackrel{\iota}{\to} & \lim\limits_{\to} (\lambda_{\widetilde{A}}, j) & \stackrel{\pi}{\to} & M & \to 0 \end{pmatrix}.$$

Let $\mathcal{C} = (0 \to \Lambda \stackrel{\iota}{\to} C \stackrel{\theta}{\to} M \to 0)$ be the extension classified by $c \in R \cong \mathrm{Ext}^1(M, \Lambda)$ and consider the extension

$$\Sigma^n(\mathcal{C}) = (0 \to \Lambda \oplus \Lambda^n \stackrel{\iota \oplus \mathrm{Id}}{\longrightarrow} C \oplus \Lambda^n \stackrel{\theta_n}{\longrightarrow} M \to 0)$$

where $\theta_n = \theta \circ \pi$ and $\pi : C \oplus \Lambda^n \to C$ is the projection. Then $A_*(\mathcal{F})$ and $\Sigma^n(\mathcal{C})$ are both classified by

$$\begin{pmatrix} c \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in R^{n+1} \cong \mathrm{Ext}^1(M, \Lambda^{n+1}).$$

Hence there is a congruence $\gamma : A_*(\mathcal{F}) \to \Sigma^n(\mathcal{C})$. Composing with A_* gives an isomorphism of extensions

$$\begin{matrix} \mathcal{F} \\ \\ \gamma \circ A_* \downarrow \\ \\ \Sigma^n(\mathcal{C}) \end{matrix} \quad = \quad \begin{pmatrix} 0 \to & \Lambda^{n+1} & \stackrel{j}{\to} & \Lambda^{m+n} & \stackrel{q}{\to} & M & \to 0 \\ \\ & \widetilde{A} \downarrow & & \gamma\natural \downarrow & & \| \ \mathrm{Id}_M & \\ \\ 0 \to & \Lambda \oplus \Lambda^n & \stackrel{\iota \oplus \mathrm{Id}}{\longrightarrow} & C \oplus \Lambda^n & \stackrel{\theta_n}{\longrightarrow} & M & \to 0 \end{pmatrix}.$$

As Λ has property SFC it follows that $C \cong \Lambda^m$. Applying (9.1) to we see that $[\mathcal{C}] = u\,[\mathcal{E}]$ for some unit $u \in R^*$. Putting $v = u^{-1}$ we obtain an isomorphism of extensions

$$\begin{matrix} \mathcal{C} \\ v_* \downarrow \\ \mathcal{E} \end{matrix} \quad = \quad \begin{pmatrix} 0 \to & \Lambda & \stackrel{j}{\to} & C & \stackrel{q}{\to} & M & \to 0 \\ & \lambda_{i(v)} \downarrow & & \widetilde{q} \downarrow & & \| \ \mathrm{Id}_M & \\ 0 \to & \Lambda & \stackrel{i}{\to} & \Lambda^m & \stackrel{p}{\to} & M & \to 0 \end{pmatrix}.$$

Stabilising kernels gives a corresponding isomorphism $\Sigma^n(v_*) : \Sigma^n(\mathcal{C}) \to \Sigma^n(\mathcal{E})$. Composing we obtain an isomorphism of extensions $\varphi = \Sigma^n(v_*) \circ \gamma \circ A_*$ thus

$$\begin{matrix} \mathcal{F} \\ \varphi \downarrow \\ \Sigma^n(\mathcal{E}) \end{matrix} \quad = \quad \begin{pmatrix} 0 \to & \Lambda^{n+1} & \stackrel{j}{\to} & \Lambda^{m+n} & \stackrel{q}{\to} & M & \to 0 \\ & \varphi_- \downarrow & & \varphi \downarrow & & \| \ \mathrm{Id}_M & \\ 0 \to & \Lambda \oplus \Lambda^n & \stackrel{i}{\to} & \Lambda^m \oplus \Lambda^n & \stackrel{p_n}{\to} & M & \to 0 \end{pmatrix}$$

giving the required isomorphism $\varphi : (\Lambda^{m+n}, q) \stackrel{\simeq}{\to} (\Lambda^m \oplus \Lambda^n, p_n)$. □

To apply this, consider the given free resolution of R

$$0 \to \Lambda \stackrel{\partial_n}{\to} F_{n-1} \stackrel{\partial_{n-1}}{\to} F_{n-2} \stackrel{\partial_{n-2}}{\to} \dots \stackrel{\partial_2}{\to} F_1 \stackrel{\partial_1}{\to} \Lambda \stackrel{\epsilon}{\to} R \to 0.$$

Then $M = \mathrm{Ker}(\partial_{n-2})$ is a representative of $\Omega_{n-1}(R)$; it satisfies $\mathcal{R}(\mathbf{I})$ from the exact sequence $(0 \to \Lambda \stackrel{\partial_n}{\to} F_{n-1} \stackrel{\partial_{n-1}}{\longrightarrow} M \to 0)$; it satisfies $\mathcal{R}(\mathbf{II})$ as $\mathrm{Ext}^1(M, \Lambda) \cong R$; hence:

Corollary 8.8 : Let Λ be an involuted PD^n-algebra augmented over the principal ideal domain R; if Λ has property SFC then $\mathcal{A}\mathrm{bs}_\Lambda(\Omega_{n-1}(R))$ holds. In particular, if $\mathcal{A}\mathrm{bs}_\Lambda(R)$ holds then $\mathcal{A}\mathrm{bs}_\Lambda(\Omega_{n-1}(R))$ also holds.

§9 : Minimality conditions for PD^n-algebras:

The notion of minimal resolution originated in the classical theory of invariants. Adapted to our present context we may interpret it as follows; let

$$\mathbf{S} \;=\; (0 \to S_n \stackrel{\partial_n}{\to} \dots \to S_1 \stackrel{\partial_1}{\to} S_0 \to M \to 0)$$

be a stably free Λ-resolution of finite type. We say that $\mathbf{S}$ is *minimal* when, for any other stably free resolution $\widetilde{\mathbf{S}}$ of M there exists a commutative diagram

$$\begin{array}{c} \widetilde{\mathbf{S}} \\ \varphi \downarrow \\ \mathbf{S} \end{array} \;=\; \left(\begin{array}{ccccccc} 0 & \to \widetilde{S}_n \stackrel{\widetilde{\partial_n}}{\to} & \cdots\cdots & \stackrel{\widetilde{\partial_1}}{\to} & \widetilde{S}_0 & \stackrel{\widetilde{\eta}}{\to} & M & \to 0 \\ & \varphi_n \downarrow & & & \varphi_0 \downarrow & & \downarrow \mathrm{Id}_M & \\ 0 & \to S_n \stackrel{\partial_n}{\to} & \cdots\cdots & \stackrel{\partial_1}{\to} & S_0 & \stackrel{\epsilon}{\to} & M & \to 0 \end{array}\right)$$

in which each φ_n is surjective. It is then a consequence that $\widetilde{\mathbf{S}} \cong \mathbf{S} \oplus \mathbf{T}$ for some stably free acyclic complex $\mathbf{T}$.

The paper of Eilenberg [4] establishes the existence of minimal resolutions under conditions which, though rather more general than originally envisaged in the classical theory, are nevertheless too restrictive for our present purpose. In a companion [8] to the present paper, the author has extended Eilenberg's approach to establish existence and uniqueness criteria for minimal resolutions in rather more generality than is required here. To apply these results to the current situation, we continue to assume that Λ is an involuted PD^n-algebra augmented over a principal ideal domain R. The main result of [8] then has the following interpretation:

Theorem 9.1 : Let Λ be an involuted PD^n-algebra augmented over a principal ideal domain R; then R has a minimal resolution over Λ if and only if $\mathrm{Abs}(\Omega_k(R))$ holds for $0 \le k \le n$.

Note that, by (8.4), the condition that $\mathrm{Abs}(\Omega_0(R))$ holds is equivalent to requiring that Λ should satisfy SFC, in which case, by (5.7), the condition that $\mathrm{Abs}(\Omega_r(R))$ should hold is equivalent to requiring that $\chi_r \equiv 1$. Additionally, it follows from (8.8) that if $\mathrm{Abs}(\Omega_0(R))$ holds then $\mathrm{Abs}(\Omega_{n-1}(R))$ also holds. In view of the duality relation $\chi_n = \chi_{n-r-1}$ it suffices to satisfy these conditions 'up to the middle dimension'. Moreover, the relation $c_r = \chi_{r-1}$ for $2 \leq r$ allows us to express the argument in terms of straightness conditions on $\Omega_r(R)$ to obtain the following, which is Theorem C of the Introduction:

Theorem 9.2: Let Λ be an involuted PD^n-algebra augmented over a principal ideal domain R where $n \geq 2$; then the following conditions are equivalent:

(i) R has a minimal free resolution over Λ;

(ii) Λ satisfies SFC and $\mathrm{Abs}(\Omega_k(R))$ holds for $1 \leq k \leq [\frac{n+1}{2}] - 1$;

(iii) Λ satisfies SFC and $\Omega_k(R)$ is straight for $2 \leq k \leq [\frac{n+1}{2}]$.

In low dimensions these conditions simplify greatly; in dimension 2 one obtains:

Corollary 9.3: Let Λ be an involuted PD^2-algebra augmented over a principal ideal domain R; then R has a minimal free Λ-resolution if and only if Λ satisfies SFC.

As an example take Σ_g to be the fundamental group of an orientable surface of genus $g \geq 1$ and let $\Lambda = \mathbf{Z}[\Sigma_g]$ be the integral group ring. When $g = 1$, Λ satisfies SFC; see for example ([10] p. 189, Cor. 4.12). However, it appears to be unknown whether Λ has SFC in any case $g \geq 2$, although we may note by [1] that, in the nonorientable case, $\mathbf{Z}[G]$ fails to have SFC when G is the fundamental group of the Klein bottle.

In dimensions 3 and 4 one needs also to consider the condition $\mathrm{Abs}(\Omega_1(R))$:

Corollary 9.4: Let Λ be an involuted PD^n-algebra augmented over a principal ideal domain R; if $3 \leq n \leq 4$ then R has a minimal free Λ-resolution if and only if Λ satisfies SFC and $\mathrm{Abs}(\Omega_1(R))$ holds.

This last condition on $\Omega_1(R)$ may be expressed more directly by asking whether the augmentation ideal $\mathrm{Ker}(\epsilon)$ admits an absolutely minimal epimorphism.

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