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Applying Tverberg Type Theorems to Geometric Problems

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I hereby declare that the work presented in this thesis is my own.

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Abstract

In this thesis three main problems are studied. The first is a generalization of a well known question by P. McMullen on convex polytopes:

'Determine the largest number $\nu(d,k)$ such that any set of $\nu(d,k)$ points lying in general position in $\mathbb{R}^d$ can be mapped, by a permissible projective transformation, onto the vertices of a k-neighbourly polytope.'

Bounds for $\nu(d,k)$ are obtained. The upper bound is attained using oriented matroid techniques. The lower bound is proved indirectly, by considering a partition problem equivalent to McMullen's question.

The core partition problem, mentioned above, can be modified in the following manner:

'Let X be a set of n points in general position in $\mathbb{R}^d$ then, what is the minimum k such that for all A, B partition of X there is always a set $\{x_1, \ldots, x_k\} \subset X$, such that

$$conv(A \backslash \{x_1, \ldots x_k\}) \cap conv(B \backslash \{x_1, \ldots x_k\}) = \emptyset\text{?'}$$

For this question, through an asymptotical analysis, a relationship between the number of points in the set (n) , and the number to be removed (k) , is shown.

Finally, another problem in convex polytopes proposed by von Stengel is considered:

'Consider a polytope, $\mathcal{P}$, in dimension d with 2d facets, which is simple. Two vertices form a complementary pair, (x,y), if every facet of $\mathcal{P}$ is incident with x or y. The $d-cube$ has 2^{d-1} complementary vertex pairs. Is this the maximal number among the simple $d-polytopes$ with 2d facets?',

It is shown that the conjecture stated above holds up to dimension seven; and extra conditions, under which the theorem holds in general, are exposed. A nice interpretation of von Stengel's question, in terms of coloured Radon partitions, is also introduced.

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Contents

Introduction

Among all theorems in Discrete Geometry, Radon's theorem has proved one of the most useful in applications within the subject. Together with its most natural generalization, Tverberg's theorem, both have sown the seed for a fruitful study on geometric partitions. Both have endless branches and forms, from coloured and conical, to those that study the geometry of the intersections of their partitions.

This thesis exposes several variations on the study of precisely that type of theorems. Most of the time, the variations have been inspired by geometric theorems, helping in their solution. But also, those variations have accepted re-stylings which, despite having no straightforward geometric meaning, result just as interesting as those who have.

The first and second chapters are completely introductory. Chapter 1, Convexity and Geometry, standardizes definitions and properties of concepts to be used later on the thesis. It also states four of the quintessential theorems of discrete geometry, which provide the foundation of several of the arguments contained in the forthcoming chapters. Chapter 2, however, provides a brief summary of Oriented Matroid theory, with an especial focus on the development of the Matroid Polytopes tools. The latter will be crucial for the proofs revealed in chapter 3.

Chapter 3 deals with a generalization of the following problem on convex polytopes, by P. McMullen :

'Determine the largest number $\nu(d)$ such that any set of $\nu(d)$ points lying in general position in $\mathbb{R}^d$ can be mapped, by a permissible projective transformation, onto the vertices of a convex polytope.'

Up to date, only bounds for $\nu(d)$, in the original McMullen's problem are known. It has also been proved that the bounds are sharp for some low dimensions. All the methods used for achieving the bounds and the bounds themselves, are reviewed at the beginning of chapter 3.

Recalling that a convex d-polytope, $\mathcal{P}$, with vertex set, V, is *k-neighbourly* for some $k \leq \lfloor \frac{d}{2} \rfloor$, if every $S \subset V$ such that $|S| \leq k$ is contained in a proper facet of $\mathcal{P}$; one might modify the question above, in the succeeding manner:

'Determine the largest number $\nu(d,k)$ such that any set of $\nu(d,k)$ points lying in general position in $\mathbb{R}^d$ can be mapped, by a permissible projective transformation, onto the vertices of a k-neighbourly polytope.'

Aided by the progress made in the solving of McMullen's problem, section 3.3 provides original bounds for the generalized problem, shows examples of when the bounds are sharp and proposes a further generalization of the problem.

The generalized McMullen's problem, has a nice equivalence in terms of partitions, namely:

'Determine the smallest number $\lambda(d,k)$ such that for any set X of $\lambda(d,k)$ points in R^d there exists a partition of X into two sets A_1, A_2 such that

$$conv(A_1 \backslash \{x_1, x_2, ..., x_k\}) \cap conv(A_2 \backslash \{x_1, x_2, ..., x_k\}) \neq \emptyset$$

for all $\{x_1, x_2, ..., x_k\} \subset X$',

Considering the equivalence above, and introducing a new direction which still keeps the same *core* partition problem, chapter 4 deals with the solution of the question:

'Let X be a set of n points in general position in $\mathbb{R}^d$ then, what is the

minimum k such that for all A, B partition of X there is always a set $\{x_1, \ldots, x_k\} \subset X$, such that

$$conv(A \backslash \{x_1, \ldots x_k\}) \cap conv(B \backslash \{x_1, \ldots x_k\}) = \emptyset ?'$$

This chapter parts completely from the methods employed in chapter 3. Here purely geometric arguments and constructions are used to prove that k has to be roughly half the size of the cardinality of the set of points.

Chapter 5 introduces a fresh question, proposed by von Stengel, which reads:

'Consider a polytope, $\mathcal{P}$, in dimension d with $2d$ facets, which is simple. Two vertices form a complementary pair, (x, y), if every facet of $\mathcal{P}$ is incident with x or y. The $d-cube$ has 2^{d-1} complementary vertex pairs. Is this the maximal number among the simple $d-polytopes$ with $2d$ facets?'.

This question relates to the previous two only in that they deal with partition problems that can have geometric interpretations.

Taking the dual problem, on simplicial polytopes, several conditions whose assumption make the conjecture above true, are discovered. Among them, $2-$*neighbourliness* seems to play an important role in their natural fulfillment. Given that in low dimensions polytopes are not $2-$*neighbourly*, an induction tool, built within the chapter, grants that the conjecture holds up to dimension seven.

Finally, a coloured partition equivalence of von Stengel's question is exhibited, as a corollary of the above, in section 5.5.

Chapter 1

Geometry and Convexity

The beauty of combinatorial problems relies not only in their often surprising and elegant solutions but on the simplicity of their statement. Simplicity that has created a need for defining mathematically a plethora of very intuitive concepts, often the hardest to describe. The expert reader might not need the following survey concepts but they will help pave the way for what is to come.

1.1 Affine Geometry

This first section consists of a study of some basic concepts, vital in the description of the geometrical phenomena that are the matter of this thesis. Not only do the subsequent definitions provide the notation for the objects in the study space, they also help define what is a very useful tool in discrete geometry; the Gale transform. This tool is absolutely necessary for the development of the coming chapters.

The natural setting for the problems to be studied in this thesis is the $d-$ *dimensional* real space with the usual metric, $\mathbb{R}^d$. Unless otherwise stated, x denotes a point and X usually denotes a finite set of points in $\mathbb{R}^d$.

A point $x' \in \mathbb{R}^d$ is an **affine combination** of X if there is a linear combination $x' = \sum_{x \in X} \alpha_x x$ where $\alpha_x \in \mathbb{R}$ and $\sum_{x \in X} \alpha_x = 1$. The set of all affine combinations of X is called the **affine span** of X, denoted by $aff(X)$. If one drops the condition $\sum_{x \in X} \alpha_x = 0$ and only allows $\alpha_x > 0$, the resulting space is called the *positive affine span* of X, denotedby $aff(X)^+$. The *negative affine span* is analogously defined, and denoted by $aff(X)^-$. The *dimension of the affine span* is denoted by $dim\, aff(X)$ and the affine span of a set $X \subset \mathbb{R}^d$ such that $dim\, aff(X) = d - 1$ is called an **affine hyperplane**.

In the case where X is finite, say $X = \{x_1, \ldots, x_n\}$ with $dim\, aff(X) = r$, the set $D(X) \subset \mathbb{R}^n$ of **affine dependences** of X is defined as the set of all points $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{n} \alpha_i x_i = 0 \quad and$$
$$\sum_{i=1}^{n} \alpha_i = 0.$$

The dimension of $D(X)$ is $dimD(X) = n - r - 1$.

A set of points X with cardinality $|X| = n$ is said to be in **general position** if no subset $S \subset X$ with $|S| = k$ is such that $dim\, aff(S) < k - 1$ for all $1 \leq k \leq d$. It is said of a **hyperplane**, H, that it is in **general position** in respect to a set of points X, if and only if H is neither parallel nor perpendicular to any plane in the set $\{aff(S) | S \subset X,\ dim\, aff(S) = d-1\}$, of hyperplanes spanned by X.

All the concepts reviewed, up to this point, arise naturally in Euclidean geometry. However, what comes next deals with the building blocks of what is possibly the most surprising tool in combinatorial geometry.

A **projective transformation** P is a transformation from $\mathbb{R}^d$ into itself which, has the form :

$$P = \frac{Ax + b}{\langle c, x \rangle + \delta},$$

where A is a linear transformation of $\mathbb{R}^d$ into itself, b and c are vectors in $\mathbb{R}^d$ and $\delta \in \mathbb{R}$, with at least one of $c \neq 0$ or $\delta \neq 0$. Furthermore, P is **permissible** for a set $X \subset \mathbb{R}^d$ iff $\forall x \in X$, $\langle c, x \rangle + \delta \neq 0$. P is **non-singular** iff the matrix T' is non-singular, where

$$T' = \begin{pmatrix} A' & b' \\ c & \delta \end{pmatrix},$$

A' denotes the matrix of the linear transformation A and b' is the transpose of b.

Now, if X and $D(X)$ are as before and the $n-r-1$ vectors $\{a_1, \dots, a_{n-r-1}\}$ of $\mathbb{R}^n$ form a basis of $D(X)$, consider the following matrix:

$$D = \left(a_1', a_2', \dots a_{n-r-1}'\right) = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n-r-1} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n-r-1} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n,1} & \alpha_{n,2} & \cdots & \alpha_{n,n-r-1} \end{pmatrix}$$

The rows of D may be considered as vectors in $\mathbb{R}^{n-r-1}$. Let the $j-th$ row of D be denoted as $\overline{x_j} = (\alpha_{j,1}, \alpha_{j,2}, \dots, \alpha_{j,n-r-1})$ for all $j = 1, \dots, n$. The set $\overline{X} = \{\overline{x_1}, \dots, \overline{x_n}\}$ is a **Gale transform** of X.

It is emphasized that $\overline{X}$ is *a* Gale transform of X, rather than *the* Gale transform of X, because the resulting points depend on the specific choice of basis for $D(X)$. Still, the Gale transforms of the same set of points using two different bases are linearly equivalent.

Once introduced, some important and relevant properties of the Gale transform are to follow.

Proposition 1.1.1. *Let $X = \{x_1, \dots, x_n\}$ be a set of n points in $\mathbb{R}^d$ and let $\overline{X} = \{\overline{x_1}, \dots, \overline{x_n}\}$ be its Gale transform, then the following statements hold:*

1. $\sum_{i=1}^{n} \overline{x_i} = 0$.
2. *$\overline{X}$ linearly and positively spans $\mathbb{R}^{n-r-1}$.*
3. *The n points of X are in general position in $\mathbb{R}^d$ if and only if the n-tuple $\overline{X}$ consists of n points in linearly general position in $\mathbb{R}^{n-d-1}$.*

4. *Let* $P = \frac{Ax+b}{\langle c,x\rangle+\delta}$ *be a nonsingular projective transformation of* $\mathbb{R}^d$ *into itself, permissible for* X. *Let* $Y = \{P(x_1), \ldots, P(x_n)\}$. *Then* $\overline{Y}$ *is linearly equivalent to the* $n-tuple$ $\{(\langle c, x_1\rangle + \delta)\overline{x_1}, \ldots, (\langle c, x_n\rangle + \delta)\overline{x_n}\}$. *Conversely if* X, Y *are two* $n-tuples$ *in* $\mathbb{R}^d$ *such that there exist nonzero numbers* $\{\lambda_1, \ldots \lambda_n\}$ *with the property* $\overline{y_i} = \lambda_i \overline{x_i}$ *for* $i = 1, \ldots, n$, *then there exist* $c \in \mathbb{R}^d$ *and* $\delta \in \mathbb{R}$ *such that* $\lambda_i = \langle c, x_i\rangle + \delta$ *for all* $i = 1, \ldots, n$, *and a linear transformation* A *and a vector* $b \in \mathbb{R}^d$ *such that,* $P = \frac{Ax+b}{\langle c,x\rangle+\delta}$ *is a non-singular projective transformation, permissible for* X, *satisfying* $y_i = P(x_i)$.

Lastly, if in the preceding definition of a Gale transform one considers the set $\overline{X'} = \left\{x' \middle| \, x' = \frac{\overline{x_i}}{\|\overline{x_i}\|} \text{ if } \overline{x}_i \neq 0 \text{ and } x' = 0 \text{ otherwise, } \forall\ i = 1, \ldots n\right\}$, $0 \in conv(\overline{X'})$ and properties 2, 3 and 4, stated in proposition 1.1.1, hold. Sometimes it is more convenient to use this newly defined set, named a **Gale diagram** of X.

1.2 Convex Polytopes

The most basic units of discrete convexity will be listed subsequently, using the concepts in the previous section, and the usual conventions on polytopes will be restated. As the subject of study throughout this thesis consists only of convex polytopes, after this section they will only be referred to as *polytopes*.

A set $K \subset \mathbb{R}^d$ is **convex** if for any two points $x, y \in K$ then the line segment $[x, y] = \{\lambda x + (1-\lambda),\ y\ 0 \leq \lambda \leq 1\}$ is contained in K.

If X is a set of points in $\mathbb{R}^d$; a point $x' \in \mathbb{R}^d$ is a **convex combination** of X if there is an affine combination $x' = \sum_{x\in X} \alpha_x x$ where $\alpha_x \geq 0$. The set of all convex combinations of X is the **convex hull** of X, denoted $conv(X)$.

When K is a convex set, an element $x \in K$ such that $x \notin conv(K \backslash x)$ is called an **extreme point** of K.

Still, defining analogous concepts to those in the affine spaces section, one might define a new kind of special position. A set of points $X \subset \mathbb{R}^d$ is said to be in **convex position** if $\forall x \in X$, x is an extreme point of $conv(X)$.

Before entering into establishing terminology for polytopes, another property of the Gale transform should be mentioned.

Proposition 1.2.1. *Let $X \subset \mathbb{R}^d$ be a set of points, then $0 \in relint\, conv(\overline{X})$, where $\overline{X}$ is the Gale transform (diagram) of X.*

Back to the subject matter, a **convex polytope** is the convex hull of a finite set of points in convex position, X. The vertices of a polytope, $\mathcal{P}$, are the points of the form $x = \mathcal{P} \cap H$, where H is any hyperplane. It is also true that the set of vertices of a polytope, $vert(\mathcal{P})$ or $V(\mathcal{P})$ is precisely X. If $\mathcal{P}$ is such that $dim\, aff(\mathcal{P}) = d$ then $\mathcal{P}$ is called a **d-polytope**.

It is necessary to establish terminology for the several important parts of a polytope.

A hyperplane $H = \{x \in X | \langle x, c \rangle = \delta\}$ is said to be a **support hyperplane** of $\mathcal{P}$ if $\langle x, c \rangle = \delta$ for some $x \in \mathcal{P}$, and $\langle x, c \rangle \leq \delta$ for all $x \in \mathcal{P}$ or $\langle x, c \rangle \geq \delta$ for all $x \in \mathcal{P}$. According to these terms a **face** of $\mathcal{P}$ can be described as any set of the form $F = \mathcal{P} \cap H$ where H is a support hyperplane of $\mathcal{P}$. If one considers the set of vertices of any given face, $V(F)$, and its complement in the set of vertices of the polytope, $V(\mathcal{P}) \backslash V(F)$; a face can also be defined as $conv(V(F))$. Furthermore, one can define a new object, a **coface**, as $conv(V(\mathcal{P}) \backslash V(F))$. The dimension of a face is precisely $dim(\mathcal{P} \cap H)$, and the dimension of a coface is the dimension of the smallest affine subspace that contains it.

In contrast with $3-$ *dimensional* polytopes (polyhedra), in higher dimensions the word *face* does not denote the highest dimensional face properly contained in the polytope. As these highest dimensional blocks are often referred to; a face, F, of a d-polytope, $\mathcal{P}$, such that $dim(F) = d - 1$ is called a **facet** and, as a consequence, its $conv(V(\mathcal{P}) \backslash V(F))$ is a **cofacet** of $\mathcal{P}$.

Two very important types of polytopes normally arise when studying general problems, mainly because they have nice properties that often make their analysis easier: simplicial and simple polytopes. A polytope $\mathcal{P}$ is **simplicial** if all its faces are $k-$ *dimensional* simplices. A $d-$ *dimensional* polytope P is **simple** if it is the dual of a simplicial polytope, or it can also be described as a polytope such that all of its vertices are incident to exactly d $(d-1)-$ *dimensional* faces.

In three dimensional space it is obvious that, apart from the tetrahedron (3-simplex), other simplicial polytopes do exist. For example, the octahedron and the icosahedron. Although both the tetrahedron and octahedron can be generalized to higher dimensions, and be baptized as simplex and cross-polytope respectively, the reader might wonder if there are any more *interesting* simplicial (and by duality, simple) polytopes in higer dimensions. The answer is yes. As an example, what is possibly the most influential polytope in convex geometry will be described.

Consider the $d-$ *dimensional* **moment curve** $C(t) = \{(t, t^2, t^3, \ldots t^d) | t \in \mathbb{R}\}$ and let $T = \{x_1 < \cdots < x_n\}$ be a set of real numbers. The set of points $X = \{C(x_1), C(x_2), \ldots, C(x_n)\}$ is in general position. Furthermore, it is in convex position, and is therefore the set of vertices of a d-dimensional simplicial polytope, denoted $C(n,d)$. Any polytope constructed in such a way is called a **cyclic polytope**.

Cyclic polytopes can be constructed with other curves, nevertheless, the parametrization given above is the most recurrent within the subject. This type of polytopes have many very important characteristics; among them, a $d-$ *dimensional* cyclic polytope has the property that any set of k of its vertices is contained in one of its facets, for all $1 \leq k \leq \lfloor \frac{d}{2} \rfloor$. Such a property among polytopes is called **k-neighbourliness**. Also, among all polytopes in $\mathbb{R}^d$, $C(n,d)$ is the polytope with n vertices, with the highest number of facets.

To finish off this section, it is important to show several results that offer a connection between the special properties of polytopes and their Gale transforms (diagrams), and how to tell when, based on its Gale transform, a set of points is actually a the vertex set of a polytope.

Proposition 1.2.2. *Let X be the set of vertices of a $d-$dimensional polytope $\mathcal{P}$, and let $\overline{X}$ be its Gale transform. Then the following statements hold:*

1. *Let $Y \subset X$, Y is a coface of X if and only if either*

   $$Y = \emptyset \text{ or } 0 \in relint\, conv(\overline{Y_X}).$$

   Here $\overline{Y_X}$ represents the points of $\overline{X}$ that correspond to Y in X.

2. *$\mathcal{P}$ is simplicial if and only if $dim\, conv(\overline{Y_X}) = dim\, conv(\overline{X})$ for every non empty coface Y.*

Proposition 1.2.3. *Let $X \subset \mathbb{R}^d$, $aff(X) = \mathbb{R}^d$, and $\overline{X}$ be the Gale transform of X. Then X is the set of vertices of a polytope if and only if one of the two following conditions holds:*

1. *$\overline{x} = 0$ for all $x \in X$ and $\mathcal{P}$ is a simplex, or*

2. *for every hyperplane $H \subset \mathbb{R}^{n-d-1}$ such that $0 \in H$, $|\{i|\overline{x_i} \in H^+\}| \geq 2$.*

1.3 Discrete Geometry

Although discrete geometry contains, as a field, the subject of polytopes, the distinction made here is justified on the basis that none of the material presented in this section belongs inherently to the theory of convex polytopes, but to its complement in discrete geometry. Nevertheless, describing the ideas that follow as complementary to those enclosed by the study of convex polytopes is certainly an abuse of ideology. The classic results about configurations of points that follow have been pivotal in the proof of several theorems in this thesis which, mainly deal with properties of convex polytopes.

Proposition 1.3.1. ***(Radon's Theorem)*** *Let X be a set of $d+2$ points in $\mathbb{R}^d$. Then there exist two disjoint subsets A, B of X such that*

$$conv(A) \cap conv(B) \neq \emptyset.$$

A partition, A, B, of a set X, with the property above, is called a **Radon partition**.

Proposition 1.3.2. ***(Tverberg's Theorem)*** *Let d and r be given positive integers. For any set X of at least $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ there are $A_1, \ldots A_r$ disjoint subsets of X such that $\bigcap_{i=1}^{r} conv(A_i) \neq \emptyset$.*

Proposition 1.3.3. ***(Erdös-Szekeres Theorem)*** *There is a number $N(t)$ such that if $\{x_1, x_2, ..., x_n\}$, where $n \geq N(t)$, are n points in the plane in general position, then $\{x_1, x_2, ..., x_n\}$ contains a subset of t points in convex position.*

Proposition 1.3.4. *Let X be a set of n points in general position in $\mathbb{R}^d$. For any $x_0 \in \mathbb{R}^d \backslash X$ define $S_{x_0} = \{S \subset X | \, |S| = d+1, x_0 \in conv(S)\}$ and $\nu_d(X) = max_{x \in \mathbf{R}^d \backslash X} |S_x|$, and let $\nu_d = max_{X \in \mathbf{R}^d} \nu_d(X)$. Then*

$$\nu_d = \begin{cases} \binom{n-\lfloor \frac{n-d-1}{2} \rfloor}{\lfloor \frac{n-d-1}{2} \rfloor} + \binom{n-\lfloor \frac{n-d-1}{2} \rfloor - 1}{\lfloor \frac{n-d-1}{2} \rfloor - 1} & \text{if } n-d-1 \text{ is even} \\ 2\binom{n-\lfloor \frac{n-d-1}{2} \rfloor - 1}{\lfloor \frac{n-d-1}{2} \rfloor} & \text{if } n-d-1 \text{ is odd.} \end{cases}$$

This is the end of a first chapter, full of rather standard definitions and results. It is intended to be a very brief introduction to affine spaces, convex polytopes and configurations of points and, as such, it is imperative to mention some references. All of the concepts and results contained in this chapter can be found in [7], [13] and [20], maybe except proposition 1.3.4, due to Bárány, which proof can be found in [1].

Chapter 2

Oriented Matroids

The aim of this chapter is to introduce a minimum amount of notions in Matroid and Oriented Matroid theory. The focus throughout is to select those concepts and results which will be used to prove the main theorems in chapter 3. The latter will become apparent as, together with the definitions and propositions, the geometrical interpretations of all objects and descriptions of the phenomena are presented.

The first section deals with Matroids, and uses two axiomatization systems; one which highlights their similarity to linear spaces, and another one which will help introduce Oriented Matroids. For other axiomatizations of matroids the reader may refer to [19] and [14].

2.1 Matroids

The study of matroids is an analysis of an abstract theory of dependence. The term matroid was first used by Whitney in a 1935 paper entitled *'On the abstract properties of linear dependence'*. There, he conceived matroids as a generalization of matrices. As a consequence, some of the matroid terminology is based on the language of linear algebra.

However, his approach was also motivated by his earlier work in graph theory, ergo, the terminology also bears some resemblance to that in the subject: there lies the connection to combinatorics. Matroids, in many cases, have proven extremely useful to solve difficult combinatorial problems, and to simplify the solution of some already solved ones; always laying bare the essential information about the problem.

2.1.1 Independent Set Axiomatization

Definition 2.1.1. *A* ***matroid*** $\mathcal{M}$ *consists of a finite set* E *and a collection* I *of subsets of* E *such that they satisfy the following conditions:*

I1. $\emptyset \in I$;

I2. if $X \in I$ *and* $Y \subset X$ *then* $Y \in I$; *and*

I3. if $X, Y \in I$ *and* $|X| = |Y| + 1$ *then there exists* $x \in X \backslash Y$ *such that* $Y \cup \{x\} \in I$.

The sets in I are called **independent sets** and subsets of E that do not belong to I are called **dependent sets**. One can think of these sets as sets of independent and non-independent vectors in a linear space. Therefore property I3 makes it natural to introduce the concept of rank, motivated by linear algebra.

Definition 2.1.2. *The* ***rank*** *of a set* $X \in I$ *is*

$$r(X) = max\{|Y| : Y \subset X \text{ and } Y \in I\}.$$

The ***rank of a matroid*** *is therefore* $r(I)$.

Just as rank has its equivalence in matroids, also a base, a hyperspace and spanning set have theirs.

Definition 2.1.3. *A* ***base*** *of* $\mathcal{M}$ *is a maximal independent subset of* I.

Definition 2.1.4. *A* ***flat*** *or subspace of* $\mathcal{M}$ *is a subset* $X \subset E$ *such that for all* $x \in E\backslash X$, $r(X \cup \{x\}) = r(X) + 1$.

Definition 2.1.5. *A subset $X \subset E$ is a **spanning subset** in $\mathcal{M}$ if it contains a base.*

Definition 2.1.6. *A **circuit** of $\mathcal{M}$ is a minimal dependent set in E. The set of all circuits in E is denoted $\mathcal{C}$.*

Just as one can have minimal linearly dependent sets in a vector space, their analogue in this context is defined as circuits.

2.1.2 Circuit Axiomatization

As it was mentioned at the beginning of this chapter, a matroid can be described in several ways. The following one is by its sets of circuits, as defined above, using a suitable equivalent set of axioms.

Definition 2.1.7. *A collection $\mathcal{C}$ of subsets of E is the set of circuits of a matroid on E if and only if conditions $C1$ and $C2$ are satisfied:*

> *C1. for all $X, Y \in \mathcal{C}$ if $X \neq Y$ then, $X \not\subset Y$; and*
>
> *C2. if $X, Y \in \mathcal{C}$, $X \neq Y$ and $\exists\ z \in X \cap Y$, then there is $Z \in \mathcal{C}$ such that $Z \subset (X \cup Y) \backslash z$.*

Still drawing the parallel to linear spaces, a circuit $X = \{x_1, \ldots, x_k\}$ can be viewed as a minimal set of linear dependences.

Suppose $X = \{x_1, \ldots, x_n\}$ is a set of n points in $\mathbb{R}^k$, with $n \geq k+1$, and consider the matroid of linear dependences of X. It is known that there is a vector $\{\alpha_1, \ldots, \alpha_k\} \in \mathbb{R}^k$ with at least one $\alpha_i \neq 0$, such that

$$\sum_{i=1}^{n} \alpha_i \cdot x_i = \bar{0}.$$

Thus, signs can be assigned to each x_i in X in the following manner:

$$X(x_i) = sgn(\alpha_i).$$

All through this chapter and the next, $X(x_i)$ will denote the **sign** of x_i **in**

the circuit X, where the context so requires. This assignment motivates the next definitions, which will be clarified in the next section.

Definition 2.1.8. *A* ***signed set*** *X of E is a set $\underline{X} \subset E$ together with a partition (X^+, X^-) of $\underline{X}$ into two different subsets (where X^+ or X^- may be empty) such that $X^+ \cup X^- = \underline{X}$. The set $\underline{X}$ is called the* ***support*** *of X.*

Definition 2.1.9. *The* ***opposite*** *signed set of a set X, $-X$, is the set with support $\underline{X}$ and signed set $(-X^+, -X^-)$,where $(-X)^+ = X^-$ and $(-X)^- = X^+$.*

Definition 2.1.10. *The* ***reoriented set*** $_{-S}X$*, where $S \subset X$, is the set with*

$$_{-S}X^+ = (X^+ \backslash S) \cup (X^- \cap S) \text{ and } _{-S}X^- = (X^- \backslash S) \cup (X^+ \cap S).$$

Definition 2.1.11. *Given any two signed sets X_1, X_2, the* ***composition*** *$Y = X_1 \circ X_2$, is defined as the signed set with ground set $\underline{Y} = \underline{X}_1 \cup \underline{X}_2$, and signs defined by $Y(x) = X_i(x)$, where $i = \min_{k=1,2}\{x \in X_k\}$.*

Note that compositions, as defined above, can be formed by several signed sets. Furthermore, a composition of several sets is associative but in general not commutative.

2.2 Oriented Matroids

In the previous section the structure of a matroid was completely given by its independent sets or by its circuits. The notion of signed sets was only artificially imposed over the matroid structure, which could potentially provoke having two different systems of signs for the circuits of the same matroid: and therefore, the extra information available wouldn't be exploited to its full extent. But, in any case, according to historical notes in [2], that is not the motivation behind the theory of oriented matroids.

It seems reasonable to assume, after studying a portion of the extensive oriented matroid bibliography, that the motivation behind different oriented matroid axiomatizations lies behind a desire of abstraction. The main contributors to the origination of the theory had been pursuing very different results in the areas of Convex Polytopes (Jon Folkman and Jim Lawrence), Graph Theory and Combinatorics (Michel Las Vergnas), and Abstract Linear Programming Duality (Robert Bland). It is consequently not surprising that this theory yields much of the latest progress on McMullen's problem.

Precisely in the context of McMullen's problem, it is that one can interpret oriented matroids as an abstraction of configurations of points over a real space. It is thus pertinent that, along with the basic definitions behind the ideas in Ramírez-Alfonsín's paper [16] and theorems 3.3.1 and 3.3.10, a parallel to properties of configurations of points is outlined.

2.2.1 Circuit Axioms and Chirotopes

Here an oriented matroid will only be defined by its sets of circuits. However, just as for matroids, there are several equivalent sets of possible axiomatizations, all conceived with very different motivations and all having different virtues. All the systems of axioms and their equivalences can be found in [2].

After the circuit axiom system, there is also a review of the concept of a chirotope, or basis orientation, that is closely related to the signature of the circuits of a matroid, and constitutes a foundation stone for the proof in proposition 3.2.7.

Definition 2.2.1. *A collection $\mathcal{C}$ of signed subsets of E is an oriented matroid $\mathcal{M}$ if and only if it satisfies the following:*

C0. $\emptyset \notin \mathcal{C}$;

C1. $\mathcal{C} = -\mathcal{C}$;

C2. for all $X, Y \in \mathcal{C}$, *if* $\underline{X} \subset \underline{Y}$ *then* $X = Y$ *or* $X = -Y$; *and*

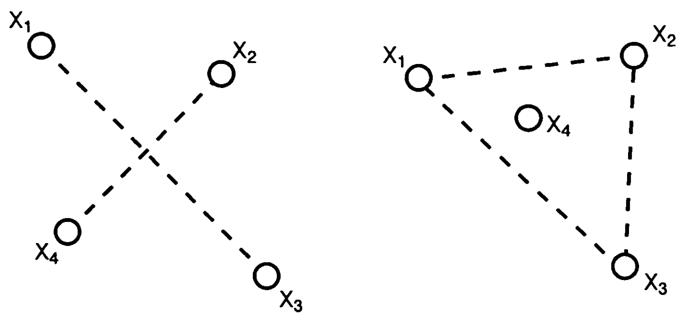

Figure 2.1: The signed circuits of a matroid of affine dependences on a ground set $\{x_1, x_2, x_3, x_4\}$, which consists of the configurations in the figure are: $(+,-,+,-)$ or $(-,+,-,+)$ for the configuration on the left hand side; $(+,+,+,-)$ or $(-,-,-,+)$ for the configuration on the right hand side.

> *C3. for all $X, Y \in \mathcal{C}$ where $X \neq -Y$ and $x \in X^+ \cap Y^-$ there is a $Z \in \mathcal{C}$ such that $Z^+ \subset (X^+ \cup Y^+)\backslash x$ and $Z^- \subset (X^- \cup Y^-)\backslash x$.*

The set of **circuit supports**, $\underline{\mathcal{C}} = \{\underline{X} | X \in \mathcal{C}\}$, of an oriented matroid $\mathcal{M}$, together with E, is the **underlying matroid**, $\underline{\mathcal{M}} = (E, \underline{\mathcal{C}})$, which is a matroid.

Also, in the case of oriented matroids, the definitions of base, rank, flat and spanning subset are those corresponding to their underlying matroid.

A very useful oriented matroid is $\mathcal{M} = (E, \mathcal{C})$ where $E = \{e_1, \ldots, e_n\}$ is a set of points in the affine space $\mathbb{A}^d$, and a circuit is a minimal affinely dependent point set in E. The signs of elements in a circuit are defined by the sign of the coefficients on the equation of the affine dependence. So, if X is a circuit in $\mathcal{M}$, X^+ and X^- define a Radon partition of the points in $\underline{X}$. Figure 2.1 shows this for two different configurations in the plane.

In this case, a basis of $\mathcal{M}$ is a maximal affinely independent subset of E, the rank of the matroid is the dimension of the space $aff(E)$, and a flat is any maximal subset of E, of a given dimension.

Definition 2.2.2. *An oriented matroid $\mathcal{M}$ is* **acyclic** *if it does not contain positive circuits.*

Note that, with the signature inherent to matroids produced from an affine point configuration, they are acyclic.

Definition 2.2.3. *A rank r* **uniform oriented matroid** *$\mathcal{M}$ on a ground set E is a matroid such that all their minimal dependence sets (circuits) have the same cardinality, $r+1$. Or, equivalently, all their maximal independent sets (bases) have cardinality r.*

Matroids of affine dependences of a set of points in general position in $\mathbb{A}^d$ are obviously uniform of rank $d+1$.

The following definitions remind of the nature of oriented matroids as geometrical objects, some of which are very familiar.

Definition 2.2.4. *A* **vector** *of an oriented matroid is any composition of signed circuits. The set of vectors of an oriented matroid is denoted with the character $\mathcal{V}$.*

Vectors in a matroid of affine dependencies are signed subsets of E that form dependent sets, but are not necessarly minimal. The signature of a vector is one of the many affine dependences possible among its points. In such affine dependencies some of the vertices might not be used. Hence, the sign of an element in a signed vector is one of $\{-, 0, +\}$.

Definition 2.2.5. *Two sets $X, Y \subseteq E$ are* **orthogonal** *(denoted $X \perp Y$) if $\underline{X} \cap \underline{Y} = \emptyset$ or there are $x, y \in \underline{X} \cap \underline{Y}$ such that $X(x) \cdot Y(x) = -X(y) \cdot Y(y)$.*

Definition 2.2.6. *Let $\mathcal{M} = (E, \mathcal{C})$ be an oriented matroid. The set of signed sets Y of E such that $Y \perp X$ for all $X \in \mathcal{C}$ is the set of* **cocircuits**, *$\mathcal{C}^*$, of $\mathcal{M}$.*

In a configuration of points X, cocircuits are the points outside of hyperplanes formed by subsets of X, and their signature reflects whether the points are in the positive or negative half-space defined by such hyperplanes. An

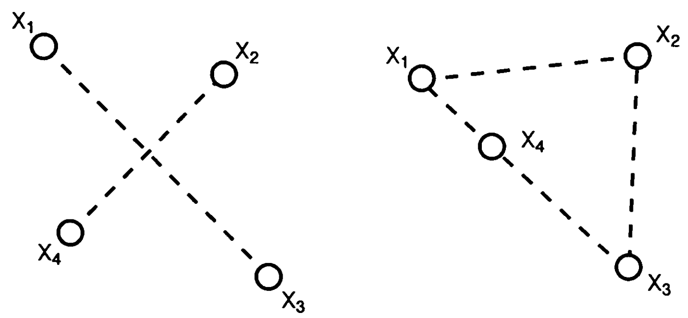

Figure 2.2: In a matroid of affine dependences on a set $\{x_1, x_2, x_3, x_4\}$, which consists of any of the configurations in the figure, the set of its covectors includes: on the right hand side, a cocircuit with ground set $\{x_1, x_2\}$, has signature $(+, -)$ or $(-, +)$ and a cocircuit with ground set$\{x_1, x_4\}$, has signature$(+, +)$ or $(-, -)$; on the left hand side, a cocircuit with ground set $\{x_1, x_2\}$, has signature $(0, +)$ or $(0, -)$ and a cocircuit with ground set $\{x_1, x_4\}$, has signature$(+, +)$ or $(-, -)$.

instance of this, can be observed in figure 2.2.

Definition 2.2.7. *A **covector** of an oriented matroid is any composition of signed cocircuits.*

The familiar looking notation for the set of cocircuits, suggests that they are in fact the set of circuits of another matroid, $\mathcal{M}^*$, which is called the *dual* of $\mathcal{M}$. As cocircuits encode approximately the same information about the matroid as the set of circuits, there is an analogous axiomatization for an oriented matroid in terms of its set of cocircuits.

Given a configuration of points, sometimes it might be useful to apply projective transformations to it. The application of a projective transformation will result in a reorientation of the circuits of the oriented matroid. Proving these facts needs of the following definitions.

Definition 2.2.8. *Let $\mathcal{M} = (E, \mathcal{C})$ be an oriented matroid. The* ***reorientation*** *of a matroid $\mathcal{M}$ over a set $S \subseteq E$, $_{-S}\mathcal{M}$, is the oriented matroid over the set of reoriented circuits $_{-S}\ \mathcal{C} = \{_{-S}X | X \in \mathcal{C}\}$.*

It is visible now that a lot of information about a configuration of points can be encrypted in an oriented matroid's circuits. But, just as in geometry, rather than thinking about the dependent sets, one can think about sets that are independent: the bases of $\mathcal{M}$, and the information they encode. It might then be useful to know whether each of these bases has a 'left hand side' or a 'right hand side' orientation. Precisely that is the geometrical interpretation of the next definition.

Definition 2.2.9. *A* ***basis orientation*** *of an oriented matroid $\mathcal{M}$ is a mapping $\mathcal{X}$ of the set of ordered bases of $\mathcal{M}$ onto the set $\{-1, 1\}$ satisfying:*

> *B1. $\mathcal{X}$ is alternating; and*
>
> *B2. for any two ordered bases of $\mathcal{M}$ of the form $(x_1, x_2, \ldots, x_r)$ and $(x_1', x_2, \ldots, x_r)$, where $x_1 \neq x_1'$,*
>
> $$\mathcal{X}(x_1, x_2, \ldots, x_r) = -X(x_1) * X(x_1') * \mathcal{X}(x_1', x_2, \ldots, x_r),$$
>
> *where $X(x_1)$ and $X(x_1')$ are the signs of x_1 and x_1', respectively, in one of the two opposite signed circuits of $\mathcal{M}$ on the set $\{x_1, x_1', x_2, \ldots, x_r\}$.*

Therefore, the signature of the circuits can be read off from the basis orientation of $\mathcal{M}$ in the following way:

Let $X = \{x_1, \ldots, x_{r+1}\} \subset E$ be a circuit such that $\{x_1, \ldots, x_r\}$ is a base of the underlying matroid, $\underline{\mathcal{M}}$. Then the sign of an element x_i in X is

$$X(x_i) = (-1)^i * \mathcal{X}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r).$$

Now that a definition of basis orientation has been given, it is natural to define the set of basis orientations of a matroid; and conversely, given some assignment of signs, find rules that ensure they provide a coherent orientation of the set of basis of a matroid.

Definition 2.2.10. *Let E be a finite set. A* ***chirotope*** *of rank r on E is a mapping, $\mathcal{X} : E^r \longrightarrow \{-1, 0, 1\}$, that satisfies the following properties:*

B1. $\mathcal{X} \neq 0$;

B2. $\mathcal{X}$ *is alternating; and*

B3. for all $\{x_1, \ldots, x_r, y_1, \ldots, y_r\} \subset E$ *such that*

$$\mathcal{X}(y_i, x_2, \ldots, x_r) * \mathcal{X}(y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_r) \geq 0,$$

$$\mathcal{X}(x_1, x_2, \ldots, x_r) * \mathcal{X}(y_1, \ldots, y_r) \geq 0.$$

Considering the close connection between circuit signatures and basis orientations, it is relevant to define a chirotope reorientation.

Definition 2.2.11. *The* ***reoriented chirotope*** $_{-S}\mathcal{X}$ *of a reoriented matroid* $_{-S}\mathcal{M}$ *is:*

$$_{-S}\mathcal{X} : E \longrightarrow \{-1, 0, 1\},$$

$$\text{where } (x_1, \ldots, x_r) \longmapsto \mathcal{X}(x_1, \ldots, x_r) * (-1)^{|S \cap \{x_1, \ldots, x_r\}|}.$$

The following subsection deals with a lot of important constructions that enrich the family of oriented matroids. These will lead to the definition of a *Lawrence Oriented Matroid*, the core of theorem 3.2.7.

2.2.2 Direct Sum and Union of Oriented Matroids

The direct sum and union of oriented matroids are techniques used to build an oriented matroid from two smaller ones. Before proceeding to this construction, some preliminary notions are needed. First, the single element extensions, which are particularly useful when proving properties by induction on the cardinality of the ground set of a matroid. Secondly, the lexicographic extensions as compositions of single element extensions.

Definition 2.2.12. *A* ***single element extension*** *on a matroid $\mathcal{M}$ is an oriented matroid, $\tilde{\mathcal{M}}$, on the ground set $\tilde{E} = E \cup p$ for some $p \notin E$, where*

E is the ground set of $\mathcal{M}$, and such that the restriction to E of $\tilde{\mathcal{M}}$ is $\mathcal{M}$.

Once a matroid has been extended, it is also important to investigate how its signature extends to a signature of its single element extension. The following proposition deals with such matter.

Proposition 2.2.13. *Let $\tilde{\mathcal{M}}$ be a single element extension of $\mathcal{M}$. Then for every cocircuit $Y \in \mathcal{C}^*$ there is a unique way of extending Y to a cocircuit of $\tilde{\mathcal{M}}$. Also, there is a unique function $\sigma : \mathcal{C}^* \longrightarrow \{-, 0, +\}$ such that*

$$\{(Y, \sigma(Y)) : Y \in \mathcal{C}^*\} \subseteq \tilde{\mathcal{C}^*},$$

where Y is represented by its signed vector.

In the case of a matroid on a configuration of points, this function indicates for each cocircuit, Y, the side of the complementary hyperplane, spanned by $E \backslash Y$, where the new point p is.

Definition 2.2.14. *Let σ^1 and σ^2 be two extending functions, their **composition** over the set of cocircuits of the matroid is:*

$$\sigma^1 \circ \sigma^2(Y) = \begin{cases} \sigma^1(Y) & \text{if } \sigma^1(Y) \neq 0 \\ \sigma^2(Y) & \text{otherwise.} \end{cases}$$

Geometrically, $\sigma^1 \circ \sigma^2$ defines a point in the configuration near to the one defined by σ^1, perturbed in the direction of the point described by σ^2.

Definition 2.2.15. *Let $\mathcal{M}$ be an oriented matroid and $\mathcal{C}^*$ its set of cocircuits. The function $\sigma[e^+](Y) = Y(e)$, $\forall\ Y \in \mathcal{C}^*$, where $\mathcal{M}$ has ground set E and $e \in E$, defines an extension by an element that is **parallel** to e. Similarly, $\sigma[e^-](Y) = -Y(e)\ \forall\ Y \in \mathcal{C}^*$, is an extension by an element that is **antiparallel** to e.*

So far, only extensions by points which are not in E had been made. However, one can choose one of the points $e \in E$ to generate an extending function. In this case, if $\mathcal{M}$ is a matroid of affine dependences, the new point in the configuration will be very near to e.

Definition 2.2.16. *Let $\mathcal{M}$ be an oriented matroid and $I = \{e_1, \dots, e_k\}$ an ordered subset of E, also let $\alpha = [\alpha_1, \dots, \alpha_k] \in \{+,-\}^k$. The* ***lexicographic extension*** *of $\mathcal{M}$ by $p := [I^\alpha]$ is the single element extension,*

$$\tilde{\mathcal{M}} = \mathcal{M} \cup \{p\} = \mathcal{M}[I^\alpha],$$

given by $\sigma[I^\alpha] = \sigma[{e_1}^{\alpha_1}] \circ \cdots \circ \sigma[{e_k}^{\alpha_k}]$.

In the case of a realizable oriented matroid given by any configuration in $\mathbb{R}^n$, the relative position of the new point p can be determined as:

$$p = \alpha_1 \cdot e_1 + \epsilon \cdot \alpha_2 \cdot e_2 + \cdots + \epsilon^{k-1} \cdot \alpha_k \cdot e_k.$$

Proposition 2.2.17. *Let $\mathcal{M}$ be an oriented matroid, $I = [e_1, \dots, e_k]$ an ordered subset of E and $\alpha = [\alpha_1, \dots, \alpha_k] \in \{+,-\}^k$. Then the lexicographic extension $\mathcal{M}[I^\alpha]$ is given by:*

$$\sigma(Y) = \begin{cases} \alpha_i \cdot Y(e_i) & \text{if } i \text{ is minimal and such that } Y(e_i) \neq 0 \\ 0 & \text{if } Y(e_i) = 0 \ \forall\ 1 \leq i \leq k. \end{cases}$$

Since i can be minimal with $Y(e_i) \neq 0$ only if e_i is not in the flat spanned by $\{e_1, \dots, e_{i-1}\}$, the lexicographic extension is determined by the lexicographically first basis contained in I, together with the corresponding signs. It is, therefore, customary to require I to be an independent set.

Proposition 2.2.18. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two oriented matroids on ground sets E_1 and E_2, with $E_1 \cap E_2 = \emptyset$. For $i = 1,2$ let $\mathcal{C}_i$ be the set of circuits, $\mathcal{C}_i^*$ the set of cocircuits, $\mathcal{X}_i$ the chirotope and r_i the rank of $\mathcal{M}_i$. There exists an oriented matroid $\mathcal{M}$ of rank $r_1 + r_2$ on the disjoint union $E_1 \cup E_2$ such that:*

1. $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$;

2. $\mathcal{C}^* = \mathcal{C}_1^* \cup \mathcal{C}_2^*$; *and*

3. $\mathcal{X}(e_1, \ldots, e_{r_1}, f_1, \ldots, f_{r_2}) = \mathcal{X}_1(e_1, \ldots, e_{r_1}) \cdot \mathcal{X}_2(f_1, \ldots, f_{r_2})$ *for all* $\{e_1, \ldots, e_{r_1}\} \subset E_1$ *and* $\{f_1, \ldots, f_{r_2}\} \subset E_2$.

$\mathcal{M}$ *is called the* ***direct sum,*** $\mathcal{M}_1 \oplus \mathcal{M}_2$ *of* $\mathcal{M}_1$ *and* $\mathcal{M}_2$.

In the realizable case, the direct sum of matroids of affine dependences, corresponds to the placement of the two configurations that realize $\mathcal{M}_1$ and $\mathcal{M}_2$ into orthogonal subspaces of dimensions r_1 and r_2 in $\mathbb{R}^{r_1+r_2}$.

The following matroid construction is needed as a preparation for the definition of the union of oriented matroids.

Let $E_1 = \{e_1, \ldots, e_n\}$ and $E_2 = \{f_1, \ldots, f_m\}$ be disjoint linearly ordered sets, and let G be any bipartite graph on $E_1 \cup E_2$. For each $f_i \in E_2$, consider the set $G_i = \{e_{i_1}, \ldots, e_{i_{d_i}}\}$ consisting of all $e_{i_j} \in E_1$ such that (e_{i_j}, f_i) is an edge of G. Now, suppose $\mathcal{M}_1$ is any oriented matroid on E_1. G induces an oriented matroid, $\mathcal{M}_2$, on E_2 as follows:

Define $\mathcal{M}'_{m+1} = \mathcal{M}_1$, and for $i \in \{m, m-1, \ldots, 1\}$, let $\mathcal{M}'_i$ be the oriented matroid on $E_1 \cup \{f_m, \ldots, f_i\}$ obtained from $\mathcal{M}'_{i+1}$ by the lexicographic extension $f_i := [e_{i_1}{}^+, \ldots, e_{i_{d_i}}{}^+]$. Finally, let $\mathcal{M}_2$ be the oriented matroid obtained from $\mathcal{M}'_1$ by deleting the set E_1 : $\mathcal{M}_2$ is the **matroid induced** from $\mathcal{M}_1$ by the graph G.

Definition 2.2.19. *Let* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be oriented matroids on the same ground set* $E = \{e_1, \ldots, e_n\}$. *Identify* E *with disjoint sets* E_1 *and* E_2 *and consider the direct sum,* $\mathcal{M}_1 \oplus \mathcal{M}_2$. *Now, let* G *be the bipartite graph on* $(E_1 \cup E_2) \cup E$ *whose edges are* (e_{1_i}, e_i) *and* (e_{2_i}, e_i) *for all* $e_i \in E$. *The* ***union*** *of* $\mathcal{M}_1$ *and* $\mathcal{M}_2$, $\mathcal{M}_1 \cup \mathcal{M}_2$, *is defined then as the oriented matroid induced from* $\mathcal{M}_1 \oplus \mathcal{M}_2$ *by the graph* G.

Before, it was said that the direct sum of two oriented matroids is equivalent to placing the two point configurations E_1 and E_2 in orthogonal spaces of adequate rank. The union will then be formed by elements $e_i \in E$ such that each e_i is in the segment formed, in the direct sum, by $[e_{1_i}, e_{2_i}]$, but close to e_{1_i}.

It is important to observe that the union of oriented matroids is not commutative, as the geometric annotation above hints at. It is also worth mentioning that the union of two realizable oriented matroids is again realizable. Moreover, as done before, it is relevant to study the chirotope of this class of matroids in respect to its parts.

Proposition 2.2.20. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be oriented matroids of rank r_1 and r_2, respectively, on a linearly ordered set E, such that $\mathcal{M}_1 \cup \mathcal{M}_2$ has rank $r_1 + r_2$. Suppose that $\{e_1 < \cdots < e_{r_1}\}$ and $\{f_1 < \cdots < f_{r_2}\}$ are bases of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, such that for all lexicographic earlier permutations $(e'_1, \ldots, e'_{r_1}, f'_1, \ldots, f'_{r_2})$ of $(e_1, \ldots, e_{r_1}, f_1, \ldots, f_{r_2})$ either $\{e'_1, \ldots, e'_{r_1}\}$ is not a basis of $\mathcal{M}_1$ or $\{f'_1, \ldots, f'_{r_2}\}$ is not a basis of $\mathcal{M}_2$. Then*

$$\mathcal{X}_{1,2}(e_1, \ldots, e_{r_1}, f_1, \ldots, f_{r_2}) = \mathcal{X}_1(e_1, \ldots, e_{r_1}) \cdot \mathcal{X}_2(f_1, \ldots, f_{r_2}),$$

*and $\mathcal{X}_{1,2}$ is called **the chirotope of the union** $\mathcal{M}_1 \cup \mathcal{M}_2$.*

2.2.3 Lawrence Oriented Matroids

All necessary tools have been gathered for unveiling the definition of a *Lawrence oriented matroid.*

Definition 2.2.21. *A rank r uniform oriented matroid, $\mathcal{M}$, on a ground set, E, is a **Lawrence Oriented Matroid** if there exist rank one uniform oriented matroids $\mathcal{M}_1, \ldots, \mathcal{M}_r$ on E, such that $\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_r$.*

As a Lawrence oriented matroid is a union of realizable oriented matroids, it is realizable [2]. Additionally, by 2.2.20, it can be concluded that in this instance:

$$\mathcal{X}(e_1, \ldots, e_r) = \mathcal{X}_1(e_1) \cdots \mathcal{X}_r(e_r),$$

where $E = \{e_1, \ldots, e_n\}$, $\mathcal{X}$ is the chirotope of $\mathcal{M}$, and $\mathcal{X}_i$ is the chirotope of $\mathcal{M}_i$.

Thus, one can represent a uniform rank r Lawrence oriented matroid, on an n element set E, as a matrix $A = (a_{i,j})$, where $a_{i,j} = \mathcal{X}_i(e_j)$ for all

$j \in E = \{1, \ldots, n\}$ and $1 \leq i \leq r$.

Finally, it would be nice to know under what circumstances a matrix A, with entries in $\{+1, -1\}$, is the chirotope of a Lawrence Oriented Matroid. The following proposition answers this question.

Proposition 2.2.22. *A chirotope, $\mathcal{X}$, corresponds to some Lawrence oriented matroid, $\mathcal{M}$, if and only if there exists a matrix $A = (a_{i,j})$, where $1 \leq i \leq r$, $1 \leq j \leq n$ and $a_{i,j} \in \{+1, -1\}$ $\forall$ i, j, and such that*

$$\mathcal{X}(B) = \prod_{i=1}^{r} a_{i,j_i},$$

for all $B = \{j_1 < \cdots < j_r\} \subset E$.

Under this representation, the opposite chirotope $-\mathcal{X}$ can be obtained inversing the signs of one row in the matrix A; and the reoriented matroid $_{-S}\mathcal{M}$, for any $S \subset E$, is obtained by inversing the sign of all the coeficients of the columns $j \in S$.

As mentioned before, this last section constitutes the foundation stone of the techniques that lead to the latest improvement on the upper bound for McMullen's problem, in particular proposition 2.2.22 is extremely important in the proof of proposition 3.2.7.

2.3 Matroid Polytopes

This section focuses on the Oriented Matroid results which have a clear motivation on convex polytope theory, and hence of great importance for chapter 3. The matroid equivalent of convexity notions are defined, and an understanding of basic properties of convex polytopes, from the point of view of oriented matroid theory, developed. It also offers a round up of the geometrical illustrations outlined in the preceding sections.

For the whole of this section, and subsequent chapters, a rank r oriented matroid $\mathcal{M}$ will be interpreted as the matroid of affine dependences of a configuration of points in general position, E, in (r-1)-space.

Given any circuit $X \in \mathcal{C}$, the partition (X^+, X^-), defined by the signature of X, can be interpreted as a Radon partition of the points in $\underline{X}$. Now, let Y be a cocircuit in $\mathcal{C}^*$ and consider (Y^+, Y^-), the partition consisting of sets contained in $aff(E \backslash \underline{Y})^+$ and $aff(E \backslash \underline{Y})^-$, respectively. By definition, Y is a cocircuit if for every $X \in \mathcal{C}$ one of the next two conditions hold:

1. $\underline{Y} \cap \underline{X} = \emptyset$; or
2. if $\underline{Y} \cap \underline{X} \neq \emptyset$, there are $x, y \in \underline{X} \cap \underline{Y}$ such that $X(x) \cdot Y(x) = -X(y) \cdot Y(y)$.

Suppose $\underline{Y} \cap \underline{X} \neq \emptyset$. Then either

$$aff(E \backslash \underline{Y}) \cap conv(\underline{X}) = \emptyset, \quad \text{or} \quad aff(E \backslash \underline{Y}) \cap conv(\underline{X}) \neq \emptyset.$$

In the first case, one can suppose that $\underline{X} \in Y^+$, and condition two holds. In the second case, $\exists\, x \in \underline{X} \cap Y^+$ and $y \in \underline{X} \cap Y^-$. If $X(x) = X(y)$, condition two holds. If $\forall x \in \underline{X} \cap Y^+$ and $\forall y \in \underline{X} \cap Y^-$, but $X(x) \neq X(y)$, then (X^+, X^-) would not be a Radon partition of X.

Hence cocircuits of $\mathcal{M}$ can be interpreted as the complements of sets of points which span a hyperplane.

Definition 2.3.1. *Let $\mathcal{M}$ be an acyclic oriented matroid on a set E. Given any cocircuit Y of $\mathcal{M}$, the set Y^+ is said to be an* ***open half space*** *of $\mathcal{M}$. If $E \backslash \underline{Y} = Y^0$, $Y^+ \cup Y^0$ is a* ***closed halfspace*** *of $\mathcal{M}$.*

From the definition above and the usual definitions of facets and faces, it is easy to translate these concepts into oriented matroid language.

Definition 2.3.2. *A* ***facet*** *of $\mathcal{M}$ is a set of points, H, which forms a hyperplane, such that $E \backslash H$ is an open half-space. In other words, the facets of $\mathcal{M}$ are the complements of its positive cocircuits.*

Definition 2.3.3. *Any intersection of facets of $\mathcal{M}$ is a* ***face*** *of $\mathcal{M}$. The* *faces of $\mathcal{M}$ are the complements of its positive covectors.*

In particular, if a positive covector has ground set $E\backslash\{x\}$, then x is called a **vertex** of $\mathcal{M}$.

Definition 2.3.4. *An oriented matroid $\mathcal{M}$, on a ground set E, is a* ***matroid polytope*** *if all the one element subsets of E are vertices of $\mathcal{M}$.*

To finish off this section, the notions of convex hull and extreme point of a matroid are highlighted; and a very important proposition is mentioned. This proposition will constitute the most important tool for the translation of McMullen's problem into an oriented matroid setting.

Definition 2.3.5. *The* ***convex hull****, $conv_{\mathcal{M}}(S)$, of a set $S \subseteq E$, relative to $\mathcal{M}$, is:*

$$S \cup \{x \in E\backslash S \mid \exists X = (X^+, X^-) \in \mathcal{C} \text{ with } X^- = \{x\} \text{ and } X^+ \subseteq S\}.$$

Definition 2.3.6. *An* ***extreme point*** *x in the matroid $\mathcal{M}$ is an element $x \in E$ such that $x \notin conv_{\mathcal{M}}(E\backslash x)$.*

It is also important to be able to tell, from an oriented matroid, when a point is contained in the convex hull of the others.

Definition 2.3.7. *An* ***interior point*** *x of the matroid $\mathcal{M}$ is an element $x \in E$ such that there is a circuit, X, with $\{x\} = X^+$.*

Finally, recalling that one of the different equivalent settings of McMullen's problem states,

'Determine the largest integer $n = \nu(d)$ such that any set of n points in general position in the affine $d-space$, $\mathbb{R}^d$, can be mapped by a permissible projective transformation on to the vertices of a convex polytope';

it is necessary to investigate what is the matroid equivalent of a projective transformation. This can be easily derived from the next result on separation of matroid polytopes.

Proposition 2.3.8. *Let E be a finite set in $\mathbb{R}^d$, $\mathcal{M}$ the oriented matroid of affine dependences of E and $S \subset E$. The following statements are equivalent:*

1. $_{-S}\mathcal{M}$ *is acyclic.*

2. $conv_{\mathcal{M}'}(S) \cap conv_{\mathcal{M}'}((E \backslash S) \cup p) = \emptyset$ *for every $\mathcal{M}'$ single element exension of $\mathcal{M}$.*

3. $conv(S) \cap conv(E \backslash S) = \emptyset$.

4. *There is a hyperplane H in $\mathbb{R}^d$ separating S strictly from $E \backslash S$.*

Proposition 2.3.9. *Let E be a finite set in $\mathbb{R}^d$, $\mathcal{M}$ the oriented matroid of affine dependences of E and $S \subset E$. Then there is an acyclic reorientation $_{-S}\mathcal{M}$, by reversing the signs of S, if and only if there is a nonsingular projective transformation P, permissible for E, such that $_{-S}\mathcal{M}$ is precisely the matroid of affine dependences of the set of points $P(E)$.*

The proofs of the last two propositions are due to Cordovil and da Silva and can be found in [5].

All the terminology needed to show Ramírez-Alfonsín's proof of the best known bound for McMullen's problem, and the newly proven bounds for the generalized McMullen's problem, are contained in this chapter.

There are many more concepts in the study of Matroid Polytopes. For example, a lot of useful tools have been built for the study of cyclic polytopes from an oriented matroid point of view. Nevertheless, the concepts presented in this section should be enough to preserve the self-conteindness of this thesis.

Lastly, let this paragraph be the final reminder that the exposition given in this chapter is not even intended as an abridgement of the rich oriented matroid theory. It represents a very small fraction of the axiomatization systems, main properties and applications needed to comprehend fully chapter number 3. For further reference one should go to [2].

Chapter 3

McMullen's Problem on Convex Polytopes

Since it emerged in the seventies, McMullen's problem has occupied the minds and efforts of several outstanding mathematicians. Its beauty resides in the simplicity of its statement and, the elusiveness of a definitive answer, which will undoubtedly enrich our understanding of the theory of convex polytopes. The question reads as follows:

Q 1. *Determine the largest number $\nu(d)$ such that any set of $\nu(d)$ points lying in general position in $\mathbb{R}^d$ can be mapped, by a permissible projective transformation, onto the vertices of a convex polytope.*

The beginning of this chapter deals with the progress achieved up to date by several researchers, and presents a very detailed development of the tools which helped the proof of the best known upper bound for $\nu(d)$.

The last section will focus on proving upper and lower bounds for a version of McMullen's problem, the *Generalized McMullen's problem*, which provides information about neighbourliness in polytopes.

McMullen's property for a set of points, could be interpreted as an attempt to find sets of points that can be transformed into vertices of a *1-neighbourly* *polytope* (one in which every point is a vertex). Considering the past observation, one might ask:

Q 2. *What is the largest number $\nu(d,k)$ such that any set points lying in general position in $\mathbb{R}^d$ can be mapped, by a permissible projective transformation, onto the vertices of a k-neighbourly polytope?*

Due to its nice geometrical interpretation, this will be deemed to be McMullen's generalized problem. The question can be stated in a more general form, for which a nice geometrical interpretation has not yet been found. Initially, it was deduced from an attempt to study partitions of points in space, with a certain generalized separability property. Such a property is the translation of McMullen's property into a configuration of points setting, using Gale Diagrams. That original setting will be carefully looked at in the forthcoming sections.

3.1 Chronological Development of a Solution

3.1.1 Larman's Paper

McMullen's problem was first presented by Peter McMullen to David Larman. Using Gale diagrams, Larman proved that the question could be reformulated as follows:

Q 3. *Determine the smallest number $\lambda(d)$ such that for any set X of $\lambda(d)$ points in R^d there exists a partition of X into two sets, A, B, such that*

$$conv(A\backslash x) \cap conv(B\backslash x) \neq \emptyset, \quad \forall \ x \epsilon X.$$

The relationship between ν and λ is,

$$\lambda(d) = \min_{w \in N}\{w | w \leq \nu(w-d-2)\}.$$

The next definition is taken from this question.

Definition 3.1.1. *Let $X \subset \mathbb{R}^d$ be a set of points such that there exists a partition of it into two disjoint sets, A, B, where*

$$conv(A\backslash x) \cap conv(B\backslash x) \neq \emptyset, \quad \forall \ x \epsilon X,$$

then X is said to be ***divisible****.*

Using this reformulation, Larman found the lower bound $2d + 1 \leq \nu(d)$ by proving that any set of $2d + 3$ points is divisible. He also proved that this bound is sharp in the cases where $d = 1, 2$ and 3, by constructing sets of six and eight points which are not divisible. This supports his conjecture that the lower bound is sharp for higher dimensions. According to him, an example of a divisible set for each dimension will be provided by a non-divisible configuration of points formed by the vertices of two simplices, one on top of the other, not necessarily regular, and slightly twisted. [9]

Using computational methods, in the year 2001 D.Forge, M. Las Vergnas and P.Schuchert [6] found a divisible configuration of 10 points in dim 4, confirming the validity of the conjecture. Furthermore, a close analysis of the set of coordinates they found, shows that the example is precisely the convex hull of a pair of simplices, one on top of the other.

In the paper referred in [9], Larman also found a set of $(d + 1)^2$ points in $\mathbb{R}^d$ such that no projective transformation takes them into the vertices of a convex polytope. So, his bounds were as follows,

$$2d + 1 \leq \nu(d) \leq (d + 1)^2.$$

3.1.2 Oriented Matroid Approach.

Chapter 2 is a broad introduction to all the topics needed to understand the work contained in a 1985 paper by I. Da Silva and R. Cordovil [5], where Oriented Matroids were used for the first time in an attempt to solve McMullen's Problem. They studied what the matroid equivalent of a projective transformation is, and what the exact equivalent of Q1 would be in an oriented matroid setting.

Using that the matroid of a set E, of points in general position spanning $\mathbb{R}^d$, is a uniform oriented matroid of rank $d+1$, McMullen's question can be stated, as follows,

Q 4. *Determine the largest integer $n = \gamma(r)$ such that for any realizable oriented matroid of uniform rank $r = d+1$ on a set of of $\gamma(r)$ elements, $\mathcal{M}$, there is an acyclic reorientation, ${}_{-S}\mathcal{M}$, where all the points are extreme points.*

or in its dual version,

Q 5. *Determine the smallest integer $n = \nu(r)$ for which there exists a realizable oriented matroid $\mathcal{M}$ on a set of $\nu(r)$ elements, of rank $r = d+1$, having the following property: every reorientation, ${}_{-S}\mathcal{M}$, has no interior points.*

Not long after the aforementioned equivalencies were proven, in 1986 M. Las Vergnas [10] proved that:

$$\nu(d) < \frac{(d+1)(d+2)}{2}.$$

In order to construct an oriented matroid large enough to fulfill the condition in Q5, he constructed the matroid of oriented cycles of a tournament on $d+2$ vertices. He then lifted its edges into the vertices of a cube, perturbed it slightly to achieve general position (in order to show it was indeed realizable as a point configuration); and, jumping back an forth between the properties of a tournament and the point configuration, concluded that such a matroid will always contain an interior point.

After these achievements, there was no further progress for more than a decade. Until, in 2001, J. Ramírez Alfonsín published a paper with a new upper bound:

$$\nu(d) < 2d + \left\lceil \frac{d+1}{2} \right\rceil. \tag{3.1}$$

He proved this bound by building a family of oriented matroids such that a reorientation which contains an interior point can always be found. This is the subject matter of the next section.

3.2 The Best Upper Bound.

All this section is based in work by Ramírez-Alfonsín contained in [16]. In subsection 2.2.3, a Lawrence oriented matroid was defined. It was also observed that any union of r uniform rank one oriented matroids on a ground set $E = \{e_1 < \cdots < e_n\}$, $\mathcal{M}$, is a Lawrence oriented matroid, hence realizable. Furthermore it can be represented by a matrix $\mathcal{A} = (a_{i,j})$, with $1 \leq i \leq r$ and $1 \leq j \leq n$, whose entries are in the set $\{1, -1\}$.

Also, note that if $\mathcal{A} = (a_{i,j})$ is as before, the matrix corresponding to the reorientation over an element $c \in E$ of the matroid $\mathcal{M}$, ${}_c\mathcal{M}_{\mathcal{A}}$, is obtained by inverting the sign of all the coefficients in the column c of $\mathcal{A}$, denoted ${}_c\mathcal{A}$.

With the purpose of finding an example of a realizable oriented matroid, such that there is a reorientation with an interior element, the search will first be reduced to the class of Lawrence oriented matroids described above. Inside that class, it is needed that the matroid is acyclic, and a way to tell when it has interior elements.

So, the aim now is finding conditions over a family of matrices with entries in the set $\{+1, -1\}$, such that any acyclic reorientation contains something that can be interpreted as an interior element in the matroid.

By definition, an interior element of an oriented matroid, $\mathcal{M}$, is an element $c \in E$ such that there is a circuit X with $X^+ = \{c\}$. If $\mathcal{M}$ is then reoriented on c, the matroid, ${}_c\mathcal{M}_{\mathcal{A}}$, would be cyclic. This same reasoning holds even if there are sets of interior elements of higher cardinality.

It is then imperative to find out how to tell when a matrix $\mathcal{A}$ represents an acyclic matroid. With this purpose, it is practical to think of the matrix as a grid with crossings in the elements $a_{i,j}$.

3.2.1 Travels and Acyclic Reorientations.

In an $r \times n$ grid, imagine a path that starts from the upper left hand side corner and travels from left to right following only edges of the grid, in such a way that it can only move right or down the grid until either the last column or the last row of the grid are reached. If, additionally, the movements possible are restricted further by asking that the maximum number of steps down to be taken at any time, between steps to the right, is one, this path is referred to as a *Travel.*

If in $\mathcal{A}$, we consider the grid formed by the elements $a_{i,j}$, one can further restrict the movement of the travel by asking it to go one step down if and only if it has just passed trough two crossings with opposite signs in the same row. A travel made with such restrictions is a *Top Travel.* Now, rotate the grid, together with the top travel, 180 degrees. One would end up with a path, which follows a similar logic, but opposite movement. Such a travel is a *Bottom Travel.* An example of the Top and Bottom travels for a small matrix is in figure 3.1. Or, more formally:

Definition 3.2.1. . *A* ***Plain Travel*** *(PT) in $\mathcal{A}$ is the following subset of the entries of $\mathcal{A}$,*

$$PT = \{[a_{1,1}, a_{1,2}, ..., a_{1,j_1}], [a_{2,j_1}, a_{2,j_1+1}, ..., a_{2,j_2}], ..., [a_{s,j_{s-1}}, a_{s,j_{s-1}+1}, ..., a_{s,j_s}]\}$$

with $2 \leq j_{i-1} < j_i \leq n \quad \forall \quad 1 \leq i \leq r, \;\; 1 \leq s \leq r$ *and* $j_s = n$.

Definition 3.2.2. *A* ***Top Travel*** *(TT) in $\mathcal{A}$ is a PT with the following additional constraints:*

1. $a_{i,j_{i-1}} \times a_{i,j} = 1, \quad \forall \quad j_{i-1} \leq j < j_i$;
2. $a_{i,j_{i-1}} \times a_{i,j_i} = -1$; *and*
3. *either*

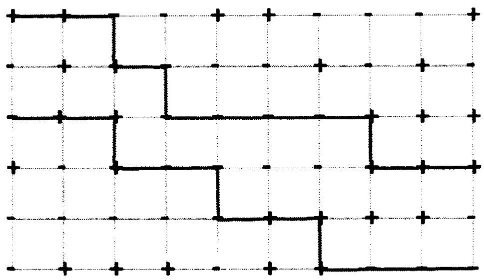

Figure 3.1: Top and Bottom travels for a 6×10 matrix.

- $1 \leq s < r$ *then* $j_s = n$ *or*

- $s = r$ *and* $j_s \leq n$.

Definition 3.2.3. *A **Bottom Travel** (BT) in $\mathcal{A}$ is defined as a TT starting from the bottom left corner of the matrix, i.e.*

1. $a_{i,j_{i+1}} \times a_{i,j} = 1, \quad \forall \quad j_i < j \leq j_{i+1}$;

2. $a_{i,j_{i+1}} \times a_{i,j_i} = -1$; *and*

3. *either*

   - $1 < s \leq r$ *then* $j_s = 1$ *or*

   - $s = 1$ *and* $1 \leq j_s$.

Remember that, in subsection 2.2.3, it has been shown that the matrix $\mathcal{A}$ encodes the chirotope of the matroid in the following way:

$$\mathcal{X}(B) = \prod_{i=1}^{r} a_{i,j_i},$$

where $E = \{e_1, \ldots, e_n\}$ is the ground set of the matroid and $B = \{j_1 < \cdots < j_r\} \subset \{1, \ldots, n\}$.

Also, the signature of every circuit in a union of uniform oriented matroids of rank one, can be read from the chirotope as:

$$\mathcal{X}(B) = \mathcal{X}(e_{j_1}, \ldots, e_{j_r}) = \mathcal{X}_{j_1}(e_{j_1}) \cdots \mathcal{X}_{j_r}(e_{j_r}),$$

where $\mathcal{X}$ is the chirotope of $\mathcal{M} = \cup_{i=1}^{r} \mathcal{M}_i$ and $\mathcal{X}_i$ is the chirotope of $\mathcal{M}_i$.

It is also known that the sign of the element x_i in a circuit C is

$$C(e_{j_i}) = (-1)^i \cdot \mathcal{X}(e_{j_1}, \ldots, e_{j_{i-1}}, e_{j_{i+1}}, \ldots, e_{j_r}),$$

so

$$C(e_{j_i}) = (-1)^i \cdot \mathcal{X}_{j_1}(e_{j_1}) \cdots \mathcal{X}_{j_{i-1}}(e_{j_{i-1}}) \cdot \mathcal{X}_{j_{i+1}}(e_{j_{i+1}}) \cdots \mathcal{X}_{j_r}(e_{j_r}).$$

In the matrix representation this means that if $C = \{e_{j_1}, \ldots, e_{j_r}\}$ is a circuit with $j_i \in \{1, \ldots, n\}$ then,

$$C(e_{j_i}) = (-1)^i \cdot a_{1,j_1} \cdots a_{i-1,j_{i-1}} \cdot a_{i,j_{i+1}} \cdots a_{r-1,j_r}.$$

Hence $C(e_{j_i}) \cdot C(e_{j_{i+1}}) = -a_{i,j_{i+1}} \cdot a_{i,j_i}$. So $C(e_{j_i}) = C(e_{j_{i+1}})$ if and only if $a_{i,j_{i+1}} = -a_{i,j_i}$. But, that is precisely the way in which travels detect when to take a step downwards. So PT can be associated with circuits of the matroid. Hence in order to study cyclicity in the matroid, one only needs to study the behavior of travels.

Using these definitions, in [16] the following propositions have been proved:

Proposition 3.2.4. *Let $\mathcal{A} = (a_{i,j})$ with $1 \leq i \leq r$, $1 \leq j \leq n$, be a matrix with entries from $\{1, -1\}$, $\mathcal{M}_{\mathcal{A}}$ its corresponding Lawrence oriented matroid, and TT and BT the top and bottom travels constructed on $\mathcal{A}$. Then the following conditions are equivalent:*

1. *$\mathcal{M}_{\mathcal{A}}$ is* cyclic*;*
2. *TT ends at $a_{r,s}$ for some $1 \leq s < n$; and*

3. BT ends at $a_{1,s'}$ for some $1 < s' \leq n$.

Proposition 3.2.5. *Let $\mathcal{A} = (a_{i,j})$ with $1 \leq i \leq r$, $1 \leq j \leq n$, be a matrix with entries from $\{1, -1\}$ and $\mathcal{M}_{\mathcal{A}}$ its corresponding Lawrence oriented matroid. Then there is a bijection between the set of all plain travels of $\mathcal{A}$ and the set of all acyclic reorientations of $\mathcal{M}_{\mathcal{A}}$.*

In order to show the bound in equation 3.1, in [16], a suitable family of matroids of rank $r = d + 1$ on a set E, with cardinality $n = 2d + \lceil \frac{d+1}{2} \rceil$, is constructed. For such family of matroids it is always enough to reorient one of the elements to make them cyclic. That is, after just one column reorientation, in the matrix which represents the matroid, either the TT ends at the last row or the BT ends at the first row.

3.2.2 Chessboards

The construction of the desired families of matrices will mainly consist of restricting the pattern of the signs in the grid. A graphical way of defining how the signs of the grid form patterns is the chessboard of the matrix.

Definition 3.2.6. *The **chessboard** of the matrix $\mathcal{A}$ is a black and white grid, with size $(r - 1) * (n - 1)$, where the square $s(i, j)$ has its upper left corner in the intersection of the row i and the column j. A square $s(i, j)$, with $1 \leq i \leq r - 1$ and $1 \leq j \leq n - 1$, will be black if the product given by the entries $a_{i,j}, a_{i,j+1}, a_{i+1,j}, a_{i+1,j+1}$ is -1, and white otherwise.*

Chessboards are invariant under reorientations of $\mathcal{A}$. They provide a mainframe of how to study the information encoded in $\mathcal{A}$ because, despite the many combinations of patterns of signs possible in a matrix, the analysis can be reduced to types of chessboards.

Observe that a chessboard has the following property: if there is one black square between TT and BT, their behavior is opposite. In other words, if TT makes a single horizontal movement from $a_{i,j}$ to $a_{i,j+1}$ and continues its movement forward, in the same row, then BT goes from $a_{i+h,j+1}$

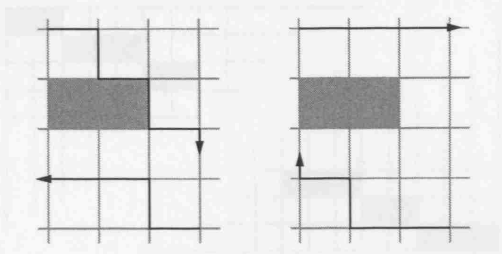

Figure 3.2: Here the top and bottom travel can be observed to have opposite behavior when there are black blocks between them and same if there are not.

to $a_{i+h,j}$ and moves vertically to $a_{i+h-1,j}$ (with $h \geq 1$), and the other way around. Figure number 3.2 illustrates the opposite behavior of travels described above.

Ramírez-Alfonsín uses a $r \times \left(2(r-1) + \lceil \frac{r}{2} \rceil\right)$ chessboard, as the one in figure 3.3, to prove that any matrix with that type of chessboard has at least one column whose reorientation will lead to either TT ending at row r or BT ending at row 1. Or, geometrically, he proved:

Proposition 3.2.7. *Let $\nu(d)$ be the largest number such that any set of $\nu(d)$ points lying in general position in $\mathbb{R}^d$ can be mapped by a permissible projective transformation onto the vertices of a convex polytope. Then*

$$\nu(d) < 2d + \left\lceil \frac{d+1}{2} \right\rceil$$

Define UD and LD as the following sets of elements of $\mathcal{A}$:

$$UD = \{a_{i,j} \mid s(i,j) \text{ or } s(i,j-1) \text{ is black}\}$$

$$LD = \{a_{i,j} \mid s(i-1,j-1) \text{ or } s(i-1,j) \text{ is black}\}.$$

That is, UD consists of all the elements delimiting the black diagonal from above, and LD are the elements delimiting the diagonal from below.

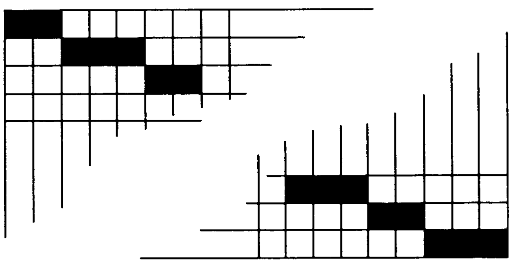

Figure 3.3: Corners of a chessboard which consists of two-block and three-block steps alternating along the diagonal, until as many steps as the dimension are placed.

The advantage of the *black diagonal* structure is that, in an acyclic matrix, it helps define an induction tool for the proofs of the theorems in this chapter. Also, it has been observed TT and BT have opposite behaviors for as long as the black diagonal is between them, this provides insight in the behavior of the travels.

3.3 The generalized McMullen's Problem

Considering the equivalence of McMullen's problem with Larman's problem given in Q3, one might try to generalize in two directions: the number of parts in the partition made or the number of removable points. For a further twist, one might try to generalize in both directions:

Q 6. *Determine the smallest number $\lambda(d, s, k)$ such that for any set X of $\lambda(d, s, k)$ points in R^d there exists a subdivision of X into s sets $A_1, A_2, ..., A_s$ such that*

$$\bigcap_{i=1}^{s} conv(A_i \backslash \{x_1, x_2, ..., x_k\}) \neq \emptyset, \quad \forall \ \{x_1, x_2, ..., x_k\} \subset X.$$

But, the focus of this section is the proof of the following theorem, which is a special case of Q 6:

Theorem 3.3.1. *Let $k \geq 2$ and let $\lambda(d,k)$ be the smallest number such that for any set, X, of $\lambda(d,k)$ points in $\mathbb{R}^d$ there exists a subdivision of X into two sets A, B, such that*

$$conv(A \backslash \{x_1, x_2, ..., x_k\}) \cap conv(B \backslash \{x_1, x_2, ..., x_k\}) \neq \emptyset,$$

$\forall \{x_1, x_2, ..., x_k\} \subset X$. *Then* $2d + k + 3 \leq \lambda(d,k) \leq (k+1)d + (k+2)$.

Before fully entering into the build up of the proof of the theorem above, a couple of equivalences of Q6, in the case where $r = 2$ are necessary.

Lemma 3.3.2. *The following two statements are equivalent:*

> *Determine the largest number $\nu(d,k)$ such that any set of $\nu(d,k)$ points lying in general position in $\mathbb{R}^d$ can be mapped, by a permissible projective transformation, onto the vertices of a* k-neighbourly *polytope.*
>
> *Determine the smallest number $\mu(d,k)$ such that for any set, X, of $\mu(d,k)$ points lying in linearly general position on S^{d-1}, it is possible to choose a sequence $E = (\epsilon_1, \ldots, \epsilon_{\mu(d,k)}) \in \{1,-1\}^{\mu(d,k)}$ such that for every k-membered subset of X_E, X_E^k, $0 \in conv(X_E \backslash X_E^k)$, where $X_E = \{\epsilon_1 x_1, \ldots, \epsilon_{\mu(d,k)} x_{\mu(d,k)}\}$.*

The relationship between $\nu(d,k)$ and $\mu(d,k)$ is:

$$\nu(d,k) = \max_{w \in \mathbb{N}} \{w \geq \mu(w - d - 1, k)\},$$

$$\mu(d,k) = \min_{w \in \mathbb{N}} \{w \leq \nu(w - d - 1, k)\}.$$

Proof. Let X be a set of points in general position in $\mathbb{R}^d$, such that $|X| = \nu \leq \nu(d,k)$. By hypothesis and 1.1.1, there is a nonsingular projective transformation, permissible for X, $P(x) = \frac{Ax+b}{\langle c,x \rangle + \delta}$, such that P(X) is the set of vertices of a k-neighbourly convex polytope. Then the Gale diagram of X, $\overline{X}$, is linearly equivalent to the set $\overline{X}_E = \{\epsilon_1 \overline{x}_1, \ldots, \epsilon_\nu \overline{x}_\nu\}$, where $\epsilon_i = sgn(\langle c, x_i \rangle + \delta)$ for all $i = 1, \ldots \nu$. Hence, by 1.2.2, for all k-membered subset of $\overline{X}_E$, $\overline{X}^k_E$, $0 \in conv(\overline{X}_E \backslash \overline{X}^k_E)$. So $\nu \geq \mu(\nu - d - 1, k)$.

Conversely, let $\overline{X}$ in $\mathbb{R}^d$, such that $|\overline{X}| = \mu \geq \mu(d,k)$, be the Gale diagram of a set $X \subset \mathbb{R}^{\mu-d-1}$. Then there is a sequence $E = (\epsilon_1, \ldots, \epsilon_\mu) \in \{1,-1\}^\mu$ such that $\overline{X_E} = \{\epsilon_1\overline{x_1}, \ldots, \epsilon_\mu\overline{x_\mu}\}$ is the Gale diagram of a k-neighbourly polytope, where $\epsilon_i \in \{1,-1\}$. By 1.1.1 (4), there are $c \in \mathbb{R}^d$ and $\delta \in \mathbb{R}$ such that $\epsilon_i = \langle c, x_i \rangle + \delta$ for all $i = 1, \ldots, \mu$, a linear transformation A and a vector $b \in \mathbb{R}^d$ such that the projective transformation $P(x) = \frac{Ax+b}{\langle c,x \rangle + \delta}$ is regular and permissible for X, and such that $P(X) = X_E$, where X_E is the Gale transform of $\overline{X_E}$. Hence $\mu \leq \nu(\mu - d - 1, k)$. □

Lemma 3.3.3. *The following two statements are equivalent:*

> *Determine the smallest number $\lambda(d,k)$ such that for any set X of $\lambda(d,k)$ points in $\mathbb{R}^d$ there exists a subdivision of X into two sets A, B such that $conv(A\backslash\{x_1, x_2, ..., x_k\}) \cap conv(B\backslash\{x_1, x_2, ..., x_k\}) \neq \emptyset$, for all $\{x_1, x_2, ..., x_k\} \subset X$.*
>
> *Determine the smallest number $\mu(d,k)$ such that for any set, X, of $\mu(d,k)$ points lying in linearly general position on S^{d-1}, it is possible to choose a sequence $E = (\epsilon_1, \ldots, \epsilon_{\mu(d,k)}) \in \{1,-1\}^{\mu(d,k)}$ such that for every k-membered subset of X_E, X^k_E, $0 \in conv(X_E \backslash X^k_E)$, where $X_E = \{\epsilon_1 x_1, \ldots, \epsilon_{\mu(d,k)} x_{\mu(d,k)}\}$.*

So, $\mu(d+1,k) = \lambda(d,k)$.

Proof. Let X be a set of $\mu < \mu(d+1,k)$ points lying in linearly general position in S^d, then for all sequences $E = (\epsilon_1, \ldots, \epsilon_\mu) \in \{1,-1\}^\mu$ there is a k-membered set, $X^k = \{x_{i_1} \ldots x_{i_k}\} \subset X$, such that $0 \notin relintconv(X_E \backslash X^k_E)$, where $X_E = \{\epsilon_1 x_1, \ldots, \epsilon_\mu x_\mu\}$ and $X^k_E = \{\epsilon_{i_1} x_{i_1}, \ldots, \epsilon_{i_k} x_{i_k}\}$

Therefore, there is a hyperplane, H' that weakly separates the origin from $conv(X_E \backslash X^k_E)$. However, as the points in X are in linear general position, there is a hyperplane, H, through the origin, such that

$$X_E \backslash X^k_E \subset S^{\nu-d-2} \cap H^+.$$

Then, given any $E \in \{1, -1\}^{\mu}$, consider the partition of X formed by the sets,

$$A = \{x_i | \epsilon_i \in E \text{ is such that } \epsilon_i = +\}$$

$$B = \{x_i | \epsilon_i \in E \text{ is such that } \epsilon_i = -\}.$$

For each E, the set X^k induces a hyperplane, H, as above, such that H separates $conv(A \backslash X^k)$ from $conv(B \backslash X^k)$. This implies that

$$\lambda(d, k) \geq \mu(d + 1, k).$$

Conversely, if a set of points, $X = \{x_1, \ldots, x_\lambda\}$, lies in an open hemisphere of S^d, and is not $k - divisible$, then there exists $\eta > 0$ such that every set $X' = \{x'_1, \ldots, x'_\lambda\}$ with $\|x_i - x'_i\| < \eta$ is not $k - divisible$ and lies in the same hemisphere. Consequently, it can be supposed that X is in linearly general position.

Given any sequence $E = (\epsilon_1, \ldots, \epsilon_\lambda) \in \{1, -1\}^{\lambda}$, with $\lambda < \lambda(d, k)$ consider the partition into two sets given by:

$$A = \{x_i | \epsilon_i \in E \text{ is such that } \epsilon_i = +\}$$

$$B = \{x_i | \epsilon_i \in E \text{ is such that } \epsilon_i = -\}.$$

By hypothesis there are points $X^k = \{x_{i_1}, \ldots, x_{i_k}\} \in X$ such that,

$$conv(A \backslash \{x_{i_1}, \ldots, x_{i_k}\}) \cap conv(B \backslash \{x_{i_1}, \ldots, x_{i_k}\}) = \emptyset.$$

Thus, there is a hyperplane through the origin, H, that separates

$$conv(A \backslash \{x_{i_1}, \ldots, x_{i_k}\}) \text{ from } conv(B \backslash \{x_{i_1}, \ldots, x_{i_k}\}).$$

Hence $A_E \backslash \{\epsilon_{i_1} x_1, \ldots, \epsilon_{i_k} x_{i_k}\}$ and $B_E \backslash \{\epsilon_{i_1} x_1, \ldots, \epsilon_{i_k} x_{i_k}\}$ are contained in the same open half space, with $A_E = \{\epsilon_i x_i | x_i \in A\}$ and $B_E = \{\epsilon_i x_i | x_i \in B\}$. Which proves that for all $E \in \{1, -1\}^{\lambda}$ there is a set $\{x_{i_1}, \ldots x_{i_k}\} \subset X$ such that $0 \notin conv(X_E \backslash \{x_{i_1}, \ldots, x_{i_k}\})$. Then $\mu(d + 1, k) \geq \lambda(d, k)$. □

From the two lemmas above, we have the final relationship between λ and ν.

Corollary 3.3.4.

$$\nu(d,k) = \max_{w \in \mathbb{N}} \{w \geq \lambda(w-d-2,k)\},$$

$$\lambda(d,k) = \min_{w \in \mathbb{N}} \{w \leq \nu(w-d-2,k)\}.$$

3.3.1 Proof of the Upper Bound

Recall that a polytope is $k-$ *neighbourly* if every $k \leq \lfloor \frac{d}{2} \rfloor$ vertices are contained in a facet, and that matroid polytopes are always acyclic.

As the goal of this section is to use matroids to prove theorem 3.3.1, a translation of geometrical neighbourliness into matroid neighbourliness is needed:

Definition 3.3.5. *A matroid polytope is* ***k-neighbourly*** *iff* $k+1 \leq |C^+|$ *and* $k+1 \leq |C^-|$, $\forall\ C \in \mathcal{C}$, *where* $\mathcal{C}$ *is the set of circuits of the matroid* $\mathcal{M}$.

Also, a matroid is cyclic iff there is at least one $C \in \mathcal{C}$ such that $C^+ = C$. So a matroid polytope has *an acyclic reorientation with at most k interior points* iff there is at least one $C \in \mathcal{C}$ such that $|C^+| \leq k$ or $|C^-| \leq k$.

As before, consider the set of Lawrence Oriented Matroids of unions of r uniform rank one oriented matroids over a set E, with cardinality n. All these matrices have a matrix representation $\mathcal{A} = (a_{i,j})$ where every $a_{i,j}$ is in the set $\{+1, -1\}$.

In order to find an upper bound, it is therefore sufficient to find families of realizable matroids (as in the previous section) such that any acyclic reorientation of them contains at least one $C \in \mathcal{C}$ such that $|C^+| \leq k$ (or $|C^-| \leq k$). That is, by propositions 3.2.4 and 3.2.5, one only needs to guarantee that for one family of acyclic matrices there is always a set, $S \subset E$, with $|S| \leq k$ such that the reorientation, $_{-S}\mathcal{M}$, is cyclic. This will be done

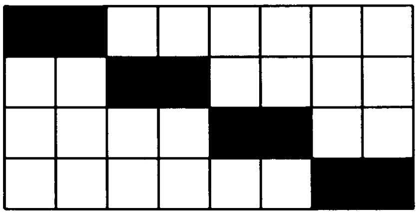

Figure 3.4: Chessboard for a 5×9 matrix

by considering the class of acyclic matroids whose matrix representation has a specific chessboard, and proving that a suitable set S can always be found. From now, using a slight abuse of notation, the matrices corresponding to cyclic (acyclic) matroids will be referred to as cyclic (acyclic).

First, take the case when k=2.

Let $\mathcal{A} = (a_{i,j})$, with $1 \leq i \leq r$ and $1 \leq j \leq n = 2(r-1)+1$, be a matrix with entries from $\{1, -1\}$, with the following chessboard:

- $s(i,j)$ is black if
  - $j = 2(i-1)+1$, or
  - $j+1 = 2(i)+1$;
- $s(i,j)$ is white otherwise.

The chessboard above, is just a chessboard consisting of length two black steps in the diagonal, as figure 3.4 shows.

Lemma 3.3.6. *A matrix $\mathcal{A}$ of size $r \times n$ with $n = 2(r-1)+1$, and the chessboard defined above, has a cyclic reorientation, ${}_S\mathcal{A}$, where $|S| \leq 2$.*

Proof. Let $r = 3$ then $n = 5$. There are the five different cases where $\mathcal{A}$ is acyclic, shown in figure 3.5. In those five cases, $\mathcal{A}$ always has a cyclic reorientation, where the reoriented set has cardinality less or equal to 2. Working from the top left hand corner in clockwise order in the figure, the columns that can be reoriented to make the chessboards cyclic are as follows,

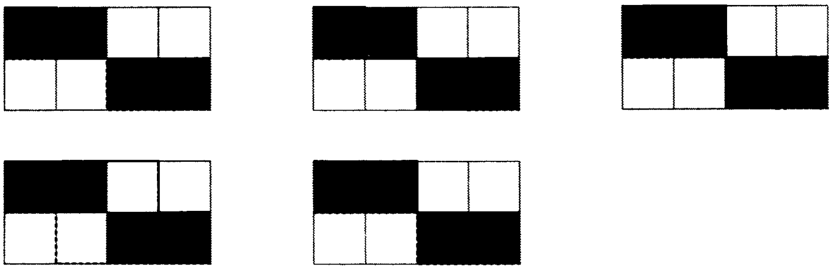

Figure 3.5: Five cases where $\mathcal{A}$ is acyclic

chessboard	columns to be reoriented
1st	4th
2nd	3rd
3rd	2nd
4th	1st and 4th
5th	2nd.

Suppose that for all $r < r^*$ the $r \times 2(r-1)+1$ matrix, $\mathcal{A}$, has a cyclic reorientation of less than 2 elements.

Let $r = r^*$ and assume that TT last intersects $UD \cap LD$ in $a_{i,j}$ with $j = 2(i-1)+1$ and $i < r$. If $2 \leq i$, by the induction hypothesis, the lemma holds. Equally, suppose BT last intersects $UD \cap LD$ (from right to left) at an element $a_{i',j'}$ with $j' = 2(i'-1)+1$. If $i' \leq r-1$, again the lemma holds.

Then TT has to go through elements $\{a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}\}$ and it always travels above UD. Also BT always travels below LD. As $\mathcal{A}$ is acyclic, TT finishes at an element $a_{i',n}$. These imply that if TT makes $2(r-1)+1-3$ horizontal movements and $i'-1$ vertical movements in order to reach column $2(r-1)+1$ from column 3, as BT always passes strictly below UD, it has opposite behaviour. Hence, BT has to do precisely $2r - i' - 3$ vertical movements before column 3. But $i' < r$, so $2r - i' - 3 > r - 3$. That is, at column 3, BT is already in row 2, and the result follows. □

In the case when $k \geq 3$, a chessboard which is suited for proving a lemma

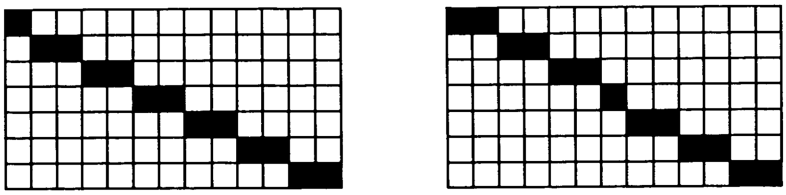

Figure 3.6: Two different types of chessboards valid for rank 8 matrices.

equivalent to 3.3.6, is constructed in the following manner:

Let $\mathcal{A} = (a_{i,j})$ with $1 \leq i \leq r$ and $1 \leq j \leq n = 2(r-1) - (k-2) + 1$, be a matrix with entries from $\{1, -1\}$, with the following chessboard:

- $s(i,j)$ is black if
  - $j = 2(i) - \lceil \frac{(i-1)+l}{s} \rceil$, or
  - $j = 2(i) - \lceil \frac{(i-1)+l}{s} \rceil + 1$ and $i + s - l \not\equiv 0 \mod s$;
- $s(i,j)$ is white otherwise.

where $s = \lceil \frac{r-1}{k-2} \rceil$, $3 \leq k < \frac{r-1}{2}$ and $1 \leq l \leq s$ are fixed.

This chessboard incorporates single blocks, evenly distributed along the diagonal, between double blocks. Figure 3.6 illustrates this chessboard in the cases where $r = 8$, $k = 3$ and $l = 1, 4$.

Lemma 3.3.7. *A matrix $\mathcal{A}$ of size $r \times n$ with $n = 2(r-1) - (k-2) + 1$ and the chessboard defined above, has a cyclic reorientation $_S\mathcal{A}$ with $|S| \leq k$ for all $3 \leq k \leq \lfloor \frac{r}{2} \rfloor$.*

Proof. As before, this proof will also work by induction, except this time it will be needed for both k and r. Let $k = 3$, although the matrix $\mathcal{A}$ represents a matroid and therefore $r > 7$, the purely combinatorial property holds for chessboards with $3 \leq r$. The proof will follow by induction on r, so first consider the case $r = 3$. In this case the chessboard has four columns and three rows, and it is easily seen that for any TT, of an acyclic matrix, three reorientations are more than enough, to make the travel end at row 3.

Now suppose the lemma holds for all $r < r^*$. Let $r = r^*$, so $n = 2r - 2$ and $s = r - 1$. Then there is precisely one single black block in the diagonal. Both $TT \cap UD \cap LD \neq \emptyset$ and $LT \cap UD \cap LD \neq \emptyset$. Let i be the largest $1 \leq i \leq r$ such that $a_{i,j} \in TT \cap UD \cap LD$ or i the smallest such that $a_{i,j} \in LT \cap UD \cap LD$.

If $1 < i$ for TT or $i < r$ for LT, by the induction hypothesis, the lemma holds.

Suppose $i = 1$. If $l = 1$, then TT takes the elements $\{a_{1,1}, a_{1,2}, a_{1,3}\}$. Hence, if column one is reoriented, the new top travel, TT', takes the elements $\{a_{1,1}, a_{1,2}, a_{2,2}, a_{2,3}\}$. But $a_{2,2} \in UD \cap LD$, and after column two there are only double blocks in the diagonal so, by lemma 3.3.6, the lemma holds.

If $l > 1$, then TT takes elements $\{a_{1,1}, a_{1,2}, a_{1,3}a_{1,4}\}$ and reorienting column one, the new top travel, TT', takes elements $\{a_{1,1}, a_{1,2}, a_{2,2}, a_{2,3}, a_{3,3}, a_{3,4}\}$, hence traveling below UD.

If TT' never crosses UD again the lemma holds.

Therefore, suppose $TT' \cap UD \cap LD \neq \emptyset$. Let i be the smallest $1 \leq i \leq r$ such that $a_{i,j} \in TT' \cap UD \cap LD$. By 3.3.6, if $i > l$, the lemma holds. Thus, it is only left to suppose $3 \leq i \leq l$ and $j = 2i - 1$.

The original TT passes through an element $a_{i',2i-1}$ with with $i' < i$. So, between column 3 and column $2i-1$, TT makes $2i-4$ horizontal movements and $i'-1$ vertical movements. Given that TT' and TT are strictly separated by the diagonal of black blocks, between columns 3 and $2i - 1$; the number of vertical movements TT' makes, between those columns, equals $2i - i' - 3$. On the other hand, by hypothesis, the number of vertical movements TT' makes between columns 3 and $2i-1$, is precisely $i-3$. Then $i-3 = 2i-i'-3$, so $i = i'$, a contradiction.

Then if $a_{i,j} \in TT' \cap UD \cap LD$, necessarily $i > l$, and, by lemma 3.3.6, the lemma holds for $k = 3$.

Suppose now that for each $k < k*$, the lemma holds for all $2k \leq r$. Let $k = k*$. Both $TT \cap UD \cap LD \neq \emptyset$ and $LT \cap UD \cap LD \neq \emptyset$. Let i be the largest $1 \leq i \leq r$ such that $a_{i,j} \in TT \cap UD \cap LD$ or i the smallest such that $a_{i,j} \in LT \cap UD \cap LD$.

If $l < i$ for TT or $i < (k-1)s + l$ for LT, by the induction hypothesis for k, the lemma holds.

Suppose $i = l$, then TT the takes elements $\{a_{i,j}, a_{i,j+1}, a_{i,j+2}\}$. Hence, if column one is reoriented, the new top travel, TT' takes the elements

$$\{a_{i,j}, a_{i,j+1}, a_{i+1,j+1}, a_{i+1,j+2}\}.$$

But $a_{i+1,j+1} \in UD \cap LD$ and after column two there are only $k-3$ single blocks in the diagonal so, by the induction hypothesis, the lemma holds.

If $i < l$ then TT takes the elements $\{a_{i,j}, a_{i,j+1}, a_{i,j+2}, a_{i,j+3}\}$ and reorienting column j, the new top travel, TT', takes elements

$$\{a_{i,j}, a_{i,j+1}, a_{i+1,j+1}, a_{i+1,j+2}, a_{i+2,j+2}, a_{i+2,j+3}\}.$$

If TT' never crosses UD again the lemma holds.

Therefore, suppose $TT' \cap UD \cap LD \neq \emptyset$. Let i' be the smallest $1 \leq i' \leq r$ such that $a_{i',j'} \in TT' \cap UD \cap LD$. By the induction hypothesis, if $i' > l$, the lemma holds. Thus, it is only left to suppose $3 \leq i' \leq l$ and $j' = 2i' - 1$.

The original TT passes through an element $a_{i'',2i'-1}$ with with $i'' < i'$. So, between column $j+2$ and column $2i'-1$, TT makes $2i'-j-3$ horizontal movements and $i''-1$ vertical movements. Given that TT' and TT are strictly separated by the diagonal of black blocks, between columns $j+2$ and $2i'-1$; the number of vertical movements TT' makes, between those columns, equals $2i'-j-i''-2$. On the other hand, by hypothesis, the number of vertical movements TT' makes between columns $j+2$ and $2i'-1$, is precisely $i'-j-2$. Then $i'-j-2 = 2i'-j-i''-2$, so $i' = i''$, a contradiction.

Therefore, the lemma holds for all $3 \leq k$ and $2k \leq r$. □

Summarizing, the previous two lemmas have proved:

$$\nu(d,k) \leq 2d - k + 1 \quad \forall k \geq 2,$$

the upper bound.

3.3.2 Proof of the Lower Bound

For the proof of the lower bound it is better to use the setting of the problem in terms of partitions of points. In [9], Larman proved the following:

Proposition 3.3.8. *Let X be a set of $2d+3$ points in general position in $\mathbb{R}^d$. Then there is a partition of X into two sets, A, B, with the following property:*

$$conv(A\backslash\{x\}) \cap conv(B\backslash\{x\}) \neq \emptyset \ \ \forall \ \{x\} \in X.$$

It is enough, in order to obtain the lower bound, to prove a generalization of 3.3.8:

Lemma 3.3.9. *Let X be a set of $(k+1)d+(k+2)$ points in general position in $\mathbb{R}^d$. Then there is a partition of X into two sets, A, B, with the following property:*

$$conv\,(A\backslash\{x_1,x_2,...,x_k\}) \bigcap conv\,(B\backslash\{x_1,x_2,...,x_k\}) \neq \emptyset,$$

$$\forall \ \{x_1,x_2,...,x_k\} \subset X.$$

Proof. The proof will follow by induction on k. Let $k=1$, by proposition 3.3.8 the lemma holds. Suppose the statement of the lemma is true for all $k < k^*$. Then it has to be proven that the lemma is true for $k = k^*$.

For brevity, if a set of points X has the property stated in the lemma 3.3.9, then X will be called *k-divisible*.

Let $S' = \{s'_1, \ldots, s'_{d+1}\}$ be the set of vectors pointing at the vertices of a regular simplex, centred at the origin. Consider the following set, $Y = \bigcup_{i=0}^{k+1} S'_i$, where $S'_i = \{i \times s'_1, \ldots, i \times s'_{d+1}\}$. Now let X be the set of points in general position obtained by perturbing Y. S_i is the set of vertices obtained from S'_i. For each k, let $A = \bigcup_{i \text{ even}} S_i$ and $B = \bigcup_{i \text{ odd}} S_i$.

The set X with such a partition is $k-$*divisible*. The removal of any set of k vertices, leaves at least one of the S_i untouched $(i \geq 1)$. Without loss of generality it can be supposed that $S_i \subset A$ and there must be a vertex $y \in S_{i-1}$, which is in B, such that $y \in conv(S_i)$.

So there are $(k+1)d+(k+2)$ points in general position which are $k-$*divisible*. The property of being k-divisible is closed among all sets of $(k+1)d+(k+2)$ points in general position in $\mathbb{R}^d$. Let $\{x_1, x_2, ..., x_n\}$ be a k-divisible set. It is therefore enough to prove that if $\{y, x_1, x_2, ..., x_n\}$, where $n = (k+1)d + (k+2)$, is a set of points in general position in $\mathbb{R}^d$, then the set $\{y, x_2, ..., x_n\}$ is also k-divisible.

Let T be the set of real numbers t such that

$$X(t) = \{(1-t)x_1 + ty, x_2, ..., x_n | \, 0 \leq t \leq 1\}$$

is k-divisible. T is a non empty closed subset of $[0, 1]$. Suppose

$$t_0 = \sup_{t \in T} t < 1$$

and let $x_1(t) = (1-t)x_1 + ty$ for all $t \in \mathbb{R}$. Then the set $X(t_0)$ is k-divisible with a subdivision $A(t_0) = \{x_1(t_0), x_2, ..., x_r\}$ and $B(t_0) = \{x_{r+1}, x_2, ..., x_n\}$ (with some relabeling possibly needed).

By definition, for each $t > t_0$ there exist points

$$\{x_{j_1}(t), x_{j_2}(t), ..., x_{j_k}(t)\} \subset X(t)$$

such that if $A(t) = \{x_1(t), x_2, ..., x_r\}$ and $B(t) = \{x_{r+1}, x_2, ..., x_n\}$, then

$$conv\,(A(t)\backslash\{x_{j_1}(t), ..., x_{j_k}(t)\}) \cap conv\,(B(t)\backslash\{x_{j_1}(t), ..., x_{j_k}(t)\}) = \emptyset.$$

Since there are only finitely many combinations of n points in subsets of size k, there is a sequence $t_n \to t_0^+$ as $n \to \infty$ such that $\{x_{j_1}(t_n), x_{j_2}(t_n), ..., x_{j_k}(t_n)\}$ is fixed and equal to $\{x_{j_1}, x_{j_2}, ..., x_{j_k}\}$. Also, for each $t > t_0$ there is a hyperplane $H(t)$ such that

$$conv\,(A(t)\backslash\{x_{j_1}(t), x_{j_2}(t), ..., x_{j_k}(t)\}) \subset H(t)^+$$

and

$$conv\,(B(t)\backslash\{x_{j_1}(t), x_{j_2}(t), ..., x_{j_k}(t)\}) \subset H(t)^-.$$

So, there is a subsequence of the sequence of hyperplanes $\{H(t_n)\}$ that converges to a hyperplane H, which necessarily weakly separates

$$conv\,(A(t_0)\backslash\{x_{j_1}, x_{j_2}, ..., x_{j_k}\}) \text{ from } conv\,(B(t_0)\backslash\{x_{j_1}, x_{j_2}, ..., x_{j_k}\}).$$

By hypothesis,

$$conv\,(A(t_0)\backslash\{x_{j_1}, x_{j_2}, ..., x_{j_k}\}) \cap conv\,(B(t_0)\backslash\{x_{j_1}, x_{j_2}, ..., x_{j_k}\}) \neq \emptyset,$$

which implies that

$$conv\,(A(t_0)\backslash\{x_{j_1}, x_{j_2}, ..., x_{j_k}\}) \cap conv\,(B(t_0)\backslash\{x_{j_1}, x_{j_2}, ..., x_{j_k}\}) \cap H \neq \emptyset.$$

Since the points of $X(0)$ are in general position, the plane H has to contain $d+1$ points of $X(t_0)$, one of which has to be the point $x_1(t_0)$ and none of which are in the set $\{x_{j_1}, x_{j_2}, ..., x_{j_k}\}$.

By Radon's theorem (proposition 1.3.1), the points in $X(t_0) \cap H$ can be divided into two sets $A'(t_0)$ and $B'(t_0)$ such that

$$conv(A'(t_0)) \cap conv(B'(t_0)) \neq \emptyset.$$

Consequently there are $kd+k+2$ points in general position outside the plane H for which we can find a partition $A''(t_0)$, $B''(t_0)$ such that

$$conv\,\left(A''(t_0)\backslash\{x_{i_1}, x_{i_2}, ..., x_{i_{k-1}}\}\right) \cap conv\,\left(B''(t_0)\backslash\{x_{i_1}, x_{i_2}, ..., x_{i_{k-1}}\}\right) \neq \emptyset$$

$$\forall\{x_{i_1}, x_{i_2}, ..., x_{i_{k-1}}\} \subset X(t_0).$$

Suppose w.l.o.g. that $x_1(t_0) \in A'(t_0)$, then for $t_0 \le t \le 1$ say $x_1(t) \in H^+$. Now consider the following partiton for $X(t)$:

$$A(t) = A''(t_0) \cup (A'(t_0)\backslash x_1(t_0)) \cup x_1(t),\ B(t) = B''(t_0) \cup B'(t_0).$$

For all $X_k = \{x_{i_1}, x_{i_2}, ..., x_{i_k}\}$, subsets of $X(t)$, if $|(X_k \cap H) \cup \{x(t)\}| \ge 1$, the lemma holds. So the only case remaining to be dealt with is when $X_k \subset \{A''(t_0) \cup B''(t_0)\}$. Observe that if there is $x_a \in \{A''(t_0)\backslash X_k \cap H^-\}$ or $x_b \in \{B''(t_0)\backslash X_k \cap H^+\}$, then for some $t = t_0 + \epsilon$,

$$\emptyset \ne conv\left(A'(t) \cup \{x_a\}\right) \cap conv\left(B'(t)\right) \subset conv\left(A(t)\backslash X_k\right) \cap conv\left(B(t)\backslash X_k\right) \tag{3.2}$$

or

$$\emptyset \ne conv\left(A'(t)\right) \cap conv\left(B'(t) \cup \{x_b\}\right) \subset conv\left(A(t)\backslash X_k\right) \cap conv\left(B(t)\backslash X_k\right). \tag{3.3}$$

If there is X_k such that $\{A''(t_0)\backslash X_k \cap H^-\} = \emptyset$ and $\{B''(t_0)\backslash X_k \cap H^+\} = \emptyset$, then $B''(t_0)\backslash X_k \subset H^-$ and $A''(t_0)\backslash X_k \subset H^+$. As at least one of

$$B''(t_0)\backslash X_k \ne \emptyset \text{ or } A''(t_0)\backslash X_k \ne \emptyset,$$

$A''(t_0)$ and $B''(t_0)$ can be swapped in the partition, and one of 3.2 or 3.3 will hold. □

Together, lemmas 3.3.9, 3.3.7 and 3.3.6 constitute the proof of the following theorems:

Theorem 3.3.1 *Let* $2 \le k \le \lfloor \frac{d}{2} \rfloor$ *and* $\lambda(d, k)$ *be the smallest number such that for any set* X *of* $\lambda(d, k)$ *points in* R^d *there exists a subdivision of* X *into two sets* A, B *such that*

$$conv(A\backslash\{x_1, x_2, ..., x_k\}) \cap conv(B\backslash\{x_1, x_2, ..., x_k\}) \ne \emptyset,$$

$$\forall \ \{x_1, x_2, ..., x_k\} \subset X.$$

$$\textit{Then } 2d + k + 3 < \lambda(d,k) \leq (k+1)d + (k+2).$$

Tracing back through the equivalences, in the geometric setting, it has also been proven that:

Theorem 3.3.10. *Let $2 \leq k \leq \lfloor \frac{d}{2} \rfloor$ and $\nu(d,k)$ be the largest number such that any set of $\nu(d,k)$ points lying in general position in $\mathbb{R}^d$ can be mapped by a permissible projective transformation onto the vertices of a k-neighborly polytope.*

$$\textit{Then } d + \lceil \tfrac{d}{k} \rceil + 1 \leq \nu(d,k) < 2d - k + 1.$$

The upper bounds presented in this section for the general McMullen's question, look considerably better than the upper bounds obtained for the original McMullen's problem. This is mainly due to the fact that the reorientation of more than two columns of the matrix in the proof, exploits further the structure of the chessboard. In most cases the first reorientation is used in the first column, pushing the TT below the diagonal of black blocks and initiating an induction argument, which isn't possible to follow through when only one reorientation is available.

3.3.3 Bound Sharpness

This subsection contains a pair of lemmas which prove that, in the partition setting, the upper bound, $\lambda(d,k) \leq (k+1)d + (k+2)$, is sharp for $d = 2$ and $k = 2, 3$. In such cases $\lambda(2,2) \leq 10$ and $\lambda(2,3) \leq 13$. Thus, sets of 9 and 12 points which are not *2-divisible* and not *3-divisible*, respectively, are exhibited.

Lemma 3.3.11. *Let X be a set of 9 points in the plane, such that the points in the set $P = \{p_1, p_2, p_3, p_4, p_5\}$ form a regular pentagon and the points in $\{p_a, p_b, p_c, p_d\}$ are placed close to some of the crossings of the pentagon's diagonals, as in figure 3.7. Then X is not $2-divisible$*

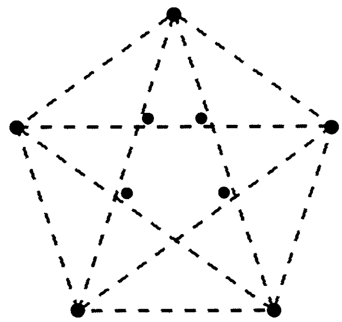

Figure 3.7: Configuration of points for which $\lambda(d,k) = (k+1)d + (k+2)$ when d=2 and k=2.

Proof. Let p_1 be the highest vertex of the pentagon, and label the rest of the vertices, $\{p_2, p_3, p_4, p_5\}$, in successive clockwise order. The point p_a *is near* ($\sim$) the intersection of the segment $[p_1, p_4] \cap [p_2, p_5]$, $p_b \sim [p_1, p_3] \cap [p_2, p_5]$, $p_c \sim [p_1, p_3] \cap [p_2, p_4]$, and $p_d \sim [p_1, p_4] \cap [p_3, p_5]$.

Suppose the sets, A and B, form a partition of X that makes it $2-$*divisible*. If, say, $|A| = 3$ and $|B| = 6$ then, when A contains at least one of the points in the pentagon, then the removal of such point always separates the convex hull of the remaining sets. If A is a set of three points contained in the pentagon. Then the vertices in the pentagon all belong to B, and the removal of some two consecutive vertices will produce disjoint remaining partitions. Hence, both $|A| > 3$ and $|B| > 3$, and without loss of generality, it can be assumed that $|A| = 5$ and $|B| = 4$.

The set B can't contain two consecutive points of P, and can't either be equal to the set $\{p_a, p_b, p_c, p_d\}$. So, B takes at least one vertex in the outer pentagon.

If B takes precisely one vertex in the outer pentagon, then A contains one of the following triangles, $\{p_1, p_2, p_3\}$, $\{p_2, p_3, p_4\}$, $\{p_3, p_4, p_5\}$, or $\{p_1, p_4, p_5\}$; and the removal of the remaining two vertices in A would make the partition non-divisible.

Thus, B has to contain precisely two vertices in P. If that is the case, the two vertices in B belong to the set of *diagonals*,

$$\{\delta_1 = \{p_1, p_3\}, \delta_2 = \{p_1, p_4\}, \delta_3 = \{p_2, p_4\}, \delta_4 = \{p_2, p_5\}, \delta_5 = \{p_3, p_5\}\};$$

and their complements $\{P\backslash\delta_1, P\backslash\delta_2, P\backslash\delta_3, P\backslash\delta_4, P\backslash\delta_5\}$, are always in A. Due to the symmetry of the configuration, the case $\delta_1 \subset B$ is equivalent to $\delta_2 \subset B$, and $\delta_3 \subset B$ is equivalent to $\delta_5 \subset B$.

In each of the different remaining valid cases, a list of the remaining possibilities and the points which removal separates the convex hulls is given:

1. $\delta_1 \subset B$ and

$$\{p_a, p_b\} \in A \implies conv(A\backslash\{p_4, p_1\}) \cap conv(B\backslash\{p_4, p_1\}) = \emptyset$$

$$\{p_a, p_c\} \in A \implies conv(A\backslash\{p_2, p_d\}) \cap conv(B\backslash\{p_2, p_d\}) = \emptyset$$

$$\{p_a, p_d\} \in A \implies conv(A\backslash\{p_2\}) \cap conv(B\backslash\{p_2\}) = \emptyset$$

$$\{p_b, p_c\} \in A \implies conv(A\backslash\{p_3, p_d\}) \cap conv(B\backslash\{p_3, p_d\}) = \emptyset$$

$$\{p_b, p_d\} \in A \implies conv(A\backslash\{p_3, p_c\}) \cap conv(B\backslash\{p_3, p_c\}) = \emptyset$$

$$\{p_c, p_d\} \in A \implies conv(A\backslash\{p_1\}) \cap conv(B\backslash\{p_1\}) = \emptyset$$

2. $\delta_3 \subset B$ and

$$\{p_a, p_b\} \in A \implies conv(A\backslash\{p_3\}) \cap conv(B\backslash\{p_3\}) = \emptyset$$

$$\{p_a, p_c\} \in A \implies conv(A\backslash\{p_3, p_c\}) \cap conv(B\backslash\{p_3, p_c\}) = \emptyset$$

$$\{p_a, p_d\} \in A \implies conv(A\backslash\{p_3\}) \cap conv(B\backslash\{p_3\}) = \emptyset$$

$$\{p_b, p_c\} \in A \implies conv(A\backslash\{p_2, p_5\}) \cap conv(B\backslash\{p_2, p_5\}) = \emptyset$$

$$\{p_b, p_d\} \in A \implies conv(A\backslash\{p_3, p_a\}) \cap conv(B\backslash\{p_3, p_a\}) = \emptyset$$

$$\{p_c, p_d\} \in A \implies conv(A\backslash\{p_1, p_4\}) \cap conv(B\backslash\{p_1, p_4\}) = \emptyset$$

3. $\delta_4 \subset B$ and

$$\{p_a, p_b\} \in A \implies conv(A\backslash\{p_3, p_4\}) \cap conv(B\backslash\{p_3, p_4\}) = \emptyset$$

$$\{p_a, p_c\} \in A \implies conv(A\backslash\{p_1, p_a\}) \cap conv(B\backslash\{p_1, p_a\}) = \emptyset$$

$$\{p_a, p_d\} \in A \implies conv(A\backslash\{p_3, p_5\}) \cap conv(B\backslash\{p_3, p_5\}) = \emptyset$$

$$\{p_b, p_c\} \in A \implies conv(A\backslash\{p_2, p_4\}) \cap conv(B\backslash\{p_2, p_4\}) = \emptyset$$

$$\{p_b, p_d\} \in A \implies conv(A\backslash\{p_1, p_b\}) \cap conv(B\backslash\{p_1, p_b\}) = \emptyset$$

$$\{p_c, p_d\} \in A \implies conv(A\backslash\{p_1\}) \cap conv(B\backslash\{p_1\}) = \emptyset$$

□

Lemma 3.3.12. *Let X be a set of 12 points in the plane, such that the set of points $\{p_1, p_2, p_3, p_4, p_5, p_6\}$ form a regular hexagon and above each of the sides $[p_1, p_2]$, $[p_3, p_4]$ and $[p_5, p_6]$ there are two points $\{q_1, q_2\}$, $\{q_3, q_4\}$ and $\{q_5, q_6\}$, respectively, placed as in figure 3.8. Then X is not $3 -$ divisible.*

Proof. Let p_1 be the point at the upper left hand corner of the hexagon and, from there, label the rest of the vertices, $\{p_2, p_3, p_4, p_5, p_6\}$, in successive clockwise order. The point q_2 is in $conv(\{p_1, p_2, q_1\})$, $q_4 \in conv(\{p_3, p_4, q_3\})$ and $q_6 \in conv(\{p_5, p_6, q_5\})$.

Suppose the sets, A, B, form a partition of X which makes it $3 -$ *divisible*. If, say, $|A| = 4$ then A contains at least one of the vertices in the outer enneagon. Hence, the removal of all the vertices in A except one in the outer enneagon, separates the convex hull of the remaining sets.

Now, suppose both $|A| > 4$ and $|B| > 4$. It is clear that none of the sets of four points, $\{p_i, p_{i+1}, q_i, q_{i+1}\}$ with i odd, formed by one of the edges of the pentagon and the two points above it, can be contained in a set of the partition. Considering the line $l = aff(q_1, q_2)$, it is true that each of the open half-spaces l^+ and l^- has to contain points in both sets of the partition. If, out of the five points in l^+ only one belongs to A then,

1. if $|A| = 5$ then $|B| = 7$, and

   $|B \cap l| = 2$, $|B \cap l^-| = 1$, $|A \cap l^+| = 1 \implies$
   $conv(B\backslash(B \cap l^-)) \cap conv(A\backslash(A \cap l^+)) = \emptyset$,

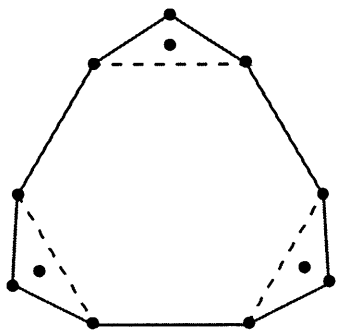

Figure 3.8: Configuration of points for which $\lambda(d,k) = (k+1)d + (k+2)$ when d=2 and k=3.

$$|B \cap l| = 1,\ |B \cap l^-| = 2,\ |A \cap l^+| = 1 \implies$$
$$conv(B \backslash (B \cap l^-)) \cap conv(A \backslash (A \cap l^+)) = \emptyset,$$

$$|B \cap l| = 0,\ |A \cap l^-| = 2,\ |A \cap l^+| = 1 \implies$$
$$conv(B) \cap conv(A \backslash ((A \cap l^+) \cup (A \cap l^-))) = \emptyset :$$

2. if $|A| = 6$ then $|B| = 6$, so

$$|B \cap l^-| = 2,\ |A \cap l^+| = 1 \implies$$
$$conv(B \backslash (B \cap l^-)) \cap conv(A \backslash (A \cap l^+)) = \emptyset :$$

3. if $|A| = 7$ then $|B| = 5$, so

$$|B \cap l^-| = 1,\ |A \cap l^+| = 1 \implies$$
$$conv(B \backslash (B \cap l^-)) \cap conv(A \backslash (A \cap l^+)) = \emptyset.$$

The case when only one of the five points in l^+ belongs to B is analogous. So, it can be assumed that for all lines formed by the affine span of the sets $\{q_1, q_2\}$, $\{q_3, q_4\}$ and $\{q_5, q_6\}$, there are two points of one set and three points of the other on each side of it.

Considering that no set $\{p_i, p_{i+1}, q_i, q_{i+1}\}$, with i odd, is a subset of either A or B, both A and B contain two points in $\{p_i, p_{i+1}, q_i, q_{i+1}\}$. Thus, $|A| = 6 = |B|$ and, either $\{p_i, q_i\}$ or $\{p_{i+1}, q_i\}$ are contained in one same set of the partition. Suppose $\{p_2, q_1\} \subset A$, then $conv(A) \subset aff(\{p_2, q_1\})^+$ and $conv(B \backslash (B \cap aff(\{p_2, q_1\})^+)) \subset aff(\{p_2, q_1\})^-$. The other cases are analogous. □

3.3.4 The *Truly* General McMullen's Problem on Partitions

McMullen's problem was originally posed as a geometrical property of a configuration of points, and even the generalization dealt with in this chapter, turns out to have a geometrical interpretation. However, the partition problem, to which it is equivalent, is very interesting in itself and does not need to have any restriction on the number, k, of points removed. So, as mentioned before, one could aim to answer the Tverberg type question that reads:

Q 6. *Determine the smallest number $\lambda(d,s,k)$ such that for any set, X, of $\lambda(d,s,k)$ points in R^d there exists a subdivision of X into s sets $A_1, A_2, ..., A_s$ such that*

$$\bigcap_{i=1}^{s} conv(A_i \backslash \{x_1, x_2, ..., x_k\}) \neq \emptyset, \quad \forall \ \{x_1, x_2, ..., x_k\} \subset X.$$

This problem sits among many different questions, like Reay's conjecture, that rather than studying *when* the partitions of the sets intersect, focuses on *how* the partitions intersect. Superficially, it seems that the dimension of the intersection of the convex hulls of partitions, might bear a relationship with $k-divisibility$. For now, all there is is the following very loose bound.

Lemma 3.3.13. *Each set, X, of $\lambda(d,s,k) \leq (k+1)((s-1)(d-1)+1)$ points in $\mathbb{R}^d$ can be divided into s pairwise disjoint sets $A_1, A_2, ..., A_s$ such that $\bigcap_{i=1}^{s} conv(A_i \backslash \{x_1, x_2, ..., x_k\}) \neq \emptyset$, for all subsets $\{x_1, x_2, ..., x_k\} \subset X$.*

The lemma above is a direct consequence of proposition 1.3.2, Tverberg's theorem.

To end this chapter, it is pertinent to point out that the problems solved in the next chapter were originally derived from variations of **Q6**. But, due to the fact that they are approached in a completely different manner, they were deemed to deserve their very own chapter.

Chapter 4

A Related Problem on Partitions

In the previous chapter a version of the generalized McMullen's problem, when the study is restricted to the case of partitions into two sets, was proved. In this section that same problem is studied, but in an asymptotic and dual manner:

Q 7. *Let X be a set of n points in general position in $\mathbb{R}^d$, then what is the minimum k such that for all A, B partitions of X there is always a set $\{x_1, \ldots, x_k\} \subset X$, such that*

$$conv(A \backslash \{x_1, \ldots x_k\}) \cap conv(B \backslash \{x_1, \ldots x_k\}) = \emptyset ?$$

The focus now is in the conditions which provoke *non-divisibility* rather than *divisibility*, as presented in chapter 3. The answer to this question will be exposed in two parts. First, it will be proved that for all configurations of points in the plane there are partitions for which k has to be roughly half n. Secondly, a configuration of points will be inductively constructed for every dimension, such that any set which separates the convex hull of the remaining partitions, has cardinality k, roughly equal to $\frac{n}{2}$.

4.1 Planar Case

In general, if one takes a set of points in convex position and colours them with two different colours, it is obvious that removing the points with the least frequent color is sufficient to separate the convex hulls of what remains. This is, k is less or equal than $\frac{n}{2}$, for all configurations and all partitions. The next lemma proves that for points in convex position this bound is almost sharp.

Lemma 4.1.1. *Let X be t points in convex position in $\mathbb{R}^2$. Divide X into two sets A, B alternately. If one removes s points $x_1, x_2, ..., x_s$ so that*

$$conv(A\backslash\{x_1, x_2, ..., x_s\}) \cap conv(B\backslash\{x_1, x_2, ..., x_s\}) = \emptyset$$

$$\textit{then } s \geq \lfloor \tfrac{t}{2} \rfloor - 1.$$

Proof. Remove a set $Y \subset X$ so that $conv(A\backslash Y) \cap conv(B\backslash Y) = \emptyset$. If $|A\backslash Y| \leq 1$ or $|B\backslash Y| \leq 1$, then, since $|A| \geq \lfloor \frac{t}{2} \rfloor$ and $|B| \geq \lfloor \frac{t}{2} \rfloor$, $|Y| \geq \lfloor \frac{t}{2} \rfloor - 1$. Otherwise $A^* = A\backslash Y$ and $B^* = B\backslash Y$ each contain at least two points.

The line segment joining any two members of A^* cannot meet the line segment joining any two members of B^*. Consequently A^* lies entirely between some consecutive pair of points of B^*, and B^* lies entirely between some consecutive pair of points of A^*.

Let $|A^*| = \alpha$ and $|B^*| = \beta$. Then, between any two consecutive members of A^*, at least one $b \in B$ lies in X. So $|B\backslash B^*| \geq \alpha - 1$. Similarly $|A\backslash A^*| \geq \beta - 1$.

The equation $\alpha + \beta + |B\backslash B^*| + |A\backslash A^*| = t$ holds, so $\alpha + \beta \leq \frac{t}{2} + 1$, and $|B\backslash B^*| + |A\backslash A^*| \geq \frac{t}{2} - 1$, therefore $|Y| \geq \lfloor \frac{t}{2} \rfloor - 1$. □

In the case where X is a set of points in general position, the previous lemma has little application. Nevertheless, if given a set X, one could split it into several sets in convex position for any partition, the result could be applied by parts to each one of the convex sets. The result that guarantees such splitting is possible, is the well known Erdös-Szekeres theorem, stated in proposition 1.3.3.

Together, the previous lemma and proposition 1.3.3 constitute the core of the proof of the main theorem in this section:

Theorem 4.1.2. *In $\mathbb{R}^2$ let X be a subset of n points in general position. Let $\mu(X)$ be the smallest k such that for all partitions A, B of X,*

$$conv(A \backslash \{x_1, x_2, ..., x_k\}) \cap conv(B \backslash \{x_1, x_2, ..., x_k\}) = \emptyset,$$

for some $\{x_1, x_2, ..., x_k\} \subset X$, and let

$$\mu(n) = \min_{\{X \subset \mathbb{R}^2 | |X| = n\}} \mu(X)$$

then

$$\lim_{n \to \infty} \frac{\mu(n)}{n} = \frac{1}{2}.$$

Proof. Consider a large set X of points in general position in the plane. By lemma 1.3.3 one can remove from X successively subsets $X_1, X_2, ..., X_r$, each comprising t points in convex position, until $|X \backslash X_1 \backslash X_2 \backslash ... \backslash X_r| < N(t)$, where N(t) is as required by 1.3.3.

Suppose that for a partition A, B of X, which alternates vertices of X_i for $i = 1, \ldots, r$, there exists a subset Y with $conv(A \backslash Y) \cap conv(B \backslash Y) = \emptyset$. Then from Lemma 4.1.1, $|Y \cap X_j| \geq \frac{t}{2} - 1$. So

$$|Y| \geq |Y \cap X_1| + ... + |Y \cap X_r| \geq \frac{r}{2}t - r,$$

and as $rt = |X_1| + ... + |X_r| > |X| - N(t)$, $|Y| > (|X| - N(t)) \left(\frac{1}{2} - \frac{1}{t}\right)$. Hence, $\mu(n) > (n - N(t)) \left(\frac{1}{2} - \frac{1}{t}\right)$.

On the other hand, at least one of A and B has cardinality less or equal than $\frac{1}{2}|X|$ and so $\mu(n) < \frac{1}{2}n$, consequently

$$\lim_{n \to \infty} \frac{\mu(n)}{n} = \frac{1}{2}.$$

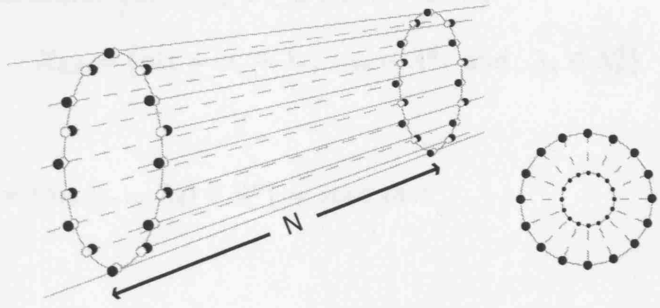

Figure 4.1: In the figure on the left, two cycles of points are placed in a cylinder and perturbed without leaving the walls of the cylinder. The figure on the right shows that from the point of view of each cycle, the other looks planar.

□

4.2 General Construction

This section *lifts* the idea presented in the previous section, in order to prove that in any dimension there is a set with many points and a partition which requires the removal of roughly half of the points to make it *non-divisible*. However intricate the following construction and proofs may seem, behind them lies a very simple idea.

From the previous theorem it has been observed that cycles in the plane with a large number of points need at least half of them to be removed, in order to separate the convex hulls of any of their partitions. Based on that, several copies of a cycle are placed in a cylindric arrangement in dimension 3, and perturbed slightly in order to achieve general position; so that from any single point, the points in a different cycle seem to be in a plane (as in figure 4.1). The proof continues by reproducing this idea in further dimensions by lifting the cylindric arrangement in a similar way.

Definition 4.2.1. *For every $d \geq 3$ and $k \geq 10$ let*

$$X_{d,k} = \{x | x = \alpha_x + \lambda_x, \quad \alpha_x \in A^d \quad and \quad \lambda_x \in \Lambda_k^d\}$$

where,

$\Lambda_k^d = \{\lambda_x = (\lambda_1, \lambda_2, ..., \lambda_d) \in \mathbb{R}^d\}$ *is such that:*

$\lambda_1 = 0$
$\lambda_2 = 0$
$\lambda_h \in \{(l-1)N | 1 \leq l \leq k\}$ *for* $3 \leq h \leq d$.

$A^d = \{\alpha_x = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{R}^d\}$ *is such that:*

$\alpha_1 = \cos\left(\frac{2\pi}{M} s_x\right) + (-1)^{s_x} t_x$
$\alpha_2 = \sin\left(\frac{2\pi}{M} s_x\right) + (-1)^{s_x} {t_x}^2$
$\alpha_h = (-1)^{s_x} {t_x}^h$ *for* $3 \leq h \leq d$
with $s_x \in \{1, \ldots, M\}$ $\left|(t_x, t_x^2, ..., t_x^d)\right| < \epsilon$ *for all* x, $t_x \neq t_{x'}$ *if* $x \neq x'$, N *is large,* $M = 2k$ *and one of the following two restrictions hold:*
(i) $\lambda_x \neq \lambda_y$ *or*
(ii) $\lambda_x = \lambda_y$ *but* $s_x \neq s_y$.

As sketched in figure 4.2, Λ_k^d is a lattice where the *almost* two dimensional cycles in A^d are hung.

By construction, all the vertices of $X_{d,k}$ are in general position. Also observe that $|X_{d,k}| = k^{d-2}M$ for all k, $\Lambda_{d,k-1} \subset \Lambda_{d,k}$, and that the projection into $x_d = c$ of $\Lambda d, k$ has the same divisibility properties as $\Lambda_{d-1,k}$.

For each $X_{d,k}$ choose the following balanced partition in to two sets:

$$A_{d,k} = \{x \in X_{d,k} | s_x = 2i - 1\} \text{ and } B_{d,k} = \{x \in X_{d,k} | s_x = 2i\}.$$

Note $|A_{d,k}| = \frac{k^{d-2}M}{2} = |B_{d,k}|$. Define for k fixed,

$$Y_{d,k,i} = \{x \in X_{d,k} | \lambda_x \cdot e_d = (i-1)N, \text{ where } e_d = (0, 0, \ldots, 1)\}.$$

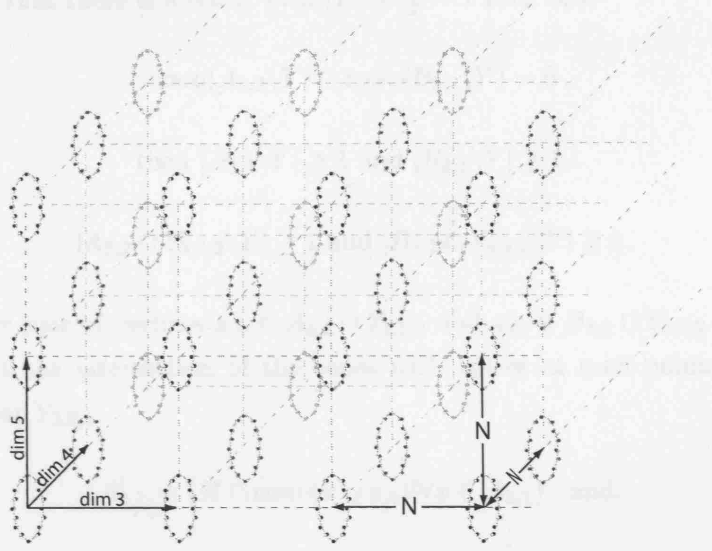

Figure 4.2: A corner of $X_{5,k}$ *collapsed* to dimension 3.

Each of the $Y_{d,k,i}$ has the same divisibility properties as $X_{d-1,k}$ and $|Y_{d,k,i}| = |X_{d-1,k}|$. The set and its partitions, as defined above, are used to prove the following theorem:

Theorem 4.2.2. *For all d, there exists a family of configurations of points, in general position, $\mathscr{X} = \{X_{d,k}\}_{k\in\mathbb{N}} \in \mathbb{R}^d$, and there is partition, $A_{d,k}, B_{d,k}$, for each $X_{d,k}$, such that if $\mu(X_{d,k})$ is the smallest cardinality of a set $X \subset X_{d,k}$ with the following property:*

$$conv(A_{d,k}\backslash X) \cap conv(B_{d,k}\backslash X) = \emptyset$$

then

$$\lim_{k\to\infty} \frac{\mu(X_{d,k})}{|X_{d,k}|} = \frac{1}{2}.$$

Proof. It is trivial to see that $\mu(X_{3,1}) = 0$. Take $X_{3,2}$, there are planes H_1, H_2 and H_3 parallel to the plane $x_3 = 0$ such that $H_1{}^- \cap X_{3,2} \subset A_{3,2}$, $H_1{}^+ \cap H_2{}^- \cap X_{3,2} \subset B_{3,2}$, $H_2{}^+ \cap H_3{}^- \cap X_{3,2} \subset A_{3,2}$ and $H_3{}^+ \cap X_{3,2} \subset B_{3,2}$. Therefore $\mu(X_{3,2}) \leq \frac{M}{2}$.

Suppose that there is a set Y with $|Y| \leq \frac{M}{2} - 1$ such that

$$conv(A_{3,2} \backslash Y) \cap conv(B_{3,2} \backslash Y) = \emptyset$$

$$\text{then } |A_{3,1} \backslash Y| \geq 1 \text{ and } |B_{3,1} \backslash Y| \geq 1,$$

$$|A_{3,2} \cap Y_{3,2,2} \backslash Y| \geq 1 \text{ and } |B_{3,2} \cap Y_{3,2,2} \backslash Y| \geq 1.$$

For every pair of vertices $x_A \in A_{3,2} \cap Y_{3,2,2}$ and $x_B \in B_{3,2} \cap Y_{3,2,2}$ one can construct the intersection of the cones with apices at such points, and a plane near $Y_{3,2,1}$:

$$A'_{3,1} = \{H \cap conv(x \cup x_A) | \forall x \in A_{3,1}\} \quad \text{and}$$

$$B'_{3,1} = \{H \cap conv(x \cup x_B) | \forall x \in B_{3,1}\}$$

where H is the plane defined by $x_3 = 1$, which is between $Y_{3,2}$ and $Y_{3,1}$.

The rays between x_A and any vertex $x \in A_{3,1}$ have parametrization:

$$x(t) = [\cos((\frac{2\pi}{M})s_x) + (-1)^{s_x} t_x](1-t) + t[\cos(\frac{2\pi}{M})s_{x_A}) + (-1)^{s_{x_A}} t_{x_A}]$$

$$y(t) = [\sin((\frac{2\pi}{M})s_x) + (-1)^{s_x} t_x^2](1-t) + t[\sin(\frac{2\pi}{M} s_{x_A}) + (-1)^{s_{x_A}} t_{x_A}^2]$$

$$z(t) = [(-1)^{s_x} t_x^3](1-t) + t\left[(-1)^{x_A} t_{x_A}^3 + N\right]$$

and the rays between x_B and a vertex $x \in B_{3,1}$ have parametrization:

$$x(t) = [\cos((\frac{2\pi}{M})s_x) + (-1)^{s_x} t_x](1-t) + t[\cos(\frac{2\pi}{M} s_{x_B}) + (-1)^{s_{x_B}} t_{x_B}$$

$$y(t) = [\sin((\frac{2\pi}{M})s_x) + (-1)^{s_x} t_x^2](1-t) + t \sin(\frac{2\pi}{M} s_{x_B}) + (-1)^{s_{x_B}} t_{x_B}^2$$

$$z(t) = [(-1)^{s_x} t_x^3](1-t) + t\left[(-1)^{x_B} t_{x_B}^3 + N\right]$$

Therefore, for every pair $x \in A_{3,1}$, x_A (or $x \in B_{3,1}$, x_B) there is a t such that $x' = x + tx_A$ (or $x' = x + tx_B$) and $x' \in A'_{3,1}$ (or $B'_{3,1}$).

As N is large, every x' is almost an orthogonal projection image of x into the plane $x_3 = 1$. Hence $X'_{3,1} = A'_{3,1} \cup B'_{3,1}$ is a convex set in the plane $x_3 = 1$. Also notice

$$\emptyset \neq conv(A'_{3,1}) \cap conv(B'_{3,1}) \subset conv(A_{3,2}) \cap conv(B_{3,2}) \cap H \tag{4.1}$$

Symmetrically, if one takes points $x_A \in A_{3,1}$ and $x_B \in B_{3,1}$ and produces sets $A'_{3,2}$, $B'_{3,2}$ using a plane near $Y_{3,2}$ and the partition of its points, as before:

$$\emptyset \neq conv(A'_{3,2}) \cap conv(B'_{3,2}) \subset conv(A_{3,2}) \cap conv(B_{3,2}) \cap H$$

Then, by lemma 4.1.1, $conv(A'_{3,2} \backslash Y') \cap conv(B'_{3,2} \backslash Y') \neq \emptyset$ for some $Y' \subset X'$, with $X' = A'_{3,2} \cup B'_{3,2}$, iff $|Y'| \geq \frac{M}{2} - 1$. This implies that if there are still vertices $x_A \in A_{3,2} \cap (Y_{3,2,2} \backslash Y)$ and $x_B \in B_{3,2} \cap (Y_{3,2,2} \backslash Y)$, and $Y_{3,2,1} \cap Y = \emptyset$ then, by equation 4.1, $|Y| \geq \frac{M}{2}$. Consequently, $\mu(X_{3,2}) \geq \frac{M}{2} - 1$ and $\frac{\mu(X_{3,2})}{X_{3,2}} = \frac{1}{4}$.

It easy to see that $\mu(X_{3,k}) \leq \frac{M}{2}(k-1)$ for all k. Suppose $\exists Y \subset X_{3,k}$ such that:

$$conv(A_{3,k} \backslash Y) \cap conv(B_{3,k} \backslash Y) = \emptyset$$

and $|Y| < (k-1)\frac{M}{2}$. Then there is an i such that the $|Y_{3,k,i} \backslash Y| \geq \frac{M}{2} + 1$, this implies that there are two points

$$x_A \in (Y_{3,k,i} \backslash Y \cap A_{3,k}) \quad and \quad x_B \in (Y_{3,k,i} \backslash Y) \cap B_{3,k}. \tag{4.2}$$

So, if one defines for all $j \neq i$ a special *partition cone* with apices in such points :

$$A'_{3,j} = \{H_j \cap conv(x \cup x_A) | \forall x \in A_{3,k} \cap Y_{3,k,j}\}$$

and

$$B'_{3,j} = \{H_j \cap conv(x \cup x_B) | \forall x \in B_{3,k} \cap Y_{3,k,j}\},$$

where H_j is the plane near $Y_{3,j}$, defined by $x_3 = (j-1)N + (-1)^{sgn(i-j)}$, then the intersection of the convex hulls of the sets defined above and the intersection of the convex hulls of the original partition are such that:

$$\emptyset \neq conv(A'_{3,j}) \cap conv(B'_{3,j}) \subset conv(A_{3,k}) \cap conv(B_{3,k}) \cap H_j. \tag{4.3}$$

Once again, by lemma 4.1.1 $conv(A'_{3,j} \backslash Y') \cap conv(B'_{3,j} \backslash Y') = \emptyset$ for some $Y' \subset X'$, with $X' = A'_{3,j} \cup B'_{3,j}$, iff $|Y'| \geq \frac{M}{2}$. But this has to hold for every $j \neq i$, so $|Y \cap Y_{3,j}| \geq \frac{M}{2}$ and 4.3 implies $|Y| \geq k(\frac{M}{2})$. This proves the theorem for dimension 3.

In order to prove the theorem for dimension $d \geq 4$, proceed by induction.

Suppose that $\forall d < d'$, $\mu(X_{d,k}) \geq k^{d-2}(\frac{M}{2})$ has been proved. Let $d' = d$, and suppose $\exists Y$ such that

$$conv(A_{d,k} \backslash Y) \cap conv(B_{d,k} \backslash Y) = \emptyset$$

and $|Y| < k^{d-2}(\frac{M}{2} - 1)$. Therefore there is $Y_{d,k,i}$ such that $|Y_{d,k,i} \cap Y| < k^{d-3}(\frac{M}{2} - 1)$.

Then, there exist vertices

$$x_A \in (Y_{d,k,i} \backslash Y) \cap A_{d,k} \quad and \quad x_B \in (Y_{d,k,i} \backslash Y) \cap B_{d,k}.$$

For every $i \neq j$ consider,

$$A'_{d,j} = \{H_j \cap conv(x \cup x_A) | x \in A_{d,k} \cap Y_{d,k,j}\}$$

and

$$B'_{d,j} = \{H_j \cap conv(x \cup x_B) | x \in B_{d,k} \cap Y_{d,k,j}\},$$

where H_j is the plane defined by $x_d = (j-1)N + (-1)^{sgn(i-j)}$. Then

$$\emptyset \neq conv(A'_{d,j}) \cap conv(B'_{d,j}) \subset conv(A_{d,k}) \cap conv(B_{d,k}) \cap H.$$

By induction on the $d-1$ dimensional set, $X'_{d,j} = A'_{d,j} \cup B'_{d,j}$ which is of type $X_{d-1,k}$, it is known that if there is $Y' \subset X'_{d,j}$ such that

$$conv(A'_{d,j} \backslash Y') \cap conv(B'_{d,j} \backslash Y') \neq \emptyset$$

then, $|Y'| \geq k^{d-3}(\frac{M}{2})$. Therefore $|Y \cap Y_{d,j}| \geq k^{d-3}(\frac{M}{2})$ for all j. It follows $|Y| \geq k^{d-2}(\frac{M}{2})$.

So, it has been proved that $\forall d \geq 3$ and $\forall k \geq 1$,

$$k^{d-2}(\frac{M}{2} - 1) \leq \mu(X_{d,k}) \leq k^{d-2}\frac{M}{2}.$$

Hence, the result follows:

$$\lim_{k \to \infty} \frac{\mu(X_{d,k})}{X_{d,k}} = \frac{1}{2}.$$

□

By the previous lemma, the next theorem holds:

Theorem 4.2.3. *In $\mathbb{R}^d$, let X be a subset of n points in general position. Let $\mu(X)$ be the smallest k such that for all partitions A, B of X, there is a set $\{x_1, \ldots, x_k\} \subset X$ such that*

$$conv(A \backslash \{x_1, x_2, ..., x_k\}) \cap conv(B \backslash \{x_1, x_2, ..., x_k\}) = \emptyset,$$

and

$$\mu(n) = \max_{\{X \subset \mathbb{R}^d \mid |X|=n\}} \mu(X)$$

then

$$\lim_{n \to \infty} \frac{\mu(n)}{n} = \frac{1}{2}.$$

This is, in the case of partitions in to two sets, in the presence of a large number of points, there are always configurations and partitions of them *entangled* enough to require the removal of roughly half of the points, in order to *untangle* the convex hulls of the partititions.

In order to prove a theorem similar to 4.1.2 for higher dimensions, not only do we need a generalization of Erdós-Szekeres' result which finds convex polytopes in higher dimensions, but also a partition of the points of the polytopes, such that the convex hulls are entangled enough to require the removal of about half of the points to separate them.

Also, recalling lemma 3.3.1, we know that for a set of points in general position ,X, such that $|X| = n \leq (k+1)d + (k+2)$, X is $k-$*divisible*. Thus for all $d \geq 3$,

$$\frac{|X|-1}{d+1} - 1 \leq \mu(X) \leq \frac{1}{2}|X|.$$

Then if $\mu(n) = \min_{\{X \subset \mathbb{R}^d \mid |X|=n\}} \mu(X)$,

$$\frac{1}{d+1} \leq \lim_{n \to \infty} \frac{\mu(n)}{n} \leq \frac{1}{2}.$$

In the case where $d = 2$ the upper bound is sharp, as proved in this chapter. For $d \geq 3$, there exists no evidence which points towards any conjecture about the sharpness of the upper or lower bound, except for $\frac{1}{d} = \frac{1}{2}$ if d equals two.

To conclude this chapter, it seems natural to conjecture on a generalization of the above. Namely, that if $\mu(X, s)$ is the smallest number such that for all partitions of X into s sets, $A_1, \ldots, A_s$, there is always a set of $\mu(X, s)$

points, $X_\mu \subset X$ such that $\bigcap_{i=1}^{s} conv(A_i \backslash X_\mu) = \emptyset$ and

$$\mu(n,s) = \max_{\{X \subset \mathbb{R}^d \,||X|=n\}} \mu(X)$$

then

$$\lim_{n\to\infty} \frac{\mu(n,s)}{n} = \frac{1}{s}.$$

More questions than answers, it seems. The next chapter follows the same inquisitive fashion of the previous two, except this time, in a completely unrelated topic. But also, as the title of the thesis indicates,it is a geometrical problem where partitions are relevant.

Chapter 5

Polytopes with Many Pairs of Facets

In 1999 B. von Stengel proposed, in his paper about the maximal number of Nash equilibria in $d \times d$ bimatrix games [17], the following problem:

Q 8. *Consider a polytope $\mathcal{P}$ in dimension d with $2d$ facets which is simple. Two vertices form a comlementary pair, (x, y), if every facet of $\mathcal{P}$ is incident with x or y. The $d-cube$ has 2^{d-1} complementary vertex pairs. Is this the maximal number among the simple $d-polytopes$ with $2d$ facets?*

He did not answer the question above, instead he found a *more* general question that would somehow answer his problem, and solved it instead. However, he accurately noticed that in the dual setting Q8 can be rephrased as,

Q 9. *Let $\mathcal{P}$ be a $d-dimensional$ simplicial polytope with $2d$ vertices. Two facets form a complementary pair, (F_1, F_2), if every vertex of $\mathcal{P}$ is incident with F_1 or F_2. The d $cross-polytope$ has 2^{d-1} complementary facet pairs. Is this the maximal number among all the simplicial $d-polytopes$ with $2d$ vertices?*

Considering simplicial polytopes and pairs of facets instead of pairs of ver-

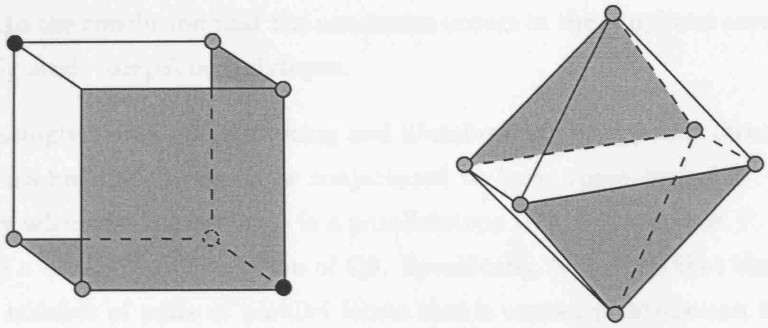

Figure 5.1: The two black vertices, on the image on the left, cover all the faces of a cube. The two shaded faces of the octahedron, on the right, cover all its vertices.

tices, no cyclic polytope has more than 2^{d-1} pairs of *complementary facets.* Figure 5.1 shows complementary pairs for the cube, and the octahedron.

Nonetheless, von Stengel's problem has proved fruitful, not only because it has contributed with a new question in convex polytopes, but also because it has inspired research in a slightly different but nevertheless close direction.

In 1999, Bremner and Klee published a paper which is concerned with pairs of vertices that form *inner diagonals.* [4] An inner diagonal of a polytope $\mathcal{P}$ is a segment whose extremes are two vertices of $\mathcal{P}$ and that lies in the relative interior of the polytope.

Given that complementary pairs of vertices in the cube form inner diagonals, they seem to be a natural way of finding complementary pairs in simplicial polytopes.

Bremner and Klee's paper focuses on finding the maximum number of inner diagonals for simplicial and simple polytopes with a fixed number of vertices or facets. Their findings, unfortunately, have no helpful repercussion in the solving of Q8. The latter is mainly because the highest numbers of inner diagonals, for 3 dimensional polytopes, occur when the polytopes have more than $2d$ vertices: and in the case when the number of facets is fixed, as they arrive to the conclusion that the maximum occurs in the simplicial case, they do only study simplicial polytopes.

Interestingly, many other covering and illumination (by opposite directions) problems have been proven or conjectured to have upper bound 2^d, sharp exactly when the convex body is a parallelotope [11]. For instance, P. Erdös proved a *non-projective* version of Q9. Specifically, he proved that the maximum number of pairs of parallel facets that a convex polytope can have is precisely 2^{d-1}. [3]

Another connection is found linking covering pairs of vertices to affine antipodals. A pair of points a, b in a set $X \subset \mathbb{R}^d$ is called (affinely) *antipodal* provided there are distinct parallel hyerplanes H_a and H_b through a and b, respectively, such that X lies in a slab between them. Moreover, the pair $\{a, b\}$ is called *strictly antipodal* if $X \cap H_a = \{a\}$ and $X \cap H_b = \{a\}$. It is easy to prove that if X is the vertex set of a polytope then a covering pair is also a strict antipodal of X.

Let $A_d(X_n)$ be the number of antipodal pairs in a set $X_n \subset \mathbb{R}^d$, and

$$A_d(n) = \max_{X_n \subset \mathbb{R}^d} \{A_d(X_n)\}.$$

Clearly $A_d(n)$ is an upper bound over the maximum number of covering pairs of vertices.

In 1963, Grünbaum posed the problem on the upper bound for $A_d(V_n)$, where V_n is the vertex set of a convex $d - polytope$, and proved $A_2(V_n) = \lfloor \frac{3n}{2} \rfloor$. Also, if

$$V_d(n) = \max_{V_n \subset \mathbb{R}^d} \{A_d(V_n)\},$$

Makai and Martini proved that $\lfloor \frac{n^2}{3} \rfloor \leq V_3(n) \leq \frac{7n^2}{16}$ and for all $d \geq 4$, $(1 - \frac{1}{3 \times 2^{d-3}})\frac{n^2}{2} - O(1) \leq V_d(n) \leq (1 - \frac{1}{2^d})\frac{n^2}{2}$. [12]

Although $V_d(n)$ is even better than $A_d(n)$ as an upper bound for the number of covering pairs of vertices, in all cases, it is very far off von Stengel's conjectured bound.

There seems to be nothing else in the literature resembling even an attempt to directly solve this problem. Therefore, starting from zero, this chapter describes all the progress made up to date. Q8 is answered in the affirmative, up to dimension seven, using just basic geometric tools.

But first, some definitions and preliminary results are needed.

5.1 Preliminaries

If F_1, F_2 is a covering pair of facets of a $d-$*dimensional* simplicial polytope, $\mathcal{P}$, then $V(F_1) \cap V(F_2)$ might be empty or nonempty. If $V(F_1) \cap V(F_2) = \emptyset$ then $|V(\mathcal{P})| = 2d$. This is the case which will be considered in this chapter. So, for convenience, most of the definitions in this section make full advantage of this fact. However, most of them can be adjusted to accommodate the other case.

Also, as $\mathcal{P}$ is a simplicial polytope, it might be assumed that its set of vertices is in general position. This will not affect its face structure.

Definition 5.1.1. *A* ***balanced partition*** *of a set X is a partition (A, B) of X such that $A \cap B = \emptyset$, $A \cup B = X$ and $A = \left\lceil \frac{|X|}{2} \right\rceil$, $B = \left\lfloor \frac{|X|}{2} \right\rfloor$.*

Definition 5.1.2. *Let $\mathcal{P}$ be a d-dimensional polytope. Let $\mathcal{F}(\mathcal{P}) = \mathcal{F}$ be its set of facets. A pair of facets $F_1, F_2 \in \mathcal{F}$ is a* ***complemetary pair of facets*** *if its respective sets of vertices, $V(F_1)$ and $V(F_2)$, are a balanced partition of $V(\mathcal{P})$. Denote the number of complementary pairs of facets of $\mathcal{P}$ as $\nu(\mathcal{P})$.*

So Q9 can be written in the following shortened form:

Q 10. *Let $\mathcal{S}^d$ be the set of d-dimensional simplicial polytopes. What is the value of*

$$\nu(d) = \max_{\mathcal{P} \in \mathcal{S}^d} \nu(\mathcal{P})?$$

Back in the case where rather than pairs of facets one is interested in pairs of vertices, one can define:

Definition 5.1.3. *Let $\mathcal{P}$ be a d-dimensional polytope. Let $\mathcal{F}(\mathcal{P}) = \mathcal{F}$ be its set of facets. A pair of vertices $v_1, v_2 \in V(\mathcal{P})$ is a* ***complementary pair of vertices*** *if $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}$, where $\mathcal{F}_i = \{F \in \mathcal{F} | v_i \in V(F)\}$, the set of facets incident to v_i, for $i = 1, 2$. Denote the number of complementary pairs of vertices of $\mathcal{P}$ by $\lambda(\mathcal{P})$.*

Q8 can be written in the following way:

Q 11. *Let $\mathcal{T}^d$ be the set of d-dimensional simple polytopes. What is the value of*

$$\lambda(d) = \max_{\mathcal{P} \in \mathcal{T}^d} \lambda(\mathcal{P})?$$

Now, preparing for a further equivalence, suppose X is a Gale diagram of a simplicial polytope $\mathcal{P}$ with $2d + 2$ vertices. It is known that $0 \in conv(X)$, and X has the property that for every H hyperplane through the origin both $|H^+ \cap X| \geq 2$ and $|H^- \cap X| \geq 2$. It is also true that a set A, $A \subset X$, represents the vertices of a face of the polytope if and only if $0 \in relint\, conv(X \backslash A)$.

If $\mathcal{P}$ is simplicial and in general position in $\mathbb{R}^d$, then X is in linear general position in $\mathbb{R}^{|V(\mathcal{P})|-d-1}$. Therefore, no subset of X with less than $|V(\mathcal{P})| - d$ points contains the origin in its convex hull. Hence, if (A, B) is a partition of X such that both A and B represent facets of a simplicial $d - polytope$ with $2d$ vertices, necessarily (A, B) is a balanced partition of X.

Definition 5.1.4. *A balanced partition of a Gale transform of X, (A, B), such that $0 \in relint\, conv(A)$ and $0 \in relint\, conv(B)$ will be referred to as an* ***embracing partition****.*

If $\mathcal{P}$ is a simple polytope and X is its Gale diagram then a covering pair of vertices $\{v_1, v_2\}$ such that $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$, do not form an edge. Therefore in X, there is a hyperplane which separates them from the rest of the points in the diagram.

Definition 5.1.5. *A pair of vertices of a Gale transform of X, $\{a, b\}$, such that there is a hyperplane through the origin, $H_{a,b}$, that separates $\{a, b\}$ from $X \backslash \{a, b\}$ will be referred to as an* ***isolated pair****.*

So, as complementary pairs of facets of a polytope $\mathcal{P}$ correspond to embracing partitions of its Gale diagram, X, in this context, question 9 can be reformulated as:

Q 12. *Let X be a set of $2d+2$ points in general position in the $(d-1)$-unit sphere such that for every H, hyperplane through the origin, both $|H^+ \cap X| \geq 2$ and $|H^- \cap X| \geq 2$. What is the maximum number of embracing partitions, (A, B), that X can have?*

or, using Q 8 and a Gale transform,

Q 13. *Let X be a set of points in the $(d-1)$-unit sphere such that for every H, hyperplane through the origin, both $|H^+ \cap X| \geq 2$ and $|H^- \cap X| \geq 2$ and, given any $x \in X$, there are at most d other points, $\{x_1, \ldots, x_d\}$, such that $0 \in relint\ conv(X \backslash \{x, x_i\})$, for all $i = 1, \ldots, d$. What is the maximum number of isolated pairs $\{a, b\}$ that X can have?*

The setting in Q12 will be the one used to study the problem. With such a purpose, the following definition is introduced.

Definition 5.1.6. *Let X be a set of points in S^{d-1}, such that $|X| > d$. Then for every $A \subset X$ such that $|A| \leq d$, define the* ***spherical hull*** *as follows:*

$$sph(A) := relint\ cone^+(\{0\} \cup A) \cap S^{d-1}$$

A set of A of $d+1$ points in a sphere centred at the origin contains it in the interior of its convex hull if and only if for every $x \in A$, $\overline{x} \in sph(A \backslash \{x\})$. Here $\overline{x}$ represents the *antipodal point* to x in the sphere. Similarly $\overline{A}$ represents the set of antipodal points to A.

Notice that if (A, B) is an embracing partition of X, for all $x \in A$ there is a unique point $y \in B$ such that $\overline{x} \in sph(B \backslash y)$.

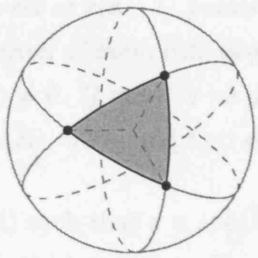

Figure 5.2: The shaded area is the spherical hull of three points on a 2-dimensional sphere.

To finish this section let us analyze what the answer to Q 8 is in low dimensions.

All polygons are simplicial and, in this case, $2d = 4$. Therefore, only quadrilaterals can have a pair of covering facets, which implies $\nu(2) = 2^1$. Hence, the conjecture holds for d=2.

Lemma 5.1.7. *Let $\mathcal{P}$ be a d-dimensional polytope then $\nu(d) \leq 2^{d-1}$, for $d = 3, 4$.*

Proof. If d=3, and X is a Gale diagram of $\mathcal{P}$, then X is a set of six points in the unit circle. By 1.3.4 the maximum number of simplices which contain the origin, with vertices in X, is eight. Therefore there are at most four pairs of covering facets of $\mathcal{P}$.

If d=4 , one can proceed through two different routes. First, by scrutiny over Grünbaum's list [8] of all 4-dimensional simplicial polytopes, one can confirm that there is no simplicial polytope with more than 8 pairs of facets. Or, by a Radon-like argument which will be discussed in section 5.5.

□

To finish this section, a results about the geometry of spherical hulls of sets and their diametrically opposite sets, will be shown.

Lemma 5.1.8. *Let Y be a set of $2(d-1)$ points in linearly general position in S^{d-2}, let A, B, be a partition of them into two sets, such that $|A| = |B| = d-1$ and $sph(A) \cap sph(B) \neq \emptyset$. If there is no $y \in B$ such that $y \in sph(A)$ or $x \in A$ such that $x \in sph(B)$, then $sph(A) \cap sph(\overline{B}) = \emptyset$.*

Proof. Suppose $\exists z \in sph(A)$ such that $z \in sph(\overline{B})$, then $0 \in conv(\{z\} \cup B)$. Also, by hypothesis $\forall x \in A$, the hyperplane $H_x = aff(0 \cup A \backslash x)$ is such that $H_x^+ \cap B \neq \emptyset$ and $H_x^- \cap B \neq \emptyset$.

Fix $x \in A$ such that $aff^+(A \backslash x \cup 0) \cap sph(B)$ and consider the following cone, $C_{z,x} = aff^+(0 \cup conv(z \cup [aff^+(A \backslash x) \cap sph(B)]))$. Observe

$$conv(z \cup [aff^+(A \backslash x) \cap sph(B)]) \subset conv(\{z\} \cup B),$$

also $aff(conv(z \cup [aff^+(A \backslash x) \cap sph(B)]))$ separates the origin from some vertices in B, suppose y is one of those vertices. Then

$$aff(0 \cup y) \cap conv(z \cup [aff^+(A \backslash x) \cap sph(B)]) \neq \emptyset,$$

this implies $y \in C_{z,x}$. Therefore y can be expressed as a positive affine combination of $0, z$ and points in $aff^+(A \backslash x \cup 0)$, and $y \in aff^+(A \cup 0)$, a contradiction.

□

Lemma 5.1.9. *Let Y be a set of $2(d-1)$ points in linearly general position in S^{d-2}, let A, B be a partition of them into two sets such that $|A| = |B| = d-1$ and $sph(\overline{A}) \cap sph(\overline{B}) \neq \emptyset$. Then $sph(\overline{A}) \cap sph(B) \neq \emptyset$ if and only if there is $x \in A$ such that $\overline{x} \in sph(B)$ or $y \in B$ such that $y \in sph(\overline{A})$.*

Proof. The reverse implication is trivial.

Suppose $sph(\overline{A}) \cap sph(B) \neq \emptyset$ but there is no $x \in A$ such that $\overline{x} \in sph(B)$ or $y \in B$ such that $y \in sph(\overline{A})$.

Let $A' = A$ and $B' = \overline{B}$. Then, $sph(\overline{A'}) \cap sph(\overline{B'}) \neq \emptyset$ and there is no $x \in A'$

such that $\overline{x} \in sph(\overline{B'})$ or $y \in B'$ such that $\overline{y} \in sph(\overline{A'})$. Therefore by lemma 5.1.8 $sph(\overline{A'}) \cap sph(\overline{B'}) = \emptyset$, but this is $sph(\overline{A}) \cap sph(\overline{B}) = \emptyset$, and the lemma holds.

□

5.2 An almost Proof of the Conjecture

Before fully entering the general analysis, which will lead to the main theorem in this section, one might wonder which hypothesis are necessary to make the statement of Q8 easier to prove. The next (and only) lemma of this section deals with a specific additional condition. However, the proof of the lemma is very important in understanding the main issues to be faced when solving the general case.

Lemma 5.2.1. *Let $\mathcal{P}$ be a $d-$dimensional simplicial polytope such that all of its faces are part of a complementary pair, then $\mathcal{P}$ has at most 2^{d-1} pairs of complementary facets.*

Proof. Let X be a Gale diagram of the vertices of $\mathcal{P}$ and let

$$F(X) = \{S \subset X | 0 \in conv(S)\}.$$

Suppose (A, B) is an embracing partition then, by hypothesis, for each $x \in A$ there is a unique $y \in B$ such that $\overline{x} \in sph(B \backslash y)$, and $\overline{y} \in sph(A \backslash x)$. The latter is because if $x \cup B \backslash y$ is a cofacet then, $y \cup A \backslash x$ is also a cofacet. This is, $sph(\overline{A \backslash \{x\}}) \cap X = \{x, y\}$ and $sph(\overline{B \backslash \{y\}}) \cap X = \{x, y\}$. Hence, there are d pairs of points $\{x_i, y_i\}$ with $x_i \in A$ and $y_i \in B$ such that $X \cap sph(\overline{A \backslash \{x_i\}}) \cap sph(\overline{B \backslash \{y_i\}}) = \{x_i, y_i\}$ for all $i \in \{1, \ldots, d\}$.

Take a fixed pair $\{x_i, y_i\}$, as above, then

$$sph(A \backslash \{x_i\}) \cap sph(B \backslash \{y_i\}) \neq \emptyset$$

and

$$sph(\overline{A\backslash\{x_i\}}) \cap sph(\overline{B\backslash\{y_i\}}) \neq \emptyset.$$

Also, no $x \in X\backslash\{x_i, y_i\}$ is such that $x \in sph(\overline{A\backslash\{x_i\}})$ or $x \in sph(\overline{B\backslash\{y_i\}})$. Then there is a hyperplane through the origin which separates

$$A\backslash\{x_i\} \cup B\backslash\{y_i\} \quad \text{from} \quad \overline{A\backslash\{x_i\}} \cup \overline{B\backslash\{y_i\}}.$$

To prove the later suppose $sph(A\backslash\{x_i\}) \cap sph(\overline{B\backslash\{y_i\}}) \neq \emptyset$. By 5.1.9 there is either $x \in sph(A\backslash\{x_i\})$ such that $\overline{x} \in sph(\overline{B\backslash\{y_i\}})$ or $\overline{y} \in sph(\overline{B\backslash\{y_i\}})$ such that $y \in sph(A\backslash\{x_i\})$, both of which lead to a contradiction. Therefore

$$sph(A\backslash\{x_i\}) \cap sph(\overline{B\backslash\{y_i\}}) = \emptyset,$$

so there is a hyperplane H, through the origin, which separates $sph(A\backslash\{x_i\})$ form $sph(\overline{B\backslash\{y_i\}})$, and such a hyperplane separates

$$sph(\overline{A\backslash\{x_i\}}) \cup sph(\overline{B\backslash\{y_i\}}) \quad \text{from} \quad sph(A\backslash\{x_i\}) \cup sph(B\backslash\{y_i\}).$$

This implies that for all $i \in \{1, \ldots, d\}$ there is a hyperplane H_i that separates $\{x_i, y_i\}$ from $X\backslash\{x_i, y_i\}$.

So the vertices $\{x_i, y_i\}$, of the Gale diagram, never represent a face of the polytope, and for all embracing pairs(A, B) of X either $x_i \in A$ or $y_i \in A$. □

5.3 Non-2-neighbourly Polytopes

$2-$ *neighbourliness* is relevant because if there are pairs of vertices that are never contained in the same facet, then the members of an embracing pair contain either one of the vertices or the other, providing an induction argument. All of this will be clarified within this and the next sections.

Within this section, the set of polytopes studied is reduced to the set of simplicial polytopes which are not 2-neighbourly.

Lemma 5.3.1. *Let $\mathcal{X}_1^d$ be the set of Gale diagrams of not $2-$ neighbourly $(d+1)-$ dimensional, simplicial polytopes with $2d+2$ vertices. For any $X \in \mathcal{X}_1^d$ define $\rho(X,d)$ as the number of embracing partitions of X. Then*

$$\rho(1,d) = \max_{X \in \mathcal{X}_1^d} \rho(X,d) \leq 2 \times \rho(d-1).$$

Here $\rho(l) = \max_{X \in \mathcal{X}^l} \rho(X,l)$ and $\mathcal{X}^l$ is the set of Gale diagrams of simplicial, $(l+1)-$ polytopes with $2l+2$ vertices.

Proof. Let $X \in \mathcal{X}_1^d$. then there are two points $x, y \in X$ and a hyperplane H such that $\{x,y\} \subset H^+$ and $X \backslash \{x,y\} \subset H^-$. Thus, for every embracing partition (A,B) one can suppose w.l.o.g. $x \in A$ and $y \in B$. So that $x \in sph(\overline{A \backslash x})$ and $y \in sph(\overline{B \backslash y})$, this is because as X is the Gale Diagram of a simplicial polytope, A and B correspond to vertices of simplicial facets. So $A \backslash x$ and $B \backslash y$ represent subfacets of the polytope. This implies that there is $z \in B$ such that $z \neq x$ and $0 \in relint\ conv((A \backslash x) \cup z)$. But as both $A \backslash x$ and $B \backslash y$ are in the same side of H, $z = y$. Hence, necessarily, $y \in sph(\overline{A \backslash x})$ and $x \in sph(\overline{B \backslash y})$, so $\{x,y\} \in sph(\overline{A \backslash x}) \cap sph(\overline{B \backslash y})$ for all embracing partitions.

Let $z = \frac{1}{2}(x+y)$ then $z \in sph(\overline{A \backslash x}) \cap sph(\overline{B \backslash y})$ for all embracing partitions of X. Now consider a hyperplane H_0 in general position with respect to X such that $0 \in H_0^+$ and $X \backslash \{x,y\} \subset H_0^-$. Consider the central projection $p : \{\overline{z}\} \cup X \backslash \{x,y\} \to H_0$ such that $w \mapsto aff(w) \cap H_0$. Define

$$X' = \left\{ \frac{w - p(\overline{z})}{\|w - p(\overline{z})\|} \middle| w \in p(\{\overline{z}\} \cup X \backslash \{x,y\}) \right\}.$$

Then a balanced partition (A,B) of X is an embracing pair only if the partition $(A',B') = \left(\frac{w - p(\overline{A \backslash x})}{\|w - p(\overline{A \backslash x})\|}, \frac{w - p(\overline{B \backslash y})}{\|w - p(\overline{B \backslash y})\|} \right)$ of X' is an embracing pair, and the result follows. □

Instead of considering 2-neighbourly polytopes now, a way of untangling the conditions necessary for the conjecture to hold is by asking under what circumstances the polytope that we are dealing with is precisely non 2-neighbourly. The following two lemmas provide conditions on the Gale diagrams of simplicial polytopes which imply that the polytope is not 2-neighbourly.

Lemma 5.3.2. *Let X be the Gale diagram of a $d-$ dimensional simplicial polytope $\mathcal{P}$, with $2d$ vertices and at least one pair of complementary facets. Let (A, B) be an embracing pair of X, and suppose that there is $x \in A$ such that*

$$sph(\overline{A \backslash x}) \cap X = \{x\},$$

then $\mathcal{P}$ is not $2-$ neighbourly.

Proof. There is a unique point $y_x \in B$ such that $x \in sph(\overline{B \backslash y_x})$. Also for some $y \in B$ it is true that $sph(B \backslash y) \cap sph(\overline{A \backslash x}) = \emptyset$. Note that if this holds there is a hyperplane through the origin such that $sph(B \backslash y) \subset H^+$ and $sph(\overline{A \backslash x}) \subset H^-$, hence $sph(\overline{B \backslash y}) \subset H^-$ and $sph(A \backslash x) \subset H^+$. This is $X \cap H^- = \{x, y\}$, and the result follows.

By lemma 5.1.9, for $y \in B$, $sph(B \backslash y) \cap sph(\overline{A \backslash x}) \neq \emptyset$, occurs only if either of the following holds:

(a) there is $y' \in B \backslash y$ such that $y' \in sph(\overline{A \backslash x})$

(b) there is $x' \in A \backslash x$ such that $x' \in sph(\overline{B \backslash y})$.

By hypothesis, (a) is not possible. However (b) can always hold, maybe except for y_x. Once again, there are two instances.

First, suppose there is some $x' \in A \backslash x$ such that $x' \in sph(\overline{B \backslash y_x})$, this implies that there is $y_{x'} \in B \backslash y_x$ such that $X \cap sph(\overline{B \backslash y_{x'}}) = \{y_{x'}\}$. Then $sph(B \backslash y_{x'}) \cap sph(\overline{A \backslash x}) = \emptyset$.

Secondly, suppose $sph(\overline{B \backslash y_x}) \cap X = \{y_x, x\}$ and $sph(\overline{A \backslash x}) \cap X = \{x\}$. Then $sph(B \backslash y_x) \cap sph(\overline{A \backslash x}) = \emptyset$, and the result follows. □

Lemma 5.3.3. *Let X be the Gale diagram of a $d-$ dimensional simplicial polytope $\mathcal{P}$ with $2d$ vertices and at least one pair of complementary facets. Let (A, B) be an embracing pair in X, and suppose there are $x \in A$ and*

$y \in B$ such that

$$sph(\overline{A\backslash x}) \cap X = \{x, y\} \quad and \quad sph(\overline{B\backslash y}) \cap X = \{x, y\},$$

then $\mathcal{P}$ is not $2-$ neighbourly.

Proof. Given that $sph(\overline{A\backslash x}) \cap X = \{x, y\}$ and $sph(\overline{B\backslash y}) \cap X = \{x, y\}$, as in the previous lemma, by 5.1.8, $sph(B\backslash y) \cap sph(\overline{A\backslash x}) \neq \emptyset$, occurs only if either of the following theorem holds:

(a) there is $y' \in B\backslash y$ such that $y' \in sph(\overline{A\backslash x})$

(b) there is $x' \in A\backslash x$ such that $x' \in sph(\overline{B\backslash y})$.

By hypothesis, neither of the cases is possible.

Hence, $sph(\overline{A\backslash x}) \cap sph(B\backslash y) = \emptyset$. So, there is a hyperplane through the origin such that $sph(B\backslash y) \subset H^+$ and $sph(\overline{A\backslash x}) \subset H^-$, hence $sph(\overline{B\backslash y}) \subset H^-$ and $sph(A\backslash x) \subset H^+$. This is $X \cap H^- = \{x, y\}$. □

5.4 Main Results

As the title suggests, all the instruments needed for concluding the results contained in this section have been developed. So, without further ado, it is observed that the lemmas in the preceding section particularly imply that not all facets of a cyclic polytope can belong to a pair (which can also be inferred using Gale's evenness condition), and consequently the following theorem holds:

Theorem 5.4.1. *Let $\mathcal{S}_2^d$ be the set of $d-$dimensional at least $2-$neighbourly simplicial polytopes with $2d$ vertices, and let $\nu(\mathcal{P}, d)$ the number of complementary pairs of facets of $\mathcal{P}$. If $\mathcal{P} \in \mathcal{S}_2^d$ then*

$$\max_{\mathcal{P} \in \mathcal{S}_2^d} \nu(\mathcal{P}, d) \leq \frac{1}{4} |\mathcal{F}(C(2d, d))|.$$

Proof. If X is the Gale diagram of a polytope which is at least $2-neighbourly$ and has at least one pair of embracing pairs then the hypotheses of lemmas 5.3.2 and 5.3.3 must not hold. That is, if (A, B) is an embracing pair of X then for all $x \in A$ there is $y \in B$ such that $y \in sph(\overline{A\backslash x})$ but $x \notin sph(\overline{B\backslash y})$. So given x and y, as before, the partition $(y \cup A\backslash x, x \cup B\backslash y)$ is never an embracing pair. Hence none of the $2d$ $d-dimensional$ facets incident to the pair, represented by A, B, in the Gale transform, is part of a complementary pair. Then

$$\frac{2d}{d}\nu(\mathcal{P}, d) + 2\nu(\mathcal{P}, d) \leq |\mathcal{F}(\mathcal{P})|$$

holds for all $\mathcal{P} \in \mathcal{S}_2^d$. □

The above is,

$$\max_{\mathcal{P}\in\mathcal{S}_2^d} \nu(\mathcal{P}, d) \leq \begin{cases} \frac{1}{2}\binom{2d-\lfloor\frac{d}{2}\rfloor-1}{\lfloor\frac{d}{2}\rfloor} & \text{if } d \text{ is odd, and} \\ \frac{1}{4}\left[\binom{2d-\lfloor\frac{d}{2}\rfloor}{\lfloor\frac{d}{2}\rceil} + \binom{2d-\lfloor\frac{d}{2}\rfloor-1}{\lfloor\frac{d}{2}\rfloor-1}\right] & \text{if } d \text{ is even.} \end{cases}$$

Finally, the most anticipated theorem of this chapter.

Theorem 5.4.2. *Let $\mathcal{S}^d$ be the set of $d-dimensional$ simplicial polytopes with $2d$ vertices and $\nu(\mathcal{P}, d)$ is the number of complementary pairs of $\mathcal{P} \in \mathcal{S}^d$. Then*

$$\max_{\mathcal{P}\in\mathcal{S}^d} \nu(\mathcal{P}, d) \leq 2^{d-1}$$

for $d = 2, \ldots, 7$.

Proof. Trivially the theorem holds for $d = 2$, and up to $d = 3$, a simplicial polytope $\mathcal{P}$ can't be $2-neighbourly$. So by 5.3.1 and simple induction, the theorem holds for all simplicial polytopes up to dimension 4.

In the cases where d=5,6,7, if the polytope is at least $2-neighbourly$ the bounds given by theorem 5.4.1 are 10, 28 and 60, respectively. So if the polytope is not $2-neighbourly$ by 5.3.1 and using the bounds above it can be concluded that the theorem holds. □

5.5 A Coloured Radon-type problem

As announced in lemma 5.1.7, in this section another equivalence of Q8 will be introduced. In the proof of the lemma, a projection into a hyperplane through the origin is made. Such a projection produces a set of points where instead of finding embracing partitions, the aim is to find partitions with a coloured Radon sub-partition. That same idea is now reconsidered, to obtain a question on coloured Radon partitions.

Let $\mathcal{P}$ be a simplicial $(d+2)-$*polytope* with $2d+4$ vertices and X be its Gale diagram in $\mathbb{R}^{d+1}$. Then X can be assumed to be a set of points in general position. If H' is any hyperplane through the origin in *general position* with respect to X, then $|X \cap H'^{+}| \geq 2$.

Let H be a hyperplane parallel to H' such that $0 \in H^+$,

$X^+ = H^+ \cap X = X^+ = (H')^+ \cap X$ and $X^+ = H^+ \cap X = X^- = (H')^- \cap X$.

Now, a balanced partition A, B of X represents a covering pair of facets of the polytope if and only if $0 \in$ *relint conv*(A) and $0 \in$ *relint conv*(B). Then, necessarily $A \cap X^+ \neq \emptyset$, $A \cap X^- \neq \emptyset$, $B \cap X^+ \neq \emptyset$, and $B \cap X^- \neq \emptyset$.

Take $\overline{X}^-$, the set of vertices diametrically opposite to X^-, and let Y^+ be the *central* projection of X^+ into H and Y^- be the central projection of $\overline{X}^-$. Here for every point on the sphere, z, its central projection is the point $aff(z \cup \{0\}) \cap H$.

Then, a balanced partition of X, (A, B), is an embracing pair if and only if A and B induce coloured Radon partitions on $Y = Y^+ \cup Y^-$, in the following way: if (A_Y, B_Y) is the partition in Y induced by (A, B), then A_Y has sub-partition $(A_Y \cap Y^+, A_Y \cap Y^-)$ and B_Y has sub-partition $(B_Y \cap Y^+, B_Y \cap Y^-)$, where both $(A_Y \cap Y^+, A_Y \cap Y^-)$ and $(B_Y \cap Y^+, B_Y \cap Y^-)$ are Radon partitions of A_Y and B_Y, respectively.

This last observation motivates the following definition.

Definition 5.5.1. *Let X be a set of at least $2d+4$ points in R^d, with a colouring X^+ and X^- such that $|X^+| \geq 2$ and $|X^-| \geq 2$. A subset S of X, with $|S| = 2d+4$, has a* ***coloured balanced partition****, (A_S, B_S), if $(A_S \cap X^+, A_S \cap X^-)$ and $(B_S \cap X^+, B_S \cap X^-)$ are Radon partitions of A_S and B_S, respectively.*

Thus, Q12 can be reformulated as follows:

Q 14. *Let X be a set of $2d+4$ points in general position in $\mathbb{R}^d$ and let X^+ and X^- be a colouring of X such that $|X^+| \geq 2$ and $|X^-| \geq 2$, and suppose that for all $x \in X$ the set $X \backslash \{x\}$ contains a coloured Radon partition. What is the maximum number of different coloured balanced partitions (A, B) that X can have?*

Therefore, the coming corollaries are a consequence of theorems 5.4.1 and 5.4.2.

Corollary 5.5.2. *Let X be a set of $2d+4$ points in general position in $\mathbb{R}^d$, X^+ and X^- be a colouring of X such that $|X^+| \geq 3$ and $|X^-| \geq 3$ such that for all $x \in X$ the set $X \backslash \{x\}$ contains a coloured Radon partition. Let $\eta(X)$ be the maximum number of coloured balanced partitions in X. Then*

$$\eta(d) \leq \begin{cases} \frac{1}{2}\binom{2d-\lceil\frac{d}{2}\rceil+2}{\lceil\frac{d}{2}\rceil+1} & \text{if } d \text{ is odd, and} \\ \frac{1}{4}\left[\binom{2d-\lceil\frac{d}{2}\rceil+3}{\lceil\frac{d}{2}\rceil+1} + \binom{2d-\lceil\frac{d}{2}\rceil+2}{\lceil\frac{d}{2}\rceil}\right] & \text{if } d \text{ is even.} \end{cases}$$

where $\eta(d) = \max_{\{X \in \mathbb{R}^d \| X| = 2d+4|\}} \eta(X)$.

Corollary 5.5.3. *Let X be a set of $2d+4$ points in general position in $\mathbb{R}^d$, X^+ and X^- be a colouring of X such that $|X^+| \geq 2$ and $|X^-| \geq 2$, and suppose that for all $x \in X$ the set $X \backslash \{x\}$ contains a coloured Radon partition. Let $\eta(X)$ be the maximum number of coloured balanced partitions in X. Then*

$$\eta(d) \leq 2^{d+1},$$

where $\eta(d) = \max_{\{X \in \mathbb{R}^d \| X| = 2d+4|\}} \eta(X)$ *for* $d = 1, \ldots, 5$.

These corollaries shed some light into the study of numbers of Radon partitions of a given configuration of points. In the two-coloured case, the Gale diagram approach looks like a good way of studying this type of partitions. This will be further explored in the note contained in the Appendix A.

Finally we might remark that the assumptions made throughout the chapter are,

- two facets in a pair do not intersect in vertices; and
- the polytopes are simplicial.

Both conditions were imposed because they seem natural in the original statement of the problem. However, studying the theorem when any of the assumptions are dropped should be equally interesting, and might even result in the same bound.

It also remains to answer von Stengel's question for $d > 7$. By the evidence exposed in this chapter, it is enough to prove the following statement:

Let X be a set of $2d$ points lying in general position in S^{d-2} such that for every hyperplane H through the origin $|X \cap H^+| \geq 2$ and $|X \cap H^-| \geq 2$. Suppose that for all embracing partitions of A, B of X, if $x \in A$ and $y_x \in B$ is such that $x \in sph(B \backslash y_x)$ then $y_x \notin sph(A \backslash x)$; and also if $y \in B$ and $x_y \in A$ is such that $y \in sph(B \backslash x_y)$ then $x_y \notin sph(B \backslash y)$. Then

$$|\nu(X)| \leq 2^{d-1},$$

where $\nu(X)$ is the number of embracing partitions of X.

If the statement above proves to be correct, using lemma 5.3.1, theorem 5.4.2 can be extended to every dimension.

Appendix A

Another Coloured Radon Type Theorem

In the proof of lemma 5.1.7, proposition 1.3.4, on the maximum number of simplices with vertices on a fixed set X, containing one same point, has been used strongly.

The obvious connection to Q12 is that it might seem that maximizing the number of embracing partitions could be achieved by maximizing the number of simplices embracing the origin, except that X has the restriction of been the Gale diagram of a polytope.

Also, maximizing the number of simplices that contain the origin, when X is the Gale Diagram of a polytope, has been done before. It is the Upper Bound Theorem.

The question of finding the maximum number of simplices with vertices on a set X, over the unit sphere, containing the origin, has been studied by Wendel in a probabilistic setting. Given any $d + 1$ randomly selected points, independent and identically distributed, according to the uniform distribution in the unit (d-1)-sphere centered at the origin, he proved that the probability of the convex hull of those points containing the origin is $\frac{1}{2^d}$.

More generally, Wagner and Welzl [18] proved that:

'For any absolutely continuous probability distribution in d-space, the probability that the convex hull of d+1 randomly and independently selected points contains the origin is at most $\frac{1}{2^d}$, and this bound is tight.'

For proving the statement above they introduced an interesting continuous analogue of the Upper Bound Theorem. They raised the question of whether their theorem could be established by a simpler and 'more illuminating' direct argument. In [15] Pach provides a couple of such arguments when considering the problem in dimension 2. Both arguments tackle discrete variants of the problem, from where the Wagner-Welzl result follows by passing to the limit. He claims that his arguments can be extended at least to 3-space if the following planar problem is solved:

Q 15. *Given n points in general position in the plane, coloured red and blue, what is the maximum number of multicoloured $4-$ tuples with the property that the convex hull of its red elements and the convex hull of its blue elements have at least one point in common?*

In particular, he wants to confirm that, when the maximum is attained, the number of red and blue elements are roughly the same.

Question Q15 sounds incredibly familiar to Q14 and a closer look at it uncovers that, by identical arguments to those used in the beginning of this section, Q15 might be equivalent to:

Q 16. *Given a set X of n points in linearly general position in S^d, what is the maximum number of subsets S of X with $|S| = d+2$ such that $0 \in conv(S)$?*

The question above is, of course, answered by the Upper Bound Theorem. Through a Gale transform, the points in S^d can be taken into a $n-d-2$ dimensional polytope. Therefore the maximum number of simplices of X embracing the origin is $|\mathcal{F}(C(n-d-2,n))|$. As $C(n-d-2,n)$ is $\lfloor\frac{n-d-2}{2}\rfloor-$ *neighbourly*, for any hyperplane through the origin, H,

$$|H^+ \cap X| \geq \lceil\tfrac{n-d-2}{2}\rceil \quad \text{and} \quad |H^- \cap X| \geq \lceil\tfrac{n-d-2}{2}\rceil.$$

Therefore, when n is large, any set, X, achieving the maximum, originates from a set Y with a colouring which paints roughly half of the vertices with each colour.

However, it is true that a condition has been overseen. Namely, a set X is the Gale diagram of a polytope iff for all H, hyperplanes through the origin, $|H^+ \cap X| \geq 2$ and $|H^- \cap X| \geq 2$. Such condition is translated into the Radon partition problem setting as follows. If X is a set of points in general position in $\mathbb{R}^d$, coloured red and blue, in such a way that both colours appear more than once, and $\forall x \in X$ there is still a coloured Radon partition with vertices in $X \backslash x$.

It seems likely that any configuration X, for which all coloured Radon partitions use one same fixed point, $x \in X$, will not attain the maximum. But this has yet to be studied carefully.

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