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LOW DIMENSIONAL ALGEBRAIC COMPLEXES OVER INTEGRAL GROUP RINGS

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I, Wajid Hassan Mannan, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

The realization problem asks: When does an algebraic complex arise, up to homotopy, from a geometric complex? In the case of 2- dimensional algebraic complexes, this is equivalent to the D2 problem, which asks when homological methods can distinguish between 2 and 3 dimensional complexes.

We approach the realization problem (and hence the D2 problem) by classifying all possible algebraic 2- complexes and showing that they are realized. We show that if a dihedral group has order 2^n, then the algebraic complexes over it are parametrized by their second homology groups, which we refer to as algebraic second homotopy groups. A cancellation theorem of Swan ([11]), then allows us to solve the realization problem for the group D_8.

Let X be a finite geometric 2- complex. Standard isomorphisms give $\pi_2(X) \cong H_2(\tilde{X};\mathbb{Z})$, as modules over $\pi_1(X)$. Schanuel's lemma may then be used to show that the stable class of $\pi_2(X)$ is determined by $\pi_1(X)$. We show how $\pi_3(X)$ may be calculated similarly. Specifically, we show that as a module over the fundamental group, $\pi_3(X) = S^2(\pi_2(X))$, where $S^2(\pi_2(X))$ denotes the symmetric part of the module $\pi_2(X) \otimes_{\mathbb{Z}} \pi_2(X)$. As a consequence, we are able to show that when the order of $\pi_1(X)$ is odd, the stable class of $\pi_3(X)$ is also determined by $\pi_1(X)$.

Given a closed, connected, orientable 5- dimensional manifold, with finite fundamental group, we may represent it, up to homotopy equivalence, by an algebraic complex. Poincare duality induces a homotopy equivalence between this algebraic complex and its dual. We consider how similar this homotopy equivalence may be made to the identity, (through appropriate choice of algebraic complex). We show that it can be taken to be the identity on 4 of the 6 terms of the chain complex. However, by finding a homological obstruction, we show that in general the homotopy equivalence may not be written as the identity.

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In memory of my late father, who inspired all his children to value study and learning over other worldly endeavors,

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Part I

Introduction

There are three main theorems in Chapter One. Theorem 1.1.1 states that any two projective resolutions, of a module, having equal finite length, may be stabilized to the same homotopy type. This is stronger than Schanuel's lemma, which states that they have the same final homology group.

Theorem 1.4.11 states that the only groups of period 2 are cyclic. This result is well known though an explicit proof is hard to find in the literature. Swan makes a comment outlining an argument in [9]. We give a more direct proof which is elementary and avoids the technical difficulties of Swan's argument.

Theorem 1.5.1 says that an initial segment of a partial free resolution of a module may be changed without altering homotopy type, possibly at the cost of introducing a stably free module to the resolution. This will allow us to parametrize certain algebraic complexes by the last map in their sequence.

The question of when an $n+1$- dimensional CW- complex is homotopic to an n dimensional one has been addressed by C.T.C. Wall [14]. He showed that for $n \neq 2$, the vanishing of $n+1^{th}$ cohomology over all coefficient bundles is sufficient.

Wall's methods are not effective for the case $n = 2$, which is called the D(2) problem. The problem is parametrized by the fundamental group of the CW- complex in question.

Chapter Two is concerned with the D(2) property for dihedral groups. In [3] (62.3) it is shown that the dihedral groups D_{4n+2} satisfy the D(2) property. The smallest dihedral group not covered by this is D_8. Theorem 2.3.4 states that the D(2) property does hold for D_8.

More generally, for dihedral groups of order $4n$, we show that a minimal element of $\Omega_3(\mathbb{Z})$ is realized as the π_2 of a presentation (proposition 2.3.2). In §2.4 we parametrize all possible minimal elements of $\Omega_3(\mathbb{Z})$ by a finite group.

In the case of dihedral groups of order 2^n, $n \in \mathbb{Z}$, we are further able to show that up to chain homotopy equivalence there is a unique algebraic 2- complex with a "standard" π_2 (theorem 2.2.11).

In Chapter Three, theorem 3A states that given a geometric 2- complex, X, with finite fundamental group G, we have $\pi_3(X) \cong S^2(J)$, where $J = \pi_2(X)$. We will define a module over $\mathbb{Z}[G]$, V_G and show that $\pi_3(X)$ is determined by G, up to stabilization by copies of $\mathbb{Z}[G]$ and copies of V_G (theorem 3.5.5). Rationally, we show that $\pi_3(X) \otimes \mathbb{Q} \cong \mathbb{Q}[G]^a \oplus (V_G \otimes \mathbb{Q})^b$ for integers a, b (theorem 3.6.5).

In the case where G is a group of odd order, we have $V_G \cong 0$. Hence in this case, the stable class of $\pi_3(X)$ is determined by G, and $\pi_3(X)$ is rationally free (corollaries 3.5.6 and 3.6.6).

Let M be a closed, connected, orientable 5- dimensional manifold, with finite fundamental group G (we assume manifolds to be without boundary). In chapter four we consider algebraic complexes $C_*(\tilde{M'})$, where M' is a finite CW- complex, with $M \sim M'$. $C_*(\tilde{M'})$ must satisfy Poincare duality. We use this to show that up to chain homotopy equivalence, we may represent it by an algebraic 2- complex, $\mathcal{A}$, connected to its dual via a G- invariant bilinear form, β, on $(\pi_2(\mathcal{A}))^*$. We denote the resulting algebraic 5- complex $(\mathcal{A}, \beta)$.

We next consider the homotopy equivalence induced by Poincare Duality. In particular we are interested in how similar it can be made to the identity. We show that it can be taken as the identity on 4 of the 6 terms of the chain complex. However, we find a homological obstruction to this homotopy equivalence actually being the identity. In particular, certain manifolds described in [1] do not satisfy the homological condition necessary, for being able to write the homotopy equivalence as the identity.

We now make some notational points:

All rings are assumed to contain a multiplicative identity. Modules are assumed to be right modules over the relevant ring. A map between modules is assumed to be linear over the relevant ring, unless we describe it as a map of sets.

Suppose A and B are modules. If $\alpha : A \to B$ is a map, and B is a summand of a third module C, then $\alpha : A \to C$ will denote $\alpha : A \to B$ composed with inclusion of the summand B, in C.

Again, let A, B, C be modules. If $f : A \to C$ and $g : B \to C$ are maps, then $f \oplus g : A \oplus B \to C$ is defined by $(f \oplus g)(a \oplus b) = f(a) + g(b)$.

The abbreviation f.g. will be used to denote finitely generated, whether in the context of a module or a ring.

Contrary to some conventions, the subscript on a group will denote its order. Hence D_n will denote the dihedral group of order n, S_n will denote the symmetric group of order n and A_n will denote the alternating group of order n.

Let R be a ring and G a group. $R[G]$ denotes the free R- module with group elements as basis. It is a ring, with product structure given by group multiplication, for the basis elements, and extended linearly over R, for the remaining elements. The "group ring" of a group G, will refer to $\mathbb{Z}[G]$.

In the context of a $\mathbb{Z}[G]$- module, $\mathbb{Z}$ will denote the $\mathbb{Z}[G]$- module whose underlying Abelian group is $\mathbb{Z}$, and on which the action of G is trivial.

The map sending $\sum \lambda_i g_i$ to $\sum \lambda_i$, for $\lambda_i \in \mathbb{Z},\ g_i \in G$ will be referred to as augmentation, and will usually be denoted ϵ. Its kernel, denoted IG will be called "the augmentation ideal".

The dual of a module, M, denoted M^* is the set of $\mathbb{Z}$- linear maps $M \to \mathbb{Z}$. M^* has the structure of an Abelian group with respect to point-wise addition. In fact, it is a module over $\mathbb{Z}[G]$, with G action given by $\alpha g(m) = \alpha(mg^{-1})$ for all $m \in M$, $\alpha \in M^*$ and $g \in G$. Similarly we define the dual of a map, f, to be precomposition with f. The dual of a map, f, will be denoted by f^*.

As we work over finite groups, this definition is consistent with the one where M^* is defined as the set of $\mathbb{Z}[G]$- linear maps $M \to \mathbb{Z}[G]$.

If $\beta : M \times M \to \mathbb{Z}$ is a G- invariant bilinear form on a module M, it will also be regarded as a map $M \to M^*$, which sends $x \in M$ to the element of M^* defined by $\beta(x)(y) = \beta(x, y)$, for all $y \in M$.

Part II

Chapter 1

Algebraic methods

There are three main theorems in this chapter. Theorem 1.1.1 states that any two projective resolutions of a module, having equal finite length, may be stabilized to the same homotopy type. Theorem 1.4.11 states that the only groups of period 2 are cyclic. Although this result is well known, the proof is technically simpler than any in publication. Theorem 1.5.1 says that an initial segment of a partial free resolution of a module may be changed without altering homotopy type, possibly at the cost of introducing a stably free module to the resolution. This will allow us to parametrize certain algebraic complexes by the last map in their sequence.

§1.1 Schanuel's Lemma

Schanuel's lemma plays a key role in our algebraic study of homotopy. We adapt the proof to give us a stronger result; any two projective resolutions of a module may be stabilized to the same homotopy type.

Let R be a ring. An algebraic complex over R consists of a sequence of modules C_i and maps $\delta_i : C_i \to C_{i-1}$, such that $\delta_i\delta_{i-1} = 0$ for each i. It may be denoted (C_i, δ_i).

A chain map $f : (C_i, \delta_i) \to (D_i, \partial_i)$, consists of a sequence of maps $f_i : C_i \to D_i$ such that for each i, $f_{i-1}\delta_i = \partial_i f_i$.

A chain homotopy, I, between two chain maps $f : (C_i, \delta_i) \to (D_i, \partial_i)$ and $g : (C_i, \delta_i) \to (D_i, \partial_i)$ consists of a sequence of maps $I_i : C_i \to D_{i+1}$ such that for each i, $\partial_{i+1} I_i + I_{i-1}\delta_i = f_i - g_i$.

A chain homotopy equivalence, $f : (C_i, \delta_i) \to (D_i, \partial_i)$, is a chain map for which there exists a chain map $g : (D_i, \partial_i) \to (C_i, \delta_i)$ and a chain homotopy, I, between the identity and fg, and a chain homotopy J, between the identity and gf. Two algebraic complexes, (C_i, δ_i) and (D_i, ∂_i) are said to be homotopy equivalent precisely when there exists a homotopy equivalence between them. In this case, we may write $(C_i, \delta_i) \sim (D_i, \partial_i)$.

An important notational point is that whenever we write down an algebraic chain complex, any maps denoted by dotted arrows or modules connected to the complex by such arrows, are not part of the complex.

Let A be a left module over R and let $C_* = (C_i, \delta_i)$ be an algebraic complex (of right modules as usual). Let $(C_i \otimes_R A, \delta_{i*})$ denote the tensor product of (C_i, δ_i) with A over R. Then $H_i(C_*; A)$ denotes the kernel of δ_{i*} quotiented by the image of δ_{i+1_*}.

Let A be a right module over R and let $C_* = (C_i, \delta_i)$ be an algebraic complex (of right modules as usual). Let $(\mathrm{Hom}_R(C_i, A), \delta_i^*)$ denote the functor $\mathrm{Hom}_R(\_, A)$ applied to (C_i, δ_i). Then $H^i(C_*; A)$ denotes the kernel of δ_{i+1}^* quotiented by the image of δ_i^*.

This notation for homology and cohomology is slightly non-standard.

We say a sequence of modules and maps is exact when the image of each map is the kernel of the next.

In this section, we work over a fixed ring R, with unit. All maps, modules, algebraic chain complexes and chain homotopies will be assumed to be over R. Modules denoted by "P_i" and "Q_i" may be assumed to be projective.

Suppose we have an algebraic complex

$$\cdots \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \cdots \tag{1}$$

Replacing it with the complex

$$\cdots \xrightarrow{\partial_{n+1}} F_n \oplus F \xrightarrow{\partial_n \oplus 1} F_{n-1} \oplus F \xrightarrow{\partial_{n-1}} \cdots \tag{2}$$

is called a simple homotopy equivalence. We may define a chain map i from (1) to (2) which is the identity on F_r, for $r \neq n, n-1$ and the natural inclusion for $r = n, n-1$. Similarly we may define a chain map j from (2) to (1) which is the identity on F_r, for $r \neq n, n-1$ and the natural projection for $r = n, n-1$. Then ji is the identity and ij differs from the identity by 0 on the F_i and the identity on the copies of F.

Let $I_r : F_r \to F_{r+1}$ be 0 for $r \neq n-1$ and let $I_n : F_{n-1} \oplus F \to F_{n-1} \oplus F$ be 0 on F_{n-1} and the identity on F. Then we have $1 - ij = \partial' I + I\partial'$ where ∂' agrees with ∂_r for $r \neq n$, and is the map $\partial_n \oplus 1$ on $F_n \oplus F$. From this we see that a simple homotopy equivalence is indeed a homotopy equivalence.

Let

$$P_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \overset{\epsilon}{\dashrightarrow} M \dashrightarrow 0$$

and

$$Q_n \xrightarrow{\partial'_n} \cdots \xrightarrow{\partial'_2} Q_1 \xrightarrow{\partial'_1} Q_0 \overset{\epsilon'}{\dashrightarrow} M \dashrightarrow 0$$

be exact sequences.

Let $R_0 = P_0$ and $S_0 = Q_0$. Define R_i, S_i, for $i = 1, \ldots n$ by

$R_i = S_{i-1} \oplus P_i$
$S_i = R_{i-1} \oplus Q_i$

Note R_i and S_i are projective, as they are constructed as direct sums of projective modules.

Theorem 1.1.1 *The complexes*

$$P_n \oplus S_n \xrightarrow{\partial_n \oplus 0} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \tag{1}$$

and

$$Q_n \oplus R_n \xrightarrow{\partial'_n \oplus 0} Q_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \xrightarrow{\partial'_2} Q_1 \xrightarrow{\partial'_1} Q_0 \tag{2}$$

are chain homotopy equivalent.

Chain homotopies induce isomorphisms on homology groups. In particular, a chain homotopy between (1) and (2) will induce an isomorphism between $\mathrm{Ker}(\partial_n \oplus 0)$ and $\mathrm{Ker}(\partial'_n \oplus 0)$, so we have the following corollary:

Corollary 1.1.2 *(Schanuel)*$\mathrm{Ker}\partial_n \oplus \mathrm{S_n} = \mathrm{Ker}\partial'_n \oplus \mathrm{R_n}$

Proof of theorem: We perform a series of simple homotopy equivalences, u_i on (1), for $i = 1, \ldots, n$. Recall $R_0 = P_0$. Define u_1 to consist of replacing

$$\xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} R_0 \overset{\epsilon}{\dashrightarrow} M$$

with

$$\xrightarrow{\partial_2} P_1 \oplus S_0 \xrightarrow{\delta_1} R_0 \oplus S_0 \overset{\epsilon \oplus 0}{\dashrightarrow} M \tag{3}$$

where δ_1 is defined by

$$\delta_1 = \begin{pmatrix} \partial_1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note $R_1 = P_1 \oplus S_0$, so we can write (3) as,

$$\xrightarrow{\partial_2} R_1 \xrightarrow{\delta_1} R_0 \oplus S_0 \overset{\epsilon\oplus 0}{\dashrightarrow} M$$

Define u_i to consist of replacing

$$\xrightarrow{\partial_{i+1}} P_i \xrightarrow{\partial_i} R_{i-1} \xrightarrow{\delta_{i-1}}$$

with

$$\xrightarrow{\partial_{i+1}} P_i \oplus S_{i-1} \xrightarrow{\delta_i} R_{i-1} \oplus S_{i-1} \xrightarrow{\delta_{i-1}\oplus 0} \quad (4)$$

where δ_i is defined by

$$\delta_i = \begin{pmatrix} \partial_i & 0 \\ 0 & 1 \end{pmatrix}$$

Note $R_i = P_i \oplus S_{i-1}$, so we can write (4) as,

$$\xrightarrow{\partial_{i+1}} R_i \xrightarrow{\delta_i} R_{i-1} \oplus S_{i-1} \xrightarrow{\delta_{i-1}\oplus 0}$$

Finally, note that u_n replaces

$$P_n \oplus S_n \xrightarrow{\partial_n \oplus 0} R_{n-1} \xrightarrow{\delta_{n-1}}$$

with

$$R_n \oplus S_n \xrightarrow{\delta_n \oplus 0} R_{n-1} \oplus S_{n-1} \xrightarrow{\delta_{n-1}\oplus 0}$$

Hence applying $u_1, \ldots, u_n$ to (1) gives

$$R_n \oplus S_n \xrightarrow{\delta_n \oplus 0} R_{n-1} \oplus S_{n-1} \xrightarrow{\delta_{n-1}\oplus 0} \ldots\ldots \xrightarrow{\delta_2 \oplus 0} R_1 \oplus S_1 \xrightarrow{\delta_1 \oplus 0} R_0 \oplus S_0 \quad (5)$$

Similarly we perform a series of simple homotopy equivalences v_i on (2), where v_i consists of replacing

$$\xrightarrow{\partial'_{i+1}} Q_i \xrightarrow{\partial'_i} S_{i-1} \xrightarrow{\delta'_{i-1}}$$

with

$$\xrightarrow{\partial'_{i+1}} Q_i \oplus R_{i-1} \xrightarrow{\delta'_i} S_{i-1} \oplus R_{i-1} \xrightarrow{\delta'_{i-1}\oplus 0} \tag{6}$$

where δ'_i is defined by

$$\delta'_i = \begin{pmatrix} \partial'_i & 0 \\ 0 & 1 \end{pmatrix}$$

Note $S_i = Q_i \oplus R_{i-1}$, so we can write (6) as,

$$\xrightarrow{\partial'_{i+1}} S_i \xrightarrow{\delta'_i} S_{i-1} \oplus R_{i-1} \xrightarrow{\delta'_{i-1}\oplus 0}$$

Also v_n replaces

$$Q_n \oplus R_n \xrightarrow{\partial'_n\oplus 0} S_{n-1} \xrightarrow{\delta'_{n-1}}$$

with

$$S_n \oplus R_n \xrightarrow{\delta'_n\oplus 0} S_{n-1} \oplus R_{n-1} \xrightarrow{\delta'_{n-1}\oplus 0}$$

Hence applying $v_1, \ldots, v_n$ to (2) gives

$$S_n \oplus R_n \xrightarrow{\delta'_n\oplus 0} S_{n-1} \oplus R_{n-1} \xrightarrow{\delta'_{n-1}\oplus 0} \cdots\cdots \xrightarrow{\delta'_2\oplus 0} S_1 \oplus R_1 \xrightarrow{\delta'_1\oplus 0} S_0 \oplus R_0 \tag{7}$$

Recall R_i, S_i are projective. As (5) and (7) are chain homotopy equivalent to (1) and (2) respectively, they are both exact. Further, they extend to exact sequences

$$R_n \oplus S_n \xrightarrow{\delta_n \oplus 0} R_{n-1} \oplus S_{n-1} \xrightarrow{\delta_{n-1}\oplus 0} \cdots\cdots \xrightarrow{\delta_2 \oplus 0} R_1 \oplus S_1 \xrightarrow{\delta_1 \oplus 0} R_0 \oplus S_0 \dashrightarrow^{\epsilon \oplus 0} M \dashrightarrow 0 \quad (8)$$

and

$$S_n \oplus R_n \xrightarrow{\delta'_n \oplus 0} S_{n-1} \oplus R_{n-1} \xrightarrow{\delta'_{n-1}\oplus 0} \cdots\cdots \xrightarrow{\delta'_2 \oplus 0} S_1 \oplus R_1 \xrightarrow{\delta'_1 \oplus 0} S_0 \oplus R_0 \dashrightarrow^{\epsilon' \oplus 0} M \dashrightarrow 0 \quad (9)$$

We complete the proof of theorem 1.1.1 by constructing a pair of inverse chain isomorphisms, h, k, between (8) and (9).

As R_0, S_0 are projective, we may pick f_0, g_0 so that the following diagrams commute:

$$\begin{array}{ccc} R_0 & \xrightarrow{\epsilon} & M \\ \downarrow f_0 & & \downarrow 1 \\ S_0 & \xrightarrow{\epsilon'} & M \end{array} \qquad\qquad \begin{array}{ccc} R_0 & \xrightarrow{\epsilon} & M \\ \uparrow g_0 & & \uparrow 1 \\ S_0 & \xrightarrow{\epsilon'} & M \end{array} \tag{10}$$

Define $h_0 : R_0 \oplus S_0 \to S_0 \oplus R_0$ and $k_0 : S_0 \oplus R_0 \to R_0 \oplus S_0$ by

$$h_0 = \begin{pmatrix} f_0 & 1 - f_0 g_0 \\ 1 & -g_0 \end{pmatrix} \qquad\qquad k_0 = \begin{pmatrix} g_0 & 1 - g_0 f_0 \\ 1 & -f_0 \end{pmatrix}$$

Direct calculation shows that $h_0 k_0 = 1$ and $k_0 h_0 = 1$.

Also from commutativity of (10), we deduce

$$(\epsilon' \quad 0)\begin{pmatrix} f_0 & 1 - f_0 g_0 \\ 1 & -g_0 \end{pmatrix} = (\epsilon' f_0 \quad \epsilon'(1 - f_0 g_0)) = (\epsilon \quad 0)$$

and

$$(\epsilon \quad 0)\begin{pmatrix} g_0 & 1-g_0f_0 \\ 1 & -f_0 \end{pmatrix} = (\epsilon g_0 \quad \epsilon(1-g_0f_0)) = (\epsilon' \quad 0)$$

Hence the following diagrams commute:

$$\begin{array}{ccc} R_0 \oplus S_0 & \xrightarrow{\epsilon\oplus 0} & M \\ \downarrow h_0 & & \downarrow 1 \\ S_0 \oplus R_0 & \xrightarrow{\epsilon'\oplus 0} & M \end{array} \qquad \begin{array}{ccc} R_0 \oplus S_0 & \xrightarrow{\epsilon\oplus 0} & M \\ \uparrow k_0 & & \uparrow 1 \\ S_0 \oplus R_0 & \xrightarrow{\epsilon'\oplus 0} & M \end{array}$$

Now suppose that for some $i \leq n$, we have defined $h_j : R_j \oplus S_j \to S_j \oplus R_j$ and $k_j : S_j \oplus R_j \to R_j \oplus S_j$ for $j = 0, \ldots, i-1$, so that for each j, we have $h_jk_j = 1$ and $k_jh_j = 1$. We proceed by induction.

As before, pick f_i, g_i so that the following diagrams commute:

$$\begin{array}{ccc} R_i & \xrightarrow{\delta_i} & R_{i-1} \oplus S_{i-1} \\ \downarrow f_i & & \downarrow h_{i-1} \\ S_i & \xrightarrow{\delta'_i} & S_{i-1} \oplus R_{i-1} \end{array} \qquad \begin{array}{ccc} R_i & \xrightarrow{\delta_i} & R_{i-1} \oplus S_{i-1} \\ \uparrow g_i & & \uparrow k_{i-1} \\ S_i & \xrightarrow{\delta'_i} & S_{i-1} \oplus R_{i-1} \end{array} \tag{11}$$

Define $h_i : R_i \oplus S_i \to S_i \oplus R_i$ and $k_i : S_i \oplus R_i \to R_i \oplus S_i$ by

$$h_i = \begin{pmatrix} f_i & 1-f_ig_i \\ 1 & -g_i \end{pmatrix} \qquad k_i = \begin{pmatrix} g_i & 1-g_if_i \\ 1 & -f_i \end{pmatrix}$$

Direct calculation shows that $h_ik_i = 1$ and $k_ih_i = 1$.

Recall $h_{i-1}k_{i-1} = 1$ and $k_{i-1}h_{i-1} = 1$. From commutativity of (11) we deduce

$$(\delta_i' \quad 0)\begin{pmatrix} f_i & 1-f_ig_i \\ 1 & -g_i \end{pmatrix} = (\delta_i' f_i \quad \delta_i'(1-f_ig_i)) = h_{i-1}(\delta_i \quad 0)$$

and

$$(\delta_i \quad 0)\begin{pmatrix} g_i & 1-g_if_i \\ 1 & -f_i \end{pmatrix} = (\delta_i g_i \quad \delta_i(1-g_if_i)) = k_{i-1}(\delta_i' \quad 0)$$

Hence the following diagrams commute:

$$\begin{array}{ccc} R_i \oplus S_i & \xrightarrow{\delta_i \oplus 0} & R_{i-1} \oplus S_{i-1} \\ \downarrow h_i & & \downarrow h_{i-1} \\ S_i \oplus R_i & \xrightarrow{\delta_i' \oplus 0} & S_{i-1} \oplus R_{i-1} \end{array} \qquad \begin{array}{ccc} R_i \oplus S_i & \xrightarrow{\delta_i \oplus 0} & R_{i-1} \oplus S_{i-1} \\ \uparrow k_i & & \uparrow k_{i-1} \\ S_i \oplus R_i & \xrightarrow{\delta_i' \oplus 0} & S_{i-1} \oplus R_{i-1} \end{array}$$

Together with the identity on M, the h_i, k_i are therefore a pair of mutually inverse chain maps, between (8) and (9). Hence (1) is chain chain homotopy equivalent to (5), which is chain isomorphic to (7) which in turn is chain chain homotopy equivalent to (2). □

If the modules P_i and Q_i are finitely generated and free, then R_n and S_n are also finitely generated and free. Let $\mathrm{Free}_n(M)$ denote the set of homotopy types of f.g. free n- term resolutions, of a module M. We may give $\mathrm{Free}_n(M)$ the structure of a tree, by placing an edge between the homotopy types of any pair of resolutions of the form

$$F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \stackrel{\epsilon}{\dashrightarrow} M$$

and

$$F_n \oplus R \xrightarrow{\partial_n \oplus 0} F_{n-1} \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \stackrel{\epsilon}{\dashrightarrow} M$$

where R is regarded as a module over itself. From theorem 1.1.1, we may conclude

Theorem 1.1.3 *Free$_n(M)$ is a connected tree.*

Another tree of interest is the stable tree of a module.

Definition 1.1.4 *Stable equivalence* Two modules over R, L and N, are stably equivalent if there is an isomorphism $L \oplus R^a \to N \oplus R^b$, for integers a and b.

The stable class of a module K is the set of modules stably equivalent to K. We may give this the structure of a tree, by assigning an edge to any pair of modules L, N, where N is isomorphic to $L \oplus R$.

From corollary 1.1.2, we know that if (A_i, δ_i) and (B_i, ∂_i) are elements of Free$_n(M)$, then $\ker(\delta_n)$ and $\ker(\partial_n)$ are in the same stable class. We denote this class $\Omega_{n+1}(M)$.

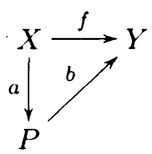
We may conclude that we have a map of trees Free$_n(M) \to \Omega_{n+1}(M)$, which sends an element of Free$_n(M)$, (A_i, δ_i), to $\ker(\delta_n)$.

§1.2 The Derived Category

The previous section provides some motivation for considering a category where stably equivalent modules are isomorphic objects. We follow Johnson[4], in constructing such a category. All the results of this section are explained in greater detail in [4], §19. We provide a summary, for narrative purposes.

Modules over the group ring, $\mathbb{Z}[G]$, of a group form the objects of a category, whose morphisms are $\mathbb{Z}[G]$- linear maps.

A map $f : X \to Y$ is said to factor through a projective if there exists a projective module P and maps a, b such the following diagram commutes:



Lemma 1.2.1 *A map factors through a projective module if and only if it factors through a free module.*

Proof: Suppose $f = ba$ as before. We have some projective module Q such that $P \oplus Q$ is free. Let $i : P \hookrightarrow P \oplus Q$ be the natural inclusion and $p : P \oplus Q \twoheadrightarrow P$ be the natural projection. Then $pi = 1$. Hence $f = ba = (bp)(ia)$

□

For any pair of modules, M and N, define an equivalence relation $\sim$ on $\mathrm{Hom}_{\mathbb{Z}[G]}(M, N)$ given by $f \sim g$ if and only if $f - g$ factors through a projective module. To see this is transitive, note that if $f - g$ factors through P and $g - h$ factors through Q then both $f - g$ and $g - h$ factor through $P \oplus Q$ so $f - h = (f - g) + (g - h)$ factors through $P \oplus Q$.

If f factors through a projective then clearly any map composed with f also does. By linearity of composition we therefore have that composition of maps under $\sim$ is well defined.

Hence we can define the derived category of the category of $\mathbb{Z}[G]$- modules, $\mathrm{Der}(\mathbb{Z}[G])$, to be the category whose objects are (left) $\mathbb{Z}[G]$- modules and whose morphisms are $\mathbb{Z}[G]$-linear homomorphisms under the relation $\sim$.

Lemma 1.2.2 *In the derived category, $0 \cong P$ for all P projective.*

Proof: We have unique maps $i : 0 \to P$ and $p : P \to 0$. pi is the identity on 0 and $ip - 1$ factors through $1 : P \to P$.

□

Lemma 1.2.3 *In the derived category $M \cong M \oplus P$ for P projective.*

Proof: Let $i : M \to M \oplus P$, $p : M \oplus P \to P$ be the natural inclusion, and natural projection between M and $M \oplus P$ respectively. Clearly $pi = 1$. Also $1 - ip$ restricts to 0 on the summand M and $1 : P \to P$, so $1 - ip$ factors through projection onto P.

□

Proposition 1.2.4 *Suppose we have an exact sequence of modules over $\mathbb{Z}[G]$, G a finite group:*

$0 \longrightarrow A \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ *where the F_i are free.*

Then $\mathrm{End}_{\mathrm{Der}}(A) \cong \mathrm{End}_{\mathrm{Der}}(M)$.

Proof: See [3]. The isomorphism is given by taking taking any endomorphism $f : M \to M$ and using projectivity of the F_i to extend to a commutative diagram such as

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & A & \longrightarrow & F_n & \longrightarrow & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle D(f)} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & A & \longrightarrow & F_n & \longrightarrow & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

Then if f' is equivalent in the derived category to f, any choice of $D(f)$ will be equivalent in the derived category to any choice of $D(f')$.

The inverse is constructed dually, using the relative injectivity of the F_i. See [3] for details of the diagram chases.

□

Definition 1.2.5 *Algebraic n- complex* An algebraic n- complex is an algebraic complex (F_i, ∂_i), $i = 1, \cdots, n$, over $\mathbb{Z}[G]$, satisfying:

i)F_i is free and finitely generated, $i = 1, \cdots, n$.

ii)coker$(\partial_1) = \mathbb{Z}$.

iii)$\ker(\partial_1) = \mathrm{im}(\partial_2)$.

Let G be a finite group of order n and let $J = H_2(X;\mathbb{Z}[G])$ for some algebraic 2-complex, X, over $\mathbb{Z}[G]$.

Definition 1.2.6 *k- invariant* We define a map $k : \mathrm{End}(J) \to \mathbb{Z}_n$. Given any map $\alpha : J \to J$, one may extend it to a chain map, $X \to X$. The chain map will induce a map on the last cokernel, $\mathbb{Z}$. This map will be multiplication by m, for some integer m. A map $\mathbb{Z} \to \mathbb{Z}$ factors through a projective module if and only if it factors through a free module, hence if and only if it is some sum of maps which factor through $\mathbb{Z}[G]$. As any such map will be multiplication by a number divisible by n, the number m is determined up to congruence modulo n. The k- invariant of α, $k(\alpha)$, is then defined to be the congruence of m in $\mathbb{Z}_n$.

If α is an automorphism, $J \to J$, then $k(\alpha)$ will be a unit, modulo n.

Definition 1.2.7 *Swan Map* The Swan map, $\mathrm{Aut}(J) \to \mathbb{Z}_n^*$, sends an automorphism, α, to its k- invariant, $k(\alpha)$.

Note that $\mathbb{Z}_n$ is being identified here with $\mathrm{End}_{\mathrm{Der}}(J)$. Hence if the Swan map was defined with respect to a different algebraic 2- complex, it would be the same map, as the only ring isomorphism $\mathbb{Z}_n \to \mathbb{Z}_n$ is the identity.

Theorem 1.2.8 *If the Swan map,* $\mathrm{Aut}(J) \to \mathbb{Z}_n^*$, *is surjective, then any algebraic* 2-*complex,* Y, *with* $H_2(Y;\mathbb{Z}[G]) = J$ *satisfies* $X \sim Y$.

Proof: Any element of Aut(J) will induce a chain map X to Y. Let k be the multiplication induced on $\mathbb{Z}$. Now suppose that the Swan map is surjective, restricted to a map from $\mathrm{Aut}_{\mathbb{Z}[G]}(J) \to \mathrm{Aut}_{\mathrm{Der}}(J)$. Then we have a chain map $X \to X$ with k-invariant the inverse of k modulo n, and which induces an isomorphism on J. Composing this chain map with the chain map $X \to Y$ gives a chain map which induces an automorphism on J and has k-invariant 1. We can replace this with a chain map that induces an automorphism on J and actually induces multiplication by 1 on $\mathbb{Z}$. This will necessarily be a homotopy equivalence. Hence X and Y will be chain homotopy equivalent. See [2] for details of the diagram chases. $\square$

If the Swan map is surjective for a module J, then it is also surjective for $J \oplus \mathbb{Z}[G]$, as if $\alpha \in \mathrm{Aut}(J)$ then we have an automorphism $J \oplus \mathbb{Z}[G] \to J \oplus \mathbb{Z}[G]$ with the same k- invariant, given by the matrix:

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

Definition 1.2.9 *Minimal module* A module is minimal in its stable tree if it does not contain a summand isomorphic to $\mathbb{Z}[G]$.

We may conclude that it is sufficient to check that this property holds for all minimal modules in the stable class of J, in order to deduce that it holds for all modules in that stable class. Again see [2] for details of this method.

§1.3 Group Cohomology

A resolution for a module M over a ring is an exact sequence of modules, E_i, ∂_i, such that the cokernel of ∂_1 is M.

When $\mathbb{Z}$ is regarded as a module over $\mathbb{Z}[G]$ for some group G, we will assume the trivial action. Let G be a finite group. There exists a resolution of finitely generated free modules, F_i, over $\mathbb{Z}[G]$, for $\mathbb{Z}$ (see [5]):

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \twoheadrightarrow \mathbb{Z}$$

Definition 1.3.1 *For a left $\mathbb{Z}[G]$ module A, $H_n(G; A)$ is given by tensoring the resolution for $\mathbb{Z}$ with A and taking the kernel of ∂_n quotiented out by the image of ∂_{n+1}*

Definition 1.3.2 *For a right $\mathbb{Z}[G]$ module A, $H^n(G; A)$ is given by applying $\mathrm{Hom}_{\mathbb{Z}[G]}(\bullet, A)$ to the resolution for $\mathbb{Z}$ with A and taking the kernel of ∂^*_{n+1} quotiented out by the image of ∂^*_n*

We note the following:

Proposition 1.3.3 *Let A be a module over $\mathbb{Z}[G]$. The elements of $H^2(G; A)$ parametrize short exact sequences of groups of the form*
$1 \to A \to E \to G \to 1$,
where the conjugation action of G on A is given by the ZG- action on A.
(See [12])

We may generalize cohomology groups to general rings:

Definition 1.3.4 *Given modules M and A, $\mathrm{Ext}^n(M, A)$ is the n'th cohomology group of a resolution of M with coefficient module A.*

Proposition 1.3.5 $\mathrm{Ext}^1(M, A)$ *parametrizes short exact sequences of modules of the form $A \to X \to M$, up to chain isomorphism with identities at both ends.*
(See [5])

Combining the previous two propositions we see that there is a correspondence between short exact sequences of the form

$$1 \to A \to ? \to G \to 1$$

and short exact sequences of modules of the form

$$0 \to A \to ? \to IG \to 0$$

where IG is the kernel of the map augmentation map $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$, sending $1 \in ZG$ to $1 \in \mathbb{Z}$. The kernel of the augmentation map is called the augmentation ideal.

As a result of this, any module, A which occurs in an exact sequence of the form

$$0 \to A \to F \to \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

corresponds to some surjection $E \to G$, for some group E.

Specifically, given a short exact sequence of groups

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{j} G \longrightarrow 1$$

we have a $\mathbb{Z}[G]$- module $(IE)_G$, which is the augmentation ideal of E, quotiented out by the ideal generated by elements of the form $x(e_1 - e_2)$ where $x \in IE$ and $e_1, e_2 \in E$ both map to the same element of G, via j. The action of an element $g \in G$ on $(IE)_G$ is the action of any preimage of g. The choice of preimage does not effect the action.

The map j induces a map from $(IE)_G \to IG$. The kernel of this map is isomorphic to A with G- action given by conjugation, as before.

Conversely, we may construct an inverse to this operation: Given any short exact sequence of modules

$$0 \longrightarrow A \xrightarrow{i} M \xrightarrow{j} G \longrightarrow 0$$

we let E_M denote the subset of M which maps to an element of the form $g-1, g \in G$, via j. We define a product on E_M by setting for each $e_1, e_2 \in E_M$, the product $e_1 \circ e_2$ is equal to $e_1 g + e_2$, where e_2 maps to $g-1$. We have a map $E_M \to G$ given by sending e_1 to g where j sends e_1 to $g-1$. The kernel of this map is A, so we have a short exact sequence of groups

$$1 \longrightarrow A \xrightarrow{i} E_M \xrightarrow{j} G \longrightarrow 1$$

For any group G, we may denote by F_G the free group generated by the underlying set of G. There is a natural surjection $F_G \twoheadrightarrow G$. Let K_G denote the kernel of this surjection, and let K'_G denote the commutator subgroup of K_G. We have a short exact sequence:

$$1 \longrightarrow K_G/K'_G \longrightarrow F_G/K'_G \longrightarrow G \longrightarrow 1$$

(1)

Proposition 1.3.6 *This sequence splits on the right, if and only if $H^2(G; A)$ vanishes for all coefficient modules A.*

Proof: K_G/K'_G is a $\mathbb{Z}[G]$- module, with G action given by conjugation as before. If $H^2(G; K_G/K'_G) = 0$ then by proposition 1.3.3 (1) must split on the right.

Conversely, suppose (1) splits on the right, with $k : G \to F_G/K'_G$ as the splitting map and we have any short exact sequence

$$1 \longrightarrow A \xrightarrow{i} E \xrightarrow{j} G \longrightarrow 1$$

(2)

with A abelian. We may construct maps $f_2 : F_G/K'_G \to E$ and $f_1 : K_G/K'_G \to A$ to make the following diagram commute:

$$\begin{array}{ccccc} K_G/K'_G & \longrightarrow & F_G/K'_G & \longrightarrow & G \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow 1 \\ A & \xrightarrow{i} & E & \xrightarrow{j} & G \end{array}$$

This is done by sending each generator of F_G to some element of E which lies in the preimage of the corresponding element of G. This is well defined as K_G maps into A and A is abelian.

We have $j \circ (f_2 \circ k) = (j \circ f_2) \circ k = 1$. Hence (2) splits. But A, together with the conjugation action of G on it,was chosen arbitrarily, hence $H^2(G; A) = 0$, for all coefficient modules A.

□

§1.4 Periodicity

The main result of this section is well known to those working in the field, though an explicit proof is hard to find in the literature. The result states that the only groups which are (homologically) of period 2 are cyclic. Swan makes a comment outlining an argument in [9]. We give a more direct proof which is elementary and avoids the technical difficulties of Swan's argument. This result will be used in the next section.

We say a finite group, G, has period 2 if and only if there exists an exact sequence of $\mathbb{Z}[G]$ modules

$$0 \to \mathbb{Z} \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

with P_0, P_1 finitely generated projective. (We will assume the G - action on $\mathbb{Z}$ to be trivial throughout).

ϵ will denote the augmentation map $\mathbb{Z}[G] \to \mathbb{Z}$ which takes $1 \in \mathbb{Z}[G]$ to $1 \in \mathbb{Z}$. ϵ^* will denote its dual, which sends $1 \in \mathbb{Z}$ to $\sum_{g \in G} g \in \mathbb{Z}[G]$.

Proposition 1.4.1 *Let G be a finite group which has period 2. Then there exists an exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow S \longrightarrow \mathbb{Z}[G] \stackrel{\epsilon}{\longrightarrow} \mathbb{Z} \longrightarrow 0$$

where S is projective.

Proof: If G has period 2 then there exists an exact sequence

$$0 \to \mathbb{Z} \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

with P_0, P_1 finitely generated and projective. Choosing Q such that $Q \oplus P_0 \cong \mathbb{Z}[G]^n$ for some n, and performing a simple congruence, we get the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{a} P \xrightarrow{b} \mathbb{Z}[G]^n \xrightarrow{c} \mathbb{Z} \to 0 \tag{1}$$

where $P = P_1 \oplus Q$ and is finitely generated, projective.

Let a $\mathbb{Z}[G]$- basis be chosen, $< e_1, ..., e_n >$, for $\mathbb{Z}[G]^n$. As c is a surjection, and $c(e_i g) = c(e_i)$ for all g and i, some $\mathbb{Z}$- linear combination of the $c(e_i)$, $\quad i = 1, .., n$, must equal 1. Let E denote the $\mathbb{Z}$- linear span of the e_i, $\quad i = 1, ..., n$. It follows that c restricts to a surjection of Abelian groups $E \twoheadrightarrow \mathbb{Z}$.

Clearly $\mathbb{Z}$ is a free Abelian group, so this surjection splits, as a map of Abelian groups. Therefore a $\mathbb{Z}$- basis of E, f_i, $\quad i = 1, ..., n$ may be chosen, such that $c(f_1) = 1$ and $c(f_i) = 0$, $\quad i = 2, ..., n$.

Lemma 1.4.2 *The* f_i, $\quad i = 1, ..., n$ *are a* $\mathbb{Z}[G]$- *basis for* $\mathbb{Z}[G]^n$.

Proof of lemma: Any element $x \in \mathbb{Z}[G]^n$ can be uniquely written as $\sum_{g \in G} x_g g$ with $x_g \in E$. Each x_g may be uniquely written as a $\mathbb{Z}$- linear combination of the f_i. Therefore x may be written uniquely as a $\mathbb{Z}[G]$- linear combination of the f_i.

□

(Proof of Proposition continued)

The module, $\mathbb{Z}[G]^n$ splits as the direct sum of the $\mathbb{Z}[G]$- linear span of f_1 and the $\mathbb{Z}[G]$- linear span of f_i, $\quad i = 2, ..., n$. Hence (1) may be rewritten as

$$0 \longrightarrow \mathbb{Z} \xrightarrow{a} P \xrightarrow{b_1 \oplus b_2} \mathbb{Z}[G]^{n-1} \oplus \mathbb{Z}[G] \xrightarrow{0,\, \epsilon} \mathbb{Z} \longrightarrow 0 \tag{2}$$

By exactness, b_1 must surject onto $\mathbb{Z}[G]^{n-1}$, hence b_1 splits. (Note $\mathbb{Z}[G]^{n-1}$ projective). So $P = S \oplus \mathbb{Z}[G]^{n-1}$ where S is the kernel of b_1. S is a summand of P, hence projective.

By exactness, a maps $\mathbb{Z}$ into S. Also b_2 restricts to a map β from S into the kernel of ϵ in $\mathbb{Z}[G]$. The following sequence is obtained:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{a} S \xrightarrow{\beta} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \qquad (3)$$

Lemma 1.4.3 *This sequence is exact.*

Proof of lemma: The image of β must equal the kernel of ϵ, as for any $x \in ker(\epsilon)$, $b_1 \oplus b_2$ must map some element of P to $(0, x) \in \mathbb{Z}[G]^{n-1} \oplus \mathbb{Z}[G]$. This element of P must lie in S. (Note S was defined as the kernel of b_1).

The kernel of β in S is by definition the intersection of the kernels of b_1 and b_2 in P. This equals the kernel of $b_1 \oplus b_2$ in P, and hence is equal to the image of a in S. Therefore (3) is exact.

□

This completes the proof of the proposition.

□

Lemma 1.4.4 *All modules and cokernels in (3) are torsion free and of finite $\mathbb{Z}$- rank.*

Proof: $\mathbb{Z}$ and $\mathbb{Z}[G]$ are torsion free and of finite $\mathbb{Z}$- rank. S is finitely generated and projective, hence torsion free and of finite $\mathbb{Z}$- rank. By exactness, the cokernels are submodules of torsion free modules of finite $\mathbb{Z}$- rank, hence themselves torsion free and of finite $\mathbb{Z}$- rank.

□

Corollary 1.4.5 *The sequence (3) may be dualized to get the exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\epsilon^*} \mathbb{Z}[G] \xrightarrow{\beta^*} T \xrightarrow{a^*} \mathbb{Z} \longrightarrow 0 \qquad (4)$$

where T is the dual of S.

T is projective, as dualizing commutes with taking direct sums.

Lemma 1.4.6 *There is no surjective homomorphism from T to $\mathbb{Z} \oplus \mathbb{Z}$.*

Proof: Tensoring (4) with $\mathbb{Q}$, the rationals, yields

$$0 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}[G] \longrightarrow T \otimes \mathbb{Q} \longrightarrow \mathbb{Q} \longrightarrow 0$$

This is an exact sequence of finitely generated modules over $\mathbb{Q}[G]$. As $\mathbb{Q}[G]$ is semi-simple, the 'Whitehead trick' may be performed: $T \otimes \mathbb{Q} \oplus \mathbb{Q} \cong \mathbb{Q}[G] \oplus \mathbb{Q}$.

Cancellation gives: $T \otimes \mathbb{Q} \cong \mathbb{Q}[G]$.

Suppose f was a surjection $T \twoheadrightarrow \mathbb{Z} \oplus \mathbb{Z}$. Tensoring f with $\mathbb{Q}$ would yield a $\mathbb{Q}[G]$-linear surjection $\mathbb{Q}[G] \twoheadrightarrow \mathbb{Q} \oplus \mathbb{Q}$. This is impossible because surjections between finitely generated $\mathbb{Q}[G]$ modules split and $\mathbb{Q}[G]$ does not contain a copy of $\mathbb{Q} \oplus \mathbb{Q}$.

$\square$

For any $\mathbb{Z}[G]$ module, M, let M_G denote the module resulting from quotienting M by the submodule generated by elements of the form
$m(1-g),\ m \in M,\ g \in G$.

Let $p : T \twoheadrightarrow T_G$ denote the natural surjection. For some $\mathbb{Z}[G]$ module $W, T \oplus W \cong \mathbb{Z}[G]^l$, for some integer l. So $T_G \oplus W_G \cong (T \oplus W)_G \cong (\mathbb{Z}[G]^l)_G \cong \mathbb{Z}^l$.
Therefore, $T_G \cong \mathbb{Z}^j$ for some $j \leq l$. But as p surjects onto it, T_G must equal $\mathbb{Z}$ or 0, by lemma 1.4.6.

Any map from T to a $\mathbb{Z}[G]$ module with trivial G-action must factor through p. So a^* in (4) factors through p.

$$\begin{array}{ccccccccc} & & & & T_G & & & & \\ & & & {}^{p}\nearrow & \downarrow q & & & & \\ 0 \longrightarrow \mathbb{Z} & \xrightarrow{\epsilon^*} & \mathbb{Z}[G] & \xrightarrow{\beta^*} & T & \xrightarrow{a^*} & \mathbb{Z} \longrightarrow 0 \end{array}$$

α^* is a surjection, so $T_G \neq 0$, hence $T_G \cong \mathbb{Z}$. q is a surjection $\mathbb{Z} \to \mathbb{Z}$. Hence q must be an isomorphism and $ker(\alpha^*) = ker(p)$. Consequently, the following sequence is exact

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\epsilon^*} \mathbb{Z}[G] \xrightarrow{\beta^*} T \xrightarrow{p} \mathbb{Z} \longrightarrow 0 \tag{5}$$

Lemma 1.4.7 *Let G be a finite group, of order k, having period 2. For any $\mathbb{Z}[G]$- module A, with trivial G-action, $H^1(G; A) = ker(\times k : A \to A)$, the kernel of multiplication by k.*

Proof: Combining (3) and (5) over $\mathbb{Z}$ gives the first few terms of a resolution for $\mathbb{Z}$ over $\mathbb{Z}[G]$:

$$\begin{array}{c} \cdots\cdots\cdots \longrightarrow S \xrightarrow{\beta} \mathbb{Z}[G] \xrightarrow{\epsilon^*\circ\epsilon} \mathbb{Z}[G] \xrightarrow{\beta^*} T \overset{p}{\twoheadrightarrow} \mathbb{Z} \\ \epsilon \searrow \quad \nearrow \epsilon^* \\ \mathbb{Z} \end{array} \tag{6}$$

A map from $\mathbb{Z}[G]$ to A is determined by the element of A to which the identity is sent. Hence $\mathrm{Hom}_{ZG}(\mathbb{Z}[G], A)$ may be identified with A.

$\epsilon^* \circ \epsilon$ sends $1 \in \mathbb{Z}[G]$ to $\sum_{g \in G} g \in \mathbb{Z}[G]$. So if $f : \mathbb{Z}[G] \to A$ sends $1 \in \mathbb{Z}[G]$ to $a \in A$, then

$$f\epsilon^* \circ \epsilon(1) = f(\sum_{g \in G} g) = f(1) \sum_{g \in G} g = ak$$

Suppose f is a map from T to A. As the G- action on A is trivial, f must factor through p, hence $f\beta^* = 0$.

$$\cdots\cdots\cdots \longrightarrow S \xrightarrow{\beta} \mathbb{Z}[G] \xrightarrow{\epsilon^* \circ \epsilon} \mathbb{Z}[G] \xrightarrow{\beta^*} T \xrightarrow{p} \mathbb{Z}$$

$$f \downarrow \quad \swarrow$$

$$A$$

Hence the cochain obtained by applying $\mathrm{Hom}(\bullet, A)$ to (6) begins

$$\cdots\cdots\cdots \longleftarrow A \xleftarrow{\times k} A \xleftarrow{0} \mathrm{Hom}_{ZG}(T, A)$$

where k is the order of G. From this we see that $H^1(G; A)$ is the kernel of multiplication by k on A.

□

The following is a standard result:

Proposition 1.4.8 *Let G be a finite group and A an Abelian group. Regarding A as a $\mathbb{Z}[G]$-module with trivial G-action* $H^1(G; A) \cong \mathrm{Hom}(G, A)$.

Proposition 1.4.9 *Let G be a finite group and A an Abelian group. Let G' denote the commutator subgroup of G. Then* $\mathrm{Hom}(G, A) \cong \mathrm{Hom}(G/G', A)$.

Proof: Given a homomorphism from G to A, G' must lie in its kernel. Hence any homomorphism from G to A factors uniquely through a homomorphism from G/G' to A.

□

Let $\mathbf{C}^*$ denote the Abelian group, consisting of unit complex numbers, with addition given by complex multiplication.

Proposition 1.4.10 *If B is a finite Abelian group then* $\mathrm{Hom}(B, \mathbb{C}^*) \cong B$

Proof: Let $\mathbb{Z}_r$ denote the cyclic group of order r and let b be a generator of it. A map $\mathbb{Z}_r \rightarrow \mathbf{C}^*$ is determined by which r^{th} root of unity b is mapped to. So

$\text{Hom}(\mathbb{Z}_r, \mathbf{C}^*)$ is generated by any map sending b to a primitive r^{th} root of unity. Hence $\text{Hom}(\mathbb{Z}_r, \mathbf{C}^*) \cong \mathbb{Z}_\mathbf{r}$

Now any finite Abelian group B is the product of finite cyclic groups: $B \cong \bigoplus_{i=1}^m \mathbb{Z}_{r_i}$

So

$$\text{Hom}(B, \mathbb{C}^*) \cong \text{Hom}(\bigoplus_{i=1}^m \mathbb{Z}_{r_i}, \mathbb{C}^*) \cong \bigoplus_{i=1}^m \text{Hom}(\mathbb{Z}_{r_i}, \mathbb{C}^*) \cong \bigoplus_{i=1}^m \mathbb{Z}_{r_i} \cong B$$

□

$\mathbb{C}^*$ can be regarded as a $\mathbb{Z}[G]$- module, assuming the trivial action of G on $\mathbb{C}^*$. Note that the kernel of multiplication by an integer, k on $\mathbb{C}^*$, is the cyclic group of k^{th} roots of unity.

Theorem 1.4.11 *Let G be a finite group having period 2. Then G is cyclic.*

Proof: Given a finite group, G, which has period 2, let k denote the order of G and let G' denote its commutator subgroup.

$$G/G' \cong \text{Hom}(G/G', \mathbb{C}^*) \cong \text{Hom}(G, \mathbb{C}^*) \cong H^1(G, \mathbb{C}^*) \cong \mathbb{Z}_k$$

(by propositions 1.4.10, 1.4.9, 1.4.8, and lemma 1.4.7)

As $\mathbb{Z}_k$ has the same order as G,

$$G \cong G/G' \cong \mathbb{Z}_k$$

□

The converse is also true:

Proposition 1.4.12 *If a finite group G is cyclic, then it has period 2.*

Proof: Let t be a generator of G and let n be the order of t. IG is the submodule of $\mathbb{Z}[G]$ consisting of elements of the form $\sum_{i=0}^{n-1} t^i\lambda_i$ with $\lambda_i \in \mathbb{Z}$, $\sum_{i=0}^{n-1} \lambda_i = 0$. It is generated by the element $t-1$.

Let σ denote $\sum_{i=0}^{n-1} t^i$.

$(t-1)\sigma = 0$. If $(t-1)\mu = 0$ where $\mu = \sum_{i=0}^{n-1} \mu_i t^i$, then equating coefficients gives $\mu_i = \mu_{i+1}$. Hence σ divides μ. Therefore $IG \cong \mathbb{Z}[G]/\sigma\mathbb{Z}[G]$.

There exist exact sequences

$$0 \longrightarrow IG \longrightarrow \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \tag{7}$$

and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\epsilon^*} \mathbb{Z}[G] \longrightarrow IG^* \longrightarrow 0 \tag{8}$$

where (8) is the dual of (7).

As $\epsilon^*(1) = \sigma$, from (8) it is observed that IG^* is isomorphic to $\mathbb{Z}[G]/\sigma\mathbb{Z}[G]$, which is isomorphic to IG. Combining (8) and (7) over the isomorphism $IG \equiv IG^*$ gives

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathbb{Z} \xrightarrow{\epsilon^*} \mathbb{Z}[G] & & \longrightarrow & & \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \\ & \searrow & & \nearrow & \\ & & IG^* \cong IG & & \end{array}$$

This is exact

□

So a finite group is cyclic if and only if it has period 2.

If G is a finite group which is not cyclic, there can be no isomorphism between IG and IG^*. In fact they cannot be stably equivalent, as if they were, we would have

$IG \oplus \mathbb{Z}[G]^n \cong IG^* \oplus \mathbb{Z}[G]^m$, for some n, m. It would then be possible to construct the following exact sequence:

$$0 \longrightarrow \mathbb{Z} \stackrel{\epsilon^*\oplus 0}{\longrightarrow} \mathbb{Z}[G] \bigoplus \mathbb{Z}[G]^m \longrightarrow \mathbb{Z}[G] \bigoplus \mathbb{Z}[G]^n \stackrel{\epsilon,\ 0}{\longrightarrow} \mathbb{Z} \longrightarrow 0$$

$$\searrow \qquad \nearrow$$

$$IG^* \bigoplus \mathbb{Z}[G]^m \cong IG \bigoplus \mathbb{Z}[G]^n$$

contradicting theorem 3.1.11.

Note that no infinite group can have period 2 in the sense that

Proposition 1.4.13 *Let H be any infinite group. Let $f : \mathbb{Z} \to P$ be a map for some projective $\mathbb{Z}[H]$- module, P. Then f=0.*

Proof: Suppose we have a non- zero map $f : \mathbb{Z} \to P$. A module Q may be chosen, such that $P \oplus Q$ is free. Then $f \oplus 0 : \mathbb{Z} \to P \oplus Q$ may be composed with projection onto a copy of $\mathbb{Z}[H]$, to yield a non-zero map $g : \mathbb{Z} \to \mathbb{Z}[H]$. Let $x = g(1)$. By $\mathbb{Z}[H]$- linearity, we know that $xh = x$ for all $h \in H$. However, as x is non-zero, it is expressible as a finite sum $\sum_i g_i\lambda_i$, with λ_i integers and λ_j non-zero for some j. Then, choosing h such that $g_jh \neq g_k$ for any k, we have $xh \neq x$, giving the desired contradiction. □

Proposition 1.4.14 *If a finite group G acts freely, on a circle, then it is cyclic.*

Proof: By lifting the cells of the quotient manifold to the circle, we would get a cellular resolution of the circle:

$$\mathbb{Z} \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z}$$

Any fixed point free map, f, on a circle is homotopic to the antipodal map, as for each $x \in S^1$, we can choose a unique shortest arc between $f(x)$ and $-x$. f must therefore preserve orientation.

The group action on the left hand copy of the integers is therefore trivial. F_1 and F_0 are clearly finitely generated and free, so G has period 2 and is cyclic.

□

Finally, in this section, we apply theorem 1.4.11 to obtain a lemma which we use in the following section.

Let G be a finite group.

Definition 1.4.15 *Quaternionic* We say a real representation of G is quaternionic if its endomorphism ring is $\mathbb{H}$, the quaternions.

Let $V_1, \cdots, V_k$ be the irreducible real representation of G. Given a $\mathbb{Z}[G]$- module, M, of finite rank, we have the decomposition:

$$M \otimes \mathbb{R} = \bigoplus_{i=1}^{k} V_i^{n_i}$$

Definition 1.4.16 *Eichler* M satisfies the Eichler condition precisely when V_i quaternionic implies that $n_i \neq 1$, for $i = 1, \cdots, k$.

Theorem 1.4.17 *(Swan-Jacobinski) Let M be a torsion free $\mathbb{Z}[G]$- module of finite $\mathbb{Z}$-rank. Suppose $M \oplus \mathbb{Z}[G]$ satisfies the Eichler condition. Let L be a $\mathbb{Z}[G]$- module with* $\mathrm{rk}_{\mathbb{Z}}(L) \geq \mathrm{rk}_{\mathbb{Z}}(M \oplus \mathbb{Z}[G])$*, and L stably equivalent to M. Then we have $L \cong M \oplus \mathbb{Z}[G]^r$ for some $r \geq 1$. (See [3] §15)*

In particular, note that for any finite group, $\mathbb{Z}[G]^2 \otimes \mathbb{R}$ will contain more than one copy of any irreducible module. Hence $\mathbb{Z}[G]^2$ satisfies the Eichler condition and has the form $\mathbb{Z}[G] \oplus \mathbb{Z}[G]$. Therefore any stably free module of $\mathbb{Z}[G]$ -rank greater than 1, must be free.

Note also that if G is cyclic, G does not have any irreducible real representations with endomorphism ring $\mathbb{H}$ so $0 \oplus \mathbb{Z}[G]$ satisfies the Eichler condition and all finitely generated stably free modules are free.

Lemma 1.4.18 *Let G be a finite group. Suppose there exists an exact sequence over $\mathbb{Z}[G]$*

$$SF \xrightarrow{\partial_1} \mathbb{Z}[G]^a \longrightarrow \mathbb{Z} \longrightarrow 0$$

where SF is a finitely generated stably free module. Then SF is free.

Proof: If the $\mathbb{Z}[G]$- rank of SF is greater than 1, the result would follow from the Swan-Jacobinski Theorem. Suppose the $\mathbb{Z}[G]$- rank of SF is 1. It is sufficient to prove that G is cyclic as then all finitely generated stably free modules over it would be free. We will show that G has period 2, as then it must be cyclic, by theorem 1.4.11.

Exactness and consideration of $\mathbb{Z}$- rank imply that the kernel of ∂_1, K, must have $\mathbb{Z}$- rank congruent to 1 modulo the order of the group. The $\mathbb{Z}$- rank of K must be less than that of SF. Hence if SF has $\mathbb{Z}[G]$- rank 1, then K, must have $\mathbb{Z}$- rank 1. By the 'Whitehead Trick', $K \otimes \mathbb{Q} \cong \mathbb{Q}$ as $\mathbb{Q}[G]$- modules. Hence the G- action on K is trivial and we have an exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow SF \xrightarrow{\partial_1} \mathbb{Z}[G]^a \longrightarrow \mathbb{Z} \longrightarrow 0$$

□

Lemma 1.4.19 *Let G be a finite group. Suppose there exists an exact sequence over $\mathbb{Z}[G]$*

$$SF \xrightarrow{\partial_2} \mathbb{Z}[G]^b \xrightarrow{\partial_1} \mathbb{Z}[G]^a \longrightarrow \mathbb{Z} \longrightarrow 0$$

where SF is a finitely generated stably free module. Then SF is free.

Proof: If the $\mathbb{Z}[G]$- rank of SF is greater than 1, the result would follow from the Swan-Jacobinski Theorem. Suppose the $\mathbb{Z}[G]$- rank of SF is 1. It is sufficient to prove that G is cyclic as then all finitely generated stably free modules over it would be free. We will show that G has period 2, as then it must be cyclic, by theorem 1.4.11.

Exactness and consideration of $\mathbb{Z}$- rank imply that the kernel of ∂_1, K', must have $\mathbb{Z}$- rank congruent to 1 modulo the order of the group. The $\mathbb{Z}$- rank of K' must be less than that of SF. Hence if SF has $\mathbb{Z}[G]$- rank 1, then K', must have $\mathbb{Z}$- rank 1. By the 'Whitehead Trick', $K \otimes \mathbb{Q} \cong \mathbb{Q}$ as $\mathbb{Q}[G]$- modules. Hence the G- action on K' is trivial and we have an exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G]^b \xrightarrow{\partial_1} \mathbb{Z}[G]^a \longrightarrow \mathbb{Z} \longrightarrow 0$$

□

§1.5 Free Resolutions

The main theorem of this section, theorem 1.5.1, essentially shows that the homotopy type of any resolution may be represented by a resolution with a prespecified initial segment:

Theorem 1.5.1 *Let*

$$F_m \xrightarrow{\partial_m} F_{m-1} \xrightarrow{\partial_{m-1}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots F_1 \xrightarrow{\partial_1} F_0 \dashrightarrow M \dashrightarrow 0 \qquad [1]$$

and

$$G_n \xrightarrow{\partial'_n} G_{n-1} \xrightarrow{\partial'_{n-1}} \ldots\ldots G_1 \xrightarrow{\partial'_1} G_0 \dashrightarrow M \dashrightarrow 0$$

be exact sequences, over a ring R with $n < m$ and the F_i, G_i finitely generated free modules. Then there exists an exact sequence over R:

$$F_m \xrightarrow{\partial_m} F_{m-1} \xrightarrow{\partial_{m-1}} \ldots\ldots \xrightarrow{\partial_{n+3}} F_{n+2} \longrightarrow SF \longrightarrow G_n \xrightarrow{\partial'_n} \ldots\ldots\ldots G_1 \xrightarrow{\partial'_1} G_0 \dashrightarrow M \dashrightarrow 0$$

which is chain homotopy equivalent to [1] with SF finitely generated, stably free.

Proof: By Schanuel's Lemma, there exist finitely generated free modules K, L, such that

$$F_n \oplus L \xrightarrow{\partial_n \oplus 0} F_{n-1} \xrightarrow{\partial_{n-1}} \ldots\ldots F_1 \xrightarrow{\partial_1} F_0 \qquad [2]$$

is chain homotopy equivalent to

$$G_n \oplus K \xrightarrow{\partial'_n \oplus 0} G_{n-1} \xrightarrow{\partial'_{n-1}} \ldots\ldots G_1 \xrightarrow{\partial'_1} G_0 \qquad [3]$$

Let f denote a chain homotopy equivalence from [2] to [3] and let g denote a chain homotopy inverse to f, from [3] to [2]. Let I denote a chain homotopy from $1_{[2]}$ to gf and let J denote a chain homotopy from $1_{[3]}$ to fg. So $I_{r-1}\partial_r + \partial_{r+1}I_r = g_r f_r - 1_{[2]}$ and $J_{r-1}\partial'_r + \partial'_{r+1}J_r = f_r g_r - 1_{[3]}$.

[1] is chain homotopy equivalent to

$$F_m \xrightarrow{\partial_m} F_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_{n+2}} F_{n+1} \oplus L \xrightarrow{\partial_{n+1} \oplus 1_L} F_n \oplus L \xrightarrow{\partial_n \oplus 0} F_{n-1} \cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0$$

[4]

Proposition 1.5.2 *The following sequence is exact:*

$$F_m \xrightarrow{\partial_m} F_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_{n+2}} F_{n+1} \oplus L \xrightarrow{f_n \circ (\partial_{n+1} \oplus 1_L)} G_n \oplus K \xrightarrow{\partial'_n \oplus 0} G_{n-1} \xrightarrow{\partial'_{n-1}} \ldots G_1 \xrightarrow{\partial'_1} G_0$$

[5]

Proof of Proposition: The image of $\partial_{n+1} \oplus 1_L$ is the kernel of $\partial_n \oplus 0$ and f_n restricts to an isomorphism from the kernel of $\partial_n \oplus 0$ to the kernel of $\partial'_n \oplus 0$, because it is the last term in a chain homotopy. Hence the image of $f_N \circ (\partial_{n+1} \oplus 1_L)$ is the kernel of $\partial'_n \oplus 0$ and the kernel of $f_N \circ (\partial_{n+1} \oplus 1_L)$ is the kernel of $\partial_{n+1} \oplus 1_L$. □

Let $\hat{f}_r = f_r$ for $r \leq n$ and $\hat{f}_r = 1$ for $r > n$. Let $\hat{g}_r = g_r$ for $r \leq n$ and $\hat{g}_r = 1$ for $r > n$.

Proposition 1.5.3 *$\hat{f}$ is a chain map* [4] *to* [5] *and $\hat{g}$ is a chain map* [5] *to* [4].

Proof of Proposition: For commutativity, it is sufficient to check that $g_n f_n \partial_n = \partial_n$:

$$g_n f_n \partial_n = \partial_n + I_{n-1} \partial_{n-1} \partial_n = \partial_n$$ □

Proposition 1.5.4 *$\hat{f}$ and $\hat{g}$ are homotopy inverse to each other.*

Proof of Proposition: Let $\hat{I}_r = I_r$ for $r \leq n-1$ and $\hat{I}_r = 0$ for $r > n-1$. Let $\hat{J}_r = J_r$ for $r \leq n-1$ and $\hat{J}_r = 0$ for $r > n-1$. Then $\hat{I}$ is a homotopy from $\hat{g}\hat{f}$ to 1 and $\hat{J}$ is a homotopy from $\hat{f}\hat{g}$ to 1. □

So [1] is chain homotopy equivalent to [4] which is chain homotopy equivalent to [5].

K is in the kernel of $\partial'_n \oplus 0$ so it is in the image of $f_n \circ (\partial_{n+1} \oplus 1_L)$. Therefore composing $f_n \circ (\partial_{n+1} \oplus 1_L)$ with projection onto K gives a surjection. Let SF denote the kernel of this surjection. As K is free, this surjection splits. Hence [5] can be written as

$$F_m \xrightarrow{\partial_m} F_{m-1} \xrightarrow{\partial_{m-1}} \ldots\ldots \xrightarrow{\partial_{n+2}} SF \oplus K \xrightarrow{D} G_n \oplus K \xrightarrow{\partial'_n} G_{n-1} \xrightarrow{\partial'_{n-1}} \ldots\ldots G_1 \xrightarrow{\partial'_1} G_0 \qquad [6]$$

Note $SF \oplus K = F_{n+1} \oplus L$, so SF is finitely generated, stably free. With respect to the above decompositions, let D be represented by

$$D = \begin{bmatrix} d & \phi \\ 0 & 1_K \end{bmatrix}$$

The image of ∂_{n+2} is contained in the kernel of D, hence in SF. We have a sequence

$$F_m \xrightarrow{\partial_m} F_{m-1} \xrightarrow{\partial_{m-1}} \ldots\ldots \xrightarrow{\partial_{n+2}} SF \xrightarrow{d} G_n \xrightarrow{\partial'_n} G_{n-1} \xrightarrow{\partial'_{n-1}} \ldots\ldots G_1 \xrightarrow{\partial'_1} G_0 \qquad [7]$$

Proposition 1.5.5 [7] *is exact.*

Proof of Proposition: If an element of SF is in the kernel of d it is in the kernel of D, hence the image of ∂_{n+2}. If an element of G_n is in the kernel of ∂'_n, then it is in the image of D. However if $Dx \in G_n$ then $x \in SF$ and $Dx = dx$. □

[7] is chain homotopy equivalent to

$$F_m \xrightarrow{\partial_m} F_{m-1} \xrightarrow{\partial_{m-1}} \ldots\ldots \xrightarrow{\partial_{n+2}} SF \oplus K \xrightarrow{D'} G_n \oplus K \xrightarrow{\partial'_n} G_{n-1} \xrightarrow{\partial'_{n-1}} \ldots\ldots G_1 \xrightarrow{\partial'_1} G_0 \qquad [8]$$

where D' is represented by $D = \begin{bmatrix} d & 0 \\ 0 & 1_K \end{bmatrix}$

So to prove the theorem, it is sufficient to show that [6] is chain isomorphic to [8].

The image of ϕ is contained in the image of d and K is free, so there exists a map ψ which makes the following diagram commute:

$$\begin{array}{ccc} K & & \\ \downarrow \psi & \searrow & \phi \\ SF & \xrightarrow{d} & G_n \end{array}$$

Define a chain map h from [6] to [8] by h is the identity on all terms except $SF \oplus K$, where it is represented by the matrix

$$\begin{bmatrix} 1_{SF} & \psi \\ 0 & 1_K \end{bmatrix}$$

Define a chain map k from [8] to [6] by k is the identity on all terms except $SF \oplus K$, where it is represented by the matrix

$$\begin{bmatrix} 1_{SF} & -\psi \\ 0 & 1_K \end{bmatrix}$$

Then $hk = 1_{[8]}$ and $kh = 1_{[6]}$.

This completes the proof of Theorem 1.5.1. □

Let G be a finite group. We present two corollaries to theorem 1.5.1, lemma 1.4.18, and lemma 1.4.19.

Corollary 1.5.6 *Suppose we have an exact sequence*

$$0 \longrightarrow K \longrightarrow \mathbb{Z}[G]^b \xrightarrow{\partial_1} \mathbb{Z}[G]^a \longrightarrow \mathbb{Z} \longrightarrow 0$$

Then we have a short exact sequence of the form

$$0 \longrightarrow K \longrightarrow \mathbb{Z}[G]^{b'} \xrightarrow{\partial_1'} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

such that

$$ZG^b \xrightarrow{\partial_1} \mathbb{Z}[G]^a$$

is chain homotopy equivalent to

$$\mathbb{Z}[G]^{b'} \xrightarrow{\partial_1'} \mathbb{Z}[G]$$

Corollary 1.5.7 *Suppose we have exact sequences*

$$0 \longrightarrow J \longrightarrow \mathbb{Z}[G]^c \xrightarrow{\partial_2} \mathbb{Z}[G]^b \xrightarrow{\partial_1} \mathbb{Z}[G]^a \longrightarrow \mathbb{Z} \longrightarrow 0$$

and

$$0 \longrightarrow K \longrightarrow \mathbb{Z}[G]^{b'} \xrightarrow{\partial_1'} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

Then we have a short exact sequence of the form

$$0 \longrightarrow J \longrightarrow \mathbb{Z}[G]^{c'} \xrightarrow{\partial_2'} \mathbb{Z}[G]^{b'} \xrightarrow{\partial_1'} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

such that

$$\mathbb{Z}[G]^c \xrightarrow{\partial_2} \mathbb{Z}[G]^b \xrightarrow{\partial_1} \mathbb{Z}[G]^a$$

is chain homotopy equivalent to

$$\mathbb{Z}[G]^{c'} \xrightarrow{\partial_2'} \mathbb{Z}[G]^{b'} \xrightarrow{\partial_1'} \mathbb{Z}[G]$$

In particular, one may fix an exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}[G]^b \xrightarrow{\partial_1} \mathbb{Z}[G] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

with K a minimal element of its stable class, so that any algebraic 2-complex is chain homotopy equivalent to one of the form

$$\mathbb{Z}[G]^c \xrightarrow{\partial_2} \mathbb{Z}[G]^b \xrightarrow{\partial_1} \mathbb{Z}[G]$$

So we may parametrize algebraic 2-complexes, by maps from finitely generated free modules to K.

Chapter 2

The D2 problem for Dihedral groups

This chapter will be concerned with the D(2)- problem for dihedral groups of order $4n$. In the first section we state and briefly discuss the general D(n)- problem, before focusing on the D(2) problem.

In §2.2 we show that for dihedral groups whose order is a power of 2, the Swan map (definition 1.2.7) is surjective. In §2.3 we use this result to show that D_8 satisfies the D2 property.

In §2.4 we give an exhaustive list of candidates for minimal elements of $\Omega_3(Z)$, over any dihedral group of order $4n$. Finally, in §2.5 we note that whilst not periodic over $\mathbb{Z}$, the dihedral groups of order $8n+4$ are periodic over $\mathbb{Z}[\frac{1}{2}]$.

§2.1 The D(n) problem

In this section we introduce the D(n) problem by outlining some of the content of [3].

We work in the category of CW- complexes and continuous maps. (See [15], chapter II). In particular, geometric complex will refer to a connected finite dimensional, finite, CW complex, and geometric n- complex will refer to an n dimensional geometric complex. If geometric complexes X and Y are homotopy equivalent, we write $X \sim Y$.

Following the standard convention, we regard elements of $\tilde{X}$ as equivalence classes of paths in X, based at b. Let $p : \tilde{X} \to X$ denote the covering map, which sends a path to its endpoint. .

If X is a geometric complex, with base point b, then we may give $\tilde{X}$ the structure of a geometric complex in the following way. For each cell, D, of X, select a point $x_D \in \mathrm{Int}(D)$ and let $\phi_D : D \to X$ denote the natural insertion.

Then for each $y \in \tilde{X}$ such that $p(y) = \phi_D(x_D)$, we may define a cell of $\tilde{X}$ which we denote $\tilde{D_y}$. As a closed ball we identify its points with those of D. We define a map $\phi_y : \tilde{D_y} \to \tilde{X}$ which sends $t \in \tilde{D_y}$ to the path $y \circ \phi_D(z)$, where z is a path in $\tilde{D_y}$ connecting x_D to t. We say $\tilde{D_y}$ lies above D.

To reconstruct $\tilde{X}$ as a geometric complex, we need the cells $\tilde{D_y}$ together with attaching maps. If the $n-1$ skeleton has been constructed, then $\phi_y|_{\partial\tilde{D_y}}$ may be regarded as the attaching map for $\tilde{D_y}$ for each n- cell D, once we have shown $\phi_y(\partial\tilde{D_y})$ is contained in the $n-1$ skeleton of $\tilde{X}$. To see this, note that it is contained in the universal cover of the $n-1$ skeleton of X. Let q be an element of this. Then $p(q) = \phi_D(s)$ for some $n-1$ cell, D. Let z be a path in D, from s to x_D. Then let $y = q \circ \phi_D(z)$. Then q is in the $n-1$ cell $\tilde{D_y}$.

We denote the corresponding complex of abelian groups $C_*(\tilde{X})$. Let $G = \pi_1(X)$. Then G acts transitively on the cells which lie above D, for each cell D of X. Hence the abelian groups in $C_*(\tilde{X})$ have the structure of $\mathbb{Z}[G]$ modules. These modules are free, as no non-trivial element of $\pi_1(X)$ fixes any cell. The boundary maps are linear with respect to $\mathbb{Z}[G]$. Hence $C_*(\tilde{X})$ is an algebraic complex over $\mathbb{Z}[G]$, which we will refer to as the associated algebraic complex of X.

If B^n denotes an n- dimensional ball, then $C_*(\widetilde{X \vee B^n})$ is given by applying a simple homotopy equivalence to $C_*(\tilde{X})$, whereby a copy of $\mathbb{Z}[G]$ is added to $C_n(\tilde{X})$ and $C_{n-1}(\tilde{X})$.

From our definition, any geometric n- complex X is connected. Hence $\tilde{X}$ is necessarily connected and simply connected. As a result $C_*(\tilde{X})$ is exact at C_1 and the cokernel of the last boundary map ∂_1 is $\mathbb{Z}$. Hence $C_*(\tilde{X})$ is an algebraic n- complex.

Given a geometric complex X and a module, A, over $\pi_1(X)$, we define $H^n(X;A) = H^n((C_*\tilde{X}, \partial_*); A)$ and $H_n(X;A) = H_n((C_*\tilde{X}, \partial_*); A)$. When a module is used in this context, we refer to it as a coefficient bundle. Note that to avoid ambiguity, we must specify whether A is an abelian group (in which case we are referring to the (co)homology of $C_*(X)$), or a module over the fundamental group. Unless otherwise stated, in the context of coefficients for a (co)homology group, $\mathbb{Z}$ will denote the abelian group, rather than the module with trivial group action.

The D(n) problem asks when a geometric $n+1$- complex, X, is homotopy equivalent to a geometric n complex. C.T.C.Wall has shown that for $n \geq 3$, a necessary and sufficient condition is that $H^{n+1}(X;A) = 0$ for all coefficient bundles (see [14]). Note that this condition is equivalent to saying that $(C_*\tilde{X}, \partial_*)$ is chain homotopy equivalent to an algebraic n- complex, as if $H^{n+1}(X; C_{n+1}(\tilde{X})) = 0$ then the following diagram commutes:

$$\begin{array}{ccc} C_{n+1}(\tilde{X}) & \xrightarrow{\partial_{n+1}} & C_n(\tilde{X}) \\ {\scriptstyle 1}\downarrow & \swarrow{\scriptstyle f} & \\ C_{n+1}(\tilde{X}) & & \end{array}$$

for some map f. Hence ∂_{n+1} splits and $C_{n+1}(\tilde{X})$ has some stably free complement S in $C_n(\tilde{X})$. Finally note a simple homotopy equivalence connects the complexes

$$C_{n+1}(\tilde{X}) \xrightarrow{\partial_{n+1}} C_{n+1}(\tilde{X}) \oplus S \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C_0(\tilde{X})$$

and

$$0 \longrightarrow S \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C_0(\tilde{X})$$

If $n > 0$, then in fact this complex will be chain homotopy equivalent to

$$S \oplus C_{n+1}(\tilde{X}) \xrightarrow{\partial_n \oplus 1} C_{n-1}(\tilde{X}) \oplus C_{n+1}(\tilde{X}) \longrightarrow \cdots \xrightarrow{\partial_1} C_0(\tilde{X})$$

This is a complex containing only free modules.

An example of why the weaker condition, $H_2(X;\mathbb{Z}) = 0$ is not sufficient is the Klein bottle, K. It is constructed by identifying the ends of an oriented cylinder, so that the induced orientations on the ends agree. Whilst $H_2(K;\mathbb{Z}) = 0$, it is not homotopy equivalent to a geometric 1- complex, because $H^2(K;\mathbb{Z}) = \mathbb{Z}_2$.

Lemma 2.1.1 *Let X be a geometric 1- complex and suppose $H^1(X;A) = 0$ for all coefficient bundles A. Then X is homotopy equivalent to a set of points.*

Proof: We know that the associated algebraic complex of X is chain homotopy equivalent to a 0- algebraic complex. Hence we have a short exact sequence:

$$0 \to F \to \mathbb{Z}$$

where F is free over $\pi_1(X)$. Hence the augmentation ideal of $\pi_1(X)$ must be 0. We may conclude that $\pi_1(X)$ is trivial. Hence X must be a geometric 1- complex with no cycles. It is therefore a set of trees, each of which is contractible to a point.

□

Now suppose that X is a geometric 2- complex and suppose $H^2(X;A) = 0$ for all coefficient bundles A. Then the associated algebraic complex of X is an algebraic 2- complex. In particular $H_2(X;\mathbb{Z}) = 0$, so the associated algebraic complex of X is exact. Consequently X is homotopy equivalent to $k(\pi_1(X), 1)$. If $\pi_1(X)$ is free, then $k(\pi_1(X), 1)$ must be homotopy equivalent to a wedge of circles which is a geometric 1- complex.

In fact $\pi_1(X)$ must be free as the following theorem implies:

Theorem 2.1.2 *(Stallings-Swan) Suppose that we have an exact sequence over $\mathbb{Z}[G]$, for some group G, of the form:*

$$0 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

where P_0 and P_1 are projective. Then G is free

We know that $\pi_1(X)$ satisfies the hypothesis of this theorem, as the associated algebraic complex of X is chain homotopy equivalent to an algebraic 1- complex.

Note also, that the theorem may be stated group theoretically. The existence of the exact sequence

$$0 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

is equivalent to $H^2(G; A)$ vanishing for all coefficient bundles A. By proposition 1.3.3, this in turn is equivalent to saying that every group surjection, $j : E \to G$, with $\ker(j)$ abelian, will split. So the theorem could be stated

Theorem 2.1.3 *If every group surjection, with abelian kernel, to a group G, splits, then G is free.*

We may conclude that the D(0), D(1), and D(n), $n \geq 3$ problems all have the same solution. In each case, a geometric complex is homotopy equivalent to one of dimension 1 less than it, precisely if its top cohomology group vanishes with respect to all coefficient bundles. We now discuss the extent to which this is known to hold for D(2).

Due to the relationship between 2- complexes and presentations of fundamental groups, it is natural to parametrize the D(2) problem over the fundamental group. So for a finite group G, the D(2) problem is:

Let X be a finite geometric 3- complex, with fundamental group G, and with vanishing third cohomology group for every coefficient bundle. Must X be homotopy equivalent to a finite geometric 2- complex?

If the answer is yes, then we say G satisfies the D(2) property.

We now outline some of the arguments in [3] which show that for each finite group G, the D(2) property for G is equivalent to every algebraic 2- complex over $\mathbb{Z}[G]$ being realized by a geometric 2- complex.

If X satisfies the hypothesis of the D(2) problem, then $C_*(\tilde{X}) \sim Y$ for some algebraic 2- complex, Y. Suppose every such algebraic 2- complex, was chain homotopy equivalent to $C_*(\tilde{Z})$, for some geometric 2- complex, Z. Then we would have $C_*(\tilde{X}) \sim C_*(\tilde{Z})$. From [3], theorem 59.4, we would then have $X \sim Y$ would then have.

Conversely, let (F_*, d_*) be an algebraic 2- complex of free modules, over $\mathbb{Z}[G]$, which is not realizable up to homotopy as a geometric 2- complex. Take any finite presentation of G and let Y denote its Cayley complex. By theorem 1.1.1 there exists a free finitely generated module F and a number n, such that $Y \vee \bigvee_{1=1}^{n} S^2$, is chain homotopy equivalent to

$$F_2 \oplus F \xrightarrow{d_2 \oplus 0} F_1 \xrightarrow{d_1} F_0$$

Let f denote the homotopy equivalence between this complex and $C_*(\tilde{Y'})$. Let the e_i be a basis for F. As F lies in the kernel of $d_2 \oplus 0$ and f_2 restricts to a map between second homology groups, for each i, $f_2(e_i)$ is an element of $\pi_2(Y')$. Starting with Y', attach one 3- cell, B_i for each i, where the attaching map from ∂B_i, is given by $f_2(e_i)$, the element of $\pi_2(Y')$. Let Z denote the resulting geometric 3- complex. Note that $C_3(\tilde{Z}) \cong F$ and $C_i(\tilde{Z}) \cong C_i(\tilde{Y'})$ for $i = 0, 1, 2$.

The algebraic complex (F_*, d_*) is related, via a simple homotopy equivalence to

$$F \xrightarrow{(0,1)} F_2 \oplus F \xrightarrow{d_2 \oplus 0} F_1 \xrightarrow{d_1} F_0$$

This is in turn chain homotopy equivalent to $C_*(\tilde{Z})$ via the following homotopy equivalence:

$$\begin{array}{ccccccc}
F & \xrightarrow{(0,1)} & F_2 \oplus F & \xrightarrow{d_2 \oplus 0} & F_1 & \xrightarrow{d_1} & F_0 \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
C_3(\tilde{Z}) & \xrightarrow{f_2} & C_2(\tilde{Z}) & \xrightarrow{\partial_2} & C_1(\tilde{Z}) & \xrightarrow{\partial_2} & C_0(\tilde{Z})
\end{array}$$

where the ∂_i are the boundary maps for Y.

Z cannot be homotopy equivalent to a geometric 2- complex, as the associated algebraic complex of such a complex would be chain homotopy equivalent to (F_*, d_*), contradicting the hypotheses that (F_*, d_*) is not realizable up to homotopy as a geometric 2- complex. However, Z clearly satisfies the hypotheses of the general D(2) problem as its associated algebraic complex is chain homotopy equivalent to (F_*, d_*).

Hence a finite group G satisfies the D(2) property if and only if every algebraic 2- complex is realizable up to homotopy, as a geometric 2- complex.

If G is a finite group then we also have the following characterization of homologically 2 dimensional complexes.

Lemma 2.1.4 *Over $\mathbb{Z}[G]$ an algebraic 3- complex, (F_i, d_i) is chain homotopy equivalent to an algebraic 2- complex if and only if $H_2((F_i, d_i), \mathbb{Z})$ is torsion free and $H_3((F_i, d_i), \mathbb{Z}) = 0$.*

Proof: As F_3^* is projective, d_3^* has a right inverse. $d_3 = d_3^{**}$ because the cokernel of d_3 is torsion free. Hence d_3 is the inclusion of a free summand into F_2.

□

A finite geometric 2- complex will always be homotopy equivalent to the Cayley complex of a finite presentation. Hence our problem reduces to a purely algebraic one. Following Fox, we may define an algorithm for constructing the associated algebraic complex of a finite Cayley complex directly from the finite presentation, and our question becomes:

Is every algebraic 2- complex chain homotopy equivalent to one constructed from a finite presentation?

Let $F_2 \stackrel{\partial_2}{\to} F_1 \stackrel{\partial_1}{\to} F_0$ be an algebraic 2- complex. The kernel of ∂_2 will be referred to as π_2 of the algebraic 2- complex. As the kernel of ∂_1 is equal to the image of ∂_2 and the cokernel of ∂_1 is isomorphic to the $\mathbb{Z}[G]$- module $\mathbb{Z}$, we have that $\pi_2(F_*, \partial_*) \in \Omega_3(\mathbb{Z})$.

Proposition 2.1.5 *Let X be a geometric 2- complex, with $\pi_1(X) = G$, for some finite group G. Then $\pi_2(C_*(\tilde{X})) \cong \pi_2(X)$ as modules over $\mathbb{Z}[G]$.*

Proof: We first define an isomorphism of abelian groups, and then show that it is $\mathbb{Z}[G]$- linear. From the definition, we have a natural identification of $H_2(\tilde{X}; \mathbb{Z})$ with $\pi_2(C_*(\tilde{X}))$. As $\tilde{X}$ is simply connected, the Hurewicz isomorphism theorem tells us that the Hurewicz homomorphism induces an isomorphism $h : \pi_2(\tilde{X}) \to H_2(\tilde{X}; \mathbb{Z})$. The covering map $p : \tilde{X} \to X$ induces an isomorphism $p_* : \pi_2(\tilde{X}) \to \pi_2(X)$.

Hence we have an isomorphism of Abelian groups $hp_*^{-1} : \pi_2(X) \to \pi_2(C_*(\tilde{X}))$, which lifts any element of $\pi_2(X)$ to $\tilde{X}$ and then applies the resulting map $H_2(S^2; Z) \to H_2(\tilde{X}; \mathbb{Z})$ to the positive generator of $H_2(S^2; \mathbb{Z})$. We must check that this isomorphism respects the action of G.

Given $g \in G$, and $\alpha \in \pi_2(X)$, by construction, we have that αg lifts to $(p_*^{-1}\alpha)g$, attached to the base point of $\tilde{X}$ via a lift of g. Hence the element of $H_2(\tilde{X};\mathbb{Z})$ which it sends the positive generator of $H_2(S^2;Z)$ to, is $(hp_*^{-1}\alpha)g$. So $hp_*^{-1}(\alpha g) = (hp_*^{-1}\alpha)g$.

□

Definition 2.1.6 *The Euler characteristic of an algebraic complex of free modules is the alternating sum of the dimensions of its modules.*

Our principal method of showing that a finite group, G, satisfies the D(2) property will be to show that any algebraic 2- complex over G, is determined up to chain homotopy equivalence by its Euler characteristic. Once that is done, it is only necessary to show that the algebraic complex with minimal Euler characteristic is chain homotopy equivalent to one arising from a presentation. Then all the others will necessarily be chain homotopy equivalent to the algebraic complex arising from the presentation, together with the appropriate number of trivial relations, "$e = e$" added.

A necessary precursor to implementing this method is to show that $\pi_2(X)$ for any algebraic two-complex, X, over a finite group G, is determined by its $\mathbb{Z}$- rank. Clearly, the Swan - Jacobinski theorem is a useful tool in this endeavor.

Given an algebraic 2- complex X, we may tensor it with $\mathbb{R}$. Then the "Whitehead trick" together with semi-simplicity of $\mathbb{R}[G]$ tell us that $\mathbb{R} \otimes_{\mathbb{Z}} \pi_2(X) = \mathbb{R} \otimes_{\mathbb{Z}} IG \oplus \mathbb{R}[G]^k$ for some integer k. Hence given a minimal element, $J \in \Omega_3(\mathbb{Z})$, we know that $J \oplus \mathbb{Z}[G]$ satisfies the Eichler condition. Any non-minimal element of $\Omega_3(\mathbb{Z})$ is therefore equal to $J \oplus \mathbb{Z}[G]^r$ for some integer r. The remaining problem is to prove that J is the unique minimal element of $\Omega_3(\mathbb{Z})$.

If for some group it is shown that there is a unique potential π_2 for each Euler characteristic, it remains to show that the π_2 determines the algebraic 2- complex.

From theorem 1.2.8, we know that the surjectivity of the Swan map is sufficient to show this.

We end this section with an example of a group satisfying the D(2) property.

Lemma 2.1.7 *The group $\{e\}$ satisfies the D(2) property.*

Proof: Over $\mathbb{Z}$, any finitely generated torsion free module is isomorphic to $\mathbb{Z}^k$ for some k. Also $\mathbb{Z}$ is a Noetherian ring. Consequently, any torsion free map $\mathbb{Z}^a \to \mathbb{Z}^b$, is the composition of projection onto a free summand of $\mathbb{Z}^a$, with inclusion into a free summand of $\mathbb{Z}^b$.

Hence any algebraic 2- complex over $\mathbb{Z}$, is related through simple homotopy equivalences to

$\mathbb{Z}^k \to 0 \to \mathbb{Z}$

for some k. This is realized geometrically as the wedge of k copies of S^2.

□

§2.2 Surjectivity of the Swan map.

The D(2) property has been verified for dihedral groups of order $4n+2$ in [3]. The remainder of this chapter is concerned with the D(2) property for dihedral groups of order $4n$. In the next section we prove that D_8 satisfies the D(2) property. We begin with more general considerations.

Let D_{4n} be the group given by the presentation, $< a, b \,|a^{2n} = b^2 = e,\ aba = b >$. Σ will denote $\sum_{i=0}^{2n-1} a^i$. This presentation has a Cayley complex, which in turn has an associated algebraic complex. This is an exact sequence over $\mathbb{Z}[D_{4n}]$:

$$J \hookrightarrow \mathbb{Z}[D_{4n}]^3 \xrightarrow{\partial_2} \mathbb{Z}[D_{4n}]^2 \xrightarrow{\partial_1} \mathbb{Z}[D_{4n}] \xrightarrow{\epsilon} \mathbb{Z} \quad (1)$$

ϵ is determined by mapping $1 \in \mathbb{Z}[D_{4n}]$ to $1 \in \mathbb{Z}$. J is the kernel of ∂_2. Let e_1, e_2 denote basis elements of $\mathbb{Z}[D_{4n}]^2$. Then $\partial_1 e_1 = a - 1,\ \partial_1 e_2 = b - 1$.

Let E_1, E_2, E_3 be basis elements of $\mathbb{Z}[D_{4n}]^3$, which correspond to the relations in the presentation so that:

$$\partial_2 E_1 = e_1\Sigma$$
$$\partial_2 E_2 = e_2(1+b)$$
$$\partial_2 E_1 = e_1 + e_2 a + e_1 ba - e_2 = e_1(1+ba) + e_2(a-1)$$

With respect to the basis $\{E_1, E_2, E_3\}$ and the basis $\{e1, e2\}$, ∂_2 is given by ;

$$\begin{bmatrix} \Sigma & 0 & 1+ba \\ 0 & 1+b & a-1 \end{bmatrix}$$

Let

$$\alpha_0 = 1 + a + b$$

$$\alpha_1 = \begin{bmatrix} 1+a-ba & b-1 \\ 0 & 1 \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} 1+a-ba & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is easily verified that:

Proposition 2.2.1 *The following diagram commutes:*

$$\begin{array}{ccccccccc} J & \hookrightarrow & \mathbb{Z}[D_{4n}]^3 & \xrightarrow{\partial_2} & \mathbb{Z}[D_{4n}]^2 & \xrightarrow{\partial_1} & \mathbb{Z}[D_{4n}] & \xrightarrow{\epsilon} & \mathbb{Z} \\ \downarrow \theta & & \downarrow \alpha_2 & & \downarrow \alpha_1 & & \downarrow \alpha_0 & & \downarrow 3 \\ J & \hookrightarrow & \mathbb{Z}[D_{4n}]^3 & \xrightarrow{\partial_2} & \mathbb{Z}[D_{4n}]^2 & \xrightarrow{\partial_1} & \mathbb{Z}[D_{4n}] & \xrightarrow{\epsilon} & \mathbb{Z} \end{array}$$

where θ is the restriction of α_2.

For the remainder we will assume 3 coprime to n. Note that if we regard the above diagram as a diagram of commutative $\mathbb{Z}$- modules and $\mathbb{Z}$- linear maps, there are well defined integer determinants for all the maps in the chain map. A map is an isomorphism if and only if it has determinant ± 1. (As the property of being an isomorphism is dependent only on surjectivity and injectivity, it does not depend on whether we are regarding modules as being over $\mathbb{Z}[D_{4n}]$, or $\mathbb{Z}$).

Note also that over $\mathbb{Z}$, all the maps in the exact sequences above are quotienting of a summand, followed by inclusion of a summand. Consequently, the following proposition holds;

Proposition 2.2.2 $3\text{Det}(\theta)\text{Det}(\alpha_1) = \text{Det}(\alpha_2)\text{Det}(\alpha_0)$

Proof: Let u be the restriction of α_1 to the kernel of ∂_1 and let v be the restriction of α_0 to the kernel of ϵ. Then by the previous discussion, we have

$$3\text{Det}(\theta)\text{Det}(\alpha_1) = \text{Det}(\theta)\text{Det}(u)\text{Det}(v)3 = \text{Det}(\alpha_2)\text{Det}(\alpha_0)$$

□

Proposition 2.2.3 $\text{Det}(1 + a + b) = -3$

Proof: Let A be the matrix for left multiplication by $1 + a + b$ in the regular representation, with basis $\{a^{2n-1}, a^{2n-2}, \cdots, a, 1, ba^{2n-1}, ba^{2n-2}, \cdots, ab, b\}$. Then the upper right quadrant of A and the lower left quadrant of A are copies of the identity matrix. The upper left quadrant has 1's along the diagonal and immediately above as well as a 1 in the bottom left corner. The lower right quadrant has 1's along the diagonal and immediately below, as well as a 1 in the top right corner. All the other entries in A are 0.

For example, if n were equal to 4, the matrix A would be

$$\left[\begin{array}{cccccccc|cccccccc}
1&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1\\
\\
1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1
\end{array}\right]$$

Label the rows of A, $v_1, v_2 .....v_{4n}$. We will perform row operations.

First let $v'_{2n} = v_{2n} - v_1 + v_2 - v_3 .... - v_{2n-1}$. Now let $v''_{2n} = v_{4n}$ and $v''_{4n} = v'_{2n}$. Let the remaining $v''_i = v_i$. This swap causes a change of sign in the determinant, so the matrix with rows v''_i has determinant -Det$A$. In the case $n = 4$, this matrix is

$$\left[\begin{array}{cccccccc|cccccccc}
1&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1\\
\\
1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&-1&1&-1&1&-1&1&-1&1
\end{array}\right]$$

For each $2n + 1 \leq i \leq 4n - 2$, let $v'''_i = v''_i + v''_{i+1} - v''_{i-2n}$.

Let $v'''_{4n-1} = v''_{4n-1} + v''_{2n} - v_{2n-1}''$.

For $i \leq 2n$ let $v'''_i = v''_i$.

When $n = 4$, the matrix with rows v_i''' is

$$
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{bmatrix}
$$

In general, the matrix with rows v_i''' has an upper triangular top left quadrant, with 1's along the diagonal and a lower left quadrant with no non-zero entries. Let B denote the lower right quadrant. Then $\mathrm{Det}(1 + a + b) = -\mathrm{Det}(B)$.

Cycle the top $2n - 1$ rows of B upwards to get the matrix B'. As this is a cycle of odd length, $\mathrm{Det}(B') = \mathrm{Det}(B)$. When $n = 4$, B' is

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{bmatrix}
$$

Label the rows of B' as $w_1, ..., w_{2n}$. Set $u_i = w_i - w_{i+1}$ for $i = 1, 2, \cdots, 2n-3$. Let B'' denote the matrix with rows u_i. After these row operations, we have $\mathrm{Det}(1+a+b) = -\mathrm{Det}(B'')$

When $n = 4$, B'' is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

We must consider two cases: n congruent to 1 modulo 3 and n congruent to 2 modulo 3.

If $n = 1$ modulo 3 then replace u_{2n-1} with

$u_{2n-1} - u_1 - u_2 - u_4 - u_5 - u_7 - u_8 \cdots - u_{2n-3}$.

Also, replace u_{2n} with

$u_{2n} + (u_1 - u_2 + u_3) + (u_7 - u_8 + u_9) + (u_{13} - u_{14} + u_{15}) \ldots + (u_{2n-7} - u_{2n-6} + u_{2n-5})$.

We are left with a matrix with 1's along the diagonal and 0's below, except in the last four columns. The 4 by 4 matrix in the bottom right corner is

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\mathrm{Det}(1 + a + b) = -\mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix} = -\mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} = -3$$

If $n = 2$ modulo 3 then replace u_{2n-1} with

$u_{2n-1} - u_1 - u_2 - u_4 - u_5 - u_7 - u_8 \ldots - u_{2n-5}$.

Also, replace u_{2n} with

$u_{2n} + (u_1 - u_2 + u_3) + (u_7 - u_8 + u_9) + (u_{13} - u_{14} + u_{15}) \ldots + (u_{2n-9} - u_{2n-8} + u_{2n-7})$.

We are left with a matrix with 1's along the diagonal and 0's below, except in the last four columns. The 4 by 4 matrix in the bottom right corner is

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

$$\mathrm{Det}(1+a+b) = -\mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} = -\mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$= -\mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & -1 \end{bmatrix} = -\mathrm{Det}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} = -3$$

□

Proposition 2.2.4 $\mathrm{Det}(2-b) = 3^{2n}$

Proof: Let A be the matrix for $2-b$ in the regular representation, with basis $\{1, b, a, ba, a^2, ba^2 ...... a^{2n-1}, ba^{2n-1}\}$. Then A consists of $2n$ two by two blocks of the form

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

along the diagonal. Hence $\mathrm{Det}(A) = 3^{2n}$

□

Proposition 2.2.5 $\mathrm{Det}(1 + a - ba) \neq 0$

Proof: Let $\alpha_2' = \begin{bmatrix} 2-b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The following diagram commutes:

$$\begin{array}{ccccccccc} J & \hookrightarrow & \mathbb{Z}[D_{4n}]^3 & \xrightarrow{\partial_2} & \mathbb{Z}[D_{4n}]^2 & \xrightarrow{\partial_1} & \mathbb{Z}[D_{4n}] & \xrightarrow{\epsilon} & \mathbb{Z} \\ \downarrow \eta & & \downarrow \alpha_2' & & \downarrow \alpha_1 & & \downarrow \alpha_0 & & \downarrow 3 \\ J & \hookrightarrow & \mathbb{Z}[D_{4n}]^3 & \xrightarrow{\partial_2} & \mathbb{Z}[D_{4n}]^2 & \xrightarrow{\partial_1} & \mathbb{Z}[D_{4n}] & \xrightarrow{\epsilon} & \mathbb{Z} \end{array}$$

where η is the restriction of α_2'

Therefore $3\mathrm{Det}(\eta)\mathrm{Det}(\alpha_1) = \mathrm{Det}(\alpha_2')\mathrm{Det}(\alpha_0)$

So $3 * \mathrm{Det}(\eta)\mathrm{Det}(1 + a - ba) = -3 * 3^{2n}$. Hence $\mathrm{Det}(1 + a - ba)$ cannot be 0. □

Proposition 2.2.6 *θ is an isomorphism*

Proof: $3\mathrm{Det}(\theta)\mathrm{Det}(\alpha_1) = \mathrm{Det}(\alpha_2)\mathrm{Det}(\alpha_0)$

Therefore $3\mathrm{Det}(\theta)\mathrm{Det}(1 + a - ba) = -3\mathrm{Det}(1 + a - ba)$. As $\mathrm{Det}(1 + a - ba)$ is non-zero, we can conclude that

$\mathrm{Det}(\theta) = -1$.

Hence θ is an isomorphism.

□

Recall the definition of the Swan map (Definition 1.2.7).

Corollary 2.2.7 *If $3 \in (\mathbb{Z}_{4n})^*$ then 3 is in the image of the Swan Map* $\mathrm{Aut}(\mathrm{J}) \to (\mathbb{Z}_{4n})^*$

Let us now consider dihedral groups of order 2^m for $m \geq 2$. Clearly 2^m is divisible by 4 and coprime to 3. Hence we know that 3 is in the image of the Swan Map.

Lemma 2.2.8 *2^m divides $3^{2^{m-3}} - 1 + 2^{m-1}$ for $m \geq 4$.*

Proof: We proceed by induction. $3^{2^{4-3}} - 1 + 2^{4-1} = 16$. So the proposition holds for $m = 4$. Now suppose it holds for some m. Then $2^m z = 3^{2^{m-3}} - 1 + 2^{m-1}$ for some z. Rearranging gives

$$3^{2^{m+1-3}} = (3^{2^{m-3}})^2 = (2^m z + 1 - 2^{m-1})^2$$

So

$$3^{2^{m+1-3}} - 1 + 2^{m+1-1} = (2^m z + 1 - 2^{m-1})^2 - 1 + 2^m$$

$$= 2^{2m} z^2 + 2^{2m-2} + 2^{m+1} z - 2^{2m} z = 2^{m+1}(2^{m-1}(z^2 - z) + 2^{m-3} + z)$$

So the proposition holds for $m + 1$. Hence by induction it holds for all m. □

Proposition 2.2.9 *The elements 3, -3 generate $(\mathbb{Z}/2^m)^*$ for $m \geq 2$.*

Proof: The order of $(\mathbb{Z}/2^m)^*$ is 2^{m-1}. $(\mathbb{Z}/4)^* = \{1, 3\}$ and $(\mathbb{Z}/8)^* = \{1, -1, 3, -3\}$, so only the case $m \geq 4$ remains. We know that the order of 3 in $(\mathbb{Z}/2^m)^*$ is a power of 2. The previous lemma shows us that for $m \geq 4$ it is at least 2^{m-2}, as

$3^{2^{m-3}} \equiv 1 + 2^{m-1}$ Mod 2^m.

It remains to show that -1 is not a power of 3, as then the $\pm 3^k$ give us all 2^{m-1} elements of $(\mathbb{Z}/2^m)^*$.

Suppose $3^k = -1 \quad \text{Mod } 2^m$ for some $m \geq 4$. Then $3^k = -1 \quad \text{Mod } 8$ which is impossible as 3^k only takes the values 1 and 3 modulo 8.

□

Combining this result with corollary 2.2.7 we obtain

Corollary 2.2.10 *The Swan Map* $Aut(J) \to (\mathbb{Z}_{2^m})^*$ *is surjective for all* $m \geq 2$.

From theorem 1.2.8 we may conclude

Theorem 2.2.11 *An algebraic 2- complex,* X*, over* $\mathbb{Z}[D_{2^m}]$*, with* $\pi_2(X) \cong J \oplus \mathbb{Z}[G]^k$*, is uniquely determined up to chain homotopy equivalence.*

§2.3 The D(2) property for $\mathbb{Z}[D_8]$

Let F_2 denote the two element module over $\mathbb{Z}[D_{4n}]$, on which the action of $\mathbb{Z}[D_{4n}]$ is trivial.

Proposition 2.3.1 *(i)*$H^0(D_{4n}, F_2) = F_2$

(ii)$H^1(D_{4n}, F_2) = F_2^2$

(iii)$H^2(D_{4n}, F_2) = F_2^3$

Proof: We have the following resolution for $\mathbb{Z}$ over $\mathbb{Z}[D_{4n}]$:

$$\cdots \longrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

where C_0 is the free module generated by $*$; C_1 is the free module generated by e_1, e_2; C_2 is the free module generated by E_1, E_2, E_3, and C_3 is the free module generated by D_1, D_2, D_3, D_4.

The maps $\partial_1, \partial_2, \partial_3$ are given by

$\partial_1 e_1 = *(a-1)$

$\partial_1 e_2 = *(b-1)$

$\partial_2 E_1 = e_1 \Sigma$

$\partial_2 E_2 = e_2(b+1)$

$\partial_2 E_3 = e_1(1+ba) + e_2(a-1)$

$\partial_3 D_1 = E_1(a-1)$

$\partial_3 D_2 = E_2(b-1)$

$\partial_3 D_3 = E_1(b+1) - E_3\Sigma$

$\partial_3 D_4 = E_2(a-1) - E_3(1-ba)$

As all the coefficients above have even augmentation, applying $\mathrm{Hom}_{Z[D_{4n}]}(\bullet, F_2)$ to this resolution gives:

$\cdots \longleftarrow F_2^4 \overset{0}{\longleftarrow} F_2^3 \overset{0}{\longleftarrow} F_2^2 \overset{0}{\longleftarrow} F_2$

from which we deduce (i), (ii) and (iii) immediately.

□

Recall the sequence(1), from §2.2.

Proposition 2.3.2 *J has minimal Z- rank in its stable class.*

Proof: Given any finite algebraic 2-complex, consider the cochain obtained by applying $\mathrm{Hom}_{Z[D_{4n}]}(\bullet, F_2)$:

$$F_2^{d_2} \overset{v_2}{\leftarrow} F_2^{d_1} \overset{v_1}{\leftarrow} F_2^{d_0}$$

where d_0, d_1, d_2, are the $\mathbb{Z}[D_{4n}]$ ranks of the zeroth, first and second modules in the complex. As $H^0(D_{4n}, F_2) = F_2$, the kernel of v_1 has F_2- rank 1. Consequently, the image of v_1 has F_2- rank $d_0 - 1$. $H^1(D_{4n}, F_2) = F_2^2$ so v_2 has kernel of F_2- rank $2+d_0-1 = d_0+1$. The image of v_2 is then seen to have rank d_1-d_0-1. $H^2(D_{4n}, F_2) = F_2^3$ so we know that $d_2 \geq 3 + d_1 - d_0 - 1$. Rearranging gives $d_2 - d_1 + d_0 \geq 2$.

Exactness implies that the $\mathbb{Z}$- rank of the algebraic π_2 of the algebraic complex must be $4n(d_2 - d_1 + d_0) - 1$. Hence our inequality implies that this is at least $8n - 1$, which is the $\mathbb{Z}$- rank of J.

□

We now restrict to the case $n = 2$.

Proposition 2.3.3 *The only elements in the stable class of J are modules of the form $J \oplus \mathbb{Z}[D_8]^k$.*

Proof: We refer to [11], Theorem 6.1. This states that over $\mathbb{Z}[D_8]$, $A \oplus C = B \oplus C$ implies $A = B$ for torsion free, finitely generated modules A, B, C.

If a module M is in the stable class of J then $M \oplus \mathbb{Z}[D_8]^r = J \oplus \mathbb{Z}[D_8]^s$. From proposition 2.3.2 we have $s \geq r$. From the theorem, we deduce that $M = J \oplus \mathbb{Z}[D_8]^{s-r}$.

□

The only modules that can turn up as the second homology group of an algebraic 2-complex over $\mathbb{Z}[D_8]$ are ones of the form $J \oplus \mathbb{Z}[D_8]^k$ for some $k \geq 0$. From corollary 2.2.10 we know that for these modules, the Swan map is surjective. Hence theorem 1.2.8 implies that for each k, up to chain homotopy, there is a unique algebraic 2-complex with second homology group $J \oplus \mathbb{Z}[D_8]^k$. Given any k, the homotopy class of this algebraic 2-complex is realized by the Cayley complex of the presentation

$$< a, b \; | a^{2n} = b^2 = e, \; aba = b, \; r_1 = e, \, r_2 = e, \ldots r_k = e >$$

where $r_i = e$ for $i = 1, \ldots, k$. We conclude

Theorem 2.3.4 *The group D_8 satisfies the D(2) property.*

In the next section we examine minimal elements of the stable class $\Omega_3(\mathbb{Z})$ for general groups D_{4n}, where we cannot immediately apply the torsion free cancellation which we used to prove proposition 2.3.3.

§2.4 Minimal elements of $\Omega_3(\mathbb{Z})$

As before, we work over the group ring $\mathbb{Z}[D_{4n}]$. Consider the resolution (1), in §2.2:

$$J \hookrightarrow \mathbb{Z}[D_{4n}]^3 \xrightarrow{\partial_2} \mathbb{Z}[D_{4n}]^2 \xrightarrow{\partial_1} \mathbb{Z}[D_{4n}] \xrightarrow{\epsilon} \mathbb{Z} \quad (1)$$

From the proposition 3.3.2 we know that J has minimal $\mathbb{Z}$- rank in its stable class. From the Swan - Jacobinski Theorem, we know that the only non-minimal modules in the stable class of J are ones of the form $J \oplus \mathbb{Z}[D_{4n}]^k$. It remains to investigate those modules, stably equivalent to J and of the same $\mathbb{Z}$- rank.

In this section, we take an arbitrary minimal element of $\Omega_3(\mathbb{Z})$ and show that it must be isomorphic to one of a finite set of modules, which we parametrize by the group ${\mathbb{Z}_{2n}}^7 \oplus {\mathbb{Z}_2}^2$.

Let W_2 denote the image of ∂_2. We have a short exact sequence:

$J \to \mathbb{Z}[D_{4n}]^3 \to W_2$

Suppose K is stably equivalent to J and of the same $\mathbb{Z}$- rank. Then we have a short exact sequence:

$K \oplus F \to \mathbb{Z}[D_{4n}]^3 \oplus F' \to W_2$

where F and F' are free module of the same rank. $(\mathbb{Z}[D_{4n}]^3 \oplus F')/F$ is stably free and of the same $\mathbb{Z}$- rank as $\mathbb{Z}[D_{4n}]^3$. In fact it is $\mathbb{Z}[D_{4n}]^3$, as dihedral groups satisfy the Eichler condition, hence stably free modules over them are free.

Hence we have the following short exact sequence:

$K \xrightarrow{i} \mathbb{Z}[D_{4n}]^3 \xrightarrow{j} W_2$

As W_2 is the image of ∂_2, it is the submodule of $\mathbb{Z}[D_{4n}]^2$, generated by

$e_1\Sigma$

$e_2(1+b)$

$e_1(1+ba) + e_2(a-1)$

where, as before, e_1, e_2 are a basis of $\mathbb{Z}[D_{4n}]^2$.

Define $p : W_2 \to \mathbb{Z}[D_{4n}]$ by projection onto the factor generated by e_2. Let W_1^T denote the image of the map. Clearly, W_1^T is generated by $e_2(1+b)$ and $e_2(a-1)$.

Lemma 2.4.1 *The kernel of p is generated by* $e_1\Sigma$.

Proof: Clearly $e_1\Sigma$ is in the kernel of p. Its span has $\mathbb{Z}$- rank 2. W_2 has $\mathbb{Z}$- rank $4n+1$ by exactness of (1). Also W_1^T has $\mathbb{Z}$- rank $4n-1$, as it is generated by $e_2(1+b)$ and $e_2(a-1)$. Hence the $\mathbb{Z}$- rank of the kernel of p is 2.

Therefore, given any element, α in the kernel of p, there exists some integer, r, such that αr is in the span of $e_1\Sigma$. Hence α itself must be in the span of $e_1\Sigma$.

□

The kernel of p will be denoted by by ZC_2.

Let f_1, f_2 be a basis for a module isomorphic to $\mathbb{Z}[D_{4n}]^2$. Let W_2^T be the submodule generated by

$f_1\Sigma$

$f_2(1-b)$

$f_1(1-ba) - f_2(a-1)$

Let s denote the natural inclusion of W_2^T in $\mathbb{Z}[D_{4n}]^2$, and let $t : \mathbb{Z}[D_{4n}]^2 \to W_1^T$ be the map which sends f_1 to $e_2(a-1)$ and f_2 to $e_2(1+b)$.

Lemma 2.4.2 *We have a short exact sequence:*

$$W_2^T \xrightarrow{s} \mathbb{Z}[D_{4n}]^2 \xrightarrow{t} W_1^T$$

.

Let M denote the kernel of $p \circ j$.

Lemma 2.4.3 *M is isomorphic to $W_2^T \oplus \mathbb{Z}[D_{4n}]$.*

Proof: We have short exact sequences:

$M \to \mathbb{Z}[D_{4n}]^3 \xrightarrow{p \circ j} W_1^T$

and

$W_2^T \oplus \mathbb{Z}[D_{4n}] \xrightarrow{s \oplus 1} \mathbb{Z}[D_{4n}]^3 \xrightarrow{t \oplus 0} W_1^T$

Hence M is stably equivalent to $W_2^T \oplus \mathbb{Z}[D_{4n}]$, by Schanuel's Lemma. Dihedral groups satisfy the Eichler condition and clearly $W_2^T \oplus \mathbb{Z}[D_{4n}]$ contains a free copy of the group ring as a summand, so by the Swan-Jacobinski theorem, M must be isomorphic to $W_2^T \oplus \mathbb{Z}[D_{4n}]$.

□

K includes in M and the cokernel of this inclusion is the kernel of p, ZC_2. As K was chosen arbitrarily, we may conclude that any module, stably equivalent to J and of the same $\mathbb{Z}$- rank, occurs as the kernel of some surjection $M \to ZC_2$.

As the action of a on ZC_2 is trivial, the kernel of any map $M \to ZC_2$, must contain $M(a-1)$.

Let $\mathbb{Z}^T$ denote the $\mathbb{Z}[D_{4n}]$ module whose underlying Abelian group is isomorphic to the integers, and on which a acts trivially and b acts as multiplication by -1.

Lemma 2.4.4 *$W_2^T/W_2^T(a-1)$ has $\mathbb{Z}$- rank 3.*

Proof: $(W_2^T/W_2^T(a-1)) \otimes \mathbb{Q} = (W_2^T \otimes \mathbb{Q})/((W_2^T \otimes \mathbb{Q})(a-1)$. It is sufficient to show that this has dimension 3 over $\mathbb{Q}$.

We have an exact sequence:

$$0 \to W_2^T \xrightarrow{s} \mathbb{Z}[D_{4n}]^2 \xrightarrow{t} \mathbb{Z}[D_{4n}] \to \mathbb{Z}^T \to 0$$

Tensoring this sequence with $\mathbb{Q}$ and applying "Whitehead's trick", yields $W_2^T \otimes \mathbb{Q} \oplus \mathbb{Q}[D_{4n}] = \mathbb{Q}[D_{4n}]^2 \oplus \mathbb{Q}^T$, where $\mathbb{Q}^T$ is the $\mathbb{Q}$- rank - 1 module on which a acts trivially and b acts as multiplication by -1. Hence $W_2^T \otimes \mathbb{Q} = \mathbb{Q}[D_{4n}] \oplus \mathbb{Q}^T$.

So $(W_2^T \otimes \mathbb{Q})/((W_2^T \otimes \mathbb{Q}) = \mathbb{Q}[D_{4n}]/\mathbb{Q}[D_{4n}](a-1) \oplus \mathbb{Q}^T/Q^T(a-1) = ZC_2 \otimes \mathbb{Q} \oplus \mathbb{Q}^T$ which has $\mathbb{Q}$ rank 3.

□

Lemma 2.4.5 *The cokernel of the natural inclusion*
$M(a-1) \hookrightarrow M$
is $\mathbb{Z} \oplus \mathbb{Z}^T \oplus \mathbb{Z}^T \oplus ZC_2$

Proof: M is isomorphic to $W_2^T \oplus \mathbb{Z}[D_{4n}]$. We first observe that
$\mathbb{Z}[D_{4n}]/\mathbb{Z}[D_{4n}](a-1) \cong ZC_2$.

W_2^T is generated by
$f_1\Sigma$
$f_2(1-b)$
$f_1(1-ba) - f_2(a-1)$

Let w_1, w_2, w_3 denote the images of $f_1\Sigma, f_2(1-b), f_1(1-ba)-f_2(a-1)$ respectively, under quotienting by $W_2^T(a-1)$. w_1 generates a copy of ZC_2 and w_2 generates a copy of $\mathbb{Z}^T$.

Note that $w_3 2n = w_3\Sigma = w_1(1-b)$. Hence $w_1 b = (w_1 - w_3 n) - w_3 n$. Also $w_1 = (w_1 - w_3 n) + w_3 n$. So $W_2^T/W_2^T(a-1)$ is generated over $\mathbb{Z}$ by $w_2, w_3, w_3 b$ and $w_1 - w_3 n$.

We will show that $w_3 b{=}-w_3$ and hence that $W_2^T/W_2^T(a-1)$ is generated over $\mathbb{Z}$ by w_2, w_3, and $w_1 - w_3 n$:
$w_3(1+b)$ is equal to the image of $f_1(1-ba)(1+b) - f_2(a-1)(1+b)$ under quotienting by $W_2^T(a-1)$. But
$f_1(1-ba)(1+b) - f_2(a-1)(1+b) = (f_2(b-1) - (f_1(1-ba) - f_2(a-1))b)(a-1)$
which is in $W_2^T(a-1)$. Hence $w_3(1+b) = 0$.

The elements w_2, w_3, and $w_1 - w_3 n$ must be $\mathbb{Z}$- linearly independent, in order for their span to have $\mathbb{Z}$- rank 3.

Hence we know that $W_2^T/W_2^T(a-1)$ is torsion free. We know that $(w_3)(1+b)2n = 0$ and $(w_1 - w_3 n)(1-b)2 = 0$, and are able to conclude that $w_3 b = -w_3$ and $(w_1 - w_3 n)b = (w_1 - w_3 n)$.

Hence $w_1 - w_3 n$, w_2 and w_3 generate $\mathbb{Z} \oplus \mathbb{Z}^T \oplus \mathbb{Z}^T$ and $M/M(a-1) \cong \mathbb{Z} \oplus \mathbb{Z}^T \oplus \mathbb{Z}^T \oplus ZC_2$.

□

Any surjection $M \to ZC_2$ must therefore factor through $\mathbb{Z} \oplus \mathbb{Z}^T \oplus \mathbb{Z}^T \oplus ZC_2$. Hence we have a surjection
$\phi : \mathbb{Z} \oplus \mathbb{Z}^T \oplus \mathbb{Z}^T \oplus ZC_2 \to ZC_2$
and the cokernel of the natural inclusion of $M(a-1)$ in K, is naturally identified with the kernel of ϕ.

Lemma 2.4.6 *The kernel of ϕ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}^T \oplus \mathbb{Z}^T$.*

Proof: Let u_1, u_2, u_3, u_4 generate copies of $\mathbb{Z}$, $\mathbb{Z}^T$, $\mathbb{Z}^T$ and ZC_2, respectively in the direct sum $\mathbb{Z} \oplus \mathbb{Z}^T \oplus \mathbb{Z}^T \oplus ZC_2$. v will generate the image of ϕ.

$\phi(u_1)$ is an integer multiple of $v(1+b)$ and $\phi(u_2)$, $\phi(u_3)$ are integer multiples of $v(1-b)$. Consequently $\phi(u_4) = v(x+yb)$, where x and y are integers with odd sum, as otherwise ϕ would not be surjective.

If some linear combination of the u_i is in the kernel of ϕ, the coefficient on u_4 must have even augmentation, as the augmentation multiplied by $x + y$ must be even. Hence the kernel of ϕ is contained in the $\mathbb{Z}$- linear span of u_1, u_2, u_3, $u_4(1+b)$, $u_4(1-b$. Also the image of this span under ϕ must be the whole of the span of $v(1+b)$, $v(1-b)$, as $x+y$ is odd.

ϕ restricts to a surjection from the span of $u_1, u_4(1+b)$ to the span of $v(1+b)$. The kernel of a surjection $\mathbb{Z}^2 \to Z$ must be isomorphic to $\mathbb{Z}$.

ϕ restricts to a surjection from the span of $u_2, u_3, u_4(1-b)$ to the span of $v(1-b)$. The kernel of a surjection $(\mathbb{Z}^T)^3 \to Z^T$ must be isomorphic to $\mathbb{Z}^T \oplus \mathbb{Z}^T$.

Hence the kernel of ϕ is $\mathbb{Z} \oplus \mathbb{Z}^T \oplus \mathbb{Z}^T$

□

Hence we know that K occurs in a short exact sequence of the form:

$$M(a-1) \to K \to \mathbb{Z} \oplus \mathbb{Z}^T \oplus \mathbb{Z}^T$$

Hence all candidates for minimal elements of $\Omega_3(\mathbb{Z})$, are parametrized by the group
$\mathrm{Ext}^1_{\mathbb{Z}[D_{4n}]}(\mathbb{Z} \oplus \mathbb{Z}^T \oplus \mathbb{Z}^T, M(a-1))$

In order to calculate this extension group, we will require a better description of $W_2^T(a-1)$. We know that W_2^T is generated by

$f_1\Sigma$
$f_2(1-b)$
$f_1(1-ba) - f_2(a-1)$

Lemma 2.4.7 *If $f_1 x \in W_2^T$ then Σ divides x.*

Proof: Suppose $f_1 x = f_1\Sigma\mu_1 + f_2(1-b)\mu_2 + (f_1(1-ba) - f_2(a-1))\mu_3$. We could replace$\mu_2$ with a polynomial in a, which must be divisible by (a-1). We will denote the polynomial $(a-1)p$ where p is a polynomial in a. So
$f_1 x = f_1\Sigma\mu_1 + f_2(1-b)(a-1)p + (f_1(1-ba) - f_2(a-1))\mu_3 =$
$= f_1\Sigma\mu_1 + f_2(a-1)(1+ba)p + (f_1(1-ba) - f_2(a-1))\mu_3$

$= f_1\Sigma\mu_1 + (f_1(1-ba) - f_2(a-1))(\mu_3 - (1+ba)p)$

As $(a-1))(\mu_3 - (1+ba)p) = 0$, we know that $(\mu_3 - (1+ba)p) = \Sigma y$, for some element y of $\mathbb{Z}[D_{4n}]$. So $x = \Sigma\mu_1 + \Sigma(1-b)y$.

□

Let W_1 denote the augmentation ideal of $\mathbb{Z}[D_{4n}]$.

Lemma 2.4.8 $W_2^T(a-1) \cong W_1(a-1)$

Proof: We have a surjective homomorphism $W_2^T(a-1) \twoheadrightarrow W_1(a-1)$ given by projection onto the f_2 component. It is sufficient to show that it has 0 kernel.

Suppose $f_1 x$ is in $W_2^T(a-1)$, for some $x \in \mathbb{Z}[D_{4n}]$. Clearly $(a-1)$ divides x. By the previous lemma, Σ also divides x. Hence $x = 0$.

□

Recall we defined ZC_2 to be the quotient $\mathbb{Z}[D_{4n}]/\mathbb{Z}[D_{4n}](a-1)$. Let L denote the quotient $\mathbb{Z}[D_{4n}]/ID(a-1)$.

$\mathrm{Ext}^1_{\mathbb{Z}[D_{4n}]}(\mathbb{Z} \oplus \mathbb{Z}^T \oplus \mathbb{Z}^T,\ M(a-1))$

$\cong \mathrm{Hom}_{\mathrm{Der}}(W_1 \oplus W_1^T \oplus W_1^T,\ W_2^T(a-1) \oplus \mathbb{Z}[D_{4n}](a-1))$

By dimension shifting, we have that this is equal to

$\mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z} \oplus \mathbb{Z}^T \oplus \mathbb{Z}^T,\ L \oplus ZC_2)$.

$\cong \mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z}, ZC_2) \oplus \mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z}^T,\ ZC_2)^2 \oplus \mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z},\ L) \oplus \mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z}^T,\ L)^2$

The module ZC_2 is generated freely over $\mathbb{Z}$ by the images of 1 and b. $1b = b$ and $bb = 1$. The action of a is trivial. A map from $\mathbb{Z}$ to ZC_2 is determined by where $1 \in \mathbb{Z}$ is sent to. It must go to an element which is fixed by the action of b. Hence it must go to some multiple of the image of $(1+b)$.

Any map from $\mathbb{Z}$ to ZC_2 which factors through a projective will send $1 \in \mathbb{Z}$ to $x\Sigma(1+b)$ for some element $x \in ZC_2$. Thus $1 \in \mathbb{Z}$ will go to some multiple of $(1+b)2n$. So $\mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z}, ZC_2) \cong \mathbb{Z}_{2n}$.

Similarly, a map from $\mathbb{Z}^T$ to ZC_2 is determined by where $1 \in \mathbb{Z}^T$ is sent to. It must go to an element which is sent to its negative by the action of b. Hence it must go to some multiple of the image of $(1-b)$.

Any map from $\mathbb{Z}^T$ to ZC_2 which factors through a projective will send $1 \in \mathbb{Z}^T$ to $x\Sigma(1-b)$ for some element $x \in ZC_2$. Thus $1 \in \mathbb{Z}^T$ will go to some multiple of $(1-b)2n$. So $\mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z}^T, ZC_2) \cong \mathbb{Z}_{2n}$.

Given any element of $\mathbb{Z}[D_{4n}]$, by subtracting an appropriate integral combination of 1 and b we have a multiple of $(a-1)$. By further subtracting an appropriate integral multiple of $(a-1)$, we have an element of $W^1(a-1)$. Hence L is generated over $\mathbb{Z}$ by the images of 1, b, and $(a-1)$, denoted hereafter by u_1, u_2 and u_3. Note $u_3 2n$ is equal to the image of $(a-1)\Sigma + (2n-\Sigma)(a-1)$ which equals the image of $(2n-\Sigma)(a-1)$ which is 0. So $u_3 2n = 0$.

Now suppose $u_1 r + u_2 s + u_3 t = 0$ for integers r, s, t. Clearly $r = 0$ and $s = 0$. Also $(a-1)t = x(a-1)$ for some $x \in W^1$. Hence t differs from something of augmentation 0 by some multiple of Σ. Hence $2n$ divides t. So as Abelian group, L is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}_{2n}$.

Next we calculate the group action on L.

$a = 1 + (a-1)$

$ba = b + b(a-1) = b + (a-1) + (b-1)(a-1)$

$(a-1)a = (a-1) + (a-1)(a-1)$

So

$(u_1 r + u_2 s + u_3 t)a = u_1 r + u_2 s + u_3(r+s+t)$

Also

$1b = b$

$bb = 1$

$(a-1)b = -ba^{-1}(a-1) = (1-ba^{-1})(a-1) - (a-1)$

So

$(u_1r + u_2s + u_3t)b = u_1s + u_2r - u_3t$

A map from $\mathbb{Z}$ to L is determined by where $1 \in \mathbb{Z}$ goes. It must go to an element which is fixed by the action of b. Hence it goes to an element of the form $(u_1r + u_2r + u_3nt)$. This element must also be fixed by the action of a. Hence it has the form $(u_1ns + u_2ns + u_3nt)$.

Any such map which factors through a projective must send $1 \in \mathbb{Z}$ to an element of the form

$(u_1r + u_2s + u_3t)(1+b)\Sigma = (u_1(r+s) + u_2(r+s))\Sigma =$

$u_1(r+s)2n + u_2(r+s)2n + u_3(2(r+s)(2n-1)n = u_1(r+s)2n + u_2(r+s)2n$

Hence $\mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z}, L) \cong {\mathbb{Z}_2}^2$.

A map from $\mathbb{Z}^T$ to L is determined by where $1 \in \mathbb{Z}^T$ goes. It must go to an element which is sent to its negative by fixed by the action of b. Hence it goes to an element of the form $(u_1s - u_2s + u_3t)$. This element will automatically be fixed by the action of a.

Any such map which factors through a projective must send $1 \in \mathbb{Z}^T$ to an element of the form

$(u_1r + u_2s + u_3t)(1-b)\Sigma = (u_1(r-s) + u_2(s-r) + u_32t)\Sigma =$

$u_1(r-s)2n + u_2(s-r)2n + u_34tn = u_1(r-s)2n + u_2(s-r)2n$

Hence $\mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z}^T, L) \cong {\mathbb{Z}_{2n}}^2$.

Hence the extension group is

$\mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z}, ZC_2) \oplus \mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z}^T, ZC_2)^2 \oplus \mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z}, L) \oplus \mathrm{Hom}_{\mathrm{Der}}(\mathbb{Z}^T, L)^2$

$\cong \mathbb{Z}_{2n} \oplus {\mathbb{Z}_{2n}}^2 \oplus {\mathbb{Z}_2}^2 \oplus {\mathbb{Z}_{2n}}^4 \cong {\mathbb{Z}_{2n}}^7 \oplus {\mathbb{Z}_2}^2$

In conclusion, we have found an exhaustive list of candidates for minimal modules in the stable class $\Omega_3(\mathbb{Z})$. These candidates are parametrized by the elements of the finite group, ${\mathbb{Z}_{2n}}^7 \oplus {\mathbb{Z}_2}^2$. This gives us an upper bound of $512n^7$ for the number of minimal modules in $\Omega_3(\mathbb{Z})$.

Not all of the candidates will be in $\Omega_3(\mathbb{Z})$. If one could show that those which are in $\Omega_3(\mathbb{Z})$, are all isomorphic, then one would have proved that cancellation of free modules holds in $\Omega_3(\mathbb{Z})$ for dihedral groups of order $4n$.

§2.5 D_{8n+4}

The dihedral groups D_{4n+2} have balanced presentations and period 4 resolutions over $\mathbb{Z}$. This makes them easier to work with in certain respects than the dihedral groups D_{4n}, which have neither. We show in this section that the groups D_{8n+4} have period 4 resolutions if one works over the ground ring $\mathbb{Z}[x]/< 2x - 1>$, which we denote $\mathbb{Z}[\frac{1}{2}]$. This raises the possibility that the methods used to prove the D(2) property for the groups D_{4n+2} (see [2] and [3]) may be generalized to the groups D_{8n+4}.

Lemma 2.5.1 $D_{8n+4} \cong D_{4n+2} \times C_2$

Proof: Take $D_{8n+4} =< a, b |\, aba = a,\ b^2 = a^{4n+2} = e >$. Then a^{2n+1} is central and generates a copy of C_2. The elements a^2, b generate a copy of D_{4n+2}. As $2n + 1$ is odd, any element can be uniquely written as a product of an element in the copy of C_2 and an element in the copy of D_{4n+2}.

□

We work over the ring, $\mathbb{Z}[\frac{1}{2}][D_{8n+4}]$, which we will denote R. Let

$$\nabla_1 = (1 + a^{2n+1})/2, \qquad \nabla_2 = (1 - a^{2n+1})/2$$

Then ∇_1, ∇_2 are central idempotents, which sum to 1. Consequently, we have

$$R \cong R\nabla_1 \times R\nabla_2$$

as rings. Let T denote the ring $R\nabla_1$ and S denote the ring $R\nabla_2$.

Regarding a^2 and b as generators of the subgroup D_{4n+2}, as before, we have an isomorphism $T \cong \mathbb{Z}[\frac{1}{2}][D_{4n+2}]$. From [3], $41(ii) we obtain a free period 4 resolution of $\mathbb{Z}$ over $\mathbb{Z}[D_{4n+2}]$. This naturally yields a period 4 resolution of $\mathbb{Z}[\frac{1}{2}]$ over T. We write this resolution in our notation, regarding T as a subring of R:

$$0 \to \mathbb{Z}[\frac{1}{2}] \xrightarrow{\epsilon^*} T \xrightarrow{\partial_3} T^2 \xrightarrow{\partial_2} T^2 \xrightarrow{\partial_1} T \xrightarrow{\epsilon} \mathbb{Z}[\frac{1}{2}] \to 0 \tag{1}$$

Going from left to right, let the basis elements of the free modules in this sequence be $\{c\}$, $\{f_1, f_2\}$, $\{e_1, e_2\}$ and $\{v\}$. ϵ is the augmentation map, which takes $v \in T$ to $1 \in \mathbb{Z}[\frac{1}{2}]$ and ϵ^* is its dual.

For any integer k let $\Sigma_k = \sum_{i=0}^{k-1} a^{2i}$. With these conventions we set:

$$\partial_3 c = f_1\nabla_1(1 + a^2 - a^{2(n+1)} - b) + f_2\nabla_1(a^2 - a^{2n}b)$$

$$\partial_2 f_1 = e_1\nabla_1\Sigma_{2n+1} - e_2\nabla_1(1 + b)$$
$$\partial_2 f_2 = e_1\nabla_1(\Sigma_n(b - 1) + a^{2n}b) + e_2\nabla_1(1 - a^{2n})$$

$$\partial_1 e_1 = v\nabla_1(a^2 - 1)$$
$$\partial_1 e_2 = v\nabla_1(b - 1)$$

We also have an exact sequence over S:

$$0 \longrightarrow 0 \longrightarrow S \xrightarrow{p_3} S^2 \xrightarrow{p_2} S^2 \xrightarrow{p_1} S \longrightarrow 0 \longrightarrow 0$$

Again, going from left to right, let the basis elements of the free modules in this sequence be $\{c'\}$, $\{f_1', f_2'\}$, $\{e_1', e_2'\}$ and $\{v'\}$.

With respect to this basis we set

$$p_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

We can take the direct product of these two resolutions, to get a resolution over the product ring, R:

$$0 \to \mathbb{Z}[\frac{1}{2}] \xrightarrow{\epsilon^* \times 0} R \xrightarrow{\partial_3 \times p_3} R^2 \xrightarrow{\partial_2 \times p_2} R^2 \xrightarrow{\partial_1 \times p_1} R \xrightarrow{\epsilon \times 0} \mathbb{Z}[\frac{1}{2}] \to 0$$

Let δ denote the augmentation map $R \to \mathbb{Z}[\frac{1}{2}]$ sending $1 \in R$ to $1 \in \mathbb{Z}[\frac{1}{2}]$. Then $\delta(\nabla_1) = 1$ and $\delta(\nabla_2) = 0$. Hence $\delta = \epsilon \times 0$. Note also that $\delta^*(1) = 2\nabla_1\epsilon(1)$.

Hence, if we let $d_i = \partial_i \times p_i$ we get an exact sequence:

$$0 \to \mathbb{Z}[\frac{1}{2}] \xrightarrow{\epsilon^*} R \xrightarrow{d_3} R^2 \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \xrightarrow{\epsilon} \mathbb{Z}[\frac{1}{2}] \to 0 \qquad (2)$$

Let $c'' = c + c'$, $f_1'' = f_1 + f_1'$, $f_2'' = f_2 + f_2'$, $e_1'' = e_1 + e_1'$, $e_2'' = e_2 + e_2'$, $v'' = v + v'$. We have

$$d_3 c'' = f_1'' \nabla_1 (1 + a^2 - a^{2(n+1)} - b) + f_2'' (\nabla_1 (a^2 - a^{2n} b) + \nabla_2)$$

$$d_2 f_1'' = e_1'' \nabla_1 \Sigma_{2n+1} - e_2'' (\nabla_1 (1 + b) + \nabla_2)$$
$$d_2 f_2'' = e_1'' \nabla_1 (\Sigma_n (b - 1) + a^{2n} b) + e_2'' \nabla_1 (1 - a^{2n})$$

$$d_1 e_1'' = v'' (\nabla_1 (a^2 - 1) + \nabla_2)$$
$$d_1 e_2'' = v'' \nabla_1 (b - 1)$$

We have demonstrated a period 4 resolution over $\mathbb{Z}[\frac{1}{2}][D_{8n+4}]$. We now consider how this relates to the D(2) problem for D_{8n+4}.

Let J denote the image of ∂_3. Let ι denote its inclusion into T^2. The image of d_3 is equal to $J \times S$. Let ι_S denote the the inclusion of S into S^2, induced by p_3.

We have a chain map:

$$\begin{array}{ccccccccc} S & \xrightarrow{\iota_S} & S^2 & \xrightarrow{p_2} & S^2 & \xrightarrow{p_1} & S & \longrightarrow & 0 \\ \downarrow 1 & & \downarrow 1 & & \downarrow 1 & & \downarrow 1 & & \downarrow 0 \\ S & \xrightarrow{\iota_S} & S^2 & \xrightarrow{p_2} & S^2 & \xrightarrow{p_1} & S & \longrightarrow & 0 \end{array}$$

So given $k \in \mathbb{Z}[\frac{1}{2}]$ if there exists an automorphism α, and a chain map

$$\begin{array}{ccccccccc} J & \xrightarrow{\iota} & T^2 & \xrightarrow{\partial_2} & T^2 & \xrightarrow{\partial_1} & T & \xrightarrow{\epsilon} & \mathbb{Z}[\frac{1}{2}] \\ \downarrow \alpha & & \downarrow m_2 & & \downarrow m_1 & & \downarrow m_0 & & \downarrow \times k \\ J & \xrightarrow{\iota} & T^2 & \xrightarrow{\partial_2} & T^2 & \xrightarrow{\partial_1} & T & \xrightarrow{\epsilon} & \mathbb{Z}[\frac{1}{2}] \end{array}$$

we have the product chain map

$$\begin{array}{ccccccccc} J \times S & \xrightarrow{\iota \times \iota_S} & R^2 & \xrightarrow{d_2} & R^2 & \xrightarrow{d_1} & R & \xrightarrow{\epsilon} & \mathbb{Z}[\frac{1}{2}] \\ \downarrow \alpha \times 1 & & \downarrow m_2 \times 1 & & \downarrow m_1 \times 1 & & \downarrow m_0 \times 1 & & \downarrow \times k \\ J \times S & \xrightarrow{\iota \times \iota_S} & R^2 & \xrightarrow{d_2} & R^2 & \xrightarrow{d_1} & R & \xrightarrow{\epsilon} & \mathbb{Z}[\frac{1}{2}] \end{array}$$

However, even if the m_i are chosen to be representable by matrices with coefficients all in $\mathbb{Z}[D_{4n+2}] \subset T$, the maps $m_i \times 1$ may not be. Consequently, we have not completely reduced the problem of realizing k- invariants via automorphisms over D_{8n+4}, to the technically simpler problem over D_{4n+2} (see [2] and [3]). However, the above construction does provide a link between the two problems.

§2.6 Summary of results concerning the D(2) problem for dihedral groups

In [3] (62.3) it is shown that the dihedral groups D_{4n+2} satisfy the D(2) property. The smallest dihedral group not covered by this is D_8. We show that the D(2) property does hold for D_8 (theorem 2.3.4).

More generally, for dihedral groups of order $4n$, we show that a minimal element of $\Omega_3(\mathbb{Z})$ is realized as the π_2 of a presentation (proposition 2.3.2). We parametrize all possible minimal elements of $\Omega_3(\mathbb{Z})$ by the group ${\mathbb{Z}_{2n}}^7 \oplus {\mathbb{Z}_2}^2$ (§2.4).

In the case of dihedral groups of order 2^n, $n \in \mathbb{Z}$, we are further able to show that up to chain homotopy equivalence, there is a unique algebraic 2- complex with "standard" algebraic π_2 (theorem 2.2.11).

Chapter 3

π_3 of geometric 2- complexes

Let X be a finite geometric 2- complex with finite fundamental group G. Lemma 2.1.5 implies that $\pi_2(X) = H_2(C_*(\tilde{X}); \mathbb{Z})$ as modules over $\mathbb{Z}[G]$. Hence $\pi_2(X)$ is determined by $C_*(\tilde{X})$.

In fact, $C_*(\tilde{X})$ determines the homotopy type of X, (see [3], theorem 49.5) and hence it determines all the homotopy groups, as modules over $\mathbb{Z}[G]$. In this chapter we compute $\pi_3(X)$ from $C_*(\tilde{X})$. We also show that if G has even order, then G determines the stable class of the module $\pi_3(X)$.

Let G be a finite group and let J be a finitely generated, torsion free, module over it. There is a G- action on $J \otimes_{\mathbb{Z}} J$, given by $(a \otimes b)g = ag \otimes bg$, making it a $\mathbb{Z}[G]$ module. Let t be the $\mathbb{Z}[G]$- linear automorphism of $J \otimes_{\mathbb{Z}} J$ defined by $t(a \otimes b) = b \otimes a$.

Definition $S^2(J) = \{x \in J \otimes J | tx = x\}$

The main theorem of this chapter is now stated.

Theorem 3A *Let X be a finite geometric 3- complex, with finite fundamental group G. If $C_*(\tilde{X}) \sim \mathcal{A}$, for some finite algebraic 2- complex $\mathcal{A}$, then $\pi_3(X) \cong S^2(J)$, where $J = H_2(C_*(\tilde{X});\ \mathbb{Z}[G])$.*

In particular,

Corollary *Let X be any finite geometric 2- complex, with finite fundamental group G. Let $J \cong \pi_2(C_*(\tilde{X}))$. Then $\pi_3(X) \cong S^2(J)$.*

The claim that $\pi_3(X)$ may be computed directly from J is non - trivial in the sense that J does not by itself determine the homotopy type of $C_*(\tilde{X})$.

Let $\mathcal{A}$ be a finite algebraic 2- complex.

Definition $\pi_3(\mathcal{A}) = S^2(J)$, where $J \cong \pi_2(\mathcal{A})$.

From the theorem, we see that this definition is consistent, in that if $C_*(\tilde{X}) \sim \mathcal{A}$, for a finite geometric 3- complex X, then $\pi_3(\mathcal{A}) \cong \pi_3(X)$.

§3.1 Higher Covers

We computed π_2 of a geometric 2- complex by noting that passing to the universal cover preserved it, and then applying the Hurewicz isomorphism theorem to compute it from the homology of the algebraic 2- complex. Our first step in computing π_3 is to generalize the notion of universal cover.

We say a X space is r- connected if $\pi_i(X) = 0$, for $i = 0, 1, ..., r$.

Definition 3.1.1 For $r \geq 1$, an r- cover of an $r-1$- connected space X, is a fibre bundle map $f : Y \to X$, with fibre $k(\pi_r(X), r-1)$ and with $\delta_r : \pi_r(X) \to \pi_{r-1}(k(\pi_r(X), r-1))$ an isomorphism. Here δ_r is the boundary operator in the long exact sequence associated to the fibre bundle.

Lemma 3.1.2 *We have:*

$\pi_i(Y) = 0, \qquad i \leq r$

$\pi_i(Y) = \pi_i(X), \qquad i > r$

Proof: X is $r-1$- connected, and $k(\pi_r(X), r-1)$ is an Eilenberg- Mac lane space. Consequently, the long exact sequence associated to the fibre bundle has the form:

$$\cdots \xrightarrow{0}$$

$$0 \longrightarrow \pi_{r+2}(Y) \longrightarrow \pi_{r+2}(X) \xrightarrow{0}$$

$$0 \longrightarrow \pi_{r+1}(Y) \longrightarrow \pi_{r+1}(X) \xrightarrow{0}$$

$$0 \longrightarrow \pi_{r}(Y) \xrightarrow{0} \pi_{r}(X) \xrightarrow{\delta_r}$$

$$\pi_r(X) \xrightarrow{0} \pi_{r-1}(Y) \longrightarrow 0 \xrightarrow{0}$$

$$0 \xrightarrow{0} \pi_{r-2}(Y) \longrightarrow 0 \xrightarrow{0}$$

$$\cdots$$

So Y is r- connected and we have isomorphisms $\pi_i(Y) \to \pi_i(X)$ for $i > r$.

□

Note also that the universal cover of a connected space, together with the associated covering map, are a 1- cover. In fact all 1- covers have that form.

Suppose we have a connected space X and a tower of maps

$$\cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X$$

where for each i, f_i is an i- cover. Then each X_i is i- connected and by induction on r, $\pi_s(X_r) \cong \pi_s(X)$ as long as $s > r$.

In particular, $\pi_{r+1}(X_r) \cong \pi_{r+1}(X)$, and X_r is r- connected, for $r \geq 1$, so by the Hurewicz isomorphism theorem, we may conclude:

Lemma 3.1.3 $\pi_{r+1}(X) \cong H_{r+1}(X_r; \mathbb{Z})$, $r \geq 1$.

We will use this identity in the next section to calculate π_3 of a geometric 2- complex.

Note that the fibre of the map f_r is $k(\pi_r(X_{r-1}), r-1)$, which is equal to $k(\pi_r(X), r-1)$. The inclusion of this fibre in X_r induces a map on homology: $H_{r+1}(k(\pi_r(X), r-1); \mathbb{Z}) \to H_{r+1}(X_r; \mathbb{Z})$.

We already have $\pi_{r+1}(X) \cong H_{r+1}(X_r; \mathbb{Z})$, so a map is induced

$$H_{r+1}(k(\pi_r(X), r-1); \mathbb{Z}) \to \pi_{r+1}(X)$$

for each $r \geq 1$. Both the domain and codomain of this map are invariants of X.

§3.2 The Hopf fibration

Our first example of a 2- cover will be the Hopf fibration. In this section we define it and look at the map of sets it induces: $\pi_2(X) \to \pi_3(X)$. Crucially, for our main theorem, will show that this map respects the action of $\pi_1(X)$.

We may regard S^3 as the set of pairs $(z, w) \in \mathbb{C}^2$, satisfying $z\bar{z} + w\bar{w} = 1$. Define a relation $\sim$, by setting $(z, w) \sim (z\lambda, w\lambda)$ whenever $\lambda \in \mathbb{C}$ satisfies $\lambda\bar{\lambda} = 1$.

Definition 3.2.1 *Hopf fibration* The Hopf fibration, $h : S^3 \to S^2$ is the natural map $S^3 \to S^3/\sim$ composed with the topological identifications $S^3/\sim \cong \mathbb{C}P^1 \cong S^2$.

This map is a fibre bundle map with fibre S^1. If for some $x \in S^2$ we pick a point y in $h^{-1}(x)$, we may parametrize the elements of $h^{-1}(x)$ as λy for $\lambda \in \mathbb{C}$, satisfying $\lambda\bar{\lambda} = 1$.

Let $+ = h(0, 1) \in S^2$. We take $+$ as base point and identify the complement of $+$ with the interior of a closed disk D^2. We have a map $I : D^2 \to S^2$, which restricts to the identification on the interior, and maps the boundary to $+$. Then $I : (D_2, \partial D_2) \to (S^2, +)$ represents a generator of $\pi_2(S^2)$.

I lifts to a map sending D^2 to $\{(\surd(1 - w\bar{w}), w) | w\bar{w} \leq 1\}$. This restricts to a homeomorphism $\partial D^2 \to h^{-1}(+)$.

Hence the map $\pi_2(S^2) \to \pi_1(S^1)$, associated to the fibre bundle, sends a generator of $\pi_2(S^2)$ to a generator of $\pi_1(S^1)$, and is an isomorphism. Consequently, this map is a 2- cover, so by lemma 3.1.3, we have $\pi_3(S^2) \cong \pi_3(S^3) \cong H_3(S^3; \mathbb{Z})$.

Lemma 3.2.2 $H_3(S^3;\mathbb{Z}) \cong \mathbb{Z}$

Proof: S^3 may be constructed from a single point and a single 3- cell, with associated algebraic complex:

$$\mathbb{Z} \to 0 \to 0 \to \mathbb{Z}$$

□

Let X be a topological space with base point $*$. We consider $(0, 1)$ to be the base point of S^3.

Definition 3.2.3 $h^* : \pi_2(X) \to \pi_3(X)$ is the map which sends an element of $\pi_2(X)$, represented by a map $\alpha : (S^2, +) \to (X, *)$ to the element of $\pi_3(X)$ represented by $\alpha \circ h$.

We will now define a construction, which takes an arbitrary map $\alpha : (D^2, \partial D_2) \to (X, *)$ to a map $\alpha^T : (S^3, (0, 1)) \to (X, *)$.

We may identify D^2 with the set of complex numbers $\{w|w\bar{w} \leq \frac{1}{2}\}$.

Consider the solid torus in S^3, given by, $\{(z, w) \in S^3|w\bar{w} \leq \frac{1}{2}\}$. We denote this torus N. Points in N may be parametrized by the argument of z and the element of D^2, w. Let R_θ denote a rotation of D^2, about θ radians anti clockwise.

Given an arbitrary map $\alpha : (D^2, \partial D_2) \to (X, *)$, the map $\alpha^T : (S^3, (0, 1)) \to (X, *)$ is defined as follows:

α^T sends the complement of N to the base point.

If (θ, w) is an element of N (with respect to the above parametrization), then

$$\alpha^T(\theta, w) = \alpha(R_\theta w)$$

Pictorially then, α^T maps the solid torus N to X by rotating the map α as one goes round N.

If α_t, $t \in [0,1]$ is a homotopy from α_0 to α_1, then α_t^T, $t \in [0,1]$ is a homotopy from α_0^T to α_1^T. Hence the map sending α to α^T may regarded as a map of sets $\pi_2(X) \to \pi_3(X)$. In fact, we will show that $h^*(\alpha) = \alpha^T$. (We abuse notation by denoting elements of homotopy groups by maps representing them.)

Let 1 denote the generator of $\pi_2(S^2)$ represented by the identity map
$\mathrm{Id} : (D^2, \partial D^2) \to (D^2, \partial D^2)$, composed with the natural collapse
$c : (D^2, \partial D^2) \to (S^2, +)$. We may regard h as a map $(S^3, (0,1)) \to (S^2, +)$.

Lemma 3.2.4 *As elements of $\pi_3(S^2)$, we have $h = 1^T$.*

Proof: We may deform h to h' by thickening the preimage of the base point and pushing the other fibers out accordingly. Consider the preimage, under h', of a point other than the base point. Following it round a circle in the z - plane, centered on the origin of the z- plane, takes one round a circle in the w- plane, centered on the origin of the w- plane. Hence h' is homotopic to 1^T.

□

Note that a map, $\alpha : (D^2, \partial D_2) \to (X, *)$, represents an element of $\pi_2(X)$, so $h^*(\alpha)$ is a well defined element of $\pi_3(X)$. Specifically, $h^*(\alpha) = \alpha' \circ h$, where
$\alpha' : (S^2, +) \to (X, *)$ satisfies $\alpha' \circ c = \alpha$.

Lemma 3.2.5 *For any map $\alpha : (D^2, \partial D_2) \to (X, *)$, we have $h^*(\alpha) = \alpha^T$ as elements of $\pi_3(S^2)$.*

Proof: $h^*(\alpha)$ is represented by $\alpha' \circ h$, where $\alpha' \circ c = \alpha$. By the previous lemma $\alpha' \circ h$ is homotopic to $\alpha' \circ 1^T$. Clearly $\alpha' \circ 1^T$ maps the complement of N to $*$. Given $(\theta, w) \in N$ we have

$$\alpha' \circ 1^T(\theta, w) = \alpha' c(R_\theta w) = \alpha(R_\theta w) = \alpha^T(\theta, w)$$

So $\alpha' \circ 1^T = \alpha^T$.

□

The element $0 \in \pi_2(X)$ is represented by the map which sends D^2 to $*$. Therefore the map 0^T sends every point of S^3 to the $*$, and we have $0^T = 0$. Hence $h^*(0) = 0$.

Given $\alpha : (D^2, \partial D_2) \to (X, *)$, we define $-\alpha : (D^2, \partial D_2) \to (X, *)$, by setting $-\alpha(w) = \alpha(\bar{w})$ for all $w \in D^2$. Clearly as elements of $\pi_2(X)$, we have $\alpha + (-\alpha) = 0$, so there is no ambiguity in the minus sign.

If we let $z = z_1 + iz_2$, $w = w_1 + iw_2$ for $(z_1, z_2, w_1, w_2) \in \mathbb{R}^4$, then we may regard S^3 as sitting in $\mathbb{R}^4$. The base point $(0, 1)$ is $(0, 0, 1, 0)$, with respect to this parametrization. The following matrix represents a rotation of S^3, fixing the base point:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t) & 0 & \sin(t) \\ 0 & 0 & 1 & 0 \\ 0 & -\sin(t) & 0 & \cos(t) \end{bmatrix}$$

For $t \in [0, \pi]$, we denote this rotation L_t. We now have a homotopy, $\alpha^T L_t$, from α^T to $\alpha^T L_\pi$, which keeps the image of the base point fixed.

Lemma 3.2.6 *Given $\alpha \in \pi_2(X)$ we have $h^*(-\alpha) = h^*(\alpha)$.*

Proof: We will show that as maps, $(-\alpha)^T = \alpha^T L_\pi$. The map L_π preserves the solid torus, $\{(z, w) \in S^3 | w\bar{w} \leq \frac{1}{2}\}$, sending (θ, w) to $(-\theta, \bar{w})$. Hence

$$\alpha^T L_\pi(\theta, w) = \alpha^T(-\theta, \bar{w}) = \alpha(R_{-\theta}\bar{w}) = \alpha(\bar{R_\theta w}) = (-\alpha)^T(\theta, w)$$

So $(-\alpha)^T = \alpha^T L_\pi$, which we know is homotopic to α^T. Hence $h^*(-\alpha) = h^*(\alpha)$.

□

Lemma 3.2.7 *Let g be an element of $\pi_1(X)$ and let α be an element of $\pi_2(X)$. Then $h^*(\alpha)g = h^*(\alpha g)$.*

Proof: We must show that $\alpha^T g$ is homotopic to $(\alpha g)^T$. We regard α^T as a map from the ball $\{p \in \mathbb{R}^3 | \ |p| \leq 1\} \to X$, which maps the boundary to $*$. We interpret g as a map $I \to X$, mapping endpoints to $*$. We regard $\alpha^T g$ as a map from the ball $\{p \in \mathbb{R}^3 | \ |p| \leq 3\} \to X$ defined as follows:

$\alpha^T g(p) = \alpha^T(p)$, if $|P| \leq 1$,

$\alpha^T g(p) = g(2 - p)$, if $1 \leq |P| \leq 2$,

$\alpha^T g(p) = *$, if $|P| \geq 2$.

Consider the the cylinder $\{(a, b, c) \in \mathbb{R}^3 | a^2 + b^2 \leq 0.3, \ \ -2.5 \leq c \leq 2.5\}$, which we will denote C. Without loss of generality, we may assume that C passes through the hole of the solid torus corresponding to N.

By deforming $\alpha^T g$ slightly, to a map f, we get its restriction to C to be given by

$$f(a, b, c) = m(c)$$

where m is the path $e \cdot g^{-1} \cdot e \cdot g \cdot e$. Let m_t be a homotopy, fixing end points, with $m_0(c) = m(c)$ and $m_1(c) = *$.

We define a homotopy f_t as follows:

$f_t(p) = f(p)$, for $p \notin C$,

$f_t((a, b, c)) = m_t(c)$, for $a^2 + b^2 \leq 0.2$,

$f_t((a, b, c)) = m_{10t(0.3 - a^2 - b^2)}(c)$, for $0.3 \leq a^2 + b^2 \leq 0.2$,

Now the solid torus corresponding to N is contained in a larger solid torus, N', with the smaller cylinder $C' = \{(a, b, c) \in \mathbb{R}^3 | a^2 + b^2 \leq 0.2, \ \ -2.5 \leq c \leq 2.5\}$, in its hole. The map f_1 sends the complement of N' to $*$. The map f_1, restricted to the cross section of N', corresponding to an argument θ, is homotopic to $\alpha g \circ R_\theta$.

Hence as elements of $\pi_3(X)$, we have $h^*(\alpha)g = \alpha^T g = f = f_1 = (\alpha g)^T = h^*(\alpha g)$.

□

§3.3 π_3 of a 2- complex

In this section we will calculate the Abelian group structure of $\pi_3(X)$ where X is a finite geometric complex whose universal cover is homotopy equivalent to a finite wedge of spheres. This was first done by Milnor and Hilton (see [8]).

Let X be a finite geometric 3- complex, satisfying $C_*(\tilde{X}) \sim \mathcal{A}$, for some algebraic 2- complex $\mathcal{A}$. From the proof of lemma 2.1.7, for some integer k, we have a homotopy equivalence $\tilde{X} \sim W$ where $W = \bigvee_{i=1}^{k} W_i$, with each $W_i \cong S^2$. Let $\{w\} = \bigcap_{i=1}^{k} W_i$. As Abelian groups,
$\pi_n(X) \cong \pi_n(W)$ for $n > 1$.

We proceed to construct a 2- cover of W. The fibre of a 2- cover of W,will be $k(\pi_2(W), 1)$. We have $\pi_2(W) = H_2(W; \mathbb{Z}) = \mathbb{Z}^k$, as the associated algebraic complex of W is:

$$\mathbb{Z}^k \to 0 \to \mathbb{Z}$$

Let T^k denote the product of k copies of S^1.

Lemma 3.3.1 $k(\pi_2(X), 1) = T^k$

Proof: $\tilde{T^k} = \mathbb{R}^k$, which is contractible. Hence $\pi_n(T^k) = 0$ for $n > 1$. Using the identity $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$, we have $\pi_1(T^k) = \mathbb{Z}^k$.

□

Let S_i, $i = 1, \cdots, k$ denote 3- spheres. For each i, let $h_i : S_i \to W_i$ denote the Hopf fibration and let C_i denote the fibre over w in each fibration.

There is a natural inclusion, $\iota : \vee_{i=1}^{k} W_i \hookrightarrow \prod_{i=1}^{k} W_i$ which satisfies:
$\iota(x)_i = x$ if $x \in W_i$
$\iota(x)_i = w$ if $x \notin W_i$
for each $i \in 1, \cdots, k$.

Hence we may consider the pullback of the fibration $\prod_{i=1}^{k} h_i$ along ι:

$$\begin{array}{ccc} Y & \longrightarrow & \prod_{i=1}^{k} S_i \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \prod_{i=1}^{k} h_i} \\ \bigvee_{i=1}^{k} W_i & \xrightarrow{\ \iota\ } & \prod_{i=1}^{k} W_i \end{array}$$

The fibre of $\prod_{i=1}^{k} h_i$ (and consequently f) is $\prod_{i=1}^{k} C_i \cong T^k$

Proposition 3.3.2 *f is a 2-cover.*

Proof: The boundary operator $\pi_2(X) \to \pi_1(T^k)$ sends a generator of $\pi_2(X_i)$ to a generator of $\pi_1(C_i)$. Hence it may be represented by the identity matrix and is certainly an isomorphism.

□

From the previous section we know that $\pi_3(X) \cong H_3(Y;\mathbb{Z})$. We proceed to calculate this homology group.

As each S_i is a 3- sphere, it may be constructed as a CW- complex from the following cells;

the point w,

a 1- cell, e_i for C_i,

a pair of 2- cells U_i^+, U_i^-

a pair of 3- cells V_i^+, V_i^-

For each i, the boundary maps on these cells are:

$\partial e_i = 0$

$\partial U_i^+ = e_i$

$\partial U_i^- = -e_i$

$\partial V_i^+ = U_i^+ + U_i^-$

$\partial V_i^- = -U_i^+ - U_i^-$

The product space $\prod_{i=1}^k S_i$ naturally inherits the structure of a CW- complex with cells given by tensor products of the cells constituting the S_i. The boundary map of a tensor product is calculated using the identity

$$\partial(C \otimes D) = \partial C \otimes D + (-1)^{\deg C} \otimes \partial D$$

(See [13], p120).

Y is a subcomplex of $\prod_{i=1}^k S_i$. It is the preimage under $\prod_{i=1}^k h_i$ of $W \subset \prod_{i=1}^k S_i$. Hence $(x_1, \cdots x_k) \in \prod_{i=1}^k S_i$ is an element of Y precisely when $\#\{i | \ x_i \notin C_i\} \leq 1$. We enumerate the 3- cells which constitute Y and use the identity above to calculate the boundary of each.

The generators of $C_3(Y)$ are:

$e_i \otimes e_j \otimes e_l$, for $i < j < l$,

V_i^+ and V_i^-,

$e_i \otimes U_j^+$ for $i < j$ and $U_j^+ \otimes e_i$ for $j < i$,

$e_i \otimes U_j^-$ for $i < j$ and $U_j^- \otimes e_i$ for $j < i$,

Applying the boundary operator to each of these gives:

$\partial(e_i \otimes e_j \otimes e_l) = 0,$

$\partial V_i^+ = U_i^+ + U_i^-, \quad \partial V_i^- = -U_i^+ - U_i^-,$

$\partial(e_i \otimes U_j^+) = -e_i \otimes e_j, \quad \partial U_j^+ \otimes e_i = e_i \otimes e_j,$

$\partial(e_i \otimes U_j^-) = e_i \otimes e_j, \quad \partial U_j^- \otimes e_i = -e_i \otimes e_j.$

Hence we see that the kernel of the boundary operator is generated by

$e_i \otimes e_j \otimes e_l$, for $i < j < l$,

$V_i^+ + V_i^-$,

$e_i \otimes U_j^+ + U_i^+ \otimes e_j$ for $i < j$,

$e_i \otimes U_j^+ + e_i \otimes U_j^-$ for $i < j$,

$U_i^+ \otimes e_j + U_i^- \otimes e_j$ for $i < j$.

We now enumerate the 4- cells of Y. The generators of $C_4(Y)$ are:

$e_i \otimes e_j \otimes e_l \otimes e_m$ for $i < j < l < m$,

$e_i \otimes e_j \otimes U_l^+$ for $i < j < l$,

$e_i \otimes U_j^+ \otimes e_l$ for $i < j < l$,

$U_i^+ \otimes e_j \otimes e_l$ for $i < j < l$,

$e_i \otimes e_j \otimes U_l^-$ for $i < j < l$,

$e_i \otimes U_j^- \otimes e_l$ for $i < j < l$,

$U_i^- \otimes e_j \otimes e_l$ for $i < j < l$,

$e_i \otimes V_j^+$ for $i < j$,

$V_i^+ \otimes e_j$ for $i < j$,

$e_i \otimes V_j^-$ for $i < j$,

$V_i^- \otimes e_j$ for $i < j$.

Applying the boundary operator gives

$\partial(e_i \otimes e_j \otimes e_l \otimes e_m) = 0$

$\partial(e_i \otimes e_j \otimes U_l^+) = e_i \otimes e_j \otimes e_k$

$\partial(e_i \otimes U_j^+ \otimes e_l) = -e_i \otimes e_j \otimes e_k$

$\partial(U_i^+ \otimes e_j \otimes e_l) = e_i \otimes e_j \otimes e_k$

$\partial(e_i \otimes e_j \otimes U_l^-) = -e_i \otimes e_j \otimes e_k$

$\partial(e_i \otimes U_j^- \otimes e_l) = e_i \otimes e_j \otimes e_k$

$\partial(U_i^- \otimes e_j \otimes e_l) = -e_i \otimes e_j \otimes e_k$

$\partial(e_i \otimes V_j^+) = -e_i \otimes U_j^+ - e_i \otimes U_j^-$

$\partial(V_i^+ \otimes e_j) = U_i^+ \otimes e_j + U_i^- \otimes e_j$

$\partial(e_i \otimes V_j^-) = e_i \otimes U_j^+ + e_i \otimes U_j^-$

$\partial(V_i^- \otimes e_j) = -U_i^+ \otimes e_j - U_i^- \otimes e_j$

Hence the image of $C_4(Y)$ under the boundary operator is generated by:

$e_i \otimes e_j \otimes e_l$, for $i < j < l$,

$e_i \otimes U_j^+ + e_i \otimes U_j^-$ for $i < j$,

$U_i^+ \otimes e_j + U_i^- \otimes e_j$ for $i < j$.

This leaves:

$V_i^+ + V_i^-$,

$e_i \otimes U_j^+ + U_i^+ \otimes e_j$ for $i < j$,

to generate $H_3(Y;\mathbb{Z})$.

Let J denote the algebraic π_2 of $\mathcal{A}$. As a $\mathbb{Z}[G]$ - module, J is isomorphic to $\pi_2(X)$. This is isomorphic to $\pi_2(W)$ as an Abelian group, which in turn is equal to

$$\bigoplus_{i=1}^{k} \pi_2(W_i)$$

We regard w as the base point of each W_i. For each i, let w_i denote the generator of $\pi_2(W_i)$, induced from the identity map $W_i \to S^2$. The isomorphism $\pi_2(W) \to \pi_2(X)$ is induced by a homotopy equivalence $W \to \tilde{X}$ composed with the covering map $\tilde{X} \to X$. Let $q : W \to X$ denote this composition.

For each i, let $\alpha_i : S^2 \to X$ denote $q \circ w_i$ and let $*$ denote qw. Taking $*$ once and for all as the base point of X, we may regard α_i as representing an element of $\pi_2(X)$. In particular, α_i represents $q_* w_i \in \pi_2(X)$.

The $\alpha_i, i \in 1, \cdots, k$ are a basis for J over $\mathbb{Z}$. Therefore the $\alpha_i \otimes \alpha_j, i, j \in 1, \cdots, k$ are a basis for $J \otimes_{\mathbb{Z}} J$ over $\mathbb{Z}$. Hence, any element of $S^2(J)$ may be written uniquely as a $\mathbb{Z}$- linear combination of $\alpha_i \otimes \alpha_j$. However, in any such linear combination, the coefficient on $\alpha_i \otimes \alpha_j$ would have to equal the coefficient on $\alpha_j \otimes \alpha_i$. Consequently, $S^2(J)$ is generated, over $\mathbb{Z}$, by the $\alpha_i \otimes \alpha_i$ and the $\alpha_i \otimes \alpha_j + \alpha_j \otimes \alpha_i, i \neq j$.

We may define a $\mathbb{Z}$- linear isomorphism $\phi : \pi_3(X) \to S^2(J)$, by sending:

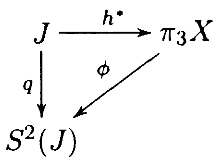
$V_i^+ + V_i^- \mapsto \alpha_i \otimes \alpha_i$,

$e_i \otimes U_j^+ + U_i^+ \otimes e_j \mapsto \alpha_i \otimes \alpha_j + \alpha_j \otimes \alpha_i$.

The purpose of the next section is to show that ϕ is $\mathbb{Z}[G]$- linear.

§3.4 Realizing elements of $\pi_3(X)$.

We have the following maps of sets:



where $q : \alpha \mapsto \alpha \otimes \alpha$, for all $\alpha \in J$.

From lemma 3.2.7, we know that h^* respects the action of G. Also q respects the action of G by construction. In order to prove that ϕ respects the action of G (and consequentially is a $\mathbb{Z}[G]$- linear isomorphism), we will show the following:

i) $h^*(J)$ generates $\pi_3(X)$ over $\mathbb{Z}$. We obtain this in corollary 3.4.8.
ii)The diagram above commutes. In other words, $\phi(h^*(\alpha)) = \alpha \otimes \alpha$ for all $\alpha \in J$. We obtain this in lemma 3.4.13.

In the previous section $\pi_3(X)$ was computed, as an Abelian group. We now realize the elements of this Abelian group as actual maps $S^3 \to X$.

Consider the identity map, $S^3 \to S_i$, taking w as base point. This map sends the generator of $H_3(S^3;\mathbb{Z})$ to the element of $H_3(Y;\mathbb{Z})$ represented by $V_i^+ + V_i^-$. Hence the Hurewicz isomorphism takes $V_i^+ + V_i^-$ to the element of $\pi_3(Y)$ represented by the identity map $S^3 \to S_i$.

The 2- cover, f, restricts to the Hopf fibration $S_i \to W_i$. So $V_i^+ + V_i^-$ is sent to the element of $\pi_3(W)$, represented by the Hopf fibration $S^3 \to W_i$. This equals $h^*(w_i)$.

Composing $h^*(w_i)$ with q gives $q \circ h^*(w_i) = q \circ w_i \circ h = \alpha_i \circ h = h^*(\alpha_i)$. Hence the isomorphism $H_3(Y;\mathbb{Z}) \to \pi_3(X)$, due to lemma 3.1.3, takes $V_i^+ + V_i^-$ to

$h^*(\alpha_i) \in \pi_3(X)$.

Lemma 3.4.1 *For each $i \in 1, \cdots, k$, we have $\phi(\alpha_i^T) = \alpha_i \otimes \alpha_i$.*

Proof: From the construction of ϕ and the discussion above, we have $\phi(h^*(\alpha_i)) = \alpha_i \otimes \alpha_i$. So by lemma 3.2.5 $\phi(\alpha_i^T) = \phi(h^*(\alpha_i)) = \alpha_i \otimes \alpha_i$

□

Once again, we may view S^3 as $\{(z, s) \in \mathbb{C}^2 | z\bar{z} + s\bar{s} = 1\}$. It decomposes into two solid tori, given by $s\bar{s} \leq \frac{1}{2}$ and $s\bar{s} \geq \frac{1}{2}$. Given $i < j$, we may identify the disk U_j^+ with the disk $s\bar{s} \leq \frac{1}{2}$ in the complex plane. We may also identify the set of arguments of z, on the complex plane, with C_i.

Similarly we may identify the disk U_i^+ with the disk $z\bar{z} \leq \frac{1}{2}$ in the complex plane. We may also identify the set of arguments of s, on the complex plane, with C_j.

Hence we have a natural identification map $I : S^3 \to C_i \times U_j^+ \cup U_i^+ \times C_j$. This takes a generator of $H_3(S^3; \mathbb{Z})$ to $e_i \otimes U_j^+ + U_i^+ \otimes e_j$. Hence we have

Lemma 3.4.2 *The Hurewicz isomorphism, $H_3(Y; \mathbb{Z}) \to \pi_3(Y)$, takes $e_i \otimes U_j^+ + U_i^+ \otimes e_j$ to the element of $\pi_3(Y)$ represented by I.*

For any $a \in C_i$ and $b \in C_j$, the map f restricts to $w_j : (a, U_j^+) \to W_j$ and to $w_i : (U_i^+, b) \to W_i$, where w_i, w_j are now regarded as maps $D^2 \to W$, which send the boundary to w.

Fix two simply linked solid tori, A, B, each parametrized $S^1 \times D^2$, in S^3. Let Q be a topological space with base point %. Given $\alpha, \beta \in \pi_2(Q)$, we construct an element of $\pi_3(Q)$:

Definition 3.4.3 $\alpha \vee \beta : S^3 \to Q$ is defined by

$\alpha \vee \beta(p) = \%$ for $p \notin A, B$

$\alpha \vee \beta(\theta, d) = \alpha(d)$ for $(\theta, d) \in A$

$\alpha \vee \beta(\theta, d) = \beta(d)$ for $(\theta, d) \in B$

By thinning the solid tori, $s\bar{s} \leq \frac{1}{2}$ and $s\bar{s} \geq \frac{1}{2}$, we have that f_*I and $w_i \vee w_j$ represent the same element of $\pi_3(W)$.

Lemma 3.4.4 $q \circ w_i \vee w_j = \alpha_i \vee \alpha_j$.

Proof:

$q \circ w_i \vee w_j(\theta, d) = q \circ w_i(d) = \alpha_i(d)$ for $(\theta, d) \in A$.

$q \circ w_i \vee w_j(\theta, d) = q \circ w_j(d) = \alpha_j(d)$ for $(\theta, d) \in B$.

Hence we have $q \circ w_i \vee w_j = \alpha_i \vee \alpha_j$.

□

So to recap: the Hurewicz isomorphism takes $e_i \otimes U_j^+ + U_i^+ \otimes e_j$ to $I \in \pi_3(Y)$. Composition with f takes this to $w_i \vee w_j$. Composition with q takes this to $\alpha_i \vee \alpha_j$.

We may conclude:

Lemma 3.4.5 *As an Abelian group, $\pi_3(X)$ is generated freely by $\alpha_i^T, i \in \{1, \cdots, k\}$ and $\alpha_i \vee \alpha_j, i < j \in \{1, \cdots, k\}$. Also,*

$$\phi(\alpha_i^T) = \alpha_i \otimes \alpha_i, \quad i \in \{1, \cdots, k\}$$

$$\phi(\alpha_i \vee \alpha_j) = \alpha_i \otimes \alpha_j + \alpha_j \otimes \alpha_i, i < j \in \{1, \cdots, k\}.$$

We now describe two homotopies which will be used in the lemmas which follow.

i) Let $\gamma \in \pi_3(X)$ agree with α^T on the solid torus $\{(z, s) \in S^3 | s\bar{s} \leq \frac{1}{2}\}$, for some $\alpha \in \pi_2(X)$. So given a point (θ, s) in the solid torus, we have $\gamma(\theta, s) = \alpha(R_\theta(s))$.

For $t \in [0, \pi]$, let

$\gamma_t(\theta, s) = \alpha(s), \quad$ for $\theta \leq t$.

$\gamma_t(\theta, s) = \alpha(R_{\frac{2\pi(\theta - t)}{2\pi - t}}(s)), \quad$ for $\theta \geq t$.

$\gamma_t(p) = \gamma(p), \quad$ for p not in the solid torus.

Intuitively, γ_π is γ with the "twist" moved round to one side of the torus.

ii) This homotopy will be referred to as "pinching" the solid torus.

Consider γ_π as before, and select a cylinder, in S^3 which intersects the solid torus in two disconnected places, on the "untwisted" side, and which does not intersect any other points which do not map to $*$, under γ_π. Parametrize this cylinder $A \times I$, where A is a disk of radius 3 about the origin in $\mathbb{C}$, and I is the interval $[-1, 1]$. Let A_1 denote the disk of radius $\frac{1}{\sqrt{2}}$ about i, in A and A_2 denote the disk of radius $\frac{1}{\sqrt{2}}$ about $-i$, in A.

By deforming γ_π to γ', fixing the base point, we may have γ' restricted to the cylinder giving:

$\gamma'(a, s) = *, \quad \text{if } a \notin A_1, A_2,$

$\gamma'(a, s) = \alpha(a - i), \quad \text{if } a \in A_1,$

$\gamma'(a, s) = \alpha(-a - i), \quad \text{if } a \in A_2.$

For each $s \in I$, we have the map $\gamma'(\_, s) : A \to X$ representing $\alpha - \alpha = 0 \in \pi_2$. Hence, fixing some s, we have a homotopy

$h_t : A \to X$, for $t \in [0, 1]$, with $h_0(a) = \gamma'(a, s)$, and $h_1(a) = *$.

Let $\gamma'_t, \quad t \in [0, 1]$ be defined as follows:

$\gamma'_t(p) = \gamma'(p), \quad \text{if } p \notin A \times I,$

$\gamma'_t(a, s) = h_t(a), \quad \text{for } |s| \leq \frac{1}{2},$

$\gamma'_t(a, s) = h_{(2-2|s|)t}(a), \quad \text{for } |s| \geq \frac{1}{2},$

So γ'_1 represents the same element of π_2 as γ_π, but it has the "twisted" part of the solid torus "pinched off" from the untwisted part.

Lemma 3.4.6 *Let* $\alpha, \beta \in \pi_2(X)$. *Then* $\alpha \vee \beta = (\alpha + \beta)^T - \alpha^T - \beta^T$.

Proof: Consider the map $(\alpha + \beta)^T : S^3 \to X$. We may partition the solid torus $\{(z, w) \in S^3 | w\bar{w} \leq \frac{1}{2}\}$, into two linked solid tori, such that altering $(\alpha + \beta)^T$ to map one solid tori to $*$ would leave α^T, and altering $(\alpha + \beta)^T$ to map the other solid tori to $*$, would leave β^T.

Hence "pinching off" the "twists" in the two solid tori gives

$$(\alpha+\beta)^T = \alpha^T + \beta^T + \alpha \vee \beta$$

□

Note the left hand side of the equality in lemma 3.4.6 is symmetric in α and β.

Hence we have:

Corollary 3.4.7 $\alpha \vee \beta = \beta \vee \alpha$

Also, as each $\alpha \vee \beta$ may now be written in terms of $(\alpha+\beta)^T$, α^T and β^T, we have:

Corollary 3.4.8 *Elements of the form α^T, for $\alpha \in \pi_2(X)$, span $\pi_3(X)$ over $\mathbb{Z}$.*

Lemma 3.4.9 *Let $\alpha, \beta, \gamma \in \pi_2(X)$. Then*

$$(\alpha+\beta+\gamma)^T = (\alpha+\beta)^T + (\beta+\gamma)^T + (\gamma+\alpha)^T - \alpha^T - \beta^T - \gamma^T$$

Proof: Consider the map $(\alpha+\beta+\gamma)^T : S^3 \to X$. We may partition the solid torus $\{(z,w) \in S^3 | w\bar{w} \leq \frac{1}{2}\}$ into three mutually linked solid tori, A, B, C so that if the complement of A, B or C was mapped to $*$, the map would be α^T, β^T, γ^T respectively.

Pinching off the twist in each solid torus gives

$$(\alpha+\beta+\gamma)^T = \alpha^T + \beta^T + \gamma^T + \Delta$$

where Δ is represented by a map $S^3 \to X$, sending the complement of three linked tori, A', B', C' to $*$, and projecting each solid tori ($= S^1 \times D^2$) onto D^2 and mapping via α, β, γ respectively.

We may pinch each solid torus between its links with the other two, to get

$$\Delta = \alpha \vee \beta + \beta \vee \gamma + \gamma \vee \alpha$$

Hence

$$(\alpha+\beta+\gamma)^T = \alpha^T + \beta^T + \gamma^T + \alpha\vee\beta + \beta\vee\gamma + \gamma\vee\alpha$$
$$= (\alpha+\beta)^T + (\beta+\gamma)^T + (\gamma+\alpha)^T - \alpha^T - \beta^T - \gamma^T$$

□

We now apply this lemma to get:

Lemma 3.4.10 *Suppose $\alpha, \beta, \gamma \in \pi_2(X)$ satisfy the following:*
$\phi(\alpha^T) = \alpha\otimes\alpha, \quad \phi(\beta^T) = \beta\otimes\beta, \quad \phi(\gamma^T) = \gamma\otimes\gamma,$
$\phi((\alpha+\beta)^T) = (\alpha+\beta)\otimes(\alpha+\beta),$
$\phi((\alpha+\gamma)^T) = (\alpha+\gamma)\otimes(\alpha+\gamma),$
$\phi((\gamma+\beta)^T) = (\gamma+\beta)\otimes(\gamma+\beta),$
Then $\phi((\alpha+\beta+\gamma)^T) = (\alpha+\beta+\gamma)\otimes(\alpha+\beta+\gamma)$.

Proof:

$$\phi((\alpha+\beta+\gamma)^T) \quad = \quad \phi((\alpha+\beta)^T + (\beta+\gamma)^T + (\gamma+\alpha)^T - \alpha^T - \beta^T - \gamma^T)$$

(by 3.4.9)

$$= (\alpha+\beta)\otimes(\alpha+\beta) + (\beta+\gamma)\otimes(\beta+\gamma) + (\gamma+\alpha)\otimes(\gamma+\alpha) - \alpha\otimes\alpha - \beta\otimes\beta - \gamma\otimes\gamma$$

(by hypothesis)

$$= \alpha\otimes\alpha + \beta\otimes\beta + \gamma\otimes\gamma + \alpha\otimes\beta + \beta\otimes\alpha + \beta\otimes\gamma + \gamma\otimes\beta + \alpha\otimes\gamma + \gamma\otimes\alpha$$
$$= (\alpha+\beta+\gamma)\otimes(\alpha+\beta+\gamma)$$

□

As we have said, the α_i form a basis for J over $\mathbb{Z}$.

Definition 3.4.11 *Norm* Given $\alpha = \sum_r \alpha_i\lambda_i$, $\lambda_i \in \mathbb{Z}$, we define the norm of α, denoted $|\alpha|$, by

$$|\alpha| = \sum_r |\lambda_i|$$

Lemma 3.4.12 *Let $\alpha \in J$ satisfy $|\alpha| \leq 2$. Then $\phi(\alpha^T) = \alpha \otimes \alpha$*

Proof: The only element of J with norm equal to 0 is 0. We have previously noted that $0^T = 0$, so we have $\phi(0^T) = \phi(0) = 0 = 0 \otimes 0$.

The only elements of J with norm equal to 1 are ones of the form α_i or $-\alpha_i$. By lemma 3.4.5, $\phi(\alpha_i^T) = \alpha_i \otimes \alpha_i$. By lemma 3.2.6, we have
$\phi(-\alpha_i^T) = \phi(\alpha_i^T) = \alpha_i \otimes \alpha_i = -\alpha_i \otimes -\alpha_i$.

Elements of J with norm equal to 2 are of the form $\alpha_i + \alpha_j$, $-\alpha_i - \alpha_j$, $\alpha_i - \alpha_j$, $\alpha_i 2$, or $-\alpha_i 2$, where $i \neq j$. By lemma 3.2.6, it is sufficient to consider $\alpha_i + \alpha_j$, $\alpha_i - \alpha_j$, and $\alpha_i 2$.

By lemma 3.4.6, for $i \neq j$, we have

$$\phi((\alpha_i + \alpha_j)^T) = \phi((\alpha_i + \alpha_j)^T - \alpha_i^T - \alpha_j^T + \alpha_i^T + \alpha_j^T)$$

$$= \phi(\alpha_i \vee \alpha_j) + \phi(\alpha_i^T) + \phi(\alpha_j^T) = \alpha_i \otimes \alpha_j + \alpha_j \otimes \alpha_i + \alpha_i \otimes \alpha_i + \alpha_j \otimes \alpha_j$$

$$= (\alpha_i + \alpha_j) \otimes (\alpha_i + \alpha_j)$$

By lemma 3.4.9, we have

$$\alpha_i^T = (\alpha_i + \alpha_j - \alpha_j)^T = (\alpha_i + \alpha_j)^T + (\alpha_i - \alpha_j)^T - \alpha_i^T - \alpha_j^T 2$$

Hence for $i \neq j$, we have

$$\phi((\alpha_i - \alpha_j)^T) = \phi(\alpha_i^T)2 + \phi(\alpha_j^T)2 - \phi((\alpha_i + \alpha_j)^T)$$

$$= \alpha_i \otimes \alpha_i + \alpha_j \otimes \alpha_j - \alpha_i \otimes \alpha_j - \alpha_j \otimes \alpha_i$$

$$= (\alpha_i - \alpha_j) \otimes (\alpha_i - \alpha_j)$$

Again, by lemma 3.4.9, we have

$$\alpha_i^T = (\alpha_i + \alpha_i - \alpha_i)^T = (\alpha_i + \alpha_i)^T - \alpha_i^T 3$$

So $\phi((\alpha_i 2)^T) = \phi(\alpha_i^T)4 = (\alpha_i \otimes \alpha_i)4 = \alpha_i 2 \otimes \alpha_i 2$.

Hence we have dealt with each element of J, with norm less than or equal to 2. □

Lemma 3.4.13 *Let $\alpha \in J$. Then $\phi(\alpha^T) = \alpha \otimes \alpha$.*

Proof: Suppose not. Then there exists some minimal number n such that there exists $\alpha \in J$, with $|\alpha| = n$ and $\phi(\alpha^T) \neq \alpha \otimes \alpha$. By lemma 3.4.12, $n \geq 3$, so we have $\alpha = \alpha' \pm \alpha_i \pm \alpha_j$, for some $i, j \in \{1, \cdots, k\}$ and $|\alpha'| = n - 2$.

The hypothesis' for lemma 3.4.10 are fulfilled, as $1, 2, n-1, n-2 < n$, so we have

$$\phi(\alpha^T) = \phi((\alpha' \pm \alpha_i \pm \alpha_j)^T)$$

$$= (\alpha' \pm \alpha_i \pm \alpha_j) \otimes (\alpha' \pm \alpha_i \pm \alpha_j)$$

$$= \alpha \otimes \alpha$$

which contradicts $\phi(\alpha^T) \neq \alpha \otimes \alpha$.

□

Hence by lemma 3.2.5, we have $\phi(h^*(\alpha)) = \alpha \otimes \alpha$.

Lemma 3.4.14 *The isomorphism $\phi : \pi_3(X) \to S^2(J)$ is $\mathbb{Z}[G]$- linear.*

Proof: Recall lemma 3.2.7, which states that given $g \in G$, and $\alpha \in J$, we have $h^*(\alpha g) = h^*(\alpha)g$. From lemma 3.4.13 we have

$$\phi(h^*(\alpha)g) = \phi(h^*(\alpha g)) = \alpha g \otimes \alpha g = (\alpha \otimes \alpha)g = \phi(h^*(\alpha))g$$

By corollary 3.4.8, given an arbitrary element $\beta \in \pi_3(X)$, we may write

$$\beta = \sum_r h^*(\beta_r)\lambda_r$$

with $\beta_r \in J$, $\lambda_r \in \mathbb{Z}$.

So

$$\phi(\beta g) = \sum_r \phi(h^*(\beta_r)g)\lambda_r = \sum_r \phi(h^*(\beta_r))g\lambda_r = \phi(\beta)g$$

□

From this lemma we may conclude:

Theorem 3A *If $C_*(\tilde{X}) \sim \mathcal{A}$, for some finite algebraic 2- complex $\mathcal{A}$, then $\pi_3(X) \cong S^2(J)$ as $\mathbb{Z}[G]$- modules.*

In particular:

Corollary 3.4.15 *If X is a geometric 2- complex, and $\pi_2(X) = J$, then $\pi_3(X) \cong S^2(J)$ as $\mathbb{Z}[G]$- modules.*

§3.5 The effect of stabilizing π_2.

Let X be a finite geometric 2- complex with finite fundamental group G. Let X' be another finite geometric 2- complex with fundamental group G. From corollary 1.1.2 and lemma 2.1.5 we know that there exist integers a, b such that

$$\pi_2(X) \oplus \mathbb{Z}[G]^a \cong \pi_2(X') \oplus \mathbb{Z}[G]^b$$

In this section we investigate the corresponding relationship between $\pi_3(X)$ and $\pi_3(X')$.

If we let $J = \pi_2(X)$ and $J' = \pi_2(X')$, then from the last section we have $\pi_3(X) = S^2(J)$ and $\pi_3(X') = S^2(J')$. We know that

$$S^2(J \oplus \mathbb{Z}[G]^a) \cong S^2(J' \oplus \mathbb{Z}[G]^b)$$

The next few lemmas give us an expansion of this.

Lemma 3.5.1 *Let A_i, $i = 1, \cdots, n$ be modules over $\mathbb{Z}[G]$, with finitely generated, free underlying Abelian groups. Then*

$$S^2(\bigoplus_{i=1}^{n} A_i) = \bigoplus_{i=1}^{n} S^2(A_i) \oplus \bigoplus_{i<j} A_i \otimes_{\mathbb{Z}} A_j$$

Proof: For each i, let the $e_{i,r}$ be a basis over $\mathbb{Z}$ for A_i. Then $S^2(\bigoplus_{i=1}^{n} A_i)$ is freely generated over $\mathbb{Z}$, by elements of the form:

$e_{i,r} \otimes e_{i,r}$,

$e_{i,r} \otimes e_{i,s} + e_{i,s} \otimes e_{i,r}, \quad r < s,$

$e_{i,r} \otimes e_{j,s} + e_{j,s} \otimes e_{i,r}, \quad i < j,$

For each i, the $\mathbb{Z}$- linear span of the $e_{i,r} \otimes e_{i,r}$, for all r, and the $e_{i,r} \otimes e_{i,s} + e_{i,s} \otimes e_{i,r}$,

$r < s$, is closed under the group action and is isomorphic over $\mathbb{Z}[G]$ to $S^2(A_i)$.

Similarly, for each pair i, j, with $i < j$, the $\mathbb{Z}$- linear span of the $e_{i,r} \otimes e_{j,s} + e_{j,s} \otimes e_{i,r}$, for all r and s, is closed under the group action. Furthermore we have an isomorphism from it to $A_i \otimes_{\mathbb{Z}} A_j$, which maps

$$e_{i,r} \otimes e_{j,s} + e_{j,s} \otimes e_{i,r} \mapsto e_{i,r} \otimes e_{j,s}$$

So over $\mathbb{Z}[G]$, the module $S^2(\bigoplus_{i=1}^n A_i)$ decomposes into the sum

$$\bigoplus_{i=1}^{n} S^2(A_i) \oplus \bigoplus_{i<j} A_i \otimes_{\mathbb{Z}} A_j$$

□

Note that for $\mathbb{Z}[G]$- modules A, B, we have a $\mathbb{Z}[G]$- linear isomorphism $A \otimes_{\mathbb{Z}} B \to B \otimes_{\mathbb{Z}} A$, given by sending $a \otimes b \mapsto b \otimes a$.

Lemma 3.5.2 *Let A be a $\mathbb{Z}[G]$ module whose underlying Abelian group is isomorphic to $\mathbb{Z}^k$. Then $A \otimes_{\mathbb{Z}} \mathbb{Z}[G] \cong \mathbb{Z}[G]^k$.*

Proof: The action of an element $g \in G$ on A is a $\mathbb{Z}$- linear isomorphism. Hence, if $e_1, \cdots, e_k$ is a $\mathbb{Z}$- linear basis for A, then so is $e_1 g, \cdots, e_k g$. Hence $A \otimes_{\mathbb{Z}} \mathbb{Z}[G]$ is freely generated over $\mathbb{Z}$ by elements of the form $e_i g \otimes g$, for $g \in G$ and $i \in \{1, \cdots, k\}$.

For each i, the $\mathbb{Z}$ linear span of the $e_i g \otimes g$, $g \in G$, is closed under the action of G. Furthermore, we have a $\mathbb{Z}[G]$- linear isomorphism from it to $\mathbb{Z}[G]$, which maps $e_i g \otimes g \mapsto g$.

Hence,

$$A \otimes_{\mathbb{Z}} \mathbb{Z}[G] \cong \bigoplus_{i=1}^{k} \mathbb{Z}[G] \cong \mathbb{Z}[G]^k$$

□

Let p denote the number of pairs, $\{g, g^{-1}\} \in G$, with $g \neq g^{-1}$.

Also, let $T = \{g \in G | g^2 = e\}$.

Definition 3.5.3 For a finite group G, we set

$$V_G = \bigoplus_{t \in T} (1+t)\mathbb{Z}[G]$$

Let n denote the order of G.

Lemma 3.5.4 $S^2(\mathbb{Z}[G]) \cong \mathbb{Z}[G]^{1+p} \oplus V_G$

Proof: Consider the $\mathbb{Z}$- linear basis for $S^2(\mathbb{Z}[G])$ given by $g \otimes g$, $g \in G$, and $g \otimes h + h \otimes g$, $\{g, h\} \subset G$.

We have $< e \otimes e > \mathbb{Z}[G] =< g \otimes g | \; g \in G > \mathbb{Z}$.

Suppose $g \neq g^{-1}$. Then $< g \otimes e + e \otimes g > \mathbb{Z}[G] =$ $< h \otimes l + l \otimes h | \; h, l \in G, \, hl^{-1} = g > \mathbb{Z}$. Further, given an element $h \otimes l + l \otimes h$, satisfying $hl^{-1} = g$, we may write $h \otimes l + l \otimes h = (g \otimes e + e \otimes g)l$. This is the unique way of writing $h \otimes l + l \otimes h$ as a $\mathbb{Z}[G]$- linear multiple of $g \otimes e + e \otimes g$, because $h \otimes l + l \otimes h \neq (g \otimes e + e \otimes g)h$, as $gh = l$ would imply $g = lh^{-1} = (hl^{-1})^{-1} = g^{-1}$.

So the $\mathbb{Z}$- linear span of the $g \otimes g$ and the $h \otimes l + l \otimes h$, $lh^{-1} \neq hl^{-1}$, is isomorphic to $\mathbb{Z}[G]^{1+p}$. The remaining elements of the basis are of the form $h \otimes l + l \otimes h$, with $hl^{-1} = lh^{-1}$. Let $t = hl^{-1}$. then $h \otimes l + l \otimes h = (e \otimes t + t \otimes e)l$. The $\mathbb{Z}$-linear span of the $h \otimes l + l \otimes h$, $hl^{-1} = lh^{-1}$, is therefore equal to

$$\bigoplus_{t \in T} (e \otimes t + t \otimes e)\mathbb{Z}[G]$$

Regarding $S^2(\mathbb{Z}[G])$ as a submodule of $\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G]$, we have $e \otimes t + t \otimes e = (e \otimes t)(1+t)$. Therefore

$$\bigoplus_{t \in T} (e \otimes t + t \otimes e)\mathbb{Z}[G] = V_G$$

and

$$S^2(\mathbb{Z}[G]) \cong \mathbb{Z}[G]^{1+p} \oplus V_G$$

□

Returning to our geometric complexes X and X', we had

$$S^2(J \oplus \mathbb{Z}[G]^a) \cong S^2(J' \oplus \mathbb{Z}[G]^b)$$

By lemma 3.5.1, this expands to:

$$S^2(J) \oplus S^2(\mathbb{Z}[G])^a \oplus (J \otimes_{\mathbb{Z}} \mathbb{Z}[G])^a \oplus (\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G])^{a(a-1)/2}$$

$$= S^2(J') \oplus S^2(\mathbb{Z}[G])^b \oplus (J' \otimes_{\mathbb{Z}} \mathbb{Z}[G])^b \oplus (\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G])^{b(b-1)/2}$$

Let k denote the $\mathbb{Z}$- rank of J, and let k' denote the $\mathbb{Z}$- rank of J'. By theorem 3A we have $S^2(J) = \pi_3(X)$ and $S^2(J') = \pi_3(X')$. Applying lemmas 3.5.2 and 3.5.4 to the equation above gives:

$$\pi_3(X) \oplus (\mathbb{Z}[G]^{1+p} \oplus V_G)^a \oplus \mathbb{Z}[G]^{ka} \oplus ZG^{na(a-1)/2}$$

$$= \pi_3(X') \oplus (\mathbb{Z}[G]^{1+p} \oplus V_G)^b \oplus \mathbb{Z}[G]^{k'b} \oplus ZG^{nb(b-1)/2}$$

Hence

$$\pi_3(X) \oplus \mathbb{Z}[G]^{(1+p+k+n(a-1)/2)a} \oplus {V_G}^a = \pi_3(X') \oplus \mathbb{Z}[G]^{(1+p+k'+n(b-1)/2)b} \oplus {V_G}^b$$

and we have the following theorem:

Theorem 3.5.5 *Let X and X' be finite geometric 2- complexes, with finite fundamental group G. Then there exist integers p, q, r, s such that:*

$$\pi_3(X) \oplus \mathbb{Z}[G]^p \oplus {V_G}^q = \pi_3(X') \oplus \mathbb{Z}[G]^r \oplus {V_G}^s$$

Note that if the order of G is odd, it does not contain any elements of order 2. Hence $V_G = 0$ and we have.

Corollary 3.5.6 *Let X and X' be finite geometric 2- complexes, with odd finite fundamental group G. Then $\pi_3(X)$ and $\pi_3(X')$ are stably equivalent.*

§3.6 The case $\pi_2 = IG^*$

In this section we consider the case of a finite geometric 2- complex, X, with finite fundamental group G and $\pi_2(X) = IG^*$. Recall that we have a short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \xrightarrow{p} IG^* \to 0$$

We now consider the $\mathbb{Z}[G]$- linear surjection $p' : S^2(\mathbb{Z}[G]) \to S^2(IG^*)$, which sends $g \otimes g \mapsto p(g) \otimes p(g)$
and $g \otimes h + h \otimes g \mapsto p(g) \otimes p(h) + p(h) \otimes p(g)$.

Let Σ denote the sum of the elements of G.

Lemma 3.6.1 *The kernel of p' is generated over $\mathbb{Z}[G]$, by $\Sigma \otimes e + e \otimes \Sigma$ and $\Sigma \otimes \Sigma$.*

Proof: We may take a $\mathbb{Z}$- linear basis for $\mathbb{Z}[G]$, given by $\{g \in G | g \neq e\}$ together with Σ. We have a $\mathbb{Z}$- linear basis for IG^* given by
$\{p(g) | g \in G, g \neq e\}$. Hence the kernel of p' is generated over $\mathbb{Z}$ by $\Sigma \otimes \Sigma$, and $g \otimes \Sigma + \Sigma \otimes g$. Clearly this is contained in the, $\mathbb{Z}[G]$ linear span of $\Sigma \otimes e + e \otimes \Sigma$ and $\Sigma \otimes \Sigma$.

□

Let S be a subset of G, containing precisely one of g or g^{-1}, for each pair g, g^{-1}, with $g \neq g^{-1}$. Let $e_g = g \otimes e + e \otimes g$ for each $g \in S$. Let $e_t = t \otimes e + e \otimes t$ for each $t \in T$. Let $e_0 = e \otimes e$. Then from the proof of lemma 3.5.4 we have:

$$S^2(\mathbb{Z}[G]) = e_0\mathbb{Z}[G] \oplus \bigoplus_{g \in S} e_g\mathbb{Z}[G] \oplus \bigoplus_{t \in T} e_t\mathbb{Z}[G]$$

Also,

$$e \otimes \Sigma + \Sigma \otimes e = e_0 2 + \sum_{g \in S} e_g(1 + g^{-1}) + \sum_{t \in T} e_t$$

and

$$\Sigma \otimes \Sigma = (e_0 2 + \sum_{g \in S} e_g(1 + g^{-1}) + \sum_{t \in T} e_t)\frac{\Sigma}{2}$$

Let

$$u = (e_0 2 + \sum_{g\in S} e_g(1+g^{-1}) + \sum_{t\in T} e_t)$$

We have

$$S^2(IG^*) = \frac{S^2(\mathbb{Z}[G])}{< u, u\frac{\Sigma}{2} >}$$

The relation $u = 0$ is equivalent to $e_0 2 = -(\sum_{g\in S} e_g(1+g^{-1}) + \sum_{t\in T} e_t)$. Let

$$M = \bigoplus_{g\in S} e_g\mathbb{Z}[G] \oplus \bigoplus_{t\in T} e_t\mathbb{Z}[G]$$

and let $u_M \in M$ be equal to $-(\sum_{g\in S} e_g(1+g^{-1}) + \sum_{t\in T} e_t)$.

Theorem 3.6.2 *If $\pi_2 X = IG^*$ then $\pi_3(X) = M[\frac{u_M}{2}]$.*

Proof: We need to check that $\frac{u_M}{2}$ satisfies the relations which e_0 was subject to:
$\frac{u_M}{2}2 + \sum_{g\in S} e_g(1+g^{-1}) + \sum_{t\in T} e_t = 0$
$(\frac{u_M}{2}2 + \sum_{g\in S} e_g(1+g^{-1}) + \sum_{t\in T} e_t)\Sigma/2 = 0$

□

Example 3.6.3 $G = C_3 =< x|x^3 = e >$

If $\pi_2(X) = IG^*$ then

$$\pi_3(X) = S^2(IG^*) = e_{x^2}\mathbb{Z}[G][(1+x)/2] \cong \mathbb{Z}[G][\frac{(1+x)}{2}]$$

Example 3.6.4 $G = Q_8 =< x, y|x^2 = y^2, xyx = y >$

There is only one element of order two in Q_8 so if $\pi_2(X) = IG^*$ we have

$$\pi_3(X) = S^2(IG^*) = (e_x\mathbb{Z}[G] \oplus e_y\mathbb{Z}[G] \oplus e_{xy}\mathbb{Z}[G] \oplus e_{y^2}\mathbb{Z}[G])[\frac{u_M}{2}]$$

where $u_m = -(e_x(1+x^3) + e_y(1+y^3) + e_{xy}(1+xy^3) + e_{x^2})$.

As $e_{y^2}\mathbb{Z}[G] \cong (1+y^2)\mathbb{Z}[G]$, (see lemma 3.5.4), we have a single relation $e_{y^2}((1-y^2) = 0$.

We remark that rationally, for some $a \in \mathbb{Z}$, $\pi_2(X) \otimes \mathbb{Q} \cong IG^* \otimes \mathbb{Q} \oplus \mathbb{Q}[G]^a$, for any finite geometric 2- complex X, with finite fundamental group, G. Consequentially, we have

$$S^2(J) \otimes \mathbb{Q} = S^2(J \otimes \mathbb{Q}) = S^2(IG^* \oplus \mathbb{Z}[G]^a) \otimes \mathbb{Q}$$

From lemma 3.6.1, the kernel of p' is generated by u and $u\Sigma/2$. Hence the kernel of $p' \otimes \mathbb{Q} : S^2(\mathbb{Q}[G]) \to S^2(IG^*) \otimes \mathbb{Q}$, is generated by u, as $u\Sigma/2$ is in the $\mathbb{Q}[G]$- linear span of u.

We therefore have a short exact sequence:

$$0 \to \mathbb{Q}[G] \to S^2(\mathbb{Q}[G]) \to S^2(IG^*) \otimes \mathbb{Q} \to 0$$

As surjections over $\mathbb{Q}[G]$ split and cancellation of finitely generated modules holds over $\mathbb{Q}[G]$, we may write

$$S^2(IG^*) \otimes \mathbb{Q} = S^2(\mathbb{Q}[G])/\mathbb{Q}[G]$$

So from lemmas 3.5.1, 3.5.2 and 3.5.4 we may conclude

Theorem 3.6.5 *There exist integers a, b, such that*

$$\pi_3(X) \otimes \mathbb{Q} = \mathbb{Q}[G]^a \oplus (V_G \otimes \mathbb{Q})^b$$

Again, if the order of G is odd, then $V_G = 0$, so

Corollary 3.6.6 *If the order of G is odd, then $\pi_3(X)$ is rationally free.*

Note that example 3.6.3 is a case in point.

§3.7 Summary

In this chapter we have shown that given a geometric 2- complex, X, with finite fundamental group G, we have $\pi_3(X) \cong S^2(J)$, where $J = \pi_2(X)$ (theorem 3A).

We have defined a module over $\mathbb{Z}[G]$, V_G and shown that $\pi_3(X)$ is determined by G, up to stabilization by copies of $\mathbb{Z}[G]$ and copies of V_G (theorem 3.5.5). Rationally, we have shown that $\pi_3(X)\otimes\mathbb{Q} \cong \mathbb{Q}[G]^a \oplus (V_G\otimes\mathbb{Q})^b$ for integers a, b (theorem 3.6.5).

In the case where G is a group of odd order, we have $V_G \cong 0$. Hence in this case, the stable class of $\pi_3(X)$ is determined and $\pi_3(X)$ is rationally free (corollaries 3.5.6 and 3.6.6).

Chapter 4

Algebraic Poincare 5- complexes

Let M be a closed, connected, orientable 5- dimensional manifold, with finite fundamental group G (we assume manifolds to be without boundary). In this chapter we consider algebraic complexes $C_*(\tilde{M'})$, where M' is a finite CW- complex, with $M \sim M'$. $C_*(\tilde{M'})$ must satisfy Poincare duality. We use this to show that up to chain homotopy equivalence, we may represent it by an algebraic 2- complex, $\mathcal{A}$, connected to its dual via a G- invariant bilinear form, β, on $(\pi_2(\mathcal{A}))^*$. We denote the resulting algebraic 5- complex $(\mathcal{A}, \beta)$.

In §4.3 we show that the algebraic 2- complex $\mathcal{A}$, is not important in the sense that any algebraic 2- complex may be stabilized to one which, together with the appropriate bilinear form, represents the homotopy type of $C_*(\tilde{M'})$.In §4.4, we describe chain homotopy equivalences between these algebraic complexes.

We next consider the homotopy equivalence induced by Poincare Duality. In particular we are interested in how similar it can be made to the identity. In §4.5 we show that it can be taken as the identity on 4 of the 6 terms of the chain complex. In §4.6 however, we find a homological obstruction to this homotopy equivalence actually being the identity. In particular, certain manifolds described in [1] do not satisfy the homological condition necessary, for being able to write the homotopy equivalence as the identity.

§4.1 The category TOP^5

Fix a finite group G. Let TOP^5 denote the category of closed connected orientable five dimensional topological manifolds with base point, with respect to which the fundamental group is identified with G. The morphisms in this category are continuous maps which preserve the base point and induce the identity on G.

Given an object of TOP^5, we may find a finite CW- complex, which is homotopy equivalent to it (see [4]). Let the following be the algebraic chain complex of the universal cover of the CW- complex:

$$C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

This is an algebraic complex of free $\mathbb{Z}[G]$- linear modules and $\mathbb{Z}[G]$- linear maps. It is exact at C_1 and the cokernel of ∂_1 is $\mathbb{Z}$.

We say that an algebraic complex satisfies Poincare Duality if it is chain homotopy equivalent to to its dual.

Proposition 4.1.1 *(Poincare Duality) The algebraic complex, (C_*, ∂_*) satisfies Poincare Duality.*

Note that Poincare duality tells us that the algebraic complex above is chain homotopy equivalent to to its dual. Therefore it is exact at C_4 and the kernel of ∂_5 is $\mathbb{Z}$. As the Euler characteristic of this complex is minus that of its dual, it must be 0.

Let ALG^5 denote the category of algebraic 5-complexes of finitely generated free $\mathbb{Z}[G]$ modules,

$$F_5 \xrightarrow{\partial_5} F_4 \xrightarrow{\partial_4} F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0$$

satisfying:

i) Poincare duality.

ii) Exactness at F_4 and F_1

iii) The cokernel of ∂_1 and the kernel of ∂_5, equaling $\mathbb{Z}$.

The morphisms in this category are homotopy equivalence classes of chain maps.

We may define a functor $C : \mathrm{TOP}^5 \to \mathrm{ALG}^5$ by choosing a homotopy equivalence, $h_M : M \to M'$ with M' a finite CW- complex, for each $M \in \mathrm{TOP}^5$. $C(M)$ is then defined to be $C_*(\tilde{M}')$.

Given a continuous map, which is a morphism in TOP^5, $f : M_1 \to M_2$, we may select a cellular map f', which is homotopic to $(h_{M_2} \circ f \circ h_{M_1}^{-1}) : M_1' \to M_2'$. Define $C(f)$ to be the equivalence class of the chain map $f'_* : C_*(\tilde{M_1'}) \to C_*(\tilde{M_2'})$.

The isomorphism class of $C(M)$ in ALG^5 is an invariant of M as by construction, different choices of M' must be homotopy equivalent to M and hence each other.

§4.2 Dual Form

Given an algebraic two complex, over $\mathbb{Z}[G]$, $J^* \to F_2 \to F_1 \to F_0 \to \mathbb{Z}$ and a G-invariant bilinear form β on J, we can associate an algebraic 5- complex:

$$\begin{array}{ccc} F_0^* \to F_1^* \to F_2^* & \longrightarrow & F_2 \to F_1 \to F_0 \\ & \searrow \quad \nearrow & \\ & J \to J^* & \end{array}$$

Let DUAL^2 denote the category whose objects consist of:

(i) An algebraic 2- complex of finitely generated free modules over $\mathbb{Z}[G]$, $J^* \dashrightarrow F_2 \to F_1 \to F_0 \dashrightarrow \mathbb{Z}$, with exactness at F_1.

(ii) A G- invariant bilinear form β, on J, such that the associated algebraic 5- complex is an element of ALG^5.

As before we define the morphisms of DUAL^2, to be homotopy equivalence classes of chain maps between the associated algebraic 5- complexes of objects in DUAL^2. If an element of ALG^5 is the associated algebraic 5- complex of an element of DUAL^2, we say it is in "dual form".

We have a functor $i : \mathrm{DUAL}^2 \to \mathrm{ALG}^5$ which sends an object to its associated algebraic 5- complex, and sends a morphism to the class of chain map which it represents. This functor is clearly full and faithful. We will show that every object in ALG^5 is isomorphic, in the category ALG^5, to an object in the image of i.

Theorem 4.2.1 *Every element of* ALG^5 *is chain homotopy equivalent to an algebraic 5-complex in dual form.*

Proof: We start with an arbitrary element of ALG^5:

$$C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \tag{1}$$

We perform three simple homotopy equivalences. Firstly, the complex (1) is chain homotopy equivalent to to

$$C_5 \oplus C_0^* \xrightarrow{\delta_5} C_4 \oplus C_0^* \xrightarrow{\partial_4 \oplus 0} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \oplus C_5^* \xrightarrow{\delta_1} C_0 \oplus C_5^* \tag{2}$$

where

$$\delta_1 = \begin{pmatrix} \partial_1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \delta_5 = \begin{pmatrix} \partial_5 & 0 \\ 0 & 1 \end{pmatrix}$$

Let $R_0 = C_0$, $R_5 = C_5$, and $R_1 = C_1 \oplus C_5^*$, $R_4 = C_4 \oplus C_0^*$. Then (2) can be written

$$R_5 \oplus R_0^* \xrightarrow{\delta_5} R_4 \xrightarrow{\partial_4 \oplus 0} C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} R_1 \xrightarrow{\delta_1} R_0 \oplus R_5^* \tag{3}$$

Again we perform a pair of simple homotopy equivalences. The complex (3) is chain homotopy equivalent to to

$$R_5 \oplus R_0^* \xrightarrow{\delta_5} R_4 \oplus R_1^* \xrightarrow{\delta_4} C_3 \oplus R_1^* \xrightarrow{\partial_3 \oplus 0} C_2 \oplus R_4^* \xrightarrow{\delta_2} R_1 \oplus R_4^* \xrightarrow{\delta_1 \oplus 0} R_0 \oplus R_5^* \tag{4}$$

where

$$\delta_2 = \begin{pmatrix} \partial_2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \delta_4 = \begin{pmatrix} \partial_4 \oplus 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Let $R_2 = C_2 \oplus R_4^*$, $R_3 = C_3 \oplus R_1^*$. Then (4) can be written

$$R_5 \oplus R_0^* \xrightarrow{\delta_5} R_4 \oplus R_1^* \xrightarrow{\delta_4} R_3 \xrightarrow{\partial_3 \oplus 0} R_2 \xrightarrow{\delta_2} R_1 \oplus R_4^* \xrightarrow{\delta_1 \oplus 0} R_0 \oplus R_5^* \qquad (5)$$

As all the modules in this complex are free and the Euler characteristic is 0, we can assume the existence of some isomorphism $\theta : R_2^* \to R_3^*$

We perform a final homotopy equivalence to get

$$R_5 \oplus R_0^* \xrightarrow{\delta_5} R_4 \oplus R_1^* \xrightarrow{\delta_4} R_3 \oplus R_2^* \xrightarrow{\delta_3} R_2 \oplus R_3^* \xrightarrow{\delta_2 \oplus 0} R_1 \oplus R_4^* \xrightarrow{\delta_1 \oplus 0} R_0 \oplus R_5^* \quad (6)$$

where

$$\delta_3 = \begin{pmatrix} \partial_3 \oplus 0 & 0 \\ 0 & \theta \end{pmatrix}$$

The algebraic complex (1) is therefore chain homotopy equivalent to to (6). We will show that (6) is chain isomorphic to an algebraic 5-complex in dual form.

Lemma 4.2.2 *There exist maps h_0, k_0, such that the following diagrams commute:*

$$\begin{array}{ccc} R_0 \oplus R_5^* & \xrightarrow{\epsilon \oplus 0} & M \\ \downarrow h_0 & & \downarrow 1 \\ R_5^* \oplus R_0 & \xrightarrow{\epsilon' \oplus 0} & M \end{array} \qquad \begin{array}{ccc} R_0 \oplus R_5^* & \xrightarrow{\epsilon \oplus 0} & M \\ \uparrow k_0 & & \uparrow 1 \\ R_5^* \oplus R_0 & \xrightarrow{\epsilon' \oplus 0} & M \end{array}$$

Proof: As the R_i are projective, we may pick f_0, g_0 so that the following diagrams commute:

$$\begin{array}{ccc} R_0 & \xrightarrow{\epsilon} & \mathbb{Z} \\ \downarrow f_0 & & \downarrow 1 \\ R_5^* & \xrightarrow{\epsilon'} & \mathbb{Z} \end{array} \qquad \begin{array}{ccc} R_0 & \xrightarrow{\epsilon} & \mathbb{Z} \\ \uparrow g_0 & & \uparrow 1 \\ R_5^* & \xrightarrow{\epsilon'} & \mathbb{Z} \end{array} \tag{9}$$

Define $h_0 : R_0 \oplus R_5^* \to R_5^* \oplus R_0$ and $k_0 : R_5^* \oplus R_0 \to R_0 \oplus R_5^*$ by

$$h_0 = \begin{pmatrix} f_0 & 1 - f_0 g_0 \\ 1 & -g_0 \end{pmatrix} \qquad k_0 = \begin{pmatrix} g_0 & 1 - g_0 f_0 \\ 1 & -f_0 \end{pmatrix}$$

Direct calculation shows that $h_0 k_0 = 1$ and $k_0 h_0 = 1$.

Also from commutativity of (9), we deduce

$$(\epsilon' \quad 0) \begin{pmatrix} f_0 & 1 - f_0 g_0 \\ 1 & -g_0 \end{pmatrix} = (\epsilon' f_0 \quad \epsilon'(1 - f_0 g_0)) = (\epsilon \quad 0)$$

and

$$(\epsilon \quad 0) \begin{pmatrix} g_0 & 1 - g_0 f_0 \\ 1 & -f_0 \end{pmatrix} = (\epsilon g_0 \quad \epsilon(1 - g_0 f_0)) = (\epsilon' \quad 0)$$

Hence the following diagrams commute:

$$\begin{array}{ccc} R_0 \oplus R_5^* & \xrightarrow{\epsilon \oplus 0} & M \\ \downarrow h_0 & & \downarrow 1 \\ R_5^* \oplus R_0 & \xrightarrow{\epsilon' \oplus 0} & M \end{array} \qquad \begin{array}{ccc} R_0 \oplus R_5^* & \xrightarrow{\epsilon \oplus 0} & M \\ \uparrow k_0 & & \uparrow 1 \\ R_5^* \oplus R_0 & \xrightarrow{\epsilon' \oplus 0} & M \end{array}$$

□

Lemma 4.2.3 *There exist a pair of inverse chain isomorphisms between the exact sequences:*

$$R_2 \oplus R_3^* \xrightarrow{\delta_2 \oplus 0} R_1 \oplus R_4^* \xrightarrow{\delta_1 \oplus 0} R_0 \oplus R_5^* \overset{\epsilon \oplus 0}{\dashrightarrow} \mathbb{Z} \to 0 \tag{7}$$

and

$$R_3^* \oplus R_2 \xrightarrow{\delta_4^* \oplus 0} R_4^* \oplus R_1 \xrightarrow{\delta_5^* \oplus 0} R_5^* \oplus R_0 \overset{\epsilon' \oplus 0}{\dashrightarrow} \mathbb{Z} \to 0 \tag{8}$$

Proof: We will construct a pair of inverse chain isomorphisms, h, k, between (7) and (8).

We have already defined h_0 and k_0. Now suppose that for $i = 0$ or 1, we have defined $h_j : R_j \oplus R_{5-j}^* \to R_{5-j}^* \oplus R_j$ and $k_j : R_{5-j}^* \oplus R_j \to R_j \oplus R_{5-j}^*$ for $j = 0, \ldots, i-1$, so that for each j, we have $h_j k_j = 1$ and $k_j h_j = 1$. We proceed by induction.

As before, pick f_i, g_i so that the following diagrams commute:

$$\begin{array}{ccc} R_i & \xrightarrow{\delta_i} & R_{i-1} \oplus R_{n-i+1}^* \\ \downarrow f_i & & \downarrow h_{i-1} \\ R_{n-i}^* & \xrightarrow{\delta_{n-i+1}^*} & R_{n-i+1}^* \oplus R_{i-1} \end{array} \qquad \begin{array}{ccc} R_i & \xrightarrow{\delta_i} & R_{i-1} \oplus R_{n-i+1}^* \\ \uparrow g_i & & \uparrow k_{i-1} \\ R_{n-i}^* & \xrightarrow{\delta_{n-i+1}^*} & R_{n-i+1}^* \oplus R_{i-1} \end{array} \tag{10}$$

Define $h_i : R_i \oplus R_{n-i}^* \to R_{n-i}^* \oplus R_i$ and $k_i : R_{n-i}^* \oplus R_i \to R_i \oplus R_{n-i}^*$ by

$$h_i = \begin{pmatrix} f_i & 1 - f_i g_i \\ 1 & -g_i \end{pmatrix} \qquad k_i = \begin{pmatrix} g_i & 1 - g_i f_i \\ 1 & -f_i \end{pmatrix}$$

Direct calculation shows that $h_i k_i = 1$ and $k_i h_i = 1$.

Recall $h_{i-1}k_{i-1} = 1$ and $k_{i-1}h_{i-1} = 1$. From commutativity of (10) we deduce

$$(\delta^*_{n-i+1} \quad 0)\begin{pmatrix} f_i & 1-f_ig_i \\ 1 & -g_i \end{pmatrix} = (\delta^*_{n-i+1}f_i \quad \delta^*_{n-i+1}(1-f_ig_i)) = h_{i-1}(\delta_i \quad 0)$$

and

$$(\delta_i \quad 0)\begin{pmatrix} g_i & 1-g_if_i \\ 1 & -f_i \end{pmatrix} = (\delta_ig_i \quad \delta_i(1-g_if_i)) = k_{i-1}(\delta^*_{n-i+1} \quad 0)$$

Hence the following diagrams commute:

$$\begin{array}{ccc} R_i \oplus R^*_{n-i} & \xrightarrow{\delta_i\oplus 0} & R_{i-1} \oplus R^*_{n-i+1} \\ \downarrow h_i & & \downarrow h_{i-1} \\ R^*_{n-i} \oplus R_i & \xrightarrow{\delta^*_{n-i+1}\oplus 0} & R^*_{n-i+1} \oplus R_{i-1} \end{array} \qquad \begin{array}{ccc} R_i \oplus R^*_{n-i} & \xrightarrow{\delta_i\oplus 0} & R_{i-1} \oplus R^*_{n-i+1} \\ \uparrow k_i & & \uparrow k_{i-1} \\ R^*_{n-i} \oplus R_i & \xrightarrow{\delta^*_{n-i+1}\oplus 0} & R^*_{n-i+1} \oplus R_{i-1} \end{array}$$

Together with the identity on $\mathbb{Z}$, the h_i, k_i are therefore a pair of mutually inverse chain isomorphisms, between (7) and (8).

$$\begin{array}{ccccccccc} R_2 \oplus R^*_3 & \xrightarrow{\delta_2\oplus 0} & R_1 \oplus R^*_4 & \xrightarrow{\delta_1\oplus 0} & R_0 \oplus R^*_5 & \overset{\epsilon\oplus 0}{\dashrightarrow} & \mathbb{Z} & \to & 0 \\ \downarrow h_2 & & \downarrow h_1 & & \downarrow h_0 & & \downarrow 1 & & \\ R^*_3 \oplus R_2 & \xrightarrow{\delta^*_4\oplus 0} & R^*_4 \oplus R_1 & \xrightarrow{\delta^*_5\oplus 0} & R^*_5 \oplus R_0 & \overset{\epsilon'\oplus 0}{\dashrightarrow} & \mathbb{Z} & \to & 0 \end{array}$$

□

Let $S_0 = R_0 \oplus R^*_5$, $S_1 = R_1 \oplus R^*_4$, $S_2 = R_2 \oplus R^*_3$.

Also let $d_1 = \delta_1 \oplus 0$ and $d_2 = \delta_2 \oplus 0$. Let $d_3 = \delta_3 k^*_2$.

We complete the proof of the theorem with the following lemma:

Lemma 4.2.4 *The complex (6) is chain isomorphic to*

$$S_0^* \xrightarrow{d_1^*} S_1^* \xrightarrow{d_2^*} S_2^* \xrightarrow{d_3} S_2 \xrightarrow{d_2} S_1 \xrightarrow{d_1} S_0 \tag{11}$$

Proof: We have the following chain isomorphism:

$$\begin{array}{ccccccccccc} R_5 \oplus R_0^* & \xrightarrow{\delta_5} & R_4 \oplus R_1^* & \xrightarrow{\delta_4} & R_3 \oplus R_2^* & \xrightarrow{\delta_3} & R_2 \oplus R_3^* & \xrightarrow{\delta_2 \oplus 0} & R_1 \oplus R_4^* & \xrightarrow{\delta_1 \oplus 0} & R_0 \oplus R_5^* \\ \downarrow h_0^* & & \downarrow h_1^* & & \downarrow h_2^* & & \downarrow 1 & & \downarrow 1 & & \downarrow 1 \\ S_0^* & \xrightarrow{d_1^*} & S_1^* & \xrightarrow{d_2^*} & S_2^* & \xrightarrow{d_3} & S_2 & \xrightarrow{d_2} & S_1 & \xrightarrow{d_1} & S_0 \end{array}$$

We need only verify that the central square commutes: $d_3 h_2^* = \delta_3 k_2^* h_2^* = \delta_3$.

□

This completes the proof of theorem 4.2.1 □

We may conclude:

Theorem 4.2.5 *The functor i : $\mathrm{DUAL}^2 \to \mathrm{ALG}^5$ is full, faithful and surjective up to isomorphism. Hence i is a natural equivalence.*

This means that when using the functor C to provide an invariant of an element of TOP^5, up to isomorphism in ALG^5, we may work in the category DUAL^2. In order to parametrize the values this invariant can take, we need only classify the forms β on elements of $\Omega_{-3}^{\mathbb{Z}[G]}(\mathbb{Z})$, which give rise to elements of ALG^5.

§4.3 Polarization

We would prefer to work over a fixed algebraic 2- complex, and and have the form completely determine the resulting 5- complex. To that end fix any algebraic 2- complex, of finitely generated free modules:

$$\mathcal{A} \quad = \quad J^* \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to \mathbb{Z}$$

with exactness at F_1. Let

$$\mathcal{A}^n \quad = \quad F_2 \oplus \mathbb{Z}[G]^n \stackrel{d_2 \oplus 0}{\to} F_1 \stackrel{d_1}{\to} F_0$$

Let J_n denote $J \oplus \mathbb{Z}[G]^n$. The following result opens the possibility of classifying algebraic 5- complexes, without in any way having to classify algebraic 2-complexes.

Theorem 4.3.1 *Any element of* ALG^5 *is chain homotopic to the associated algebraic 5-complex of the element of* DUAL^2*, represented by* $(\mathcal{A}^n, \ \gamma)$*, for some* n *and bilinear form,* γ *on* J_n.

Proof: Any element of ALG^5 is chain homotopy equivalent to to an algebraic complex of the form

$$T_0^* \xrightarrow{\Delta_1^*} T_1^* \xrightarrow{\Delta_2^*} T_2^* \xrightarrow{\Delta_3} T_2 \xrightarrow{\Delta_2} T_1 \xrightarrow{\Delta_1} T_0 \tag{12}$$

for finitely generated free modules S_i.

Lemma 4.3.2 *For some integer,* n*, and free module* T*, we may apply a pair of simple homotopy equivalences to* $\mathcal{A}^n$ *to get*

$$L_2 \xrightarrow{D_2} L_1 \xrightarrow{D_1} L_0$$

and a pair of simple homotopy equivalences to

$$T_2 \oplus T \xrightarrow{\Delta_2 \oplus 0} T_1 \xrightarrow{\Delta_1} T_0$$

to get

$$S_2 \xrightarrow{\partial_2} S_1 \xrightarrow{\partial_1} S_0$$

so that we have a chain isomorphism:

$$\begin{array}{ccccc} S_2 & \xrightarrow{\partial_2} & S_1 & \xrightarrow{\partial_1} & S_0 \\ \downarrow \theta_2 & & \downarrow \theta_1 & & \downarrow \theta_0 \\ L_2 & \xrightarrow{D_2} & L_1 & \xrightarrow{D_1} & L_0 \end{array}$$

Proof: See theorem 1.1.1.

□

Note that (12) is chain homotopy equivalent to to

$$T_0^* \xrightarrow{\Delta_1^*} T_1^* \xrightarrow{\Delta_2^*} T_2^* \oplus T \xrightarrow{\delta_3} T_2 \oplus T \xrightarrow{\Delta_2 \oplus 0} T_1 \xrightarrow{\Delta_1} T_0$$

where $\delta_3 = \begin{pmatrix} \Delta_3 & 0 \\ 0 & 1 \end{pmatrix}$. This in turn is chain homotopy equivalent to to

$$S_0^* \xrightarrow{\partial_1^*} S_1^* \xrightarrow{\partial_2^*} S_2^* \xrightarrow{\partial_3} S_2 \xrightarrow{\partial_2} S_1 \xrightarrow{\partial_1} S_0$$

where ∂_3 is induced from Δ_3.

Let $D_3 = \theta_2 \circ \partial_3 \circ \theta_2^*$. Let γ be the bilinear form induced on J_n by D_3.

We have a chain isomorphism

$$\begin{array}{ccccccccccc} S_0^* & \xrightarrow{\partial_1^*} & S_1^* & \xrightarrow{\partial_2^*} & S_2^* & \xrightarrow{\partial_3} & S_2 & \xrightarrow{\partial_2} & S_1 & \xrightarrow{\partial_1} & S_0 \\ \downarrow \theta_0^{*-1} & & \downarrow \theta_1^{*-1} & & \downarrow \theta_2^{*-1} & & \downarrow \theta_2 & & \downarrow \theta_1 & & \downarrow \theta_0 \\ L_0^* & \xrightarrow{D_1^*} & L_1^* & \xrightarrow{D_2^*} & L_2^* & \xrightarrow{D_3} & L_2 & \xrightarrow{D_2} & L_1 & \xrightarrow{D_1} & L_0 \end{array}$$

Finally note that the algebraic complex

$$L_0^* \xrightarrow{D_1^*} L_1^* \xrightarrow{D_2^*} L_2^* \xrightarrow{D_3} L_2 \xrightarrow{D_2} L_1 \xrightarrow{D_1} L_0$$

is obtained from the algebraic 5- complex associated to $(\mathcal{A}^n, \gamma)$, by performing four simple homotopy equivalences. Hence (12) is chain homotopy equivalent to to the algebraic 5- complex associated to $(\mathcal{A}^n, \gamma)$.

□

It may be convenient to regard all elements of ALG^5 as being parametrized, up to homotopy, by forms on the same algebraic 2- complex. To this end, note that for $n \leq m$ the associated algebraic complexes of $(\mathcal{A}^n, \beta)$ and $(\mathcal{A}^m, \gamma)$ are chain homotopy equivalent, where β is a bilinear form on J_n and γ is the bilinear form on J_m defined by

$$\gamma = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$$

Hence by identifying β with γ, we may regard β as a form on J_m, with no ambiguity as to which element (up to isomorphism in the category) of ALG^5 is associated to it.

§4.4 Homotopy equivalence in $DUAL^2$

We proceed to give necessary and sufficient conditions for elements of $DUAL^2$ to be chain homotopy equivalent. Let

$$\mathcal{B} = J^* \to F_2 \overset{d_2}{\to} F_1 \overset{d_1}{\to} F_0$$

$$\mathcal{C} = K^* \to E_2 \overset{\delta_2}{\to} E_1 \overset{\delta_1}{\to} E_0$$

and let β, γ be G- invariant bilinear forms on $\mathcal{B}, \mathcal{C}$ respectively.

Proposition 4.4.1 *$(\mathcal{B}, \beta)$ and $(\mathcal{C}, \gamma)$ are chain homotopy equivalent if and only if there exist maps $\phi_1, \psi_2 : J^* \to K^*$, $\phi_2, \psi_1 : K^* \to J^*$, which all augment to ± 1, and maps*

$I : J^* \to J$ and $L : K^* \to K$ which factor through projective modules, such that the following diagram commutes:

$$\begin{array}{ccc} J & \xrightarrow{\beta} & J^* \\ {\scriptstyle \phi_2^*}\downarrow & & \downarrow{\scriptstyle \phi_1} \\ K & \xrightarrow{\gamma} & K^* \\ {\scriptstyle \psi_2^*}\downarrow & & \downarrow{\scriptstyle \psi_1} \\ J & \xrightarrow{\beta} & J^* \end{array}$$

and

$1 - \psi_1\phi_1 = \beta I,$

$1 - \psi_2^*\phi_2^* = I\beta,$

$1 - \phi_1\psi_1 = \gamma L,$

$1 - \phi_2^*\psi_2^* = L\gamma.$

and ψ_1 and ϕ_1 augment to the same value and ψ_2 and ϕ_2 augment to the same value.

Proof: Firstly suppose we have a pair of inverse (up to homotopy) homotopy equivalences $f_i, g_i, i = 0, 1, 2, 3, 4, 5,$ between $(\mathcal{B}, \beta)$ and $(\mathcal{C}, \gamma)$. Then we may induce ϕ_1, ϕ_2 from f_2, f_3 and induce ψ_1, ψ_2 from g_1, g_2. These induced maps all augment to ± 1 and make the diagram commute. Also and ψ_1 and ϕ_1 will augment to the same value and ψ_2 and ϕ_2 will augment to the same value.

We also have maps I_i, L_i such that $1 - g_i f_i = I_i\partial + \partial' I_{i-1}$, and $1 - f_i g_i = L_i\partial + \partial' L_{i-1}$ where ∂, ∂' are the relevant boundary maps in each case and I_i and L_i are taken to be 0, for $i = -1, 5$. Let ι, κ denote the inclusions of J^* in F_2, K^* in E_2 respectively. Then we may construct I, L, by setting $I = \iota^* \circ I_2 \circ \iota$ and $L = \kappa^* \circ L_2 \circ \kappa$.

Clearly I and K factor through projective modules. We know

$$\iota\beta I = (\iota\beta\iota^*)I_2\iota = (1 - g_2f_2 - I_1d_2)\iota = \iota - g_2f_2\iota = \iota(1 - \psi_1\phi_1)$$

As ι is injective, we have $1 - \psi_1\phi_1 = \beta I$.

Similarly, we have

$$\kappa\gamma L = (\kappa\gamma\kappa^*)L_2\kappa = (1 - f_2g_2 - L_1\delta_2)\kappa = \kappa - f_2g_2\kappa = \kappa(1 - \phi_1\psi_1)$$

and as κ is injective we have $1 - \phi_1\psi_1 = \gamma L$.

Also

$$I\beta\iota^* = \iota^* I_2(\iota\beta\iota^*) = \iota^*(1 - g_3f_3 - d_2^*I_3) = \iota_* - \iota^* g_3f_3 = (1 - \psi_2^*\phi_2^*)\iota^*$$

As ι^* is surjective, we have $1 - \psi_2^*\phi_2^* = I\beta$.

Again,

$$L\gamma\kappa^* = \kappa^* L_2(\kappa\gamma\kappa^*) = \kappa^*(1 - f_3g_3 - \delta_2^*L_3) = \kappa_* - \kappa^* f_3g_3 = (1 - \phi_2^*\psi_2^*)\kappa^*$$

and as κ^* is surjective, we have $1 - \phi_2^*\psi_2^* = L\gamma$.

Hence all the required conditions are satisfied.

Conversely, suppose that we have $\phi_1, \phi_2, \psi_1, \psi_2, I, L$ satisfying the required conditions. We must construct a homotopy equivalence.

We may write $\mathcal{B}$ in the form

$$J^* \to F_2 \xrightarrow{d_2} S \oplus F \xrightarrow{d_1'\oplus 1_F} \mathbb{Z}[G] \oplus F$$

for some free module F, and stably free module S.

Simple homotopy equivalences connect $(\mathcal{B}, \beta)$ and $(\mathcal{B}', \beta)$ where

$$\mathcal{B}' \quad = \quad J^* \to F_2 \oplus F \xrightarrow{d_2\oplus 1_F} F_1 \xrightarrow{d_1'} \mathbb{Z}[G]$$

Similarly, we may write $\mathcal{C}$ in the form

$$K^* \to E_2 \xrightarrow{\delta_2} T \oplus E \xrightarrow{\delta_1'\oplus 1_E} \mathbb{Z}[G] \oplus E$$

for some free module E, and stably free module T.

Simple homotopy equivalences connect $(\mathcal{C}, \gamma)$ and $(\mathcal{C}', \gamma)$ where

$$\mathcal{C}' \quad = \quad K^* \to E_2 \oplus E \overset{\delta_2 \oplus 1_E}{\longrightarrow} E_1 \overset{\delta_1'}{\to} \mathbb{Z}[G]$$

From now on we will write $\mathcal{B}'$ as

$$J^* \to F_2' \overset{d_2'}{\to} F_1' \overset{d_1'}{\to} \mathbb{Z}[G]$$

and $\mathcal{C}'$ as

$$K^* \to E_2' \overset{\delta_2'}{\to} E_1' \overset{\delta_1'}{\to} \mathbb{Z}[G]$$

We may extend $\psi_1, \psi_2^*, \phi_1, \phi_2^*$ so that the following diagram commutes:

$$\begin{array}{ccccccccccccccccccccc}
\mathbb{Z} & \longrightarrow & \mathbb{Z}[G] & \xrightarrow{d_1'^*} & F_1'^* & \xrightarrow{d_2'^*} & F_2'^* & \xrightarrow{\iota^*} & J & \xrightarrow{\beta} & J^* & \xrightarrow{\iota} & F_2' & \xrightarrow{d_2'} & F_1' & \xrightarrow{d_1'} & \mathbb{Z}[G] & \longrightarrow & \mathbb{Z} \\
\downarrow{\scriptstyle \pm\pm 1} & & \downarrow{\scriptstyle f_5} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle \phi_2^*} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \pm 1} \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}[G] & \xrightarrow{\delta_1'^*} & E_1'^* & \xrightarrow{\delta_2'^*} & E_2'^* & \xrightarrow{\kappa^*} & K & \xrightarrow{\gamma} & K^* & \xrightarrow{\kappa} & E_2' & \xrightarrow{\delta_2'} & E_1' & \xrightarrow{\delta_1'} & \mathbb{Z}[G] & \longrightarrow & \mathbb{Z} \\
\downarrow{\scriptstyle \pm\pm 1} & & \downarrow{\scriptstyle g_5} & & \downarrow{\scriptstyle g_4} & & \downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle \psi_2^*} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle \pm 1} \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}[G] & \xrightarrow{d_1'^*} & F_1'^* & \xrightarrow{d_2'^*} & F_2'^* & \xrightarrow{\iota^*} & J & \xrightarrow{\beta} & J^* & \xrightarrow{\iota} & F_2' & \xrightarrow{d_2'} & F_1' & \xrightarrow{d_1'} & \mathbb{Z}[G] & \longrightarrow & \mathbb{Z}
\end{array}$$

For some projective module P, there exist maps $a : J^* \to P$ and $b : P \to J$ such that $I = ba$. As ι is torsion free and P is projective, we have maps $a' : F_2' \to P$ and $b' : P \to F_2'^*$ such that $a = a'\iota$ and $b = \iota^* b'$. Let $I_2 = b'a'$. Then we have
$\iota^* I_2 \iota = \iota^* b' a' \iota = ba = I$

Let $d_3' = \iota\beta\iota^*$ and $\delta_3' = \kappa\gamma\kappa^*$. Then

$$(1 - g_2 f_2 - d_3' I_2) d_3' = (\iota - g_2 f_2 \iota - \iota\beta\iota^* I_2 \iota))\beta\iota^* = \iota(1 - \psi_1\phi_1 - \beta I)\beta\iota^* = 0$$

As F_2' is relatively injective, we may conclude that $(1 - g_2 f_2 - d_3' I_2)$ factors through d_2'. Hence we have some map $I_1 : F_1' \to F_2'$, such that $1 - g_2 f_2 = d_3' I_2 + I_1 d_2'$.

We have

$$(1 - g_1 f_1 - d_2' I_1) d_2' \quad = \quad d_2' - g_1 f_1 d_2' - d_2'(1 - g_2 f_2 - d_3' I_2)$$
$$= d_2' - g_1 f_1 d_2' - d_2' + g_1 f_1 d_2' = 0$$

Again F_2' is relatively injective so $(1 - g_1 f_1 - d_2' I_1)$ factors through d_1'. Hence we have some map$I_0 : \mathbb{Z}[G] \to F_1'$ such that $1 - g_1 f_1 = d_2' I_1 + I_0 d_1'$.

Let $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ denote the augmentation map. From commutativity of the above diagram, we know that $\epsilon(1 - f_0 g_0) = 0$. Hence we know that the image of $(1 - f_0 g_0)$ lies in $IG \subset \mathbb{Z}[G]$. Clearly the image of $d_1' I_0$ also lies in IG. Hence the image of $(1 - f_0 g_0 - d_1' I_0)$ lies in IG.

As before we have,

$$(1 - g_0 f_0 - d_1' I_0) d_1' \quad = \quad d_1' - g_0 f_0 d_1' - d_1'(1 - g_1 f_1 - d_2' I_1)$$
$$= d_1' - g_0 f_0 d_1' - d_1' + g_0 f_0 d_1' = 0$$

Hence $(1 - g_0 f_0 - d_1' I_0)$ factors through ϵ. Let $W : \mathbb{Z} \to \mathbb{Z}[G]$ satisfy

$$(1 - g_0 f_0 - d_1' I_0) = W\epsilon.$$

As ϵ is surjective, the image of W must lie in IG. But any map $\mathbb{Z} \to IG$ is necessarily equal to 0. Hence $W = 0$ and $1 - g_0 f_0 = d_1' I_0$.

We proceed to construct I_3, I_4 in a dual fashion. We have already shown that $(1 - g_2 f_2 - d_3' I_2) d_3' = 0$. By commutativity of the diagram, we may conclude that $d_3'(1 - g_3 f_3 - I_2 d_3') = 0$. By projectivity of $F_2'^*$, we have some map I_3 satisfying $1 - g_3 f_3 = I_2 d_3' + d_2'^* I_3$.

Again,

$$d_2'^*(1 - g_4 f_4 - I_3 d_2'^*) = d_2'^* - g_3 f_3 d_2'^* - (1 - g_3 f_3 - I_2 d_3') d_2'^* = 0$$

and by projectivity of $F_1'^*$, we have a map $I_4 : F_1'^* \to \mathbb{Z}[G]$ such that $1 - g_4 f_4 = I_3 d_2'^* + d_1'^* I_4$.

Repeating the method once more, we have

$d_1'^*(1 - g_5 f_5 - I_4 d_1'^*) = d_1'^* - g_4 f_4 d_1'^* - (1 - g_4 f_4 - I_3 d_2'^*) d_1'^* = 0$

so $(1 - g_5 f_5 - I_4 d_1'^*)$ factors through ϵ^*, and we have some map $W' : \mathbb{Z}[G] \to \mathbb{Z}$ satisfying $1 - g_5 f_5 = I_4 d_1'^* + \epsilon^* W'$.

We know that $\epsilon^* W' \epsilon^* = (1 - g_5 f_5 - I_4 d_1'^*)\epsilon^* = 0$. As ϵ^* is injective, we may conclude that $W'\epsilon^* = 0$. So W' factors through $\mathbb{Z}[G]/\epsilon(\mathbb{Z}) \cong IG^*$. However, any map $IG^* \to \mathbb{Z}$ is necessarily equal to 0, so $W' = 0$ and $1 - g_5 f_5 = I_4 d_1'^*$.

Collating, we have:

$1 - g_5 f_5 = I_4 d_1'^*$

$1 - g_4 f_4 = I_3 d_2'^* + d_1'^* I_4$

$1 - g_3 f_3 = I_2 d_3' + d_2'^* I_3$

$1 - g_2 f_2 = d_3' I_2 + I_1 d_2'$

$1 - g_1 f_1 = d_2' I_1 + I_0 d_1'$

$1 - g_0 f_0 = d_1' I_0$

Hence the I_i, for $i = 0, 1, 2, 3, 4$, form a chain homotopy from the identity to $g_i f_i$.

Similarly we may construct maps L_0, L_1, L_2, L_3, L_4, which form a chain homotopy between the identity and the $f_i g_i$, $i = 0, 1, 2, 3, 4, 5$.

Hence $(\mathcal{B}, \beta)$ is chain homotopy equivalent to $(\mathcal{B}', \beta)$, which is chain homotopy equivalent to $(\mathcal{C}', \gamma)$, which is chain homotopy equivalent to $(\mathcal{C}, \gamma)$, as required.

□

Note that this proposition does not imply that the underlying algebraic 2- complex is not important in defining the isomorphism class of an object in $DUAL\pm^2$, as whether or not a map $J^* \to K^*$ augments to ± 1, is dependent on the precise inclusions of J^*, K^*, in F_2, E_2 respectively.

Lemma 4.4.2 *Let $(\mathcal{B}, \beta)$, be as before. Suppose we have some homotopy equivalence f_*, from $(\mathcal{B}, \beta)$ to its dual. Then $\pm f_*$ is chain homotopic to a chain map of the form:*

$$\begin{array}{cccccc}
F_0^* \xrightarrow{d_1^*} & F_1^* \xrightarrow{d_2^*} & F_2^* \xrightarrow{d_3} & F_2 \xrightarrow{d_2} & F_1 \xrightarrow{d_1} & F_0 \\
\downarrow \pm 1 & \downarrow \pm 1 & \downarrow \theta_2 & \downarrow \theta_1 & \downarrow 1 & \downarrow 1 \\
F_0^* \xrightarrow{d_1^*} & F_1^* \xrightarrow{d_2^*} & F_2^* \xrightarrow{d_3^*} & F_2 \xrightarrow{d_2} & F_1 \xrightarrow{d_1} & F_0
\end{array}$$

for some maps $\theta_1 : F_2 \to F_2$, $\theta_2 : F_2^* \to F_2^*$.

Proof: Consider the chain homotopy equivalence:

$$\begin{array}{cccccc}
F_0^* \xrightarrow{d_1^*} & F_1^* \xrightarrow{d_2^*} & F_2^* \xrightarrow{d_3} & F_2 \xrightarrow{d_2} & F_1 \xrightarrow{d_1} & F_0 \\
\downarrow \alpha_5 & \downarrow \alpha_4 & \downarrow \alpha_3 & \downarrow \alpha_2 & \downarrow \alpha_1 & \downarrow \alpha_0 \\
F_0^* \xrightarrow{d_1^*} & F_1^* \xrightarrow{d_2^*} & F_2^* \xrightarrow{d_3^*} & F_2 \xrightarrow{d_2} & F_1 \xrightarrow{d_1} & F_0
\end{array}$$

where the α_i are equal to the f_i or the $-f_i$, depending on which is necessary to force the induced action on the cokernel of d_1 to be the identity.

The kernel of d_1^* and the cokernel of d_1 are both $\mathbb{Z}$. Hence α_0 and α_5 must induce multiplication by ± 1 on $\mathbb{Z}$. Our choice of sign forces α_0 to induce multiplication by 1.

$$\begin{array}{ccccc}
J^* \xrightarrow{l} & F_2 \xrightarrow{d_2} & F_1 \xrightarrow{d_1} & F_0 \to & \mathbb{Z} \\
\downarrow \phi_1 & \downarrow \alpha_2 & \downarrow \alpha_1 & \downarrow \alpha_0 & \downarrow 1 \\
J^* \xrightarrow{l} & F_2 \xrightarrow{d_2} & F_1 \xrightarrow{d_1} & F_0 \to & \mathbb{Z}
\end{array}$$

$$J^* \xrightarrow{l} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to \mathbb{Z}$$

$$\downarrow \phi_2 \quad \downarrow \alpha_3^* \quad \downarrow \alpha_4^* \quad \downarrow \alpha_5^* \quad \downarrow \pm 1$$

$$J^* \xrightarrow{l} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to \mathbb{Z}$$

Hence we have maps $I_0 : F_0 \to F_1$ and $I_1 : F_1 \to F_2$, such that $d_1 I_0 = 1 - \alpha_0$, and $I_0 d_1 + d_2 I_1 = 1 - \alpha_1$.

Similarly we have maps $I_4 : F_1^* \to F_0^*$ and $I_3 : F_2^* \to F_1^*$, such that $d_1 I_4^* = \pm 1 - \alpha_5^*$, and $I_4^* d_1 + d_2 I_3^* = \pm 1 - \alpha_4^*$.

Next set $\theta_1 = \alpha_2 + I_1 d_2$ and $\theta_2 = \alpha_3 + d_2^* I_3$. Then taking $I_2 = 0$, the I_i, $i = 0, 1, 2, 3, 4$ form the required chain homotopy.

□

§4.5 Poincare Duality

Let X be a CW- complex with finite fundamental group G. Let $L_*(\tilde{X})$ denote $C_*(\tilde{X})$ with coefficients restricted to $\mathbb{Z}$. As G is finite, we may naturally identify $H_p(\tilde{X}; \mathbb{Z}[G])$ with $H_p(L_*(\tilde{X}); \mathbb{Z})$ and we may identify $H^p(\tilde{X}; \mathbb{Z}[G])$ with $H^p(L_*(\tilde{X}); \mathbb{Z})$. We make frequent use of these identifications, and hence assume natural maps $H^p(\tilde{X}; \mathbb{Z}[G]) \times H_p(\tilde{X}; \mathbb{Z}[G]) \to \mathbb{Z}$. We now make a more specific statement of Poincare Duality:

Theorem 4.5.1 *(Poincare Duality) Let M be an element of* TOP^5 *and M' be homotopy equivalent to M. Then, given a generator of $H_5(\tilde{M'}; \mathbb{Z}[G])$, denoted η, there exists a chain homotopy equivalence, over $\mathbb{Z}[G]$, $\phi : C_*(\tilde{M'})^* \to C_*(\tilde{M'})$ satisfying the following: Given $\alpha \in H^p(\tilde{M'}; \mathbb{Z}[G])$, we have $\phi_*(\alpha) = \eta \frown \alpha$. Here ϕ_* denotes the induced map $H^p(\tilde{M'}; \mathbb{Z}[G]) \to H_{5-p}(\tilde{M'}; \mathbb{Z}[G])$.*

Corollary 4.5.2 *Given $p \in \{0, 1, 2, 3, 4, 5\}$ and $\alpha \in H^p(\tilde{M'}; \mathbb{Z}[G])$, $\beta \in H^{5-p}(\tilde{M'}; \mathbb{Z}[G])$ we have $\beta(\phi_*(\alpha)) = \alpha(\phi_*(\beta))$.*

Proof:

$$\beta(\phi_*(\alpha)) = \beta(\eta \frown \alpha) = (\alpha \smile \beta)\eta = -1^{p(5-p)}(\beta \smile \alpha)\eta = -1^{p(5-p)}\alpha(\eta \frown \beta) = -1^{p(5-p)}\alpha(\phi_*(\beta))$$

As either p or $5-p$ must be even, we have $\beta(\phi_*(\alpha)) = \alpha(\phi_*(\beta))$

□

Suppose now, that we have chain homotopy equivalence $f : C_*(\tilde{M}') \to \mathcal{A}$, for some algebraic 5- complex $\mathcal{A}$. Then we have a homotopy equivalence $f \circ \phi \circ f^* : \mathcal{A}^* \to \mathcal{A}$. Let $\phi' = f \circ \phi \circ f^*$. If a chain homotopy equivalence $\mathcal{A}^* \to \mathcal{A}$ is chain homotopic to one constructed in this way, starting with some generator of $H_5(\tilde{M}'; \mathbb{Z}[G])$, we say it is a duality equivalence. Note that (up to sign) the maps from cohomology to homology, induced by a duality equivalence, are determined by M (with respect to the isomorphisms f_* and f^*).

Lemma 4.5.3 *Given* $p \in \{0, 1, 2, 3, 4, 5\}$ *and* $\alpha \in H^p(\mathcal{A}; \mathbb{Z}[G])$, $\beta \in H^{5-p}(\mathcal{A}; \mathbb{Z}[G])$ *we have* $\beta(\phi'_*(\alpha)) = \alpha(\phi'_*(\beta))$.

Proof: $\beta(\phi'_*(\alpha)) = \beta(f_*\phi_*f^*(\alpha)) = f^*(\beta)(\phi_*f^*(\alpha)) = f^*(\alpha)(\phi_*f^*(\beta))$
$= \alpha(f_*\phi_*f^*(\beta)) = \alpha(\phi'_*(\beta))$

□

By theorem 4.2.1 we may choose $\mathcal{A}$ to be of the form $(\mathcal{B}, \beta)$ for some algebraic 2-complex $\mathcal{B} = J^* \xrightarrow{\iota} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to Z$ and bilinear form $\beta : J \times J \to \mathbb{Z}$. Let $d_3 = \iota^*\beta\iota$. The algebraic complex $(\mathcal{B}, \beta)$ is written:

$$\mathbb{Z} \dashrightarrow F_0^* \xrightarrow{d_1^*} F_1^* \xrightarrow{d_2^*} F_2^* \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \dashrightarrow \mathbb{Z}$$

As F_0^* is the dual of F_0, we may apply elements of $\mathbb{Z}$ (occurring on the left of this sequence), to $\mathbb{Z}$ (occurring on the right of this sequence). We follow the convention that the choice of identification of the kernel of d_1^* with $\mathbb{Z}$, forces this application to be given by multiplication.

Note that the copy of $\mathbb{Z}$ occurring on the left of the sequence may be identified with $H_5((\mathcal{B},\beta);\mathbb{Z}[G])$. Similarly, the copy of $\mathbb{Z}$ occurring on the right of the sequence may be identified with $H_0((\mathcal{B},\beta);\mathbb{Z}[G])$.

The complex $(\mathcal{B},\beta)^*$ may be written:

$$\mathbb{Z} \dashrightarrow F_0^* \stackrel{d_1^*}{\to} F_1^* \stackrel{d_2^*}{\to} F_2^* \stackrel{d_3^*}{\to} F_2 \stackrel{d_2}{\to} F_1 \stackrel{d_1}{\to} F_0 \dashrightarrow \mathbb{Z}$$

This sequence only differs from $(\mathcal{B},\beta)$ in the middle term.

Again, note that the copy of $\mathbb{Z}$ occurring on the left of this sequence may be identified with $H^0((\mathcal{B},\beta);\mathbb{Z}[G])$. Similarly, the copy of $\mathbb{Z}$ occurring on the right of this sequence may be identified with $H^5((\mathcal{B},\beta);\mathbb{Z}[G])$.

Let $\alpha \in H^0((\mathcal{B},\beta);\mathbb{Z}[G])$ be represented by the integer a, and let $\gamma \in H_0((\mathcal{B},\beta);\mathbb{Z}[G])$ be represented by the integer c. Then our conventions imply that application of elements of $H^0((\mathcal{B},\beta);\mathbb{Z}[G])$ to $H_0((\mathcal{B},\beta);\mathbb{Z}[G])$ is given by the application of $\mathbb{Z}$ to $\mathbb{Z}$, which was forced to be multiplication. Hence $\alpha(\gamma) = ac$.

Let $\alpha \in H^5((\mathcal{B},\beta);\mathbb{Z}[G])$ be represented by the integer a, and let $\gamma \in H_5((\mathcal{B},\beta);\mathbb{Z}[G])$ be represented by the integer c. Then our conventions imply that the application of elements of $H^5((\mathcal{B},\beta);\mathbb{Z}[G])$ to $H_5((\mathcal{B},\beta);\mathbb{Z}[G])$ is given by evaluation on the application of elements of $\mathbb{Z}$ to $\mathbb{Z}$, which was forced to be multiplication. Hence $\alpha(\gamma) = ca = ac$.

Let $x, y \in \mathbb{Z}$ be chosen so that the following diagram commutes:

$$\begin{array}{ccccccccccccccc}
\mathbb{Z} & \dashrightarrow & F_0^* & \stackrel{d_1^*}{\to} & F_1^* & \stackrel{d_2^*}{\to} & F_2^* & \stackrel{d_3^*}{\to} & F_2 & \stackrel{d_2}{\to} & F_1 & \stackrel{d_1}{\to} & F_0 & \dashrightarrow & \mathbb{Z} \\
y \downarrow & & \phi_5' \downarrow & & \phi_4' \downarrow & & \phi_3' \downarrow & & \phi_2' \downarrow & & \phi_1' \downarrow & & \phi_0' \downarrow & & x \downarrow \\
\mathbb{Z} & \dashrightarrow & F_0^* & \stackrel{d_1^*}{\to} & F_1^* & \stackrel{d_2^*}{\to} & F_2^* & \stackrel{d_3}{\to} & F_2 & \stackrel{d_2}{\to} & F_1 & \stackrel{d_1}{\to} & F_0 & \dashrightarrow & \mathbb{Z}
\end{array}$$

Lemma 4.5.4 x=y

Proof: Let $\alpha \in H^5((\mathcal{B},\beta);\mathbb{Z}[G])$ be represented by the integer a, and let $\gamma \in H^0((\mathcal{B},\beta);\mathbb{Z}[G])$ be represented by the integer c. By lemma 4.5.3 we know $\alpha(\phi_*(\gamma)) = \gamma(\phi_*(\alpha))$. Hence we have $ayc = \alpha(\phi_*(\gamma)) = \gamma(\phi_*(\alpha)) = cxa$.
As α and γ were picked arbitrarily, we must have $x = y$.

□

As ϕ' is a homotopy equivalence, we must have $x = \pm 1$. If $x = -1$, we can replace η with $-\eta$. Hence without loss of generality we have a duality equivalence:

$$\begin{array}{cccccccccccccc}
\mathbb{Z} & \dashrightarrow & F_0^* & \xrightarrow{d_1^*} & F_1^* & \xrightarrow{d_2^*} & F_2^* & \xrightarrow{d_3^*} & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 & \dashrightarrow & \mathbb{Z} \\
1\downarrow & & \phi_5'\downarrow & & \phi_4'\downarrow & & \phi_3'\downarrow & & \phi_2'\downarrow & & \phi_1'\downarrow & & \phi_0'\downarrow & & 1\downarrow \\
\mathbb{Z} & \dashrightarrow & F_0^* & \xrightarrow{d_1^*} & F_1^* & \xrightarrow{d_2^*} & F_2^* & \xrightarrow{d_3} & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 & \dashrightarrow & \mathbb{Z}
\end{array}$$

Hence by lemma 4.4.2, for some maps θ_1, θ_2 we have a duality equivalence:

$$\begin{array}{ccccccccccc}
F_0^* & \xrightarrow{d_1^*} & F_1^* & \xrightarrow{d_2^*} & F_2^* & \xrightarrow{d_3^*} & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 \\
\downarrow 1 & & \downarrow 1 & & \downarrow \theta_2 & & \downarrow \theta_1 & & \downarrow 1 & & \downarrow 1 \\
F_0^* & \xrightarrow{d_1^*} & F_1^* & \xrightarrow{d_2^*} & F_2^* & \xrightarrow{d_3} & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0
\end{array}$$

A natural question to ask at this stage is, whether or not we can choose $(\mathcal{B},\beta)$ such that the identity map is a duality equivalence. If this is possible for a manifold, we say that it is self dual. In the next section, we will show that not all manifolds in TOP^5 are self dual.

Note that if the identity map is a chain map $(\mathcal{B},\beta)^* \to (\mathcal{B},\beta)$, then $d_3^* = 1d_3^* = d_3 1 = d_3$. $d_3 = \iota^*\beta\iota$.

As $d_3 = \iota^*\beta\iota$, with ι injective and ι^* surjective, we have $\beta^* = \beta$. So β must in this case be symmetric.

§4.6 Linking Number

In this section we construct the linking number. We show that if a manifold is self dual, then its linking number is symmetric. We then use the fact that the linking number is always antisymmetric, to show that a self dual manifold, M, must satisfy $\mathrm{Tor}(H_2(\tilde{M}; \mathbb{Z}[G])) = \mathbb{Z}_2{}^k$, for some integer k. Finally we refer to the existence of manifolds not satisfying this homology condition, which are therefore not self dual.

Let (F_i, d_i), $i = 0, 1, 2, 3, 4, 5$, be a free, finite, algebraic 5- complex over $\mathbb{Z}[G]$. Then an element of $\mathrm{Tor}(H^3(F_i, \mathbb{Z}[G]))$ may be represented by a map $F_3 \to \mathbb{Z}$, of the form $\frac{1}{a}wd_3$, for some integer a and $w : F_2 \to \mathbb{Z}$.

Similarly, an element of $\mathrm{Tor}(H_2(F_i, \mathbb{Z}[G]))$ may be represented by an element of F_2 of the form $\frac{1}{b}d_3x$, for some integer b and $x \in F_3$.

Hence we have a bilinear map $\mathrm{Tor}(H^3(F_i, \mathbb{Z}[G])) \times \mathrm{Tor}(H_2(F_i, \mathbb{Z}[G])) \to \mathbb{Q}/\mathbb{Z}$, given by

$$(\frac{1}{a}wd_3, \frac{1}{b}d_3x) \mapsto \frac{1}{ab}w(d_3x)$$

Let Z denote this bilinear map.

Lemma 4.6.1 *This bilinear map is well defined.*

Proof: Suppose we had made a different choice of representative to $\frac{1}{a}wd_3$. Then the different choice would differ by a map of the form vd_3, for some map $v : F_2 \to \mathbb{Z}$. We have

$$(\frac{1}{a}wd_3 + vd_3, \frac{1}{b}d_3x) \mapsto \frac{1}{ab}w(d_3x) + v(\frac{1}{b}d_3x)$$

with $v(\frac{1}{b}d_3x) \in \mathbb{Z}$.

Again, suppose we had made a different choice of representative to $\frac{1}{b}d_3x$. Then the different choice would differ by an element of the form d_3y, for some $y \in: F_3$. We have

$$(\frac{1}{a}wd_3, \frac{1}{b}d_3x + d_3y) \mapsto \frac{1}{ab}w(d_3x) + \frac{1}{a}wd_3y$$

with $\frac{1}{a}wd_3y \in \mathbb{Z}$.

Either way, the value of the bilinear map, in $\mathbb{Q}/\mathbb{Z}$ is unchanged.

□.

Suppose now we have another free, finite algebraic 5- complex, (E_i, δ_i) and a homotopy equivalence $f : (F_i, d_i) \to (E_i, \delta_i)$. Let $\frac{1}{a}w\delta_3$ represent an element of $\mathrm{Tor}(H^3(E_i, \mathbb{Z}[G]))$ and let $\frac{1}{b}d_3x$ represent an element of $\mathrm{Tor}(H_2(F_i, \mathbb{Z}[G]))$. We have:

$$f^*(\frac{1}{a}w\delta_3) = \frac{1}{a}f^*(w\delta_3) = \frac{1}{a}w\delta_3 f_3 = \frac{1}{a}wf_2d_3 = \frac{1}{a}wf_2d_3$$

and

$$f_*(\frac{1}{b}d_3x) = \frac{1}{b}f_2(d_3x) = \frac{1}{b}f_*(d_3x) = \frac{1}{b}\delta_3 f_3 x$$

Hence:

$$Z(f^*(\frac{1}{a}w\delta_3), \frac{1}{b}d_3x) = Z(\frac{1}{a}(wf_2d_3, \frac{1}{b}d_3x) = \frac{1}{ab}wf_2(d_3x)$$

$$= \frac{1}{ab}w\delta_3 f_3 x = Z(\frac{1}{a}w\delta_3, \frac{1}{b}\delta_3 f_3 x) = Z(\frac{1}{a}w\delta_3, f_*(\frac{1}{b}d_3x))$$

So the maps f_* and f^* are adjoint with respect to Z.

Suppose now that M' is homotopic to some element of TOP^5, and $(\mathcal{B}, \beta)$ is chain homotopy equivalent to $C_*(\tilde{M'})$, via a homotopy equivalence $f : C_*(\tilde{M'}) \to (\mathcal{B}, \beta)$. Let ϕ denote a duality equivalence $(\mathcal{B}, \beta)^* \to (\mathcal{B}, \beta)$ and let ψ be a homotopy inverse to it.

Let Θ denote $\mathrm{Tor}(H_2((\mathcal{B}, \beta); \mathbb{Z}[G]))$. We define the linking number on $(\mathcal{B}, \beta)$ to be the bilinear map $\nu : \Theta \times \Theta \to \mathbb{Q}/\mathbb{Z}$, given by:

$$\nu(x, y) = Z(\psi_* x, y)$$

Let h denote a homotopy inverse to f. Suppose we have a homotopy equivalence $g : C_*(\tilde{M'}) \to (\mathcal{C}, \gamma)$. Let k denote a homotopy inverse to it. Then a duality equivalence $(\mathcal{C}, \gamma)^* \to (\mathcal{C}, \gamma)$ is given by $gh\phi h^* g^*$. A homotopy inverse to that is given by $k^* f^* \psi f k$. Let $m = fk$.

Now let $\Theta' = \mathrm{Tor}(H_2((\mathcal{C},\gamma);\mathbb{Z}[G]))$ and let ν' be the linking number $\Theta' \times \Theta' \to \mathbb{Q}/\mathbb{Z}$. We have:

$$\nu'(x,y) = Z(m^*\psi_* m_* x, y) = \nu'(x,y) = Z(\psi_* m_* x, m_* y) = \nu(m_* x, m_* y)$$

by the adjointness property. Hence we have that the linking number is well defined up to isomorphism. In particular, such properties as being symmetric or antisymmetric are independent of the choice of $(\mathcal{B},\beta)$.

Theorem 4.6.2 *(see [1], Lemma D(ii)) ν is antisymmetric.*

Note that [1] is concerned with simply connected, closed, orientable 5- manifolds. However, the fact that G is finite means that $\tilde{M}$ will be closed, simply connected and orientable.

We now describe ν in terms of β. We know that $H_2((\mathcal{B},\beta);\mathbb{Z}[G])$ is the cokernel of $\beta : J \to J^*$. Hence $\Theta = \mathrm{Tor}(H_2((\mathcal{B},\beta);\mathbb{Z}[G]))$ consists of maps $J \to \mathbb{Z}$ of the form $\frac{1}{a}\beta(x,\_)$, with $a \in \mathbb{Z}$, $x \in J$.

Let $H = \{h \in J\otimes\mathbb{Q} | \quad \beta(h,x) \in \mathbb{Z} \quad \forall x \in J\}$, where we take the natural extension of β to $J \otimes \mathbb{Q}$. This extension restricts to a form on H. Any element of Θ may then be written in the form $\beta(h,\_)$, for some $h \in H$.

Let $K = \{k \in H | \quad \beta(k,x) = 0 \quad \forall x \in J\}$. Then the set of maps of the form $\beta(h,\_)$, $h \in H$, are naturally identified with H/K. Two such maps are homologous precisely if they differ by a map of the form $\beta(x,\_)$, $x \in J$. Such maps are naturally identified with J/K. Consequently, we have $\Theta = H/(J+K)$.

Let $\lambda_1 : J \to J$ be the map induced by ψ_3 and let $\lambda_2 : J \to J$ be the dual of the map induced by ψ_2. We will now compute $\nu(x/a, y/b)$, for $x,y \in J$, $a,b \in \mathbb{Z}$.

x/a represents the element of Θ corresponding to the map $\frac{1}{a}\beta(x,\_)$. Hence $\psi_*(x/a) = \frac{1}{a}\beta(x,\lambda_2\_)$. Applying this to y and dividing by b gives:

$$\nu(x/a,y/b) = \frac{1}{ab}\beta(x,\lambda_2(y))$$

From the commutativity of the square

$$\begin{array}{ccc} J & \xrightarrow{\beta} & J^* \\ {\scriptstyle\lambda_1}\downarrow & & \downarrow{\scriptstyle\lambda_2^*} \\ J & \xrightarrow{\beta^*} & J^* \end{array}$$

we know that $\beta(x, \lambda_2(y)) = \beta(y, \lambda_1(x))$, for all $x, y \in J$. Hence
$\beta(x, \lambda_2(y)) + \beta(y, \lambda_2(x)) = \beta(x, (\lambda_1 + \lambda_2)(y))$.

The fact that ν is antisymmetric may therefore be stated as follows:

Lemma 4.6.3 *If $a|\beta(x, \_)$ and $b|\beta(y, \_)$ then $ab|\beta(x, (\lambda_1 + \lambda_2)(y))$.*

We now suppose that M is self dual. We may choose $(\mathcal{B}, \beta)$ such that we may take ϕ and ψ to be identity chain maps. In this case β will be symmetric and $\lambda_1 = \lambda_2 = 1$. Hence from lemma 4.6.3 we have that if $a|\beta(x, \_)$ and $b|\beta(y, \_)$ then $ab|2\beta(x, y)$.

We will denote the kernel of β by H_3. The inclusion of H_3 in J splits over $\mathbb{Z}$. Let V denote a complementary space. By combining a basis of H_3, with a basis of V, we obtain a basis of J, with respect to which, we may represent β by a matrix B. Then B will have the form

$$B = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}$$

where Det(C) $\neq 0$.

The condition m divides $\beta(x, y)$ for all $y \in J$, is equivalent to $m|Bx$. Hence we have:

Lemma 4.6.4 *$m|Bx$ and $l|By$ imply that $ml|2x^T By$ for all integers m, l and x, $y \in J$.*

Lemma 4.6.5 *$m|Cx$ and $l|Cy$ imply that $ml|2x^T Cy$ for all integers m, l and x, $y \in V$.*

Proof: If $m|Cx$ and $l|Cy$ then $m|Bx'$ and $l|By'$, where $x' = (0, x)$ and $y' = (0, y)$. Consequently $ml|2x'^T By'$. But $2x'^T By' = 2x^T Cy$ so $ml|2x^T Cy$.

□

Note that C is invertible over $\mathbb{Q}$ and symmetric. Note also, that $\Theta = \text{coker}(C)$.

Lemma 4.6.6 *If Θ has a non-trivial element of order k, then there exists some vector $x \in V$, such that $k|Cx$ and k, x are coprime.*

Proof: Some vector must have order k modulo the columns of C. Hence multiplying that vector by k, gives Cx for some x. If l were some non-trivial divisor of x and k, then Cx/l would be in the image of C and our original vector would have order less than or equal to k/l.

□

Suppose Θ has a non-trivial element of order k. Let x' denote x factored out by the highest common factor of the components of x. As k is coprime to x, we still have $k|Cx'$. We may extend the vector x' to a basis of V. Let D denote the matrix representing the bilinear form represented by C, with respect to the new basis.

We know that D is a symmetric matrix with non-zero determinant. Also we know that if $m|Dx$ and $l|Dy$ then $ml|2x^T Dy$, for any $x, y \in J$. Finally, we know that the first row and the first column of D are divisible by k.

Lemma 4.6.7 $k = 2$

Proof: Let e_1 denote $\begin{pmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ . \\ 0 \end{pmatrix}$

As D has non-zero determinant, we may choose some vector v such that $Dv = me_1$ for some positive integer m.

We have $v^T De_1 = m$. As $m|Dv$ and $k|De_1$, we know that km divides $2m$. m is positive, so $k = 2$.

□

Corollary 4.6.8 $\Theta = \mathbb{Z}_2{}^s$, *for some* s.

Proof: Θ is a finitely generated Abelian group, whose non-trivial elements all have order 2.

□

We have shown that if M is self dual, then $H_2(\tilde{M}; ZG) = \mathbb{Z}^r \oplus \mathbb{Z}_2^s$ for some integers r and s. To show that elements of TOP^5 are not always self dual, we need only show a manifold which does not satisfy this homological condition.

We refer again to [1].

Proposition 4.6.9 *(See [1], lemma 1.1(i)) For each integer* $k > 1$, *there exists a simply connected, closed, manifold* M_k, *with* $H_2(M_k; \mathbb{Z}) = \mathbb{Z}_k \oplus \mathbb{Z}_k$.

Lemma 4.5.4 tells us that for any manifold, we may choose $(\mathcal{B}, \beta)$ so that we may take both λ_1 and λ_2 to augment to 1. If the manifold is simply connected (as above), then we are working over $\mathbb{Z}$, so J is free and the augmentation condition is vacuous. However, it is possible that some condition on G could force closed, orientable, simply connected manifolds with fundamental group G, to be self dual.

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