Stably free modules over virtually free groups

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I, Seamus O'Shea, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

We study stably free modules over various group rings $\mathbf{Z}[G]$, using the method of Milnor patching. In particular, we construct infinite sets of stably free modules of rank one over various rings. Let F_n denote the free group on n generators. The two classes of group rings under consideration are:

(i) $\mathbf{Z}[G \times F_n]$, where G is finite nilpotent and of non square-free order, and $n \geq 2$;

(ii) $\mathbf{Z}[Q(12m) \times C_\infty]$, where $Q(12m)$ is the binary polyhedral group of order $12m$.

The modules in question are constructed as pullbacks arising from fibre square decompositions of the group rings.

We also study the $D(2)$-problem of low-dimensional topology. We give an affirmative answer to the $D(2)$-problem for the dihedral group of order $4n$, assuming the group ring $\mathbf{Z}[D_{4n}]$ satisfies torsion free cancellation. By results of Swan, Endo, and Miyata, this happens for a number of small primes n. Johnson has shown that the groups D_{4n+2} satisfy the $D(2)$-property, but his result relies on the fact that D_{4n+2} has periodic cohomology, a property not shared by D_{4n}. This forces us to introduce the torsion free cancellation hypothesis, and to explicitly realize the group of k-invariants $(\mathbf{Z}/4n)^*$.

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To Cassie

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Chapter 1

Introduction

Let Λ be a ring; we say that a Λ-module P is *stably free* of rank $m - n$ when

$$P \oplus \Lambda^n \cong \Lambda^m$$

for some n, m. We shall be interested in stably free modules over integral group rings. Let F_n denote the free group on n generators. The main result of this thesis is:

Theorem A. Let G be a finite nilpotent group of non square-free order, and let F be a group which maps surjectively onto F_n for some $n \geq 2$. Then $\mathbf{Z}[G \times F]$ admits infinitely many isomorphically distinct stably free modules of rank 1.

In contrast Johnson [20] has shown that both $\mathbf{Z}[C_p \times F_m]$ and $\mathbf{Z}[D_{2p} \times F_m]$ admit no non-free stably free modules when p is prime, C_p is the cyclic group of order p and D_{2p} is the dihedral group of order $2p$ ($p \neq 2$). Johnson [21] has also shown that $k[G \times F_m]$ admits no non-free stably free modules, where k is any field and G is any finite group.

It is natural to ask whether the hypothesis that G be nilpotent can be dropped from Theorem A. The smallest non square-free number for which there exists a non nilpotent group of that order is 12. In the second part of this thesis we show that the conclusion of Theorem A holds for D_6^*, the dicyclic group of order 12. In fact we shall show something stronger - let $Q(4m)$ denote the group with presentation

$$Q(4m) = \langle x, y \mid x^m = y^2,\ yx = x^{-1}y \rangle,$$

so that $D_6^* \cong Q(12)$. Then we shall prove:

Theorem B. Let F be a group which maps surjectively onto F_n for some $n \geq 1$. Then for every $m \geq 1$, $\mathbf{Z}[Q(12m) \times F]$ admits infinitely many isomorphically distinct stably free modules of rank 1.

Johnson [20] has previously shown that the conclusion of Theorem B holds for the groups $Q(8m)$[1]; in light of this it seems likely that Theorem B holds for the groups $Q(4m)$, but the details for prime $m \geq 5$ become intractable. Notice that the hypothesis of Theorem B is satisfied when F is a finitely generated group with $H_1(F; \mathbf{Q}) \neq 0$; for then $F^{ab}/\text{torsion} \cong C_\infty^n$ for some n and hence there is a surjective mapping $F \to C_\infty = F_1$.

The study of these modules is motivated by the D(2)-problem of low dimensional topology (see chapter 2). In the case where $G = C_n \times F_m$ we can, by constructing an explicit free resolution of $\mathbf{Z}$ by $\mathbf{Z}[G]$-modules, show that $\Omega_1(\mathbf{Z}) = \Omega_3(\mathbf{Z})$. Let $SF(\mathbf{Z}[G])$ denote the set of isomorphism classes of finitely generated stably free modules over $\mathbf{Z}[G]$. In many cases the structure of $\Omega_1(\mathbf{Z})$ is essentially determined by that of $SF(\mathbf{Z}[G])$ (see [19], [20]). Unfortunately, the techniques for parameterizing $\Omega_1(\mathbf{Z})$ by the stably free modules over $\mathbf{Z}[G]$ do not extend to the case $G = C_n \times F_m$. Nevertheless Theorem A provides some hope of interesting structure in $\Omega_1(\mathbf{Z}) = \Omega_3(\mathbf{Z})$ over $\mathbf{Z}[C_n \times F_m]$, when n is not square-free. Every group G for which the D(2)-problem has currently been confirmed has the property that $\mathbf{Z}[G]$ has no non-trivial stably free modules.

If one could find an appropriate surjective correspondence between $\Omega_1(\mathbf{Z})$ and $SF(\mathbf{Z}[G])$ in the case $G = C_n \times F_m$ for $m \geq 2$ one would obtain an infinite collection of pairwise homotopically distinct algebraic 2-complexes over $\mathbf{Z}[G]$ none of which obviously arise from a geometric cell complex. In the case $m = 1$, Edwards [15] has provided a parameterization of $\Omega_1(\mathbf{Z})$ using matrices over $(\mathbf{Z}/n)[t, t^{-1}]$. Using this and the fact that $\mathbf{Z}[C_n \times C_\infty]$ has no non-trivial stably free modules he was able to solve the D(2)-problem in the affirmative for $C_n \times C_\infty$. His techniques

[1] In his thesis [32], Kamali had previously shown that $\mathbf{Z}[Q(8m) \times C_\infty]$ admits infinitely many stably free modules when m is not a power of two.

can be generalized to the case $C_n \times F_m$ (see [20]) to provide a parameterization of $\Omega_1(\mathbf{Z})$ by certain equivalence classes of matrices over $(\mathbf{Z}/n)[F_m]$. However, it has so far proved impossible to compute these equivalence classes and so the parameterization in this case is of purely theoretical intrest.

We can place Theorem B in context by comparing it with Swan's paper [34]. He shows that the rings $\mathbf{Z}[Q(4m)]$ have a finite number of non-trivial stably frees except in a few exceptional cases. Despite the fact that $Q(12)$ fails the Eichler condition its integral group ring nevertheless has no non-trivial stably frees. Theorem B says that although $R[Q(12)]$ has no non-trivial stably frees for $R = \mathbf{Z}$, this breaks down when we take $R = \mathbf{Z}[t, t^{-1}]$, the ring of Laurent polynomials over $\mathbf{Z}$. In fact, essentially the same proof as that of Theorem B shows that $R[Q(12)]$ has infinitely many isomorphically distinct stably free modules of rank 1 when $R = \mathbf{Z}[t]$, the ring of ordinary polynomials over $\mathbf{Z}$. Combining results of Johnson [20] with Theorem B shows that the minimal level of $\Omega_1(\mathbf{Z})$, over $\mathbf{Z}[Q(12m) \times F_m]$, is infinite, but the structure of $\Omega_3(\mathbf{Z})$ is not currently known.

Finally, we study the D(2)-problem for dihedral groups of order $4n$. Say that a group ring $\mathbf{Z}[G]$ satisfies *torsion free cancellation* when $X \oplus N \cong X \oplus M \implies N \cong M$ for any $\mathbf{Z}[G]$-modules X, N, M which are torsion free over $\mathbf{Z}$. We shall show:

Theorem C. Suppose that $\mathbf{Z}[D_{4n}]$ satisfies torsion free cancellation. Then the D(2)-property holds for D_{4n}.

The calculations of Swan [35] and Endo and Miyata [13] show that torsion free cancellation holds for $\mathbf{Z}[D_{4p}]$ when p is prime and $2 \leq p \leq 31$, $p = 47, 179$ or 19379. To date the only finite non-abelian, non-periodic groups for which the D(2)-property is known to hold are those of the form D_{4p}, where p is prime. Mannan [29] has previously shown that Theorem C holds for $n = 2$. Johnson [18] has shown that the $D(2)$-property holds for the groups D_{4n+2} for any $n \geq 1$; however his result relies on the fact that D_{4n+2} has periodic cohomology, a property not shared by D_{4n} .

Latiolais [27] has previously shown that the homotopy type of a CW-complex with fundamental group D_{4n} is determined by the Euler characteristic. This result was extended by Hambleton and Kreck [16], to include those complexes whose fundamental groups are finite subgroups of $SO(3)$. Latiolais achieves this by realizing all values of the Browning obstruction group [5], [22], [23]; as originally defined by Browning in his unpublished preprints [6], [7], [8], this obstruction group classifies algebraic 2-complexes over certain finite groups. However, Latiolais works in an entirely geometric context, reworking Browning's approach and only explicitly showing that his version of the obstruction group classifies *geometric* 2-complexes. It is not clear that the approaches of Browning and Latiolais are compatible, but nevertheless it may be possible to give an alternative proof of Theorem C, without the torsion free cancellation condition, using this approach.

1.1 Structure of the thesis

The structure of this thesis is as follows: chapters 2 - 6 are mainly expositions of standard material on rings and modules, with chapters 7, 8 and 9 containing the original work. The main exception to this is 3.0.4, which could be considered implicit in appendix A of [34], but does not appear explicitly in the literature.

Chapter 2 gives general background on the D(2)-problem needed to prove theorem C - a general reference for this material is [18]. Chapter 3 shows how to construct projective modules over various rings by decomposing them as fibre products. Chapters 4 and 5 contain accounts of stably free cancellation and weakly Euclidean rings, both of which prove useful concepts when analyzing projective modules over a fibre product ring in terms of projective modules over its factors. Chapter 6 gives the necessary background for the calculation of the various lower K-groups in chapter 8. The remaining chapters then proceed to prove theorems A, B and C in sequence. The main technical result needed for theorem A is 3.0.4, and theorem B is proved by relating the stably free modules in question to the Johnson's fibre square 3.2.1. Theorem C is independent from the material in chapters 3-6.

1.2 Conventions

Many of the statements in this thesis are stated for arbitrary rings; in practice we shall only be concerned with quite well behaved rings (mainly group algebras and their quotients). We can, therefore, make some simplifying assumptions. Say that a ring Λ satisfies the *invariant basis number property* (abbreviated to IBN) when, for integers $n, m \geq 1$, $\Lambda^n \cong \Lambda^m \iff n = m$. Throughout this thesis we shall assume that every ring under consideration has IBN.

Considering an isomorphism $\Lambda^n \cong \Lambda^m$ as an invertible $m \times n$ matrix, it is clear that having IBN is equivalent to satisfying the condition:

> If $A \in M_{n\times m}(\Lambda)$ and $B \in M_{m\times n}(\Lambda)$ are such that $AB = I_n$ and $BA = I_m$ then $n = m$.

Now suppose that $f : \Lambda \to \Lambda'$ is a ring homomorphism and suppose that Λ does not have IBN. Choose matrices $A \in M_{n\times m}(\Lambda)$ and $B \in M_{m\times n}(\Lambda)$ such that $AB = I_n$ and $BA = I_m$ with $n \neq m$. Applying f to A and B we have $f(A)f(B) = I_n$ and $f(B)f(A) = I_m$ and thus Λ' does not have IBN either. In contrapositive form this says:

> If $f : \Lambda \to \Lambda'$ is a ring homomorphism and Λ' has IBN then so does Λ.

The following argument, due to Cohn [11], shows that commutative rings have IBN. Let Λ be commutative and suppose that $A \in M_{n\times m}(\Lambda)$ and $B \in M_{m\times n}(\Lambda)$ are such that $AB = I_n$ and $BA = I_m$ with $n \neq m$; say $m < n$. Extend A and B by adjoining $n - m$ columns of zeros to A and $n - m$ rows of zeros to B as follows:

$$\hat{A} = \begin{pmatrix} A & 0 \end{pmatrix} \qquad \hat{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}$$

Then clearly $\hat{A}\hat{B} = AB = I_n$ and thus $\det(\hat{A})\det(\hat{B}) = 1$. However the presence of zero rows/columns implies that $\det(\hat{A}) = \det(\hat{B}) = 0$ which is a contradiction. Thus any commutative ring has IBN. For any group G the augmentation map provides a homomorphism from the group algebra $\Lambda[G]$ to Λ. Therefore any group algebra over a commutative ring has IBN.

Suppose that P is a stably free module over a ring Λ with $P \oplus \Lambda^m \cong \Lambda^n$ and $P \oplus \Lambda^r \cong \Lambda^s$. Adding r copies of Λ to the first equation we have $\Lambda^r \oplus P \oplus \Lambda^m \cong \Lambda^s \oplus \Lambda^m \cong \Lambda^r \oplus \Lambda^n$. Thus if Λ has IBN then $n - m = s - r$; in this case we define the *stably free rank* of P to be the unique integer $n - m$. We denote the set of isomorphism classes of rank n stably free modules over a ring Λ with IBN by $\mathrm{SF}_n(\Lambda)$.

All modules are right modules unless otherwise stated. For any ring Λ, Λ^* will denote the group of units.

Chapter 2

The D(2)-problem

Let X be a finite connected CW-complex, with universal cover $\tilde{X}$. We say that X is *cohomologically* 2*-dimensional* when $H_3(\tilde{X};\mathbf{Z}) = H^3(X;\mathcal{B}) = 0$ for all coefficient systems $\mathcal{B}$ on X. The $D(2)$-problem asks:

> Let X be a finite connected CW complex of geometrical dimension 3 which is cohomologically 2-dimensional. Is X homotopy equivalent to a finite CW complex of geometrical dimension 2?

The $D(2)$-problem was first posed by Wall [36], in connection with a general attempt to formulate conditions on a space which guarantee that it is homotopy equivalent to a space with given characteristics. In general, the $D(n)$-problem asks if a cohomologically n-dimensional complex is homotopy equivalent to a finite n-dimensional complex; Wall answered this in the affirmative for each $n \neq 2$.

Clearly the $D(2)$-problem is parameterized by the fundamental group: just restrict the question to those cell complexes X with a particular fundamental group $\pi_1(X)$. Results of Johnson and Mannan show that the $D(2)$-problem is equivalent to another, the realization problem. Let X be a finite 2-dimensional CW-complex with $\pi_1(X) = G$. Consider the cellular 2-complex of $\tilde{X}$, the universal cover of X:

$$C_2 \to C_1 \to C_0$$

where $C_n = H_n(\tilde{X}^n, \tilde{X}^{n-1};\mathbf{Z})$ is the free $\mathbf{Z}[G]$-module on the basis consisting of the n-cells of X. Since $\tilde{X}$ is simply-connected we have, by Hurewitz's theorem,

$H_2(\tilde{X};\mathbf{Z}) \cong \pi_2(\tilde{X}) \cong \pi_2(X)$. Also $H_1(\tilde{X};\mathbf{Z}) \cong 0$ and $H_0(\tilde{X};\mathbf{Z}) \cong \mathbf{Z}$ and so we may extend the above chain complex to an exact sequence of finitely generated $\mathbf{Z}[G]$-modules

$$C_*(X) = (0 \to \pi_2(X) \to C_2 \to C_1 \to C_0 \to \mathbf{Z} \to 0) \tag{2.1}$$

in which each C_i is free. More generally define an *algebraic* 2*-complex over* $\mathbf{Z}[G]$ to be an exact sequence of finitely generated $\mathbf{Z}[G]$-modules

$$\mathbf{P} = (0 \to \pi_2(\mathbf{P}) \to P_2 \to P_1 \to P_0 \to \mathbf{Z} \to 0) \tag{2.2}$$

in which each P_i is stably free. The realization problem asks:

> Is every algebraic 2-complex homotopy equivalent to a complex of the form (2.1) arising from a 2-dimensional CW-complex?

The realization problem is obviously parameterized by the choice of group G. Johnson [18] proved the following, subject to a mild technical condition on G, later shown to be unnecessary by Mannan [30]:

> Let G be a finitely presented group. Then the realization problem holds for G if and only if the $D(2)$-problem holds for G.

A chain map $f : \mathbf{P} \to \mathbf{Q}$, between two algebraic 2-complexes over $\mathbf{Z}[G]$, is said to be a *weak homotopy equivalence* if the induced maps $f_* : \pi_2(\mathbf{P}) \to \pi_2(\mathbf{Q})$ and $f_* : \mathbf{Z} \to \mathbf{Z}$ are isomorphisms. Weak homotopy equivalence corresponds to ordinary chain homotopy equivalence in the sense that f is a weak homotopy equivalence if and only if the induced map

$$\begin{array}{ccccc} P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 \end{array}$$

is a chain homotopy equivalence. Write $\mathbf{Alg}_G$ for the set of weak homotopy classes of algebraic 2-complexes over $\mathbf{Z}[G]$.

Let $f : X \to Y$ be a homotopy equivalence between two CW-complexes with $\pi_1(X) \cong \pi_1(Y) \cong G$; it is well known that the induced map $f_* : C_*(X) \to C_*(Y)$ is then a weak homotopy equivalence. Thus the correspondence $X \mapsto C_*(X)$ gives a faithful representation of 2-dimensional homotopy theory in the algebraic homotopy category determined by $\mathbf{Alg}_G$. The realization question asks if this correspondence induces an equivalence of categories.

2.1 Realizing algebraic complexes

Evidently the realization problem is parameterized by the isomorphism class of the module $\pi_2(\mathbf{P})$ appearing in (2.2). Thus a first step in solving the realization problem for G is determining which $\mathbf{Z}[G]$-modules $\pi_2(\mathbf{P})$ may occur in a sequence of the form (2.2). Let Λ be a ring and suppose we are given two exact sequences of Λ-modules:

$$0 \to I \to \Lambda^n \to M \to 0 \ \ ; \ \ 0 \to I' \to \Lambda^m \to M \to 0.$$

Recall Schanuel's lemma which states that $I \oplus \Lambda^m \cong I' \oplus \Lambda^n$. We say that Λ-modules I, I' are *stably equivalent* if $I \oplus \Lambda^m \cong I' \oplus \Lambda^n$ for some n, m. Given two exact sequences

$$0 \to J \to F_n \to \ldots \to F_1 \to F_0 \to M \to 0;$$

$$0 \to J' \to F'_n \to \ldots \to F'_1 \to F'_0 \to M \to 0$$

where each F_i is finitely generated free, then iteratively applying Schanuel's lemma shows that J and J' are stably equivalent; write $\Omega_{n+1}(M)$ for the class of modules stably equivalent to J. We call $\Omega_n(M)$ the n^{th} syzygy of M over Λ. Therefore, a first step in solving the realization problem for G is to determine $\Omega_3(\mathbf{Z})$ over $\mathbf{Z}[G]$.

Given a finite presentation $\mathcal{G}$ of a group G, the *presentation complex* $X(\mathcal{G})$ associated to $\mathcal{G}$ is a 2-dimensional CW complex, with a single vertex, one loop at the vertex for each generator, and one 2-cell for each relator in $\mathcal{G}$, with the boundary of the 2-cell attached along the appropriate word. Applying the functor $C_*(-)$

to $X(\mathcal{G})$ now gives a representative of $\Omega_3(\mathbf{Z})$ over $\mathbf{Z}[G]$, namely $\pi_2(C_*(X(\mathcal{G})))$. For any group G we can define an augmentation homomorphism $\varepsilon : \mathbf{Z}[G] \to \mathbf{Z}$ by setting $\varepsilon(g) = 1$ for each $g \in G$. We have an exact sequence

$$0 \to I \to \mathbf{Z}[G] \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

in which $I = \ker(\varepsilon)$ is the augmentation ideal, and so $\Omega_1(\mathbf{Z})$ is just the stable class of I.

The next step in solving the $D(2)$-problem for G is to describe the fibres of the map $\mathbf{Alg}_G \to \Omega_3(\mathbf{Z})$ given by $\mathbf{P} \mapsto \pi_2(\mathbf{P})$. For finite groups G, this is achieved via the Swan map [18]. Fix a finite group G and put $\Lambda = \mathbf{Z}[G]$. Let $\mathbf{P} = (0 \to \pi_2(\mathbf{P}) \to P_2 \to P_1 \to P_0 \to \mathbf{Z} \to 0)$ be an algebraic 2-complex over G and let $\mathbf{E} = (0 \to \pi_2(\mathbf{P}) \to E_2 \to E_1 \to E_0 \to \mathbf{Z} \to 0) \in \mathrm{Ext}^3_\Lambda(\mathbf{Z}, \pi_2(\mathbf{P}))$ be an arbitrary extension of $\mathbf{Z}$ by $\pi_2(\mathbf{P})$. Then by the universal property of projective modules, there exists a commutative diagram

$$\begin{array}{ccccccccccccc}
\mathbf{P} & = & (0 & \longrightarrow & \pi_2(\mathbf{P}) & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & \mathbf{Z} \longrightarrow 0) \\
\downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle\alpha_+} & & \downarrow{\scriptstyle\alpha_2} & & \downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\alpha_0} & & \downarrow{\scriptstyle\mathrm{Id}} \\
\mathbf{E} & = & (0 & \longrightarrow & \pi_2(\mathbf{P}) & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & \mathbf{Z} \longrightarrow 0)
\end{array}$$

We may extend α_+ thus:

$$\begin{array}{ccccccccccc}
0 \longrightarrow & \pi_2(\mathbf{P}) & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & \mathbf{Z} & \longrightarrow 0 \\
 & \downarrow{\scriptstyle\alpha_+} & & \downarrow{\scriptstyle\alpha_2'} & & \downarrow{\scriptstyle\alpha_1'} & & \downarrow{\scriptstyle\alpha_0'} & & \downarrow{\scriptstyle\tilde{\alpha}} & \\
0 \longrightarrow & \pi_2(\mathbf{P}) & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & \mathbf{Z} & \longrightarrow 0
\end{array}$$

Then $\tilde{\alpha}$ is unique up to congruence modulo $|G|$ and we have a well-defined map $\kappa : \mathrm{End}_\Lambda(\pi_2(\mathbf{P})) \to \mathbf{Z}/|G|$ given by $\kappa(\alpha_+) = \tilde{\alpha}$. The *k-invariant of the transition* $\alpha : \mathbf{P} \to \mathbf{E}$ is defined to be $k(\mathbf{P} \to \mathbf{E}) = \kappa(\alpha_+)$. Given $\alpha \in \mathrm{End}_\Lambda(\pi_2(\mathbf{P}))$ we have a k-invariant $k(\mathbf{P} \to \alpha_*(\mathbf{P})) = \kappa(\alpha)k(\mathbf{P} \to \mathbf{P}) = \kappa(\alpha)$, where $\alpha_*(\mathbf{P})$ is the pushout extension. Since $\kappa(\alpha)$ is a unit if α is an isomorphism, this induces a mapping

$$\mathrm{Aut}_\Lambda(\pi_2(\mathbf{P})) \to (\mathbf{Z}/|G|)^*$$

called the Swan map, which is independent of the choice of algebraic complex $\mathbf{P}$ in which $\pi_2(\mathbf{P})$ appears. We have (see [20], Theorems 54.6 and 54.7):

Theorem 2.1.1. Fix a module $\pi_2(\mathbf{P}) \in \Omega_3(\mathbf{Z})$, and suppose that the Swan map $\mathrm{Aut}(\pi_2(\mathbf{P})) \to (\mathbf{Z}/|G|)^*$ is surjective. Then for each $n \geq 0$ there is, up to chain homotopy equivalence, a unique algebraic 2-complex of the form

$$0 \to \pi_2(\mathbf{P}) \oplus \Lambda^n \to F_2 \to F_1 \to F_0 \to \mathbf{Z} \to 0.$$

Chapter 3

Projective modules over fibre products

In [31], Milnor introduced techniques for analysing the structure of projective modules over a fibre product ring in terms of its factors. These techniques were further developed by Swan in [34] to investigate the structure of stably free modules over various group rings. Suppose we are given two ring homomorphisms $\psi_+ : A_+ \to A_0$ and $\psi_- : A_- \to A_0$. The fibre product of A_+ and A_- over A_0 is the ring

$$A_+ \times_{A_0} A_- = \{(a_+, a_-) \in A_+ \times A_- \mid \psi_+(a_+) = \psi_-(a_-)\},$$

where addition and multiplication are defined component wise. If $A = A_+ \times_{A_0} A_-$ we often represent this situation as a *fibre square*:

$$\mathcal{A} = \left\{ \begin{array}{ccc} A & \xrightarrow{\pi_-} & A_- \\ \big\downarrow{\scriptstyle \pi_+} & & \big\downarrow{\scriptstyle \psi_-} \\ A_+ & \xrightarrow{\psi_+} & A_0 \end{array} \right. \tag{3.1}$$

where π_+ and π_- are the projections from A to A_+ and A_- respectively. It is easy to show that the condition $A = A_+ \times_{A_0} A_-$ is equivalent to requiring that the following is an exact sequence (of additive groups):

$$0 \to A \xrightarrow{\begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix}} A_+ \oplus A_- \xrightarrow{(\psi_+, -\psi_-)} A_0.$$

Let M be a right module over A; then M determines a triple

$$(M_+, M_-; \alpha(M))$$

where $M_\sigma = M \otimes_{\pi_\sigma} A_\sigma$ for $\sigma = +, -$ and $\alpha(M)$ is the canonical A_0-isomorphism making the following commute:

$$\begin{array}{ccc} (M \otimes_{\pi_+} A_+) \otimes_{\psi_+} A_0 & \xrightarrow{\alpha(M)} & (M \otimes_{\pi_-} A_-) \otimes_{\psi_-} A_0 \\ \downarrow & & \downarrow \\ M \otimes_{\psi_+\pi_+} A_0 & \xrightarrow{\mathrm{Id}} & M \otimes_{\psi_-\pi_-} A_0 \end{array}$$

where the vertical maps are the canonical isomorphisms. Conversely, suppose we are given a triple $(M_+, M_-; \alpha)$, where M_σ is a right module over A_σ and $\alpha : M_+ \otimes_{\psi_+} A_0 \to M_- \otimes_{\psi_+} A_0$ is an A_0-module isomorphism. Then we obtain an A-module $\langle M_+, M_-; \alpha \rangle$ given by

$$\langle M_+, M_-; \alpha \rangle = \{(m_+, m_-) \in M_+ \times M_- \mid \alpha(m_+ \otimes 1) = m_- \otimes 1\}$$

with A-action given by $(m_+, m_-) \cdot a = (m_+ \cdot \pi_+(a), m_- \cdot \pi_-(a))$. To see that $(m_+, m_-) \cdot a \in \langle M_+, M_-; \alpha \rangle$, note that

$$\begin{aligned} \alpha(m_+ \cdot \pi_+(a) \otimes 1) = \alpha(m_+ \otimes \psi_+\pi_+(a)) &= m_- \otimes \psi_+\pi_+(a) \\ &= m_- \otimes \psi_-\pi_-(a) \\ &= m_- \cdot \pi_-(a) \otimes 1. \end{aligned}$$

The following shows that every finitely generated projective A-module arises in this way.

Proposition 3.0.1. Let P be a finitely generated projective A-module. Then

$P \cong \langle P_+, P_-; \alpha(P)\rangle$.

Proof. Given an A-module M, define a homomorphism $\delta_M : M \to \langle M_+, M_-; \alpha(M)\rangle$ by $\delta_M(x) = (x \otimes_{\pi_+} 1, x \otimes_{\pi_-} 1)$. Since A is the fibre product of A_+ and A_- over A_0, then δ_A is an isomorphism. Suppose that $P \oplus Q \cong A^n$; then clearly

$$\delta_{P\oplus Q} = \begin{pmatrix} \delta_P & 0 \\ 0 & \delta_Q \end{pmatrix}$$

and since $\delta_{P\oplus Q} = \delta_{A^n} = \delta_A \oplus \ldots \oplus \delta_A$ is an isomorphism, so are δ_P and δ_Q. Alternatively, we could apply the exact functor $P \otimes -$ to the exact sequence

$$0 \to A \xrightarrow{\begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix}} A_+ \oplus A_- \xrightarrow{(\psi_+, -\psi_-)} A_0$$

and observe that $\langle P_+, P_-; \alpha(P)\rangle = \mathrm{Ker}(\mathrm{Id} \otimes (\psi_+, -\psi_-))$. $\square$

An A-module M is said to be *locally projective* (resp. *locally free*) if M_σ is projective (resp. free) for $\sigma = +, -$. Clearly every projective A-module is locally projective. Under certain conditions the converse is true.

The stable linear group $GL(R)$ of a ring R is the direct limit of the inclusions $GL_n(R) \to GL_{n+1}(R)$ as the upper left block matrix. The stable elementary subgroup $E(R) \subset GL(R)$ is defined as the direct limit of the groups $E_n(R)$, where $E_n(R) \subset GL_n(R)$ is the subgroup of $n \times n$ elementary matrices over R. It is a consequence of Whitehead's lemma that $E(R) = [GL(R), GL(R)]$. Let $\mathcal{A}$ be a fibre square as in (3.1). Say that $\mathcal{A}$ is *E-surjective* if the double coset $\psi_-(E(A_-))\backslash E(A_0)/\psi_+(E(A_+))$ consists of a single point; that is, if every $[N] \in E(A_0)$ can be written as a product $[N] = [\psi_-(N_-)][\psi_+(N_+)]$ for some $N_- \in E_n(A_-)$ and $N_+ \in E_m(A_+)$ for some n, m.

Lemma 3.0.2. Let $\mathcal{A}$ be a fibre square as in (3.1). If $\mathcal{A}$ is E-surjective then, for each integer $n \geq 1$ and each $\alpha \in GL_n(A_0)$, there exists $m \geq 1$ and $\beta \in GL_m(A_0)$ such that $\alpha \oplus \beta = [h_+][h_-]$ for some $h_+ \in GL_{n+m}(A_+)$, $h_- \in GL_{n+m}(A_-)$. Here $[h_\sigma] = h_\sigma \otimes \mathrm{Id}_{A_0}$.

Proof. Let $\alpha \in GL_n(A_0)$. Then by Whitehead's lemma $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in E_{2n}(A_0)$. Since $\mathcal{A}$ is E-surjective, we may choose an integer m such that

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \oplus [I_m] = [I_{2n+m}] \in \psi_-(E(A_-))\backslash E(A_0)/\psi_+(E(A_+)).$$

Therefore $\alpha \oplus (\alpha^{-1} \oplus I_m) = [h_+][h_-]$ for some $h_\sigma \in E_{2n+m}(A_\sigma) \subset GL_{2n+m}(A_\sigma)$. □

The following is essentially proved in [31]:

Theorem 3.0.3. Let $\mathcal{A}$ be a fibre square as in (3.1). Suppose that $\mathcal{A}$ is E-surjective; then a finitely generated A-module $\langle P_+, P_-; \alpha \rangle$ is projective if and only if it is locally projective.

Proof. Clearly a projective A-module is locally projective. Suppose that $P = \langle P_+, P_-; \alpha \rangle$ is locally projective over

$$\mathcal{A} = \left\{ \begin{array}{ccc} A & \xrightarrow{\pi_-} & A_- \\ \downarrow{\scriptstyle \pi_+} & & \downarrow{\scriptstyle \psi_-} \\ A_+ & \xrightarrow{\psi_+} & A_0 \end{array} \right.$$

Then we may choose Q_+, Q_- such that $P_+ \oplus Q_+ \cong A_+^n$, $P_- \oplus Q_- \cong A_-^n$ for some n. Set $K_\sigma = Q_\sigma \oplus A_\sigma^n$. We have exact sequences

$$0 \to Q_+ \otimes A_0 \to (Q_+ \otimes A_0) \oplus (P_+ \otimes A_0) \to P_+ \otimes A_0 \to 0;$$

$$0 \to Q_- \otimes A_0 \to (Q_- \otimes A_0) \oplus (P_- \otimes A_0) \to P_- \otimes A_0 \to 0.$$

Observing that $P_+ \otimes A_0 \cong P_- \otimes A_0$, and $(Q_\sigma \otimes A_0) \oplus (P_\sigma \otimes A_0) \cong A_0^n$, we see by Schanuel's lemma that $(Q_+ \otimes A_0) \oplus A_0^n \cong (Q_- \otimes A_0) \oplus A_0^n \implies K_+ \otimes A_0 \cong K_- \otimes A_0$. Choose an isomorphism $\beta : K_+ \otimes A_0 \to K_- \otimes A_0$; then the module

$$\langle P_+ \oplus K_+, P_- \oplus K_-; \alpha \oplus \beta \rangle$$

is locally free. Therefore $\alpha \oplus \beta \in GL_{2n}(A_0)$ and by (3.0.2) we may choose

$\gamma \in GL_m(A_0)$ and $h_\sigma \in GL_m(A_\sigma)$ such that $\alpha \oplus \beta \oplus \gamma = [h_+][h_-]$. Define a map

$$f : \langle P_+ \oplus K_+ \oplus A_+^m, P_- \oplus K_- \oplus A_-^m; \alpha \oplus \beta \oplus \gamma\rangle \to \langle A_+^{2n+m}, A_-^{2n+m}; I_{2n+m}\rangle$$

by $f(x, y) = (h_+(x), h_-^{-1}(y))$. To see that $f(x, y) \in \langle A_+^{2n+m}, A_-^{2n+m}; I_{2n+m}\rangle$, note that

$$(\alpha \oplus \beta \oplus \gamma)(x \otimes 1) = [h_+][h_-](x \otimes 1) = y \otimes 1$$

which is true if and only if $h_+(x)\otimes 1 = h_-^{-1}(y)\otimes 1$. Since h_+, h_- are isomorphisms, so is f. By (3.0.1) $\langle A_+^{2n+m}, A_-^{2n+m}; I_{2n+m}\rangle \cong A^{2n+m}$ and hence

$$\langle P_+, P_-; \alpha\rangle \oplus \langle K_+ \oplus A_+^m, K_- \oplus A_-^m; \beta \oplus \gamma\rangle \cong A^{2n+m}$$

as required. □

Now suppose we are given an E-surjective fibre square $\mathcal{A}$. Then a locally free module over A is automatically projective; the question of when it is stably free is rather more delicate. Denote the set of isomorphism classes of finitely generated locally free modules of rank n over A by $\mathrm{LF}_n(A)$. Let P be a locally free A-module; the isomorphism $\alpha(P)$ in the triple $(A_+^n, A_-^n; \alpha(P))$ associated to P is not uniquely determined by the isomorphism class of P. Since $\alpha(P) : A_+^n \otimes_{\psi_+} A_0 \to A_-^n \otimes_{\psi_-} A_0$, and $A_\sigma^n \otimes_{\psi_+} A_0 \cong A_0^n$ $(\sigma = +, -)$, we may regard $\alpha(P)$ as belonging to $GL_n(A_0)$ for some n. However, if $\beta \in GL_n(A_+)$ and $\gamma \in GL_n(A_-)$, we may define an isomorphism of A-modules

$$f : \langle A_+^n, A_-^n; \alpha(P)\rangle \to \langle A_+^n, A_-^n; [\gamma] \circ \alpha(P) \circ [\beta]\rangle$$

by setting $f(m_+, m_-) = (\beta^{-1}(m_+), \gamma(m_-))$, where $[\beta] = \beta\otimes_{\psi_+} \mathrm{Id}$ and $[\gamma] = \gamma\otimes_{\psi_-} \mathrm{Id}$. Conversely, suppose that $f : \langle A_+^n, A_-^n; \alpha\rangle \to \langle A_+^n, A_-^n; \beta\rangle$ is an isomorphism. If we define

$$\begin{aligned} f_+ =& f \otimes_{\psi_+} \mathrm{Id} : \langle A_+^n, A_-^n; \alpha\rangle \otimes_{\psi_+} A_+ \to \langle A_+^n, A_-^n; \beta\rangle \otimes_{\psi_+} A_+; \\ f_- =& f \otimes_{\psi_-} \mathrm{Id} : \langle A_+^n, A_-^n; \alpha\rangle \otimes_{\psi_-} A_- \to \langle A_+^n, A_-^n; \beta\rangle \otimes_{\psi_-} A_- \end{aligned}$$

then it is clear that $f_\sigma \in GL_n(A_\sigma)$ for $\sigma = +, -$ and $\beta = [f_-]\circ\alpha\circ[f_+^{-1}]$. Therefore there exists a bijection

$$\mathrm{LF}_n(A) \leftrightarrow \psi_-(GL_n(A_-))\backslash GL_n(A_0)/\psi_+(GL_n(A_+)) \tag{3.2}$$

Abbreviate the double coset on the right to $\overline{GL_n}(\mathcal{A})$. For each pair of integers $n, k \geq 1$ define a stabilization map

$$\begin{aligned} s_{n,k} : \overline{GL_n}(\mathcal{A}) &\to \overline{GL_{n+k}}(\mathcal{A}) \\ [\alpha] &\mapsto [\alpha \oplus I_k] \end{aligned}$$

Then, since the free module A^n determines the triple $(A_+^n, A_-^n; I_n)$, we have:

> If a locally free A-module M of rank n determines the triple $(M_+, M_-; \alpha)$, then M is stably free if and only if $s_{n,k}[\alpha] = [I_{n+k}]$ for some k.

The following theorem allows us to construct the non-trivial stably free modules of Theorem A.

Theorem 3.0.4. Let $\mathcal{A}$ be a fibre square as in (3.1). Suppose that $\mathcal{A}$ is E-surjective and that $\psi_-(A_-^*)\backslash[A_0^*, A_0^*]/\psi_+(A_+^*)$ is infinite: then $\mathrm{SF}_1(A)$ is infinite.

Proof. Let $\{a_i\}_{i\in I}$ be an infinite set of coset representatives in $\psi_-(A_-^*)\backslash[A_0^*, A_0^*]/\psi_+(A_+^*)$. For each $i \in I$ form the locally free A-module $P_i = (A_+, A_-; a_i)$. Then by (3.2) $P_i \cong P_j \iff i = j$. To see that each P_i is stably free, consider

$$s_{1,1}[a_i] = \begin{bmatrix} a_i & 0 \\ 0 & 1 \end{bmatrix}.$$

Each $a_i \in [A_0^*, A_0^*]$ and so by Whitehead's lemma $s_{1,1}[a_i] \in E_2(A_0) \implies s_{1,j}[a_i] \in E_{1+j}(A_0)$ for each $j \geq 1$. Since $\mathcal{A}$ is E-surjective, there exists $k \geq 2$, $X_+ \in E_k(A_+)$ and $X_- \in E_k(A_-)$ such that $s_{1,k-1}[a_i] = [\psi_-(X_-)][\psi_+(X_+)] = [I_k] \in \overline{GL_k}(\mathcal{A})$ and so $P_i \oplus A^{k-1} \cong A^k$.

□

The following 'Mayer - Vietoris' type theorem is often useful when computing lower K-groups (see [28]):

Theorem 3.0.5. Let $\mathcal{A}$ be a fibre square as in (3.1). If $\mathcal{A}$ is E-surjective then there exists an exact sequence

$$K_1(A) \xrightarrow{f_1} K_1(A_+) \oplus K_1(A_-) \xrightarrow{f_2} K_1(A_0) \xrightarrow{\delta} K_0(A) \xrightarrow{f_3} K_0(A_+) \oplus K_0(A_-) \xrightarrow{f_4} K_0(A_0)$$

in which

$$\begin{aligned}
&f_1[X] = ([(\pi_+)_*(X)], [(\pi_-)_*(X)]),\\
&f_2([X_+],[X_-]) = (\psi_+)_*[X_+]((\psi_-)_*[X_-])^{-1},\\
&f_3[P] = ([(\pi_+)_*(P)], [(\pi_-)_*(P)]),\\
&f_4([P_+],[P_-]) = (\psi_+)_*[P_+] - (\psi_+)_*[P_+],\\
&\delta[\alpha] = [\langle A_+^n, A_-^n; \alpha\rangle] \text{ (where } \alpha \in GL_n(A_0)).
\end{aligned}$$

3.1 Constructing fibre squares

The following construction provides a common source of fibre squares:

Proposition 3.1.1. Let I, J be ideals of a ring R. Then there is a fibre square of canonical maps

$$\begin{array}{ccc}
R/(I \cap J) & \longrightarrow & R/I \\
\downarrow & & \downarrow \\
R/J & \longrightarrow & R/(I+J)
\end{array}$$

Proof. The square is obviously commutative. If $[r]_I$ and $[s]_J$ have the same image in $R/(I+J)$ then $r - s = x + y$ where $x \in I$, $y \in J$. Define $t := r - x = s + y$; then $[t]_{I\cap J}$ has image $[r]_I$ in R/I and image $[s]_J$ in R/J. However if $[t']_{I\cap J}$ has the same images in R/I and R/J then $t - t' \in I$ and $t - t' \in J$ and hence $[t]_{I\cap J} = [t']_{I\cap J}$. □

Clearly all of the maps in the above fibre square are surjective; in fact any fibre square with all maps surjective is isomorphic to one of this form. Fibre squares of this form are obviously E-surjective, since the induced map $E(R/J) \to E(R/(I+J))$ is surjective. Similarly:

Proposition 3.1.2. Let

$$\mathcal{A} = \left\{ \begin{array}{ccc} A & \xrightarrow{\pi_-} & A_- \\ \downarrow{\scriptstyle \pi_+} & & \downarrow{\scriptstyle \psi_-} \\ A_+ & \xrightarrow{\psi_+} & A_0 \end{array} \right.$$

be a fibre square in which either ψ_+ or ψ_- is surjective (such a square is called a *Milnor square*). Then $\mathcal{A}$ is E-surjective.

Milnor [31] originally considered projective modules over these squares only, but his techniques extend easily to the wider class of E-surjective squares.

Proposition 3.1.3. Let I be an ideal in R, and suppose that $f : R \to S$ is a ring homomorphism such that $f|_I : I \to f(I)$ is bijective. Then

$$\begin{array}{ccc} R & \xrightarrow{[\,]} & R/I \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_*} \\ S & \xrightarrow{[\,]} & S/f(I) \end{array}$$

is a Milnor square.

Proof. We must show that

$$0 \to R \xrightarrow{\begin{pmatrix} [\,] \\ f \end{pmatrix}} R/I \oplus S \xrightarrow{\begin{pmatrix} f_*, & -[\,] \end{pmatrix}} S/I$$

is exact. Suppose that $\begin{pmatrix} [\,] \\ f \end{pmatrix}(r) = 0$. Then $[r] = 0 \implies r \in I \implies f(r) \in I$ and thus $r = 0$ since f is bijective on I and $f(r) = 0$. Therefore the sequence is exact at R. Now, if $r \in R$, then $\begin{pmatrix} f_*, & -[\,] \end{pmatrix} \begin{pmatrix} [\,] \\ f \end{pmatrix}(r) = \begin{pmatrix} f_*, & -[\,] \end{pmatrix} \begin{pmatrix} [r] \\ f(r) \end{pmatrix} = f_*[r] - [f(r)] = 0$. Finally, if $\begin{pmatrix} f_*, & -[\,] \end{pmatrix} \begin{pmatrix} [r] \\ s \end{pmatrix} = 0$, then $[f(r)] - [s] = 0 \implies f(r) - s \in f(I)$. Choose (unique) $t \in I$ such that

$f(t) = f(r) - s$. Then $\begin{pmatrix} [\,] \\ f \end{pmatrix}(r-t) = \begin{pmatrix} [r-t] \\ f(r-t) \end{pmatrix} = \begin{pmatrix} [r] \\ s \end{pmatrix}$, and so the sequence is exact. □

Corollary 3.1.4. Let G be a finite group and let H be a normal subgroup of G. Then the following is a Milnor square:

$$\begin{array}{ccc} \mathbf{Z}[G] & \longrightarrow & \mathbf{Z}[G]/(\Sigma_H) \\ \downarrow & & \downarrow \\ \mathbf{Z}[G/H] & \longrightarrow & (\mathbf{Z}/|H|)[G/H] \end{array}$$

where $\Sigma_H = \sum_{h \in H} h$.

Proof. Apply (3.1.3) with $f : \mathbf{Z}[G] \to \mathbf{Z}[G/H]$ given by $f(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g[g]$, and $I = (\Sigma_H) \cdot \mathbf{Z}[G]$ (so that $f(I) = |H| \cdot \mathbf{Z}[G/H]$). We need to show that $f|_I : I \to f(I)$ is bijective. Clearly it is surjective; suppose that

$$f\left(\Sigma_H \cdot \sum_{g\in G} a_g g\right) = |H| \cdot \sum_{g\in G} a_g[g] = 0$$

Then $\sum_{g\in G} a_g g \in \ker(f)$. Since $\ker(f) = \operatorname{im}(h_1 - 1, \ldots, h_m - 1)$, where $h_1, \ldots, h_m$ generate H, we have

$$\sum_{g\in G} a_g g = (h_1 - 1)\lambda_1 + \ldots + (h_m - 1)\lambda_m$$

for some $\lambda_1, \ldots, \lambda_m \in \mathbf{Z}[G]$. But this implies that $\Sigma_H \cdot \sum_{g\in G} a_g g = 0$, and hence f is bijective on I. □

Another example of a fibre square is provided by Karoubi squares: let S be a multiplicative submonoid of a ring R. Then S is said to be *regular* if it is central in R and contains no zero divisors. When S is a regular submonoid of R we may form the localization $S^{-1}R$ in the usual way.

Proposition 3.1.5. Let $f : A \to B$ be a ring homomorphism. Suppose that S is a regular submonoid of A and that $f(S)$ is a regular submonoid of B. Suppose also that, for each $s \in S$, the canonical mapping $f_* : A/sA \to B/f(s)B$ is an isomorphism. Then the following is a fibre square:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ S^{-1}A & \xrightarrow{f_*} & f(S)^{-1}B \end{array}$$

where i and j denote the canonical inclusions.

Such a fibre square is called a *Karoubi square.*

Theorem 3.1.6. Every Karoubi square is E-surjective.

For a proof see ([34], appendix A). More fibre squares may be generated by applying left exact functors to existing squares:

Proposition 3.1.7. Let $\mathcal{F}$: *Rings* $\to$ *Rings* be a left exact functor (that is, a functor which is left exact when considered as a functor on the underlying abelian groups). If $\mathcal{A}$ is a fibre square as in (3.1) then so is

$$\mathcal{F}(\mathcal{A}) = \left\{ \begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(\pi_-)} & \mathcal{F}(A_-) \\ \downarrow{\scriptstyle \mathcal{F}(\pi_+)} & & \downarrow{\scriptstyle \mathcal{F}(\psi_-)} \\ \mathcal{F}(A_+) & \xrightarrow{\mathcal{F}(\psi_+)} & \mathcal{F}(A_0) \end{array} \right.$$

For example, any group algebra $\mathbf{Z}[G]$ is free as an additive group; thus $- \otimes_{\mathbf{Z}} \mathbf{Z}[G]$ is a functor *Rings* $\to$ *Rings*, which is left exact as a functor on the underlying abelian groups. Hence:

Proposition 3.1.8. Let $\mathcal{A}$ be a fibre square as in (3.1). Then

$$\begin{array}{ccc} A \otimes \mathbf{Z}[G] & \xrightarrow{\pi_- \otimes Id} & A_- \otimes \mathbf{Z}[G] \\ \downarrow{\scriptstyle \pi_+ \otimes Id} & & \downarrow{\scriptstyle \psi_- \otimes Id} \\ A_+ \otimes \mathbf{Z}[G] & \xrightarrow{\psi_+ \otimes Id} & A_0 \otimes \mathbf{Z}[G] \end{array}$$

is also a fibre square.

3.2 A fibre square calculation

In this section we present a fibre square calculation due to Johnson [20]. We will explicitly construct an infinite class of rank 2 locally free modules over a certain pullback ring. This calculation is the basis of the proof of Theorem B.

Let p be prime and let

$$\begin{aligned} \hat{\mathbf{Z}}_{(p)} &= \text{the ring of } p\text{-adic integers;} \\ \hat{\mathbf{Q}}_{(p)} &= \text{the ring of } p\text{-adic numbers, i.e. the field of fractions of } \hat{\mathbf{Z}}_{(p)}. \end{aligned}$$

Consider the fibre square

$$\mathcal{T}(p) = \left\{ \begin{array}{ccc} X & \longrightarrow & \hat{\mathbf{Z}}_{(p)}[t, t^{-1}] \\ \downarrow & & \downarrow j \\ \hat{\mathbf{Q}}_{(p)} & \xrightarrow{i} & \hat{\mathbf{Q}}_{(p)}[t, t^{-1}] \end{array} \right.$$

with the obvious maps i, j and where X is defined to be the pullback

$$X = \hat{\mathbf{Q}}_{(p)} \times_{\hat{\mathbf{Q}}_{(p)}[t,t^{-1}]} \hat{\mathbf{Z}}_{(p)}[t, t^{-1}].$$

We are interested in $\overline{GL}_2(\mathcal{T}(p)) = GL_2(\hat{\mathbf{Q}}_{(p)}) \backslash GL_2(\hat{\mathbf{Q}}_{(p)}[t, t^{-1}]) / GL_2(\hat{\mathbf{Z}}_{(p)}[t, t^{-1}])$; put

$$\mathcal{Z}(n) = \begin{pmatrix} 1 & \frac{t^n}{p} \\ 0 & 1 \end{pmatrix} \in GL_2(\hat{\mathbf{Q}}_{(p)}[t, t^{-1}]).$$

Theorem 3.2.1. (F. E. A. Johnson [20]) : The matrices $\mathcal{Z}(n)$ represent pairwise distinct classes in $\overline{GL}_2(\mathcal{T}(p))$.

Proof. Suppose for contradiction that $[\mathcal{Z}(n)] = [\mathcal{Z}(m)]$ for $n \neq m$; write $\mathcal{Z}(n) = X\mathcal{Z}(m)Y$ for some $X \in GL_2(\hat{\mathbf{Q}}_{(p)})$ and some $Y \in GL_2(\hat{\mathbf{Z}}_{(p)}[t, t^{-1}])$. Write

$$X^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; ; \; Y = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

Then expanding we have:

$$\begin{pmatrix} a & b + a\frac{t^n}{p} \\ c & d + c\frac{t^n}{p} \end{pmatrix} = \begin{pmatrix} e + g\frac{t^m}{p} & f + h\frac{t^m}{p} \\ g & h \end{pmatrix}$$

Equating entries in the $(2,1)$-position we see that $g = c$ is a constant polynomial. From the $(2,2)$ and $(1,1)$-entries we have

$$h = \frac{c}{p}t^n + d \quad ; \quad e = a - \frac{g}{p}t^m$$

and hence from the $(1,2)$-entry:

$$f = b + \frac{a}{p}t^n - \frac{h}{p}t^m = b + \frac{a}{p}t^n - \frac{d}{p}t^m - \frac{c}{p^2}t^{n+m}.$$

Now, since $Y \in GL_2(\hat{\mathbf{Z}}_{(p)}[t, t^{-1}])$, we have $\det(Y) \in (\hat{\mathbf{Z}}_{(p)}[t, t^{-1}])^*$. Since $\hat{\mathbf{Z}}_{(p)}$ is an integral domain, $\hat{\mathbf{Z}}_{(p)}[t, t^{-1}]$ has only trivial units and hence $\det(Y) = ut^r$ for some r and some $u \in \hat{\mathbf{Z}}^*_{(p)}$. Since $\det(\mathcal{Z}(n)) = \det(\mathcal{Z}(m)) = 1$, we have $\det(Y) = eh - fg = ad - bc$.

However, since $f \in \hat{\mathbf{Z}}_{(p)}[t, t^{-1}]$, from the above calculation we know that $b \in \hat{\mathbf{Z}}_{(p)}$, $a/p \in \hat{\mathbf{Z}}_{(p)}$, $d/p \in \hat{\mathbf{Z}}_{(p)}$ and $c/p^2 \in \hat{\mathbf{Z}}_{(p)}$. Write $a = p\alpha$, $c = p^2\gamma$ and $d = p\delta$ for $\alpha, \gamma, \delta \in \hat{\mathbf{Z}}_{(p)}$; then

$$\det(Y) = p^2\alpha\delta - p^2 b\gamma,$$

and therefore p divides $\det(Y)$. This is a contradiction since p is not a unit in $\hat{\mathbf{Z}}_{(p)}$.

□

Chapter 4

Stably free cancellation

A ring Λ is said to have stably free cancellation (abbreviated to SFC) when every stably free module over Λ is actually free. All principal ideal domains have SFC, as do all local rings. It is a famous theorem of Quillen-Suslin that a polynomial ring over a field has SFC (see Lam's excellent account [26]). It is possible for a ring to have SFC yet fail the IBN condition: Cohn [10] has constructed a ring Λ over which every projective module is isomorphic to Λ. Rings with SFC are sometimes called Hermite rings; however this term has been used in different senses by some authors (see [26], p.37) and so we avoid it.

4.1 Cancellation over free algebras

For any ring Λ denote by $\mathrm{rad}(\Lambda)$ the Jacobson radical of Λ (so that $\mathrm{rad}(\Lambda)$ is the intersection of all right ideals in Λ). Recall that an ideal $\mathbf{m}$ of Λ is said to be *radical* when $\mathbf{m} \subset \mathrm{rad}(\Lambda)$. The following is known as Nakayama's lemma:

Proposition 4.1.1. Let M be a finitely generated Λ-module, $N \subset M$ be a submodule and let $\mathbf{m}$ be a radical ideal in Λ. Then

$$M = N + M \cdot \mathbf{m} \implies M = N.$$

Proposition (Bass [2] Prop. 2.12) 4.1.2. Let $\mathbf{m}$ be a two sided radical ideal in Λ and let P, Q be finitely generated projective modules over Λ. Set $\bar{\Lambda} = \Lambda/\mathbf{m}$

and write $\bar{M} = M \otimes_\Lambda (\Lambda/\mathbf{m}) \cong M/M \cdot \mathbf{m}$ for Λ-modules M. If $\bar{f} : \bar{P} \to \bar{Q}$ is an isomorphism, then there exists a Λ-isomorphism $f : P \to Q$.

Proof. Let $\bar{f} : \bar{P} \to \bar{Q}$ be an isomorphism. Then we may choose $f : P \to Q$ such that the following commutes

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow{[\,]} & & \downarrow{[\,]} \\ \bar{P} & \xrightarrow{\bar{f}} & \bar{Q} \end{array}$$

This is because $[\,] : Q \to \bar{Q}$ is surjective and P is projective. Now, $\bar{f}[p] = [q] \iff [f(p)] = [q] \iff f(p) - q \in Q \cdot \mathbf{m}$. Therefore

$$\mathrm{Im}(\bar{f}) = (\mathrm{Im}(f) + Q \cdot \mathbf{m})/Q \cdot \mathbf{m}$$

and since $\bar{f}$ is surjective, $\mathrm{Im}(\bar{f}) = Q/Q \cdot \mathbf{m}$, and therefore $\mathrm{Im}(f) + Q \cdot \mathbf{m} = Q \implies \mathrm{Im}(f) = Q$ by (4.1.1). Since Q is projective, we have $P \cong \ker(f) \oplus Q$, and hence $\ker(f)$ is finitely generated. However, $\overline{\ker(f)} = 0$ as $\bar{f}$ is injective, and so applying (4.1.1) with $M = \ker(f)$, $N = 0$ shows that $\ker(f) = 0$. □

Corollary 4.1.3. Let $\mathbf{m}$ be a two sided radical ideal in Λ. Then

$$\Lambda/\mathbf{m} \text{ has SFC} \implies \Lambda \text{ has SFC.}$$

Proof. Suppose that $P \oplus \Lambda^m \cong \Lambda^n$; then $\bar{P} \oplus \bar{\Lambda}^m \cong \bar{\Lambda}^n$ and so $\bar{P} \cong \bar{\Lambda}^{n-m}$ by hypothesis. Therefore by (4.1.2) $P \cong \Lambda^{n-m}$. □

The following is a result of Dicks-Sontag [14]:

Theorem 4.1.4. If D is a division ring then $D[F_m]$ has SFC for every $m \geq 1$.

Proposition 4.1.5. Let Λ be a ring with SFC. Then the full matrix ring $M_n(\Lambda)$ has SFC for each n.

Proof. If M is a Λ-module define

$$R_n(\Lambda) = \{(m_1, \ldots, m_n) \mid m_i \in M\} \;\; ; \;\; C_n(\Lambda) = \{ \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \mid m_i \in M\}.$$

Then $R_n(M)$ becomes a Λ-$M_n(\Lambda)$ bimodule and $C_n(\Lambda)$ becomes a $M_n(\Lambda)$-Λ bimodule via matrix multiplication. Define functors

$$F : \mathcal{M}\mathrm{od}_\Lambda \to \mathcal{M}\mathrm{od}_{M_n(\Lambda)} \; ; \; F(M) = R_n(M)$$
$$G : \mathcal{M}\mathrm{od}_{M_n(\Lambda)} \to \mathcal{M}\mathrm{od}_\Lambda \; ; \; G(N) = N \otimes_{M_n(\Lambda)} C_n(\Lambda).$$

There is a natural isomorphism $M \cong G(F(M)) = R_n(M) \otimes C_n(\Lambda)$ given by

$$m \mapsto (m, \ldots, m) \otimes \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Therefore F is an equivalence of categories. If S is stably free over $M_n(\Lambda)$ then $G(S)$ is stably free, and hence free, over Λ. Therefore $S \cong F(G(S)) \cong F(\Lambda^r) \cong M_n(\Lambda)^r$ for some r.

□

Wedderburn's theorem now shows that $\Lambda[F_m]$ has SFC for any right semisimple ring Λ. (Note that a product $\Lambda = \Lambda_1 \times \Lambda_2$ has SFC if and only if both Λ_1 and Λ_2 have SFC.) Now suppose that Λ is a right artinian ring. The canonical mapping $\phi : \Lambda \to \Lambda/\mathrm{rad}(\Lambda)$ induces a surjective ring homomorphism $\phi_* : \Lambda[F_m] \to \Lambda/\mathrm{rad}(\Lambda)[F_m]$ in which $\ker(\phi_*) = \mathrm{rad}(\Lambda)[F_m]$. In general $\mathrm{rad}(\Lambda)[F_m]$ is not a radical ideal in $\Lambda[F_m]$; however it is if $\mathrm{rad}(\Lambda)$ is nilpotent. This may be seen by using the following characterization of $\mathrm{rad}(\Lambda)$: $\mathrm{rad}(\Lambda)$ is the set of all elements $x \in \Lambda$ such that, for all $y \in \Lambda$, $1 - xy$ is a unit in Λ. If $x \in \mathrm{rad}(\Lambda)[F_m]$ and $\mathrm{rad}(\Lambda)$ is nilpotent then it is easy to check that

$$(1 - xy)^{-1} = 1 + xy - (xy)^2 + \ldots \pm (xy)^n$$

for any $y \in \Lambda[F_m]$ and where $(xy)^{n+1} = 0$. Since Λ is right artinian, $\mathrm{rad}(\Lambda)$ is nilpotent (see Lam [24], Theorem 4.12) and hence $\mathrm{rad}(\Lambda)[F_m]$ is a radical ideal in $\Lambda[F_m]$. Applying (4.1.3) with $\mathbf{m} = \mathrm{rad}(\Lambda)[F_m]$ now shows:

Corollary 4.1.6. Let Λ be a right artinian ring. Then $\Lambda[F_m]$ has SFC.

A ring A is said to be a *retract* of a ring B when there exist homomorphisms $i : A \to B$ and $r : B \to A$ such that $r \circ i = \mathrm{Id}_A$. Given a ring homomorphism $f : R \to S$ and an R-module M, we can define an S-module $M \otimes_f S = M \otimes_R S$ by considering S as a left R module via the action $r \cdot s = f(r)s$.

Let M be stably free over A; say $M \oplus A^n \cong A^m$. Then $(M \oplus A^n) \otimes_i B \cong A^m \otimes_i B$ and thus $(M \otimes_i B) \oplus B^n \cong B^m$, since $A \otimes_i B \cong B$. Suppose now that B has SFC; then necessarily $M \otimes_i B \cong B^{n-m}$. Applying the functor $- \otimes_r A$ we have $(M \otimes_i B) \otimes_r A \cong A^{n-m}$. The map

$$\begin{aligned} \psi : (M \otimes_i B) \otimes_r A &\to M \otimes_{r \circ i} A \\ m \otimes b \otimes a &\mapsto m \otimes r(b)a \end{aligned}$$

is an isomorphism; since $r \circ i = \mathrm{Id}_A$ we have $M \otimes_{r \circ i} A \cong M$ and thus $M \cong A^{n-m}$. We have shown:

Proposition 4.1.7. Let A be a retract of a ring B with SFC. Then A also has SFC.

Suppose that S_1 and S_2 are two stably free module of rank n over A; say $S_1 \oplus A^m \cong S_2 \oplus A^m \cong A^{n+m}$ for some m. We obtain two stably free B-modules: $(S_1 \otimes_i B) \oplus B^m \cong (S_2 \otimes_i B) \oplus B^m \cong B^{n+m}$. Suppose that $(S_1 \otimes_i B) \cong (S_2 \otimes_i B)$; then

$$S_1 \cong S_1 \otimes_{r \circ i} A \cong (S_1 \otimes_i B) \otimes_r A \cong (S_2 \otimes_i B) \otimes_r A \cong S_1 \otimes_{r \circ i} A \cong S_2.$$

Therefore we have:

Proposition 4.1.8. Let A be a retract of B. Then for each $n \geq 1$ there is an injective map $i_* : \mathrm{SF}_n(A) \to \mathrm{SF}_n(B)$ given by $i_*(S) = S \otimes B$.

Chapter 5

Weakly Euclidean rings

For any ring Λ, let $M_n(\Lambda)$ denote the ring of $n \times n$ matrices over Λ and let $GL_n(\Lambda)$ denote the ring of invertible $n \times n$ matrices over Λ. If $\epsilon(i,j)$ denotes the matrix whose entries are given by $\epsilon(i,j)_{r,s} = \delta_{ir}\delta_{js}$ then for any $\lambda \in \Lambda$ we may define an elementary invertible matrix $E(i,j;\lambda)$ by

$$E(i,j;\lambda) = I_n + \lambda\epsilon(i,j).$$

$E_n(\Lambda)$ will denote the subgroup of $GL_n(\Lambda)$ generated by the elementary matrices $E(i,j;\lambda)$ $(\lambda \in \Lambda)$. Denote by $D_n(\Lambda)$ the subgroup of $GL_n(\Lambda)$ consisting of all diagonal matrices. Recall that, over a Euclidean domain Λ, any matrix $M \in M_n(\Lambda)$ has a Smith normal form; that is, M may be written as a product $M = E_1 D E_2$, where $D \in D_n(\Lambda)$ and $E_1, E_2 \in E_n(\Lambda)$. Equivalently,

$$M_n(\Lambda) = E_n(\Lambda) \cdot D_n(\Lambda) \cdot E_n(\Lambda).$$

If M is invertible then so is D; let $D = \mathrm{Diag}(\delta_1, \ldots \delta_n)$ for $\delta_1, \ldots \delta_n \in \Lambda^*$. Then

$$E(i,j;\lambda)D = DE(i,j;\delta_i^{-1}\lambda\delta_j).$$

and so $E_n(\Lambda) \cdot \Delta_n(\Lambda) = \Delta_n(\Lambda) \cdot E_n(\Lambda)$, where $\Delta_n(\Lambda) = D_n(\Lambda)^*$. Therefore over a Euclidean domain we have $GL_n(\Lambda) = E_n(\Lambda) \cdot \Delta_n(\Lambda)$. Write $GE_n(\Lambda)$ for the product $E_n(\Lambda) \cdot \Delta_n(\Lambda)$. More generally, we say that a ring Λ is *weakly Euclidean* if $GL_n(\Lambda) = GE_n(\Lambda)$; in other words every invertible matrix over Λ is reducible to a diagonal matrix by means of elementary row and column operations. Weakly Euclidean rings have been extensively studied by Cohn [9]. Our notion of weakly Euclidean rings coincides with Cohn's generalized Euclidean rings; Johnson [20] prefers to reserve the term generalized Euclidean for the wider class of rings for which the equation $M_n(\Lambda) = E_n(\Lambda) \cdot D_n(\Lambda) \cdot E_n(\Lambda)$ holds, and we adopt his terminology here.

5.1 Weakly Euclidean free algebras

The following is due to Cohn [9]:

Theorem 5.1.1. Let k be a (possibly skew) field. Then $k[F_m]$ is weakly Euclidean.

We shall now give a useful lifting criterion due to Johnson [20]. We say that a ring homomorphism $\psi : A \to B$ has the lifting property for the identity when for all $a \in A$, $\psi(a) = 1 \implies a \in A^*$. First we need a lemma:

Lemma 5.1.2. Let $\psi : A \to B$ be a surjective ring homomorphism with the lifting property for the identity. If $X \in GL_n(A)$ is such that $\psi(X) = I_n$ then $X \in GE_n(A)$.

Proof. The proof is by induction on n. First suppose that $X \in GL_2(A)$ is such that $\psi(X) = I_2$. Then clearly $\psi(X_{12}) = \psi(X_{21}) = 0$ and $\psi(X_{22}) = 1$; therefore by hypothesis $X_{22} \in A^*$. It is clear that

$$E(1,2; -X_{12}X_{22}^{-1})XE(2,1; -X_{22}^{-1}X_{21}) = \begin{pmatrix} Y & 0 \\ 0 & X_{22} \end{pmatrix}$$

where $Y \in A$. Since X is invertible, $Y \in A^*$ and so $X \in GE_2(A)$. Now suppose that, for each $Y \in GL_{n-1}(A)$ such that $\psi(Y) = I_{n-1}$, we have $Y \in GE_{n-1}(A)$. Let $X \in GL_n(A)$ be such that $\psi(X) = I_n$. Then $X_{nn} \in A^*$ and we have

$$\prod_{i=1}^{n-1} E(i,n; -X_{in}X_{nn}^{-1}) \cdot X \cdot \prod_{i=1}^{n-1} E(n,i; -X_{nn}^{-1}X_{ni}) = \begin{pmatrix} X' & 0 \\ 0 & X_{nn} \end{pmatrix}$$

for some $X' \in GL_n(A)$. Applying ψ we have $\psi(X') = I_{n-1}$ since $\psi(X) = I_n$ and $\psi(E(i,n; -X_{in}X_{nn}^{-1})) = \psi(E(n,i; -X_{nn}^{-1}X_{ni})) = 0$ for each i. Therefore by induction hypothesis $X' \in GE_{n-1}(A)$ and hence $X \in GE_n(A)$. □

Proposition 5.1.3. Let $\psi : A \to B$ be a surjective ring homomorphism with the lifting property for the identity. If B is weakly Euclidean then so is A.

Proof. For any $X \in GL_n(A)$ we may write $\psi(X) = ED$ for some $E \in E_n(B)$ and some $D \in \Delta_n(B)$. Since ψ is surjective we may choose $\hat{E} \in E_n(A)$ and $\hat{D} \in D_n(A)$ such that $\psi(\hat{E}) = E^{-1}$ and $\psi(\hat{D}) = D^{-1}$. Therefore $\psi(\hat{D}\hat{E}X) = I_n$ and so by (5.1.2) $\hat{D}\hat{E}X \in GE_n(\Lambda)$. Since $\hat{E}$ and X are invertible so is $\hat{D}$; therefore $D \in \Delta(\Lambda)$ and $X \in GE_n(A)$. □

Let Λ be weakly Euclidean and consider $R = M_n(\Lambda)$. Any $m \times m$ matrix M over R may be considered as a matrix over Λ. Writing $M = ED$ for some $E \in E_{nm}(\Lambda)$ and $D \in D_{nm}(\Lambda)$ and now viewing E and D as matrices in $M_n(R)$ it is clear that D is diagonal as matrix over R and that E is a product of elementary matrices over R. Thus:

Proposition 5.1.4. Let Λ be weakly Euclidean. Then the full matrix ring $M_n(\Lambda)$ is weakly Euclidean.

Since weakly Euclidean rings are closed under products, then by Wedderburn's theorem, (5.1.1) and (5.1.4) we have that $\Lambda[F_m]$ is weakly Euclidean whenever Λ is right semi-simple.

Proposition 5.1.5. If Λ is a ring such that Λ/I is weakly Euclidean for some two-sided radical ideal I, then Λ is also weakly Euclidean.

Proof. By (5.1.3), we must show that the canonical mapping $f : \Lambda \to \Lambda/I$ has the lifting property for the identity. Suppose that $f(\lambda) = [\lambda] = [1]$. Then $1 - \lambda \in I \subset \text{rad}(\Lambda)$. Since $\text{rad}(\Lambda)$ consists of those elements x such that, for all $y \in \Lambda$, $1 - xy$ is a unit, we have $1 - (1 - \lambda) = \lambda$ is a unit in Λ. □

As noted above, $\text{rad}(\Lambda)[F_m]$ is radical in $\Lambda[F_m]$ whenever Λ is right artinian; applying (5.1.5) with $I = \text{rad}(\Lambda)[F_m]$ gives:

Corollary 5.1.6. Let Λ be a right artinian ring. Then $\Lambda[F_m]$ is weakly Euclidean.

5.2 Lifting stably free modules

Let $A = A_1 \times A_2$ be a direct product of rings. The relationship between stably free modules over the factors and those over A is straightforward: if S is stably free of rank n over A_1 then the A-module $S \times A_2^n$ (with the obvious A-action) is stably free over A. For a fibre product $A = A_+ \times_{A_0} A_-$ the relationship is less clear; consider an E-surjective fibre square

$$\mathcal{A} = \left\{ \begin{array}{ccc} A & \xrightarrow{\pi_-} & A_- \\ \downarrow{\scriptstyle \pi_+} & & \downarrow{\scriptstyle \psi_-} \\ A_+ & \xrightarrow{\psi_+} & A_0 \end{array} \right.$$

Any ring homomorphism $\phi : \Lambda_1 \to \Lambda_2$ induces a mapping $\phi_* : \mathrm{SF}_n(\Lambda_1) \to \mathrm{SF}_n(\Lambda_2)$ given by $\phi_*(S) = S \otimes_\phi \Lambda_2$. We would like to be able to assert that $(\pi_\sigma)_* : \mathrm{SF}_n(A) \to \mathrm{SF}_n(A_\sigma)$ is surjective for $\sigma = +, -$ but without some further conditions on A_0 this does not happen.

Let S_+ be a stably free module over A_+ of rank n; then $S_+ \otimes A_0$ is stably free of rank n over A_0. Now suppose that A_0 has SFC, so that $S_+ \otimes A_0 \cong A_0^n$. Then there exists an isomorphism

$$\alpha : S_+ \otimes A_0 \to A_-^n \otimes A_0$$

and so we may form the A-module $P = \langle S_+, A_-^{n+m}; \alpha \rangle$. Choose an isomorphism $\beta : S_+ \oplus A_+^m \to A_+^{n+m}$ for some m and define

$$f : P \oplus A^m = \langle S_+ \oplus A_+^m, A_-^{n+m}; \alpha \oplus I_m \rangle \to \langle A_+^{n+m}, A_-^{n+m}; (\alpha \oplus I_m) \circ (\beta^{-1} \otimes \mathrm{Id}) \rangle$$

by $f(m_+, m_-) = (\beta(m_+), m_-)$. Then f is an isomorphism of A-modules. Therefore, up to isomorphism, $P \oplus A^m$ is locally free for some m and since $\mathcal{A}$ is E-surjective $P \oplus A^m$ is a locally free projective. However, without further assumptions on A_0 we cannot guarantee that $P \oplus A^m$ is stably free; that is, we cannot guarantee that

$$[(\alpha \oplus I_m) \circ (\beta^{-1} \otimes \mathrm{Id}) \oplus I_k] = [I_{n+m+k}] \in \overline{GL}_{n+m+k}(\mathcal{A}) \tag{5.1}$$

for some k. The following is due to Johnson [20]:

Theorem 5.2.1. Let $\mathcal{A}$ be an E-surjective fibre square as above. If A_0 has SFC and is weakly Euclidean then the induced map $\pi_+ \times \pi_- : \mathrm{SF}_1(A) \to \mathrm{SF}_1(A_+) \times \mathrm{SF}_1(A_-)$ is surjective.

The condition that A_0 be weakly Euclidean is not necessary however. Since $\mathcal{A}$ is E-surjective we have $GL(A_-)\backslash GL(A_0)/GL(A_+) = K_1(A_-)\backslash K_1(A_0)/K_1(A_+)$ and so stabilizing (5.1) gives:

Proposition 5.2.2. Let $\mathcal{A}$ be an E-surjective fibre square as above. If A_0 has SFC and $K_1(A_-)\backslash K_1(A_0)/K_1(A_+)$ consists of a single element then for each n the induced map $\pi_+ \times \pi_- : \mathrm{SF}_n(A) \to \mathrm{SF}_n(A_+) \times \mathrm{SF}_n(A_-)$ is surjective.

Chapter 6

Orders and algebras

Let R be a commutative integral domain of characteristic zero, with field of fractions k. An R-*order* is an R-algebra whose underlying module is finitely generated and free. We may embed any R-order Λ in $k \otimes \Lambda$ via

$$\begin{aligned} \Lambda &\to k \otimes \Lambda \\ \lambda &\mapsto 1 \otimes \lambda \end{aligned}$$

and thus speak of Λ as an R-order in the k-algebra $k \otimes \Lambda$. To give some examples:

(i) The matrix ring $M_n(R)$ is an R-order in $M_n(k)$.

(ii) Let k be an algebraic number field (i.e. a finite extension of $\mathbf{Q}$). Then the ring of algebraic integers in k is a $\mathbf{Z}$-order in k.

(iii) Let G be a finite group. Then $\mathbf{Z}[G]$ is a $\mathbf{Z}$-order in $\mathbf{Q}[G]$.

An R-order Λ in A is said to be *maximal* if it is not properly contained in another R-order in A. Maximal orders always exist, but may not be unique. A ring Λ is said to be left (resp. right) hereditary if every left (resp. right) ideal of Λ is projective. If Λ is Noetherian then Λ is left hereditary if and only if it is right hereditary (see [1]). When this is the case we simply say that Λ is hereditary. We note the following result, which will be useful in a later chapter (see [33] Theorem 21.4):

Proposition 6.0.1. Let Λ be a maximal R-order in A. Then Λ is hereditary.

As a consequence, when Λ is maximal all submodules of free Λ-modules are projective (see [33] Corollary 10.7), and hence Λ has global dimension 1. An example of a maximal order in the real quaternions $\mathbf{H}$ is given by the Hurwitz quaternions

$$H = \{a + bi + cj + dk \in \mathbf{H} \mid a, b, c, d \in \mathbf{Z} \text{ or } a, b, c, d \in \mathbf{Z} + \frac{1}{2}\} \subset \mathbf{H}.$$

This shows that the set ordinary integral quaternions (or Lipschitz quaternions) is not a maximal order in $\mathbf{H}$.

6.1 The discriminant

Given an R-order Λ, and $x \in \Lambda$, denote by $\hat{x}$ the homomorphism of right modules $\hat{x} : \Lambda \to \Lambda$ given by

$$\hat{x}(y) = xy.$$

There is a symmetric bilinear form $\beta_\Lambda : \Lambda \times \Lambda \to R$ given by $\beta_\Lambda(x, y) = \mathrm{Tr}(\hat{x}\hat{y})$. Say that Λ is *nondegenerate* as an R-algebra when β_Λ is nondegenerate as a bilinear form; that is, when the map

$$\beta_\Lambda^* : \Lambda \to \Lambda^*; \quad \beta_\Lambda^*(x)(y) = \beta_\Lambda(x, y)$$

is injective, where $\Lambda^* = \mathrm{Hom}_R(\Lambda, R)$ is the R-dual of Λ.

Proposition ([20] Corollary 5.5) 6.1.1. Let R be a commutative integral domain with field of fractions k, and let Λ be an R-order in $A = k \otimes \Lambda$. Then

$$\Lambda \text{ is nondegenerate} \iff A \text{ is semisimple.}$$

Suppose that $\{\epsilon_1, \ldots, \epsilon_n\}$ is an R-basis for Λ. The *discriminant* $\mathrm{Disc}(\epsilon_1, \ldots, \epsilon_n)$ is defined by

$$\mathrm{Disc}(\epsilon_1, \ldots, \epsilon_n) = \det((\beta_{i,j})_{1 \le i,j \le n})$$

where $\beta_{i,j} = \beta_\Lambda(\epsilon_i, \epsilon_j)$. If $\{e_1, \ldots, e_n\}$ is another R-basis for Λ, and if $A =$

$(\alpha_{i,j})_{1\leq i,j\leq n}$ is the change of basis matrix

$$\epsilon_i = \sum_{j=1}^{n} \alpha_{i,j} e_j$$

then $\text{Disc}(\Lambda) = \det(A)^2 \text{Disc}(e_1, \ldots, e_n)$. It is now easy to see:

Proposition 6.1.2. The following conditions on an R-order Λ are equivalent:

(i) Λ is nondegenerate;

(ii) $\text{Disc}(\epsilon_1, \ldots, \epsilon_n) \neq 0$ for some R-basis $\{\epsilon_1, \ldots, \epsilon_n\}$ of Λ;

(iii) $\text{Disc}(\epsilon_1, \ldots, \epsilon_n) \neq 0$ for all R-bases $\{\epsilon_1, \ldots, \epsilon_n\}$ of Λ.

This now gives an invariant of R-orders as follows; if $\{\epsilon_1, \ldots, \epsilon_n\}$ is an R-basis for an R-order Λ, we define the *discriminant* $\text{Disc}(\Lambda)$ to be the image of $\text{Disc}(\epsilon_1, \ldots, \epsilon_n)$ in the multiplicative monoid $R/(R^*)^2$. Since $\mathbf{Z}^* = \{1, -1\}$, the discriminant of a $\mathbf{Z}$-order is a well-defined integer.

Now suppose that Λ is an R-order in a simple algebra $A = k \otimes \Lambda$ whose centre is k (the field of fractions of R). Then we may make the identification $A \otimes_k \bar{k} \cong M_n(\bar{k})$, where $\bar{k}$ is the algebraic closure of k.

Proposition 6.1.3. Let $x \in A$. Then $\text{Tr}(\hat{x}) = \text{Tr}(\widehat{x \otimes 1}) = n\text{tr}(x \otimes 1)$, where $\text{tr}(x \otimes 1)$ refers to the trace of the element $x \otimes 1 \in M_n(\bar{k})$.

Proof. Let $\{\epsilon(1,1), \ldots, \epsilon(1,n), \epsilon(2,1), \ldots, \epsilon(n,n)\}$ be the standard basis for $M_n(\bar{k})$. If $x \otimes 1 = (x_{i,j})_{1\leq i,j\leq n}$, then we have

$$\widehat{x \otimes 1}(\epsilon(i,j)) = x_{1,i}\epsilon(1,j) + x_{2,i}\epsilon(2,j) + \ldots + x_{n,i}\epsilon(n,j)$$

and consequently the coefficient of the diagonal element (that is, the coefficient of $\epsilon(i,j)$) is $x_{i,i}$. Therefore there are n diagonal entries of $\widehat{x \otimes 1}$ equal to $x_{i,i}$, and hence $\text{Tr}(\widehat{x \otimes 1}) = nx_{1,1} + \ldots + nx_{n,n} = n\text{tr}(x \otimes 1)$ □

We may now define the *reduced trace*, $\text{tr}(\Lambda) \in R/(R^*)^2$ of an R-order in a central simple algebra A as follows: let $\{\epsilon_1, \ldots, \epsilon_n\}$ be a basis for Λ over R and put

$$\text{disc}(\Lambda) = \det((b_{i,j})_{1\leq i,j\leq n}),$$

where $b_{i,j} = \mathrm{tr}(\epsilon_i\epsilon_j\otimes 1)$. Then above proposition shows that $\mathrm{Disc}(\Lambda) = n^{n^2}\mathrm{disc}(\Lambda)$, where $\Lambda = R^n$ as an R-module.

Proposition 6.1.4. Suppose that Λ and Γ are $\mathbf{Z}$-orders in the same $\mathbf{Q}$-algebra A such that $\Lambda \subset \Gamma$. Then $\mathrm{Disc}(\Gamma)$ divides $\mathrm{Disc}(\Lambda)$, and $\Lambda = \Gamma$ if and only if $\mathrm{Disc}(\Lambda) = \mathrm{Disc}(\Gamma)$. If A is central simple, the same statements hold for the reduced discriminant disc.

Proof. Let $\{\epsilon_1 \dots \epsilon_n\}$ be a $\mathbf{Z}$-basis for Λ and let $\{\phi_1 \dots, \phi_n\}$ be a $\mathbf{Z}$-basis for Γ. Write $\epsilon_i = \sum \alpha_{ij}\phi_j$ for $\alpha_{ij} \in \mathbf{Z}$; then

$$\mathrm{Disc}(\Lambda) = \mathrm{Disc}(\epsilon_1, \dots, \epsilon_n) = \det(A)^2\mathrm{Disc}(\phi_1, \dots, \phi_n) = \det(A)^2\mathrm{Disc}(\Gamma)$$

and so $\mathrm{Disc}(\Gamma) \mid \mathrm{Disc}(\Lambda)$. Clearly $\mathrm{Disc}(\Gamma) = \mathrm{Disc}(\Lambda) \iff \det(A) = \pm 1 \iff \{\epsilon_1, \dots, \epsilon_n\}$ is also a basis for Γ. □

6.2 Quaternion algebras

Let R be a (commutative) integral domain with characteristic $\neq 2$: given two non-zero elements $a, b \in R$, the *quaternion algebra determined by a and b*, $\left(\frac{a,b}{R}\right)$, is the free R-algebra on two generators i, j modulo the defining relations

$$i^2 = a,\ j^2 = b,\ ij = -ji.$$

For $k := ij$ we have $k^2 = (ij)(ij) = -i^2j^2 = -ab$. For example, Hamilton's quaternions are given by $\mathbf{H} = \left(\frac{-1,-1}{\mathbf{R}}\right)$.

Proposition 6.2.1. Let R be a commutative ring in which $2a$ is invertible. Suppose that there exists $z, w \in R$ such that $z^2 - bw^2 = a$; then there is an isomorphism of R algebras

$$\left(\frac{a,b}{R}\right) \cong M_2(R)$$

Proof. It is easy to check that the R-linear map $f : \left(\frac{a,b}{R}\right) \to M_2(R)$ defined by

$$f(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ f(i) = \begin{pmatrix} z & w \\ -bw & -z \end{pmatrix},\ f(j) = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$$

is a ring homomorphism. To see that f is bijective when $2a$ is invertible note that, if $z^2 - bw^2 = a$ then

$$a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} z & w \\ -bw & -z \end{pmatrix} + w\begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}\begin{pmatrix} z & w \\ -bw & -z \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix}.$$

□

Quaternion algebras over a field are either division algebras or 2×2 matrix algebras (see [25], p.58):

Proposition 6.2.2. Let K be a field. Then $\left(\frac{a,b}{K}\right)$ is a division algebra if and only if the equation $ax^2 + by^2 = 1$ has no solution in K. If $\left(\frac{a,b}{K}\right)$ is not a division algebra then it is isomorphic to $M_2(K)$.

In a later chapter we shall need to consider the quaternion algebra $\left(\frac{-1,-3}{\mathbf{Q}}\right)$, which is a division algebra since the equation $-x^2 - 3y^2 = 1$ obviously has no solution in $\mathbf{Q}$.

Define a *bi-pointed ring* to be an ordered triple (R, a, b) where R is a ring and a, b are two non-zero elements of R. A morphism of bi-pointed rings $f : (R, a, b) \to (S, c, d)$ is just a ring homomorphism $f : R \to S$ such that $f(a) = c$, $f(b) = d$. The quaternion algebra construction gives a functor from the category of bi-pointed rings (R, a, b) with R a commutative integral domain with characteristic $\neq 2$ to itself. It is clear that this functor is exact when considered as a functor on the underlying additive groups. Thus by (3.1.7) we have:

Proposition 6.2.3. Suppose that

$$\begin{array}{ccc} A & \xrightarrow{\pi_-} & A_- \\ \downarrow{\scriptstyle \pi_+} & & \downarrow{\scriptstyle \psi_-} \\ A_+ & \xrightarrow{\psi_+} & A_0 \end{array}$$

is a fibre square with A, A_+, A_- and A_0 all commutative integral domains of characteristic $\neq 2$. Denote $a_\sigma = \pi_\sigma(a)$, $b_\sigma = \pi(b_\sigma)$ for $\sigma = +, -$ and $a_0 = \psi_+(\pi_+(a))$, $b_0 = \psi_+(\pi_+(b))$. If $a, b, a_+, b_+, a_-, b_+, a_0, b_0 \neq 0$ then the following is also a fibre square

$$\begin{array}{ccc} \left(\frac{a,b}{A}\right) & \xrightarrow{(\pi_-)_*} & \left(\frac{a_-,b_-}{A_-}\right) \\ \downarrow{\scriptstyle (\pi_+)_*} & & \downarrow{\scriptstyle (\psi_-)_*} \\ \left(\frac{a_+,b_+}{A_+}\right) & \xrightarrow{(\psi_+)_*} & \left(\frac{a_0,b_0}{A_0}\right) \end{array}$$

6.3 Cyclic algebras

Let R be a commutative ring and suppose that $\theta : R \to R$ satisfies $\theta^n = Id$ for some n. Choose an element $a \in R$ such that $\theta(a) = a$. We define the *cyclic algebra* $\mathcal{C}_n(R, \theta, a)$ to be the free two sided R-module of rank n with basis $\{1, y, \dots, y^{n-1}\}$ and multiplication determined by the relations

$$y^n = a, \quad yr = \theta(r)y \quad \text{for all } r \in R.$$

Some group rings can occur as cyclic algebras over group rings of a normal subgroup. For example, the group ring of the dihedral group of order 6 may be written $\mathbf{Z}[D_6] = \mathcal{C}_2(\mathbf{Z}[C_3], \theta, 1)$, where θ is induced by the non-trivial element of $\mathrm{Aut}(C_3)$.

Define a *pointed n-ring* to be a triple (R, θ, a) where $\theta : R \to R$ satisfies $\theta^n = Id$ and $a \in R$ is such that $\theta(a) = a$. A morphism of pointed n-rings $f : (R, \theta, a) \to (S, \psi, b)$ is a ring homomorphism $f : R \to S$ such that $f(a) = b$ and $f \circ \theta = \psi \circ f$. Then the cyclic algebra construction defines a functor from the category of commutative pointed n-rings to the category of rings. This functor is clearly exact when considered as a functor on the category of underlying additive groups. Thus by (3.1.7) we have:

Proposition 6.3.1. Suppose that

$$\begin{array}{ccc} (A, \theta, a) & \xrightarrow{\pi_-} & (A_-, \theta_-, a_-) \\ \downarrow{\scriptstyle \pi_+} & & \downarrow{\scriptstyle \psi_-} \\ (A_+, \theta_+, a_+) & \xrightarrow{\psi_+} & (A_0, \theta_0, a_0) \end{array}$$

is a commutative square of pointed n-rings such that $A = A_+ \times_{A_0} A_-$. Then

$$\begin{array}{ccc} \mathcal{C}_n(A,\theta,a) & \xrightarrow{(\pi_-)_*} & \mathcal{C}_n(A_-,\theta_-,a_-) \\ \downarrow{\scriptstyle (\pi_+)_*} & & \downarrow{\scriptstyle (\psi_-)_*} \\ \mathcal{C}_n(A_+,\theta_+,a_+) & \xrightarrow{(\psi_+)_*} & \mathcal{C}_n(A_0,\theta_0,a_0) \end{array}$$

is also a fibre square.

6.4 The cyclic algebra $\mathcal{C}$

Let ζ_6 denote the sixth root of unity, and let $\theta : \mathbf{Z}[\zeta_6] \to \mathbf{Z}[\zeta_6]$ denote the homomorphism given by

$$\theta(1) = 1;\ \theta(\zeta_6) = \zeta_6^{-1} = \zeta_6^5.$$

Clearly -1 is a fixed point of θ; let $\mathcal{C}$ denote the cyclic algebra $\mathcal{C}_2(\mathbf{Z}[\zeta_6], \theta, -1)$. We shall embed $\mathcal{C}$ in the quaternion algebra $\left(\frac{-1,-3}{\mathbf{Q}}\right)$. The cyclic algebra $\mathcal{C}$ has two generators ζ_6, y with defining relations

$$\zeta_6^2 - \zeta_6 + 1 = 0;\ y^2 = -1;\ y\zeta_6 = \zeta_6^{-1}y = -\zeta_6^2 y.$$

Define $\psi : \mathcal{C} \to \left(\frac{-1,-3}{\mathbf{Q}}\right)$ by $\psi(y) = i$, $\psi(\zeta_6) = (j+1)/2$. It is easy to check that

$$\left(\frac{j+1}{2}\right)^2 - \left(\frac{j+1}{2}\right) + 1 = 0 \quad \text{and} \quad i\left(\frac{j+1}{2}\right) = -\left(\frac{j+1}{2}\right)^2 i$$

and so ψ is a well-defined ring homomorphism. Clearly we have

$$\mathcal{C} \cong \psi(\mathcal{C}) = \mathbf{Z} + \mathbf{Z}i + \mathbf{Z}\left(\frac{j+1}{2}\right) + \mathbf{Z}i\left(\frac{j+1}{2}\right)$$

which is a $\mathbf{Z}$-order in $\left(\frac{-1,-3}{\mathbf{Q}}\right)$. The following calculation is due to Swan [34].

Proposition 6.4.1. $\mathcal{C}$ is a maximal $\mathbf{Z}$-order in $\left(\frac{-1,-3}{\mathbf{Q}}\right)$.

Proof. Take the basis $E = \{1, i, \frac{1+j}{2}, \frac{i+k}{2}\}$ for $\mathcal{C}$ over $\mathbf{Z}$. Then $\mathbf{Q}(i)$ is a splitting field for $\left(\frac{-1,-3}{\mathbf{Q}}\right)$; we have

$$\mathbf{Q}(i) \otimes \left(\frac{-1,-3}{\mathbf{Q}}\right) \cong M_2(\mathbf{Q}(i))$$

$$1 \otimes 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ 1 \otimes i \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ 1 \otimes j \mapsto \begin{pmatrix} 2i & 1 \\ 1 & -2i \end{pmatrix}$$

Computing the reduced discriminant gives

$$\mathrm{disc}(\mathcal{C}) = \det \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 \end{pmatrix} = -9,$$

Suppose that $\mathcal{C}$ is properly contained in Γ, where Γ is another $\mathbf{Z}$-order in $\left(\frac{-1,-3}{\mathbf{Q}}\right)$. Then we may choose a basis $\{\phi_1, \ldots, \phi_4\}$ for Γ such that $1 = \alpha_1\phi_1$, $i = \alpha_2\phi_2$, $\frac{j+1}{2} = \alpha_3\phi_3$, $\frac{i+k}{2} = \alpha_4\phi_4$ for some $\alpha_1 \ldots, \alpha_4 \in \mathbf{Z}$. By (6.1.4) we have $-9 = \mathrm{disc}(\mathcal{C}) = (\alpha_1\alpha_2\alpha_3\alpha_4)^2\mathrm{disc}(\Gamma)$ and so $\mathrm{disc}(\Gamma) = -1$, and $\alpha_1\alpha_2\alpha_3\alpha_4 = \pm 3$. This shows that $3\mathcal{C} \subset 3\Gamma \subset \mathcal{C}$; let I be the two sided ideal in $\mathcal{C}/3\mathcal{C}$ given by $I = 3\Gamma/3\mathcal{C}$. Then I is a vector space over $\mathbf{F}_3$, and since all but one of the α_is are units, $\dim_{\mathbf{F}_3}(I) = 1$. Moreover, if $\alpha_i = \pm 3$, then $\alpha_i^2 = 9 \implies \alpha_i^2 x \in 3\mathcal{C}$ if $x \in \Gamma$ and so $I^2 = 0$.

In $\mathcal{C}/3\mathcal{C}$ we have $\left(\frac{j+1}{2} + 1\right)^2 = 0$. Therefore, if we put $J = \mathbf{F}_3\left(\frac{j+1}{2} + 1\right) + \mathbf{F}_3 i\left(\frac{j+1}{2} + 1\right)$ it is easy to check that $J^2 = 0$. Also

$$(\mathcal{C}/3\mathcal{C})/J \cong \mathbf{F}_3[i] \cong \mathbf{F}_9.$$

Let $\mathrm{rad}(\Lambda/3\Lambda)$ denote the Jacobson radical of $(\mathcal{C}/3\mathcal{C})$; then since J is nilpotent we have $J \subset \mathrm{rad}(\mathcal{C}/3\mathcal{C})$ and $(\mathcal{C}/3\mathcal{C})/\mathrm{rad}(\mathcal{C}/3\mathcal{C}) \subset (\mathcal{C}/3\mathcal{C})/J \implies J = \mathrm{rad}(\mathcal{C}/3\mathcal{C})$ since $\mathbf{F}_9$ is simple. Therefore $I \subset J$ as I is nilpotent. As a module clearly $J \cong \mathbf{F}_9$; therefore J is simple and we must have $I = J$ or $I = 0$. However, $\dim_{\mathbf{F}_3}(I) = 1$ and so we have a contradiction. Therefore $\mathcal{C}$ is maximal.

□

Chapter 7

Stably free modules over $\mathbf{Z}[G \times F_n]$

Denote by $\mathrm{SF}_1(\Lambda)$ the set of isomorphism classes of stably free modules of rank 1 over a ring Λ. Theorem A states that $\mathrm{SF}_1(\mathbf{Z}[G \times F])$ is infinite when F maps surjectively onto a non-abelian free group and where G is finite nilpotent of non square-free order.

7.1 The prime squared case

Theorem A will be deduced from the following two special cases:

(I) $\mathrm{SF}_1(\mathbf{Z}[C_{p^2} \times F_m])$ is infinite for every prime p and $m \geq 2$;

(II) $\mathrm{SF}_1(\mathbf{Z}[C_p \times C_p \times F_m])$ is infinite for every prime p and $m \geq 2$.

For any positive integer d let $c_d(x)$ denote the d^{th} cyclotomic polynomial. From the factorization $(x^{p^2} - 1) = c_{p^2}(x)c_p(x)c_1(x) = c_{p^2}(x)(x^p - 1)$ we obtain the Milnor square

$$\begin{array}{ccc} \mathbf{Z}[x]/(x^{p^2}-1) & \longrightarrow & \mathbf{Z}[x]/(c_{p^2}(x)) \\ \downarrow & & \downarrow \\ \mathbf{Z}[x]/(x^p-1) & \longrightarrow & \mathbf{Z}[x]/I \end{array}$$

where I is the sum of the ideals $(x^p - 1)$ and $(c_{p^2}(x))$. However, since $c_{p^2}(x) = (x^{p(p-1)} + x^{p(p-2)} + \ldots + x^p + 1)$, we have

$$p = c_{p^2}(x) - (x^{p(p-2)} + 2x^{p(p-3)} + \ldots + (p-2)x^p + (p-1))(x^p - 1),$$

and hence $I = (p, x^p - 1)$. Therefore we may rewrite the above square as

$$\begin{array}{ccc} \mathbf{Z}[x]/(x^{p^2}-1) & \longrightarrow & \mathbf{Z}[x]/(c_{p^2}(x)) \\ \downarrow & & \downarrow \\ \mathbf{Z}[x]/(x^p-1) & \longrightarrow & \mathbf{F}_p[x]/(x^p-1) \end{array}$$

Applying the functor $- \otimes \mathbf{Z}[F_m]$ we obtain another Milnor square:

$$\mathcal{A} = \left\{ \begin{array}{ccc} \mathbf{Z}[C_{p^2} \times F_m] & \longrightarrow & \mathbf{Z}[\zeta_{p^2}][F_m] \\ \downarrow & & \downarrow \psi_- \\ \mathbf{Z}[C_p \times F_m] & \xrightarrow{\psi_+} & \mathbf{F}_p[C_p \times F_m] \end{array} \right.$$

where ζ_{p^2} is a primitive p^2-th root of unity. Since $\mathbf{Z}[\zeta_{p^2}]$ is an integral domain, $\mathbf{Z}[\zeta_{p^2}][F_m]$ has only trivial units; that is

$$\mathbf{Z}[\zeta_{p^2}][F_m]^* = (\mathbf{Z}[\zeta_{p^2}])^* \times F_m$$

Proposition 7.1.1. $\mathbf{Z}[C_p \times F_m]^* = (\mathbf{Z}[C_p])^* \times F_m$

Proof. Consider the projections $\pi_- : \mathbf{Z}[C_p \times F_m]^* \to \mathbf{Z}[\zeta_p][F_m]^*$ and $\pi_+ : \mathbf{Z}[C_p \times F_m]^* \to \mathbf{Z}[F_m]^*$ given by $\pi_-(x) = \zeta_p$ and $\pi_+(x) = 1$. Let $u \in \mathbf{Z}[C_p \times F_m]^*$. Then $\pi_+(u) \in \mathbf{Z}[F_m]^*$ which has only trivial units; thus

$$u = aw \; + \sum_{g \in F_m - \{w\}} a_g g$$

where $a(1) = \pm 1$, $w \in F_m$ and $a_g \in \mathbf{Z}[C_p] = \mathbf{Z}[x]/(x^p - 1)$. Each a_g is divisible by $(x-1)$ since $a_g \in \ker(\pi_+)$. Now

$$\pi_-(u) = a(\zeta_p)w \; + \sum_{g \in F_m - \{w\}} a_g(\zeta_p) g$$

and since $\mathbf{Z}[\zeta_p][F_m]$ has only trivial units we must have $a_g(\zeta_p) = 0$. (Note that we cannot have $a(\zeta_p) = 0$, for then $a(x) = (1 + \ldots + x^{p-1})b(x)$, and $a(1) = \pm 1 \implies b(1) = \pm 1/p$, contradicting the fact that $a(x) \in \mathbf{Z}[x]$.) Therefore both $(1 + x + \ldots + x^{p-1})$ and $(x - 1)$ divide each a_g and so each $a_g = 0$. ☐

Proposition 7.1.2. If $m \geq 2$ then

$$X = \mathbf{Z}[\zeta_{p^2}][F_m]^* \backslash [\mathbf{F}_p[C_p \times F_m]^*, \mathbf{F}_p[C_p \times F_m]^*]/\mathbf{Z}[C_p \times F_m]^*$$

is infinite.

Proof. Define $y = (1-x) \in \mathbf{F}_p[C_p]^*$; then $y^p = 0$. Choose two generators t and s of F_m and define

$$\delta_n = (1+yt)s^n(1+yt)^{-1}s^{-n} \in [\mathbf{F}_p[C_p \times F_m]^*, \mathbf{F}_p[C_p \times F_m]^*].$$

We claim that $\{\delta_n \mid n \in \mathbf{N}\}$ are a set of distinct coset representatives in X. Suppose that $[\delta_n] = [\delta_m]$; then there exists $u \in \mathbf{Z}[\zeta_{p^2}][F_m]^*$ and $u' \in \mathbf{Z}[C_p \times F_m]^*$ such that $\delta_n = \psi_-(u)\delta_m\psi_+(u')$. In fact since u and u' are necessarily trivial units

$$\delta_n = \psi_-(a)\psi_+(b)w\delta_m v$$

for some $a \in \mathbf{Z}[\zeta_{p^2}]^*$, $b \in \mathbf{Z}[C_p]^*$ and some $w, v \in F_m$. The units of $\mathbf{F}_p[C_p]$ are of the form $c + d$ where $c \in \mathbf{F}_p^*$ and $d \in (y)$, as $\mathbf{F}_p[C_p]$ is a local ring with maximal ideal (y). Therefore

$$(*)\ (1+yt)s^n(1+yt)^{-1}s^{-n} = (c+dy)w(1+yt)s^m(1+yt)^{-1}s^{-m}v$$

Using the fact that

$$(**)\ (1+yt)s^k(1+yt)^{-1}s^{-k} = 1 + y(t - s^kts^{-k}) - y^2(ts^kts^{-k} - s^kt^2s^{-k}) + \dots$$
$$\dots \pm y^{p-1}(ts^kt^{p-2}s^{-k} - s^kt^{p-1}s^{-k})$$

we can expand both sides $(*)$ to obtain

$$1 + e = cwv + dwv + cwfv + dwfv$$

where $1+e$ is given by setting $k = n$ in $(**)$ and $1+f$ is given by setting $k = m$ in $(**)$ (so that $e, f \in (y)$). Every element of $\mathbf{F}_p[C_p \times F_m]$ has a unique representation of the form

$$\sum_{i=0}^{p-1} \alpha_i y^i \qquad (\alpha_i \in \mathbf{F}_p[F_m]).$$

Therefore we may compare coefficients of y in $(*)$. The coefficient of $y^0 = 1$ shows that $1 = cwv \implies c = 1$ and $v = w^{-1}$. Writing $d = \sum_{i=1}^{p-1} d_i y^i$ and comparing coefficients of y gives $d_1 = 0$ and

$$t - s^n t s^{-n} = wtw^{-1} - ws^m t s^{-m} w^{-1}$$

and so we must have

$$t = wtw^{-1} \text{ and } s^n t s^{-n} = ws^m t s^{-m} w^{-1}$$

The first equation shows that $w = 1$ and the second shows that $m = n$. □

(7.1.2) and (3.0.4) together prove (**I**):

Theorem 7.1.3. For every prime p and every $m \geq 2$, $\mathrm{SF}_1(\mathbf{Z}[C_{p^2} \times F_m])$ is infinite.

The proof of (**II**) is very similar. Let A be a ring and consider the Milnor square

$$\begin{array}{ccc} A[x]/(x^p - 1) & \longrightarrow & A[x]/(1 + x + \ldots + x^{p-1}) \\ \downarrow & & \downarrow \\ A & \longrightarrow & A/p \end{array} \tag{7.1}$$

Setting $A = \mathbf{Z}[y]/(y^p - 1)$ we have

$$\begin{array}{ccc} \mathbf{Z}[x,y]/(x^p - 1, y^p - 1) & \longrightarrow & \mathbf{Z}[x,y]/(\Sigma_x, y^p - 1) \\ \downarrow & & \downarrow \\ \mathbf{Z}[y]/(y^p - 1) & \longrightarrow & \mathbf{F}_p[y]/(y^p - 1) \end{array}$$

Making the identifications $\mathbf{Z}[x,y]/(x^p - 1, y^p - 1) = \mathbf{Z}[C_p \times C_p]$, $\mathbf{Z}[x,y]/(\Sigma_x, y^p -$

$1) = \mathbf{Z}[\zeta_p][C_p]$ and $B[y]/(y^p - 1) = B[C_p]$ and tensoring with $\mathbf{Z}[F_m]$ we have

$$\mathcal{B} = \begin{cases} \mathbf{Z}[C_p \times C_p \times F_m] \longrightarrow \mathbf{Z}[\zeta_p][C_p \times F_m] \\ \qquad\downarrow \qquad\qquad\qquad\qquad \downarrow \\ \mathbf{Z}[C_p \times F_m] \longrightarrow \mathbf{F}_p[C_p \times F_m] \end{cases}$$

We first need to show that $\mathbf{Z}[\zeta_p][C_p \times F_m]$ has only trivial units:

Proposition 7.1.4. $\mathbf{Z}[\zeta_p][C_p \times F_m]^* = (\mathbf{Z}[\zeta_p][C_p])^* \times F_m$.

Proof. Consider the Milnor square formed by setting $A = \mathbf{Z}[y]/(1 + y + \ldots + y^p)$ in (7.1), tensoring with $\mathbf{Z}[F_m]$ and then taking unit groups:

$$\begin{array}{ccc} \mathbf{Z}[x,y]/(x^p - 1, \Sigma_y)[F_m]^* & \longrightarrow & \mathbf{Z}[x,y]/(\Sigma_x, \Sigma_y)[F_m]^* \\ \downarrow & & \downarrow \\ \mathbf{Z}[y]/(\Sigma_y)[F_m]^* & \longrightarrow & \mathbf{F}_p[y]/(\Sigma_y)[F_m]^* \end{array}$$

Since both $\mathbf{Z}[x,y]/(\Sigma_x, \Sigma_y)$ and $\mathbf{Z}[y]/(\Sigma_y)$ are integral domains the corresponding corners have only trivial units. A similar proof to that of (7.1.1) now applies. $\square$

Essentially the same proof as that of (7.1.2) proves:

Proposition 7.1.5. $\mathbf{Z}[\zeta_p][C_p \times F_m]^* \backslash [\mathbf{F}_p[C_p \times F_m]^*, \mathbf{F}_p[C_p \times F_m]^*] / \mathbf{Z}[C_p \times F_m]^*$ is infinite.

Together (7.1.5) and (3.0.4) prove (**II**):

Theorem 7.1.6. For every prime p and every $m \geq 2$, $\mathrm{SF}_1(\mathbf{Z}[C_p \times C_p \times F_m])$ is infinite.

7.2 Proof of Theorem A

Let G be a finite group and let H be a normal subgroup of G. By (3.1.4), we may form the Milnor square:

$$\begin{array}{ccc} \mathbf{Z}[G] & \longrightarrow & \mathbf{Z}[G]/(\Sigma_H) \\ \downarrow & & \downarrow \\ \mathbf{Z}[G/H] & \longrightarrow & (\mathbf{Z}/N)[G/H] \end{array}$$

where $\Sigma_H = \sum_{h \in H} h$ and $N = |H|$. Tensoring with $\mathbf{Z}[F_m]$ we have:

$$\begin{array}{ccc} \mathbf{Z}[G \times F_m] & \longrightarrow & \mathbf{Z}[G \times F_m]/(\Sigma_H) \\ \downarrow & & \downarrow \\ \mathbf{Z}[G/H \times F_m] & \longrightarrow & (\mathbf{Z}/N)[G/H \times F_m] \end{array}$$

Now by (4.1.6) and (5.1.6), $(\mathbf{Z}/N)[G/H \times F_m]$ has SFC and is weakly Euclidean. Hence by (5.2.1) the induced map $\mathrm{SF}_1(\mathbf{Z}[G \times F_m]) \to \mathrm{SF}_1(\mathbf{Z}[G/H \times F_m]) \times \mathrm{SF}_1(\mathbf{Z}[G \times F_m]/(\Sigma))$ is surjective and thus:

Proposition 7.2.1. Let G be a finite group with normal subgroup $H \lhd G$. If $\mathrm{SF}_1(\mathbf{Z}[G/H \times F_m])$ is infinite then so is $\mathrm{SF}_1(\mathbf{Z}[G \times F_m])$.

Let G be a finite group of order p^k where p is prime and $k \geq 2$. Then there exists a normal subgroup $H \lhd G$ such that $|H| = p^{k-2}$ (see [17] p.24). Hence either $G/H \cong C_{p^2}$ or $G/H \cong C_p \times C_p$; in either case by (7.1.3), (7.1.6) and (7.2.1) $\mathrm{SF}_1(\mathbf{Z}[G \times F_m])$ is infinite.

Now let G be a finite nilpotent group of non square-free order. Since G is nilpotent, G is the direct product of its Sylow subgroups (see [17], p.24) say $G \cong H_1 \times \ldots \times H_r$. As $|G|$ is non square-free we may choose a prime p such that p^k is the largest power of p dividing $|G|$ and where $k \geq 2$. Therefore at least one of the H_i has order p^k — assume without loss of generality that $|H_1| = p^k$. Then $|G/(H_2 \times \ldots \times H_r)| = p^k$ and so we have:

Theorem 7.2.2. Let G be a finite nilpotent group of non square-free order and let $m \geq 2$. Then $\mathbf{Z}[G \times F_m]$ admits infinitely many isomorphically distinct stably free modules of rank 1.

Now let F be a group and suppose there exists a surjective map $f : F \to F_n$ for some $n \geq 2$. Then if $t_1, \ldots t_n$ generate F_n choose $x_1, \ldots, x_n \in F$ such that $f(x_i) = t_i$ for each $1 \leq i \leq n$. We may define a right inverse $g : F_n \to F$ by $g(t_i) = x_i$ for each i. For any group G we have homomorphisms $f_* : \mathbf{Z}[G \times F] \to \mathbf{Z}[G \times F_m]$ and $g_* : \mathbf{Z}[G \times F_n] \to \mathbf{Z}[G \times F]$ such that $f_* \circ g_* = \mathrm{Id}$, and so $\mathbf{Z}[G \times F_n]$ is a retract of $\mathbf{Z}[G \times F]$. Therefore by (4.1.8) there is an injective mapping $\mathrm{SF}_1(\mathbf{Z}[G \times F_n]) \to \mathrm{SF}_1(\mathbf{Z}[G \times F])$. Together with (7.2.2) this proves Theorem A of the introduction:

Theorem 7.2.3. Let G be a finite nilpotent group of non square-free order, and let F be a group which maps surjectively onto F_2 for some $n \geq 2$. Then $\mathbf{Z}[G \times F]$ admits infinitely many isomorphically distinct stably free modules of rank 1.

Chapter 8

Stably free modules over $\mathbf{Z}[Q(12m) \times C_\infty]$

The question now arises: does the conclusion of Theorem A hold for any finite group of non square-free order? The smallest groups for which this question arises are those of order 12; then A_4 the alternating group on 4 elements, D_6^* the dicyclic group of order 12 and D_{12} the dihedral group of order 12 are all soluble but not nilpotent.

In this chapter we shall show that the conclusion of Theorem A holds for $G = D_6^*$. In fact we shall show something stronger, namely:

> There are infinitely many isomorphically distinct stably free modules of rank 1 over the integral group ring $\mathbf{Z}[D_6^* \times C_\infty]$.

Denote by $Q(4m)$ the group

$$Q(4m) = \langle x, y \mid x^m = y^2,\ yx = x^{-1}y \rangle.$$

Notice that $Q(12) = D_6^*$. In [20], Johnson shows that, for even values of m, there are infinitely many isomorphically distinct stably free modules of rank one over $\mathbf{Z}[Q(4m) \times C_\infty]$. We shall obtain an analogous result for $\mathbf{Z}[Q(4m) \times C_\infty]$ when m is a multiple of 3.

From the factorization $(x^{2m} - 1) = (x^m - 1)(x^m + 1)$ and (3.1.1) we obtain a fibre square:

$$\begin{array}{ccc} \mathbf{Z}[C_{2m}] & \longrightarrow & \mathbf{Z}[x]/(x^m+1) \\ \downarrow & & \downarrow \\ \mathbf{Z}[C_m] & \longrightarrow & \mathbf{F}_2[C_m] \end{array}$$

(since $\mathbf{F}_2[C_m] = \mathbf{Z}[x]/((x^m+1)+(x^m-1))$). The canonical involution θ : $\mathbf{Z}[C_{2m}] \to \mathbf{Z}[C_{2m}]$ given by $\theta(x) = x^{-1}$ induces involutions on each of the corners of the above square (with fixed points equal to images of x^m). So we may apply the cyclic algebra construction to the above fibre square of involuted rings to obtain another fibre square (see (6.3.1)):

$$\begin{array}{ccc} \mathbf{Z}[Q(4m)] & \longrightarrow & \mathcal{C}_2(\mathbf{Z}[x]/(1+x^m), \theta, -1) \\ \downarrow & & \downarrow \\ \mathbf{Z}[D_{2m}] & \longrightarrow & \mathbf{F}_2[D_{2m}] \end{array}$$

Applying the exact functor $- \otimes \mathbf{Z}[t, t^{-1}]$ gives another fibre square:

$$\begin{array}{ccc} \mathbf{Z}[Q(4m) \times C_\infty] & \longrightarrow & \mathcal{C}_2(\mathbf{Z}[x]/(1+x^m), \theta, -1)[t, t^{-1}] \\ \downarrow & & \downarrow \\ \mathbf{Z}[D_{2m} \times C_\infty] & \longrightarrow & \mathbf{F}_2[D_{2m} \times C_\infty] \end{array}$$

We proceed to study stably free modules over $\mathcal{C}_2(\mathbf{Z}[x]/(1+x^m), \theta, -1)[t, t^{-1}]$ in the case $m = 3$. From the factorization $x^3 + 1 = (x+1)(x^2 - x + 1)$ we obtain the following fibre square:

$$\begin{array}{ccc} \mathbf{Z}[x]/(x^3+1) & \longrightarrow & \mathbf{Z}[x]/(x^2-x+1) \\ \downarrow & & \downarrow \\ \mathbf{Z}[x]/(x+1) & \longrightarrow & \mathbf{F}_3[x]/(x+1) \end{array}$$

The involution θ on $R = \mathbf{Z}[x]/(1+x^3)$ ($\theta(x) = x^{-1} = -x^2$ with fixed point $x^3 = -1$) induces involutions on the remaining corners of the above square; applying the cyclic algebra construction to the fibre square of involuted rings gives another fibre square as follows

$$\begin{array}{ccc} \mathcal{C}_2(\mathbf{Z}[x]/(1+x^3), \theta, -1) & \longrightarrow & \mathcal{C}_2(\mathbf{Z}[\zeta_6], \theta, -1) \\ \downarrow & & \downarrow \\ \mathbf{Z}[i] & \longrightarrow & \mathbf{F}_3[i] \end{array}$$

Here we are identifying $\mathbf{Z}[x]/(x^2 - x + 1) = \mathbf{Z}[\zeta_6]$. Writing $\mathcal{C} = \mathcal{C}(\mathbf{Z}[\zeta_6], \theta, -1)$ and tensoring with $\mathbf{Z}[C_\infty]$ now gives another fibre square

$$\begin{array}{ccc} \mathcal{C}_2(\mathbf{Z}[x]/(1+x^3), \theta, -1)[t, t^{-1}] & \longrightarrow & \mathcal{C}[t, t^{-1}] \\ \downarrow & & \downarrow \\ \mathbf{Z}[i][t, t^{-1}] & \longrightarrow & \mathbf{F}_3[i][t, t^{-1}] \end{array}$$

In the next section we shall construct projective modules over a localization of $\mathcal{C}[t, t^{-1}]$.

8.1 Projective modules over $\mathcal{C}_{(p)}[t, t^{-1}]$

Choose an odd prime p and let $\mathcal{C}_{(p)}$ denote the local ring obtained from $\mathcal{C}$ by inverting all primes $q \neq p$. From section 6.4 we known that $\mathcal{C}$ can be expressed as a $\mathbf{Z}$-order in $\left(\frac{-1,-3}{\mathbf{Q}}\right)$:

$$\mathcal{C} = \mathbf{Z} + \mathbf{Z}i + \mathbf{Z}\left(\frac{1+j}{2}\right) + \mathbf{Z}\left(\frac{i+k}{2}\right)$$

Since p is an odd prime, $2^{-1} \in \mathcal{C}_{(p)}$, and so we have

$$\mathcal{C}_{(p)} = \mathbf{Z}_{(p)} + \mathbf{Z}_{(p)}i + \mathbf{Z}_{(p)}j + \mathbf{Z}_{(p)}k = \left(\frac{-1,-3}{\mathbf{Z}_{(p)}}\right)$$

Let $\hat{\mathbf{Z}}_{(p)}$ denote the ring of p-adic integers; then the canonical inclusion $\mathbf{Z}_{(p)} \to \hat{\mathbf{Z}}_{(p)}$ induces an inclusion

$$\mathcal{C}_{(p)} = \left(\frac{-1,-3}{\mathbf{Z}_{(p)}}\right) \to \left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}}\right).$$

Let S be the regular sub-monoid of $\mathcal{C}_{(p)}$ given by $S = \{p^r \mid r \geq 0\}$. Then for all $p^r \in S$ we have

$$\mathcal{C}_{(p)}/(p^r) \cong \left(\frac{-1,-3}{\mathbf{Z}_{(p)}}\right)/(p^r) \cong \left(\frac{-1,-3}{\mathbf{Z}/p^r}\right) \cong \left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}}\right)/(p^r).$$

Therefore by (3.1.5) we obtain a Karoubi square

$$\mathcal{A}_{(p)} = \left\{ \begin{array}{ccc} \mathcal{C}_{(p)} & \longrightarrow & \left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}}\right) \\ \downarrow & & \downarrow \\ \left(\frac{-1,-3}{\mathbf{Q}}\right) & \longrightarrow & \left(\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}}\right) \end{array} \right.$$

where $\hat{\mathbf{Q}}_{(p)}$ is the field of p-adic numbers. Applying the exact functor $-\otimes \mathbf{Z}[t, t^{-1}]$ gives another fibre square:

$$\mathcal{A}_{(p)}[t, t^{-1}] = \left\{ \begin{array}{ccc} \mathcal{C}_{(p)}[t, t^{-1}] & \longrightarrow & \left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}[t,t^{-1}]}\right) \\ \downarrow & & \downarrow \\ \left(\frac{-1,-3}{\mathbf{Q}[t,t^{-1}]}\right) & \longrightarrow & \left(\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}[t,t^{-1}]}\right) \end{array} \right.$$

We shall show that the two p-adic quaternion algebras in the above square are isomorphic to rings 2×2 matrices. By (6.2.1) it suffices to show that the equation $z^2 + w^2 = -3$ has a solution in both $\hat{\mathbf{Z}}_{(p)}[t, t^{-1}]$ and $\hat{\mathbf{Q}}_{(p)}[t, t^{-1}]$. We begin by showing that it has a solution in $\mathbf{Z}/p$.

Proposition 8.1.1. Let k be a finite field. Then any element in k is expressible as a sum of two squares in k.

Proof. Let k be a finite field with $|k| = p^n$ for some prime p. Define $\psi : k^* \to k^*$ by $\psi(x) = x^2$. If $p = 2$ then, for $a, b \in k$, we have $a^2 = b^2 \iff (a-b)^2 = 0 \iff a = b$. Therefore ψ is injective, and hence surjective, so that every element in k is trivially expressible as a sum of itself and zero. If $p \neq 2$ then $\ker(\psi) = \{1, -1\}$ and so $|\mathrm{Im}(\psi)| = (p^n - 1)/2$. Let $m = (p^n - 1)/2 + 1$; then we may choose distinct elements $a_1^2, \ldots, a_m^2 \in \mathrm{Im}(\psi) \cup \{0\} \subset k$ (taking $a_1 = 0$). Let $x \in k$; then the elements $x - a_1^2, \ldots, x - a_m^2$ are obviously distinct, and since $2m > p^n$ we have $x - a_i^2 = a_j^2$ for some i, j and thus $x = a_i^2 + a_j^2$.

□

Proposition 8.1.2. For every prime $p \neq 2, 3$ the equation $z^2 + w^2 = -3$ has a solution in $\hat{\mathbf{Z}}_{(p)}$.

Proof. By (8.1.1) $z^2 + w^2 = -3$ has a solution in $\mathbf{F}_p$. Suppose that there exist $z \in \mathbf{Z}/(p^r)$, $w \in (\mathbf{Z}/(p^r))^*$ such that $z^2 + w^2 = -3$ and consider the canonical mapping $\phi_r : \mathbf{Z}/(p^{r+1}) \to \mathbf{Z}/(p^r)$. Clearly we may choose $\hat{z} \in \mathbf{Z}/(p^{r+1})$ and $w' \in (\mathbf{Z}/(p^{r+1}))^*$ such that $\phi_r(\hat{z}) = z$ and $\phi_r(w') = w$. Therefore

$$\phi_r(\hat{z}^2 + w'^2 + 3) = 0$$

Define $v = \frac{1}{2w'}(\hat{z}^2 + w'^2 + 3)$; since $v \in \ker(\phi_r)$ we have $v^2 = 0$. Now put $\hat{w} = w' - v$. Then

$$\begin{aligned} \hat{z}^2 + \hat{w}^2 + 3 &= \hat{z}^2 + w'^2 - 2w'v + 3 \\ &= \hat{z}^2 + w'^2 - (\hat{z}^2 + w'^2 + 3) + 3 \\ &= 0. \end{aligned}$$

Therefore, inductively, $z^2 + w^2 = -3$ has a solution in $\mathbf{Z}/(p^r)$ for all $r \geq 1$.

Since $\hat{\mathbf{Z}}_{(p)}$ is the inverse limit $\varprojlim(\phi_r)$, it follows that $z^2 + w^2 = -3$ has a solution in $\hat{\mathbf{Z}}_{(p)}$. □

Therefore by (6.2.1):

Corollary 8.1.3. There is an isomorphism $\left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}}\right) \cong M_2(\hat{\mathbf{Z}}_{(p)})$.

Now, since $\mathcal{A}_{(p)}[t, t^{-1}]$ is a Karoubi square, then by (3.2) the locally free modules of rank 1 over $\mathcal{C}_{(p)}$ are in one-to-one correspondence with the double coset

$$\overline{GL}_1(\mathcal{A}_{(p)}[t, t^{-1}]) = \left(\frac{-1, -3}{\hat{\mathbf{Z}}_{(p)}[t, t^{-1}]}\right)^* \backslash \left(\frac{-1, -3}{\hat{\mathbf{Q}}_{(p)}[t, t^{-1}]}\right)^* / \left(\frac{-1, -3}{\mathbf{Q}[t, t^{-1}]}\right)^*.$$

By (3.0.3) and (3.1.6) each locally free module over $\mathcal{A}_{(p)}[t, t^{-1}]$ is projective. We shall show that $\overline{GL}_1(\mathcal{A}_{(p)}[t, t^{-1}])$ is infinite by relating it to Johnson's calculation (3.2.1).

Proposition 8.1.4. When $p \neq 2, 3$ the double coset $\overline{GL}_1(\mathcal{A}_{(p)}[t, t^{-1}])$ is infinite. Hence there are an infinite number of isomorphically distinct rank 1 projective modules over $\mathcal{C}_{(p)}[t, t^{-1}]$.

Proof. Notice that $\left(\frac{-1,-3}{\mathbf{Q}[t,t^{-1}]}\right) \cong \left(\frac{-1,-3}{\mathbf{Q}}\right)[t, t^{-1}]$. Since $-z^2 - 3w^2 = 1$ does not have a solution in $\mathbf{Q}$, then by (6.2.2) $\left(\frac{-1,-3}{\mathbf{Q}}\right)$ is a division ring. Therefore $\left(\frac{-1,-3}{\mathbf{Q}}\right)[t, t^{-1}]$ has only trivial units (that is, the only units are non-zero monomials). However, each power $t^k \in \left(\frac{-1,-3}{\mathbf{Q}[t,t^{-1}]}\right)^*$ commutes with each element of $\left(\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}[t,t^{-1}]}\right)^*$, and hence may be regarded as originating in $\left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}[t,t^{-1}]}\right)^*$. Thus

$$\overline{GL}_1(\mathcal{A}_{(p)}[t, t^{-1}]) = \left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}[t, t^{-1}]}\right)^* \backslash \left(\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}[t, t^{-1}]}\right)^* / \left(\frac{-1,-3}{\mathbf{Q}}\right)^*.$$

By (8.1.2) we may choose $z, w \in \hat{\mathbf{Z}}_{(p)} \subset \hat{\mathbf{Q}}_{(p)}$ such that $z^2 + w^2 = -3$. Hence by (8.1.1) we may choose an isomorphism $f_0 : \left(\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}[t,t^{-1}]}\right) \to M_2(\hat{\mathbf{Q}}_{(p)}[t, t^{-1}])$. Then f_0 induces homomorphisms $f_- : \left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}[t,t^{-1}]}\right) \to M_2(\hat{\mathbf{Z}}_{(p)}[t, t^{-1}])$ and $f_+ : \left(\frac{-1,-3}{\mathbf{Q}}\right) \to M_2(\hat{\mathbf{Q}}_{(p)})$. In (3.2.1) we saw that

$$\overline{GL}_2(\mathcal{T}(p)) = GL_2(\hat{\mathbf{Z}}_{(p)}[t, t^{-1}]) \backslash GL_2(\hat{\mathbf{Q}}_{(p)}[t, t^{-1}]) / GL_2(\hat{\mathbf{Q}}_{(p)})$$

is infinite. Since f_0 is an isomorphism the induced map $f_* : \overline{GL}_1(\mathcal{A}_{(p)}[t, t^{-1}]) \to \overline{GL}_2(\mathcal{T}(p))$ is surjective and so $\overline{GL}_1(\mathcal{A}_{(p)}[t, t^{-1}])$ is also infinite.

□

8.2 Computing $\tilde{K}_0(\mathcal{C}_{(p)}[t, t^{-1}])$

In fact each of the projective modules constructed above are stably free; to see this we must first compute $\tilde{K}_0(\mathcal{C}_{(p)}[t, t^{-1}])$, the reduced projective class group of $\mathcal{C}_{(p)}[t, t^{-1}]$. Recall the following theorem of Grothendieck (see [3]):

Theorem 8.2.1. Let Λ be a left regular ring (i.e. a left Noetherian ring of finite global dimension). Then there is an isomorphism

$$\widetilde{K}_0(\Lambda[t, t^{-1}]) \cong \widetilde{K}_0(\Lambda).$$

Proposition 8.2.2. $\mathcal{C}_{(p)}$ is left regular for every odd prime p.

Proof. Every $\mathbf{Z}$-order is a finitely generated over $\mathbf{Z}$ and hence both left and right Noetherian. Therefore $\mathcal{C}$ is Noetherian and hence so is its localization $\mathcal{C}_{(p)}$. By (6.0.1), $\mathcal{C}$ is hereditary. Suppose that I is a left ideal of $\mathcal{C}_{(p)}$; then since $\mathcal{C}_{(p)}$ is Noetherian, I is finitely generated by (say) $x_1, \ldots, x_n$. Since $\mathcal{C}$ is a subring of $\mathcal{C}_{(p)}$ we can consider I as a $\mathcal{C}$-module. Let $\tilde{I}$ be the $\mathcal{C}$-submodule of I generated by $x_1, \ldots, x_n$; then it is easy to see that $\tilde{I} \otimes_{\mathcal{C}} \mathcal{C}_{(p)} \cong I$. Then as $\mathcal{C}$ is hereditary, $\tilde{I}$ is projective and hence so is I.

Thus every ideal in $\mathcal{C}_{(p)}$ is projective and hence $\mathcal{C}_{(p)}$ is hereditary. Every hereditary ring has global dimension 1, and so $\mathcal{C}_{(p)}$ is regular. □

Corollary 8.2.3. For every prime $p \neq 2, 3$ there is an isomorphism

$$\widetilde{K}_0(\mathcal{C}_{(p)}[t, t^{-1}]) \cong \widetilde{K}_0(\mathcal{C}_{(p)}).$$

Consider the fibre square

$$\begin{array}{ccc} \mathcal{C}_{(p)} & \longrightarrow & \left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}}\right) \\ \downarrow & & \downarrow \\ \left(\frac{-1,-3}{\mathbf{Q}}\right) & \longrightarrow & \left(\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}}\right) \end{array}$$

We shall calculate $\widetilde{K}_0(\mathcal{C}_{(p)})$ from the 'Mayer - Vietoris' exact sequence (3.0.5) associated to the above square:

$$\begin{array}{ccccc} K_1(\mathcal{C}_{(p)}) & \xrightarrow{f_1} & K_1\left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}}\right) \oplus K_1\left(\frac{-1,-3}{\mathbf{Q}}\right) & \xrightarrow{f_2} & K_1\left(\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}}\right) \\ & & \xrightarrow{\delta} & & \\ K_0(\mathcal{C}_{(p)}) & \xrightarrow{f_3} & K_0\left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}}\right) \oplus K_0\left(\frac{-1,-3}{\mathbf{Q}}\right) & \xrightarrow{f_4} & K_0\left(\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}}\right) \end{array}$$

where the maps are as in (3.0.5). Let R denote either $\hat{\mathbf{Z}}_{(p)}$ or $\hat{\mathbf{Q}}_{(p)}$; by Morita equivalence we have

$$K_0\left(\frac{-1,-3}{R}\right) \cong K_0(M_2(R)) \cong K_0(R) \cong \mathbf{Z}.$$

Since R is a principal ideal domain, there are no non-free projectives over R and so $K_0\left(\frac{-1,-3}{R}\right) \cong \mathbf{Z}$ with generator given by the right module $\begin{pmatrix} R & R \\ 0 & 0 \end{pmatrix}$. As $\left(\frac{-1,-3}{\mathbf{Q}}\right)$ is a division ring, it has no non-trivial projectives and thus the generator of $K_0\left(\frac{-1,-3}{\mathbf{Q}}\right)$ is $\left[\frac{-1,-3}{\mathbf{Q}}\right]$. Clearly we have

$$f_4\left(\begin{bmatrix} \hat{\mathbf{Z}}_{(p)} & \hat{\mathbf{Z}}_{(p)} \\ 0 & 0 \end{bmatrix}, [0]\right) = (\psi_+)_* \begin{bmatrix} \hat{\mathbf{Z}}_{(p)} & \hat{\mathbf{Z}}_{(p)} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{Q}}_{(p)} & \hat{\mathbf{Q}}_{(p)} \\ 0 & 0 \end{bmatrix},$$

and

$$f_4\left([0], \left[\frac{-1,-3}{\mathbf{Q}}\right]\right) = -(\psi_-)_* \left[\frac{-1,-3}{\mathbf{Q}}\right] = -\left[\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}}\right].$$

Therefore $\ker(f_4)$ is generated by $\left(2\begin{bmatrix} \hat{\mathbf{Z}}_{(p)} & \hat{\mathbf{Z}}_{(p)} \\ 0 & 0 \end{bmatrix}, -\left[\frac{-1,-3}{\mathbf{Q}}\right]\right)$ and $\ker(f_4) \cong \mathbf{Z}$. Therefore

$$K_0(\mathcal{C}_{(p)})/\mathrm{Im}(\delta) = K_0(\mathcal{C}_{(p)})/\ker(f_3) \cong \mathrm{Im}(f_3) = \ker(f_4) \cong \mathbf{Z}.$$

We now turn to the computation of the K_1 groups.

Proposition 8.2.4. For any prime $p \neq 2, 3$ the map

$$f_2 : K_1\left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}}\right) \oplus K_1\left(\frac{-1,-3}{\mathbf{Q}}\right) \to K_1\left(\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}}\right)$$

is surjective.

Proof. There is a commutative diagram

$$\begin{array}{ccc}
K_1\left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}}\right) \oplus K_1\left(\frac{-1,-3}{\mathbf{Q}}\right) & \xrightarrow{f_2} & K_1\left(\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}}\right) \\
\downarrow & & \downarrow \\
K_1(M_2(\hat{\mathbf{Z}}_{(p)})) \oplus K_1\left(\frac{-1,-3}{\mathbf{Q}}\right) & & K_1(M_2(\hat{\mathbf{Q}}_{(p)})) \\
\downarrow & & \downarrow \\
K_1(\hat{\mathbf{Z}}_{(p)}) \oplus K_1\left(\frac{-1,-3}{\mathbf{Q}}\right) & & K_1(\hat{\mathbf{Q}}_{(p)}) \\
\downarrow & & \downarrow \\
(\hat{\mathbf{Z}}_{(p)})^* \oplus \left(\left(\frac{-1,-3}{\mathbf{Q}}\right)^*\right)^{ab} & & (\hat{\mathbf{Q}}_{(p)})^* \\
\downarrow & & \downarrow \\
(\hat{\mathbf{Z}}_{(p)})^* \oplus \mathbf{Q}^* & \xrightarrow{f_2'} & (\hat{\mathbf{Q}}_{(p)})^*
\end{array}$$

in which all the vertical arrows are isomorphisms and f_2' corresponds to canonical inclusion followed by multiplication. Since each non-zero p-adic number is expressible in the form $a = p^{-n}b$ for some $n \in \mathbf{Z}$ and some $b \in (\hat{\mathbf{Z}}_{(p)})^*$, f_2' is surjective and hence so is f_2. □

Corollary 8.2.5. For any prime $p \neq 2, 3$ there are isomorphisms

$$K_0(\mathcal{C}_{(p)}[t, t^{-1}]) \cong K_0(\mathcal{C}_{(p)}) \cong \mathbf{Z}$$

and

$$\widetilde{K}_0(\mathcal{C}_{(p)}[t, t^{-1}]) \cong \widetilde{K}_0(\mathcal{C}_{(p)}) \cong 0.$$

Proof. We know that $K_0(\mathcal{C}_{(p)})/\mathrm{Im}(\delta) \cong \mathbf{Z}$. However, f_2 is surjective and thus $\mathrm{Im}(f_2) = \ker(\delta) = K_1\left(\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}}\right)$. Therefore $\mathrm{Im}(\delta) = 0$ and hence $K_0(\mathcal{C}_{(p)}) \cong \mathbf{Z}$.

Now let P be a projective module over $\mathcal{C}_{(p)}$; then P is locally projective, say $P = \langle P_+, P_-; \alpha\rangle$ where $P_+ = P \otimes \left(\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}}\right)$, $P_- = P \otimes \left(\frac{-1,-3}{\mathbf{Q}}\right)$, and

$$\alpha : P_+ \otimes \left(\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}}\right) \to P_- \otimes \left(\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}}\right)$$

is an isomorphism. We may choose n, m such that

$$[P_+] = \begin{bmatrix} \hat{\mathbf{Z}}_{(p)} & \hat{\mathbf{Z}}_{(p)} \\ 0 & 0 \end{bmatrix}^n \text{ and } [P_-] = \left[\frac{-1,-3}{\mathbf{Q}}\right]^m.$$

But now α is an isomorphism

$$\alpha : \begin{bmatrix} \hat{\mathbf{Q}}_{(p)} & \hat{\mathbf{Q}}_{(p)} \\ 0 & 0 \end{bmatrix}^n \xrightarrow{\sim} \left[\frac{-1,-3}{\hat{\mathbf{Q}}_{(p)}}\right]^m = \begin{bmatrix} \hat{\mathbf{Q}}_{(p)} & \hat{\mathbf{Q}}_{(p)} \\ 0 & 0 \end{bmatrix}^{2m}$$

and hence $n = 2m$. But now we have $[P_+] = \begin{bmatrix} \hat{\mathbf{Z}}_{(p)} & \hat{\mathbf{Z}}_{(p)} \\ 0 & 0 \end{bmatrix}^{2m} = \left[\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}}\right]^m$ and so

$$f_3[P] = \left(\left[\frac{-1,-3}{\hat{\mathbf{Z}}_{(p)}}\right]^m, \left[\frac{-1,-3}{\mathbf{Q}}\right]^m\right) = f_3[\mathcal{C}_{(p)}^m].$$

Therefore $f_3([P] - [\mathcal{C}_{(p)}^m]) = 0$ and so

$$[P] - [\mathcal{C}_{(p)}^m] \in \ker(f_3) = \mathrm{Im}(\delta) \implies [P] - [\mathcal{C}_{(p)}^m] = 0 \implies [P] = [\mathcal{C}_{(p)}^m]$$

and thus $[P] = 0$ in $\widetilde{K}_0(\mathcal{C}_{(p)})$. The corresponding statements for $\mathcal{C}_{(p)}[t, t^{-1}]$ now follow by (8.2.3). □

Since $\widetilde{K}_0(\mathcal{C}_{(p)}[t, t^{-1}]) \cong 0$ every projective module over $\mathcal{C}_{(p)}[t, t^{-1}]$ is stably free and so by (8.1.4):

Corollary 8.2.6. Let $p \neq 2, 3$ be an odd prime. Then $\mathrm{SF}_1(\mathcal{C}_{(p)}[t, t^{-1}])$ is infinite.

8.3 Proof of Theorem B

Let p be a prime and consider the Karoubi square

$$\mathcal{B} = \begin{cases} \mathcal{C}[t, t^{-1}] \longrightarrow \mathcal{C}_{(p)}[t, t^{-1}] \\ \downarrow \qquad\qquad\qquad \downarrow \\ \mathcal{C}[t, t^{-1}] \otimes \mathbf{Z}[\frac{1}{p}] \longrightarrow \left(\frac{-1,-3}{\mathbf{Q}[t,t^{-1}]}\right). \end{cases}$$

Here the uppermost map is the canonical inclusion and we are taking $\{p^r \mid r \geq 1\}$ as our multiplicative submonoid. Since $\left(\frac{-1,-3}{\mathbf{Q}}\right)$ is a division ring then $\left(\frac{-1,-3}{\mathbf{Q}[t,t^{-1}]}\right) = \left(\frac{-1,-3}{\mathbf{Q}}\right)[t, t^{-1}]$ has SFC by (4.1.4) and is weakly Euclidean by (5.1.1). Thus by (5.2.1) the induced map

$$\mathrm{SF}_1(\mathcal{C}[t, t^{-1}]) \to \mathrm{SF}_1(\mathcal{C}_{(p)}[t, t^{-1}]) \times \mathrm{SF}_1(\mathcal{C}[t, t^{-1}] \otimes \mathbf{Z}[\frac{1}{p}])$$

is surjective. For $p \neq 2, 3$ we have:

Proposition 8.3.1. $\mathrm{SF}_1(\mathcal{C}[t, t^{-1}])$ is infinite.

Consider the fibre square

$$\begin{array}{ccc} \mathcal{C}_2(\mathbf{Z}[x]/(1 + x^3), \theta, -1)[t, t^{-1}] & \longrightarrow & \mathcal{C}[t, t^{-1}] \\ \downarrow & & \downarrow \\ \mathbf{Z}[i][t, t^{-1}] & \longrightarrow & \mathbf{F}_3[i][t, t^{-1}]. \end{array}$$

Since x^2+1 is irreducible over $\mathbf{F}_3$ then $\mathbf{F}_3[i] \cong \mathbf{F}_9$ and hence by (4.1.4) and (5.1.1) $\mathbf{F}_3[i][t, t^{-1}]$ is weakly Euclidean and has SFC. Thus by (5.2.1), the mapping

$$\mathrm{SF}_1(\mathcal{C}_2(\mathbf{Z}[x]/(1 + x^3), \theta, -1)[t, t^{-1}]) \to \mathrm{SF}_1(\mathcal{C}[t, t^{-1}]) \times \mathrm{SF}_1(\mathbf{Z}[i][t, t^{-1}])$$

is surjective. Thus:

Proposition 8.3.2. $\mathrm{SF}_1(\mathcal{C}_2(\mathbf{Z}[x]/(1 + x^3), \theta, -1)[t, t^{-1}])$ is infinite.

Now consider the fibre square

$$\begin{array}{ccc} \mathbf{Z}[Q(12) \times C_\infty] & \longrightarrow & \mathcal{C}_2(\mathbf{Z}[x]/(1+x^3), \theta, -1)[t, t^{-1}] \\ \downarrow & & \downarrow \\ \mathbf{Z}[D_6 \times C_\infty] & \longrightarrow & \mathbf{F}_2[D_6 \times C_\infty]. \end{array}$$

By (4.1.6), (5.1.6) and (5.2.1)

$$\mathrm{SF}_1(\mathbf{Z}[Q(12) \times C_\infty]) \to \mathrm{SF}_1(\mathcal{C}_2(\mathbf{Z}[x]/(1+x^3), \theta, -1)[t, t^{-1}]) \times \mathrm{SF}_1(\mathbf{Z}[D_6 \times C_\infty])$$

is surjective. By a result of Johnson [20], $\mathbf{Z}[D_6 \times C_\infty]$ has no non-trivial stably frees. However, since $\mathrm{SF}_1(\mathcal{C}_2(\mathbf{Z}[x]/(1+x^3), \theta, -1)[t, t^{-1}]$ is infinite we have:

Theorem 8.3.3. $\mathbf{Z}[Q(12) \times C_\infty]$ admits infinitely many isomorphically distinct stably free modules of rank 1.

We proceed to lift the stably free modules constructed in (8.3.3) to those over $\mathbf{Z}[Q(12m) \times C_\infty]$. From the factorization $(x^{6m} - 1) = (x^6 - 1)q(x)$, where $q(x) = x^{6(m-1)} + x^{6(m-2)} + \ldots + 1$ we obtain the following Milnor square:

$$\begin{array}{ccc} \mathbf{Z}[x]/(x^{6m} - 1) & \longrightarrow & \mathbf{Z}[x]/(q(x)) \\ \downarrow & & \downarrow \\ \mathbf{Z}[x]/(x^6 - 1) & \longrightarrow & (\mathbf{Z}/m)[x]/(x^6 - 1). \end{array}$$

The canonical involution on $\mathbf{Z}[x]/(x^{6m} - 1)$ given by $\theta(x) = x^{-1}$, with fixed point x^{3m} induces involutions on the other rings in the above fibre square. Applying the cyclic algebra construction gives another fibre square:

$$\begin{array}{ccc} \mathbf{Z}[Q(12m)] & \longrightarrow & \mathcal{C}_2(\mathbf{Z}[x]/(q(x)), \theta, x^{3m}) \\ \downarrow & & \downarrow \\ \mathbf{Z}[Q(12)] & \longrightarrow & (\mathbf{Z}/m)[Q(12)]. \end{array}$$

Taking the tensor product with $\mathbf{Z}[t, t^{-1}]$ gives another:

$$\begin{array}{ccc} \mathbf{Z}[Q(12m) \times C_\infty] & \longrightarrow & \mathcal{C}_2(\mathbf{Z}[x]/(q(x)), \theta, x^{3m})[t, t^{-1}] \\ \downarrow & & \downarrow \\ \mathbf{Z}[Q(12) \times C_\infty] & \longrightarrow & (\mathbf{Z}/m)[Q(12) \times C_\infty]. \end{array}$$

Now, by (4.1.6), (5.1.6) and (5.2.1), we have that the induced map

$$\mathrm{SF}_1(\mathbf{Z}[Q(12m)\times C_\infty] \to \mathrm{SF}_1(\mathbf{Z}[Q(12)\times C_\infty])\times \mathrm{SF}_1(\mathcal{C}_2(\mathbf{Z}[x]/(q(x)), \theta, x^{3m})[t, t^{-1}])$$

is surjective. Therefore:

Theorem 8.3.4. For each $m \geq 1$ the integral group ring $\mathbf{Z}[Q(12m)\times C_\infty]$ admits infinitely many isomorphically distinct stably free modules of rank 1.

Now let F be a group and suppose there exists a surjective map $f : F \to F_n$ for some $n \geq 1$. Then if $t_1, \dots t_n$ generate F_n choose $x_1, \dots, x_n \in F$ such that $f(x_i) = t_i$ for each $1 \leq i \leq n$. Define a right inverse $g : F_n \to F$ by $g(t_i) = x_i$ for each i. For any group G there are homomorphisms $f_* : \mathbf{Z}[G \times F] \to \mathbf{Z}[G \times F_m]$ and $g_* : \mathbf{Z}[G \times F_n] \to \mathbf{Z}[G \times F]$ such that $f_* \circ g_* = \mathrm{Id}$, and so $\mathbf{Z}[G \times F_n]$ is a retract of $\mathbf{Z}[G \times F]$. Therefore by (4.1.8) there is an injective mapping $\mathrm{SF}_1(\mathbf{Z}[G \times F_n]) \to \mathrm{SF}_1(\mathbf{Z}[G \times F])$. Together with (8.3.4) this proves Theorem B of the introduction:

Theorem 8.3.5. Let F be a group which maps surjectively onto F_n for some $n \geq 1$. Then for every $m \geq 1$, $\mathbf{Z}[Q(12m) \times F]$ admits infinitely many isomorphically distinct stably free modules of rank 1.

Chapter 9

The D(2)-problem for D_{4n}

9.1 k-invariants over D_{2n}

For any n the group D_{2n} may be described by the presentation

$$\langle x, y \mid x^n,\ y^2,\ y^{-1}xyx \rangle.$$

Write $\Lambda = \mathbf{Z}[D_{2n}]$ and $\Sigma = 1 + x + x^2 + \ldots + x^{n-1}$. Applying the Cayley complex construction to this presentation gives the following 2-complex:

$$0 \to J \to \Lambda^3 \xrightarrow{\partial_2} \Lambda^2 \xrightarrow{\partial_1} \Lambda \xrightarrow{\varepsilon} \mathbf{Z} \to 0, \tag{9.1}$$

where ε is the augmentation map, $\partial_1 = (x-1, y-1)$ and $\partial_2 = \begin{pmatrix} \Sigma & 0 & 1+yx \\ 0 & 1+y & x-1 \end{pmatrix}$.
The following proposition is easily verified:

Proposition 9.1.1. Fix n and let k be any odd integer with $3 \leq k \leq n-1$. If we write $m = (k-1)/2$ then the following diagram commutes

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & J & \longrightarrow & \Lambda^3 & \xrightarrow{\partial_2} & \Lambda^2 & \xrightarrow{\partial_1} & \Lambda & \xrightarrow{\varepsilon} & \mathbf{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & J & \longrightarrow & \Lambda^3 & \xrightarrow{\partial_2} & \Lambda^2 & \xrightarrow{\partial_1} & \Lambda & \xrightarrow{\varepsilon} & \mathbf{Z} & \longrightarrow & 0 \end{array}$$

where $\partial_1 = (x-1, y-1)$, $\partial_2 = \begin{pmatrix} \Sigma & 0 & 1+yx \\ 0 & 1+y & x-1 \end{pmatrix}$,

$$\alpha_0 = (1 + x^{-1} + \ldots + x^{-m} + x^{-1}y + \ldots + x^{-m}y),$$

$$\alpha_1 = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \ \alpha_2 = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$a = 1 + x^{-1} \ldots + x^{-m} - x^{-2}y - \ldots - x^{-m-1}y$ and $\theta = \alpha_2|_J$.

Our aim is to show that, for $k \in (\mathbf{Z}/2n)^*$, θ is an isomorphism, and thus show that the Swan map $\mathrm{Aut}(J) \to (\mathbf{Z}/2n)^*$ is surjective (see section 2.1). Consider the commutative diagram above as a diagram of (free) **Z**-modules and **Z**-linear maps; taking determinants we have:

Proposition 9.1.2. *$k \det(\theta) \det(\alpha_1) = \det(\alpha_2) \det(\alpha_0)$.*

Proof. Let v denote the restriction of α_0 to $\ker(\varepsilon)$ and let u denote the restriction of α_1 to $\ker(\partial_1)$. Then $v(\ker(\varepsilon)) \subset \ker(\varepsilon)$, $u(\ker(\partial_1)) \subset \ker(\partial_1)$ and we have an commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\partial_1) & \longrightarrow & \Lambda^2 & \xrightarrow{\partial_1} & \ker(\varepsilon) & \longrightarrow & 0 \\ & & \downarrow u & & \downarrow \alpha_1 & & \downarrow v & & \\ 0 & \longrightarrow & \ker(\partial_1) & \longrightarrow & \Lambda^2 & \xrightarrow{\partial_1} & \ker(\varepsilon) & \longrightarrow & 0 \end{array}$$

Considered as a diagram of (free) **Z**-modules, both exact sequences split, and so there exist α_1', u' and v' such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\partial_1) & \longrightarrow & \ker(\partial_1) \oplus \ker(\varepsilon) & \longrightarrow & \ker(\varepsilon) & \longrightarrow & 0 \\ & & \downarrow u' & & \downarrow \alpha_1' & & \downarrow v' & & \\ 0 & \longrightarrow & \ker(\partial_1) & \longrightarrow & \ker(\partial_1) \oplus \ker(\varepsilon) & \longrightarrow & \ker(\varepsilon) & \longrightarrow & 0 \end{array}$$

commutes with the obvious maps, $\det(\alpha_1') = \det(\alpha_1)$, $\det(u') = \det(u)$ and $\det(v') = \det(v)$. Therefore we have $\det(\alpha_1') = \det \begin{pmatrix} u' & w \\ 0 & v' \end{pmatrix} = \det(u) \det(v)$.

Similarly $\det(\alpha_2) = \det(\theta)\det(u)$ and $\det(\alpha_0) = \det(v)\det(k) = k\det(v)$. Thus

$$\det(\alpha_2)\det(\alpha_0) = \det(\theta)\det(u)\det(v)k = k\det(\theta)\det(\alpha_1)$$

as required.

□

Now, any Λ-homomorphism is a Λ-isomorphism if and only if it is an isomorphism as a **Z**-linear map. Thus, in order to show that $[k]$ is in the image of the Swan map, it suffices to show that $\det(\theta) = \pm 1$.

Proposition 9.1.3. Suppose that k is coprime to n. Then $\det(\alpha_0) = \pm k$.

Proof. Considered as a sequence of **Z**-modules, the exact sequence

$$0 \to \ker(\varepsilon) \to \Lambda \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

splits, and so $\Lambda \cong \mathbf{Z} \oplus \ker(\varepsilon)$ as a **Z**-module. Since $\varepsilon \circ \alpha_0 = k \circ \varepsilon$, we have $\alpha_0(\ker(\varepsilon)) \subset \ker(\varepsilon)$ and $\varepsilon(\alpha_0) = k \implies \alpha_0 = k + p$ for some $p \in \ker(\varepsilon)$. Thus $\det(\alpha_0) = k\det(\alpha_0|_{\ker(\varepsilon)})$. We shall show that $\alpha_0|_{\ker(\varepsilon)}$ is an isomorphism, and hence has determinant ± 1.

First note that $\ker(\varepsilon) = (x-1)\Lambda + (y-1)\Lambda$. We have

$$\alpha_0(y-1) = (1 + x^{-1} + \ldots + x^{-m} + x^{-1}y + \ldots + x^{-m}y)(y-1) = (y-1),$$

and thus $(y-1) \in \mathrm{im}(\alpha_0)$. Now,

$$\begin{aligned}\alpha_0(x-1) &= (1 + x^{-1} + \ldots + x^{-m} + x^{-1}y + \ldots + x^{-m}y)(x-1) \\ &= x + x^{-m-1}y - x^{-m} - x^{-1}y\end{aligned}$$

and so

$$\alpha_0((x-1)x^{-1}) = 1 + x^{-m}y - x^{-m-1} - y = (1 - x^{-m-1}) + y(x^m - 1).$$

Therefore

$$\alpha_0([(x-1)x^{-1}) + (y-1)(1-x^m)]x^{-m}) = 1 - x^{-2m-1} = 1 - x^k.$$

Since k is coprime to n, we may choose r such that $kr \equiv 1 \pmod{n}$. Then

$$(x^k - 1)(1 + x^k + \dots x^{(r-1)k}) = x^{rk} - 1 = x - 1 \in \mathrm{im}(\alpha_0).$$

Thus $\alpha_0|_{\ker(\varepsilon)}$ is surjective. Therefore $\alpha_0|_{\ker(\varepsilon)} \otimes \mathbf{Q} : \mathbf{Q}^{2n-1} \to \mathbf{Q}^{2n-1}$ is bijective and hence $\alpha_0|_{\ker(\varepsilon)}$ is injective. Therefore $\det(\alpha_0|_{\ker(\varepsilon)}) = \pm 1$ and so $\det(\alpha_0) = \pm k$. □

Proposition 9.1.4. $\det(\alpha_1) = \det(\alpha_2) \neq 0$.

Proof. The following commutes:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & J & \longrightarrow & \Lambda^3 & \xrightarrow{\partial_2} & \Lambda^2 & \xrightarrow{\partial_1} & \Lambda & \xrightarrow{\varepsilon} & \mathbf{Z} & \longrightarrow & 0 \\
 & & \downarrow \theta' & & \downarrow \alpha_2' & & \downarrow \alpha_1 & & \downarrow \alpha_0 & & \downarrow k & & \\
0 & \longrightarrow & J & \longrightarrow & \Lambda^3 & \xrightarrow{\partial_2} & \Lambda^2 & \xrightarrow{\partial_1} & \Lambda & \xrightarrow{\varepsilon} & \mathbf{Z} & \longrightarrow & 0
\end{array}$$

where $\alpha_2' = \begin{pmatrix} m+1-my & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and θ' is the restriction of α_2' to J. We proceed to calculate $\det(\alpha_2') = \det(m+1-my)$. If we represent $(m+1-my)$ with respect to the basis $\{1, x, \dots, x^{n-1}, y, \dots, x^{n-1}y\}$, then we form the matrix

$$M = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

Here A is diagonal with each diagonal entry equal to $m+1$, and B is equal to $-m$ times the permutation matrix associated to $\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & n & n-1 & \dots & 2 \end{pmatrix}$. Label the rows of M by $v_1, \dots, v_{2n}$ and let N be the matrix with rows $v_1', \dots, v_{2n}'$, where $v_1' = v_1 + v_{n+1}$, $v_i' = v_i + v_{2n-i+2}$ for $2 \leq i \leq n$, and $v_i' = v_i$ for $n+1 \leq i \leq 2n$. Now label the columns of M by $w_1, \dots, w_{2n}$ and let L be the matrix with columns $w_1', \dots, w_{2n}'$ where $w_i' = w_i$ for $1 \leq i \leq n$, $w_{n+1}' = w_{n+1} - w_1$ and

$w'_{n+i} = w_{n+1} - w_{n-i+2}$ for $2 \leq i \leq n$. For example, if $n = 4$ and $k = 3$ (so that $m = 1$) we have

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \\ -1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \\ -1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 3 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

It is easy to see that L is lower triangular with n diagonal entries equal to 1 and n diagonal entries equal to $2m+1 = k$. Therefore $\det(\alpha'_2) = \det(m+1-my) = \det(L) = k^n$. Using the identity $k\det(\theta')\det(\alpha_1) = \det(\alpha_0)\det(\alpha'_2) = \pm k^{n+1}$ we see that $\det(\alpha_1) = \det(\alpha_2) \neq 0$. □

Therefore by (9.1.2), (9.1.3) and (9.1.4):

Proposition 9.1.5. If $3 \leq k \leq n-1$ is coprime to n then $\det(\theta) = \pm 1$ and so θ is an isomorphism. Thus $[k]$ is in the image of the Swan map.

Clearly $[-1]$ is in the image of the Swan map and so:

Corollary 9.1.6. The Swan map $\mathrm{Aut}(J) \to (\mathbf{Z}/2n)^*$ is surjective for each D_{2n}.

Mannan [29] has previously shown that the Swan map is surjective for D_{2^n}.

9.2 Proof of Theorem C

We now restrict to the case D_{4n}. Take $J = \ker(\partial_2)$ in (9.1); then by Mannan's calculation ([29] Proposition 3.2) we have:

Proposition 9.2.1. *J has minimal $\mathbf{Z}$-rank in $\Omega_3(\mathbf{Z})$.*

Let Γ be an order over a Dedekind domain R. We say that *torsion free cancellation* holds for Γ if $X \oplus M \cong X \oplus N \implies M \cong N$ for lattices X, M and N over Γ (so that X, M and N are finitely generated as Γ-modules and torsion free over R). There are very few finite groups G for which $\Gamma = \mathbf{Z}[G]$ has torsion free cancellation; if G is non-abelian then the only possible candidates are A_4, A_5, S_4 and D_{2n} for certain values of n. Clearly we have:

Proposition 9.2.2. *Suppose that $\mathbf{Z}[D_{4n}]$ has torsion free cancellation. Then every $J' \in \Omega_3(\mathbf{Z})$ is of the form $J' \cong J \oplus \Lambda^m$ for some $m \geq 0$.*

For a finite group G, the integral group ring $\mathbf{Z}[G]$ is a $\mathbf{Z}$-order in the semisimple algebra $\mathbf{Q}[G]$; we may choose a maximal $\mathbf{Z}$-order Γ in $\mathbf{Q}[G]$ containing $\mathbf{Z}[G]$, and define $D(\mathbf{Z}[G]) = \ker(\widetilde{K}_0(\mathbf{Z}[G]) \to \widetilde{K}_0(\Gamma))$. A necessary condition for $\mathbf{Z}[G]$ to possess torsion free cancellation is $D(\mathbf{Z}[G]) = 0$. The following is due to Swan [35]:

Theorem 9.2.3. *Let p be a prime. Then D_{4p} satisfies torsion free cancellation if and only if $D(\mathbf{Z}[D_{4p}]) = 0$.*

Endo and Miyata [13] calculate the order of $D(\mathbf{Z}[D_{2n}])$ for various values of n. In particular they show $D(\mathbf{Z}[D_{4p}]) = 0$ for prime p when $3 \leq p \leq 31$, $p = 47, 179$ or 19379. However, there do exist values of n for which $D(\mathbf{Z}[D_{4n}]) \neq 0$, for example $n = 37$. Moreover, results of Swan show that $D(\mathbf{Z}[D_{4n}]) = 0$ is not a sufficient condition for torsion free cancellation to hold. For example, $D(\mathbf{Z}[D_{2^n}]) = 0$ for all n, yet torsion free cancellation fails when $n \geq 7$ (see [35], Theorem 8.1). Of course, although values of n exist for which $\mathbf{Z}[D_{4n}]$ does not have torsion free cancellation, it may still be the case that cancellation of finitely generated free modules holds within $\Omega_3(\mathbf{Z})$ for such n.

If torsion free cancellation holds for D_{4n} then, by (2.1.1), (9.1.6) and (9.2.2), up to congruence, the only algebraic 2-complexes over D_{4n} are of the form

$$\mathcal{E}_m = (0 \to J \oplus \Lambda^m \to \Lambda^3 \oplus \Lambda^m \xrightarrow{\partial_2 \pi_1} \Lambda^2 \xrightarrow{\partial_1} \Lambda \to \mathbf{Z} \to 0),$$

where $\pi_1 : \Lambda^3 \oplus \Lambda^m \to \Lambda^3$ denotes projection onto the first factor. If a pair of algebraic 2-complexes are congruent then they are homotopy equivalent (see Johnson [20] p.182), and so the $\mathcal{E}_m$ represent all homotopy classes of algebraic 2-complexes over D_{4n}. However, $\mathcal{E}_m$ is geometrically realized by the Cayley complex arising from the presentation

$$\mathcal{G}_m = \langle x, y \mid x^{2n},\ y^2, y^{-1}xyx, 1, \ldots, 1\rangle$$

where there are m trivial relators added to the standard presentation for D_{4n}. Therefore every homotopy class of algebraic 2-complex over D_{4n} is geometrically realized and hence we have proved Theorem C:

Theorem 9.2.4. Suppose that $\mathbf{Z}[D_{4n}]$ satisfies torsion free cancellation. Then the D(2)-property holds for D_{4n}.

Combining this with (9.2.3) gives:

Corollary 9.2.5. Let p be a prime and suppose that $D(\mathbf{Z}[D_{4p}]) = 0$. Then the D(2)-property holds for D_{4p}.

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