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Piercing intersecting convex sets

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ABSTRACT

Assume two finite families $\mathcal{A}$ and $\mathcal{B}$ of convex sets in $\mathbb{R}^3$ have the property that $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Is there a constant $\gamma > 0$ (independent of $\mathcal{A}$ and $\mathcal{B}$) such that there is a line intersecting $\gamma|\mathcal{A}|$ sets in $\mathcal{A}$ or $\gamma|\mathcal{B}|$ sets in $\mathcal{B}$? This is an intriguing Helly-type question from a paper by Martínez, Roldan and Rubin. We confirm this in the special case when all sets in $\mathcal{A}$ lie in parallel planes and all sets in $\mathcal{B}$ lie in parallel planes; in fact, one of the two families has a transversal by a single line.

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1. Introduction and main results

In a paper on extensions of the Colorful Helly Theorem, Martínez, Roldan and Rubin [7] ask the following: Suppose two finite families $\mathcal{A}$ and $\mathcal{B}$ of convex sets in $\mathbb{R}^3$ have the property that $A \cap B \neq \emptyset$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Is it true then that there is a line intersecting a fixed positive fraction of the sets in $\mathcal{A}$ or in $\mathcal{B}$? The best we know is that it is true in some special cases. For instance, Bárány [2] confirmed that when all sets in $\mathcal{A}$ and $\mathcal{B}$ are cylinders, or if all sets have a bounded aspect ratio, then it is true. In general, however, this question is wide open.

A different special case of this question, namely the case of *vertical polygons*, was raised independently by Andreas Holmsen and Géza Tóth (personal communication). A vertical polygon in $\mathbb{R}^3$ is a convex polygon that lies in a plane orthogonal to the xy-plane. Is there a real number $\gamma > 0$ such that, whenever both $\mathcal{A}$ and $\mathcal{B}$ consist of vertical polygons, there is a line intersecting a γ-fraction of the sets in $\mathcal{A}$ or a γ-fraction of those in $\mathcal{B}$? One motivation for studying this question is that the aspect ratio of a vertical polygon is infinite, so it is as far as possible from the case of convex bodies with bounded aspect ratio; perhaps a solution in this case could shed light on the general problem. Even this special case, however, remains open.

Our main result, Theorem 1 below, states there is a line intersecting *all* sets of $\mathcal{A}$ or of $\mathcal{B}$, provided the vertical polygons in $\mathcal{A}$ lie in parallel planes and the vertical polygons in $\mathcal{B}$ also lie in parallel planes. We also prove, under the same condition, that we can restrict the location of the piercing line: Either there is a line in the plane of some $A \in \mathcal{A}$ intersecting $\frac{1}{6}|\mathcal{B}|$ sets in $\mathcal{B}$, or there is a line in the plane of some $B \in \mathcal{B}$ intersecting $\frac{1}{6}|\mathcal{A}|$ sets in $\mathcal{A}$. This is Theorem 2, which is stated and proved in Section 3.

We conclude the paper in Section 4 with a partial extension of our results to higher dimensions.

2. Theorem 1 and its proof

We start with two families, $\mathcal{A}$ and $\mathcal{B}$, consisting of vertical polygons such that all the polygons in $\mathcal{A}$ are in parallel planes, all the polygons in $\mathcal{B}$ are in parallel planes, and $A \cap B \neq \emptyset$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$. For each pair (A, B), take an arbitrary point $P(A, B)$ in $A \cap B$. If we replace A by $\text{conv}\{P(A, B) : B \in \mathcal{B}\}$ and replace B by $\text{conv}\{P(A, B) : A \in \mathcal{A}\}$, this new collection of vertical polygons is still parallel and pairwise intersecting. If one of these new families has a line transversal, then the corresponding original family does, as well. Moreover, the question is invariant under non-degenerate affine transformations, so we can assume that the planes containing sets in $\mathcal{A}$ are parallel with the yz-plane, and the planes containing sets in $\mathcal{B}$ are parallel with the xz-plane.

Based on these reductions, we formulate our problem in a more convenient notation. Let $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ and $Z \in \mathbb{R}^{n \times m}$, and for each $i \in [n], j \in [m]$, let $P_{ij} := (x_i, y_j, z_{ij}) \in \mathbb{R}^3$. We form convex sets $A_i := \text{conv}(\{P_{ij} : j \in [m]\})$ and $B_j := \text{conv}(\{P_{ij} : i \in [n]\})$.

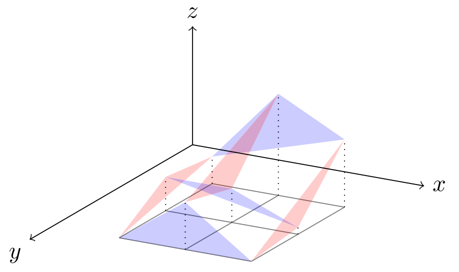


Fig. 1. An illustration of the $n = m = 3$ case. The red sets form the family $\mathcal{A}$ and the blue sets form the family $\mathcal{B}$. (For interpretation of the colors in the figures, the reader is referred to the web version of this article.)

By construction, every A_i is contained in a plane parallel to the yz-plane, every B_j is contained in a plane parallel to the xz-plane, and every A_i intersects every B_j. (In this setup, the point P_{ij} corresponds to the point $P(A_i, B_j)$ in the previous paragraph.) This setup is illustrated in Fig. 1.

Theorem 1. *With the setup of the previous paragraph, either*

- *there is a real number $x_0 \in \mathbb{R}$ and a line ℓ_x in the plane $(x_0, \cdot, \cdot)$ intersecting all sets B_j, or*
- *there is a real number $y_0 \in \mathbb{R}$ and a line ℓ_y in the plane $(\cdot, y_0, \cdot)$ intersecting all sets A_i.*

Because the problem is affine-invariant, this proves the theorem for any two collections of vertical polygons that live in parallel planes.

Proof. We start by writing up the intended conclusions as a system of linear inequalities which has a solution if and only if a piercing line exists.

Let us do this for the first possibility. Let the line ℓ_x be parametrized as $\{(x_0, y, ay + z_0) : y \in \mathbb{R}\}$. The claim that ℓ_x pierces B_j is equivalent to the existence of barycentric coordinates $(\beta_{ij} : i \in [n])$ such that $\sum_i \beta_{ij} P_{ij} = (x_0, y_j, ay_j + z_0)$.

Our system of linear inequalities specifying a piercing line $(x_0, y, ay + z_0)$ is then

$$\sum_i \beta_{ij} x_i = x_0 \qquad \forall j \in [m], \tag{x-piercing}$$

$$\sum_i \beta_{ij} z_{ij} = ay_j + z_0 \qquad \forall j \in [m], \tag{z-piercing}$$

$$\sum_i \beta_{ij} = 1 \qquad \forall j \in [m], \tag{barycentric}$$

$$\beta_{ij} \geq 0 \qquad \forall i \in [n], j \in [m]. \tag{nonnegative}$$

We now apply Farkas' Lemma to write up a system that is unsolvable in dual variables $U \in \mathbb{R}^{3\times m}$ if and only if the above system is solvable in (β, a, x_0, z_0).

$$u_{1j}x_i + u_{2j}z_{ij} + u_{3j} \geq 0 \qquad \forall i \in [n], j \in [m], \tag{β}$$

$$\sum_j u_{2j}y_j = 0, \tag{a}$$

$$\sum_j u_{1j} = 0, \tag{x_0}$$

$$\sum_j u_{2j} = 0, \tag{z_0}$$

$$\sum_j u_{3j} < 0. \tag{infeasible}$$

Below we describe the correspondence between the parts of the two systems of inequalities. The real vectors u_{1j}, u_{2j}, u_{3j} correspond to equalities (x-piercing), (z-piercing), (barycentric), respectively. The $n \times m$ inequalities of (β) correspond to the $n \times m$ non-negative variables of β. The equalities (a), (x_0), and (z_0) correspond to the real variables a, x_0, z_0, respectively.

Our next step is to combine the above dual system with the analogous system written up for the ℓ_y line. The combined system has x, y, Z, U, V as variables, and is bilinear, non-semidefinite. This is in contrast with its two constituents that are linear when x, y, Z are treated as parameters.

The complete system we get is:

$\exists x, y \in \mathbb{R}^n,\ \exists Z \in \mathbb{R}^{n\times m},\ \exists U \in \mathbb{R}^{3\times m}, V \in \mathbb{R}^{3\times n}$:

$$u_{1j}x_i + u_{2j}z_{ij} + u_{3j} \geq 0 \qquad \forall i \in [n], j \in [m], \tag{x:β}$$

$$v_{1i}y_j + v_{2i}z_{ij} + v_{3i} \geq 0 \qquad \forall i \in [n], j \in [m], \tag{y:β}$$

$$\sum_j u_{2j}y_j = 0, \tag{x:a}$$

$$\sum_i v_{2i}x_i = 0, \tag{y:a}$$

$$\sum_j u_{1j} = \sum_i v_{1i} = \sum_j u_{2j} = \sum_i v_{2i} = 0, \tag{x:x_0, y:x_0, x:z_0, y:z_0}$$

$$\sum_j u_{3j} < 0, \tag{x:infeasible}$$

$$\sum_i v_{3i} < 0. \tag{y:infeasible}$$

Having the combined system of inequalities at hand, we now finish the proof. We demonstrate that this system is unsolvable rather directly, by writing up weighted sums of our inequalities until a contradiction is reached. This implies that one of the original systems must be solvable; in other words, either $\mathcal{A}$ or $\mathcal{B}$ has a piercing line.

Remark. The argument could be formulated in the framework of Farkas' lemma: if we treat U, V as parameters and x, y, Z as variables, we can give a closed-form solution to the dual of the resulting linear system of inequalities. In an interesting contrast with this, if we treat x, y, Z as parameters and U, V as variables, the dual formulation is equivalent to presenting a piercing line, and we are not aware of any simple formula for a dual solution.

Let us introduce the three variables

$$x'_i := v_{2i}x_i, \qquad y'_j := u_{2j}y_j, \qquad z'_{ij} := u_{2j}v_{2i}z_{ij}$$

and the four sets

$$\begin{aligned} I^+ &= \{i : v_{2i} \geq 0\} \qquad & J^+ &= \{j : u_{2j} \geq 0\} \\ I^- &= \{i : v_{2i} < 0\} \qquad & J^- &= \{j : u_{2j} < 0\}. \end{aligned}$$

By multiplying (x:β) by v_{2i} we obtain

$$\forall i \in I^+,\ \forall j : u_{1j}x'_i + z'_{ij} + v_{2i}u_{3j} \geq 0, \tag{1}$$

$$\forall i \in I^-,\ \forall j : u_{1j}x'_i + z'_{ij} + v_{2i}u_{3j} \leq 0. \tag{2}$$

Similarly, by multiplying (y:β) by u_{2j} we obtain

$$\forall j \in J^+,\ \forall i : v_{1i}y'_j + z'_{ij} + u_{2j}v_{3i} \geq 0, \tag{3}$$

$$\forall j \in J^-,\ \forall i : v_{1i}y'_j + z'_{ij} + u_{2j}v_{3i} \leq 0. \tag{4}$$

By comparing (1) and (4), we obtain

$$\forall i \in I^+,\ \forall j \in J^- : u_{1j}x'_i - v_{1i}y'_j + v_{2i}u_{3j} - u_{2j}v_{3i} \geq 0. \tag{5}$$

Similarly, from (2) and (3), we obtain

$$\forall i \in I^-,\ \forall j \in J^+ : -u_{1j}x'_i + v_{1i}y'_j - v_{2i}u_{3j} + u_{2j}v_{3i} \geq 0. \tag{6}$$

We now sum the last two inequalities over each of their ranges and add them together; this will give us a contradiction. To find it, we look separately at each of the four parts of this sum, one for each summand in the inequalities (5) and (6).

The first part is

$$\sum_{i\in I^+}\sum_{j\in J^-} u_{1j}x'_i - \sum_{i\in I^-}\sum_{j\in J^+} u_{1j}x'_i,$$

which we deal with by introducing the shorthand

$$u_1^+ = \sum_{j\in J^+} u_{1j} \qquad x^+ = \sum_{j\in J^+} x'_j$$
$$u_1^- = \sum_{j\in J^-} u_{1j} \qquad x^- = \sum_{j\in J^-} x'_j.$$

From (x_0), we have $u_1^+ = -u_1^-$, and from (y:a), we have $x^+ = -x^-$. Thus, the first part of the sum is $u_1^- x_1^+ - u_1^+ x_1^- = u_1^- x_1^+ - u_1^- x_1^+ = 0$.

The second part of the sum also vanishes; the proof is the same, using the variables

$$v_1^+ = \sum_{i\in I^+} v_{1i} \qquad y^+ = \sum_{j\in J^+} y'_j$$
$$v_1^- = \sum_{i\in I^-} v_{1i} \qquad y^- = \sum_{j\in J^-} y'_j.$$

To deal with the third part of the sum, we introduce

$$v_2^+ = \sum_{i\in I^+} v_{2i} \qquad u_3^+ = \sum_{j\in J^+} u_{3j}$$
$$v_2^- = \sum_{i\in I^-} v_{2i} \qquad u_3^- = \sum_{j\in J^-} u_{3j};$$

thus $v_2^+ + v_2^- = 0$ and $u_3^+ + u_3^- = 0$. The third part of the sum is $v_2^+ u_3^- - v_2^- u_3^+ = v_2^+ u_3^- + v_2^+ u_3^+$, which is negative by (x:infeasible), unless $v_2^+ = 0$. If $v_2^+ = 0$, then $v_{2i} = 0$ for all i, in which case the sum of (y:β) over all i is $\sum_i v_{3i} = 0$, which contradicts (y:infeasible).

In a similar way, we can obtain that the fourth part of the sum is also negative. But this contradicts the fact that the sum of the four parts should be nonnegative; therefore the system is unsolvable. □

2.1. Comments on the proof

In this section, we present some remarks on the peculiarity of the above proof: It is a non-constructive existence proof, which proceeds by using linear programming duality twice.

To highlight the unusual structure of this proof, we now abstract away the concrete details. In what follows, a corresponds to a configuration of convex sets $(\mathcal{A}, \mathcal{B})$, while b roughly corresponds to the piercing line whose existence is stated in Theorem 1. We will clarify the exact semantics of b after the proof outline.

Rather than constructing a b for every a, its existence is non-constructively proven in the following way: Let c be a potential witness to the fact that b does not exist for a given a. For a given c, we construct a witness d to the fact that c is not in fact a witness for any given a. To elaborate:

- We would like to prove that $\forall a : \exists b : S_1(a, b)$. ($a$ and b are real vectors; S_1 is a bilinear system of inequalities.)
- We treat a as a parameter and b as a variable, and apply Farkas' lemma to get equivalent statement $\forall a : \neg\exists c : S_2(a, c)$.
- We switch quantifiers: $\forall c : \forall a : \neg S_2(a, c)$.
- We now treat c as a parameter and a as a variable, and apply Farkas' lemma again: $\forall c : \exists d : S_3(c, d)$.

- The above steps were all equivalences. Hence $\forall a : \exists b : S_1(a,b)$ if and only if $\forall c : \exists d : S_3(c,d)$.
- We prove $\forall c : \exists d : S_3(c,d)$ by explicitly constructing a witness d for any c.

The correspondence between the above scheme and the actual notation in the proof of Theorem 1 is as follows:

- a corresponds to our configuration of sets $(\mathcal{A}, \mathcal{B})$, parametrized by (x, y, Z).
- b corresponds to the pair of lines (ℓ_x, ℓ_y). Its exact parametrization appears in the proof only implicitly, but it can be described by (x_0, a, z_0, β) in the x direction, (y_0, a', z_0', β') in the y direction, with an extra slack variable s whose sign determines which of the two directions is supposed to be piercing.
- $S_1(a,b)$ is the bilinear system of inequalities that states that b pierces a.
- c is the potential witness of b's non-existence, parametrized by (U, V).
- $S_2(a,c)$ is the dual system stating that c is a dual witness for a, disproving the existence of a piercing line b for the given a.
- d is a weighting on the inequalities of the dual system.
- $S_3(c,d)$ states that the d-weighted sum of the inequalities of the dual system $S_2(a,c)$ leads to a contradiction, which proves that c cannot be a dual witness for any a.

We have not succeeded in streamlining this seemingly roundabout proof structure, and we now believe that such a streamlining is, in fact, not possible. That is, we formulate the following informal meta-mathematical conjecture: Any proof of Theorem 1 is necessarily non-constructive.

Let us note, however, that we *can* construct a piercing line for any given configuration a by solving a linear program. Using Megiddo's algorithm for constant dimensional linear programming [8], we can construct the piercing line in linear time. In that sense, b can be constructed. Another sense in which b can be constructed is that for a given fixed n, the space of possible a configurations can be partitioned into finitely many simplices such that a piecewise linear map assigns a piercing b to any a. That is an explicit (gigantic) formula for b. Our informal meta-mathematical conjecture states that there is no such explicit formula independent of n. Of course, to fully formalize this conjecture, we would have to specify the set of allowed operations on vectors (such as sum, maximum, argmax, argsort, etc.).

3. Fractional line transversals

In this section, we maintain the same notation as established at the beginning of Section 2. For convenience, we also assume that $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_m$. As before, $Z \in \mathbb{R}^{n \times m}$ and $P_{i,j} = (x_i, y_j, z_{i,j})$; and $A_i = \text{conv}\{P_{i,j} : j \in [m]\}$ and $B_j = \text{conv}\{P_{i,j} : i \in [n]\}$.

Theorem 2. *With the above notation, either there is a line in the plane $(x_i, \cdot, \cdot)$ for some $i \in [n]$ intersecting $\frac{m}{6}$ sets out of $B_1, \ldots, B_m$, or there is a line in the plane $(\cdot, y_j, \cdot)$ for some $j \in [m]$ intersecting $\frac{n}{6}$ sets out of $A_1, \ldots, A_n$.*

Remark. The constant $\frac{1}{6}$ depends on the bound in the fractional Helly theorem. By applying Kalai's better bound for the fractional Helly theorem [5], we may replace 1/6 by the slightly larger number $1 - \sqrt[3]{1/2} \approx 0.206$.

Unlike Theorem 1, Theorem 2 only gives a fractional transversal; however, it guarantees that this transversal lies in one of the planes $(x_i, \cdot, \cdot)$ or $(\cdot, y_j, \cdot)$.

The rest of this section comprises a proof of Theorem 2. We will start with the case $n = m = 3$ and extend this to Theorem 2 using the Fractional Helly Theorem. To simplify notation, we let H_x^i denote the plane $(x_i, \cdot, \cdot)$ and H_y^j denote the plane $(\cdot, y_j, \cdot)$. (So $A_i \subset H_x^i$ and $B_j \subset H_y^j$.)

Lemma 3. *If $n = 3$, there is either a line ℓ_x in H_x^2 intersecting all sets B_j or there is a line ℓ_y in H_y^2 intersecting all sets A_i.*

We note that Holmsen's paper [4] extending the Montejano–Karasev theorem [9] already implies that one of the two families in Lemma 3 can be pierced by a line; however, we also prove something about the *location* of the line, which will be necessary to prove Theorem 2.

Proof. Let $L_{i,j}$ be the vertical line defined by $(x_i, y_j, \cdot)$. Let P_x be the intersection of the segment $P_{2,1}P_{2,3}$ with $L_{2,2}$ and let P_y be the intersection of the segment $P_{1,2}P_{3,2}$ with $L_{2,2}$. (So P_x is contained in A_2 and P_y is contained in B_2.)

Consider the spatial quadrangle Q with vertices at $P_{i,j}$ for $i, j \in \{1, 3\}$. We form ℓ_A as the line segment that passes through both points in $Q \cap H_x^2$; we also form ℓ_B as the line segment that passes through both points in $Q \cap H_y^2$. (See Fig. 2.) Both ℓ_A and ℓ_B intersect $L_{2,2}$; in fact, they intersect at the *same* point, given by a weighted average of the four points $P_{i,j}$ with $i, j \in \{1, 3\}$. (This is the place in the argument where we need the three planes in each family to be parallel—these lines would not necessarily intersect otherwise.) Denote this common intersection by R.

By convexity, the segment $P_{2,2}P_x$ is contained in A_2, while the interval $P_{2,2}P_y$ is contained in B_2. Consider the set $\mathcal{S}_y$ of segments in H_x^2 that intersect B_1 and B_3. The collection $\{S \cap L_{2,2} : S \in \mathcal{S}_y\}$ is convex and contains both the points R and P_x, so it contains the segment RP_x. Similarly, let $\mathcal{S}_x$ be the set of segments in H_y^2 that intersect A_1 and A_3. The set $\{S \cap L_{2,2} : S \in \mathcal{S}_x\}$ is convex and contains the points P_y and R, so it contains the segment RP_y.

If the segment RP_x intersects the segment $P_{2,2}P_y$, then there is a line contained in H_x^2 that pierces B_1, B_2, B_3. Similarly, if the segment RP_y intersects the segment $P_{2,2}P_x$, then there is a line contained in H_y^2 that pierces A_1, A_2, A_3. But one of these must occur, as in any topological embedding of $\mathbb{S}^1$ to $\mathbb{R}$ some two antipodal points are mapped to the same point; so one could say that we are using the $d = 1$ case of the Borsuk–Ulam theorem when we map onto $RP_xP_{2,2}P_y$, but of course this one dimensional case can be proved more simply by checking a few cases. (See also the proof of Lemma 6 and the remark after it.) □

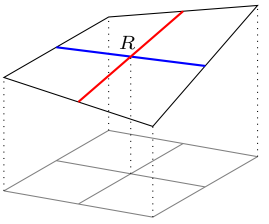
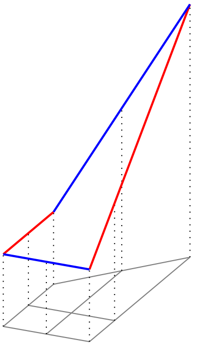


Fig. 2. The quadrangle Q, the lines ℓ_A and ℓ_B, and their intersection at R.



In Lemma 3, it is crucial that both families of planes are parallel. If this is not the case, then the claim may be false, as the following example shows. Define the points

$$x_{11} = (1,1,0) \quad x_{31} = (3,1,0) \quad x_{13} = (1,3,0)$$
$$x_{35} = (3,5,1) \quad x_{22} = (2,2,\tfrac{1}{7}),$$

then the points $x_{11}, x_{13}, x_{31}, x_{35}$ form the quadrangle shown at the right. However, for this grid, the line ℓ_B lies below ℓ_B (as defined in the proof of Lemma 3), while x_{22} lies between the two. Given

$$\begin{aligned} A_1 &= \operatorname{conv}(x_{11}, x_{13}) \qquad & B_1 &= \operatorname{conv}(x_{11}, x_{31}) \\ A_2 &= \operatorname{conv}(\ell_A, x_{22}) & B_2 &= \operatorname{conv}(\ell_B, x_{22}) \\ A_3 &= \operatorname{conv}(x_{31}, x_{35}) & B_3 &= \operatorname{conv}(x_{13}, x_{35}), \end{aligned}$$

then $A_i \cap B_j \neq \emptyset$ for every $i, j \in \{1,2,3\}$ and the sets in $\mathcal{A} = \{A_1, A_2, A_3\}$ lie in parallel planes, but the sets in $\mathcal{B} = \{B_1, B_2, B_3\}$ lie in three planes, only two of which are parallel. But ℓ_A is the only line contained in B_2's plane that can pierce A_1 and A_3, and ℓ_A does not pierce A_2. Similarly, there is no line in A_2's plane that pierces $\mathcal{B}$.

We now return to the proof of Theorem 2. A theorem of Santaló [10] from 1942 (see also [3] and [1]) states that: given a finite collection of parallel line segments in the plane such that every triple of these segments has a line transversal, there is a line that intersects all segments. The proof is based on Helly's theorem using the fact that the set of lines intersecting a fixed vertical segment (if parametrized suitably) forms a convex set in the plane. Applying the fractional Helly theorem [6] to this situation, instead of Helly's theorem, yields the following lemma.

Lemma 4 *(Fractional Helly for vertical line segments). Suppose that $\mathcal{L}$ is a collection of parallel line segments in the plane such that at least $\alpha\binom{|\mathcal{L}|}{3}$ triples of these segments can be stabbed by a line. Then there is a set of $\frac{\alpha}{3}|\mathcal{L}|$ segments in $\mathcal{L}$ that can be stabbed by a single line.*

We now have all the pieces to finish the proof.

Proof of Theorem 2. The points (x_i, y_j) form an $n \times m$ grid in the plane $z = 0$, which we will call G. A triple (j_1, j_2, j_3) with $1 \le j_1 < j_2 < j_3 \le m$ is called x_i-*good* for some $i \in [n]$ if there is a line in the plane $(x_i, \cdot, \cdot)$ intersecting $B_{j_1}, B_{j_2}, B_{j_3}$. Analogously, the triple (i_1, i_2, i_3) is y_j-*good* if there is a line in the plane $(\cdot, y_j, \cdot)$ intersecting $A_{i_1}, A_{i_2}, A_{i_3}$. Lemma 3 says that in every 3×3 subgrid $\{(x_{i_s}, y_{j_t}), s, t = 1, 2, 3\}$ of G, either (j_1, j_2, j_3) is x_{i_2}-good or (i_1, i_2, i_3) is y_{j_2}-good.

Assume next that δ is the smallest number such that the number of x_i-good triples is at most $\delta\binom{m}{3}$ for every $i \in [n]$, and the number of y_j-good triples is at most $\delta\binom{m}{3}$ for every $j \in [m]$. We show first that

$$\delta \ge \frac{1}{2}. \tag{7}$$

The proof is by double counting. Any fixed x_i-good triple, say (j_1, j_2, j_3), will appear in exactly $(i-1)(n-i)$ 3×3 subgrids. This gives at most

$$\delta\binom{m}{3}\sum_{i=2}^{n-1}(i-1)(n-i) = \delta\binom{m}{3}\binom{n}{3}$$

3×3 subgrids that contain an x_i-good triple for some i. The same argument with x and y exchanged gives the same upper bound for the number of 3×3 subgrids that contain a y_j-good triple for some j.

As the total number of 3×3 subgrids is $\binom{n}{3}\binom{m}{3}$, Lemma 3 implies that there are at least $\binom{n}{3}\binom{m}{3}$ 3×3 subgrids that are x_i- or y_j-good for some i or j. Thus $\binom{n}{3}\binom{m}{3} \le 2\delta\binom{n}{3}\binom{m}{3}$, which implies the inequality (7).

So there are at least $\frac{1}{2}\binom{m}{3}$ x_i-good triples for some $i \in [n]$ or there are at least $\frac{1}{2}\binom{n}{3}$ y_j-good triples for some $j \in [m]$. The arguments are symmetric, so assume that the latter case occurs. The plane $H = (\cdot, y_j, \cdot)$ intersects the sets $A_1, \dots, A_n$ in parallel segments, and at least half of the triples of these segment have a line transversal. Lemma 4 implies that there is a line in H intersecting $\frac{n}{6}$ of the sets $A_1, \dots, A_n$. $\square$

4. Partial extension to higher dimensions

How do we extend our results to higher dimensions? Informally, we can imagine the setup of vertical sets as taking an $n \times n$ grid in $\mathbb{R}^2$, and choosing a point in $\mathbb{R}^1$ for each intersection point of the grid. To extend to higher dimensions, we take an $n \times n \times \cdots \times n$ "base" grid in $\mathbb{R}^d$, and choose a point in $\mathbb{R}^{d-1}$ for each intersection point in the grid. We then form convex sets in $\mathbb{R}^{2d-1}$ by taking the convex hull of those points lying "above" a hyperplane in the d-dimensional grid and we group each collection of parallel sets into a family. (So we get d families overall.)

More formally, we choose vectors $x^1, \dots, x^d \in \mathbb{R}^n$ with $x^i_1 < x^i_2 < \cdots < x^i_n$ and a point $z_{\mathbf{t}} \in \mathbb{R}^{d-1}$ for each $\mathbf{t} = (t_1, \dots, t_d) \in [n]^d$. Then we set $P_{\mathbf{t}} = (x^1_{t_1}, \dots, x^d_{t_d}, z_{\mathbf{t}}) \in \mathbb{R}^{2d-1}$, and we form the convex sets

$$A^i_j = \mathrm{conv}(P_{\mathbf{t}} : t_i = j)$$

and the families $\mathcal{F}^i := \{A^i_j : 1 \le j \le n\}$.

Why do we choose a point in $\mathbb{R}^{d-1}$ for each intersection point, instead of $\mathbb{R}^1$? If we were to use $\mathbb{R}^1$, then it would be trivial to prove the analogue of Theorem 1 in higher dimensions: Restricting to a 2-dimensional subgrid of the base space recreates the 3-dimensional scenario of Section 2. To avoid a trivial reduction like this, we need $d-1$ additional dimensions.

There is an extrinsic reason for this choice of dimension, as well: According to Holmsen's extension [4] of the Montejano–Karasev theorem [9], if $n = 3$, then one of the families has a line transversal. The goal of this section is to prove a strengthened version of this statement in our setting—the analogue of Lemma 3 for higher dimensions.

Proposition 5. *Set $n = 3$ in the setup above. For some i, there is a line whose first d coordinates are $(x^1_2, x^2_2, \dots, x^{i-1}_2, \cdot, x^{i+1}_2, \dots, x^d_2)$ that pierces $\mathcal{F}^i = \{A^i_1, A^i_2, A^i_3\}$.*[1]

Proof. Let

$$B^i = \left\{(x^1_2, x^2_2, \dots, x^{i-1}_2, y, x^{i+1}_2, \dots, x^d_2) : y \in \mathbb{R}\right\} \times \mathbb{R}^{d-1}.$$

We are looking for an i such that B^i contains a line that pierces $\mathcal{F}^i$. Say that $x^i_2 = \alpha^i_1 x^i_1 + \alpha^i_3 x^i_3$ where $\alpha^i_1 + \alpha^i_3 = 1$ and $\alpha^i_1, \alpha^i_3 \ge 0$. (The values α^i_r are determined uniquely.) Given $J \subseteq [d]$ and $\mathbf{r} \in \{1,3\}^J$, define $\mathbf{t}_{J,\mathbf{r}} \in [n]^d$ by

$$(\mathbf{t}_{J,\mathbf{r}})_i = \begin{cases} 2 & \text{if } i \notin J \\ \mathbf{r}_i & \text{if } i \in J \end{cases}$$

so for example $\mathbf{t}_{[d],\mathbf{r}} = \mathbf{r}$ and $\mathbf{t}_{\emptyset,\mathbf{r}} = \mathbf{2}$. Also let $P_{J,\mathbf{r}} := P_{\mathbf{t}_{J,\mathbf{r}}}$. Now, for each $J \subseteq [d]$, define the point

$$Q_J = \sum_{\mathbf{r} \in \{1,3\}^J} \Big(\prod_{j \in J} \alpha^j_{\mathbf{r}_j}\Big) P_{J,\mathbf{r}}.$$

Since $\sum_{\mathbf{r} \in \{1,3\}^J} \left(\prod_{j \in J} \alpha^j_{\mathbf{r}_j}\right) = \prod_{j \in J} (\alpha^j_1 + \alpha^j_3) = 1$, the point Q_J is a convex combination of the points $P_{J,\mathbf{r}}$.

The coefficients are chosen in this convex combination so that the first d coordinates of Q_J are $(x^1_2, x^2_2, \dots, x^d_2)$. Moreover, we have two properties of Q_J: If $i \notin J$, then $P_{J,\mathbf{r}} \in A^i_2$ for every $\mathbf{r} \in \{1,3\}^J$, so $Q_J \in A^i_2$. On the other hand, if $i \in J$, then

$$Q_J = \alpha^i_1 \sum_{\substack{\mathbf{r} \in \{1,3\}^J \\ \mathbf{r}_i = 1}} \Big(\prod_{j \in J \setminus \{i\}} \alpha^i_{\mathbf{r}_j}\Big) P_{J,\mathbf{r}} + \alpha^i_3 \sum_{\substack{\mathbf{r} \in \{1,3\}^J \\ \mathbf{r}_i = 3}} \Big(\prod_{j \in J \setminus \{i\}} \alpha^i_{\mathbf{r}_j}\Big) P_{J,\mathbf{r}},$$

[1] In other words, the line lies "above" one of the central lines in the base $3 \times 3 \times \cdots \times 3$ grid.

which is a convex combination of a point in $B^i \cap A_1^i$ and a point in $B^i \cap A_3^i$.

If there is an i such that

$$\operatorname{conv}(Q_J : i \in J) \cap \operatorname{conv}(Q_J : i \notin J) \neq \emptyset,$$

then we are done. To see why, let Q denote the point in the intersection. On the one hand, because $Q \in \operatorname{conv}(Q_J : i \notin J)$, we know that $Q \in A_2^i$. On the other hand, because $Q \in \operatorname{conv}(Q_J : i \in J)$, Q is contained in a line that pierces A_1^i and A_3^i but is itself contained in B^i.

Since the first d coordinates of Q_J are the same for every J, the Q_J's are contained in a $(d-1)$-dimensional affine subspace. Thus, the following Lemma 6 proves that such an i always exists. □

Lemma 6. *If $Q_J \in \mathbb{R}^{d-1}$ for each $J \subseteq [d]$, there is an index i such that*

$$\operatorname{conv}(Q_J : i \in J) \cap \operatorname{conv}(Q_J : i \notin J) \neq \emptyset.$$

Proof. Suppose the conclusion is false. Then for each $i \in [d]$, there is a hyperplane H_x that separates the sets $\{Q_J : i \in J\}$ and $\{Q_J : i \notin J\}$. These hyperplanes divide $\mathbb{R}^{d-1}$ into cells, and each point Q_J must lie in a separate cell. However, we know that d hyperplanes divide $\mathbb{R}^{d-1}$ into at most

$$\sum_{i=0}^{d-1} \binom{d}{i} = 2^d - 1$$

cells, which is a contradiction. □

Remark. We can alternatively prove Lemma 6 via topology. The function $f\colon J \mapsto Q_J$ can be considered instead as a function $f\colon \{-1,1\}^d \to \mathbb{R}^{d-1}$. We can extend f to a function $\tilde{f}$ on the boundary of the unit cube in which $\tilde{f}(F) = \operatorname{conv}\big(f(F)\big)$ for every facet F. By the Borsuk–Ulam theorem, there is a pair of antipodal points $u, -u \in \partial[-1,1]^d$ such that $\tilde{f}(u) = \tilde{f}(-u)$; the facets that u and $-u$ belong to correspond exactly to a partition of vertices as described in Lemma 6.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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