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Interaction between K-mesons

and light nuclei

by

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CONTENTS

Abstract

The three-body K^-d interactions at low Kaon-lab momenta (100-300 MeV/c) are expressed in terms of two body interactions ($\bar{K}N$ and NN) by means of the Impulse Approximation method, including multiple scattering terms. The $\bar{K}N$ S-waves are derived from Ross-Humphrey's solution I and II of scattering lengths. The NN wave functions for the np and pp continuous states calculated from the NN phase shifts obtained by Breit and co-workers (1962).

K^-d elastic, inelastic and charge-exchange cross-sections, both total and differential, are calculated for Ross-Humphrey's solution I and II.

A not very successful attempt has been made of calculating independently the K^-d absorption cross-sections. To achieve this, the resonant group structure method was used, combined with $\bar{K}N$ complex Yukawa potentials, calculated from the Ross-Humphrey's sets of scattering lengths by means of a variational principle (Schwinger's and Hulthèn's). The limitations of this method are discussed.

Coulomb effects and K^-K^0 mass difference are taken into account. However, no attempt was made of including in the calculations the virtual charge-exchange processes in the multiple scattering terms, although a method to achieve this is developed.

An analysis of the numerical results is made in the light of other theoretical works on the same subject and the scant K^-d experimental data. The conclusion is reached that it is very likely that the calculated K^-d

charge-exchange cross-sections have the correct values, in spite of the inadequacy of Ross-Humphrey's parameters.

K^+d elastic, inelastic and charge-exchange cross-sections are also calculated in the same kaon-lab momenta range. The probability conservation of total flux is checked.

Chapter I

Introduction

As in the present work K^-d and K^+d scattering processes at low momenta (below 300 MeV/c in the K laboratory system) are studied in terms of two-body forces (KN and NN), it was thought convenient to discuss briefly in the first place these simpler interactions and some general properties and conservation laws of the elementary particles involved in them.

This chapter is devoted mainly to K^-N interactions. The NN interactions are treated in Chapter III. The analytical properties of the two nucleons initial (deuteron) and final state wave functions are given there in so far as they are needed for the calculation of the K^-d elastic, inelastic, charge-exchange and absorption cross-sections.

It is well-known that K^-p interactions for K^--laboratory momenta below 300 MeV/c lead to the following reactions

$$K^- + p \rightarrow K^- + p \tag{1.1}$$

$$\rightarrow \bar{K}^0 + n \tag{1.2}$$

$$\rightarrow \Sigma^- + \pi^+ \tag{1.3}$$

$$\rightarrow \Sigma^+ + \pi^- \tag{1.4}$$

$$\rightarrow \Sigma^0 + \pi^0 \tag{1.5}$$

$$\rightarrow \Lambda + \pi^0 \tag{1.6}$$

$$\rightarrow \Lambda + \pi^+ + \pi^- \tag{1.7}$$

All these channels are open to K^-p collisions covering almost of the energy interval $(0, \infty)$. Only the reaction (1.2) has a higher lower limit, since the rest mass difference between the two particles systems $\bar{K}^0 n$ and $K^- p$ is $\sim 5 \cdot 7$MeV. But the threshold energies for $\Lambda\pi$, $\Sigma\pi$ and $\Lambda(2\pi)$ systems lie respectively at 180, 100 and 40 MeV below the threshold energy for the $\bar{K}N$ system.

The reactions from (1.1) to (1.7) are due to strong nuclear forces. The 2π production in reaction (1.7) is not very important and accordingly can be ignored. But the interactions in the other channels can not be discarded in any quantitative analysis of the $K^- p$ collisions.

At first sight a complete description of the $K^- p$ system with six equally important open channels seems to be a formidable task. But a drastic simplification can be achieved when it is realised that in all these channels only strong nuclear interactions come into play, so that the principle of charge independence of nuclear forces is valid.

According to this principle and as an example, the three π-mesons π^+, π^0 and π^- are different charge states of the same particle – the π-meson.

The rest masses of the π-mesons are nearly the same ($\sim$ 140MeV). The slight differences are related to the magnitudes of the electric charges. Therefore, if the electromagnetic interactions are not taken into account and the small differences in mass are neglected, it does not matter which of the π-mesons is considered in any strong interaction, unless its state of motion is altered.

The principle of charge independence together with the principle of charge conservation (of which equations (1.1) $\rightarrow$ (1.7) are examples) can be formulated in the form of a new conservation principle, true for all strong interactions, if the concept of isotopic spin is introduced.

In isotopic-spin terms, the π-meson is a charge triplet with total isotopic-spin quantum number $I = 1$ and Z-component projections $I_Z = 1(\pi^+)$, $0(\pi^o)$, $-1(\pi^-)$; and the nucleon is a charge doublet with $I = \frac{1}{2}$ and $I_Z = \frac{1}{2}(p), -\frac{1}{2}(n)$.

Strange particles (K-mesons and the three hyperons families Λ, Σ, Ξ) can also be described as charge multiplets.

(K^+, K^o)	doublet	$I = \frac{1}{2}$;	$I_Z = \frac{1}{2}, -\frac{1}{2}$
Λ	singlet	$I = 0$;	$I_Z = 0$
$(\Sigma^+, \Sigma^o, \Sigma^-)$	triplet	$I = 1$;	$I_Z = 1, 0 - 1$
(Ξ^o, Ξ^-)	doublet	$I = \frac{1}{2}$;	$I_Z = \frac{1}{2}, -\frac{1}{2}$

It is now obvious that I is related with charge independence of the nuclear forces (strong interactions) and I_Z with charge conservation. If to this list of particles are added the charge-multiplets of the anti-particles, it can be stated that - "the total isotopic-spin I (charge independence) and its I_Z component (charge conservation) are conserved in all strong interactions".

Evidently the principle of charge conservation is more general than the I_Z invariance, since the isotopic-spin concept has a meaning only for strong interactions. Applying the I_Z-invariance to the K^-p system, two possible total isotopic-spin channels are obtained

$$I_Z = 0, \quad I = 0 \quad \phi_o^o = \frac{1}{\sqrt{2}}\{|K^-p\rangle - |\bar{K}^o n\rangle\}$$

$$I = 1 \quad \phi_1^o = \frac{1}{\sqrt{2}}\{|K^-p\rangle + |\bar{K}^o n\rangle\}$$

The states with the same I and I_Z for pion-hyperon systems are

$$I_Z = 0, \quad I = 0 \quad \phi_o = \frac{1}{\sqrt{3}}\{|\Sigma^+\pi^->-|\Sigma^o\pi^o>+|\Sigma^-\pi^+>\}$$

$$I = 1 \quad \phi_1 = \frac{1}{\sqrt{2}}\{|\Sigma^+\pi>-|\Sigma^-\pi^+>\}$$

$$\bar{\phi}_1 = |\Lambda\pi^o>$$

If T_{fi} represents the element of the transition matrix T, with initial state $\underline{i}$ and final state $\underline{f}$, it is possible to write

$$T_{fi} = C_{fi}(0)T_{fi}^{(o)} + C_{fi}(1)T_{fi}^{(1)}.$$

where $T_{fi}^{(I)}$ is the transiction matrix element for total isotopic-spin I and the $C_{fi}(I)$'s are coefficients derived from the isotopic-spin wave functions.

The following Table is easily established (Matthews and Salam, 1959)

TABLE I, 1

i	f	$C_{fi}(o)$	$C_{fi}(1)$
$K^- + p$	$K^- + p$	1/2	1/2
	$\bar{K}^o + n$	-1/2	1/2
	$\Sigma^+ + \pi^-$	$1/\sqrt{6}$	$-1/\sqrt{2}$
	$\Sigma^- + \pi^+$	$1/\sqrt{6}$	$1/\sqrt{2}$
	$\Sigma^o + \pi^o$	$-1/\sqrt{6}$	0
	$\Lambda + \pi^o$	0	$1/\sqrt{2}$

The charge independence of nuclear forces makes it clear now that the possible reactions for K^-p system can be expressed in terms of a two-by-two $(T^{(o)})$ and a three-by-three ($T^{(1)}$) transiction matrices. $T^{(o)}$ describes the reactions

$$\bar{K} + N \rightarrow \bar{K} + N$$

$$\rightarrow \Sigma + \pi$$

which occur through the isotopic-spin channel I = 0; and $T^{(1)}$ represents the interactions

$$\bar{K} + N \rightarrow \bar{K} + N$$

$$\rightarrow \Sigma + \pi$$

$$\rightarrow n + \pi$$

taking place in the I = 1 channel. With the help of the $T^{(I)}$ matrices the description of the K^-n system turns out to be very simple.

The I_Z-invariance leads in this case to only one isotopic-spin channel,

$$I_Z = -1,\ I = 1 \quad \phi_1^{-1} = |K^-n>$$

The states with same I and I_Z for pion-hyperon systems are

$$I_Z = -1,\ I = 1 \ \psi_1 = \{|\Sigma^-\pi^o> - |\Sigma^o\pi^->\}$$

$$\bar{\psi}_1 = |\Lambda\pi^->$$

and the $C_{fi}(1)$ are given in the following table

TABLE I, 2

i	f	$C_{fi}(1)$
$K^- + n$	$K^- + n$	1
	$\Sigma^- + \pi^o$	$1/\sqrt{2}$
	$\Sigma^o + \pi^-$	$-1/\sqrt{2}$
	$\Lambda + \pi^-$	1

Without going into much detail, the general properties and characteristics of the $T^{(I)}$ are now stated in the next paragraphs.

Let

$$H = Ho + V$$

be the hamiltonian of the $\bar{K}N$ system where Ho represents the non-interacting $\bar{K}$ and N and V denote the interaction energy. The solutions of the equation

$$Ho\phi_i = E_i\phi_i$$

form a complete orthonormal set of functions.

In the ϕ_i-representation, the Heitler's integral equation (see Dalitz 1962, p.53) takes the form

$$T_{fi} + i\pi\sum_m K_{fm}\rho_m(E)T_{mi} = K_{fi} \tag{1.8}$$

where K is the reaction matrix and $\rho m(Em)dEm$ is the number of states with energies between Em and Em + dEm.

In matrix form (1.8) is

$$T + i\pi K\rho T = K$$

and, after obvious algebraic matrix operations

$$K^{-1} + i\pi\rho = T^{-1} \tag{1.9}$$

or, multiplying (1.9) on the left by T and on the right by K

$$T + i\pi T\rho K = K ;$$

then

$$T = K(1 + i\pi\rho K)^{-1} = (1 + i\pi K\rho)^{-1}K . \tag{1.10}$$

Putting now

$$K_1 = (\pi\rho)^{\frac{1}{2}}K(\pi\rho)^{\frac{1}{2}}$$

$$T_1 = (\pi\rho)^{\frac{1}{2}}T(\pi\rho)^{\frac{1}{2}}$$

(1.10) gives

$$T_1 = K_1(1 + iK_1)^{-1} = (1 + iK_1)^{-1}K_1$$

The scattering matrix S is related with T_1 in the following way

$$\begin{aligned} S &= 1 + 2iT_1 \\ &= (1 + iK_1)^{-1}(1 - iK_1) \\ &= (1 - iK_1)(1 + iK_1)^{-1} \end{aligned}$$

But it is possible to prove in very general terms that K (and consequently K_1) is a hermitian matrix. In those circumstances S satisfies automatically the unitarity condition, which means that the flux of probability is conserved (Matthews and Salam, 1959):

$$SS^+ = (1 + iK_1)^{-1}(1 - iK_1)(1 - iK_1)^{-1}(1 + iK_1) = 1$$

A further step in the knowledge of the properties of the K-matrix comes from the time-reflection symmetry principle. According to this principle (which holds for strong and electromagnetic interactions, but probably not for weak forces (Christenson et al., 1964)), the elements of the K-matrix are real (see Dalitz, 1962, p.55). Since the K-matrix is hermitian its elements are real and symmetric. Therefore, the number of distinct elements of the second order matrix $K^{(o)}$ is 3 and of the third order matrix $K^{(1)}$ is 6.

Now, by a convenient normalisation of the ϕ_i, it is possible to write (Dalitz, 1962 p. 56)

$$\pi\rho = k$$

for a two-particle channels system (meson-baryon), so that (1.9) reduces to

$$T^{-1} = K^{-1} + ik \qquad (1.11)$$

Finally, if it is assumed that only S-waves are in operation in all channels at low energies, the K-matrix has constant elements. Therefore, only 9 independent parameters are necessary to describe quantitatively the $\bar{K}N$ system in this energy region - the 3 different elements of $K^{(o)}$ plus the 6 distinct elements of $K^{(1)}$.

The main features of the $T^{(I)}$ are independent of the matrix order. They can be exemplified then by considering the second order matrix $T^{(o)}$.

Writing the constant $K^{(o)}$-matrix in the form

$$K^{(o)} = \left\|\begin{matrix} s & t \\ t & u \end{matrix}\right\| .$$

and representing by k and q the two centre-of-mass momenta in channels $1(\bar{K},N)$ and $2(\Sigma\pi)$ respectively, (1.11) leads to

$$\left|T^{(o)}\right|^{-1} = \frac{1}{\Delta}\left\|\begin{matrix} u + i\Delta k & - t \\ - t & s + i\Delta q \end{matrix}\right\|$$

where $\Delta = su - t^2$

Then

$$T^{(o)} = \frac{1}{1 + i(sk + uq) - \Delta kq}\left\|\begin{matrix} s + i\Delta q & t \\ t & u + i\Delta q \end{matrix}\right\| \qquad (1.11^1)$$

The T-matrix element for elastic scattering in channel $1(\bar{K}N)$ is

$$T_{11}^{(o)} = \frac{s + 1\Delta q}{1 + 1(sk + uq) - \Delta kq}$$

or, putting

$$A_o = - s + itq \frac{1}{1 + itq} t \qquad (1.12)$$

$$T_{11}^{(o)} = - \frac{A_o^i}{A - ikA_o}$$

If δ^o is the S-wave phase shift of the elastic scattering in the two-particle channel I = 0, then

$$T^{(o)} = -\frac{e^{i\delta^o}\sin\delta^o}{k}$$

or

$$k\cot\delta^o = \frac{1}{A_o} \qquad (1.13)$$

Therefore $A_o(= a_o + ib_o)$ is the complex scattering length. Equation (1.13) represents "formally" the zero-energy approximation of the well-known effective range theory for one channel scattering

$$k\cot\delta^o = \frac{1}{A_o} + \frac{1}{2}Rok^2 + \ldots \qquad (1.14)$$

The word "formally" was stressed because in (1.14) A_o is constant and in (1.13) is energy-dependent through (1.12). However, if it is admitted that q in (1.12) has large values - and it does, because the threshold energy of the $\Sigma\pi$ system is 100 MeV below the threshold energy of the $\bar{K}N$ system - the variation of q with k is small when k is in the low energy region and consequently A_o is nearly constant.

This discussion shows clearly why A_o is a complex quantity: no elastic scattering can occur in channel 1 without some of the scattered beam being absorbed, or, what is the same thing, without suffering transi tions into channel 2.

In spite of being more difficult to handle a three-by-three matrix, a zero-effective range formula

$$k\cot\delta^1 = \frac{1}{A_1}\ , \quad A_1 = a_1 + ib_1$$

can also be obtained for the K^-p system in the isotopic spin channel $I = 1$ (Dalitz, 1962; Matthews and Salam, 1959).

The possibility of describing the K^-p interactions by a zero-effective range theory was suggested by Jackson, Ravenhall and Wyld in 1958. Some refinements dealing with the Coulomb effect and the mass difference between the $\bar{K}^0n$ and K^-p systems were introduced later. The principal developments of this theory can be found in Dalitz and Tuan (1959) and Dalitz (1962). The last work has an extensive bibliography on this subject.

CHAPTER II

K^-N Complex Potentials

1. Introduction

The aim of this chapter is to translate the strong nuclear forces involved in K^-N processes into terms of two phenomenological complex Yukawa potentials, dealing with the interactions in isotopic-spin channels I = 0 and I = 1.

$$-(u_I + iv_I)e^{-br}/br \qquad (2.1)$$

The Yukawa shape is chosen in agreement with the short range nature of the nuclear forces and the complex factor is assumed to deal with the Λ, Σ and π production in $\bar{K}N$ interactions, interpreted as absorption as it is explained in Chapter I.

In (2.1) u_I and v_I are constants for the same I, b is the reciprocal of the potential range a(= 1/b) and r is the distance between the meson and the nucleon.

The range of $\bar{K}N$ interactions is taken equal to the kaon Compton wave length

$$a = \hbar/m_K c = 0.4 \text{ Fermi} \qquad (2.2)$$

The reason for this is the following: the energy conservation principle is violated if a virtual meson (rest mass m_K, rest energy $\Delta E = m_K c^2$) is emitted by one nucleon at rest. But this violation does not matter if it occurs within the limits imposed by the time-energy uncertainty principle $\Delta E.\Delta t \sim \hbar$. This implies that the virtual meson cannot live longer than a period of time

$$t \sim \hbar | m_K c^2$$

In such conditions, the distance covered by the virtual particle during the time Δt must be smaller than $c\Delta t \sim \hbar | m_K c$, where c is the velocity of light - the maximum velocity. Therefore (2.2) is obtained.

Relativistic effects (see Appendix A) are so small for K^--laboratory momenta below 300 MeV/c (the largest momentum considered in this work) that they are discarded here and in the following chapters.

No attempt is made at this stage to consider the mass differences of the particles in the same charge multiplet ($\bar{K}$ or N), because the K^-N phenomenological potentials to be derived here are not used in the actual calculations of K^-d collisions.

Labelling $\bar{K}$ and N with indices 1 and 2 respectively, the isotopic-spin dependence can be expressed by the potential

$$Z(r) = U(r) + V(r)\, P_{12} \qquad (2.3)$$

P_{12} is the isotopic-spin exchange operator, which, as it is well-known, is related to Pauli's spin operator $\bar{\tau}$, with components

$$\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

by the equation

$$P_{12} = \frac{1}{2}\left[1 + \bar{\tau}(1)\cdot\bar{\tau}(2) \right] \qquad (2.4)$$

Representing now the isotopic-spin functions for $\bar{K}$ and N charge multiplets by

$$\chi_{I,I_z}(m), \quad m = 1, 2; \; I = 1/2; \; I_z = 1/2, -1/2,$$

The isotopic-spin functions for the $\bar{K}N$ system are

$$
\begin{aligned}
Y_{1,1}(1,2) &= Y^{(1)}_{1/2,1/2}\, Y^{(2)}_{1/2,1/2} \\
Y_{1,0}(1,2) &= \frac{1}{\sqrt{2}}\left[Y^{(1)}_{1/2,1/2}\, Y^{(2)}_{1/2,-1/2} + Y^{(1)}_{1/2,-1/2}\, Y^{(2)}_{1/2,1/2}\right] \\
Y_{1,-1}(1,2) &= Y^{(1)}_{1/2,-1/2}\, Y^{(2)}_{1/2,-1/2} \\
Y_{0,0}(1,2) &= \frac{1}{\sqrt{2}}\left[Y^{(1)}_{1/2,1/2}\, Y^{(2)}_{1/2,-1/2} - Y^{(1)}_{1/2,-1/2}\, Y^{(2)}_{1/2,1/2}\right]
\end{aligned}
\qquad (2.5)
$$

Those functions, when subjected to the operator $[\bar{\tau}(1)\cdot\bar{\tau}(2)]$ give

$$
\begin{aligned}
[\bar{\tau}(1)\bar{\tau}(2)]\, Y_{1,I_z}(1,2) &= Y_{1,I_z}(1,2)\,, \quad I_z = 1, 0, -1 \\
[\bar{\tau}(1)\bar{\tau}(2)]\, Y_{0,0}(1,2) &= -3\, Y_{0,0}(1,2)
\end{aligned}
$$

so that

$$
\begin{aligned}
P_{12}\, Y_{1,I_z}(1,2) &= Y_{1,I_z}(1,2)\,, \quad I_z = 1, 0, -1 \\
P_{12}\, Y_{0,0}(1,2) &= -Y_{0,0}(1,2)
\end{aligned}
\qquad (2.6)
$$

In K^-d scattering processes, the meson-nucleon basic interactions are K^-p and K^-n. The charge conservation principle implies selection rules on the possible isotopic-spin states given in (2.5). Thus, the pair $K^-p(I_Z = 0)$ has just two states $Y_{1,0}(1,2)$ and $Y_{0,0}(1,2)$ and the pair $K^-n(I_Z = -1)$ only one $Y_{1,-1}(1,2)$.

The wave functions and Schrodinger equations in the meson-nucleon centre-of-mass system for K^-p and K^-n interactions are:

1) for K^-p processes

$$\psi(1,2) = Y_{0,0}(1,2)\, u_0(r) + Y_{1,0}(1,2)\, u_1(r)$$

$$(\nabla^2 + k^2)\,\psi(1,2) = \frac{2\mu}{\hbar^2}\left[-\frac{e^2}{r} + Z(r)\right]\psi(1,2);$$

2) for K^-n processes

$$\psi(1,2) = Y_{1,-1}(1,2)\, u(r)$$

$$(\nabla^2 + k^2)\,\psi(1,2) = \frac{2\mu}{\hbar^2} Z(r)\,\psi(1,2)\,.$$

In the above equations k is the wave numbers and μ is the reduced mass of the K^-N system: $\mu = mm_K|(m + mK)$ where m is the nucleon mass.

In case 1) Schrodinger equation gives, taking into account (2.3) and (2.6),

$$(\nabla^2 + k^2)\,\psi(1,2) = -\frac{2\mu}{\hbar^2}\frac{e^2}{r}\psi(1,2) + $$

$$+ \frac{2\mu}{\hbar^2}\left[U(r) - V(r)\right] Y_{0,0}(1,2)\, u_0(r) +$$

$$+ \frac{2\mu}{\hbar^2}\left[U(r) + V(r)\right] Y_{1,0}(1,2)\, u_1(r)\,.$$

Multiplying both members of this equation first by $Y_{o,o}(1,2)$ and secondly by $Y_{1,o}(1,2)$, two partial differential equations are obtained:

$$(\nabla^2 + k^2)\, u_0(r) = \frac{2\mu}{\hbar^2}\left[-\frac{e^2}{r} + U(r) - V(r)\right] u_0(r) \tag{2.7}$$

$$(\nabla^2 + k^2)\, u_1(r) = \frac{2\mu}{\hbar^2}\left[-\frac{e^2}{r} + U(r) + V(r)\right] u_1(r) \tag{2.8}$$

In case 2) only one equation is obtained:

$$(\nabla^2 + k^2)\, u(r) = \frac{2\mu}{\hbar^2}\left[U(r) + V(r)\right] u(r)$$

analogous to (2.8), except for Coulomb interaction.

It is now obvious that the two nuclear potentials in equation (2.7) and (2.8) must be identified with the Yukawa Wells (2.1):

$$I = 0 \qquad U(r) - V(r) = -(u_0 + i v_0)\, e^{-br} / br \tag{2.9}$$

$$I = 1 \qquad U(r) + V(r) = -(u_1 + i v_1)\, e^{-br} / br \tag{2.10}$$

The following sections of this chapter are devoted to the calculation of these potentials from the K^-p experimental data.

2. Ross and Humphrey's sets of K^-p scattering lengths

As it was explained in Chapter I, the K^-p interactions for K^--laboratory momenta below 300 MeV/c are described, besides other elements, by two complex scattering lengths AI(= $a_I + ib_I$), I = 0, 1, where I is the label for the isotopic-spin channels.

Following these ideas, Ross and Humphrey made a complete analysis of the K^-p scattering data in this low energy region in terms of six parameters (Ross, 1961):

1) The two complex scattering lengths A_o and A_1 related to the respective δ-wave complex phase shifts δ^o and δ^1 by the zero-effective range formula

$$k \cot \delta^I = 1/A_I \quad , \quad \delta^I = \alpha_I + i \beta_I \tag{2.11}$$

2) The ratio Σ of the Λ production to the $\Lambda + \Sigma$ production in channel $I = 1$ at rest;

3) The difference in phase angle ϕ between the matrix elements for $\Sigma\pi$ production in channels $I = 1$ and $I = 0$.

Two possible solutions, I and II, were obtained by Ross and Humphrey for these parameters (the A_I are measured in fermi):

Table II. 1.

Solution	a_o	b_o	a_1	b_1	Σ	ϕ
I	−0·22 (±1·1)	2·74 (±0·3)	0·02 (±0·3)	0·38 (±0·08)	0·40 (±0·03)	96°
II	−0·59 (±0·46)	0·96 (±0·17)	1·20 (±0·06)	0·56 (±0·15)	0·39 (±0·02)	−50°

(the numbers between brackets are the errors affecting the calculated parameters)

The answer to the question of which set of possible solutions is in better agreement with the experiment is left for later chapters.

A relation between the $u_I + iv_I$ and the $\delta^I = (\alpha_I + i\beta_I)$ must be found now to calculate the nuclear potentials (2.1).

The work carried out by Ross and Humphrey on the computation of the A_I, takes into account the Coulomb effects and the mass difference between systems $\bar{K}^o n$ and $K^- p$ in such a way that the δ^I in (2.11) do not include the electromagnetic interactions. Therefore, the phase shifts δ^I are related to the asymptotic form of the S-wave radial equations derived from (2.7) and (2.8) with the Coulomb term missing.

In such conditions, a variational principle which, with the help of a trial function relates the δ^I with the constants of the short range potentials (2.1) gives the desired relationship.. In this chapter, Schwinger's variational principle is chosen.

3. Schwinger's variational principle

In the following pages the generalization of this principle (Lippmann and Schwinger, 1950) to complex potentials is carefully examined.

Let W(r) represent any one of the spherical symmetric complex potentials (2.1). Then, the L-wave radial equation derived from Shrodinger equations (2.7) or (2.8) without the Coulomb term is

$$f_L \psi_L(r) = W(r)\psi_L(r) \qquad (2.12)$$

where

$$f_L \equiv \frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + k^2 - \frac{L(L+1)}{r^2}\right] \tag{2.13}$$

$\psi_L(r)$ is the regular solution of (2.12) at the origin. Because the potentials (2.1) are such that $rW(r) \to 0$ when $r \to \infty$, $\psi_L(r)$ admits the normalisation

$$\psi_L(r) \longrightarrow \frac{e^{i\delta_L}}{kr} \sin\left(kr - L\frac{\pi}{2} + \delta_2\right) \tag{2.14}$$

(where $\delta_L = \alpha_L + i\beta_L$, α_L and β_L being constants from the same k) and its regularity at the origin implies the boundary condition

$$\begin{array}{c} r\psi_L(r) \to 0 \\ r \to 0 \end{array} \tag{2.15}$$

The linear operator f_L is associated with the Green's function (Messiah, 1962, p.818):

$$g_L^+(r,r') = -\frac{2\mu k}{\hbar^2} j_L(kr_<)\, h_L^+(kr_>) \tag{2.16}$$

(see Appendix C for the definition of $j_L(kr)$ and $h_L^+(kr)$), which satisfies the equation

$$f_L g_L^+(r,r') = \frac{\delta(r-r')}{rr'} \tag{2.17}$$

In (2.16) $r<$ $(r>)$ is the lesser (the greater) of r and r'.

With the help of (2.16) it is possible to transform (2.12) into an integral equation

$$\psi_l(r) = j_L(kr) + \int_0^\infty g_L^+(r,r')\, W(r')\, \psi_L(r')\, r'^2 dr' \tag{2.18}$$

This expression of ψ_L is obviously a solution of (2.12). To see that it is so, one only needs to apply f_L to both members of (2.18), taking into account (2.17) together with the equation

$$f_L j_L(kr) = 0 \tag{2.19}$$

Then (2.12) is obtained.

Multiplying (2.12) by $j_L(kr)$ and (2.19) by $\psi_L(r)$ and subtracting the last result from the former one gets

$$j_L(kr)\, f_L \psi_L(r) - \psi_L(r)\, f_L\, j_L(kr) = W(r)\, j_L(kr)\, \psi_L(r)$$

or

$$\frac{d}{dr}\left[r j_L \frac{d}{dr}(r\psi_L) - r\psi_L \frac{d}{dr}(r j_L)\right] = \frac{2\mu}{\hbar^2} W(r)\, j_L \psi_L r^2 \tag{2.20}$$

Integrating both members of (2.20) between 0 and ∞, and considering the boundary condition (2.15) as well as the asymptotic form (2.14), an integral expression for phase shift δ_L is obtained

$$e^{i\delta_L} \sin \delta_L = -\frac{2\mu k}{\hbar^2} \int_0^\infty j_L(kr)\, W(r)\, \psi_L(r)\, r^2\, dr \tag{2.21}$$

with (2.21) it is now possible to show that the integral solution (2.16) of $\psi_L(r)$ leads also to the asymptotic form (2.14).

In fact (2.16) gives

$$\psi_L(r) = j_L(kr) -$$

$$- \frac{2\mu k}{\hbar^2} h_L^+(kr) \int_0^r j_L(kr') W(r') \psi_L(r') r'^2 dr'$$

$$- \frac{2\mu k}{\hbar^2} j_L(kr) \int_r^\infty h_L^+(kr') W(r') \psi_L(r') r'^2 dr' \qquad (2.22)$$

where $r \to \infty$, the second integral in (2.22) vanishes and considering (2.21) and the asymptotic forms of $j_L(kr)$ and $h^+(kr)$, one gets

$$(kr)\,\psi_L(kr) \longrightarrow \sin\left(kr - L\frac{\pi}{2}\right) + e^{i\left(kr - L\frac{\pi}{2}\right)} e^{i\delta_L} \sin\delta_L =$$

$$= e^{i\delta_L} \sin\left(kr - L\frac{\pi}{2} + \delta_L\right)$$

Now, let the scalar product of two radial wave functions $f_1(r)$ and $f_2(r)$ be defined by the integral

$$< f_1, f_2 > = \int_0^\infty f_1(r) f_2(r) r^2 dr .$$

Introducing the linear operator g_L^+, so that the equation

$$\psi_L(r) = j_L(kr) + g_L^+ W \psi_L \qquad (2.23)$$

is exactly the same as the integral equation (2.18), two important functionals can be defined in very simple terms, using the notion of scalar product:

$$A[\psi] = <\psi, W j_L> \qquad (2.24)$$

$$B[\psi] = <\psi, (W - W g_L^+ W)\psi> \qquad (2.25)$$

It is clear that $A[\psi]$ and $B[\psi]$ are equal when $\psi = \psi_L$, as (2.23) shows. In this case they are related to the phase shift δ_L by

$$A[\psi_l] = B[\psi_l] \doteq -\frac{\hbar^2}{2\mu k} e^{i\delta_l} \sin\delta_l \tag{2.26}$$

With (2.24) and (2.25) it is possible now to build a third functional

$$T[\psi] = \frac{A^2}{B}$$

which is stationary for $\psi = \psi_L$, that is, for any arbitrary variation $\delta\psi$ of ψ in the vicinity of ψ_L, the corresponding variation δT of T vanishes.

δT is given by

$$\delta T = \frac{2A}{B}\delta A - \frac{A^2}{B^2}\delta B \tag{2.27}$$

where

$$\delta A = \langle\delta\psi, W\gamma_L\rangle$$

and

$$\delta B = \delta B_1 + \delta B_2$$

with

$$\delta B_1 = \langle\delta\psi, W(1 - g_L^+ W)\psi\rangle$$

$$\delta B_2 = \langle\psi, W(1 - g_L^+ W)\delta\psi\rangle$$

The two variational terms in δB are equal either for real or complex potentials. This is the essential step in the demonstration of Schwinger's variational principle, because, if $\delta B_1 = \delta B_2$, the stationarity of $T[\psi]$ for $\psi = \psi_L$ is proved. For this value of ψ one gets

$$\delta B[\psi_L] = 2\delta A[\psi_L]$$

and, from (2.26) and (2.27) it follows that $\delta T[\psi_L] = 0$.

To prove that $\delta B_1 = \delta B_2$, it is sufficient to show that

$$\langle \delta\psi, W g_L^+ W \psi \rangle = \langle \psi, W g_L^+ W \delta\psi \rangle$$

From

$$g_L^+ W \delta\psi \sim h_L^+(kr) \int_0^r j_L(kr') W(r') \delta\psi(r') r'^2 dr' +$$

$$+ j_L(kr) \int_r^\infty h_L^+(kr') W(r') \delta\psi(r') r'^2 dr' \qquad (2.28)$$

two double integrals result, when calculating $\langle \psi, W g_L^+ W \delta\psi \rangle$. The first is

$$\int_0^\infty \psi(r) W(r) h_L^+(kr) r^2 dr . \int_0^r j_L(kr') W(r') \delta\psi(r') r'^2 dr' \qquad (2.29)$$

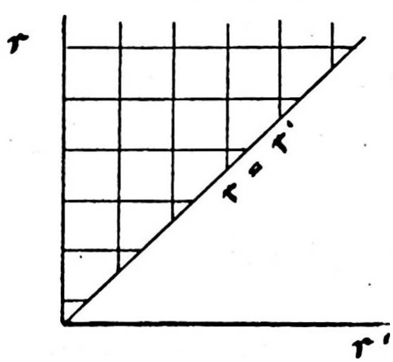


Fig. II.1

The domain of integration of (2.29) is the squared region in Fig (II.1). But (2.29) can also be calculated by integrating first with respect to r and secondly with respect to r'. In this case one gets

$$\int_0^\infty j_L(kr') W(r') \delta\psi(r') r'^2 dr' . \int_{r'}^\infty \psi(r) W(r) h_L^+(kr) r^2 dr$$

This integral is formally the same as the one that is obtained from $<\delta\psi, Wg_L^+ W\psi>$ in a development analogous to (2.28):

$$\int_0^\infty \delta\psi(r) W(r) j_l(kr) r^2 dr \cdot \int_r^\infty \psi(r') W(r') h_l^+(kr') r'^2 dr'$$

A similar proof holds for the equality between the double integral obtained from the second term in (2.28) and the first double integral in the equivalent development of $<\delta\psi, Wg_L^+ W\psi>$. Then

$$\delta B_1 = \delta B_2$$

In this proof no restriction was made on the nature (real or complex) of the nuclear potential W(r). Therefore, the extension of Schwinger's variational principle to complex potentials is achieved.

Equations (2.26) lead now to the variational formula

$$\frac{e^{-i\delta_L}}{\sin\delta_L} = -\frac{\hbar^2}{2\mu k} \cdot \frac{B[\psi_l]}{A[\psi_l]} \tag{2.30}$$

which can be transformed into a more suitable expression, if the real Green's function

$$g_l(r, r') = -\frac{2\mu k}{\hbar^2} j_L(kr_<) n_L(kr_>) \tag{2.31}$$

is introduced. In fact, because

$$h_L^+(kr) = n_l(kr) + i j_l(kr)$$

one gets

$$g_l^+(r,r') = g_l(r,r') - i\,\frac{2\mu k}{\hbar^2}\, j_l(kr)\, j_l(kr') \qquad (2.32)$$

This equation, combined with (2.24) and (2.25) gives

$$B[\psi] = \langle \psi, W(1-g_l W)\psi\rangle - i\,\frac{2\mu k}{\hbar^2}\, A^2[\psi]$$

Therefore (2.30 can be written

$$k\cot\delta_l = -\frac{\hbar^2}{2\mu}\,\frac{\langle \psi_L, W(1-g_l W)\psi_L\rangle}{\langle \psi_L, W j_L\rangle^2} \qquad (2:33)$$

This is the common form under which Schwinger's variational principle is known. Equation (2.33) has the advantage of remaining invariant for different normalizations of ψ_L and for any alteration of this function in the region where the short range potential W(r) is vanishingly small. Such properties, associated with the stationary character of (2.33) for any trial function differing slightly from ψ_L within the potential range, make Schwinger's formula very accurate.

4. The zero-energy S-wave approximation

As Table III shows, the data available for building the potentials (2.1) are the complex scattering lengths A_I ($I = 0,1$) related to the S-wave complex phase shifts δ^I by (2.11). A good S-wave trial function will solve, then, the problem of finding an expression that relates the $u_I + iv_I$ in (2.1) with the δ^I. The $u_I + iv_I$ calculated in this way depend on k. But Table III shows also that the scattering lengths are affected by large errors. Therefore, it is a reasonable assumption to calculate the potentials (2.1) only for one value of k:k = 0.

This approximateion brings several simplifications to Schwinger's variational formula (2.33). Putting

$$\lim_{k \to 0} \psi(r) = u(r)/r$$

where $\psi(r)$ is the S-wave trial function and considering that

$$j_0(kr) = \frac{\sin(kr)}{kr} \quad , \quad n_0(kr) = \frac{\cos(kr)}{kr}$$

the integrals in (2.33) give, when $k \to 0$ and $L = 0$,

$$\lim_{k \to 0} < \psi, W j_0 > = \lim_{k \to 0} \int_0^\infty \frac{\sin(kr)}{kr} W(r) \psi(r) r^2 dr =$$

$$= \int_0^\infty W(r)\, u(r)\, r\, dr \qquad (2.34)$$

$$\lim_{k \to 0} < \psi, W \psi > = \int_0^\infty W(r)\, u^2(r)\, dr \qquad (2.35)$$

The calculation of $\lim_{k \to 0} <\psi, W g_0 W \psi>$ is more involved because it is a double integral. Integration with respect to $\underline{r}'$ gives

$$-\frac{2\mu k}{\hbar^2}\cdot\frac{\cos(kr)}{kr}\int_0^r\frac{\sin(kr')}{kr'}W(r')\psi(r')r'^2dr' -$$

$$-\frac{2\mu k}{\hbar^2}\cdot\frac{\sin(kr)}{kr}\int_r^\infty\frac{\cos(kr')}{kr'}W(r')\psi(r')r'^2dr'$$

or, letting k tend to zero

$$-\frac{2\mu}{\hbar^2}\left[\frac{1}{r}\int_0^r W(r')u(r')r'dr' + \int_r^\infty W(r')u(r')dr'\right]$$

Therefore

$$\lim_{k\to 0} \langle\psi, Wq_0W\psi\rangle =$$

$$= -\frac{2\mu}{\hbar^2}\int_0^\infty W(r)u(r)dr\cdot\int_0^r W(r')u(r')r'dr' -$$

$$-\frac{2\mu}{\hbar^2}\int_0^\infty W(r)u(r)r\,dr\cdot\int_r^\infty W(r')u(r')dr' \qquad (2.36)$$

The double integrals in (2.36) are equal, as the technic, used before in proving that $\delta B_1 = \delta B_2$, shows. Then, dropping the index I in the phase shifts δ^I and the scattering lengths A_I, one has

$$A = \lim_{k\to 0}(k\cot\delta_0)^{-1} =$$

$$= -\frac{\left[\int_0^\infty W(r)u(r)r\,dr\right]^2}{2\int_0^\infty W(r)u(r)dr\int_0^r W(r')u(r')r'dr' + \frac{\hbar^2}{2\mu}\int_0^\infty W(r)u^2(r)dr} \qquad (2.37)$$

5. The zero-energy S-wave trial function for a Yukawa well

Consider the dimensionless complex constant

$$z = \frac{2\mu}{\hbar^2}(u + iv)\frac{1}{b^2}$$

where $u + iv$ represents either $u_o + iv_o$ or $u_1 + iv_1$ in (2.1) and $\underline{b}$ is the reciprocal of the potential range (2.2). From (2.18) the integral form of $\psi_o(r)$ for the Yukawa potentials (2.1) is

$$\psi_o(r) = j_o(kr) + zb\,\frac{e^{ikr}}{r}\int_0^r j_o(kr')\,e^{-br'}\psi_o(r')\,r'dr' + $$
$$+ zb\,j_o(kr)\int_r^\infty e^{ikr'}\,e^{-br'}\psi_o(r')\,dr' \qquad (2.38)$$

When r tends to zero, (2.38) gives, putting $\psi(r) = \lim_{k\to o}\psi_o(r)$,

$$\psi(r) = 1 + \frac{zb}{r}\int_0^r e^{-br'}\psi(r')\,r'dr' +$$
$$+ zb\int_r^\infty e^{-br'}\psi(r')\,dr' \qquad (2.39)$$

when $r \to \infty$, the second integral in (2.39) is vanishingly small and one gets

$$\psi(r) \to 1 + \frac{A}{r}$$
$$r \to \infty$$

where

$$A = zb\int_0^\infty e^{-br}\psi(r)\,r\,dr \qquad (2.40)$$

A, defined by (2.40), is an integral form of the scattering length. The same expression is also obtained from the definition $k\cot\delta_0 = 1/A$. This relation implies that $\delta_0 \to kA$ when $k \to 0$, so that the substitution of kA for δ_0 in (2.20) leads to (2.40).

The iteration of $\psi(r)$, using the integral equation(2.39) as the recurrence relation, gives now the trial function.

The Born approximation for ψ is obviously given by choosing $\psi^{(0)} = 1$. From

$$\int_0^r e^{-br'}\psi^{(0)}(r')\,r'\,dr' = \frac{1}{b^2} - \frac{1}{b^2}e^{-br} - \frac{1}{b}\,e^{-br}\,r$$

and

$$\int_r^\infty e^{-br'}\psi^{(0)}(r')\,dr' = \frac{1}{b}\,e^{-br}$$

one gets

$$\psi = \psi^{(0)} + \psi^{(1)} = 1 + \frac{Z}{b}\frac{1}{r}\left(1 - e^{-br}\right) \qquad (2.41)$$

where

$$\psi^{(1)} = \frac{Z}{b}\frac{1}{r}\left(1 - e^{-br}\right) \qquad (2.42)$$

The Born approximation $A^{(0)}$ for A can now be obtained in two different ways: either from the asymptotic form of (2.41) putting $r \to \infty$, which gives

$$A^{(0)} = \frac{Z}{b} \qquad (2.43)$$

or directly, calculating the integral (2.40) with $\psi = \psi^{(0)} = 1$, giving the same result (2.43).

To obtain $\psi^{(2)}(r)$, ψ in (2.39) is set equal to $\psi^{(1)}$:

$$\psi^{(2)} = \frac{Z^2}{r} \int_0^r e^{-br'} (1 - e^{-br'})\, dr' +$$

$$+ Z \int_r^\infty e^{-br'} \frac{1 - e^{-br'}}{r'}\, dr' \tag{2.44}$$

If it were possible to move the factor $(1 - e^{-br})$ outside the first integral in (2.44) and the factor $(1 - e^{-br})/r$ outside the second integral in the same equation, it is obvious that $\psi^{(2)}(r)$ would be proportional to $(1 - e^{-br})/r$, as it is $\psi^{(1)}$, so that a formally very simple trial function could be obtained.

Writing then

$$I_1(r) = \int_0^r e^{-br'} (1 - e^{-br'})\, dr', \quad I_2(r) = \int_r^\infty e^{-br'} \frac{1 - e^{-br'}}{r'}\, dr'$$

the two functions $I_1(r)$ and $I_2(r)$ must be compared respectively with

$$J_1(r) = (1 - e^{-br}) \int_0^r e^{-br'}\, dr' = \frac{1}{b} (1 - e^{-br})^2$$

$$J_2(r) = \frac{1 - e^{-br}}{r} \int_r^\infty e^{-br'}\, dr' = \frac{(1 - e^{-br}) e^{-br}}{br}$$

It is clear that

$$I_1(r) = \frac{1}{2} J_1(r) \tag{2.45}$$

The relation between $I_2(r)$ and $J_2(r)$ is not so simple as in (2.45), unless an approximation is made. However one thing is evident: $I_2(r) < J_2(r)$ for any r in the interval $(0, \infty)$. Then the limits of

$$R(r) = \frac{I_2(r)}{J_2(r)}$$

when $r \to 0$ and $r \to \infty$ give an estimate of the ratio $R(r)$ in this interval:

$$\lim_{r \to 0} R(r) = \int_0^\infty e^{-br}(1 - e^{-br})\frac{1}{r}\,dr = \log 2 = 0.67 \tag{2.46}$$

(see Appendix B for the evaluation of this integral) and

$$\lim_{r \to \infty} R(r) = \lim_{r \to \infty} \frac{I_2'(r)}{J_2'(r)} =$$

$$= \lim_{r \to \infty} \frac{1 - e^{-br}}{1 - e^{-br}/br} = 1 \tag{2.47}$$

Therefore, making the approximation

$$I_2(r) \approx \frac{1}{2}J_2(r) \tag{2.48}$$

(2.46) shows that, for small values of r, (2.48) is nearly correct. For large values of r (2.48) in the light of (2.47) is a bad approximation. However the error made in (2.48) is not so important if it is borne in mind: first, that $I_2(r)$ or $J_2(r)$ are vanishingly small when $r \to \infty$; secondly (as it is remarked at the end of §3 of this chapter), that the behaviour of the trial function in the region where $rW(r) \to 0$ does not affect the accuracy of Schwinger's variational formula.

Therefore, using relations (2.45) and (2.48), $\psi^{(2)}(r)$ becomes

$$\psi^{(2)} = \frac{Z^2}{2b}\frac{1}{r}(1 - e^{-br}) \tag{2.49}$$

and the trial function is now given by

$$\psi = \psi^{(0)} + \psi^{(1)} + \psi^{(2)} =$$

$$= 1 + \frac{Z}{b}\left(1 + \frac{Z}{2}\right)\frac{1}{r}\left(1 - e^{-br}\right) \tag{2.50}$$

Putting $r \to \infty$, (2.50· shows clearly that the scattering length is, up to the first order approximation,

$$A = A^{(o)} + A^{(1)} = \frac{Z}{b}(1 + \frac{Z}{b}) \tag{2.51}$$

But the expression for the scattering length (2.51) can be obtained directly from the definition of A. In fact putting in (2.40) $\psi = \psi^{(1)}$ one gets

$$A^{(1)} = Z^2 \int_0^\infty e^{-br}(1 - e^{-br})dr = \frac{Z^2}{2b}$$

so that the result (2.51) is reached again.

It is now clear that using the approximation (2.48) for successive iterations of ψ one gets

$$\sum_{i=0}^{n} \psi^{(i)}(r) = 1 + \left[\sum_{i=0}^{n-1} A^{(i)}\right] \cdot \frac{1}{r}(1 - e^{-br})$$

and, therefore, a good zero-energy S-wave trial function for Yukawa potentials is

$$\psi(r) = 1 + \frac{A}{r}(1 - e^{-br}) \tag{2.52}$$

Putting $\psi(r) = \frac{u(r)}{r}$ and choosing another normalization for £2.52) (Schwinger's variational principle is invariant for different normalizations of ψ; see §3) the following trial function for u(r) is obtained (Bethe, 1949):

$$u(r) = (1 - e^{-br}) + \frac{r}{A} \tag{2.53}$$

The behaviour of (2.53) is analogous to that of the regular solution of the S-wave radial equation (2.12) with k and L equal to zero and $\psi_o(r) = \frac{u_o(r)}{r}$:

$$\frac{d^2 u_0(r)}{d^2 r} + Z b \frac{e^{-br}}{r} u_0(r) = 0 ; \tag{2.54}$$

$u(r)$ and $u_o(r)$ vanishes when $r = 0$ and both have the same asymptotic form

$$U_o(r) \to (1 + \frac{r}{A}) \qquad r \to \infty \tag{2.55}$$

6. The zero-energy variational formula

From

$$W(r) = -\frac{\hbar^2}{2\mu} Zb \frac{e^{-br}}{r}$$

(by definition $Z = 2\mu(u + iv)|(\hbar b)^2$ and $W(r) = -(u + iv)e^{-br}|br)$ it is clear that the expression (2.37) for the scattering length A can be written

$$A = \frac{-I_3^Z}{2I_1 - I_2} \tag{2.56}$$

where

$$I_1 = Z^2 \int_0^\infty u(r) \frac{e^{-br}}{r} dr \int_0^r u(r') e^{-br'} dr'$$

$$I_2 = \frac{Z}{b} \int_0^\infty u^2(r) \cdot \frac{e^{-br}}{r} dr$$

$$I_3 = Z \int_0^\infty u(r) e^{-br} dr$$

Substitution of $u(r)$ by the trial function (2.53) in I_i $(i = 1,2,3)$ leads now to the expected relation between A and Z.

Putting $S = bA$ and considering that the I_i expressed in terms of S are given by (see Appendix B):

$$I_1 = \frac{Z^2}{b}\left(\log\frac{32}{27} + \frac{1}{S}\log\frac{4}{3} + \frac{1}{4}\cdot\frac{1}{S^2}\right)$$

$$I_2 = \frac{Z}{b}\left(\log\frac{4}{3} + \frac{1}{S} + \frac{1}{S^2}\right)$$

$$I_3 = \frac{Z}{b}\left(\frac{1}{2} + \frac{1}{S}\right)$$

one gets from (2.56)

$$Z = \frac{S + S^2 + \log(4/3)\,S^3}{1 + 3/2\,S + (1/4 + 2\log(4/3))\,S^2 + \log(32/27)\,S^3} \qquad (S = bA) \tag{2.57}$$

or

$$Z = \frac{S + S^2 + 0.2877\,S^3}{1 + 1.5\,S + 0.8254\,S^2 + 0.1695\,S^3} \tag{2.58}$$

The purpose of this chapter is achieved: the relationship between Z and A is given by (2.57) or (2.58), the zero-energy variational formula.

In (2.57) Z depends on the product S = bA, rather than on b and A separately. This is in good agreement with the structure of the S-wave radial equation (2.54). In fact, making the transformation $\rho = br$ in this equation and its asymptotic form (2.55) one gets

$$\frac{d^2u_0(\rho)}{d^2\rho} + Z\,\frac{e^{-\rho}}{\rho}\,u_0(\rho) = 0 \tag{2.59}$$

and

$$u_0(\rho) \underset{\rho\to\infty}{\sim} \left(1 + \frac{\rho}{S}\right) \tag{2.60}$$

Both expressions show clearly the dependence of Z on S. Furthermore, (2.60) means that for large values of ρ, $u_o(\rho)$ behaves like a (complex) straight line. Generally this straight line makes a non-zero angle with the ρ-axis. But for a particular value of Z (the one for which $S = \infty$) the asymptotic form of $u_o(\rho)$ is parallel to the ρ-axis.

It is evident from (2.59) that this value of $\underline{Z}$ is independent of $\underline{b}$. The same conclusion is reached from the variational formula (2.57), putting $S \to \infty$:

$$Z = \log \frac{4}{3} \Big| \log \frac{32}{27} = 1.693 \qquad (2.61)$$

This shows the consistency of (2.57). In the next section, the value of Z for $S = \infty$, calculated directly from the differential equation (2.59) by numerical integration is given, which is in very good agreement with (2.61).

7. Numerical results

The Ross-Humphrey's scattering lengths $A_I = a_I + ib_I$ (I = 0,1) listed in Table II.1 give by substitution in the variational formula (2.58) (with b = 2.5 Fermi^{-1}, according to (2.2)) the following values for the dimensionless constants $Z_I = X_I + iY_I$, determining the Yukawa Wells (2.1):

TABLE II.2

Ross-Humphrey's Solutions	$X_o + iY_o$	$X_1 + iY_1$
I	1.6701 + i0.3456	0.3935 + i0.7836
II	1.9326 + i1.0853	1.1821 + i0.1679

To check the accuracy of the variational formula (2.58), a numerical integration of the regular solution $u_o(r)$ of equation (2.54) was carried out in a Ferranti-Mercury computer with Z set equal to each one of Z_I $(= X_I + iY_I)$ given in this table. The results of the numerical integration are

TABLE II.3

Ross-Humphrey's Solution		$a_o + ib_o$	$a_1 + ib_1$
I	ORIGINAL	$-0.22 + i2.74$	$0.02 + i0.38$
	CALCULATED	$-0.30 + i2.66$	$0.00 + i0.38$
II	ORIGINAL	$-0.59 + i0.96$	$1.20 + i0.56$
	CALCULATED	$-0.55 + i0.93$	$1.20 + i0.57$

Inspection of Table II.3 shows that the accuracy of the variational formula (2.58) is good, so much so that the errors that affect the original A_I $(= a_I + ib_I)$ are larger by far than the errors arising from the calculated A_I.

A numerical integration of the differential equation (2.59) was also effectuated to obtain the Z-value that corresponds to $A = \infty$. As the asymptotic form (2.60) of the regular solution at the origin of (2.59) shows, $\frac{du_o(\rho)}{d\rho}$ must vanish for ρ large. The determination by trial of the best value of Z which satisfies this condition was worked out in a programme for Mercury. The result of the numerical calculation is 1.680, not very far from the variational evaluation 1.693 in (2.61).

CHAPTER III

NN Interactions

1. Introduction

This Chapter is mainly concerned with the NN interactions related with the calculation of the form factors which, as it will be seen later on, arise in the application of the Impulse Approximation method to the inelastic or charge-exchange processes in K^-d collisions.

In such processes the total spin quantum numbers ($S = 1$) of the deuteron is conserved, because the K^--meson is spinless, so that the np (inelastic) or the nn (charge-exchange) resulting systems are always in a triplet continuous state. If the wave functions of the initial (deuteron) and final states are represented respectively by $\phi_0^{\mu}(\bar{R})$ and $\phi_{\bar{K}}^{\mu}(\bar{R})$ – here $\bar{K}$ is the NN centre-of-mass wave number and μ the spin Z-component – then, the form factor is given by

$$S^{\frac{1}{2}}(\theta) = \frac{1}{3} \sum_{\mu=-1}^{1} \int [\phi_{\bar{K}}^{\mu}(\bar{R})]^{*} e^{i\bar{h}.\bar{R}} \phi_0(\bar{R})\, d\bar{R} \qquad (3.1)$$

where $\bar{h} = \left[\frac{1}{2}(\bar{k}_a - \bar{k}_b)\right]$ is the vectorial semi-difference between the initial K^-d centre-of-mass wave number $\bar{k}_a$ and its value $\bar{k}_b$ in the final state; θ is the angle defined by $\bar{k}_a$ and $\bar{k}_b$.

The problem of NN interactions is far from being solved. The lack of a successful theory of mesonic fields has switched the research in this domain to the attempt of building phenomenological potentials, whose parameters are calculated from the experimental data. Such potentials conserve purity and are invariants under rotations of the coordinate system, reflexion of the axis and time reversal. Generally speaking, they are a sum of terms, each being the product of a function of the distance between the two nucleons by a certain kind of interaction (central, spin-spin, tensor, spin-orbit, etc.), which obeys the foregoing invariance laws (Eisenbud and Wigner, 1941; Gammel and Thaler, 1960). Such potentials satisfy also the principle of charge independence of nuclear forces and reflect the exchange character manifested in them.

The first phenomenological potentials which are in a semi-quantitative agreement (Noyes, 1961) with the experimental information, are due to Gammel and Thaler (1957, 1957a) and have the general form

$$V(R) = V_C(R) + V_T(R)S_{12} + V_{LS}(R)\bar{L}.\bar{S} \qquad (3.2)$$

where S_{12} and $\bar{L}.\bar{S}$ are respectively the tensor and spin-orbit operators.

The functions $V_C(R)$, $V_T(R)$ and $V_{LS}(R)$ are energy-independent, have a hard core and take the Yukawa shape for $R > R_o$; but R_o and the other constants depend on the spin and isotopic-spin states.

Although an interesting attempt, the Gammel-Thaler model remains unsatisfactory from the point of view of giving results in close agreement with the experimental information covering a wide energy range. And the difficulty of introducing new terms in (3.2) derived from invariance arguments remains great (Noyes, 1961).

To cope with this situation Breit and Co-workers (Breit et al., 1960, 1962; Hull et al., 1961, 1962) envisaged the theoretically rigorous analysis of real phase shifts and coupling constants. It is also a phenomenological approach, but has the advantage of getting rid of a nuclear potential model and dealing directly with certain features of the NN systems, such as the triplet states with same total angular momentum (J), but different orbital angular momenta (L), the postulation of which remains yet the best explanation for well established facts. This is the case, for instance, of the measured values of the magnetic and quadrupole moments of the deuteron, which are very well interpreted if the ground state of this two-particle design is considered a mixture of $^3S_1$ and $^3D_1$ states (Sachs, 1953, 1953).

The previously cited papers give an account of different fits of the experimental data obtained for pp and np systems. In this work the best fits found by Breit and collaborators (the YALM fit for pp interactions and the YALN3M fit for np interactions) are used in the calculation of the form factor (3.1).

2. The Tensor force S_{12}

The definition of the tensor force S is

$$S_{12} = 2\left[s(\bar{S}.\bar{n}) - \bar{S}^2\right] \tag{3.3}$$

where $\bar{S}$ is the total spin operator of the two nucleons and $\bar{n}$ the unit vector ($\bar{n}^2 = 1$) of the straight line joining the two particles.

Obviously, S_{12} is a hermitian scalar operator, invariant under rotations of the ordinary and spin space, and reflexions of the coordinate and time axes. Then, if $\bar{J}$ (= $\bar{L}$ + $\bar{S}$) represent the total angular momentum ($\bar{L}$ being the orbital momentum) of the system, one has

$$\left[S_{12}, \bar{J}^2\right] = 0 \qquad \left[S_{12}, \bar{J}\right] = 0 \tag{3.4}$$

Considering the comutation relation $\left[\bar{S},\bar{S}^2\right] = 0$, (3.3) also shows that

$$\left[S_{12},\bar{S}^2\right] = 0 \tag{3.5}$$

The conservation laws (3.4) and (35) indicate that the total angular momentum eigen functions form a convenient basis to establish the transformation properties of S_{12}. With the help of the Clebsch-Gordon coefficient, such functions can be expressed in terms of the spherical harmonics $Y_L^m(\theta,\phi)$ and the two-nucleon spin functions χ_s^μ in the following way:

$$\mathcal{Y}_{LSJ}^M = \sum_{m,\mu} \langle L, S, m, \mu | J M \rangle Y_L^m(\theta,\phi)\chi_S^\mu \tag{3.6}$$

where the indices are subjected to the conditions

$$-L \leqslant m \leqslant L, \qquad -S \leqslant \mu \leqslant S$$

$$m + \mu = M, \quad |L - S| \leqslant J \leqslant L + S, \quad -J \leqslant M \leqslant J \tag{3.7}$$

Here J and M are respectively the quantum numbers of the total angular momentum and its Z-component, the pairs (L, m) and (S, μ) representing the similar quantities for the orbital momentum and total spin. If conditions (3.7) are not fulfilled the C.G. coefficients vanish (Messiah, 1962, Appendix C).

The commutation relations (3.4) and (3.5) ensure that J, S and M are good quantum numbers, but no L-conservation exists because S_{12} does not commute with $\bar{L}$. However, it is possible to add another good quantum number to the previous list. The ordinary space parity of Y_{LSJ}^M (which, by (3.6), is equal to the Y_L^m-parity, $(-1)^L$) is conserved because S_{12} is invariant under reflexions of the axes.

Therefore, when S operates on the Y_{LSJ}^M, such functions are transformed into linear combinations of themselves, with conservation of parity, J, S and M.

The L's, however, in each term of the linear combination, must differ by an even number of units, so that the parity remains the same.

Now, the states with the same J are examined. In a singlet state (S = 0), by (3.7), one always gets L = J and

$$Y^{M}_{JOJ} = Y^{M}_{L}\chi^{o}_{o} .$$

From $\bar{S}\chi^{o}_{o} = 0$ it follows that

$$S_{12}Y^{M}_{JOJ} = 0 \qquad (3.8)$$

Thus, in a singlet state, the tensor force does not contribute to the NN interactions.

In a triplet state (S = 1), if L > 1, the only possible values of L, by (3.7), are:

$$L = J - 1, \quad J, \quad J + 1 \qquad (3.9)$$

But if L = 0, by the same condition, J is necessarily equal to 1. Therefore, by (3.9) and parity conservation, $S_{12}Y^{M}_{L1J}$ is either proportional to Y^{M}_{L1J} (L = J), or a linear combination of the $Y^{M}_{L\pm1,1J}$ (the coefficient of the linear combinations are given in Rohrlich and Eisenstein, 1949, for instance).

Consider now a wave function, which satisfies the np Schrodinger equation, developed as a series of the Y^{μ}_{LSJ}. If it belongs to the singlet spin state, the tensor term vanishes by (3.8) and the radial equations derived from the Schrodinger equation are uncoupled because the spin-spin or spin-orbit terms in the NN interactions do not mix states of different J and L.

But, if the wave function represents a triplet spin state, then the operator S_{12} leads to three radial equations (Rohrlich and Eisenstein, 1949) for each value of J (except when J = 0): one uncoupled equation giving the radial function $v_J^J(KR)$, with L = J, and a system of two coupled equations defining the radial functions $v_J^{J-1}(KR)$ and $v_J^{J+1}(KR)$, with L = J-1 and L = J+1 (M is absent from the v's because the differential equations defining them are independent of this index). For J = 0, by (3.7), the system reduces to a single differential equation with L = 1.

Assuming that the radial dependence of the NN interactions has a Yukawa shape, the radial coupled equations are formally identical to equations (D.1) and (D.2) of Appendix D. But, as it is explained in this appendix, it is possible to construct two linearly independent solutions ($v_J^{1,J-1}(KR)$, $v_J^{1,J+1}(KR)$) and ($v_J^{2,J-1}(KR)$, $v_J^{2,J+1}(KR)$) of the coupled equations which vanish at the origin and are such that the ϑ's with the same numerical index (1 or 2) have equal phase shifts and obey the Wrouskian condition

$$1 + K_J^1 K_J^2 = 0 \tag{3.10}$$

where K_J^1 and K_J^2 are the coupling constants.

Therefore, since S_{12} applied to Y_{LSJ}^M leaves J, S, M and parity unchanged, a very convenient set of basic functions to expand the np triplet spin state wave function is the following (Rohrlich and Eisenstein, 1949):

$$\Psi_J^{M,J-1} = \frac{\vartheta_J^{1,J-1}}{\rho} \mathcal{Y}_{J-1,1J}^M - \frac{\vartheta_J^{1,J+1}}{\rho} \mathcal{Y}_{J+1,1J}^M \tag{3.11}$$

$$\Psi_J^{M,J} = \frac{\vartheta_J^J}{\rho} \mathcal{Y}_{J1J}^M \tag{3.12}$$

$$\Psi_J^{M,J+1} = \frac{\vartheta_J^{2,J-1}}{\rho} \mathcal{Y}_{J-1,1J}^M + \frac{\vartheta_J^{2,J+1}}{\rho} \mathcal{Y}_{J+1,1J}^M \tag{3.13}$$

with the asymptotic behaviour

$$\psi_J^{M,J-1} \sim \frac{e^{i\delta_J^{J-1}}}{\rho} \sin(\Theta_{J-1} + \delta_J^{J-1})\left[Y_{J-1,1J}^M + K_J^1 Y_{J+1,1J}^M \right] \quad (3.11')$$

$$\psi_J^{M,J} \sim \frac{e^{i\delta_J^J}}{\rho} \sin(\Theta_J + \delta_J^J)\, Y_{J1J}^M \quad (3.12')$$

$$\psi_J^{M,J+1} \sim \frac{e^{i\delta_J^{J+1}}}{\rho} \sin(\Theta_{J+1} + \delta_J^{J+1})\left[Y_{J-1,1J}^M + K_J^2 Y_{J+1,1J}^M \right] \quad (3.13')$$

Here ρ and Θ_J are given by

$$\rho = KR \,, \quad \Theta_J = \rho - \frac{\pi}{2} J$$

As (3.11') and (3.13') show, the phase shifts δ_J^{J-1} and δ_J^{J+1} are respectively associated with the mixing parameters K_J^1 and K_J^2. By definition K_y^1 is chosen to be the smallest in modulus of the two K's, so that $\psi_J^{M,J-1}$ is predominantly a L = J-1 state and $\psi_J^{M,J+1}$, a L = J+1 state ($K_J^2 = -1|K_J^1$).

The orthonormality of the Y_{L1S}^M and the Wrouskian condition (3.10) imply that the internal product of any two $\psi_y^{M,L}$, differing at least in one of the indices, vanish. Therefore, the $\psi_y^{M,L}$ are orthogonal

3. The np continuous states

Consider a np system in the spin state (S,μ), when the free-neutron wave

$$e^{i\bar{K}\bar{R}} \chi_S^\mu$$

falls upon the proton, it looses the central symmetry, because the interaction depends on the tensor force term. Thus, as in the case of the $\psi_J^{M,L}$ and for the same reasons, it is better to express the plane wave as a series of the Y_{LSJ}^M. This is achieved in the following way.

Representing respectively by $\hat{K}$ and $\hat{R}$ the spherical angular coordinates of $\bar{K}$ and $\bar{R}$, the plane wave $e^{i\bar{K}.\bar{R}}$ is given by (Messiah, 1961, p.497):

$$e^{i\bar{K}.\bar{R}} = 4\pi \sum_{L=0}^{\infty} \sum_{m=-L}^{L} i^L j_L(KR) Y_L^{m*}(\hat{K}) Y_L^m(\hat{R}) \tag{3.14}$$

If the Z-axis of the coordinate system is chosen along $\bar{K}$, so that $\theta_{\bar{K}} = 0$ and

$$Y_L^m(0, \phi_K) = \left(\frac{2L+1}{4\pi}\right)^{\frac{1}{2}} \delta_{om} ,$$

then, by the well-known expansion

$$Y_L^m(\hat{R}) \chi_S^\mu = \sum_{J=|L-S|}^{L+S} \langle L, S, m, \mu | J M \rangle \mathcal{Y}_{LSJ}^M \tag{3.15}$$

the plane wave $e^{i\bar{K}.\bar{R}} \chi_S^\mu$ assumes the form, in any triplet spin state,

$$e^{i\bar{K}.\bar{R}} \chi_1^\mu = (4\pi)^{\frac{1}{2}} \sum_{J=0}^{\infty} \sum_{L=|J-1|}^{J+1} i^L a_{\mu L}^J j_L(KR) \mathcal{Y}_{L1J}^\mu \tag{3.16}$$

where

$$a_{\mu L}^J = (2L+1)^{\frac{1}{2}} \langle L, 1, 0, \mu | J \mu \rangle \tag{3.17}$$

The development (3.16) indicates that the np wave function belonging to a triplet spin state can be expressed very easily as a series of the orthogonal functions $\psi_J^{\mu L}$:

$$\psi_{\bar{K}}^\mu(\bar{R}) = (4\pi)^{\frac{1}{2}} \sum_{J=0}^{\infty} \sum_{L=|J-1|}^{J+1} i^L C_{\mu L}^J \psi_J^{\mu L} \tag{3.18}$$

where the $C^J_{\mu L}$ are constants, which can be expressed in terms of the $a^J_{\mu L}$.

To obtain these relationships one only needs to express the condition that, for large values of R, the spherical wave falling upon the scatterer (proton) is the same in the plane wave (3.16) as well as in the total wave (3.18). Doing this, one has

$$C^J_{\mu,J-1} = \frac{a^J_{\mu,J-1} + K^1_J \, a^J_{\mu,J+1}}{1 + (K^1_J)^2} \tag{3.19}$$

$$C^J_{\mu J} = a^J_{\mu J} \tag{3.20}$$

$$C^J_{\mu,J+1} = \frac{a^J_{\mu,J-1} + K^2_J \, a^J_{\mu,J+1}}{1 + (K^2_J)^2} \tag{3.21}$$

The following relations (see Blatt and Weisskopf, 1952, for the C.G. coefficients)

$$\sum_{\mu=-1}^{1} \left(a^J_{\mu L} \right)^2 = 2J + 1, \quad L = J-1, J, J+1 \tag{3.22}$$

$$\sum_{\mu=-1}^{1} a^J_{\mu,J-1} \, a^J_{\mu,J+1} = 0 \tag{3.23}$$

are very useful in subsequent calculations. Using them and (3.19), (3.20) and (3.21) it is possible to prove that

$$\frac{1}{3} \sum_{\mu=-1}^{1} \int \left[\phi^{\mu}_{\bar{K}}(\bar{R}) \right]^{*} \left[\phi^{\mu}_{\bar{K}'}(\bar{R}) \right] d\bar{R} = \delta(\bar{K} - \bar{K}')$$

where

$$\phi^{\mu}_{\bar{K}}(\bar{R}) = (2\pi)^{-\frac{3}{2}} \, \psi^{\mu}_{\bar{K}}(\bar{R}) \tag{3.18'}$$

Therefore $(2\pi)^{-\frac{3}{2}}$ is the normalisation constant of (3.18).

4. The np bound state (deuteron)

As it was mentioned in the Introduction to this chapter, the bound state of the np system is a mixture of $^3S_1$ and $^3D_1$ states. The radial equations describing the deuteron are analogous, therefore, to equations (D.1) and (D.2) of Appendix D, if J is put equal to 1 and K is substituted by $i\alpha$ with

$\alpha^2 = -\dfrac{E_d\, m_N}{\hbar^2}$ (E_d is the binding energy of the deuteron).

However, in the calculations copied out in this work, only the predominant $^3S_1$ deuteron-state is considered, so that the previous system of coupled equations reduces to a single differential equation. Its exact solution is approximated by the usual "Hulthèn" function:

$$\phi_0^{\mu}(\bar{R}) = \phi_0(R)\,\chi_1^{\mu} \tag{3.24}$$

where

$$\phi_0(R) = N\,\frac{e^{-\alpha R} - e^{-\beta R}}{R}\,, \quad N^2 = \frac{\alpha\beta\,(\alpha+\beta)}{2\pi\,(\beta-\alpha)^2}\,, \tag{3.25}$$

N being the normalization factor.

Taking E_d = -2.225 MeV and m_N = 938.2 MeV the numerical value of α^{m}(3.25) is

$$\alpha = 0.2315\ \text{fermi}^{-1} \tag{3.26}$$

The coefficient β is calculated by means of a variational principle (Thomas, 1937) described by Sachs (1952). If the nuclear potential has the Yukawa shape and the range of the np interactions is taken equal to 1.18 fermi, then β has the numerical value

$$\beta = 1.5649\ \text{fermi}^{-1} \tag{3.27}$$

5. The form factor of K^-d inelastic collisions

By (3.18) and (3.18') the calculation of the form factor (3.1) reduces to the following integrations

$$I_J^L = (4\pi)^{\frac{1}{2}} \sum_{\mu=-1}^{1} C_{\mu L}^{J} \int [\psi_J^\mu]^\times e^{i\bar{h}.\bar{R}} \phi_o^\mu(R)\, d\bar{R} \qquad (3.28)$$

for L = J - 1, J, J + 1.

Replacing in this expression $\phi_o^\mu(\bar{R})$ by the Hulthèn function (3.24), the spin eigenfunctions χ_1^μ appear explicitly in the integrand of (3.28). But the $\psi_J^{\mu L}$ depend on the $Y_L^m(\theta,\phi)\chi_1^\mu$ through the Y_{L1J}^μ. Therefore, due to the orthonormality of the χ_s^μ, only the term

$$<L,1,0,\mu|J\mu> Y_L^0(\theta,\phi)\chi_1^\mu =$$

$$= <L,1,0,\mu|J\mu>(4\pi)^{-\frac{1}{2}}(2L+1)^{\frac{1}{2}} P_L(\cos\theta)\chi_1^\mu \qquad (3.29)$$

belonging to the expansion (3.6) of the Y_{L1J}^μ into the $Y_L^m\chi_1^\mu$, remains inside the integrals (3.28).

Introducing in (3.29) the $a_{\mu L}^J$ given by (3.17), such integrals are equal to

$$I_J^L = \int [\psi_J^L]^\times e^{i\bar{h}.\bar{R}} \phi_o(R)\, d\bar{R} \qquad (3.28')$$

where the ψ_J^L, in agreement with the definitions (3.11), (3.12) and (3.13) of the $\psi_J^{\mu L}$, are given by

$$\psi_J^{J-1} = \frac{1}{\rho} \sum_{\mu=-1}^{1} C^J_{\mu J-1} \left[a^J_{\mu J-1} v_J^{1,J-1} P_{J-1}(\cos\theta) - a^J_{\mu J+1} v_J^{1,J+1} P_{J+1}(\cos\theta) \right]$$

$$\psi_J^J = \frac{1}{\rho} \sum_{\mu=-1}^{1} C^J_{\mu J}\, a^J_{\mu J}\, v_J^J\, P_J(\cos\theta)$$

$$\psi_J^{J+1} = \frac{1}{\rho} \sum_{\mu=-1}^{1} C^J_{\mu J+1} \left[-a^J_{\mu J-1} v_J^{2,J-1} P_{J-1}(\cos\theta) + a^J_{\mu J+1} v_J^{2,J+1} P_{J+1}(\cos\theta) \right]$$

or, taking into account the expressions (3.19) to (3.23), relating the c- and a- coefficients to J and coupling constants $K_J^{(i,2)}$

$$\rho\,\psi_J^{J-1} = \frac{2J+1}{1+(K_J^1)^2} \left[v_J^{1,J-1} P_{J-1}(\cos\theta) - K_J^1\, v_J^{1,J+1} P_{J+1}(\cos\theta) \right] \qquad (3.30)$$

$$\rho\,\psi_J^J = (2J+1)\, v_J^J\, P_J(\cos\theta) \qquad (3.31)$$

$$\rho\,\psi_J^{J+1} = \frac{2J+1}{1+(K_J^2)^2} \left[-v_J^{2,J-1} P_{J-1}(\cos\theta) + K_J^2\, v_J^{2,J+1} P_{J+1}(\cos\theta) \right] \qquad (3.32)$$

By (3.18), (3.18'), (3.28) and (3.28'),the form factor (3.1) is obviously equivalent to the integration of

$$S^{\frac{1}{2}}(\theta) = \int \left[\phi_{\bar{K}}(\bar{R}) \right]^{x} e^{i\bar{h}.\bar{R}}\, \phi_o(R)\, d\bar{R} \qquad (3.33)$$

where

$$\phi_{\bar{K}}(\bar{R}) = \frac{1}{3}(2\pi)^{-\frac{3}{2}} \sum_{J=0}^{\infty} \sum_{L=|J-1|}^{J+1} i^L\, \psi_J^L \qquad (3.34)$$

Now it is possible to make two kinds of approximations for the ψ_J^L. As it is shown in the next paragraph, the K_J^1 have a very small modulus for np collisions at low energies. Therefore, powers of K_J^1 higher than the first can be ignored in (3.30) and (3.32) so that, if the relations, derived from (3.10),

$$\frac{1}{1+(K_J^2)^2}=\frac{(K_J^1)^2}{1+(K_J^1)^2}\ ,\ \frac{(K_J^2)^2}{1+(K_J^2)^2}=\frac{1}{1+(K_J^1)^2} \tag{3.35}$$

are used, the ψ_J^L for L = J - 1, J + 1 become in this approximation (see the asymptotic forms of $v_J^{(1,2)J+1}$ in (3.11') and (3.13'))

$$\rho\,\psi_J^{J-1}=(2J+1)\,\vartheta_J^{1,J-1}\,P_{J-1}(\cos\theta) \tag{3.30'}$$

$$\rho\,\psi_J^{J+1}=\frac{2J+1}{K_J^2}\,\vartheta_J^{2,J+1}\,P_{J+1}(\cos\theta) \tag{3.32'}$$

A second approximation for the ψ_J^L consists in the substitutions

$$\vartheta_J^{1,J-1}\rightarrow\rho.U_J^{J-1}\ ,\ \vartheta_J^J\rightarrow\rho\,U_J^J\ ,\ \vartheta_J^{2,J+1}\rightarrow K_J^2\,\rho\,U_J^{J+1} \tag{3.36}$$

where ($j_L(\rho)$ and $n_L(\rho)$ being respectively the spherical Bessel and Neumann functions)

$$U_J^L(\rho)=e^{i\delta_J^L}\left[\cos\delta_J^L\,j_L(\rho)+\sin\delta_J^L\,(1-e^{-ZR})\,n_L(\rho)\right] \tag{3.37}$$

the constant Z inside the dumping factor $(1-e^{-ZR})^{L+1}$ is put equal 0.8 fermi^{-1}, the reciprocal of the range of the NN interactions (1.18 fermi), so that the (ρU)'s and ϑ's are the same when R > 1.18 fermi. The exponent (2L+1) was chosen in such a way that the $U_J^L(\rho)$ behave for small values of ρ like uncoupled radial wave functions arising from short range interactions

$$U_J^L(\rho)\simeq R^L\quad(\rho=KR)\ ,\ R\rightarrow 0 \tag{3.38}$$

This is a reasonable assumption because the behaviour of the ϑ's is affected only in a region where the errors are negligible (Gourdin and Martin, 1959). To finish this paragraph, the following remark should be made on the form factor $S_2^1(\theta)$.

According to definition (3.1), the incoherent scatterings due to different spin-orientations look as if they are mixed in $S^{\frac{1}{2}}(\theta)$. However, this is not the case. In fact, the physically significant quantity is not $S^{\frac{1}{2}}(\theta)$ itself, but the square of its modulus integrated over the wave numbers ($\hat{K}$) allowed by the energy conservation principle. Considering the orthonormality of the spherical harmonics as well as the properties of the $a^J_{\mu L}$ - and $C^J_{\mu L}$ - coefficients (see (3.19) to (3.23)) a straightforward proof (although tedious) of the following relation can be established:

$$\frac{1}{9}\sum_{\mu=-1}^{1}\int \left| \int \left[\psi^{\mu}_{\bar{K}}(\bar{R})\right]^{*} e^{i\bar{h}.\bar{R}} \chi^{\mu}_{1}\, \phi_0(R)\, d\bar{R}\right|^2 d\hat{K} = \int \left| S^{\frac{1}{2}}(\theta)\right|^2 d\hat{K} \qquad (3.1')$$

where $d\hat{K}$ represents the differential of the angular part of the spherical coordinates related to $\bar{K}$.

If the np wave function $\psi^{\mu}_{\bar{K}}(\bar{R})$ is replaced by the plane wave $(2\pi)^{-\frac{3}{2}} e^{iK.R} \chi^{\mu}_{1}$, the left hand-side of (3.1') reduces to

$$(2\pi)^{-3}\int \left| \int e^{i\bar{K}.\bar{R}}\, e^{i\bar{h}.\bar{R}}\, \phi_0(R)\, d\bar{R}\right|^2 d\hat{K}.$$

This result shows, that the factor 1/3, appearing in the definition (3.1) of $S^{\frac{1}{2}}(\theta)$ has the correct value.

The relation (3.1') still holds if the plane wave $e^{i\bar{h}.\bar{R}}$ is substituted by a linear combination of terms having the form $F_{\gamma}(K,R)e^{ih_{\gamma}.\bar{R}}$, where $F_{\gamma}(K,R)$ are analytic functions of $|\bar{R}|$ and $|\bar{K}|$. This theorem will be helpful in the calculation of multiple scattering effects in $K^{-}d$ inelastic and charge-exchange collisions.

6. The YALM and YALN3M fits

The calculation of Kd inelastic and charge-exchange cross-sections for K-momenta below 300 MeV/c (in the Kd Lab system) needs the knowledge of the first (S, P, D) NN phase parameters in the NN Lab system energy range (0, 150 MeV). The parameters used in this work (see §1) are those belonging to the YALM and YALN3M fits obtained by Breit and co-workers.

Breit defines the phase shifts (θ_J^{J-1}, θ_J^{J+1}) and mixing constant (ρ_J) of two coupled states differently from those which are given by the asymptotic forms (3.11') and (3.13') of $\psi_J^{\mu,J-1}$ and $\psi_J^{\mu,J+1}$: δ_J^{J-1}, δ_J^{J+1} and K_J^1.

However, he gives the relationship between his parameters and those (δ_α, δ_β, ε) used by Blatt and Biedenharn (1952), which are related to δ_J^{J-1}, δ_J^{J+1} and K_J^1 by the equations

$$\delta_J^{J-1} = \delta_\alpha \quad , \quad \delta_J^{J+1} = \delta_\beta \quad , \quad K_J^1 = -\tan\varepsilon \, .$$

Therefore, putting $\Delta\theta_J = \theta_J^{J-1} - \theta_J^{J+1}$, $\Delta\delta_J = \delta_J^{J-1} - \delta_J^{J+1}$,

one has (see Breit et al. 1960)

$$\theta_J^{J-1} + \theta_J^{J+1} = \delta_J^{J-1} + \delta_J^{J+1} \tag{3.39}$$

$$\tan\Delta\theta_J = (\cos 2\varepsilon)\tan\Delta\delta_J \tag{3.40}$$

$$\rho_J = (\sin 2\varepsilon)\sin\Delta\delta_J \tag{3.41}$$

and, by elimination of ε in (3.40 and (3.41),

$$\tan^2 \Delta\delta_J = \frac{\rho_J^2 + \tan^2 \Delta\theta_J}{1 - \rho_J^2} \tag{3.42}$$

$$(K_J^1)^2 = \frac{\tan \Delta\delta_J - \tan \Delta\theta_J}{\tan \Delta\delta_J + \tan \Delta\theta_J} \tag{3.43}$$

Thus, from the values of the corresponding θ's and ρ's, the δ's and K's can be evaluated for the YALM and YALN3M fits (Tables (III,1) and (III,2)) by means of (3.39), (3.42) and (3.43).

Table III.1

I = 0, np Interactions, Y A L N 3 M fit

E(MeV)	δ_1^0	$(K_1^1)^2$	δ_1^2	δ_3^2	$(K_3^1)^2$	δ_3^4
5	2.0652		-0.0030			
10	1.7956		-0.0114			
15	1.6304		-0.0221			
25	1.4172	0.00	-0.0465	0.0108	0.66	-0.0090
50	1.1137		-0.1063	0.0312		-0.0261
100	0.7856	0.01	-0.2047	0.0689	0.50	-0.0561
150	0.5185	0.09	-0.2834	0.0972	0.46	-0.0808

Table III.2

I = 1, pp Interactions, Y A L M fit

E(MeV)	δ_2^1	$(K_2^1)^2$	δ_2^3
25	0.0501	0.12	-0.0041
50	0.1150		-0.0050
100	0.2138	0.05	0.0009
150	0.2704	0.04	0.0072

Tables (III.1) and (III.2) show that the errors made in discarding the powers of K_J^1 higher than the first in the coupled states with J = 1,2 do not exceed 12%. This is a reasonable approximation if one bears in mind that the error made in using the Hulthèn function $\phi_0(R)$ for the deuteron, amounts to 20% of the np system ground state (the deuteron spends 4% of its time in the D state). But for J = 3 one gets $(K_3^1) \sim 0.5$; the previous approximation, then, is no longer acceptable. However, due to the smallness of $\Delta\delta_3$ $(= \delta_3^2 - \delta_3^4)$ it is possible in this case to obtain again

$$A_{\omega\varphi} = (2J+1)\left[i^{J-1} U_J^{J-1} P_{J-1}(\cos\theta) + i^{J+1} U_J^{J+1} P_{J+1}(\cos\theta)\right] \qquad (3.44)$$

for the approximation of

$$i^{J-1}\psi_J^{J-1} + i^{J+1}\psi_J^{J+1} \qquad (3.45)$$

in $\Phi_K(R)$.

In fact, considering that $\Delta\delta_J$ is small, $\vartheta_J^{1,J+1}$ and $\vartheta_J^{2,J-1}$ are given by

$$\frac{\vartheta_J^{1,J+1}}{\rho} = K_J^1\left[U_J^{J+1} + \Delta\delta_J \frac{\partial U_J^{J+1}}{\partial\delta_J^{J+1}}\right]$$

$$\frac{\vartheta^{2,J-1}}{\rho} = U_J^{J-1} - \Delta\delta_J \frac{\partial U_J^{J-1}}{\partial\delta_J^{J-1}}$$

for R > 1.18 fermi (range of the NN interactions). Then, taking into account the relations (3.35), (3.45) gives

$$A_{\omega\varphi} = (2J+1)\left[-i^{J-1}\frac{\partial U_J^{J-1}}{\partial\delta_J^{J-1}} P_{J-1}(\cos\theta) + i^{J+1}\frac{\partial U_J^{J+1}}{\partial\delta_J^{J+1}} P_{J+1}(\cos\theta)\right]\frac{\Delta\delta_J(K_J^1)^2}{1+(K_J^1)^2} \qquad (3.46)$$

The last term in (3.46) for J=3 never exceeds 9% for NN Lab-energies below 150 MeV as the calculation of $(K_3^1)^2\Delta\delta_3$ from Table (III.1) shows. Obviously this term is valid only for R > 1.18 fermi. But it is reasonable to admit that the modulus of its exact form for R < 1.18 shall not differ widely from the modulus of the approximate expression in (3.46). Therefore, here again the approximation (3.44) for (3.45) holds well.

The pp phase shifts are calculated by Breit and collaborators in such a way that the Coulomb phases must be added to them in order to obtain the actual phases (Breit et al., 1962). Therefore, a further correction is needed when the pp triplet phase shifts $\delta_J^L(p,p)$ are applied to np systems. The relationship between the $\delta_J^L(p,p)$ and the corresponding $\delta_J^L(n,p)$ phases used in this work is the following (Jackson and Blatt, 1950):

$$\delta_J^L(n,p) \simeq \frac{1}{C_0^2}\,\delta_J^L(p,p) \tag{3.47}$$

where C_o is the Coulomb penetration factor

$$C_o^2 = \frac{2\pi n}{e^{2\pi n}-1} \tag{3.48}$$

The parameter n, given by

$$n = \frac{m_N\, e^2}{2\hbar^2 K} \tag{3.49}$$

(e is the proton charge), is expressed in terms of the NN Lab-energy, E_{Lab}, by

$$E_{Lab} = \frac{2\hbar^2 K^2}{m_N} \tag{3.50}$$

or

$$E_{Lab} = (9.1104\, K)^2 \tag{3.51}$$

if E_{Lab} is measured in MeV and K in fermi^{-1}. Therefore,

$$n = 0.1581\ E_{Lab}^{\frac{1}{2}} \qquad (3.52)$$

Table (III.3) gives the np phase shifts for P-waves:

Table III.3

E(MeV)	δ_0^1	δ_1^1	δ_2^1
5	0.0391	-0.0224	0.0552
25	0.1823	-0.1113	
50	0.2232	-0.1802	0.1233
100	0.1657	-0.2661	
150	0.0746	-0.3378	0.2815

For the actual computation of the Kd inelastic and charge-exchange cross-sections, the available values of S, P and D triplet phase shifts belonging to the NN Lab-energy interval (0, 150 MeV) were fitted by curves represented by polynomials. Apart from δ_1^0 (the S-wave phase shift), however, for energies of this interval below 25 MeV, the polynomials were replaced by

$$\delta_J^L = C_J^L K^{2L+1} \qquad (3.53)$$

The phase shifts given by (3.53) have the same behaviour for small values of K as those which are obtained from uncoupled Schrodinger radial equations for short range interactions. The relations (3.53) are the logical implications of the assumptions (3.38) relative to the behaviour of the $U_J^L(KR)$ for small values of R. The law used for the S -wave phase shift when K tends to zero is:

$$\delta_1^0 \rightarrow \pi$$
$$K \rightarrow 0$$

CHAPTER IV

Formulation of the K^-d Problem

The Elastic Scattering Amplitude

1. Introduction

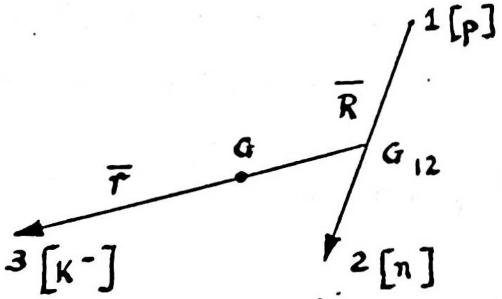


Fig. IV.1

The Schrodinger equation of the K^-d system referred to the three-body centre-of-mass G (see Fig. (IV.1)) is

$$H\Psi_a(\bar{r},\bar{R}) = E\Psi_a(\bar{r},\bar{R}) \qquad (4.1)$$

where

$$H = -\frac{\hbar^2}{2\mu}\nabla_{\bar{r}}^2 - \frac{\hbar^2}{m_N}\nabla_{\bar{R}}^2 + V_0(\bar{R}) + V_1(\bar{r}+\frac{\bar{R}}{2}) + V_2(\bar{r}-\frac{\bar{R}}{2}) \qquad (4.2)$$

In the hamiltonian H, μ represents the K^--reduced mass with respect to G, related to the meson and the nucleon masses m_K and m_N by

$$\frac{1}{\mu} = \frac{1}{2m_N} + \frac{1}{m_K} \quad , \qquad (4.3)$$

and V_0, V_1 and V_2 are two-body potentials. $V_0(\bar{R})$ gives the interaction between the two nucleons; $V_1(\bar{r}+\frac{\bar{R}}{2})$ and $V_2(\bar{r}-\frac{\bar{R}}{2})$ represent respectively the K^-p and the K^-n interactions. For the moment the isotopic spin dependence of these forces is not specified.

The wave functions $\phi_\alpha(\bar{R})$ and energies W_α of the scatterer (the np system) satisfy the equation

$$\left[-\frac{\hbar^2}{m_N}\nabla^2_{\bar{R}} + V_0(\bar{R})\right]\phi_\alpha(\bar{R}) = W_\alpha\,\phi_\alpha(\bar{R}) \tag{4.4}$$

and their normalisation is such that

$$\int \phi^*_\alpha(\bar{R})\,\phi_{\alpha'}(\bar{R})\,d\bar{R} = \delta_{\alpha\alpha'} \tag{4.5}$$

By definition (4.4) represents the deuteron for $\alpha = 0$.

In a K^-d collision, the initial and final states of the system are represented respectively by

$$\Phi_a = e^{i\bar{k}_a\cdot\bar{r}}\phi_0(\bar{R})\ ,\quad \Phi_b = e^{i\bar{k}_b\cdot\bar{r}}\phi_\alpha(\bar{R}) \tag{4.6}$$

where

$$\frac{\hbar^2 k_a^2}{2\mu} + W_0 = \frac{\hbar^2 k_b^2}{2\mu} + W_\alpha = E \tag{4.7}$$

Defining the kinetic energy operator by

$$K = -\frac{\hbar^2}{2\mu}\nabla^2_{\bar{r}} - \frac{\hbar^2}{m_N}\nabla^2_{\bar{R}} \tag{4.8}$$

and putting

$$H_0 = K + V_0, \tag{4.9}$$

both Φ_a and Φ_b are solutions of

$$(E - H_0)\Phi_{(a,b)} = 0 \tag{4.10}$$

Using the Schwinger-Lippmann (1950) formalism, the solution of equation (4.1) can be written

$$\Psi_a = \Phi_a + \frac{1}{E - H_0 + i\varepsilon}(V_1 + V_2)\Psi_a \tag{4.11}$$

where ε is the usual small positive quantity which insures the regularity of the operator $(E - H_0 + i\varepsilon)^{-1}$ and the existence of outgoing waves in the a-channels.

Considering now the transition operator T, defined by

$$T\,\Phi_a = V\,\Psi_a\ , \quad V = V_1 + V_2 \tag{4.12}$$

the transition matrix element from the initial state a to the final state b is given by

$$T_{ba} = (\Phi_b, V\Psi_a) = (\Phi_b, T\,\Phi_a) \tag{4.13}$$

Multiplying both sides of (4.11) on the left by V, using (4.12) and taking into account that the result of these operations holds for any Φ_a, one obtains

$$T = V + V\frac{1}{E - H_0 + i\varepsilon}T \tag{4.14}$$

The aim of this Chapter is to express the transition matrix elements (4.13) (with T given by the integral equations (4.14)) in terms of the two-body interactions arising in the K^-d system.

A first step in this direction is to relate T with the transition operators T_i (i = 1,2), defined by

$$V_i\Psi_{a,i} = T_i\Phi_a \tag{4.15}$$

where $\Psi_{a,i}$ is the K^--outgoing wave arising from nucleon i, "cleaned" from the waves sent by the other constituent of the deuteron. Therefore, $\Psi_{a,i}$ satisfies the equation

$$(E - H_0 - V_i)\Psi_{a,i} = 0 \tag{4.16}$$

and T_i is given by

$$T_i = V_i + V_i\frac{1}{E - H_0 + i\varepsilon}T_i$$

Putting

$$G = E - H_0 + i\varepsilon \tag{4.18}$$

T and T_i can be written

$$T = (1 - VG)^{-1}V, \quad T_i = (1 - V_iG)^{-1}T_i \tag{4.19}$$

Considering (4.19) and that

$$(1 - V_iG)^{-1} = 1 + (1 - V_iG)^{-1}V_iG,$$

a simple calculation gives (Schick, 1961)

$$\begin{aligned}
&[1 - (V_1 + V_2)G]V_1 = \\
&[(1 - V_1G)(1 - V_2G) - V_1GV_2G]^{-1} = \\
&[(1 - V_2G) - T_1GV_2G]^{-1}T_1 = \\
&(1 - V_2G)[1 - T_1GT_2G]T_1^{-1} = \\
&(1 + T_2G)(1 - T_1GT_2G)T_1
\end{aligned}$$

so that, using again (4.19), the operator T admits the following development in powers of T_1 and T_2:

$$T = T_1 + T_2 + T_1GT_2 + T_2GT_1 + T_1GT_2GT_1 + T_2GT_1GT_2 + \ldots \tag{4.20}$$

In the next paragraphs, the convergence of this series will be discussed and approximation's methods given to calculate its terms.

2. The Impulse Approximation

This method will be used in the calculation of the matrix elements $(\Phi_b, T_i\Phi_a)$, $i = 1,2$. Its fundamental assumption (Chew and Goldberger, 1952) consists in neglecting $V_0(\bar{R})$ in the operator G given by (4.18). This is equivalent to considering the two nucleons as free particles during the meson-nucleon collision's time.

But, due to the time-energy uncertainty relation, no external interaction can reveal the binding forces between the constituents of the deuteron, if it does not cover a period of time at least equal to $\hbar/|Ed|$ (Ed is the binding energy of the deuteron). Therefore, if the K^-N collision time is much shorter than $\hbar/|Ed|$, the fundamental assumption of the Impulse Approximation holds good, because the subsequent evolution of the three-body system cannot be altered very much.

Representing by $\underline{v}$ the velocity of the kaon in the K^-N centre-of-mass referencial and considering that the range of K^-N interactions is 0.4 fermi (see (2.2) in Chapter II), the condition for the validity of the fundamental assumption can be written under the form

$$\frac{10^{-13}Ed}{\hbar v} \ll 1 \tag{4.21}$$

The simplifications brought by this hypothesis on the formulation of the K^-d problem are now analysed.

Equation (4.10) reduces to

$$(E_n - K)\chi_n = 0 \tag{4.10'}$$

where

$$\frac{\hbar^2 k_a^2}{2\mu} + \frac{\hbar^2 K_a^2}{m_N} = E_n \tag{4.7'}$$

K_a being the wave number of the deuteron's internal motion.

The solutions of (4.10'), the χ_n, permit to define the approximate transition operators t_i $(i = 1,2)$ by

$$V_i\psi_{n,i} = t_i\chi_n \tag{4.15'}$$

where the $\psi_{n,i}$ satisfy the equations

$$(E_n - K - V_i)\psi_{n,i} = 0, \qquad i = 1,2 \qquad (4.16')$$

Corresponding to the exact equations (4.16). The integral equations for the t_i are, then,

$$t_i = V_i + V_i \frac{1}{E_n - K + i\varepsilon} t_i \qquad (4.17')$$

The fundamental hypothesis of the Impulse Approximation is completed by two new assumptions, complementary of each other (Chew and Goldberger, 1952):

1) The incident kaon never interacts simultaneously with the two nucleons;

2) The amplitude of the K^--wave falling upon each nucleon is only slightly altered by the presence of the other constituent of the deuteron.

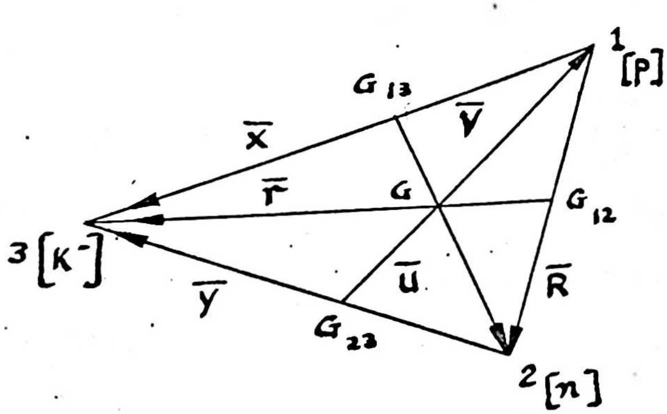


Fig. IV.2

The set of coordinates $(\overline{r},\overline{R})$ has been used in the previous formalism. However, to deal with the assumption 1), it is more convenient that the wave functions have for coordinates $(\overline{X},\overline{U})$ or $(\overline{Y},\overline{V})$ (see Fig. (IV.2)), according to whether the kaon interacts with nucleon 1 or 2.

The transformation of one set of coordinates into another implies also the knowledge of the relationships between the corresponding wave numbers. In the following pages, the transformation laws for these quantities, neglecting the relativistic effects (see Appendix A), are obtained.

Suppose, then, that (Fig. (IV.2)) G is the three-body centre-of-mass and G_{ij} represents the centre-of-mass of particles i and j. If $\bar{r}_o$, $\bar{r}_{ij}$ and $\bar{r}_i$ are respectively the coordinates of G, G_{ij} and the i-particle with respect to an inertial frame of reference, one has

$$(2m_N + m_K)\bar{r}_0 = m_K\bar{r}_3 + m_N(\bar{r}_1 + \bar{r}_2) \tag{4.22}$$

$$\bar{R} = \bar{r}_2 - \bar{r}_1 \tag{4.23}$$

$$\bar{r} = \bar{r}_3 - \bar{r}_{12} = \bar{r}_3 - \frac{\bar{r}_1 + \bar{r}_2}{2} \tag{4.24}$$

Multiplying both sides of (4.23) and (4.24) respectively by $\frac{m_N}{2}$ and μ (see (4.3)), i.e., the reduced masses related with the motions defined by $\bar{R}$ and $\bar{r}$, one obtains

$$\frac{m_N}{2}\bar{R} = \frac{1}{2}\left(m_N\bar{r}_2 - m_N\bar{r}_1\right) \tag{4.23'}$$

$$\mu\bar{r} = \frac{2m_N}{(2m_N + m_K)}\cdot m_K\bar{r}_3 - \frac{m_N\bar{r}_1 + m_N\bar{r}_2}{(2m_N + m_K)}\cdot m_K \tag{4.24'}$$

Now, if the wave numbers $\bar{k}_i$, $\bar{K}_G$, $\bar{K}$ and $\bar{k}$, associated respectively with the coordinates $\bar{r}_i$, $\bar{r}_0$, $\bar{R}$ and $\bar{r}$ (so that $m_N\dot{\bar{r}}_1 = \hbar\bar{k}_1$, etc) are introduced, one gets by differentiation with respect to time of equations (4.22), (4.23') and (4.24'),

$$\bar{K}_G = \sum_{i=1}^{3}\bar{k}_i \tag{4.25}$$

$$\bar{K} = \frac{1}{2}(\bar{k}_2 - \bar{k}_1) \tag{4.26}$$

$$\bar{k} = \frac{2m_N\bar{k}_3}{2m_N + m_K} - \frac{m_K}{2m_N + m_K}(\bar{k}_1 + \bar{k}_2) \tag{4.27}$$

as well as

$$\bar{K}_G \cdot \bar{r}_o + \bar{K}\cdot\bar{R} + \bar{k}\cdot\bar{r} = \sum_{i=1}^{3} \bar{k}_i \bar{r}_i \tag{4.28}$$

Representing respectively by $(\bar{p}, \bar{k}_u)$ and $(\bar{q},\bar{k}_v)$ the wave numbers associated with the sets of coordinates $(\bar{X},\bar{U})$ and $(\bar{Y},\bar{V})$, two relations analogous to (4.28) for these coordinates hold. Therefore, one gets

$$\bar{K}.\bar{R} + \bar{k}.\bar{r} = \bar{k}_u.\bar{U} + \bar{p}.\bar{X} = \bar{k}_v.\bar{V} + \bar{q}.\bar{Y} \tag{4.29}$$

Suppose now that particle 3 (kaon) interacts with particle 1 in such a way that particle 2 does not participate in the collision (assumption 1). In these circumstances, the motion of the three-body system must be described in terms of the coordinates $(\bar{X},\bar{U})$ rather than in terms of $(\bar{r},\bar{R})$.

From Fig. (IV.2)

$$\bar{X} = \bar{r} + \frac{\bar{R}}{2} \tag{4.30}$$

$$\bar{U} = \bar{R} + \bar{r}_1 - \bar{r}_{31} \quad , \quad \bar{r}_{31} = \frac{m_N \bar{r}_1 + m_K \bar{r}_3}{m_N + m_K}$$

or

$$\bar{U} = \bar{R} - \frac{m_K}{m_N + m_K}\bar{X} \tag{4.31}$$

Multiplying both sides of (4.30) and (4.31) respectively by the reduced masses

$$\mu_{31} = \left(\frac{1}{m_K} + \frac{1}{m_N}\right)^{-1}, \quad \mu_{2,(13)} = \left(\frac{1}{m_N} + \frac{1}{m_N + m_K}\right)^{-1} \tag{4.3'}$$

associated with $\bar{X}$ and $\bar{U}$ and applying the procedure used in the determination of $\bar{K}$ and $\bar{k}$ as functions of the $\bar{k}_i$, the following relations between $(\bar{p},\bar{k}_u)$ and $(\bar{k},\bar{K})$ are obtained:

$$\bar{p} = \frac{2m_N + m_K}{2(m_N + m_K)} . \bar{k} + \frac{m_K}{m_N + m_K} \bar{K} \qquad (4.32)$$

$$\bar{k}_U = \bar{K} - \frac{1}{2} \bar{k} \qquad (4.33)$$

Now, if $\bar{k}_i$ is the i-particle wave number in the intertial system before the collision, after this event (supposing the kaon momentum transfer equal to $\hbar\bar{\alpha}$) such wuantities become

$$\bar{k}_1 + \bar{\alpha}, \quad \bar{k}_2, \quad \bar{k}_3 - \bar{\alpha}$$

Therefore, if $\bar{K}_G^a$, $\bar{K}_a$ and $\bar{k}_a$ are the initial values of $\bar{K}_G$, $\bar{K}$ and $\bar{k}$ respectively, the final values of the same variables are, by (4.25), (4.26) and (4.27)

$$K_G^b = \bar{K}_G^a \qquad (4.34)$$

$$\bar{K}_b = \bar{K}_a - \tfrac{1}{2}\bar{\alpha} \qquad (4.35)$$

$$\bar{k}_b = \bar{k}_a - \bar{\alpha} \qquad (4.36)$$

The relation (4.34) means that the motion of the three-body centre-of-mass remains unchanged.

Finally, by (4.32) and (4.33), the initial ($\bar{p}_a$, $\bar{k}_u^a$) and the final ($\bar{p}_b$, $\bar{k}_u^b$) values of the set ($\bar{p}$, $\bar{k}_u$) are such that

$$\bar{p}_b = \bar{p}_a - \bar{\alpha} \qquad (4.37)$$

$$\bar{k}_u^b = \bar{k}_u^a \qquad (4.38)$$

In (4.37) one has

$$|\bar{p}_b| = |\bar{p}_a| \qquad (4.39)$$

when there is no excitation or break-up of the constituents of the deuteron.

Now, if T_i in the matrix elements

$$T_{ba}^{(i)} = (\Phi_b, T_i \Phi_a) \tag{4.40}$$

are substituted by the approximate transition operators t_i, given by (4.17'), one gets

$$t_i \Phi_a = t_i \mathop{S}_r |\chi_r\rangle\langle\chi_r|\Phi_a\rangle$$

or, by (4.15')

$$t_i \Phi_a = V_i \mathop{S}_r |\psi_{r,i}\rangle\langle\chi_r|\Phi_a\rangle \tag{4.41}$$

Indeed, the χ_r as well as the $\psi_{r,i}$, given respectively by (4.10') and (4.16p) form complete sets of orthogonal functions and are supposed to be normalised in such a way that

$$(\chi_{r'}, \chi_r) = (\psi_{r',i}, \psi_{r,i}) = \delta_{r',r} \tag{4.42}$$

The explicit expressions of χ_r and $\psi_{r,1}$ (i=1) are, then,

$$\chi_r(\bar{r}, \bar{R}) = \frac{1}{(2\pi)^3} e^{i(\bar{K}\cdot\bar{R} + \bar{k}\cdot\bar{r})} \tag{4.43}$$

and

$$\psi_{r,1}(\bar{x}, \bar{U}) = \frac{1}{(2\pi)^3} \psi_{\bar{p},\bar{K}}(\bar{x})\, e^{i\bar{K}_0\cdot\bar{U}} \tag{4.44}$$

where $\psi_{\bar{p},\bar{K}}(\bar{x})$ has the asymptotic form

$$\psi_{\bar{p},\bar{K}}(\bar{x}) \rightarrow e^{i\bar{p}\cdot\bar{x}} + f_{\bar{p},K}(\phi)\frac{e^{ipx}}{x} \tag{4.45}$$

This wave function represents the total wave in a meson-nucleon scattering process. The indices written explicitly in $\psi_{\bar{p},\bar{K}}(\bar{x})$ refer to the values of $\bar{p}$ and $\bar{K}$ just before the collision.

The index $\underline{r}$ in (4.43) and (4.44) stands for the set of variables $(\bar{k},\bar{K})$ or $(\bar{p},\bar{k}_{\upsilon})$ and $\displaystyle\mathop{S}_{r}$ in (4.41) should be interpreted as $\int d\bar{k}\,d\bar{K}$ or $\int d\bar{p}\,d\bar{k}_{\upsilon}$.

Now, by (4.6) and (4.43), one has

$$\langle \chi_r | \Phi_a \rangle = \frac{1}{(2\pi)^3} \int e^{-i(\bar{k}.\bar{r} + \bar{K}.\bar{R})} e^{i\bar{k}_a.\bar{r}} \phi_o(\bar{R})\, d\bar{R}$$

$$= (2\pi)^{\frac{3}{2}}\, \delta(\bar{k}-\bar{k}_a)\, g_o(\bar{K}) \qquad (4.46)$$

where g_o $(\bar{K})$ is the Fourier component

$$g_\alpha(\bar{K}) = (2\pi)^{-\frac{3}{2}} \int e^{-i\bar{K}.\bar{R}} \phi_\alpha(\bar{R})\, d\bar{R} \qquad (4.47)$$

of the NN internal motion wave function $\phi_\alpha(\bar{R})$ for $\alpha = 0$ (deuteron).

Therefore, (4.41) becomes

$$t_1 \Phi_a = (2\pi)^{-\frac{3}{2}} V_1(\bar{X}) \int \psi_{\bar{p},\bar{K}}(\bar{X})\, e^{i\bar{k}^a_\upsilon.\bar{U}} g_o(\bar{K}_a)\, d\bar{K}_a \qquad (4.48)$$

and $T^{(1)}_{ba}$ gives considering (4.6) and (4.40),

$$T^{(1)}_{ba} = (2\pi)^{-\frac{3}{2}} \int d\bar{r}\, d\bar{R}\, d\bar{K}_a\, e^{-i\bar{k}_b.\bar{r}} \phi^{*}_{\alpha}(\bar{R})\, V_1(\bar{X})\, \psi_{\bar{p},\bar{K}}(\bar{X})\, e^{i\bar{k}^a_\upsilon.\bar{U}} g_o(K_a)$$

But, by (4.49) and (4.38),

$$\bar{k}_b.\bar{r} = \bar{p}_b.\bar{X} + \bar{k}^a_\upsilon.\bar{U} - \bar{K}_b.\bar{R}$$

so that $T^{(1)}_{ba}$ can be written in the following way, if the relation

$$\bar{X} = \bar{r} + \frac{\bar{R}}{2} \qquad (4.30)$$

and (4.47) are taken into account:

$$T_{ba}^{(1)} = \int d\bar{x}\, d\bar{K}_a\, e^{-i\bar{p}_b\cdot\bar{x}} g_\alpha^{x}(\bar{K}_b) V_1(\bar{x}) \psi_{\bar{p},\bar{K}_a}(\bar{x})\, g_0(\bar{K}_a)$$

where, by (4.35) and (4.36),

$$\bar{K}_b = \bar{K}_a + \tfrac{1}{2}(\bar{k}_b - \bar{k}_a) \tag{4.49}$$

But 1') the two-body matrix element

$$\int e^{-i\bar{p}_b\cdot\bar{x}} V_1(\bar{x}) \psi_{\bar{p},\bar{K}}(\bar{x})\, d\bar{x}$$

is a slowly varying function of $\bar{K}_a$ (Chew, 1950). If assumption 1') is made, then representing by $<\bar{p}_b|t_{1,0}|\bar{p}_a>$ the value of the latter integral for $\bar{K}_a = 0$, $T_{ba}^{(1)}$ reduces to

$$T_{ba}^{(1)} = <\bar{p}_b|t_{1,0}|\bar{p}_a>\, S^{\frac{1}{2}}(\theta) \tag{4.50}$$

where

$$S^{\frac{1}{2}}(\theta) = \int g_\alpha^{x}[\bar{K}_a + \tfrac{1}{2}(\bar{k}_b - \bar{k}_a)]\, g_0(\bar{K}_a)\, d\bar{K}_a \tag{4.51}$$

The function $S^{\frac{1}{2}}(\theta)$ is the "form factor" of the Impulse Approximation. It is obvious that, by (4.47), $S^{\frac{1}{2}}(\theta)$ is equal to

$$S^{\frac{1}{2}}(\theta) = \int \phi_\alpha^{x}(\bar{R})\, e^{i(\bar{k}_b - \bar{k}_a)\cdot\bar{R}/2}\, \phi_0(\bar{R})\, d\bar{R} \tag{4.52}$$

Similarly, the matrix element $T_{ba}^{(2)}$ is given by

$$T_{ba}^{(2)} = <\bar{q}_b|t_{2,0}|\bar{q}_a> \int \phi_\alpha^{\prime x}(\bar{R})\, e^{-i(\bar{k}_b - \bar{k}_a)\cdot\bar{R}/2}\, \phi_0(\bar{R})\, d\bar{R} \tag{4.50'}$$

where, in agreement with assumption 1'), $<\bar{q}_b|t_{2,0}|\bar{q}_a>$ is the value of the integral

$$\int e^{-iq_b\cdot\bar{y}}\, V_2(\bar{y})\, \psi_{q_a,\bar{K}_a}(\bar{y})\, d\bar{y}$$

when $\bar{K}_a = 0$.

3. The deuteron's recoil

Suppose that the K^-d scattering is elastic: the final state Φ_b, defined in (4.6), is now given by

$$\Phi_b = e^{i\bar{k}_b\cdot\bar{r}}\,\phi_0(\bar{R}) \tag{4.6'}$$

where

$$|\bar{k}_b| = |\bar{k}_a| = k_a \tag{4.53}$$

Due to the deuteron's recoil, the two-body wave numbers $\bar{p}_{(a,b)}$ and $\bar{q}_{(a,b)}$ are not equal to $\bar{k}_{(a,b)}$. However, the relations between them are easily derived from equations already established. Indeed, from (4.36) and (4.37) one gets

$$\bar{p}_b - \bar{p}_a = \bar{k}_b - \bar{k}_a \tag{4.54}$$

and from (4.32) and (4.35) it follows that (putting $\bar{K}_a = 0$, in agreement with assumption 1')),

$$\bar{p}_b + \bar{p}_a = \frac{1}{\gamma}\left[(\bar{k}_b + \bar{k}_a) + \frac{m_K}{2m_N + m_K}(\bar{k}_b - \bar{k}_a)\right] \tag{4.55}$$

where

$$\gamma = \frac{2(m_N + m_K)}{2m_N + m_K} = 1.2084 \tag{4.56}$$

Therefore, squaring both sides of (4.54) and (4.55), adding the results of these operations and taking into account (4.53), one has, if θ represents the scattering angle defined by vectors $\bar{k}_a$ and $\bar{k}_b$,

$$2(\bar{p}_a^2 + \bar{p}_b^2) = \frac{4k_a^2}{\gamma^2}\left\{\cos^2\frac{\theta}{2} + \left[\gamma^2 + \left(\frac{m_K}{2m_N + m_K}\right)^2\right]\sin^2\frac{\theta}{2}\right\}$$

The coefficient $[m_K/(2m_N+m_K)]^2$ can be neglected in this expression because its value is ~ 0.04 (m_N = 2mK) and $\gamma^2 > 1$. This means that the second term in (4.55) can be discarded, so that $|\bar{p}_b| = |\bar{p}_a| = pa$, i.e., the scattering in the K^-N systems is also elastic.

In these conditions, one gets

$$p_a^2 = p_b^2 = \frac{k_a^2}{\gamma^2}\left[1 + (\gamma^2 - 1)\sin^2\frac{\theta}{2}\right] \tag{4.57}$$

or

$$p_a^2 = p_b^2 = \frac{k_a^2}{\gamma^2}\frac{\gamma^2+1}{2}\left[1 - \frac{\gamma^2-1}{\gamma^2+1}\cos\theta\right] \tag{4.57'}$$

Finally, the relation between p_a^2 (or p_b^2) and k_a^2 for K^-d elastic scattering processes adopted in this work, will be

$$p_a^2 = p_b^2 = \frac{\gamma^2+1}{2\gamma^2}\cdot k_a^2 = (0.9178\,k_a)^2 \tag{4.57''}$$

with similar relations for q_a^2 q_b^2. They can be interpreted in two different ways: either as approximations of (4.57) $[(\gamma^2-1)/(\gamma^2+1) \simeq 0.2]$ or as the result of taking the average of the scattering angle θ over all its possible values. The latter interpretation is the best when the incoming particle suffers multiple scattering.

4. Multiple Scattering

Consider the double scattering terms T_2GT_1 and T_1GT_2 in the development (4.20) for T. If the Impulse Approximation is applied to the calculation of the transition matrix elements arising from them, one has

$$T_{ba}^{(2,1)} = \langle \Phi_b | t_2 G_o t_1 | \Phi_a \rangle, \quad T_{ba}^{(1,2)} = \langle \Phi_b | t_1 G_o t_2 | \Phi_a \rangle \quad (4.58)$$

where, by (4.7'),

$$G_o = (E_n + K + i\varepsilon)^{-1}, \quad \frac{\hbar^2 k_a^2}{2\mu} + \frac{\hbar^2 K_a^2}{m_N} = E_n \quad (4.59)$$

Now, using the complete orthonormal set of functions χ_r, $T_{ba}^{(2,1)}$ admits the collowing development

$$T_{ba}^{(2,1)} = \mathop{S}_{r,r'} \langle \Phi_b | t_2 | \chi_r \rangle \langle \chi_r | G_o | \chi_{r'} \rangle \langle \chi_{r'} | t_1 | \Phi_a \rangle$$

or, considering that $\chi_{r'}$ is defined by equation (4.10') for n = r'

$$T_{ba}^{(2,1)} = \mathop{S}_{r} \frac{\langle \Phi_b | t_2 | \chi_r \rangle \langle \chi_r | t_1 | \Phi_a \rangle}{E_n - E_r + i\varepsilon} \quad (4.60)$$

Substitution of $t_1 | \Phi_a \rangle$ by (4.48) in $\langle \chi_r | t_1 | \Phi_a \rangle$ gives for this matrix element

$$(2\pi)^{-\frac{9}{2}} \int d\bar{x}\, d\bar{U}\, d\bar{K}_a\, e^{i(\bar{p}\cdot\bar{x} + \bar{K}_v\cdot\bar{U})} V_1(\bar{x})\, \psi^{-}_{\bar{p},\bar{K}_a}(\bar{x})\, e^{i\bar{K}_U^a\cdot\bar{U}} g_o(\bar{K}_a)$$

or, introducing the assumption 1') of §2),

$$\langle \chi_r | t_1 | \Phi_a \rangle = (2\pi)^{\frac{3}{2}} \int \delta(\bar{K}_a - \bar{K}_U^a) \langle \bar{p}_b | t_{1,0} | \bar{p} \rangle g_o(\bar{K}_a)\, d\bar{K}_a \quad (4.61)$$

Also, by (4.15') one has

$$\langle\phi_b|t_2|\chi_r\rangle = \langle\phi_b|v_2\psi_{r,2}\rangle$$

where, similarly to $\psi_{r,1}$ defined in (4.44),

$$\psi_{r,2}(\bar{y},\bar{v}) = \frac{1}{(2\pi)^3}\,\psi_{\bar{q},\bar{K}}(\bar{y})e^{i\bar{k}_v\cdot\bar{v}} \tag{4.62}$$

with

$$\psi_{\bar{q},\bar{K}}(\bar{y}) \underset{y\to\infty}{\to} e^{i\bar{q}\cdot\bar{y}} + f_{\bar{q},\bar{K}}(\phi)\frac{e^{iqy}}{y}$$

Therefore, using the relation $\bar{y} = \bar{r} - \frac{\bar{R}}{2}$ (see Fig. (IV.2)), one has

$$\langle\phi_b|t_2|\chi_r\rangle = \int d\bar{y}d\bar{R}\,\phi_\alpha^\times(\bar{R})e^{-i\bar{k}_b\bar{r}}v_2(\bar{y})\psi_{r,2}(\bar{y},\bar{v}) \tag{4.63}$$

Now, by (4.29), it follows that

$$\bar{k}_b\cdot\bar{r} = \bar{q}_b\cdot\bar{y} + \bar{k}_v^b\cdot\bar{v} - \bar{K}_b\cdot\bar{R} \tag{4.29'}$$

The kaon, however, strikes each time only one of the constituents of the deuteron (Assumption 1)), so that, in analogy with (4.38),

$$\bar{k}_v^b = \bar{k}_v \tag{4.38'}$$

and finally, considering assumption 1') together with (4.62), (4.29'), (4.38') and the definition of $g_\alpha(\bar{K})$ given in (4.47), one has

$$\langle\phi_b|t_2|\chi_r\rangle = \frac{1}{(2\pi)^{3/2}}g_\alpha^\times(\bar{K}_b)\langle\bar{q}_b|t_{2,0}|\bar{q}\rangle$$

Therefore, (4.60) gives

$$T_{ba}^{(2,1)} = \frac{1}{(2\pi)^3}\int\frac{g_\alpha^\times(\bar{K}_b)\langle\bar{q}_b|t_{2,0}|q\rangle\delta(\bar{k}_u-\bar{k}_u^a)\langle\bar{p}|t_{1,0}|\bar{p}_a\rangle g_0(\bar{K}_a)d\bar{K}_a d\bar{k}d\bar{K}}{E_n - E_r + i\epsilon} \tag{4.64}$$

In (4.59), E_n is expressed in terms of the initial values $\bar{k}_a$ and $\bar{K}_a$ of the wave numbers associated with the coordinates $\bar{r}$ and $\bar{R}$. But, introducing the reduced masses μ_{31} and $\mu_{2,(31)}$ defined in (4.3'), E_n can also be related to $\bar{p}_a$ and $\bar{k}_u^a$, which are the initial values of the wave numbers associated with coordinates $\bar{\chi}$ and $\bar{U}$:

$$E_n = \frac{\hbar^2 \bar{p}_a^2}{2\mu_{31}} + \frac{\hbar^2 (k_u^a)^2}{2\mu_{2,(31)}} \tag{4.3''}$$

Similarly, one gets for E_r

$$E_r = \frac{\hbar^2 p^2}{2\mu_{31}} + \frac{\hbar^2 (k_u)^2}{2\mu_{2,(31)}} \tag{4.3'''}$$

Thus, substituting $\bar{K}$ by $\bar{k}_u$ in (4.64), using the relation

$$k_u = K - \tfrac{1}{2}k \tag{4.33}$$

and integrating with respect to $\bar{k}_u$, $T_{ba}^{(2,1)}$ becomes

$$T_{b,a}^{(2,1)} = \frac{-1}{(2\pi)^3} \frac{2\mu_{31}}{\hbar^2} \int \frac{g_\alpha^{\times}(\bar{K}_b) \langle \bar{q}_b | t_{2,0} | \bar{q} \rangle \langle \bar{p} | t_{1,0} | \bar{p}_\alpha \rangle g_0(\bar{K}_a) d\bar{K}_a d\bar{k}}{p^2 - (p_a + i\varepsilon)^2} \tag{4.65}$$

where ε has been redefined.

Equation (4.33) holds also for the initial values of the wave numbers involved in it, i.e.,

$$\bar{k}_u^a = \bar{K}_a - \tfrac{1}{2}\bar{k}_a \; ; \tag{4.33'}$$

but the δ-function in (4.64) imposes the condition $\bar{k}_u = \bar{k}_u^a$, so that

$$\bar{K} = \bar{K}_a - \tfrac{1}{2}(\bar{k}_a - \bar{k}) \tag{4.66}$$

Similarly to (4.33) and (4.33') one gets for wave numbers k_v and k_v^b

$$\bar{k}_v = -\bar{K} - \tfrac{1}{2}\bar{k} \,, \qquad \bar{k}_v^b = -\bar{K}_b - \tfrac{1}{2}\bar{k}_b$$

and, from the same relations, by (4.38'),

$$\bar{K}_b = \bar{K} + \frac{1}{2}(\bar{k} - \bar{k}_b) ; \tag{4.66'}$$

then, by elimination of $\bar{K}$ in (4.66) and (4.66'),

$$\bar{K}_b = \bar{K}_a + \bar{k} - \frac{1}{2}(\bar{k}_a + \bar{k}_b) \tag{4.66''}$$

Therefore, introducing again in (4.65) the wave functions $\phi_\alpha(\bar{R})$ and $\phi_0(\bar{R})$ by means of (4.47), $T_{ba}^{(2,1)}$ gives

$$T_{ba}^{(2,1)} = - \frac{2\mu_{31}}{(2\pi)^3 \hbar^2} \int \phi_\alpha^*(\bar{R})\, \phi_0(\bar{R})\, e^{-i(\bar{k}_a + \bar{k}_b)\cdot \bar{R}/2} \, \frac{e^{i\bar{k}\cdot\bar{R}} \langle \bar{q}_b | t_{2,0} | \bar{q} \rangle \langle \bar{p} | t_{1,0} | \bar{p}_a \rangle}{p^2 - (p_a + i\varepsilon)^2} \, d\bar{k}\, d\bar{R} \tag{4.67}$$

The calculation of $T_{ba}^{(1,2)}$ using the same approximation leads to

$$T_{ba}^{(1,2)} = - \frac{2\mu_{32}}{(2\pi)^3 \hbar^2} \int \phi_\alpha^*(\bar{R})\, \phi_0(\bar{R})\, e^{i(\bar{k}_a + \bar{k}_b)\cdot \bar{R}/2} \, \frac{e^{i\bar{k}\cdot\bar{R}} \langle \bar{p}_b | t_{1,0} | \bar{p} \rangle \langle \bar{q} | t_{2,0} | \bar{q}_a \rangle}{q^2 - (q_a + i\varepsilon)^2} \, d\bar{k}\, d\bar{R} \tag{4.67'}$$

The expressions for $T_{ba}^{(1)}$, $T_{ba}^{(2)}$, $T_{ba}^{(1,2)}$ and $T_{ba}^{(2,1)}$ show that they can be interpreted as weighted means of the single and the double scattering terms arising in the collision of a particle with two moving centres, located simultaneously at $-\bar{R}/2$ and $\bar{R}/2$. The two-body scattering amplitudes at these points are respectively proportional to the matrix elements.

$$e^{-i(\bar{k} - \bar{k}')\cdot\bar{R}/2} \langle \bar{p}' | t_{1,0} | \bar{p} \rangle \quad , \quad e^{i(\bar{k} - \bar{k}')\cdot\bar{R}/2} \langle \bar{q}' | t_{2,0} | \bar{q} \rangle$$

and the "weight" is $\phi_\alpha^*(\bar{R})\, \phi_0(\bar{R})$.

In (4.67) and (4.67'), the main contributions to the integrals over $\bar{k}$ came respectively from the values of p^2 and q^2 which are close to p_a^2 and q_a^2. Thus, if in a double scattering process, the kaon is scattered elastically by the first nucleon, p_a^2 (or q_a^2) is given by (4.57"), so that the kaon-waves transmitted to the second nucleon are predominantly those which have the square of the wave number equal to

$$p^2 \simeq q^2 \simeq \frac{1+\gamma^2}{2\gamma^2} k_a^2 = p_a^2 = q_a^2 \qquad (4.68)$$

The calculation of the T-terms in (4.20) is much simpler if it is accepted the physically reasonable assumption that

2') "in any multiple scattering process all single scatterings are elastic (without break up of the deuteron) with the possible exception of the last, which can be non-elastic".

Admitting this hypothesis a straightforward extension to scattering terms of any order is open using the previous interpretation of the single and double scatterings as "weighted means". For instance, the triplet transition matrix element corresponding to $T_1GT_2GT_1$ in the development (4.20) for T is equal to

$$\frac{1}{(2\pi)^6} \cdot \frac{2\mu_{31}}{\hbar^2} \cdot \frac{2\mu_{32}}{\hbar^2} \cdot \int \langle \bar{p}_b | t_{1,0} | \bar{p} \rangle \frac{e^{-i\bar{k}\cdot\bar{R}}\, d\bar{k}}{q^2 - (q_a + i\varepsilon)^2} \cdot \langle \bar{q} | t_{2,0} | \bar{q}' \rangle \cdot$$
$$\cdot \int \frac{-e^{i\bar{k}'\cdot\bar{R}}\, d\bar{k}'}{p^2 - (p_a + i\varepsilon)^2} \cdot \langle \bar{p}' | t_{1,0} | p_a \rangle \cdot \int \phi_\alpha^*(\bar{R})\, \phi_0(\bar{R})\, e^{-i(\bar{k}_a - \bar{k}_b)\cdot\bar{R}/2}\, d\bar{R} \qquad (4.69)$$

Actually, if the calculation of $T_1GT_2GT_1$ is carried out by Impulse Approximation and assumptions 1') and 2') are taken into account, the expression (4.69) will be reached again. Therefore, in principle, all T-terms are known.

Finally, supposing that the two moving centres much heavier than the incoming particle, γ tends to 1 (see (4.56)) and $\bar{p}_a$(or $\bar{q}_a$), by (4.57") tends to $\bar{k}_a$, so that all T-terms reduce to those which are obtained in the problem dealing with one particle scattered by two fixed centres. (Drell and Verlet, 1955; Schick, 1961).

5. The off-energy shell Matrix Elements

The K^-N scattering for momenta considered in this work (see Chapter I) is isotropic. Therefore, the scattering amplitudes $f_{31}(p',p)$ and $f_{32}(q',q)$ which are associated with the two-body matrix elements by the relations

$$f_{31}(\bar{p}',\bar{p}) = -\frac{\mu_{31}}{2\pi\hbar^2}\langle\bar{p}'|t_{1,0}|\bar{p}\rangle \tag{4.70}$$

and

$$f_{32}(\bar{q}',\bar{q}) = -\frac{\mu_{32}}{2\pi\hbar^2}\langle\bar{q}'|t_{2,0}|\bar{q}\rangle \tag{4.70'}$$

have no angular dependence.

Now, to calculate the multiple scattering terms, it is essential the knowledge of the behaviour of f_{31} and f_{32} with respect to the values of p' and q' for which the inequalities $p' \neq p$ and $q' \neq q$ hold. Such values are off the energy shell or, in other words, do not respect the energy conservation principle. They correspond to processes where virtual kaons are scattered by the nucleons.

Drell and Verlet (1955) consider for S-scattering two extreme cases of behaviour (i = 1,2):

I) $f_{3i}(p_a, p) = f_{3i}(p', p) = f_{3i}(p, p_a) = f_{3i}(p_a, p_a)$; $f_{3i}(p_b, p) = f_{3i}(p_b, p_b)$;

II) $f_{3i}(p_a, p) = f_{3i}(p', p) = f_{3i}(p_a, p_a) = 0$ if $p', p \neq p_a$; $f_{3i}(p_b, p) = 0$ if $p \neq p_b$

The first approximation leads to Bruckner's model (Bruckner, 1953). In this case, the propagator e^{ikR}/R appears in the multiple scattering terms.

However, case II is more in agreement with the assumption 2') of §4) and for this reason will be adopted here.

If k^2 and k_a^2, given by (4.68), are introduced in the denominator of

$$\frac{1}{2\pi} \lim_{\varepsilon \to 0} \int \frac{f_{3i}(p, p_a)\, e^{i\bar{k}\cdot\bar{R}}}{p^2 - (p_a + i\varepsilon)^2} \cdot d\bar{k} \tag{4.71}$$

and integration with respect to the angular part of $\bar{k}$ is performed, then, considering that the f_{3i}, according to the approximation II, vanish except for $p = p_a$, the integral (4.71) is equivalent to

$$\frac{2\gamma^2}{1+\gamma^2} \cdot \frac{2}{\pi R} \lim_{\rho \to 0} \int_C \frac{k \sin(kR)\, f_{3i}(p, p_a)}{k^2 - k_a^2} \cdot dk$$

Fig. IV.3

where C is the semi-circle of radius ρ centred at point k_a, as shown in Fig. (IV.3)

Therefore, (4.71) is equal to

$$\frac{2\gamma^2}{1+\gamma^2} \cdot \frac{i}{R} \sin(kR)\, f_{3i}(p, p_a) \tag{4.72}$$

6. The K^-d Elastic Scattering Amplitude

From the previous considerations it is by now clear that the expression for the elastic scattering amplitude, $f(\theta)$, in K^-d processes, calculated by Impulse Approximation, can be written in the form

$$f(\theta) = \int |\phi_0(\bar{R})|^2 f(\theta,\bar{R})\, d\bar{R} \tag{4.73}$$

where

$$f(\theta,\bar{R}) = -\frac{2\mu}{4\pi\hbar^2} \langle \bar{k}_b | T | \bar{k}_a \rangle \tag{4.74}$$

Obviously, in these relations, μ is the reduced kaon-mass (4.3) and θ the angle defined by $\bar{k}_a$ and $\bar{k}_b$ ($|\bar{k}_a| = |\bar{k}_b|$). Considering the definition (4.56) for γ, μ can be expressed in terms of the two-body reduced masses μ_{31} and μ_{32} ($= \mu_{31}$) (see (4.3')) in the following way:

$$\mu = \gamma \mu_{31} = \gamma \mu_{32} \tag{4.75}$$

Thus, by (4.50), (4.50'), (4.70), (4.70') and (4.75), the single scattering terms in $f(\theta,\bar{R})$ (corresponding to T_1 and T_2 in the T-development (4.20)) are

$$f^{(1)}(\theta,\bar{R}) = \gamma \left[e^{-i(\bar{k}_a-\bar{k}_b)\cdot\bar{R}/2} f_{31} + e^{i(\bar{k}_a-\bar{k}_b)\cdot\bar{R}/2} f_{32} \right] \tag{4.76}$$

The double scattering, $f^{(2)}(\theta,R)$, is derived from (4.67) and (4.67') together with (4.68), (4.70), 4.70'), (4.75) and the expression (4.72) for the integral (4.71). Putting

$$\rho(R) = i \sin(kR)/R\,, \tag{4.72'}$$

one gets for $f^{(2)}(\theta,R)$:

$$f^{(2)}(\theta,\bar{R}) = \frac{2\gamma^3}{1+\gamma^2}\left[e^{-i(\bar{k}_a+\bar{k}_b)\cdot\bar{R}/2} f_{32}f_{31} + e^{i(\bar{k}_a+\bar{k}_b)\cdot\bar{R}/2} f_{31}f_{32}\right]\rho(R) \qquad (4.77)$$

From (4.69) and in the same was as for $f^{(1)}(\theta,\bar{R})$ and $f^{(2)}(\theta,\bar{R})$, the aclculation of triplet scattering gives

$$f^{(3)}(\theta,\bar{R}) = \frac{2\gamma^6}{(1+\gamma^2)^2}\left[e^{-i(\bar{k}_a-\bar{k}_b)\cdot\bar{R}/2} f_{31}f_{32}f_{31} + e^{i(\bar{k}_a-\bar{k}_b)\cdot\bar{R}/2} f_{32}f_{31}f_{32}\right]\rho^2(R)$$

and so on, for the subsequent terms of higher multiplicity.

Therefore, $f(\theta,\bar{R})$ is given by the following expression

$$f(\theta,\bar{R}) = \frac{f^{(1)}(\theta,\bar{R}) + f^{(2)}(\theta,\bar{R})}{1 - \left(\frac{2\gamma^2}{1+\gamma^2}\right)^2 f_{31}f_{32}\rho^2(R)} \qquad (4.78)$$

Substitution of (4.78) in (4.73) shows that the T-series (4.20) has been reduced to a finite expression and, under this form, is a convergent series.

When $R \to 0$, $f(\theta,\bar{R})$ tends to a limit different from zero, a result which is physically acceptable. However, if assumption I is used instead of the approximation II, $f(\theta,\bar{R})$ vanishes for $R = 0$, because now $\underline{P}(R) = e^{ikR}/R$. Therefore, assumption I is a bad approximation in the region where the scatterers are close to each other (Schick, 1961).

The elastic scattering amplitude $f(\theta)$ will be now developed into a series of partial waves, i.e.,

$$f(\theta) = \frac{1}{2ik_a}\sum_{L=0}^{\infty}(2L+1)(\eta_L-1)P_L(\cos\theta) \qquad (4.79)$$

Choosing the Hulthèn function (3.25) as the ground state wave function of the deuteron, the exponentials in $f(\theta,\bar{R})$ are the only terms in $f(\theta)$ depending on the angular part of $\bar{R}$. Representing respectively by $\hat{k}_a$, $\hat{k}_b$ and $\hat{R}$ the angular spherical coordinates of $\bar{k}_a$, $\bar{k}_b$ and $\bar{R}$, the plane waves $e^{ik_a\frac{\bar{R}}{2}}$ and $e^{-ik_b\frac{\bar{R}}{2}}$ admit the following expansions (see (3.14)):

$$e^{i\bar{k}_a\cdot\bar{R}/2} = 4\pi \sum_{L=0}^{\infty} i^L j_L(k_a R/2) \sum_{m=-L}^{L} Y_L^{*m}(\hat{k}_a) Y_L^m(\hat{R})$$

and $(|\bar{k}_a| = |\bar{k}_b| = k_a)$

$$e^{-i\bar{k}_b\cdot\bar{R}/2} = 4\pi \sum_{L=0}^{\infty} (-i)^L j_L(k_a R/2) \sum_{m=-L}^{L} Y_L^m(\hat{k}_b) Y_L^{*m}(\hat{R}) .$$

Then, using the orthonormality relations for the spherical harmonics

$$\int Y_L^{*m}(\hat{R}) Y_{L'}^{m'}(\hat{R})\, d\hat{R} = \delta_{LL'}\cdot\delta_{mm'} \qquad (4.80)$$

one has

$$\int e^{i(\bar{k}_a-\bar{k}_b)\cdot\bar{R}/2} d\hat{R} = (4\pi)^2 \sum_{L=0}^{\infty} j_L^2(k_a \tfrac{R}{2}) \sum_{m=-L}^{L} Y_L^{*m}(\hat{k}_a) Y_L^{m}(\hat{k}_b) \qquad (4.81)$$

But, by the addition theorem for the spherical harmonics (Messiah, 1961, Appendix B), the sum over <u>m</u> in (4.81) is equal to

$$\frac{2L+1}{4\pi} P_L(\cos\theta) = \sum_{m=-L}^{L} Y_L^{*m}(\hat{k}_a) Y_L^m(\hat{k}_b) ; \qquad (4.82)$$

θ is the angle defined by the unit vectors $\hat{k}_a$ and $\hat{k}_b$, so that it is equal to the scattering angle. Therefore, (4.81) becomes

$$\int e^{i(\bar{k}_a-\bar{k}_b)\cdot\bar{R}/2} d\hat{R} = 4\pi \sum_{L=0}^{\infty} (2L+1) j_L^2(k_a R/2) P_L(\cos\theta) \qquad (4.83)$$

and the same development is obtained for $\int e^{-i(\bar{k}_a-\bar{k}_b)\cdot\bar{R}/2}\, d\hat{R}$ . .

Following the same steps it can be also proved that

$$\int e^{\pm i(k_a+\bar{k}_b)\cdot\bar{R}/2}\, d\hat{R} = 4\pi \sum_{L=0}^{\infty} (-1)^L (2L+1)\, j_L^2(k_a R/2)\, P_L(\cos\theta) \qquad (4.83')$$

These relations, together with (4.73) and (4.78), lead to a new development of f(θ) in terms of $P_L(\cos\theta)$. Comparing the coefficients of such terms in this series and in (4.79), one gets

$$\frac{\eta_L - 1}{2ik_a} = 4\pi \int_0^{\infty} F_L(R)\, j_L^2(k_a R/2)\, |\phi_0(R)|^2 R^2 dR \qquad (4.84)$$

where $\phi_0(R)$ is the Hulthèn function

$$\phi_0(R) = N \frac{e^{-\alpha R} - e^{-\beta R}}{R}, \quad N^2 = \frac{\alpha\beta(\alpha+\beta)}{2\pi(\beta-\alpha)^2} \qquad (3.25)$$

and

$$F_L(R) = \frac{f_{31} + f_{32} + (-1)^L \frac{4\gamma^2}{1+\gamma^2} \cdot f_{31} f_{32} P(R)}{1 - [2\gamma^2/(1+\gamma^2)]^2 f_{31} f_{32} P^2(R)} \qquad (4.85)$$

Calculation of integrals (4.84) needs f_{31} and f_{32} expressed in terms of the K^-N S-wave elastic scattering amplitudes

$$f^I = \frac{e^{2i\delta^I} - 1}{2ip}, \quad p^2 = \frac{1+\gamma^2}{2\gamma^2} k_a^2 \qquad (4.86)$$

(I (= 0, 1) is the label for the meson-nucleon isotopic-spin channels). Actually, the phase shifts δ^I are related to Ross-Humphrey's sets of scattering lengths A_I (see Table (II.1)) by the zero-effective range formula

$$p \cot \delta^I = 1 / A_I \tag{2.11}$$

Therefore, the elimination of δ^I between (2.11) and (4.86) gives

$$f^I = \frac{A_I}{1 - i p A_I} \tag{4.86'}$$

Now, f_{32} can be identified with f^1, because f_{32} is related with $K^- n$ interactions (see Fig. (IV.2)) which occur through the isotopic-spin channel $I = 1$. Thus,

$$f_{32} = \frac{A_1}{1 - i p A_1} \tag{4.87}$$

The relations between f_{31} and f^0 and f^1 is more involved, because in this case the $K^- p$ interactions take place through the channels $I = 0$ and $I = 1$ (Chapters I and II). It is found that

$$f_{31} = \frac{1}{2} \left(\frac{A_0}{1 - i p A_0} + \frac{A_1}{1 - i p A_1} \right) \tag{4.87'}$$

7. Convergence of the Development of f(θ) into Partial Waves

The analysis of this problem is linked with the behaviour of integrals (4.84) with increasing values of L.

Making the transformation $R = 2\rho/k_a$ in those integrals and considering that $\alpha \ll \beta$ (see (3.26) and (3.27)), it is clear that the leading term of $(\eta_L - 1)$ is proportional to

$$I_L(k_a) = \int_0^\infty F_L(2\rho/k_a)\, j_L^2(\rho)\, e^{-4\alpha\rho/k_a}\, d\rho \tag{4.88}$$

The next theorem is easily proved: "when $\rho^2 < L_1 + 1{\cdot}5$, the spherical Bessel functions $j_L(\rho)$ with $L \geq L_1 + 2$ are vanishingly small for all values of ρ satisfying that inequality".

In fact, adopting the normalisation $C_{L,0} = \frac{1}{(2L+1)!!}$ and using the differential equation (C.16) defining the $j_L(\rho)$ (see Appendix C), a straightforward calculation gives the following development

$$j_L(\rho) = \rho^L \sum_{n=0}^{\infty} C_{L,n}\, \rho^{2n}, \quad C_{L,n+1} = \frac{-C_{L,n}}{2[2(L+n)+1](n+1)} \qquad (4.89)$$

Cutting this series after the third term one has, for $L = L_1$ and $\rho^2 = L + 1{\cdot}5$,

$$j_{L_1}(\sqrt{L_1 + 1.5}) = \frac{(\sqrt{L_1+1.5})^{L_1}}{(2L_1+1)!!} \cdot \left[1 - \frac{1}{4} + \frac{L_1+1.5}{8(2L_1+5)} - \dots\right]$$

where

$$\frac{1}{4} - \frac{L_1+1.5}{8(2L_1+5)} > \frac{1}{4} - \frac{1}{16} \simeq 0.19$$

Therefore, the representation of $j_L(\rho)$, for $L \geq L_1$, by the first term in the development (4.89) in the interval $0 \leq \rho \leq \sqrt{L_1 + 1{\cdot}5}$ originates an error never exceeding $\sim$ 19%.

Now, from the recurrence relation

$$j_{L-1}(\rho) + j_{L+1}(\rho) = \frac{2L+1}{\rho}\, j_L(\rho), \quad L > 0$$

one has, putting $L = L_1 + 1$ and making the previous approximation for $j_{L_1}(\rho)$ and $j_{L_1+1}(\rho)$ in the interval $0 \leq \rho \leq \sqrt{L_1 + 1{\cdot}5}$,

$$j_{L+1}(\rho) \simeq 0$$

and the proof of the theorem is completed.

Noting that, in $I_L(k_a)$, the exponential becomes vanishingly small when $\rho > \frac{k_a}{4\alpha}$, then, by the precedent theorem, if $\sqrt{L_1 + 1\cdot5} > \frac{k_a}{4\alpha}$, the $I_L(k_a)$ for $L \geq L_1 + 2$ are practically equal to zero and the partial wave series for $f(\theta)$ converges.

The region of interest in this work for the K^--Lab.-momenta, p_{Lab}, is the one that lays below 300 MeV/c (Chapter I). The K^--momentum in the K^-d centre-of-mass system, $\hbar k_a$, is related to p_{Lab} by the equation (see (A.7))

$$\hbar k_a = \frac{2m_N}{(2m_N + m_K)} p_{Lab}$$

or

$$k_{a(\text{fermi}^{-1})} = 0\cdot4012 \times 10^{-2} \cdot p_{Lab(\text{Mev/c})} \qquad (4.90)$$

Therefore to the extreme value p_{Lab} = 300 MeV/c corresponds the wave number $k_a = 1\cdot2$ fermi^{-1}, so that the inequality ($\alpha = 0\cdot2315$ fermi^{-1} by (3.26))

$$\sqrt{L_1 + 1\cdot5} \geq \frac{k_a}{4\alpha} \approx 1\cdot2$$

holds for $L_1 = 1$. So, according to the theorem stated above,

$$\eta_L - 1 \approx 0$$

for $L \geq 3$.

Incidentally, the previous discussion shows the importance of the behaviour of the $F_L(R)$ for small values of R ($0 \leq R \leq 2$) in the calculation of integrals (4.84). This is another argument in favour of assumption II rather than assumption I of §5.

8. Validity of the Impulse Approximation

The square modulus of the coefficient η_L, defined in the expression for the scattering amplitude (4.79), measures the intensity of the outgoing spherical L-wave, because, in the same expression, the intensity of the incoming L-wave is taken equal to unity. Therefore, $1 - |\eta_L|^2$ is the intensity lost by the scattered L-wave in all incoherent processes - absorption (hyperon production) inelastic and charge-exchange scattering. The physical meaning of the η_L's outlined above imposes the mathematical conditions

$$1 - |\eta_L|^2 \geqslant 0 \tag{4.91}$$

for any L.

In the following pages, it will be shown that, at very low energies, the η_L's calculated by Impulse Approximation do not satisfy the inequalities (4.91) and this method is no longer valid. The reason for this to happen is the failure of conditions for which the two complementary assumptions 1) and 2) (see §2 of this chapter) of the Impulse Approximation are acceptable. Actually, if the kaon-wave length is large when compared with the mean separation of the two nucleons, one cannot expect just a slight distortion, by one constituent of the deuteron, of the kaon-wave falling upon the other.

For closer examination of the limits of validity of Impulse Approximation, consider kaons moving with wave numbers k_a equal to or lesser than 0·4 fermi^{-1} (the K^--Lab.-momentum corresponding to this value is, by (4.90) p_{Lab} = 100 MeV/c). The main contribution for the integrals (4.84) giving the η_L's, arises in the interval $0 \leqslant R \leqslant 2$ (see §7). In this region and for those values of k_a, $P(R) \simeq ik_a$, so that it is possible to bring the $F_L(R)$ out of the integral sign and, instead of (4.84), one has

$$\frac{\eta_L - 1}{2 i k_a} = \gamma\, F_L(0)\, 4\pi \int_0^\infty j_L^2(k_a R|2)\, |\phi_o(R)|^2 R^2 dR \qquad (4.92)$$

The Mean value theorem of the Integral Calculus (Courant, 1948, Volume I, p.127) provides now a method for the definition of the mean value $\bar{R}_L$ of the separation of the two nucleons in the deuteron for a K^-d L-state. Actually one has

$$4\pi \int_0^\infty j_L^2(k_a R|2)\, |\phi_o(R)|^2 R^2 dR = j_L^2(k_a \bar{R}_L|2) \qquad (4.93)$$

because

$$4\pi \int_0^\infty |\phi_o(R)|^2 R^2 dR = 1$$

Introducing (4.93) in (4.92), the $(\eta_L - 1)$ become

$$\frac{\eta_L - 1}{2 i k_a} = \gamma\, F_L(0)\, j_L^2(k_a \bar{R}_L|2)\ ; \qquad (4.94)$$

thus, if the η_L's satisfy conditions (4.91), one gets

$$i\left[F_L(0) - F_L^*(0)\right] - 2\gamma\, |F_L(0)|^2\, j_L^2(k_a \bar{R}_L|2)$$

or, putting $F_L(0) = a_L(k_a) + i b_L(k_a)$,

$$j_L^2(k_a \bar{R}_L|2)\, b_L^2(k_a) - \frac{b_L(k_a)}{\gamma k_a} + j_L^2(k_a \bar{R}_L|2)\, a_L^2(k_a) \leq 0 \qquad (4.95)$$

The parabola

$$y = j_L^2(k_a \bar{R}_L|2)\, x^2 - \frac{x}{\gamma k_a} + j_L^2(k_a \bar{R}_L|2)\, a_L^2(k_a) \qquad (4.96)$$

has, for $X = b_L(k_a)$, the same value as the left-hand side of inequality (4.95), which can be written

$$y \leq 0 \qquad (4.95')$$

This condition means that the curve (4.96) cuts the X-axis in two points given by the "real roots" of equation $y = 0$:

$$X^{(\pm)} = \frac{(\gamma k_a)^{-1} \pm \sqrt{(\gamma k_a)^{-2} - 4 a_L^2(k_a) j_L^4(k_a \bar{R}_L|2)}}{2 j_L^2(k_a \bar{R}_L|2)} \qquad (4.97)$$

Therefore, on one hand, one has

$$(\gamma k_a)^{-1} > 2 |a_L(k_a)| j_L^2(k_a \bar{R}_L|2) \qquad (4.98)$$

On the other, a negative y means that X remains between the two roots of $y = 0$. Supposing k_a small, $X^{(-)}$ is very close to zero and $X^{(+)} \doteq [\gamma k_a j_L^2(k_a \bar{R}_L|2)]^{-1}$. Then, from $y \leq 0$, a second condition is obtained

$$(\gamma k_a)^{-1} > b_L(k_a) j_L^2(k_a \bar{R}_L|2) \qquad (4.99)$$

For a S wave, $j_0^2(k_a \bar{R}_L|2) \simeq 4(k_a \bar{R}_0)^{-2}$, so that (4.98) and (4.99) become respectively

$$k_a > \frac{8\gamma |a_0(k_a)|}{\bar{R}_0^2} \qquad (4.98')$$

and

$$k_a > \frac{4\gamma b_0(k_a)}{\bar{R}_0^2} \qquad (4.99')$$

In (4.99') it is used $b_L(k_a)$ rather than $|b_L(k_a)|$ because this coefficient belongs to the imaginary part of the scattering amplitude and so, it is an essentially positive quantity in so far it is related with the total cross-section.

The inequalities (4.98') and (4.99') show clearly that the wave number k_a cannot be less than a certain limit (the greater of the two numbers $8\gamma|a_o(k_a)|/\bar{R}_o^2$ and $4\gamma\, b_o(k_a)/\bar{R}_o^2$), if condition (4.91) for L = 0, the only one important at very low energies, should not fail. Such limit can be very low, when $\bar{R}_o$ is large. In such case, the validity of the Impulse Approximation should be discussed in terms of its fundamental hypothesis expressed by (4.21), rather than in tetms of assumptions 1) and 2).

This argument, however, does not apply to the deuteron. Actually, as it will be proved subsequently, the first $\bar{R}_L$ at very low energies, though they tend to increase with L, are independent of k_a and their common value is $\sim 3{\cdot}84$ fermi.

The analysis of the integrals $I_L(k_a)$, defined in (4.88) and the study of the convergence of the series (4.89) show that, for $k_a \lessapprox 0{\cdot}4$ fermi^{-1}, and L > 0, the approximation

$$j_L(k_a R/2) \simeq \frac{(k_a R)^L}{2^L(2L+1)!!}\ , \quad (k_a R/2)^2 < L+1.5 \qquad (4.100)$$

for the spherical Bessel functions is a good one to be used in the calculation of the $\bar{R}_L$, defined in (4.93); one has, therefore,

$$\bar{R}_L^{2L} = 4\pi\int_0^\infty R^{2L}\,(e^{-\alpha R} - e^{-\beta R})^2\, dR$$

or, considering that $\int_0^\infty R^n e^{-aR} dR = n!|a^{n+1}$;

$$\bar{R}_L^{2L} = \frac{2\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^2} \cdot \left[\left(\frac{1}{2\alpha}\right)^{2L+1} + \left(\frac{1}{2\beta}\right)^{2L+1} - 2\left(\frac{1}{\alpha+\beta}\right)^{2L+1}\right] \cdot (2L)!$$

But, $\beta = 7\alpha$; thus

$$\bar{R}_L = \frac{1}{2\alpha} \sqrt[2L]{\frac{\beta(\alpha+\beta)}{(\beta-\alpha)^2}(2L)!}$$

$$= 2.2 \sqrt{1.58\,(2L)!} \text{ fermi} \tag{4.101}$$

Obviously, the expressions (4.101) are only valid for values of k_a such that the inequalities (4.100) with $R = \bar{R}_L$ are satisfied:

$$(k_a \bar{R}_L | 2)^2 < L + 1.5 \tag{4.100'}$$

Now, using Stirling's formula for (2L)! when L is big, it can be proved that (Courant, 1958, Volume I, p.391)

$$\lim_{L\to\infty} \frac{\sqrt[2L]{1.58(2L)!}}{2L} = \frac{1}{e}$$

where <u>e</u> is the base of the natural logarithms. Therefore, the condition (4.101') becomes with increasing L:

$$L < \left(\frac{2\alpha e}{k_a}\right)^2 \simeq \left(\frac{1.25}{k_a}\right)^2$$

For $k_a = 0{\cdot}4$ fermi^{-1}, (4.101) is a good approximation for all values of $L > 0$ less than 9.

For $L = 0$ and $k_a \lesssim 0{\cdot}4$ fermi, the approximation

$$j_0(k_a R/2) \simeq 1 - \frac{(k_a R)^2}{4!}$$

of the spherical Bessel function of zero order is a convenient one. Therefore,

$$\bar{R}_0 \simeq \bar{R}_1 = 3.84 \text{ fermi}.$$

This value is very close to the mean value of the deuteron's radius ($\sim 3{\cdot}2$ fermi). Such result proves that the previous discussion is well-founded from the point of view of the physics of the deuteron.

It is also worth while to be noted the increasing of $\bar{R}_L$ with L. This means that for greater values of L than $L = 0$, the Impulse Approximation improves and should lead to a correct result. This is to be expected,because it is a well-established fact that the same thing happens with the Born Approximation.

To complete the present discussion, just one remark more: it is the violation of inequalities (4.91) that explains why Day, Snow and Sucher (1959) found, in their work on K^-d scattering reactions, σ total < σ elastic (see, in the next chapter, how these cross-sections are expressed in terms of the η_L). However, in this case, it is not the failure of assumptions 1) and 2) the reason for such violation, but the use of a pure Impulse Approximation, which, in the notation used in this chapter, amounts to writing $F_L(R) = f_{31} + f_{32}$. When, in the same paper, those authors introduce double and multiple scattering corrections they found the correct result σ total > σ elastic.

Finally, the condition (4.21) for the validity of the fundamental assumption of the Impulse Approximation can be written when k_a is expressed in fermi^{-1}

$$\frac{2\cdot 5 \times 10^{-2}}{k_a} \ll 1 \qquad (4.21')$$

If $k_a = 0\cdot 4$ fermi^{-1}, the left-hand side of (4.21') gives $\sim 0\cdot 06$ which is a reasonable result. But the condition (4.91) for L = 0,

$$1 - |\eta_0|^2 \geqslant 0,$$

starts failing for values of k_a lower than $0\cdot 4$ fermi^{-1}. Therefore, the region of investigation for K^-d scattering in this work will be the K^--Lab.-momentum interval 100 MeV/c $\leqslant p_{Lab} \leqslant$ 300 MeV/c. Evidently, it is not possible to go over the upper limit because the $\bar{K}N$ scattering for such momenta can no longer be translated in terms of Ross-Humphrey's sets of scattering lengths (chapter I).

Table IV shows the variation of the η_L with the k_a and L for Ross-Humphrey's solutions I and II. Note that everywhere the η_L's are such that

$$|\eta_L| \leqslant 1$$

Table IV

p_{Lab} MeV/c	η_0 R.Part	η_0 I.Part	η_1 R.Part	η_1 I.Part	η_2 R.Part	η_2 I.Part	η_3 R.Part	η_3 I.Part
I 100	0·0656	-0·0211	0·9252	-0·0023	0·9954	-0·0001	0·9996	-0·0000
I 200	0·0416	-0·0126	0·8042	-0·0033	0·9674	-0·0005	0·9937	-0·0001
I 300	0·1541	-0·0076	0·7350	-0·0029	0·9317	-0·0007	0·9808	-0·0002
II 100	-0·0962	0·5209	0·9212	0·0786	0·9959	0·0039	0·9996	0·0003
II 200	-0·2396	0·4568	0·7983	0·1462	0·9746	0·0149	0·9961	0·0020
II 300	-0·1446	0·2937	0·5969	0·1811	0·8966	0·0381	0·9712	0·0095

The η_i's in this table have been corrected in such a way that the Coulomb interaction is taken into account (see next chapter). They were obtained by numerical calculation carried on the Mercury-Ferranti Computor belonging to the University of London.

CHAPTER V

K^-d Elastic, Total and Absorption Cross-Sections

1. The Resonant Group Structure Method

The decomposition of the K^-d elastic scattering amplitude, $f(\theta)$, into partial waves as well as the introduction of the η_L-coefficients (Chapter IV) lead to a straightforward calculation of the K^-d elastic and total cross-sections. But the determination of the absorption cross-sections, including the whole of the production of hyperons in K^-d collisions, requires a different approach to the K^-d problem. However, such an objective can be achieved by means of the Resonant Group Structure Method (Wheeler, 1937) combined with an appropriate description of the K^-N and np nuclear forces. As will be seen in the following paragraphs, the inclusion of absorption in K^-d scattering processes is provided by the imaginary parts of the K^-N complex potentials defined in Chapter II.

If 1, 2 and 3 represent respectively the particles p, n and K^- (Fig (IV.2)), the two body nuclear interactions are conveniently described by central potentials $V_{ij}(r_{ij})$ having the following form:

$$V_{ij}(r_{ij}) = V_{ij}^{(1)}(r_{ij}) + V_{ij}^{(2)}(r_{ij})P_{ij}, \quad i \neq j \tag{5.1}$$

$$r_{ij} = r_{ji}$$

Here r_{ij} is the distance between particles i and j and P_{ij}, defined in (2.4) represents the charge-exchange operator for the same particles. Thus, P_{ij} operates only on the isotopic-spin part of the K^-d wave function.

If i (or j) = 3 (one of the particles is the kaon) the potentials $V_{ij}^{(1)}(r_{ij})$ and $V_{ij}^{(2)}(r_{ij})$ must be identified respectively with the potentials given by equations (2.9) and (2.10):

$$V_{ij}^{(1)}(r_{ij}) = -\frac{1}{2}\left[(u_1 + u_0) + i(v_1 + v_0)\right]\frac{e^{-br_{ij}}}{br_{ij}} \tag{2.9'}$$

$$V_{ij}^{(2)}(r_{ij}) = -\frac{1}{2}\left[(u_1 - u_0) + i(v_1 - v_0)\right]\frac{e^{-br_{ij}}}{br_{ij}} \tag{2.10'}$$

But, when the two interacting particles are the nucleons 1 and 2, the only nuclear force between them is, as it will be proved below

$$V_{12}(r_{12}) = V_{12}^{(1)}(r_{12}) - V_{12}^{(2)}(r_{12}) \tag{5.1'}$$

Therefore, (5.1') must be identified with the deuteron ground state potential.

In spite of the K^-n reactions through the isotopic-spin Channel I = 0 being forbidden (see Chapter II), the operator P_{ij} appears in all potentials (5.1). This is so because charge-exchange processes occur continuously between the two nucleons, leaving the isotopic-spin channels I = 0 and I = 1 open for particles 2 (n) and 3 (K^-).

Consider now the K^-d wave function.

If $Y_{I,I_z}(A)$ and $Y_{I',I'_z}(B)$ represent the isotopic-spin functions of two groups of particles A and B, the isotopic-spin functions of the whole system are (see (3.6)):

$$Y_{J,I,I'}^{J_z}(A,B) = \sum_{I_z,I'_z} \langle I,I',I_z,I'_z|J,J_z\rangle \, Y_{I,I_z}(A)\, Y_{I',I'_z}(B)$$

Putting $A \equiv (1,2)$ and $B \equiv (3)$, $Y_{I,I_Z}(1,2)$ and $Y_{I',I'_Z}(3)$ stand respectively for the np system and the kaon. By the principle of charge conservation, the only possible value for $\mathcal{J}_z$ is $-\frac{1}{2}$. To this eigenvalue correspond three eigenfunctions:

$$Y^{-\frac{1}{2}}_{\frac{3}{2},1,\frac{1}{2}}(1,2,3) = \sqrt{\frac{2}{3}}\, Y_{1,0}(1,2)\, Y_{\frac{1}{2},-\frac{1}{2}}(3) + \sqrt{\frac{1}{3}}\, Y_{1,-1}(1,2)\, Y_{\frac{1}{2},\frac{1}{2}}(3) \tag{5.2}$$

$$Y^{-\frac{1}{2}}_{\frac{1}{2},1,\frac{1}{2}}(1,2,3) = \sqrt{\frac{1}{3}}\, Y_{1,0}(1,2)\cdot Y_{\frac{1}{2},-\frac{1}{2}}(3) - \sqrt{\frac{2}{3}}\, Y_{1,-1}(1,2)\, Y_{\frac{1}{2},\frac{1}{2}}(3) \tag{5.3}$$

$$Y^{-\frac{1}{2}}_{\frac{1}{2},0,\frac{1}{2}}(1,2,3) = Y_{0,0}(1,2)\, Y_{\frac{1}{2},-\frac{1}{2}}(3) \tag{5.4}$$

But conservation of the total isotopic-spin quantum number in strong interaction processes rules out the state (5.2) because the isotopic-spin part of the deuteron wave function has $I = 0$ and, with this value for I, $\mathcal{J}$ can never be equal to $3/2$.

The resonant group structure method requires that the remaining eigenfunctions (5.3) and (5.4) be anti-symmetrized with respect to particles 1 and 2 (the two nucleons) because the deuteron's total isotopic-spin is $I = 0$. However, as (5.3) is symmetric in 1 and 2, the K^-d wave function, written in the C.M. system of the three particles reduces to

$$\rho(1,2,3)\,\psi(\bar{r},\bar{R}) \quad , \quad \psi(\bar{r},\bar{R}) = \psi(\bar{r},-\bar{R}) \tag{5.5}$$

where

$$\rho(1,2,3) = Y^{-\frac{1}{2}}_{\frac{1}{2},0,\frac{1}{2}}(1,2,3) \qquad (5.4')$$

and the coordinates $\bar{r}$ and $\bar{R}$ are linked with the r_{ij} by the following equations (see Fig. (IV.a))

$$r_{13} = \left|\bar{r} + \frac{\bar{R}}{2}\right|, \qquad r_{23} = \left|\bar{r} - \frac{\bar{R}}{2}\right|, \qquad r_{12} = |\bar{R}| \qquad (5.6)$$

Obviously (5.5) must satisfy the Schrodinger equation

$$\left[K - \frac{e^2}{r_{13}} + V_{12}(r_{12}) + V_{13}(r_{13}) + V_{23}(r_{23})\right]\rho\psi =$$

$$= (E + E_d)\rho\psi \qquad (5.7)$$

where K is the kinetic energy operator defined in chapter IV,

$$K = -\frac{\hbar^2}{2\mu}\nabla^2_{\bar{r}} - \frac{\hbar^2}{m_K}\nabla^2_{\bar{R}}\ , \qquad (4.8)$$

<u>e</u>, the electron charge, E_d, the deuteron's binding energy and E, the kaon energy in the c.m. of the K^-d system.

The following relations are easily obtained from the definition of $\rho(1,2,3)$:

$$\rho^{\times}(1,2,3)P_{12}\rho(1,2,3) = -1$$

$$\rho^{\times}(1,2,3)P_{13}\rho(1,2,3) = \tfrac{1}{2}$$

$$\rho^{\times}(1,2,3)P_{23}\rho(1,2,3) = \tfrac{1}{2}\ ,$$

so that, if both sides of (5.7) are multiplied by $\rho^{\times}(1,2,3)$ and the potentials (2.9') and (2.10') are introduced, the K^-d wave equation becomes

$$\Big\{ K - \frac{e^2}{r_{13}} + \left[V_{12}^{(1)}(r_{12}) - V_{12}^{(2)}(r_{12}) \right] -$$

$$- \frac{1}{4}\left[(u_0 + 3u_1) + i(v_0 + 3v_1) \right] \cdot \left(\frac{e^{-br_{13}}}{br_{13}} + \frac{e^{-br_{23}}}{br_{23}} \right) \Big\} \psi = (E + E_d)\psi \tag{5.8}$$

No charge-exchange operator is needed for the Coulomb potential, because the nuclear potential in (5.8) is symmetric with respect to coordinates r_{13} and r_{23}, and, therefore, it is immaterial to write either $-e^2/r_{13}$ or $-e^2/r_{23}$ in this equation.

Consider now the wave functions $\phi_\alpha(\bar{R})$ and energies W_α of the np system, defined in Chapter IV. They satisfy the equations

$$\left[-\frac{\hbar^2}{m_N} \nabla_R^2 + V_0(\bar{R}) \right] \phi_\alpha(\bar{R}) = W_\alpha \phi_\alpha(\bar{R}) \tag{4.4}$$

and the normalisation conditions (4.5). As the $\phi_\alpha(\bar{R})$ form a complete orthonormal set of functions, $\psi(\bar{r},R)$ can be developed in the following way

$$\psi(\bar{r},\bar{R}) = \sum_\alpha f_\alpha(\bar{r})\, \phi_\alpha(\bar{R})$$

Making now the approximation of supposing small the polarisation of the deuteron due to the presence of the kaon (Buckingham and Massey, 1941) this series reduces to its first term

$$\psi(\bar{r},\bar{R}) = f_0(\bar{r})\, \phi_0(\bar{R}) \tag{5.9}$$

where $\phi_0(\bar{R})$ represents the ground state of the np system (deuteron) and $W_0 = E_d$. In the approximation (5.9) the K^-d wave function is already symmetrised with respect $\bar{R}$, according to the condition (5.5), because the deuteron is always in a S + D state. Therefore, considering that $r_{12} = |\bar{R}|$ (see (5.6)) and identifying the potential $V_0(\bar{R})$ with (r.1'), the wave equation (5.8) becomes, by (4.4), (4.8) and (5.9),

$$\left\{ -\frac{\hbar^2}{2\mu}\nabla_{\bar{r}}^2 - \frac{e^2}{r_{13}} - \frac{1}{4}\left[(u_0+3u_1)+i(v_0+3v_1)\right]\left(\frac{e^{-br_{13}}}{br_{13}}+\frac{e^{-br_{23}}}{br_{23}}\right)\right\} f_0(\bar{r})\phi_0(\bar{R}) =$$
$$= E f_0(\bar{r})\,\phi_0(\bar{R})$$

Multiplying both sides of this equation by $\phi_0(\bar{R})$, integrating over $\bar{R}$ and putting

$$F(r) = \int \left(\frac{e^{-br_{13}}}{br_{13}} + \frac{e^{-br_{23}}}{br_{23}}\right) |\phi_0(\bar{R})|^2 \, d\bar{R} \qquad (5.10)$$

$$F_0(r) = \int \frac{|\phi_0(\bar{R})|^2}{r_{13}} \, d\bar{R} \qquad (5.11)$$

$$k_a^2 = \frac{2\mu E}{\hbar^2} \qquad (5.12)$$

one gets, finally the resonant group structure approximation of the K^-d wave equation

$$(\nabla^2 + k_a^2) f(\bar{r}) = -\frac{2\mu}{\hbar^2}\left\{ e^2 F_0(r) + \frac{1}{4}\left[(u_0+3u_1)+i(v_0+3v_1)\right] F(r)\right\} f(\bar{r}) \qquad (5.13)$$

The index o has been suppressed in $f(\bar{r})$.

2. The K^-d potentials $\phi(r)$ and $\Phi_0(r)$

Consider the second term belonging to F(r) in (5.10):

$$I(r) = \int \frac{e^{-br_{23}}}{br_{23}} \, |\phi_0(\bar{R})|^2 d\bar{R} \qquad (5.10')$$

Making the approximation of representing $\phi_0(\bar{R})$ by the Hulthèn function (3.25) and putting

$$\bar{\rho} = \bar{R}|2 \qquad (5.14)$$

and

$$\chi(\rho) = \frac{(e^{-2\chi\rho} - e^{-2\beta\rho})^2}{\rho} , \qquad (5.15)$$

I(r) becomes

$$I(r) = \frac{2\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^2 b} \int_0^\infty \chi(\rho)\, d\rho \int_0^\pi \frac{e^{-br_{23}}}{r_{23}} \sin\theta \, d\theta \qquad (5.16)$$

where θ is the polar angle of the spherical coordinates related to $\bar{\rho}$. But, according to the definition (5.6) of r_{23} one has

$$r_{23}^2 = r^2 + \rho^2 - 2r\rho \cos\theta,$$

so that I(r) can be written as

$$I(r) = \frac{2\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^2 b} \cdot \frac{1}{r} \int_0^\infty \chi(\rho)\, d\rho \int e^{-br_{23}} dr_{23} \qquad (5.17)$$

The limits of the second integral in (5.17) for $\theta = 0$ and $\theta = \pi$ are respectively a) $r_{23} = r - \rho$ and $r_{23} = r + \rho$ if $\rho < r$ and b) $r_{23} = \rho - r$ and $r_{23} = \rho + r$ if $\rho > r$. Therefore, introducing the hyperbolic sinus function, I(r) gives

$$I(r) = \frac{2\alpha\beta(\alpha+\beta)}{b^2(\beta-\alpha)^2} \, \Phi(r) . \qquad (5.18)$$

where

$$\phi(r) = \frac{e^{-br}}{r} \int_0^r \chi(\rho) \sinh b\rho \, d\rho + \frac{\sinh br}{r} \int_r^\infty \chi(\rho) e^{-b\rho} d\rho \tag{5.19}$$

Since the term in r_{13} of $F(r)$ is equal to $I(r)$ one has

$$F(r) = \frac{8\alpha\beta(\alpha + \beta)}{\beta^2(\beta - \alpha)^2} \phi(r) \tag{5.20}$$

Putting now

$$\psi(r) = r\phi(r) \tag{5.21}$$

and differentiating $\psi(r)$ twice with respect to $\underline{r}$, one gets from (5.19)

$$\psi''(r) - b^2\psi(r) = -b\chi(r) \tag{5.22}$$

But the Laplacian of $\phi(r)$ reduces to

$$\Delta\phi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial\phi}{\partial r}\right)$$

because $\phi(r)$ is a spherically summetric function. Thus, by (5.21) and (5.22),

$$\Delta\phi = \frac{\psi''}{r} = b^2\phi - b\,\frac{\chi(r)}{r} \tag{5.23}$$

Now, the Schrodinger equation (5.13) shows that the K^-d effective nuclear potential is given by

$$V_{eff} = \frac{b}{8} M\, F(r) \tag{5.24}$$

where

$$M = -\frac{4\left[(u_0 + 3u_1) + i(v_0 + 3v_1)\right]}{b} \tag{5.25}$$

Therefore, comparing the modulus of the Hulthèn function $|\phi_0(2r)|^2$ with $\chi(r)$ (see (3.25) and (5.15)) and considering (5.20, (5.23) and (5.24), one has that Veff(r) is a solution of

$$\Delta V_{eff} - b^2 V_{eff} = -4\pi |\phi_o(2r)|^2 \qquad (5.26)$$

This well-known partial differential equation represents a static and spinless meson field. Such result shows that the resonant group structure approximation is equivalent to the problem of obtaining the scattering of the K^--meson by a cloud of "nuclear charge". The product of the"kaon-charge" by the density of the "cloud's charge" is given by $M|\phi_0(2r)|^2$ and the range of the nuclear forces arising in this field is equal to 1/b, i.e., exactly the same as the one that was found in the two-body interactions K^-p and K^-n.

Multiplying the term in r_{13} of F(r) (see 5.10) by b and taking the limit of the result when $b \to 0$ one gets the potential $F_o(r)$ defined in (5.11). Thus, by (5.18) and (5.19),

$$F_0(r) = \frac{4\alpha\beta(\alpha + \beta)}{(\beta - \alpha)^2} \Phi_0(r) \qquad (5.18')$$

where

$$\Phi_o(r) = \lim_{b \to 0} \frac{\Phi(r)}{b} = \frac{1}{r}\int_0^r \rho\, \chi(\rho)\, d\rho + \int_r^\infty \chi(\rho)\, d\rho \qquad (5.19')$$

Putting

$$\psi_0(r) = r\Phi_0(r), \qquad (5.21')$$

the differential equation for $\psi_o(r)$ is obtained from the corresponding equation (5.22) for $\psi(r)$, using again the previous method: dividing both sides of (5.22) by <u>b</u> and taking the limit of the result when $b \to 0$, one gets

$$\psi_0''(r) = -\chi(r) \tag{5.22'}$$

Therefore, by (5.18'), (5.21') and (5.22'), the Coulomb potential

$$V_C(r) = e\, F_0(r) \tag{5.27}$$

due to the proton is a solution of the Poisson equation

$$\Delta V_C(r) = -4\pi\, \eta(r) \tag{5.26'}$$

where

$$\eta(r) = e \cdot \frac{\alpha\beta(\alpha+\beta)}{\pi(\beta-\alpha)^2} \cdot \frac{\left(e^{-2\alpha r} - e^{-2\beta r}\right)^2}{r^2} \tag{5.28}$$

is the electric charge density. Obviously one has

$$\int \eta(r)\, d\vec{r} = e \tag{5.29}$$

So, similarly to the interpretation of the K^-d nuclear potential, the Coulomb potential V_C (r), as far as it is a solution of the Poisson equation (5.26') represents a spherically shaped electric cloud, around the deuteron C.M., generated by the motion of the proton.

3. Connections between the Resonant Group Structure and the Impulse Approximation Methods

At this stage of the present work it is easy to prove that the effective nuclear potential $V_{eff}(r)$, defined in (5.24), and the Impulse Approximation without multiple scattering corrections, applied to the K^-d problem lead to the same Born approximation.

Representing by $[f]_{\bar{q}}$ the Fourier transform of the function $f(\bar{r})$, i.e.,

$$[f]_q = \int e^{-i\bar{q}.\bar{r}} f(\bar{r}) d\bar{r} \quad . \tag{5.30}$$

one can write

$$V_{eff}(r) \sim S_{\bar{q}} [V_{eff}]_{\bar{q}} e^{i\bar{q}.\bar{r}} \tag{5.31}$$

and

$$\left|\phi_0(2r)\right|^2 \sim S_{\bar{q}} \left[|\phi_0(2r)|^2\right]_{\bar{q}} e^{i\bar{q}.\bar{r}} \tag{5.32}$$

The constants of proportionality in both developments are equal. But $V_{eff}(r)$ and $|\phi_0(2r)|^2$ are linked by the partial differential equation (5.26). Therefore, one has

$$(q^2 + b^2)\left[V_{eff}\right]_{\bar{q}} = 4\pi M \left[|\phi_0(2r)|^2\right]_{\bar{q}}$$

or, putting $\bar{R} = 2\bar{r}$,

$$\left[V_{eff} \right]_{\bar{q}} = \frac{\pi M}{2(q^2 + b^2)} \int e^{-i\bar{q}.\bar{R}/2} |\phi_0(R)|^2 d\bar{R}, \tag{5.33}$$

according to (5.30)

Now, if $\bar{q}$ is chosen equal to the K^--momentum transfer in a K^-d elastic scattering process, i.e.,

$$q = 2k_a \sin\frac{\theta}{2} \tag{5.34}$$

where k_a and θ are respectively the kaon wave number and the scattering angle in the C.M. of the three-body system, the Born approximation for the K^-d elastic scattering amplitude, $f^{(B)}(\theta)$, is given by, μ being the K^-d reduced mass (see (4.3))

$$f^{(B)}(\theta) = -\frac{2\mu}{4\pi\hbar^2}\left[V_{eff}\right]_{\bar{q}}$$

or, introducing in (5.33) the definition (5.25) for M,

$$f^{(B)}(\theta) = \frac{\mu}{\hbar^2}\cdot\frac{(u_0+3u_1)+i(v_0+3v_1)}{b(q^2+b^2)}\cdot\int e^{-i\frac{\bar{q}}{2}\cdot\bar{R}}\left|\phi_0(R)\right|^2 d\bar{R} \tag{5.35}$$

The integral in (5.35) is not altered if $-\bar{q}$ is changed into $+\bar{q}$. Therefore, the Fourier transform of $|\phi_0(R)|^2$ is equal to the form factor $S^{\frac{1}{2}}(\theta)$ defined in (4.52) for $\alpha = 0$. Actually, according to (4.50) and (4.50'), the elastic scattering amplitude $f^{(I)}(\theta)$, obtained by Impulse Approximation without taking into account the multiple scattering terms, is given by

$$f^{(I)}(\theta) = -\frac{2\mu}{4\pi\hbar^2}\left(<\bar{p}_b|t_{1,0}|\bar{p}_a> + <\bar{q}_b|t_{2,0}|\bar{q}_a>\right)S^{\frac{1}{2}}(\theta)$$

It will be seen now that the Born approximation for $f^{(I)}(\theta)$ leads to the result obtained in (5.35). To achieve this, it is necessary to relate the transition matrix elements appearing in $f^{(I)}(\theta)$ with the Fourier transforms of the complex Yukawa potentials (2.1), giving the K^-N interactions in the isotopic-spin channels I = 0,1, i.e.,

$$\left[-(u_I+iv_I)\frac{e^{br}}{br}\right]_{\bar{q}_I} = -\frac{4\pi(u_I+iv_I)}{b(q_I^2+b^2)} \tag{2.1'}$$

where the $\bar{q}_I$ are the K^--momentum transfers occurring in the C.M.'s of the K^-p or K^-n systems.

The matrix elements $\langle \bar{p}b|t_{1,0}|\bar{p}a\rangle$ and $\langle \bar{q}b|t_{2,0}|\bar{q}a\rangle$ are related with K^-p and K^-n collisions (see Chapter IV). The K^-n interactions exist only in the K^-n isotopic-spin channel $I = 1$, but the K^-p interactions can occur with equal probability in the isotopic-spin channels $I = 0$ and $I = 1$. Therefore

$$\langle \bar{p}b|t_{1,0}|\bar{p}a\rangle \simeq -\frac{1}{2}\cdot\frac{4\pi}{b}\left(\frac{u_0+iv_0}{q_0^2+b^2}+\frac{u_1+iv_1}{q_1^2+b^2}\right) \tag{5.36}$$

and

$$\langle \bar{q}b|t_{2,0}|\bar{q}a\rangle \simeq -\frac{4\pi}{b}\cdot\frac{u_1+iv_1}{q_1^2+b^2} \tag{5.37}$$

are the Born approximations for $\langle \bar{p}b|t_{1,0}|\bar{p}a\rangle$ and $\langle \bar{q}b|t_{2,0}|\bar{q}a\rangle$. (Compare (5.36) and (5.37) with (4.87') and (4.87) respectively).

Consider now the momentum transfer $\bar{q}_0$. It is given by

$$q_0 = 2p \sin\frac{\theta_0}{2} \tag{5.38}$$

where p is the modulus of the wave numbers appearing in $\langle \bar{p}b|t_{1,0}|\bar{p}a\rangle$ and θ_0 is the scattering angle in C.M. of the K^-p system. But, by (4.54),

$$\bar{p}b - \bar{p}a = \bar{k}b - \bar{k}a$$

or, squaring both sides of this equation and because the collision is elastic (so that $p = |\bar{p}a| = |pb|$, $k_a = |\bar{k}a| = |\bar{k}b|$),

$$p^2 \sin^2\frac{\theta_0}{2} = k_a^2 \sin^2\frac{\theta}{2}$$

Therefore, by (5.34) and (5.38),

$$q_0 = q$$

Similarly one also has $q_1 = q$. Substituting, then, q_0 and q_1 by q into (5.36) and (5.37) the Born approximation of $f^{(I)}(\theta)$ leads to the expression (5.33).

Thus, for high energies of the kaon, when the multiple scattering is unimportant and the Born approximation becomes valid, the two methods discussed in this paragraph must agree very closely. However, for low energies, the resonant group structure, in so far as it is formulated here, breaks down. This point will be made clearer when the nature of the phase shifts derived from equation (5.13) is discussed.

4. The plot of the K^-d potentials $\phi(r)$ and $\bar{\Phi}_0(r)$

The function $\phi(r)$ is a short range potential. This property can be proved either directly from the analytical expression (5.19) for $\phi(r)$, or from the condition that $\phi(r)$ is a solution of the partial differential equation (5.23). Actually, the density $r^{-1}\chi(r)$ vanishes exponentially (see (5.15)) when $r \to \infty$ and the term $b^2\phi(r)$ in (5.23) forbids the spread of the nuclear interaction outside a sphere centred at the deuteron's C.M. The leading term of $r^{-1}\chi(r)$ is $r^{-2}e^{-4\alpha r}$, where $4\alpha \sim 1$ fermi^{-1} (see (3.26)). Therefore, one has $\phi(r) < 10^{-5}$ for $r > 7{\cdot}5$ fermi, because $e^{-7{\cdot}5} \simeq 5.10^{-4}$ and a $(= 1/b)$ is equal to $0{\cdot}4$ fermi (see (2.2)).

To plot the function $\psi(r) = r\phi(r)$ it is convenient to express the hyperbolic function sin hbr in terms of the positive and negative exponentials. Doing this, $\psi(r)$ becomes, according to (5.19):

$$2\psi(r) = e^{-br}\int_0^r \chi(\rho)e^{b\rho}d\rho + e^{br}\int_r^\infty \chi(\rho)e^{-b\rho}d\rho - e^{-br}\int_0^\infty \chi(\rho)e^{-b\rho}d\rho \tag{5.39}$$

The last integral in (5.39) gives, using the formula (B.6) and the definition (5.15) for $\chi(\rho)$,

$$\int_0^\infty \chi(\rho)\, e^{-b\rho}\, d\rho = \log \frac{[2(\alpha+\beta)+b]^2}{(4\alpha+b)(4\beta+b)} \qquad (5.40)$$

The integral $\int_r^\infty \chi(\rho) e^{-b\rho} d\rho$ can be easily expressed in terms of the exponential integral function

$$-E_i(-r) = \int_r^\infty \frac{e^{-\rho}}{\rho}\, d\rho \qquad (5.41)$$

In fact, by (5.15) and (5.41), one gets

$$\int_r^\infty \chi(\rho)\, e^{-b\rho} d\rho = -E_i[-(4\alpha+b)r] - E_i[-(4\beta+b)r] +$$

$$+ 2E_i\{-[2(\alpha+\beta)+b]r\} \qquad (5.42)$$

For values of r belonging to the interval $1 \leq r \leq +\infty$, $-E_i(-r)$ is given exactly to at least seven figures by the Hastings' approximation:

$$-E_i(-r) \simeq \frac{e^{-r}}{r} \cdot R(r), \quad R(r) = \frac{a_0 + a_1 r + a_2 r^2 + a_3 r^3 + r^4}{b_0 + b_1 r + b_2 r^2 + a_3 r^3 + r^4} \qquad (5.43)$$

where the coefficients of the rational function R(r) are numerical constants conveniently tabulated (Hastings, 1957, page 190). But, when r lays in the inverval $0 \leq r \leq 1$, (5.43) is no longer valid, so that the development of $-E_i(-r)$ into a power series must be used in this region. Leaving this new problem aside for the moment, the calculation of the integral $\int_0^r \chi(\rho)e^{b\rho}d\rho$ of (5.39) is now envisaged.

Consider the small positive constant Δ. By (5.15), one has

$$\int_\Delta^r \chi(\rho)e^{b\rho}d\rho = \int_\Delta^r \frac{e^{-(4\alpha-b)\rho}}{\rho}d\rho + \int_\Delta^r \frac{e^{-(4\beta-b)}}{\rho}d\rho - $$
$$-2\int_\Delta^r \frac{e^{-[2(\alpha+\beta)-b]\rho}}{\rho}\cdot d\rho \qquad (5.44)$$

where, by (2.2), (3.26), and (3.27), the numerical values of the exponential constants are:

$$4\alpha - b = -1\cdot5740$$
$$4\beta - b = 3\cdot7596$$
$$2(\alpha+\beta) - b = 1\cdot0928$$

Therefore, the integral $\int_\Delta^r \frac{e^{-(4\alpha-b)\rho}}{\rho}d\rho$ of (5.44) has a positive exponential. But it can be transformed into an integral with a negative exponential if ρ goes into $-\rho$:

$$\int_\Delta^r \frac{e^{-(4\alpha-b)\rho}}{\rho}d\rho = \int_{-\Delta}^{-r} \frac{e^{(4\alpha-b)\rho}}{\rho}d\rho =$$
$$= \int_\Delta^\infty \frac{e^{(4\alpha-b)\rho}}{\rho}d\rho - \left[\int_\Delta^\infty \frac{e^{(4\alpha-b)\rho}}{\rho}d\rho + \int_{-r}^{-\Delta} \frac{e^{(4\alpha-b)\rho}}{\rho}d\rho\right] \qquad (5.45)$$

Putting (5.45 into (5.44), using the identity

$$\int_\Delta^r \frac{e^{-\lambda\rho}}{\rho}d\rho = \int_\Delta^\infty \frac{e^{-\lambda\rho}}{\rho}d\rho + Ei(-\lambda r) \quad , \lambda > 0$$

for the other two integrals belonging to (5.44), taking the limit of the obtained result when $\Delta \to 0$ and making use again of (B.6), one gets, finally,

$$\int_0^r \chi(\rho)\, e^{b\rho} d\rho = \log \frac{[2(\alpha+\beta)-b]^2}{(b-4\alpha)(4\beta-b)} -$$

$$- P\int_{-r}^{\infty} \frac{e^{-(b-4\alpha)\rho}}{\rho} d\rho + E_i[-(4\beta-b)r] - 2E_i\{-[2(\alpha+\beta)-b]r\} \quad (5.46)$$

where P means that only the principal part of the integral following it has been taken. Such integral can be evaluated by means of the equation

$$P\int_{-r}^{\infty} \frac{e^{-\lambda\rho}}{\rho} d\rho = -E_i(-\lambda r) + P\int_{-r}^{r} \frac{e^{-\lambda\rho}}{\rho} d\rho \quad (5.47)$$

where the second term on the right hand side is easily developed into a rapidly converging series:

$$P\int_{-r}^{r} \frac{e^{-\lambda\rho}}{\rho} d\rho = -2\sum_{L=0}^{\infty} \frac{(\lambda r)^{2L+1}}{(2L+1)(2L+1)!} \quad (5.47')$$

However, for large values of r, it is better to have (5.47) expanded into an asymptotic series:

$$P\int_{-r}^{\infty} \frac{e^{-\lambda\rho}}{\rho} d\rho = -e^{-\lambda\rho} \sum_{L=0}^{n} \frac{L!}{(\lambda r)^{L+1}}$$

The error committed in the evaluation of the integral by taking the n first terms in the series does not exceed the term of the order n+1.

When r belongs to the interval $0 \leq r \leq 1$, the function $\psi(r)$ is obtained by a Taylor's series expansion at the origin:

$$\psi(r) = \sum_{n=0}^{\infty} \frac{1}{n!} \psi^{(n)}(0) r^n \qquad (5.48)$$

From the analytical expression (5.39) for $\psi(r)$ and from (5.40), one has

$$\psi(0) = 0, \quad \psi'(0) = b \operatorname{Log} \frac{[2(\alpha+\beta)+b]^2}{(4\alpha+b)(4\beta+b)} \qquad (5.49)$$

Putting

$$\lambda_1 = 4\alpha, \qquad \lambda_2 = 4\beta, \qquad \lambda_3 = 2(\alpha+\beta)$$

so that

$$\lambda_1 + \lambda_2 - 2\lambda_\alpha = 0$$

and considering the exponential series, the $\chi(r)$-derivatives of different orders at the origin are given by the following expressions

$$\chi(0) = 0, \quad \chi^{(n)}(0) = (-1)^{n+1} \frac{\lambda_1^{n+1} + \lambda_2^{n+1} - 2\lambda_3^{n+1}}{n+1}, \; n \geq 1 \qquad (5.50)$$

Therefore, the $\psi(r)$ derivatives at the origin of any order higher than the first are given by the recurrence relation

$$\psi^{(n+2)}(0) = b^2 \psi^{(n)}(0) - b\chi^{(n)}(0) \qquad (5.51)$$

obtained from the differential equation (5.22) for $\psi(r)$. Obviously, by (5.49), (5.50) and (5.51), one has $\psi''(0) = 0$.

The plot of $\psi_0(r) = r\Phi_0(r)$ is achieved along the same lines. From (5.19') and using the Hastings' approximation for the exponential integral, it is clear that $\psi_0(r)$ admits the following development when r is in the interval $1 \leq r \leq +\infty$:

$$\psi_0(r) = \frac{(\beta-\alpha)^2}{4\alpha\beta(\alpha+\beta)} + \left[R(\lambda_1 r)-1\right]\frac{e^{-\lambda_1 r}}{\lambda_1} + \left[R(\lambda_2 r)-1\right]\frac{e^{-\lambda_2 r}}{\lambda_2} - 2\left[R(\lambda_3 r)-1\right]\frac{e^{-\lambda_3 r}}{\lambda_3} \qquad (5.52)$$

where, as before, λ_1, λ_2 and λ_3 are respectively the constants 4α, 4β and $2(\alpha+\beta)$.

If r belongs to the interval $0 \leq r \leq 1$, a Taylor's series expension, similar to the one obtained for $\psi(r)$ can be used to plot $\psi_0(r)$. The $\psi_0(r)$ derivatives at the origin in this case are

$$\psi_0(0) = \psi_0''(0) = 0 \;,\; \psi_0'(0) = \log\frac{\lambda_3^2}{\lambda_1\lambda_2} \;,\; \psi_0^{(n+2)}(0) = -\chi^{(n)}(0) \qquad (5.51')$$

Fig. (V.1) shows the plots of $\phi(r)$ and $\frac{2\mu}{\hbar^2}\Phi_0(r)$.

5. K^-d phase shifts

Putting

$$A + iB = \frac{2\mu}{\hbar^2}\cdot\frac{2\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^2 b^2}\left[(u_0+iv_0) + 3(u_1+iv_1)\right] \qquad (5.53)$$

$$n = -\frac{\mu e^2}{\hbar^2 k_a}\;,\quad \frac{\mu e^2}{\hbar^2} = 0.0145 \text{ fermi}^{-1} \qquad (5.54)$$

$$C = \frac{4\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^2}\;,\quad \frac{C\mu e^2}{\hbar^2} = 0.0214 \text{ fermi}^{-2} \qquad (5.55)$$

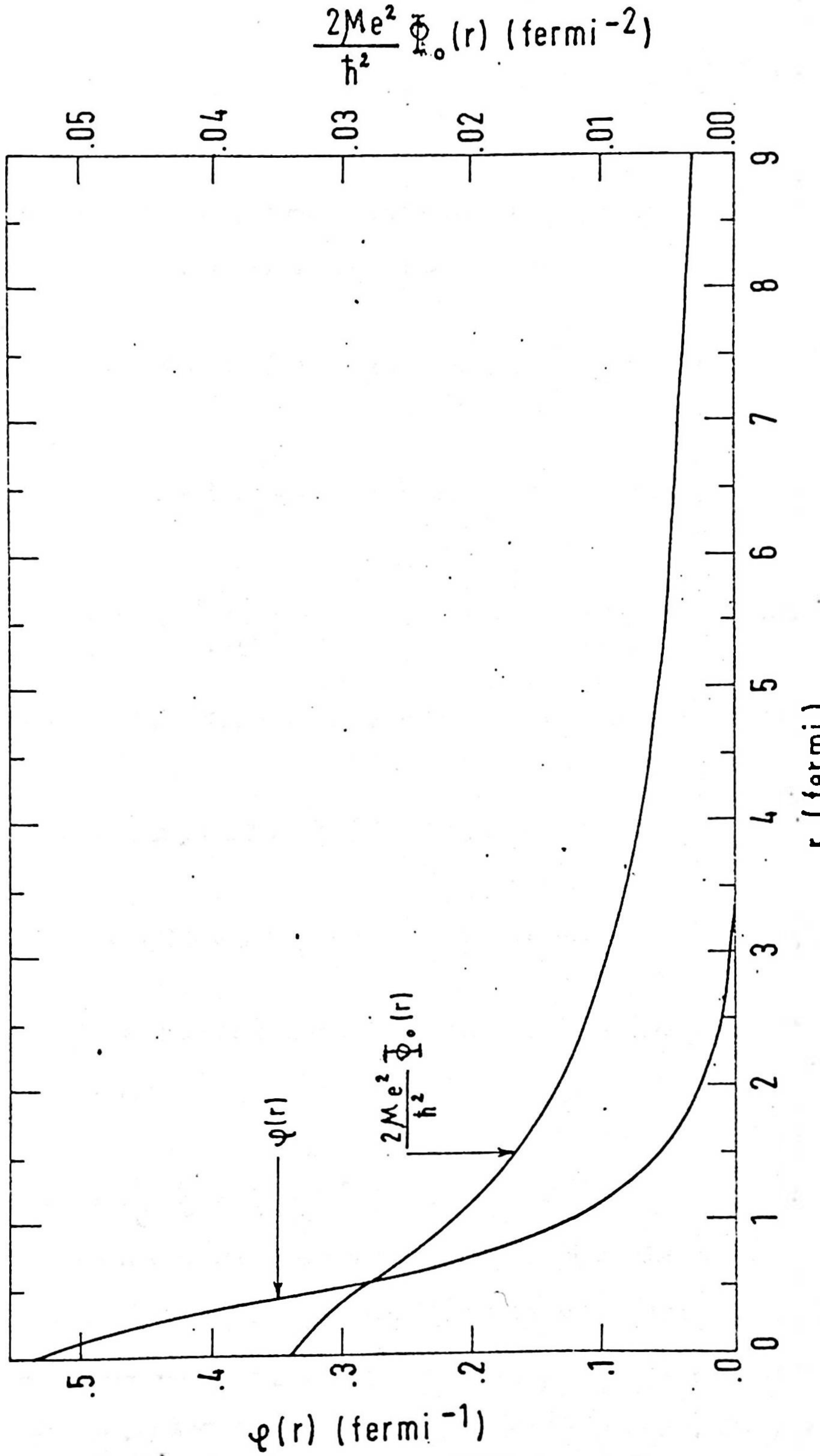

$\frac{2Me^2}{\hbar^2}\Phi_0(r)$ (fermi $^{-2}$)
$\varphi(r)$ (fermi $^{-1}$)
r (fermi)
$\varphi(r)$
$\frac{2Me^2}{\hbar^2}\Phi_0(r)$
.05
.04
.03
.02
.01
.00
.5
.4
.3
.2
.1
.0
0
1
2
3
4
5
6
7
8
9

FIG. V. 1

and introducing the expressions (5.20) and (5.18') for F(r) and $F_0(r)$ respectively, the Radial Schrodinger equations, derived from the K^-d wave equation (5.13) by the partial waves method, are given by

$$\left[\frac{d^2}{dr^2} + k_a^2 - \frac{L(L+1)}{r^2} + (A + iB)\,\phi(r) - 2Cnk_a\,\Phi_0(r) \right] \chi_L(r) = 0$$

for L = 0, 1, ... ∞ (5.56)

The complex quantity (A + iB) is easily related to the dimensionless coefficients

$$Z_I = x_I + iy_I = \frac{2\bar{\mu}}{\hbar^2 b^3} (u_I + iv_I), \quad \bar{\mu} = \frac{m_N m_K}{m_N + m_K} \quad ; I = 0,1$$

defined in §5 of Chapter II. Considering that $\mu = \bar{\mu}\gamma$, γ being defined here as in (4.56), one has

$$A + iB = (Z_0 + 3Z_1)\cdot\gamma\,\frac{2\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^2} =$$

$$= (Z_0 + 3Z_1) \times 0.8846 \text{ fermi}^{-1} \qquad (5.53')$$

Making now an obvious transformation, the potential $\Phi_0(r)$ can be written under the form (see (5.55)):

$$\Phi_0(r) = \frac{1}{C}\frac{1}{r} - \frac{1}{r}\int_r^\infty \rho\,\chi(\rho)\,d\rho + \int_r^\infty \chi(\rho)\,d\rho \qquad (5.19'')$$

showing clearly the presence of the Coulomb term. Thus, when the regular solution at the origin ($\chi_L(0) = 0$) of equation (5.56) is integrated numerically to obtain the complex phase shift $\delta_L\, (= \alpha_L + i\beta_L)$, there is a point r_0 belonging to the r-axis beyond which $\phi(r)$ and $C\,\Phi_0(r) - 1/r$ are vanishingly small. For values of r larger than r_0, the equation (5.56) becomes the pure Coulomb radial equation (C.2), so that $\chi_L(r)$ can now be expressed as a linear combination of $F_L(k_a r)$ and $G_L(k_a r)$ (see (C.3) and (C.4)), i.e.,

$$\chi_L(\rho) = (a_L + i b_L)\left[\cos\delta_L \, F_L(\rho) + \sin\delta_L \, G_L(\rho)\right] \tag{5.57}$$

where $a_L + ib_L$ is a complex constant.

The equation (5.57) together with the one that is obtained from its first derivative $\chi'_L(\rho) = \frac{d\chi_L}{d\rho}$, both calculated at the point r_0 (or $\rho_0 = k_a r_0$), determine the phase shift δ_L. In the actual calculations the value ρ_0 is obtained by taking also into account the condition that it falls into the range of validity of the asymptotic forms (C.8) and (C.9) for $F_L(\rho)$ and $G_L(\rho)$ respectively, i.e. , $\rho_0 \doteq 6$.

Now, if the functions $R_L(\rho)$ and $\theta_L(\rho)$, calculated at ρ_0 and defined by

$$F_L(\rho) = R_L(\rho) \sin\theta_L \qquad G_L(\rho) = R_L \cos\theta_L \tag{5.58}$$

$$R_L^2 = F_L^2 + G_L^2 \qquad \tan\theta_L = F_L / G_L \tag{5.59}$$

are considered, the elimination of $a_L + ib_L$ between $\chi_L(\rho_0)$ and $\chi'_L(\rho_0)$ gives

$$\tan(\theta_L + \delta_L) = \left[\frac{\chi_L}{\chi'_L R_L^2 - \chi_L R_L R'_L}\right]_{\rho=\rho_0} \tag{5.60}$$

if the equation

$$R_L^2 \theta'_L = 1,$$

derived from the Wrouskian condition (C.11) for $F_L(\rho)$ and $G_L(\rho)$, is used.

Writing

$$\rho_1 e^{i\theta_1} = i\left[\chi_L\right]_{\rho=\rho_0} , \quad \rho_2 e^{i\theta_2} = \left[\chi'_L R_L^2 - \chi_L R_L R'_L\right]_{\rho=\rho_0}$$

and expressing $\mathrm{Tan}(\theta_L + \delta_L)$ in terms of positive and negative exponentials, one has

$$e^{2i(\theta_L+\delta_L)} = \frac{\rho_2 e^{i\theta_2} + \rho_1 e^{i\theta_1}}{\rho_2 e^{i\theta_2} - \rho_1 e^{i\theta_1}} ;$$

therefore, the imaginary part β_L of δ_L $(= \alpha_L + i\beta_L)$ is given by

$$\beta_L = -\frac{1}{4} \operatorname{Log}\left[\frac{\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \operatorname{Cos}(\theta_2-\theta_1)}{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \operatorname{Cos}(\theta_2-\theta_1)}\right] \tag{5.61}$$

and its real part, α_L, by

$$\alpha_L = \left[\frac{1}{2} \tan^{-1}\left(\frac{\rho_2 e^{i\theta_2} + \rho_1 e^{i\theta_1}}{\rho_2 e^{i\theta_2} - \rho_1 e^{i\theta_1}}\right) - \theta_L\right] \pm m\pi \tag{5.62}$$

where <u>m</u> is an integer.

The pure Coulomb phase shifts ζ_L , defined in (C.13) and appearing in $F_L(\rho)$ and $G_L(\rho)$, obviously are calculated by means of (C.14) and (C.15).

The important quantities in the calculation of different sorts of K^-d cross-sections can be defined by

$$\gamma_L = e^{2i\delta_L} \tag{5.63}$$

Although similar to the η_L's obtained by Impulse Approximation (see (4.79) and (4.84)) the γ_L's should not be confused with them, as it will be seen in the next paragraph.

By (5.63), it is clear that the indeterminacy of the calculated α_L (see (5.62)) does not affect the cross-sections because on has always $e^{2im\pi} = 1$. However, the expression (5.61) giving the β_L can lead to values physically unacceptable. Actually the γ_L's must satisfy the inequality

$$1 - |\gamma_L|^2 \geqslant 0 \qquad (5.64)$$

analogous the ones obeyed by the η_L's (see (4.91)) and obtained by making the same considerations. Thus, one must have always $-1 \leqslant \cos(\theta_2 - \theta_1) \leqslant 0$ in (5.61), so that the inequalities $\beta_L \geqslant 0$ and (5.64) hold.

Equation (5.53') expresses A + iB in terms of Z_0 and Z_1. In the actual calculations, the values for these dimensionless coefficients were derived from Ross-Humphrey's scattering lengths A_0 and A_1, by means of the Hulthèn's variational formula (see Appendix E).

Table (V) gives the phase shifts δ_L for Ross-Humphrey's Solutions I and II and for the first four partial waves in the K^--Lab momentum interval 100 to 300 MeV/c. From the inspection of this table, it is clear that β_L is always positive i.e., the two sets of solutions are physically admissible in this momentum range.

The numerical determination of the δ_L was performed in the Mercury-Ferranti Computer of the University of London. The INSTEP facilities of this computer were used in the evaluation of $\chi_L(r)$, solution of the differential equation (5.56), as well as of $\chi_L'(r)$.

TABLE V

P Lab MeV/c	δ^0		δ^1		δ^2		δ^3	
	α_0	β_0	α_1	β_1	α_2	β_2	α_3	β_3
I{ 100	0·1951	0·3986	0·0163	0·0180	0·0012	0·0012	0·0001	0·0001
I{ 200	0·3367	0·4228	0·0651	0·0746	0·0129	0·0131	0·0027	0·0027
I{ 300	0·3654	0·3813	0·1173	0·1335	0·0365	0·0384	0·0116	0·0116
II{ 100	0·8572	0·3830	0·0362	0·0113	0·0024	0·0007	0·0003	0·0001
II{ 200	0·8025	0·2156	0·1498	0·0436	0·0253	0·0067	0·0073	0·0013
II{ 300	0·7183	0·1487	0·2606	0·0636	0·0750	0·0169	0·0228	0·0049

6. Limitations of the Resonant Group Structure Method. K^-d Absorption Cross-Sections

Suppose the radial wave function $\chi_L(r)$ normalised in such a way that

$$\chi_L(r) \underset{r\to\infty}{\longrightarrow} e^{i\delta_L}\sin\left(k_a r - n\log 2k_a r - L\frac{\pi}{2} + \sigma_L + \delta_L\right) \quad (5.57')$$

From the differential equation (5.56) of which $\chi_L(r)$ is a solution and from the regular spherical Coulomb function $F_L(r)$ together with its differential equation (C.2), by a procedure similar to the one used in the derivation of the integral formula (2.21), it is possible to obtain the following integral expression for δ_L:

$$e^{i\delta_L}\sin\delta_L = \frac{1}{k_c}\cdot\int_0^\infty\left\{(A+iB)\phi(r) - 2nk_a\left[c\Phi_0(r) - \frac{1}{r}\right]\right\}F_L\chi_L\,dr \quad (5.65)$$

The integral is convergent because $\phi(r)$ and $C\bar{\Phi}_0(r) - 1/r$ (see (5.19")) are short range potentials.

The effective nuclear potential $(A + iB)\phi(r)$ does not include the inelastic and charge-exchange effects in K^-d scattering. Actually, if the pion-hyperon production (absorption) is switched off in such collisions, the Ross-Humphrey's scattering lengths A_0 and A_1, as well as the coefficients Z_0 and Z_1 and the phase shifts δ_L, become real quantities leading to pure elastic scattering processes. Therefore, an energy-dependent term $W(k_\alpha, r)$ should be added to $(A + iB)\phi(r)$, so that the inelastic and charge-exchange scattering be taken into account formally.

Consider now the two differential equations

$$\left[\frac{d^2}{dr^2} + k_a^2 - \frac{L(L+1)}{r^2} + (A+iB)\phi(r) + W(k_a, r) - 2nCk_a\bar{\Phi}_0(r)\right]\bar{\chi}_L = 0 \qquad (5.56)$$

and

$$\left[\frac{d^2}{dr^2} + k_a^2 - \frac{L(L+1)}{r^2} + W(k_a, r) - \frac{2nk_a}{r}\right]\bar{F}_L = 0 \qquad (5.67)$$

where $\bar{\chi}_L(r)$ and $\bar{F}_L(r)$ are the respective regular solutions at the origin and normalised so that one has

$$\bar{\chi}_L(r) \underset{r\to\infty}{=} e^{i\bar{\delta}_L} \sin\left(k_a r - n\log 2k_a r - L\frac{\pi}{2} + \zeta_L + \bar{\delta}_L\right) \qquad (5.66')$$

$$\bar{F}_L(r) \underset{r\to\infty}{=} \sin\left(k_a r - n\log 2k_a r - L\frac{\pi}{2} + \zeta_L + \bar{\zeta}_L\right) \qquad (5.67')$$

Applying to equations (5.66) and (5.67) the method followed to obtain the integral expression (5.65) one gets

$$e^{i\bar{\delta}_L}\sin(\bar{\delta}_L - \bar{\zeta}_L) = \frac{1}{k_a}\int_0^\infty \left\{(A+iB)\,\phi(r) - 2n\,k_a\left[c\,\bar{\Phi}_0(r) - 1/r\right]\right\}\bar{\chi}_L \bar{F}_L\, dr \quad (5.68)$$

The integral formulae (5.65) and (5.68) lead to the approximate result $\delta_L = \bar{\delta}_L - \bar{\zeta}_L$. It is possible to write down an exact expression for the K^-d absorption cross-section if the functions $\bar{\chi}_L(r)$ and $\bar{F}_L(r)$ (and the respective phase shifts $\bar{\delta}_L$ and $\bar{\zeta}_L$) are considered. In fact, the initial intensity (1) of the kaon ingoing L-partial wave is reduced $|e^{2i\bar{\zeta}_L}|^2$ times by the presence of the corrective term $W(k_a,r)$ and $|e^{i(\bar{\delta}_L-\bar{\zeta}_L)}|^2$ times by the total effective potential $(A + iB)\phi(r) + W(k_a,r)$.

Thus, the exact K^-d absorption cross-section is equal to

$$\sigma_{ab} = \frac{\pi}{k_a^2}\sum_{L=0}^{\infty}(2L+1)\,|e^{2i\bar{\zeta}_L}|^2\left(1 - |e^{2i(\bar{\delta}_L-\bar{\zeta}_L)}|^2\right) \quad (5.69)$$

No attempt was made to calculate the $W(k_a,r)$ potential in this work, although a sketch of how it can be constructed is indicated in the next paragraph. Here only the approximate formula $\left(W(k_a,r) \approx 0,\ \delta_L \approx \bar{\delta}_L - \bar{\zeta}_L\right)$:

$$\sigma_{ab} \simeq \frac{\pi}{k_a^2}\sum_{L=0}^{\infty}(2L+1)\left(1 - |e^{2i\delta_L}|^2\right) \qquad (5.69')$$

for σ_{ab} has been used in the plot of the K^-d absorption curves (Ross-Humphrey's solutions I and II) shown in Fig. (V.2). This picture shows also the absorption curves obtained by subtracting from the total incoherent scattering cross-section (see its definition in §8 of this chapter) the inelastic plus charge-exchange cross-sections calculated in Chapter VII. As it was predicted in §3 of this chapter, the discrepancies between the two methods tend to increase for lower energies.

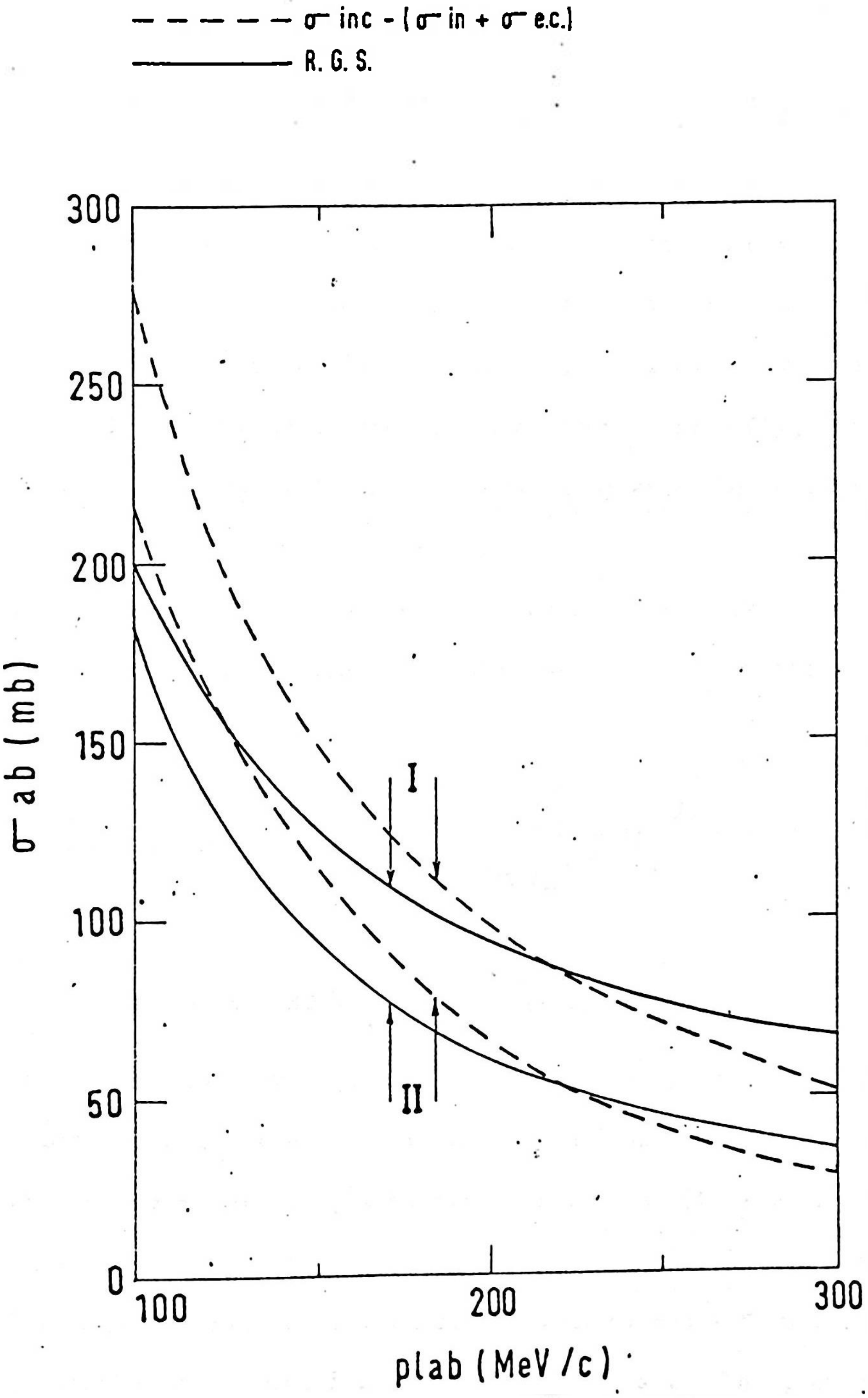

σ inc - (σ in + σ e.c.)
R. G. S.
300
250
200
150
100
50
0
100
200
300
σ ab (mb)
plab (MeV/c)
I
II

FIG. V. 2

7. Correction of the Coulomb Effects in Impulse Approximation

Two main questions must be considered when treating the problem of the Coulomb interaction between charged kaons and protons belonging to deuterons. First, an electromagnetic model of the deuteron must be chosen such that it takes into account the condition that the proton is not located at the C.M. of the two nucleons but moves around the neutron. Secondly, the nuclear parameters η_L, obtained by Impulse Approximation with the Coulomb interaction switch off (see (4.84)) must be corrected.

Making the transformation R = 2r in the integral expressions (4.84) for the η_L and defining the complex nuclear phase shifts δ_L by the relation $\eta_L = e^{2i\delta_L}$, one has

$$\frac{\eta_L - 1}{2i} = e^{i\delta_L} \sin\delta_L = -\frac{1}{k_a}\int_0^\infty U_L(r)\, j_l^2(k_a r)(k_a r)^2 dr \tag{5.70}$$

where

$$U_L(r) = -32\pi\gamma F_l(r)\,|\phi_0(2r)|^2 \tag{5.71}$$

The meaning of the integral formula (5.70) is that the exact coefficient η_L (or the exact phase δ_L) are the Born approximations of the corresponding quantities for the effective potentials $U_L(r)$. But it is well-known (Jost and Kohn, 1952) that, at low energies, a linear combination of the successive powers of a negative exponential function is an adequate form for a short range potential, capable of reproducing the phase shifts δ_L. Thus, by means of a variational principle (Schwinger's or Hulthèn's, for instance), it is possible for each value of k_a to determine a linear combination V(k,r) of the first n powers of the exponential function which leads in the usual way to the phase shifts δ_L for L = 0, 1, . . . n-1. If this is so, the regular solution $Y_L(r)$ at the origin of the radial equation

$$\left[\frac{d^2}{dr^2} + k_a^2 - \frac{L(L+1)}{r^2} - V(k,r)\right] Y_L(r) = 0 \tag{5.72}$$

can be normalised in such a manner that its asymptotic behaviour is expressed by

$$Y_L(r) \rightarrow e^{i\delta_L} \sin(k_a r - L\tfrac{\pi}{2} + \delta_L) \tag{5.73}$$

From (5.72) and the radial L-wave equation of free motion (C.16) one has, by the same procedure used in the extension of the integral expression (5.65),

$$e^{i\delta_L} \sin\delta_L = -\frac{1}{k_a}\int_0^\infty V(k_a,r)\, Y_L(r)\, j_L(k_a r)(k_a r)\, dr$$

or, by (5.70),

$$\int_0^\infty V(k_a,r)\, Y_L(r)\, j_L(k_a r)(k_a r)\, dr =$$

$$= \int_0^\infty U_L(r)\, j_L^2(k_a r)(k_a r)^2\, dr \tag{5.74}$$

Suppose now that the potential 2n ka C $\phi_0(r)$, due to an electric cloud of density $\eta(r)$ (see (5.27) and (5.28)) is introduced in equation (5.72) as an additive term to V(k,r), so that the Coulomb interaction between the particles K^- and p is taken into account. The regular solution $\bar{Y}_L(r)$ at the origin of the new radial L-wave equation, normalised in such a way that its asymptotic form is given by

$$\bar{Y}_L(r) \underset{r\to\infty}{=} e^{i\delta_L'} \sin(k_a r - n\log 2k_a r - L\tfrac{\pi}{2} + \chi_L + \delta_L')$$

leads to the corrected phase δ_L' (or η_L'). Its integral form is equal to

$$\frac{\eta'_L - 1}{2i} = e^{i\delta'_L}\sin\delta'_L = -\frac{1}{k_a}\int_0^\infty\left\{V(k_a,r) + 2n\,k_a\left[C\bar{\Phi}_o(r) - 1/r\right]\right\}\bar{y}_L F_L(k_a r)\,dr \qquad (5.75)$$

where $F_L(k_a r)$ represents the regular spherical Coulomb function (C.3).

Considering now that $|n|$ is very small for the K^--Lab. momentum range covered in this work (by (4.90) and (5.54), $|n|$ lays in the interval $0{\cdot}034 \geq |n| \geq 0{\cdot}012$ when $100 \leq p_{Lab} \leq 300$ MeV/c), one has approximately

$$\int_0^\infty\left\{V(k_a,r) + 2n\,k_a\left[C\bar{\Phi}_o(r) - 1/r\right]\right\}\bar{y}_L(r)\,F_L(k_a r)\,dr \simeq$$

$$\simeq \int_0^\infty\left\{U_L(r) + 2n\,k_a\left[C\bar{\Phi}_o(r) - 1/r\right]\right\}F_L^2(k_a r)\,dr \qquad (5.76)$$

The main contribution of $F_L^2(k_a r)$ for the second integral in (5.76) comes from the small values of r, because either $U_L(r)$ or $2n\,k_a\left[C\bar{\Phi}_o(r) - 1/r\right]$ are short range potentials. Thus, one gets (Jackson and Blatt, 1950):

$$\frac{\left[(k_a r)j_L(k_a r)\right]^2}{F_L^2(k_a r)} \simeq \frac{1}{C_o^2\prod_{\lambda=0}^{L}\left(1 + \frac{n^2}{\lambda^2}\right)} \simeq \frac{1}{C_o^2} \qquad (5.77)$$

where Co^2 is the Coulomb penetration factor

$$C_o^2 = \frac{2\pi n}{e^{2\pi n} - 1} \qquad (5.77')$$

and, by (5.75), (5.76) and (5.77),

$$\frac{\eta'_L - 1}{2i} = e^{i\delta'_L}\sin\delta'_L = -\frac{C_o^2}{k_a}\int_0^\infty\left\{U_L(r) + 2n\,k_a\left[C\bar{\Phi}_o(r) - 1/r\right]\right\}j_L^2(k_a r)(k_a r)^2\,dr \qquad (5.78)$$

The numerical computation of the coefficients η_L given in Table (IV) was obtained from the integral expression (5.78) subjected to the following simplification. The quantity $\Delta\eta_L$, defined by

$$\frac{\Delta\eta_L}{2i} = -2n C_0^2 \int_0^\infty [C\bar{\Phi}_0(r) - 1/r]\, j_L^2(k_a r)(k_a r)^2 dr \tag{5.79}$$

is the Born approximation of the difference between the η_L-coefficients due to the point charge potential and to the electric cloud interaction. The behaviour of the $j_L(k_a r)$ at the origin (see (4.100)) shows that, for the same energy the modulus of $\Delta\eta_L$ has its largest value when L = 0. Therefore if $|\eta_L|$ is reasonably small the term $2 k_a n [C\bar{\Phi}(r) - 1/r]$ can be suppressed in (5.78) except for an S-wave. In this case the integral (5.79) can be handled analytically. Actually, by (5.19"), one has

$$\Delta\eta_0 = \Delta\eta_0^{(1)} + \Delta\eta_0^{(2)} \tag{5.79'}$$

where

$$\frac{\Delta\eta_0^{(1)}}{2i} = 2nC_0^2 C \int_0^\infty \frac{\sin^2(k_a r)}{r} dr \int_r^\infty \rho\, \chi(\rho)\, d\rho \tag{5.80}$$

and

$$\frac{\Delta\eta_0^{(2)}}{2i} = -2nC_0^2 C \int_0^\infty \sin^2(k_a r)\, dr \int_r^\infty \chi(\rho)\, d\rho \tag{5.81}$$

Using the definition (5.15) of $\chi(\rho)$ and the condition $\alpha \ll \beta$ (see (3.26) and (3.27)), $\Delta\eta_0^{(1)}$ is approximately equal to

$$\frac{\Delta\eta_0^{(1)}}{2i} \simeq \frac{2n\,C_0^2 C}{4\alpha}\int_0^\infty \frac{e^{-4\alpha r}}{r}\sin^2(k_a r)\,dr =$$

$$= \frac{n C_0^2 C}{8\alpha}\log\left[1 + \left(\frac{k_a}{2\alpha}\right)^2\right] \qquad (5.80')$$

From (B.6) and since (see Courant, Volume II, page 318)

$$\int_0^\infty \frac{e^{-\lambda_1\rho}\,e^{-\lambda_2\rho}}{\rho}\sin(k\rho)\,d\rho = \tan^{-1}\frac{(\lambda_2-\lambda_1)k}{\lambda_1\lambda_2 + k^2},$$

one has

$$\frac{\Delta\eta_0^{(2)}}{2i} = -n C_0^2 C\int_0^\infty \chi(\rho)\,d\rho\int_0^\rho [1-\cos(2k_a r)]\,dr =$$

$$= -n C_0^2\left\{1 - \frac{C}{2k_a}\left[\tan^{-1}\frac{(\beta-\alpha)k_a}{2\alpha(\alpha+\beta)+k_a^2} - \tan^{-1}\frac{(\beta-\alpha)k_a}{2\beta(\alpha+\beta)+k_a^2}\right]\right\} \qquad (5.81')$$

For the extreme values 100 and 300 MeV/c of the K^--Lab. momentum range covered in this work $\Delta\eta_0/2i$ is respectively equal to -0·007 and -0·010, i.e. the term $2nk_a\left[C\phi_0(r) - 1/r\right]$ in (5.78) is practically zero when $L > 0$.

Incidentally, the construction of the potential $V(k,r)$ leads to the evaluation of the corrective term $W(k,r)$, discussed in the last paragraph. One obviously has

$$W(k,r) = -V(k,r) - (A + iB)\phi(r).$$

8. K^-d Elastic and Total Scattering Curves

The K^-d differential elastic scattering cross-sections, $d\sigma_{el}/d\Omega$, are plotted in Figs. (V.3) and (V.4) against the K^-d C.M. scattering angle θ for p_{Lab} = 100; 200; 300 MeV/c and for Ross-Humphrey's solutions I and II.

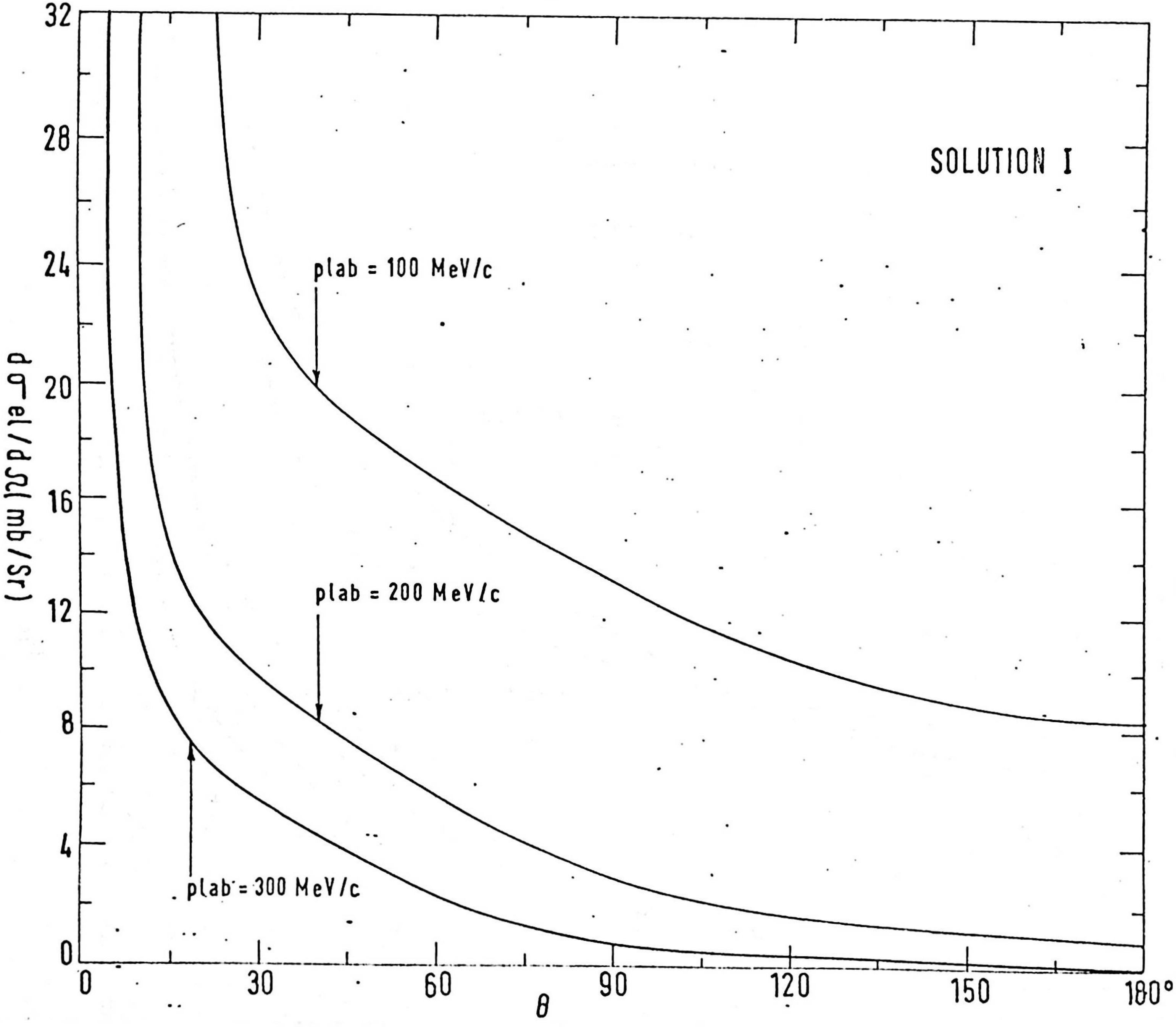

SOLUTION I
plab = 100 MeV/c
plab = 200 MeV/c
plab = 300 MeV/c
dσel/dΩ(mb/Sr)
θ
0
4
8
12
16
20
24
28
32
0
30
60
90
120
150
180°

FIG. V. 3

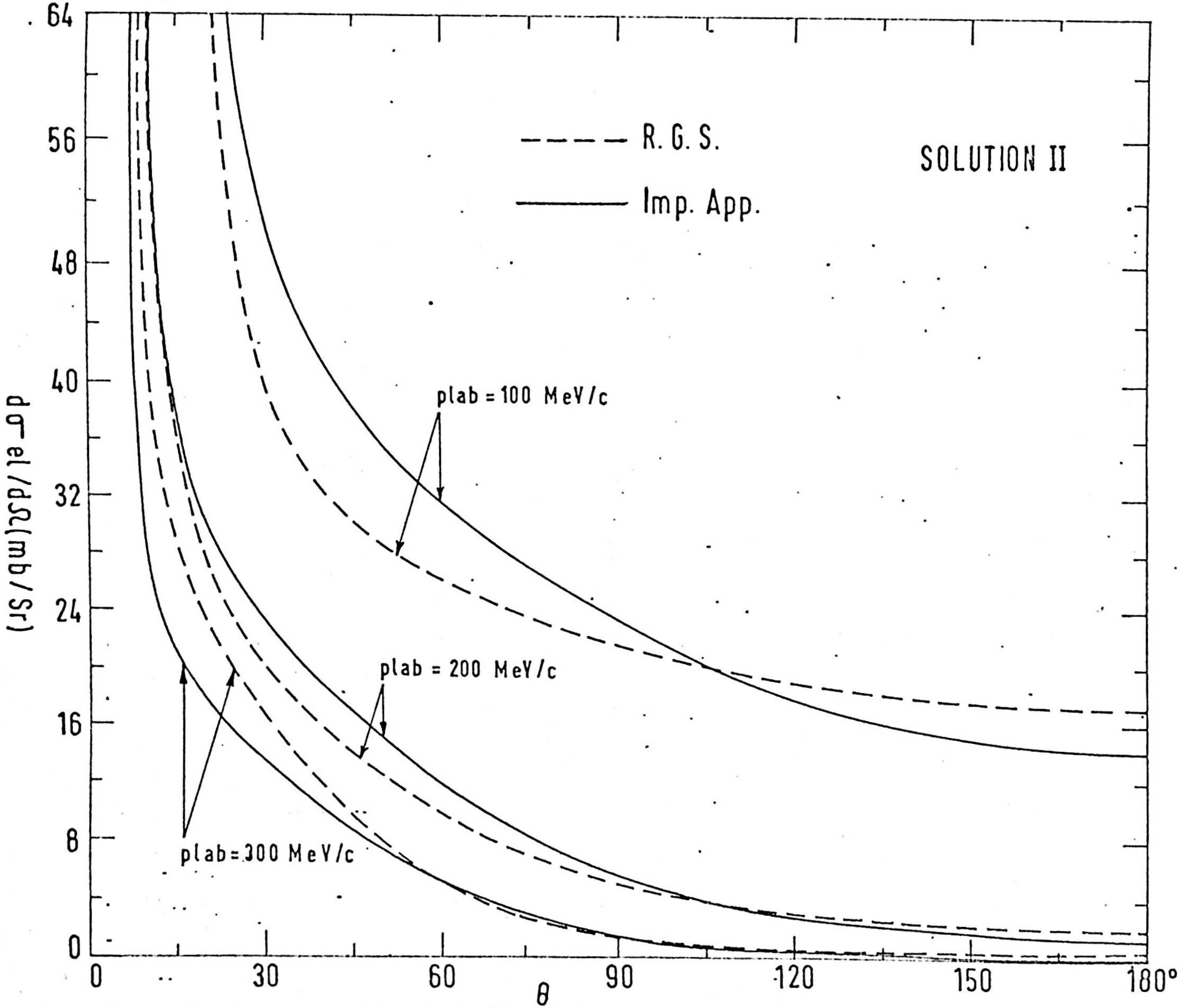

R. G. S.
Imp. App.
SOLUTION II
plab = 100 MeV/c
plab = 200 MeV/c
plab = 300 MeV/c
dσ el/dΩ(mb/Sr)
θ
0
8
16
24
32
40
48
56
64
0
30
60
90
120
150
180°

FIG. V.4

They were calculated by means of the formula

$$\frac{d\sigma_{el}}{d\Omega} = |f(\theta)|^2 ; \quad f(\theta) = \frac{1}{2ik_a} \sum_{L=0}^{\infty} (2L+1) \left[e^{2i(\chi_L + \delta'_L)} - 1 \right] P_L(\cos\theta) \tag{5.82}$$

where the χ_L are the pure Coulomb phase shifts given by (C.14) and (C.15) and the δ'_L are the nuclear phase shifts calculated by Impulse Approximation (see Table (IV)) and given by (5.78).

The numerical computation of $d\sigma_{el}/d\Omega$ was carried out by separating f(θ) into two additive terms

$$f(\theta) = f_c(\theta) + f_N(\theta) \tag{5.82'}$$

where

$$f_c(\theta) = \frac{-n}{2k_a \sin^2\frac{\theta}{2}} e^{i(-n \log \sin^2\frac{\theta}{2} + 2\chi_0)} \tag{5.83}$$

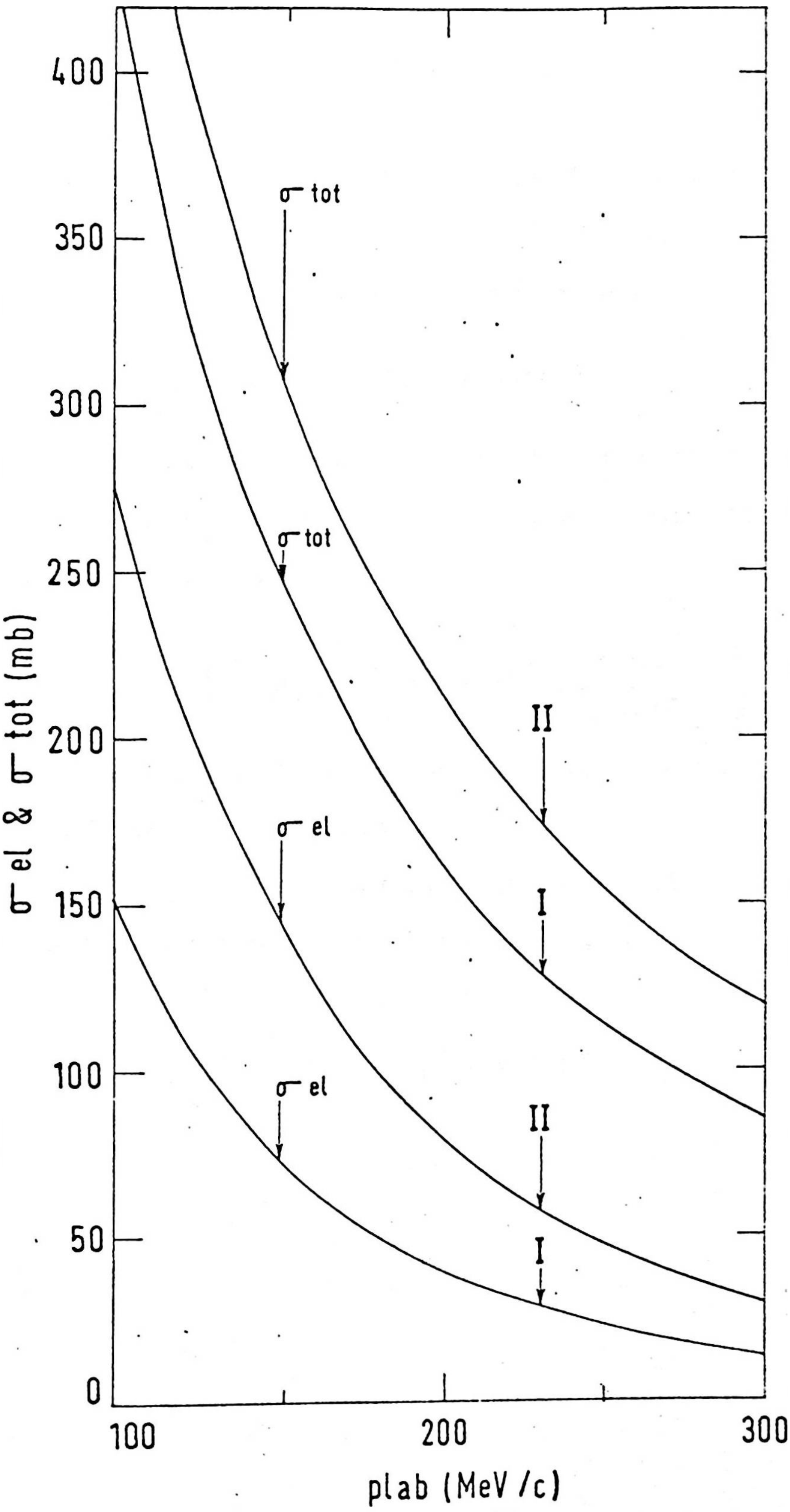
represents the pure Coulomb part of the elastic scattering and (see (5.78))

$$f_N(\theta) = \frac{1}{2ik_a} \sum_{L=0}^{\infty} (2L+1) e^{2i\chi_L} (\eta'_L - 1) P_L(\cos\theta) \tag{5.84}$$

is the term depending on the nuclear interaction.

In order to have a better assessment of the approximation involved in the application of the resonant group structure method, Fig. (V.4) also shows the K^-d differential cross-sections obtained by substituting the η'_L in (5.84) for the γ_L defined in (5.63). As in the case of the absorption curves, these graphs show that the approximation W(k,r) = 0 is poor for low energies.

Fig. (V.5) represents the K^-d total elastic scattering (σ_{el}) and total (σ_{TOT}) cross-sections curves plotted versus p_{Lab} in the interval 100 to



σ tot
σ tot
σ el
σ el
II
I
II
I
400
350
300
250
200
150
100
50
0
σ el & σ tot (mb)
100
200
300
plab (MeV /c)

FIG. V. 5

300 MeV/c and calculated by Impulse Approximation for Ross-Humphrey's solution I and II.

σ_{el} was calculated by means of the integral

$$\sigma_{el} = 2\pi \int_{\theta_0}^{\pi} |f(\theta)|^2 \sin\theta \, d\theta \tag{5.85}$$

where θ_0 represents a cut-off equal to 35^0 in the C.M. scattering angle; this value seems to be reasonable, if the accuracy of the experiments is considered. Furthermore, the approximation

$$2\,\mathrm{Real}\left[f_c(\theta) f_N^*(\theta)\right] \simeq \mathrm{Real}\left[\frac{-\eta f(0)}{k_a \sin^2\frac{\theta}{2}}\right] \tag{5.86}$$

was made in the evaluation of the interference between the Coulomb and the nuclear terms of the scattering amplitude.

Finally, σ_{TOT} was calculated by adding σ_{el} to the total incoherent scattering cross-section (σ_{inc}) , defined by

$$\sigma_{inc} = \frac{\pi}{k_a^2} \sum_{L=0}^{\infty} (2L+1)\left(1 - |\eta'_L|^2\right) \tag{5.87}$$

CHAPTER VI

K^-d Inelastic and Charge Exchange Scattering

1. Introduction

The study of the K^-d non-elastic processes

$$K^- + d \rightarrow K^- + p + n \tag{6.1}$$

$$K^- + d \rightarrow \bar{K} + n + n \tag{6.2}$$

by Impulse Approximation is the main purpose of this chapter.

In such collisions, part of the energy of the incoming kaon is absorbed by the NN system during its transition from the initial state (deuteron) to the final state (two free nucleons). Therefore, one has, using the notation of Chapter IV,

$$|\bar{k}_b| < |\bar{k}_a| \tag{6.3}$$

instead of the equality (4.53). But the condition

$$p_b = |\bar{p}_b| = |\bar{p}_a| = p_a \tag{6.4}$$

as well as the kinematic relations (4.54) and (4.55) still hold, because, according to the fundamental hypothesis and assumption 1) of Impulse Approximation (see §2 of Chapter IV), the K^-N system is conceived as being isolated from the other nucleon during the momentum transfer. Thus, if both sides of equations (4.54) and (4.55) are squared, the results of these operations added and condition (6.4) is taken into account with $p_a = p_b$ replaced by $p'_a = p'_b$, one gets

$$4p'^2_b = 4p'^2_a = \frac{1}{\gamma^2}\Big\{\Big[\gamma^2 + \left(\frac{m_K}{2m_N + m_K}\right)^2(\bar{k}_b - \bar{k}_a)^2 + (\bar{k}_b + \bar{k}_a)^2 + \frac{2m_N}{2m_N + m_K}\left(k_b^2 - k_a^2\right)\Big\} \tag{6.5}$$

Introducing now the scattering angle θ defined by the directions of vector $\bar{k}_a$ and $\bar{k}_b$, the term $(\bar{k}_b - \bar{k}_a)^2$ gives

$$(\bar{k}_b - \bar{k}_a)^2 = k_b^2 + k_a^2 - 2k_b k_a \cos\theta =$$

$$= 4k_a \sin^2\frac{\theta}{2} + (k_b^2 - k_a^2) - 2k_a(k_b - k_a)\cos\theta \quad . \qquad (6.6)$$

Substituting θ by π − θ in (6.6) one has also

$$(\bar{k}_b + \bar{k}_a)^2 = 4k_a^2\cos^2\frac{\theta}{2} + (k_b^2 - k_a^2) + 2k_a(k_b - k_a)\cos\theta \qquad (6.6')$$

so that (6.5) becomes, if the coefficient $\left(\frac{m_K}{2m_N + m_K}\right)^2$ (= 0·04) is neglected (see (4.56) for the definition of γ),

$$p_b'^2 = p_a'^2 = k_a^2 \frac{1+\gamma^2}{2\gamma^2}\left[1 - \frac{1}{k_a^2}\cdot\frac{(\gamma+1)^2 - 2}{2(1+\gamma^2)}\left(k_b^2 - k_a^2\right) - \right.$$

$$\left. - \frac{\gamma^2 - 1}{\gamma^2 + 1}\,\frac{k_b}{k_a}\cos\theta\right], \qquad (6.7)$$

or

$$p_b'^2 = p_a'^2 \simeq k_a^2 \frac{1+\gamma^2}{2\gamma^2}\left[1 - \frac{1}{k_a^2}\cdot\frac{(\gamma+1)^2 - 2}{2(1+\gamma^2)}\left(k_b^2 - k_a^2\right)\right] \qquad (6.7')$$

if the term in cosθ is suppressed.

When $k_b = k_a$, the relations (6.7) and (6.7') reduce respectively, as they should, to the expressions (4.57') and (4.57"), obtained previously for K^-d elastic collisions.

The values of $p_b' = p_a'$ adopted in this chapter are derived from (6.7'). This approximation seems reasonable because the relative error of $p_b'^2 = p_a'^2$, made by using (6.7') instead of (6.7), does not exceed the same error (0·2) for K^-d elastic collisions ($k_b = k_a$) and drops to 0·16 and 0 when k_b becomes equal to $0{\cdot}5k_a$ and 0 respectively.

Introducing now the wave number $\bar{K}$ of the NN system and constant α^2 $(= -\frac{m_N W_0}{\hbar^2}$, $W_0 = -E_d)$ (see respectively §1 and §4 of chapter III), the energy conservation principle equation (4.7) can be expressed by

$$\frac{k_a^2}{2\mu} - \frac{\alpha^2}{m_N} = \frac{k_b^2}{2\mu} + \frac{K^2}{m_N} \qquad (6.8)$$

where μ is the K^- reduced mass, defined in (4.3). The alternative form of (6.8)

$$k_a^2 - 4(\gamma-1)\alpha^2 = k_b^2 + 4(\gamma-1)K^2 \qquad (6.8')$$

is obtained if the coefficient γ (= 0·2) is used (see 4.56)).

The equation (6.8') shows that K has a maximum, K_{max}, when $k_b = 0$, i.e.,

$$K_{max}^2 = \frac{1}{4(\gamma-1)}\left[k_a^2 - 4(\gamma-1)\alpha^2\right] \qquad (6.9)$$

and that k_b is always lower than k_a if the deuteron, after the collision with the kaon, goes into a NN continuous state.

2. The K^-d inelastic transition matrix elements $(\phi_b, T\phi_a)$

The initial and final state wave functions used in the calculations of these elements are respectively equal to (see (4.6))

$$\Phi_a = e^{i\bar{k}_a\cdot\bar{r}}\,\phi_0(R), \quad \Phi_b = e^{i\bar{k}_b\cdot\bar{r}}\,\phi_{\bar{K}}(\bar{R}) \qquad (6.10)$$

where $\phi_0(R)$ is the Hulthèn function defined in (3.25) and $\phi_{\bar{K}}(\bar{R})$ is the wave function of the np continuous state $\bar{K}$, given in (3.34). Considering the approximations (3.30'), (3.32'), (3.36) and (3.37), $\phi_{\bar{K}}(\bar{R})$ can be expressed in the form

$$\phi_{\bar{K}}(\bar{R}) = \frac{1}{3}(2\pi)^{-\frac{3}{2}} \sum_{J=0}^{\infty} (2J+1) \sum_{L=|J-1|}^{J+1} i^L U_J^L(KR) P_L(\cos\theta_{\bar{K}R}) \qquad (6.11)$$

where θ_{KR} is the angle between $\bar{K}$ and $\bar{R}$.

The evaluation of $(\phi_b, T\phi_a)$ by Impulse Approximation requires the introduction of a model describing the K^-d inelastic processes. The model adopted here consists in supposing that the kaon is scattered elastically in all single processes contributing to the multiple scattering terms appearing in the development (4.20) for T, with the exception of the last K^-N collision, which is inelastic (see also §9 of this chapter on the same subject).

Such a model does not contradict the assumption 2') made in §4 of Chapter IV. According to this hypothesis, the matrix element $(\phi_b, T\phi_a)$ is equal to the "weighted mean" of the sum of all multiple scattering terms arising in the expansion of $<\bar{k}_b|T|\bar{k}_a>$, when T is replaced by its development (4.20), i.e.,

$$(\Phi_b, T\Phi_a) = \int \phi^*_{\bar{K}}(\bar{R})\, \phi_0(R) <\bar{k}_b|T|\bar{k}_a>\, d\bar{R} \qquad (6.12)$$

When the K^-d collisions are elastic, the sum of the scattering terms belonging to this expansion of $<\bar{k}_b|T|\bar{k}_a>$ is equal to (see (4.74) and (4.78))

$$-\frac{2\mu}{4\pi\hbar^2} <\bar{k}_b|T|\bar{k}_a> = \frac{f^{(1)}(\theta,\bar{R}) + f^{(2)}(\theta,\bar{R})}{1 - \left(\frac{2\delta^2}{1+\delta^2}\right)^2 f_{31} f_{32} P^2(R)} \qquad (6.13)$$

where $f^{(1)}(\theta,\bar{R})$ and $f^{(2)}(\theta,\bar{R})$ are given respectively by (4.76) and (4.77) and f_{3i} $(i = 1,2)$ are the isotropic K^-N scattering amplitudes defined in (4.87') and (4.87).

However, when the K^-d scattering process is inelastic, (6.13) is no longer valid, unless $f^{(1)}(\theta,\bar{R})$ and $f^{(2)}(\theta,\bar{R})$ are redefined in the following way:

$$f^{(1)}(\theta,\bar{R}) = \gamma\left[e^{-i(\bar{k}_a-\bar{k}_b)\cdot\bar{R}/2} f'_{31} + e^{i(\bar{k}_a-\bar{k}_b)\cdot\bar{R}/2} f'_{32}\right] \tag{6.14}$$

and

$$f^{(2)}(\theta,\bar{R}) = \frac{2\gamma^3}{1+\gamma^2}\left[e^{-i(\bar{k}_a+\bar{k}_b)\cdot\bar{R}/2} f'_{32} f_{31} + e^{i(\bar{k}_a+\bar{k}_b)\cdot\bar{R}/2} f'_{31} f_{32}\right] \rho(R) \tag{6.15}$$

The scattering amplitudes f_{3i} $(i = 1,2)$ are calculated at p_a $\left(= \sqrt{\frac{1+\gamma^2}{2\gamma^2}}\, k_a\right)$. This value of the K^-N wave number corresponds to K^-d elastic collisions. The f'_{3i} $(i = 1.2)$ in (6.14) and (6.15) are equal to the f_{3i} $(i = 1,2)$ with p_a replaced by p'_a given by (6.7'). This allows for the K^-d inelastic collisions.

From (6.13), (6.14) and (6.15) it is now clear that, in agreement with the model for K^-d inelastic collisions introduced above, the K^-N scattering contributing to the K^-d multiple scattering terms maintains the elasticity of the K^-d processes with the exception of the last K^-N collision, which breaks up the deuteron into two free nucleons.

Finally, the relations (6.13), (6.14) and (6.15) show also that $(\phi_b, T\phi_a)$ is proportional to the integral over all values of $\bar{R}$ of the product of $\phi^*_{\bar{K}}(\bar{R})\phi_0(R)$ by a linear combination of plane waves multiplied by functions of $|\bar{K}|$ and $|\bar{R}|$. Therefore, according to the remark made at the end of §4 of Chapter III, $\phi_{\bar{K}}(\bar{R})$ is an appropriate function to represent the final state of the np system.

3. Selection Rules for the expansion coefficients of $(\phi_b, T\phi_a)$.

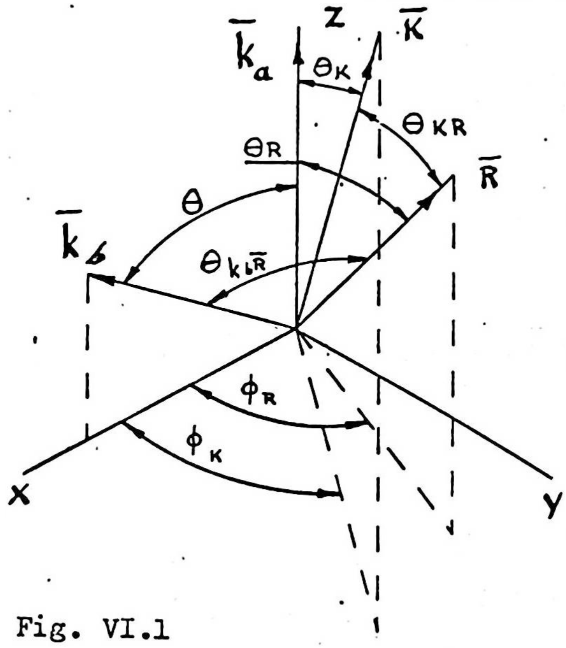


Fig. VI.1

Obviously, the matrix element $(\Phi_b, T\Phi_a)$ depends on k_a and k_b (by (6.8') K can be expressed in terms of these wave numbers) and on the angles necessary to fix the vectors $\bar{k}_a$, $\bar{k}_b$ and $\bar{K}$ with respect to a coordinate system of reference. Choosing the XZ-plane as defined by $\bar{k}_a$ and $\bar{k}_b$, with $\bar{k}_a$ along the Z-axis, only the three angles θ, θ_K and ϕ_K (see Fig. (VI.1)) are needed to achieve this purpose.

Replacing now the plane waves $e^{i\bar{k}_b\cdot\bar{R}/2}$ and $e^{i\bar{k}_a\cdot\bar{R}/2}$ in (6.14) and (6.15) by the respective expansions into spherical waves (see Fig. (VI.1) for the angles), i.e.

$$e^{i\bar{k}_b\cdot\bar{R}/2} = \sum_{L=0}^{\infty}(2L+1)i^L j_L(k_b R/2)\,P_L(\cos\theta_{k_b R}), \qquad (6.16)$$

$$e^{i\bar{k}_a\cdot\bar{R}/2} = \sum_{L'=0}^{\infty}(2L'+1)i^{L'} j_{L'}(k_a R/2)\,P_{L'}(\cos\theta_R) \qquad (6.17)$$

and, writing

$$\phi_{\bar{K}}(\bar{R}) = (2\pi)^{-\frac{3}{2}}\sum_{L''=0}^{\infty} i^{L''}\alpha_{L''}(KR)\,P_{L''}(\cos\theta_{KR}) \qquad (6.11')$$

where (see (6.11))

$$\alpha_L(KR) = \frac{1}{3}\sum_{J=|L-1|}^{L+1}(2J+1)\,U_J^L(KR) \qquad (6.11'')$$

it is clear, considering (6.12) and (6.13), that $(\Phi_b, T\Phi_a)$ admits the development

$$-\frac{2\mu}{4\pi\hbar^2}(\bar{\Phi}_b, T\bar{\Phi}_a) = \frac{1}{(2\pi)^2}\sqrt{\frac{\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^2}}\sum_{L,L',L''=0}^{\infty} C_{LL'L''} I_{LL'L''} \tag{6.18}$$

where the factors $C_{LL'L''}$ and $I_{LL'L''}$ are equal to

$$C_{LL'L''} = (-1)^{L+L'} \cdot (i)^{L+L'+L''} (2L+1)(2L'+1)\int_0^{\pi}\int_0^{2\pi} P_L(\cos k_b R)\, P_{L'}(\cos\theta_R)\, P_{L''}(\cos\theta_{KR}) \cdot \sin\theta_R \, d\theta_R \, d\varphi_R \tag{6.19}$$

and

$$I_{LL'L''} = \int_0^{\infty} F_{LL'}(k_a,k_b,R)\, j_L(k_b R|2)\, j_{L'}(k_a R|2)\, \alpha^{*}_{L''}(KR)(e^{-\alpha R} - e^{-\beta R}) R\, dR \tag{6.20}$$

with $F_{LL'}(k_a,k_b,R)$ defined as follows

$$F_{LL'}(k_a,k_b,R) = \gamma \frac{f'_{31} + (-1)^{L+L'} f'_{32} + \frac{2\gamma^2}{1+\gamma^2}(-1)^{L'}\left[f'_{32} f_{31} + (-1)^{L+L'} f'_{31} f_{32}\right] P(R)}{1 - \left(\frac{2\gamma^2}{1+\gamma^2}\right)^2 f_{31} f_{32} P^2(R)} \tag{6.21}$$

In order to evaluate the integral in (6.19), it is necessary to have $P_L(\cos\theta k_b R)$ and $P''_L(\cos\theta_{KR})$ expressed in terms of the angles θ_R and ϕ_R (see Fig. (VI.1)). This is easily achieved using the addition theorem for Legendre functions, i.e.,

$$P_L(\cos\theta_{k_b R}) = P_L(\cos\theta_R) P_L(\cos\theta) + 2\sum_{m=1}^{L}\frac{(L-m)!}{(L+m)!} P_L^m(\cos\theta_R) P_L^m(\cos\theta)\cos m\varphi_R \tag{6.22}$$

and

$$P_{L''}(\cos\Theta_{KR}) = P_{L''}(\cos\Theta_R)\,P_{L''}(\cos\Theta_K) +$$
$$+\,2\sum_{m''=1}^{L''}\frac{(L''-m'')!}{(L''+m'')!}\,P_{L''}^{m''}(\cos\Theta_R)\,P_{L''}^{m''}(\cos\Theta_K)\cos m''(\varphi_R-\varphi_K). \quad (6.23)$$

The developments (6.22) and (6.23), when considered inside the integrals defining the $C_{LL'L''}$, $\mathcal{A}$, lead to products of factors having the following forms:

1) Performing the integration with respect to ϕ_K, some of the factors belong to the type (by (6.22) and (6.23) one has $m, m'' > 1$):

$$\int_0^{2\pi}\cos m\,\phi_R\cos m''(\phi_R-\phi_K)\,d\phi_R = \begin{cases}\pi\cos m\,\phi_K\ , & \text{if } m''=m\\ 0\ , & \text{if } m''\neq m\end{cases} \quad (6.24)$$

the other products, where either $\cos\phi_R$ or $\cos(\phi_R - \phi_K)$ appear as a factor, vanish when integrated over ϕ_R. These results constitute the first selection rule in the expansion of $(\Phi_b, T\Phi_a)$.

2) Integrating now with respect to θ_R, two new types of factors are obtained (putting $\xi = \cos\theta_K$ and representing by p any positive integer):

$$\int_{-1}^{1}P_L(\xi)\,P_{L'}(\xi)\,P_{L''}(\xi)\,d\xi = 0\ , \text{ if } L+L'+L''=2p+1 \quad (6.25)$$

$$\int_{-1}^{1}P_L^m(\xi)\,P_{L'}(\xi)\,P_{L''}^m(\xi)\,d\xi = 0\ , \text{ if } L+L'+L''=2p+1 \quad (6.25')$$

The second selection rule consists in relations (6.25) and (6.25'), which are based on the parity properties of the Legendre polynomials and functions:

Therefore, defining the coefficients $a_{LL'L''}$ and $a^m_{LL'L''}$ by means of ($L + L' + L'' = 2p$)

$$a_{LL'L''} = (-1)^{p+L'}(2L+1)(2L'+1)\int_{-1}^{1} P_L(\xi)\,P_{L'}(\xi)\,P_{L''}(\xi)\,d\xi \tag{6.26}$$

$$a^m_{LL'L''} = (-1)^{p+L'}\cdot 2\,\frac{(2L+1)(2L'+1)(L-m)!\,(L''-m)!}{(L+m)!\,(L''+m)!}\cdot\int_{-1}^{1} P_L^m(\xi)\,P_{L'}(\xi)\,P_{L''}^m(\xi)\,d\xi \tag{6.27}$$

and considering (6.20), (6.26) and (6.27) as well as the selections rules obtained above, one has the following expansions for the $C_{LL'L''}$s:

$$C_{LL'L''} = 2\pi\Big[a_{LL'L''}\,P_L(\cos\theta)\,P_{L''}(\cos\theta_K) + \sum_{m=1}^{\min(L,L'')} a^m_{LL'L''}\,P_L^m(\cos\theta)\,P_{L''}^m(\cos\theta_K)\cos m\phi_K\Big] \tag{6.28}$$

where min(L,L") represents the smallest of the integers L and L".

Finally, introducing the coefficients $A_{LL''}$ and $A^m_{LL''}$, given by the developments

$$A_{LL''} = \sum_{L'=2p_{min}-(L+L'')}^{\infty}{}' a_{LL'L''}\cdot I_{LL'L''} \tag{6.29}$$

and

$$A^m_{LL''} = \sum_{L'=2p_{min}-(L+L'')}^{\infty}{}' a^m_{LL'L''}\cdot I_{LL'L''} \tag{6.30}$$

where p_{min} represents the lowest positive integer such that $2p_{min} - (L + L'') \geq 0$ and Σ' means that L' increases by steps of two units, one gets, by (6.18), (6.28), (6.29) and (6.30), the following expansion for $(\phi_b, T\phi_a)$:

$$-\frac{2\mu}{4\pi\hbar^2}(\bar{\Phi}_b, T\,\bar{\Phi}_a) = \frac{1}{2\pi}\,\frac{\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^2}\cdot\Big[\sum_{i,j=0}^{\infty} A_{ij}\,P_i(\cos\theta)\,P_j(\cos\theta_K) + \sum_{i,j=1}^{\infty}\sum_{m=1}^{\min(i,j)} A^m_{ij}\,P^m_i(\cos\theta)\,P^m_j(\cos\theta_K)\cos m\phi_K\Big] \qquad (6.31)$$

where $i = L$, $j = L''$.

4. The behaviour of $I_{LL'L''}$ with increasing values of the indices

Consider the definition (6.20) of $I_{LL'L''}$. The main contribution for the integral giving this coefficient comes from the values of R belonging to the interval $0 \leq R < 1/\alpha$ (or, introducing the new variable $\rho = k_a R/2$, from the values of ρ satisfying the condition $0 \leq \rho < k_a/2\alpha$). This is so because the leading exponential $e^{-\alpha R}$ in (6.20) (by (3.26) and (3.27) one has $\alpha \ll \beta$) is vanishingly small when $R > 1/\alpha$ (or $\rho > k_a/2\alpha$).

However, it was proved in §7 of Chapter IV that the spherical Bessel function $j_{L'}(k_a R/2) = j_{L'}(\rho)$ is practically equal to zero for all values of ρ belonging to the interval $0 \leq \rho < \sqrt{L'_1 + 1\cdot 5}$ if $L' \geq L'_1 + 1\cdot 5$. Therefore, if one has

$$L'_1 + 1\cdot 5 > \left(\frac{k_a}{2\alpha}\right)^2 \qquad (6.32)$$

all the $I_{LL'L''}$ with $L' \geq L'_1 + 2$ are vanishingly small.

By means of the inequality (6.32) and the equation (4.90) relating k_a to p_{Lab}, it is now possible to calculate L'_1 for the extremes of the range covered in this work by the K^--Lab. momentum: 100 MeV/c $\leq p_{Lab} \leq 300$ MeV/c.

When $p_{Lab} = 100$ MeV/c one has $k_a/2\alpha = 0\cdot 86$ so that $L'_1 = 1$ and the $I_{LL'L''}$s with $L' \geq 3$ have very small modulus. But, if $p_{Lab} = 300$ MeV, one gets $k_a/2\alpha = 2\cdot 6$ and by (6.32) L'_1 cannot be less than 5.

However, in this instance the conclusion is not that the $I_{LL'L''}$s become vanishingly small only when L' is greater than or equal to 7. Actually, the $I_{LL'L''}$'s depend on the wave numbers k_b and K which are related to k_a by the energy conservation principle (6.8'). Therefore, if k_b is close to k_a, one has $K \simeq 0$ and, reciprocally, when K is nearly equal to its maximum value, K_{max} (see (6.9)), k_b is close to zero. But, by (6.11") the factor $\alpha^{*}_{L''}(KR)$ in (6.20) for small values of $\rho = KR$ behaves like the functions $U^{*L''}_{J}(KR)$ ($J = L''-1, L'', L''+1$) i.e., $\alpha^{*}_{L''}(KR) \sim (KR)^{L''}$ (See (3.38)); and, similarly, the behaviour of the spherical Bessel function $j_L(k_bR/2)$ in (6.20), for small values of the argument (i.e., k_b), is given by $j_L(k_bR/2) \sim (k_bR/2)^L$. Thus, one always has a small factor - $(KR)^{L''}$ or $(k_bR/2)^L$ - in the integrand of the $I_{LL'L''}$'s, so that these quantities can be neglected for $p_{Lab} = 300$ MeV/c (and, a fortiori, for $p_{Lab} < 300$ MeV/c) when one of the indices L, L' or L" exceeds 2, as the actual numerical calculation of the integral (6.20) clearly shows.

In such conditions the sum over L' in the series (6.29) and (6.30) can be stopped at L' = 2 and the sums over i and j in the development (6.31) at i = j = 2.

Finally, due to the orthogonality relations of the Legendre polynomials and functions, i.e.,

$$\int_{-1}^{1} P_L(\xi)\, P_{L''}(\xi)\, d\xi = \frac{2}{2L+1}\,\delta_{LL''}\,;\quad \int_{-1}^{1} P_L^m(\xi)\, P_{L''}^m(\xi)\, d\xi = \frac{2(L+m)!}{(2L+1)(L-m)!}\,\delta_{LL''} \tag{6.33}$$

the developments (6.29) for the $A_{LL''}$ coefficients reduce to one term only, if L or L" is equal to zero. Thus, the $A_{LL''}$ with $L, L'' < 2$ are exactly given by

$$A_{00} = a_{000} I_{000}$$

$$A_{01} = a_{011} I_{011} \qquad A_{10} = a_{110} I_{110}$$

$$A_{02} = a_{022} I_{022} \qquad A_{20} = a_{220} I_{220}$$

The following approximations were used for the remaining $A_{LL''}$ and $A^m_{LL''}$ with $L, L'' \leq 2$:

$$A_{11} \simeq a_{101} I_{101} + a_{121} I_{121} \qquad A_{21} \simeq a_{211} I_{211}$$

$$A_{12} \simeq a_{112} I_{112} \qquad A_{22} \simeq a_{202} I_{202}$$

$$A^1_{11} \simeq a^1_{101} I_{101} + a^1_{121} I_{121} \qquad A^1_{21} \simeq a^1_{211} I_{211}$$

$$A^1_{12} \simeq a^1_{112} I_{112} \qquad A^1_{22} \simeq a^1_{202} I_{202}$$

$$A^2_{22} \simeq a^2_{202} I_{202}$$

The approximations for A_{11} and A^1_{11} are very good because $a_{141} = a^1_{141} = 0$.

5. K^-d inelastic cross-sections

The study of the dependence of the $I_{LL'L''}$'s on the indices, carried out in the last paragraph, leads to the complete knowledge of the coefficients A_{ij} and A^m_{ij} in the development (6.31) for the transition matrix element $(\Phi_b, T\Phi_a)$. The K^-d inelastic differential cross-section $d\sigma_{in}/d\Omega$ can now be easily expressed in terms of the same coefficients and the scattering angle by means of the integral (Messiah, 1962, p.836)

$$\frac{d\sigma_{in}}{d\Omega} = C_0^2 \int \frac{k_b}{k_a} \left| \frac{\mu}{2\pi\hbar^2} (\Phi_b, T\Phi_a) \right|^2 d\bar{K} \tag{6.34}$$

where $d\bar{K}$ represents the element of volume in the np wave number space, i.e.,

$$d\bar{K} = K^2 dK \sin\theta_K \, d\theta_K \, d\phi_K \qquad (6.35)$$

and Co is the Coulomb penetration factor defined in (5.77') and introduced here as a correction to allow the inclusion of the electromagnetic interaction between the negative kaon and the proton (see Landau and Lifshitz, 1958, p.439).

The integration in (6.35) is to be performed over all directions of $\bar{K}$ and over the values of the $\bar{K}$-modulus belonging to the interval $\left[0, K_{max}\right]$, where K_{max} is given by equation (6.9). Therefore, according to the orthogonality relations (6.33) for Legendre polynomials and functions and writing

$$B_{ii'} = \frac{1}{k_a} \sum_{j=0}^{\infty} \frac{1}{2j+1} \int_0^{K_{max}} k_b K^2 A_{ij} A^*_{i'j} \, dK \qquad (6.36)$$

and

$$B^m_{ii'} = \frac{1}{k_a} \sum_{j=m}^{\infty} \frac{(j+m)!}{(2j+1)(j-m)!} \int_0^{K_{max}} k_b K^2 A^m_{ij} (A^m_{i'j})^* dK \qquad (6.37)$$

(by (6.31) one always has in (6.37) $1 \leq m \leq \min(i,i')$) , One gets the following expansion for $d\sigma_{in} / d\Omega$, if $(\phi_b, T\phi_a)$ is replaced in (6.34) by its development (6.31),

$$\frac{d\sigma_{in}}{d\Omega} = \frac{C_0^2}{\pi} \frac{\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^2} \left[\sum_{i,i'=0}^{\infty} B_{ii'} P_i(\cos\theta) P_{i'}(\cos\theta) + \right.$$
$$\left. + \sum_{i,i'=1}^{\infty} \sum_{m=1}^{\min(i,i')} B^m_{ii'} P^m_i(\cos\theta) P^m_{i'}(\cos\theta) \right] \qquad (6.38)$$

In the actual numerical calculations, the summations over the indices i,i' and j in (6.36), (6.37) and (6.38) are stopped at i = i' = j = 2, as explained in §4.

By integrating (6.38) over $d\Omega$, and using again the orthogonality relations (6.33), the total K^-d inelastic cross-section is given in this approximation by

$$\sigma_{in} = C_o^2 \frac{4\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^2}\Big(B_{oo} + \frac{1}{3}B_{1,1} + \frac{1}{5}B_{22} + \\ + \frac{2}{3}B^1_{11} + \frac{6}{5}B^1_{12} + \frac{24}{5}B^2_{22}\Big) \qquad (6.39)$$

6. K^-d charge-exchange scattering

The calculation of the cross-section for charge-exchange processes, given by equation (6.2), follows similar lines to those pursued for the treatment of the K^-d inelastic scattering in the previous paragraphs.

The K^-d charge-exchange matrix element will be represented in the subsequent pages by $(\phi^1_b, T^1\phi_a)$, where ϕ_a is the initial state wave function defined in (6.10), but ϕ^1_b, i.e., the wave function for the K^-d final state is now equal to

$$\Phi^1_b = e^{i\bar{k}_b\cdot\bar{r}} \cdot \frac{\phi_{\bar{K}}(\bar{R}) - \phi_K(-\bar{R})}{\sqrt{2}} \qquad (6.40)$$

Obviously, $\phi_K(\bar{R})$ in (6.40) is the np continuous state wave function given by (6.11) and (6.11'). In fact, as the total ordinary spin of the K^-d system is a constant of the motion, the final nn state arising from the charge-exchanges between the negative kaon and the proton, is in a NN triplet state. Therefore, ϕ^1_b must be anti-symmetric with respect to $\bar{R}$, so that the nn system obeys Pauli's principle.

As in the case of the evaluation of the inelastic transition matrix elements, it is also necessary to consider a model for the K^-d charge-exchange collisions.

It will be supposed that the kaon is scattered elastically in all single processes contributing to the multiple scattering terms belonging to the T-expansion (4.20), except for the last K^-p collision, where the charge-exchange process takes place. (In §9 of this chapter a more complete model for this process is used).

The formula equivalent to (6.12) is now

$$(\Phi'_b, T\,\Phi_a) = \int \frac{\phi^*_K(\bar{R}) - \phi^*_K(\bar{R})}{\sqrt{2}}\,\phi_0(R) < \bar{k}_b | T^1 | \bar{k}_a > d\bar{R} \tag{6.41}$$

where $<\bar{k}_b|T^1|\bar{k}_a>$ is given by (6.13), if the functions of $f^{(1)}(\theta,\bar{R})$ and $f^{(2)}(\theta,\bar{R})$ are redefined in the following way:

$$f^{(1)}(\theta,\bar{R}) = \gamma\, e^{-i(\bar{k}_a-\bar{k}_b).\bar{R}/2}\, f'_{31}$$

$$f^{(2)}(\theta,\bar{R}) = \frac{2\gamma^3}{1+\gamma^2}\, e^{i(\bar{k}_a+\bar{k}_b).\bar{R}/2}\, f'_{31}\, f_{32}\, \rho(R) \tag{6.42}$$

(6.43)

The scattering amplitudes f_{3i} (i = 1,2) in (6.43) as well as in the denominator of (6.13) are equal to those defined in (4.87') and (4.87) respectively and calculated at $p_a(=\sqrt{\frac{1+\gamma^2}{2\gamma^2}}\,.k_a)$, i.e., the K^-N wave number corresponding to K^-d elastic collisions. But f'_{31} in (6.42) and (6.43) represents now the charge-exchange amplitude of the system formed by the proton (particle 1) and the negative kaon (particle 3). Thus, f'_{31} can be expressed in terms of f^0 and f^1, i.e., the K^-p scattering amplitudes corresponding respectively to the isotopic-spin channels I = 0 and I = 1 defined in (4.86'). One has, therefore,

$$f'_{31} = \frac{1}{2}\left(\frac{A_1}{1-ip'A_1} - \frac{A_0}{1-ip'A_0}\right) \tag{6.44}$$

where p'_a is given by (6.7').

The terms proportional to $e^{i(\bar{k}_a-\bar{k}_b).\bar{R}/2}$ and $e^{-i(\bar{k}_a+\bar{k}_b).\bar{R}/2}$ which appear in (6.14) and (6.15) are missing in (6.42) and (6.43) because, according to the model adopted here for the charge-exchange scattering, they would lead to K^-n charge-exchange processes which are forbidden by the conservation of the total isotopic spin of the system formed by the negative kaon (particle 3) and the neutron (particle 1) (see Chapters I and II).

It is relatively simple at this stage to obtain the $(\phi^1_b, T^1\phi_a)$-expansion equivalent to that of $(\phi_b, T\phi_a)$ given in (6.31). Actually, if the functions $F_{LL'}(k_a,k_b,R)$ in the integrals (6.20) giving the $I_{LL'L''}$s are replaced by the expressions (with f'_{31} defined now as in (6.44)!)

$$F_{LL'}(k_a,k_b,R) = (-1)^{L+L'}\gamma\,\frac{f'_{31} + (-1)^{L'}\frac{2\gamma^2}{1+\gamma^2} f'_{31} f_{32} P(R)}{1-\left(\frac{2\gamma^2}{1+\gamma^2}\right)^2 f_{31} f_{32} P^2(R)} \tag{6.45}$$

in agreement with (6.13), (6.42) and (6.43) one gets, considering (6.41), the following development

$$-\frac{2\mu}{4\pi\hbar^2}\left(\Phi^1_b, T\Phi_a\right) = \frac{1}{\pi}\sqrt{\frac{\alpha\beta(\alpha+\beta)}{2(\beta-\alpha)^2}}\left[\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}{}' A_{ij} P_i(\cos\theta) P_j(\cos\theta_K)\right.$$
$$\left. + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty}{}' \sum_{m=1}^{\min(i,j)} A^m_{ij} P^m_i(\cos\theta) P^m_j(\cos\theta_K)\cos m\phi_K\right] \tag{6.46}$$

The summation over j(= L") starts at j = 1 and increases by steps of two units each time, because only the Legendre polynomials $P_{L''}(\cos_K)$ with odd L" appear in the development of the factor $[\phi_K(\bar{R}) - \phi_K(-\bar{R})]^*$ belonging to the integrand of (6.41) (see (6.11')). Such variation of j is indicated in (6.46) by $\sum_{j=1}^{\infty}{}'$.

Finally, assuming the same approximation for the K^-d charge-exchange cross-sections (differential and total) as the one made in the calculation of the K^-d inelastic cross-sections (the indices i and j are never greater than 2), one has

$$\frac{d\sigma_{c.e.}}{d\Omega} = C_0^2 \frac{2\alpha\beta(\alpha+\beta)}{\pi(\beta-\alpha)^2}\left[\sum_{i,i'=0}^{2} B_{ii'} P_i(\cos\theta) P_{i'}(\cos\theta) + \sum_{i,i'=1}^{1} B^1_{ii'} P^1_i(\cos\theta) P^1_{i}(\cos\theta)\right] \tag{6.47}$$

and

$$\sigma_{c.e.} = C_0^2 \frac{8\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^2}\left[B_{00} + \frac{1}{3}B_{11} + \frac{1}{5}B_{22} + \frac{2}{3}B^1_{11} + \frac{6}{5}B^1_{22}\right]. \tag{6.48}$$

where $B_{ii'}$ and $B^1_{ii'}$ are equal to the following integrals

$$B_{ii'} = \frac{1}{3k_a}\int_0^{K_{max}} k_b K^2 A_{i1} A^*_{i'1}\, dK \tag{6.36'}$$

and

$$B^1_{ii'} = \frac{1}{3k_a}\int_0^{K_{max}} k_b K^2 A^1_{i1} (A^1_{i'1})^* dK \tag{6.37'}$$

7. Correction of the mass difference Δ between the $\bar{K}^0 n$ and the $K^- p$ systems

The expressions of the $K^- d$ inelastic and charge-exchange cross-sections obtained in the previous paragraphs are based on the assumption that the particles belonging to the same doublet, either $(\bar{K}^0 K^-)$ of (pn), have equal masses. But this assumption is only approximately true, so that the $K^- d$ cross-sections at low kaon momenta depend on Δ.

Let m_i (i = 1, 2, 3, 4) be respectively the masses of the p, n, K^- and $\bar{K}^0$ particles; then Δ (in MeV) is equal to (see, for instance, Dalitz, 1962, p.73)

$$\Delta = \left[(m_2 + m_4) - (m_1 + m_2)\right] c^2 = 5.7 \text{ MeV} \qquad (6.49)$$

and the C.M. total energies of the $K^- p$ and $\bar{K}^0 n$ systems are given by (see (A.11))

$$E_{K^- p} = \sqrt{m_1^2 c^4 + \hbar^2 p_a^2 c^2} + \sqrt{m_3^2 c^4 + \hbar^2 p_a^2 c^2} \qquad (6.50)$$

and

$$E_{\bar{K}^0 n} = \sqrt{m_2^2 c^4 + \hbar^2 p_0^2 c^2} + \sqrt{m_4^2 c^4 + \hbar^2 p_0^2 c^2} \qquad (6.51)$$

Here p_a denotes as before the C.M. $K^- p$ wave number and p_o is the same variable for the $\bar{K}^0 n$ system.

When the $K^- p$ state goes into the $\bar{K}^0 n$ state one has $E_{\bar{K}^0 n} = E_{K^- p}$ and (6.50) and (6.51) lead to the approximate relation for low energies:

$$p_a^2 = p_0^2 + s^2 , \quad s = \sqrt{\frac{2\bar{\mu}}{\hbar^2}}\, \Delta = 0.307 \text{ fermi}^{-1} \qquad (6.52)$$

where $\bar{\mu}$ $\left[= m_1 m_2/(m_1 + m_2)\right]$ is the $K^- p$ reduced mass.

The effect of Δ on the K^-N S-wave scattering amplitudes f^I ($I = 0, 1$) defined in (4.86) is to mix the scattering lengths A_I ($I = 0, 1$) for each K^-p isotopic-spin channel in the following way (Dalitz, 1962, p.79):

$$f^0 = \frac{A_0(1 - ip_0A_1)}{D(p_a)} \tag{6.53}$$

and

$$f^1 = \frac{A_1(1 - ip_0A_0)}{D(p_a)} \tag{6.54}$$

where

$$D(p_a) = 1 - \frac{i}{2}(p_a + p_0)(A_0 + A_1) - p_a p_0 A_0 A_1 \tag{6.55}$$

and $+i|p_0|$ must replace p_0 when $p_a < \Delta$. In the limit $p_0 = p_a$, (6.53) and (6.54) reduce to the former values of f^0 and f^1 given in (4.86).

Thus, the relations (4.87') and (4.87) for f_{31} and f_{32}, as well as (6.44) for f'_{31} must be substituted respectively by

$$f_{31} = \frac{\frac{1}{2}(A_0 + A_1) - ip_0A_0A_1}{D(p_a)} \tag{6.56}$$

$$f_{32} = \frac{A_1(1 - ip_0A_0)}{D(p_a)} \tag{6.57}$$

and

$$f'_{31} = \frac{A_1 - A_0}{2D(p'_c)} \tag{6.58}$$

In the integrals (6.36') and (6.37') for K^-d charge-exchange processes, the values of K_{max} must also be corrected. To do this, the energy conservation principle equation (6.8) (or (6.8')) is considered again. The left-hand side of this equation is the sum of the two terms: a) the rest plus the C.M. kinetic energies of the K^-d system; and b) the energy of the deuteron's internal motion. Thus, one has in the notation of this paragraph

$$E_a = \sqrt{m_3^2 c^4 + \hbar^2 k_a^2 c^2} + \sqrt{(m_1+m_2)^2 c^4 + \hbar^2 k_a^2 c^2} + E_d \qquad (6.59)$$

But the right-hand side of (6.8) for K^-d charge-exchange processes is equal to the sum of c) the rest plus the kinetic energies of the $\bar{K}^0nn$ system plus d) the nn internal motion, i.e.

$$E_b = \sqrt{m_4^2 c^4 + \hbar^2 k_b^2 c^2} + \sqrt{(m_1+m_2)^2 c^4 + \hbar^2 k_b^2 c^2} + \frac{\hbar^2 K^2}{m_1} \qquad (6.59')$$

The equality of E_a and E_b gives approximately

$$k_a^2 + \frac{2\mu}{\hbar^2} E_d = k_b^2 + \frac{2\mu}{m_1 \hbar^2} + \frac{2\mu}{\hbar^2}\Delta \qquad (6.60)$$

or, considering that the K^-d and K^-N reduced masses μ and $\bar{\mu}$ are related by the equation $\mu = \gamma\bar{\mu}$ (see (4.50) for the definition of γ),

$$k_a^2 - 4(\gamma-1)\alpha^2 = k_b^2 + 4(\gamma-1)K^2 + \gamma\delta^2 \qquad (6.60')$$

Thus K_{max} is now given by

$$K^2_{max} = \frac{1}{4(\gamma-1)}\left[k_a^2 - 4(\gamma-1)\alpha^2 - \gamma\delta^2\right] \qquad (6.61)$$

These corrections (both for K^-d inelastic and charge-exchange processes) are incorporated in the numerical calculations of the coefficients (A_{ij}, A^m_{ij}) and (B_{ii}', B^m_{ii}') defined in the previous paragraphs. Such calculations as well as the evaluation of the differential and total cross-sections for these K^-d collisions have been carried out in the Mercury-Ferranti Computer belonging to the London University. The obtained results shown in Figs (VI.2) to (VI.5) are discussed in the next paragraph.

8. Analysis of the results and conclusions

When the present work was started, it was the author's intention to establish which solution belonging to the Ross-Humphrey's sets is physically acceptable in the light of the experimental data on K^-d processes at low energies.

Unhappily, this programme cannot be carried out to the end, because of two main objections: first, the available K^-d experimental data is very scant; secondly, quite recently it has been shown that none of the sets of scattering lengths found by Ross and Humphrey explain some features of the K^-p interactions.

The first objection is illustrated by one fact that the three experimental points of $\sigma_{el} + \sigma_{in}$ K^-d total cross-section (the experimental difficulties in separating the elastic from the inelastic processes lead to consider them together) given by Alvarez (1959) in his report on K^--meson in deuterium, are still the only available in the interval 0 to 300 MeV/c of K^--Lab. momentum (see Table (VI.1)).

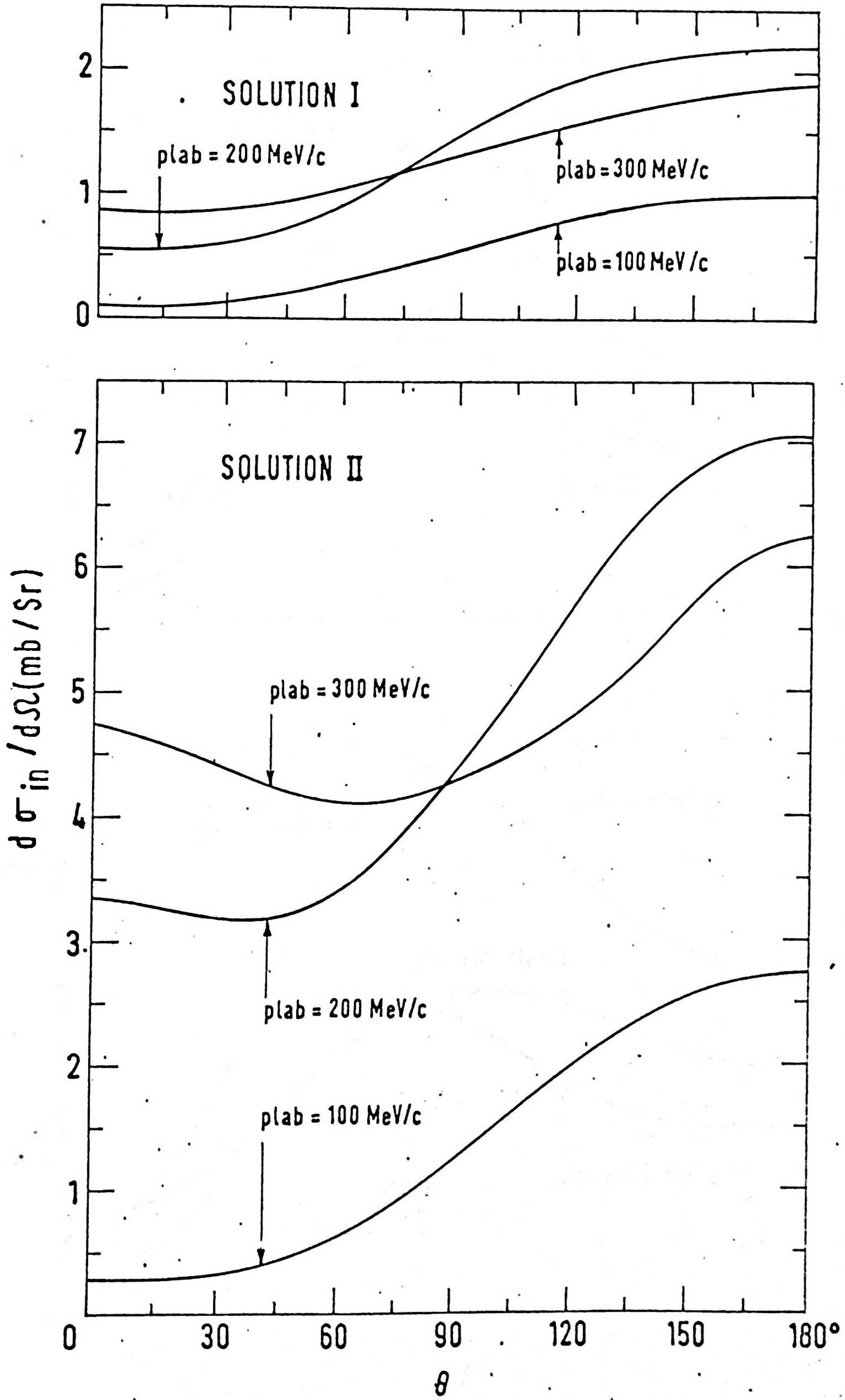

SOLUTION I
plab = 200 MeV/c
plab = 300 MeV/c
plab = 100 MeV/c
SOLUTION II
plab = 300 MeV/c
plab = 200 MeV/c
plab = 100 MeV/c
dσin / dΩ (mb / Sr)
0
1
2
3
4
5
6
7
0
30
60
90
120
150
180°
θ

FIG. VI. 2

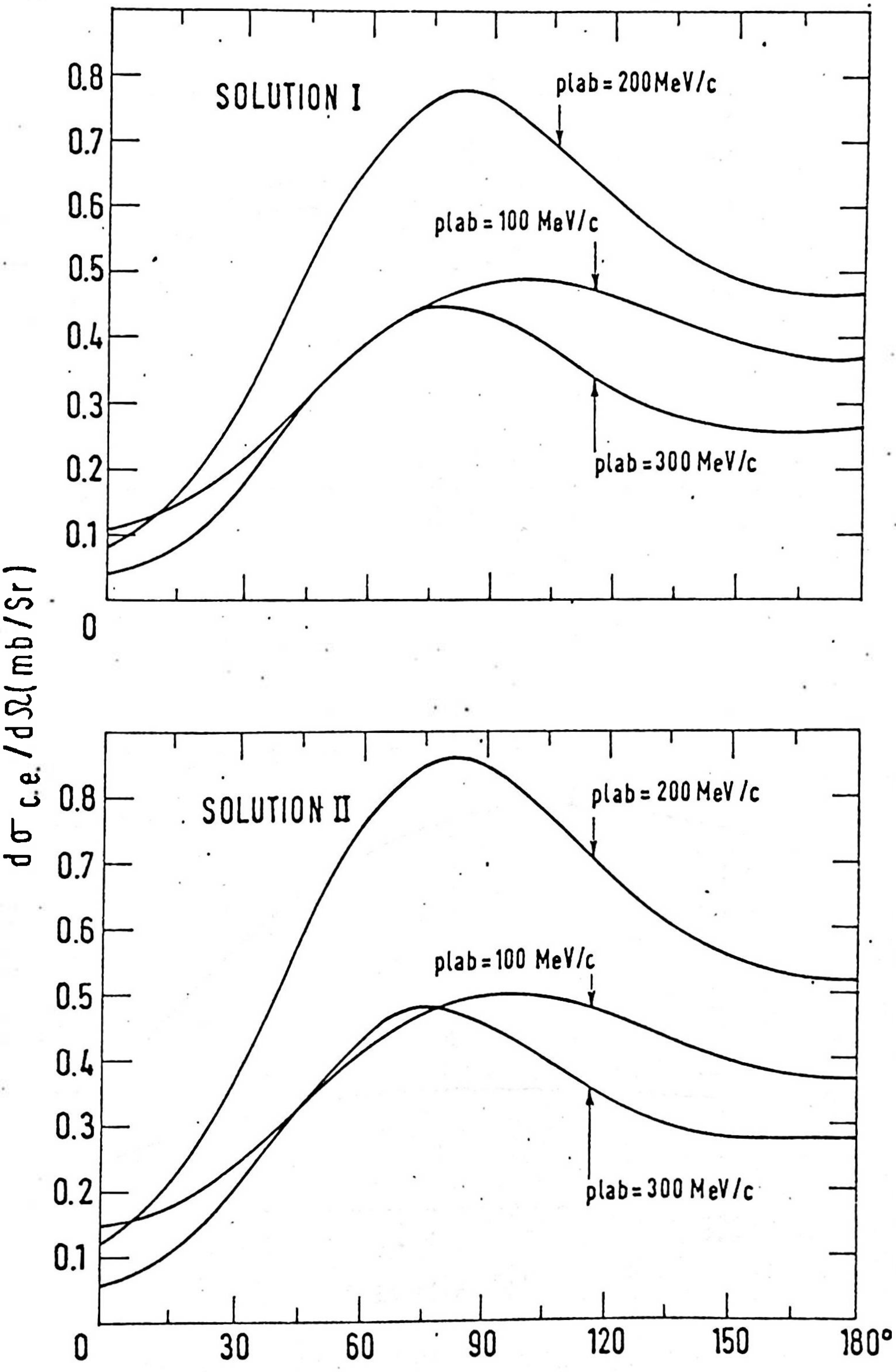
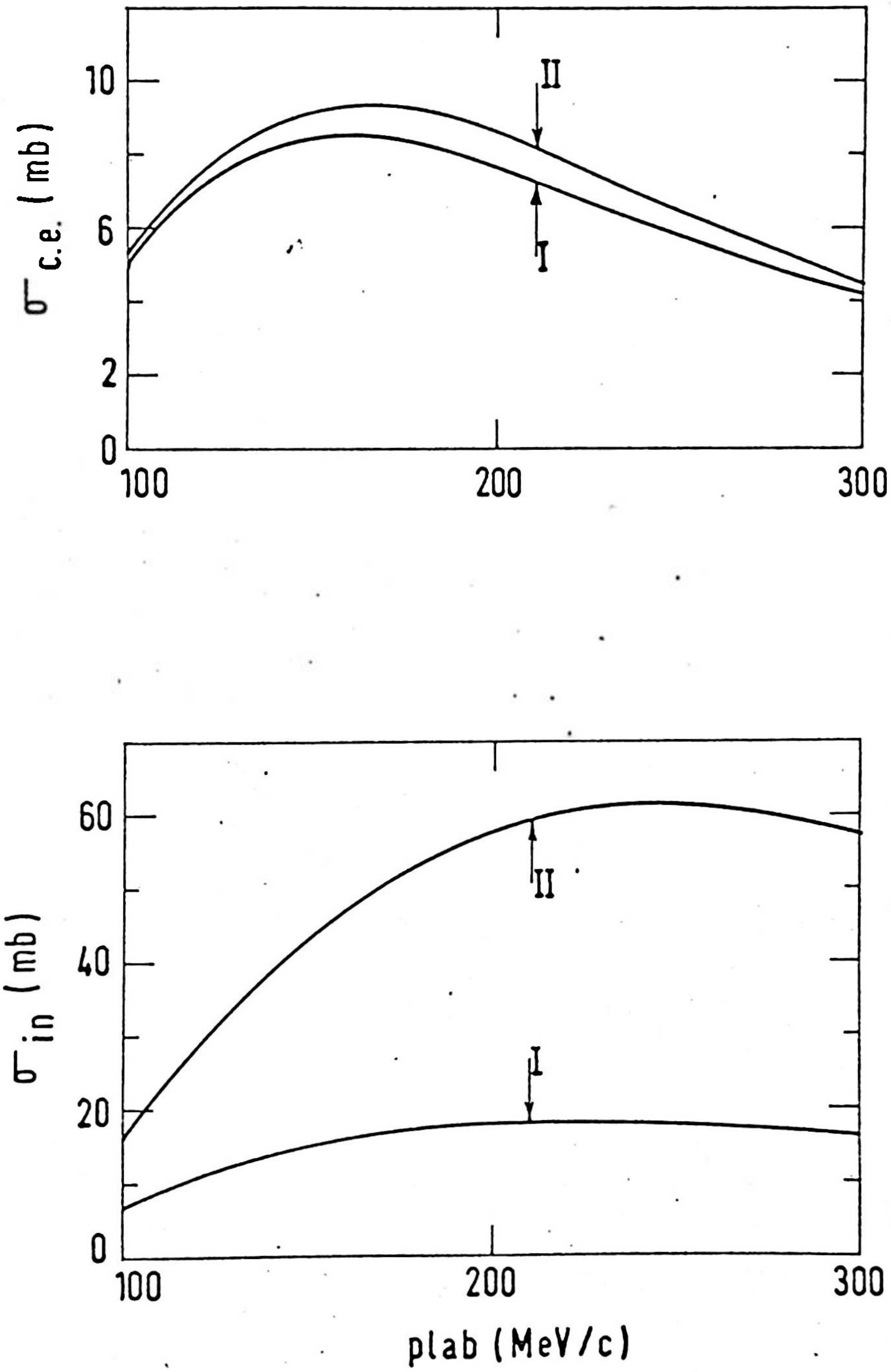

SOLUTION I
plab = 200MeV/c
plab = 100 MeV/c
plab = 300 MeV/c
SOLUTION II
plab = 200 MeV/c
plab = 100 MeV/c
plab = 300 MeV/c
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
30
60
90
120
150
180°
θ
dσ c.e / dΩ (mb/Sr)

FIG. VI. 3



10
6
2
0
σ c.e. (mb)
100
200
300
II
I
60
40
20
0
σ in (mb)
100
200
300
plab (MeV/c)
II
I

FIG. VI. 4

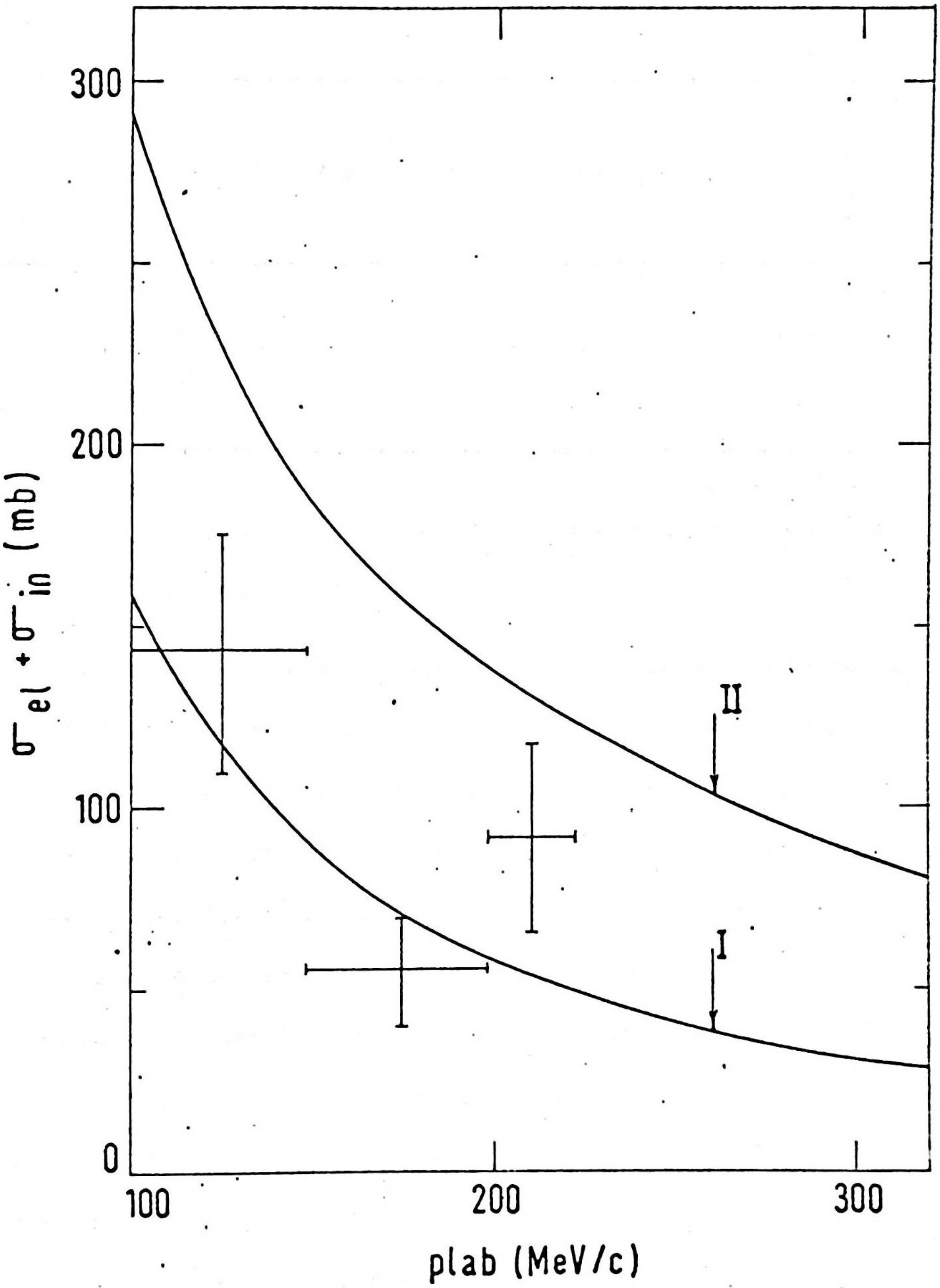


FIG. VI. 5

Table VI.1

$\sigma_{el} + \sigma_{in}$ K^-d experimental cross-sections

p_{Lab} (MeV/c)	125 ± 25	175 ± 25	210 ± 13
$\sigma_{el} + \sigma_{in}$ (mb)	145 ± 35	55 ± 15	99 ± 15

These points are plotted in Fig. (VI.5) and seem to favour Ross-Humphrey's solution I rather than solution II.

Another piece of experimental information can be put into terms of the general behaviour of the K^-d differential cross-sections: Dahl et al. (1960) showed that most of the K^- collisions in deuterium through Lab. angles greater than 45° ($\sim 54^\circ$ in C.M. scattering angle) are inelastic. Comparing the graphs of Fig. (VI.2) for K^-d inelastic processes with those of Fig. (V.3) and (V.4) for elastic scattering, the general trend observed by Dahl et al. is in better agreement (at least in the range 200 to 300 MeV/c of the K^- Lab momentum) with Ross-Humphrey's solution II, than with solution I.

Finally, the fractional absorption rates $R_\Sigma(\pi^+)$, $R_\Sigma(\pi^-)$ and $R_\Lambda(\pi^-)$ for the processes (Y stands for the hyperons Σ and Λ):

$$K^- + d \rightarrow \pi + Y + N$$

has been calculated for Ross-Humphrey's solution I and II at different values of K^--Lab. momentum in the interval (0, 300 MeV/c) by Chand and Dalitz (1962) and Chand (1963); the two solutions give approximately the same rates, so that no clear distinction can be drawn between them. It is interesting to note that the fitting of solution I at 300 MeV/c with experimental data is very good. At rest, however, the discrepancies between the calculated and the experimental absorption rates increase and are of the same order of magnitude for both Ross-Humphrey's sets of solutions.

The second objection, i.e., the inability of both Ross-Humphrey's solutions in reproducing some features of the K^-p interactions, is based on the following considerations:

a) Argument of Akiba and Capps (1962). To account for the interference of the K^-p S-waves amplitudes with the 395-MeV/c $D_{3/2}$ resonance, Tripp et al. (1962) were led to admit a negative phase difference $\phi = \phi_0 - \phi_1$ between the matrix elements for $\Sigma\pi$ production in the I = 0, 1 K^-p isotopic-spin channels.

Akiba and Capps argue that since no violent fluctuation of the Σ^-/Σ^+ ratio is observed between 175 MeV/c and 400 MeV/c, ϕ must be also negative below the former K^-p Lab. momentum. In these conditions, solution I is not acceptable and solution II is possible, because they have respectively positive and negative phase differences (see Table (II.1)).

b) The large difference in the $(\Sigma^-\pi^+)/(\Sigma^+\pi^-)$ ratio observed when stopped negative kaons are absorbed in hydrogen ($\sim$ 2) and in deuterium (1) lead Schult and Capps (1961, 1962) to assume that the $\langle \bar{K}N|T^{(0)}|\Sigma\pi\rangle$ transition matrix elements (see (1.11')) depend on the Σ,π energy. This hypothesis requires a negative ϕ and a $a_0 \lesssim -1{\cdot}3$ fermi (a_0 is the real part of the I = 0 K^-p

scattering length). Therefore, according to this assumption, solution II is also ruled out.

At the same time, Schult and Capps, although such an assumption was not necessary, explained this rapid energy dependence of $<\bar{K}N|T^{(0)}|\Sigma\pi>$ in terms of a S-wave $\Sigma\pi$ resonance of the Dolitz-Tuan type (Dalitz and Tuan, 1959) in the K^-p isotopic-spin channel I = 0 at an energy 0-20 MeV below the K^-p threshold. Later on (1962), the same authors identified the assumed resonance with the Y_0*-resonance found experimentaly by Alston et al. (Capps and Schult, 1962). The Y_0* can be interpreted as a K^-p bound state and this permitted Dalitz (1961) to derive its mass E_r and width Γ in terms of the A_0 (= $a_0 + ib_0$) scattering length. Dalitz used a linear approximation of the Breit and Wigner formula to the denominator of the I = 0 K^-p scattering amplitude. The result is, written in a system of units in which $\hbar = c = 1$,

$$E_r = m_K^2 + m_N^2 - (2\bar{\mu}a_0)^{-1}, \quad \Gamma = 2b_0/(\bar{\mu}|a_0|^3) \tag{6.62}$$

where $\bar{\mu}$ is the K^- reduced mass.

c) Solution II is also inconsistent with the data for K_2^0p interactions (Lauers et al., 1961). This happens because the large positive value of the real part of A_1 (= $a_1 + ib_1$). The discrepancy is reduced if a_1 is small and positive.

The contradictions between theory and experiment stated above lead to the conclusion that both Ross-Humphrey's sets of solutions are possibly inadequate.

In fact, quite recently, a systematic χ^2-search, similar to that carried out by Ross and Humphrey, was performed by Kim (1965) in an experimental sample ten times as large as that used by those authors in their analysis.

Kim's solution II has a very poor fit to the experimental results, so that it can be ruled out. However, solution I (see Table (VI.2) where the symbols have the same meaning as in Table (II.1)) agrees quite well (see Kim, 1965 and Burhop et al. 1965) with the K^-p experimental data for K^--Lab. momenta below 300 MeV/c. Furthermore, Table (VI.2) shows that the requisites formulated in a), b) and c) are satisfied by this solution.

Table VI.2

Kim's Solution I

a_0 (fermi)	b_0	a_1 (fermi)	b_1	ε	ϕ
-1·674 (±0·038)	0·722 (±0·040)	-0·003 (±0·058)	0·688 (±0·033)	0·318 (±0·021)	-53·8°

Finally Kim's solution I not only agrees with Sakitt's solution I (Sakitt, 1964) but also is a good approximation to the mass and. possibly, to the width of the Y_0* resonance. Actually, the relations (6.62) applied to this solution lead to

$$E_r = 1{\cdot}410 \pm 1{\cdot}0 \text{ MeV}, \qquad \Gamma = 37{\cdot}0 \pm 3{\cdot}2 \text{ MeV}$$

which should be compared with the experimental results (see Kim, 1965)

$$E_r = 1405 \text{ MeV}, \qquad \Gamma = 50 \text{ MeV or } 35 \pm 5 \text{ MeV}.$$

In the light of the previous discussion of the inadequacy of the Ross-Humphrey's sets of solutions, the practical value of the present work is a lot lessened, unless the calculations are repeated with Kim's solution I. However, from a theoretical point of view, something has been achieved if the results obtained by the author (heading A of Table (VI.3)) are compared with those worked out by Chand (1963) for the K^-d problem, using also the Ross-Humphrey's scattering lengths (heading C of the same Table).

The main differences between the two sets of values A and C, for each K^-d cross-section occur below the K^--Lab. momentum of 200 MeV/c and in the K^--Lab. momentum range 100 to 300 MeV/c for the σ_{in} cross-sections of solution I. The explanation of these discrepancies lies in the different approaches to the K^-d problem employed by Chand and the author.

In his work Chand uses the boundary condition model introduced by Jackson et al. (1958) for the discussion of $\bar{K}N$ scattering and developed by Dalitz and Tuan in many of their papers on the K^-p interactions. This method (Chand and Dalitz, 1962) insures time-reversal invariance and probability conservation of the total flux in all possible channels, i.e., the unitarity of the scattering matrix.

When the model is applied to the K^-d problem (Chand, 1963), the general properties satisfied by the boundary conditions lead to a scattering amplitude with single and multiple scattering terms. The propagator of these terms has the form e^{ik_aR}/R and include virtual charge-exchange processes arising from the $\bar{K}^0nn$ states. However the two scatterers (the nucleons) are treated as fixed centres.

In Chand's paper, the difference between σ_{tot}, calculated either as a sum $(\sigma_{el} + \sigma_{in} + \sigma_{c.e} + \sigma_{ab})$, or by means of the optical theorem, never exceeds

Table (VI.3)

K^-d cross-sections, expressed in mb

P_{Lab} (MeV/c)	σ_{el} A	σ_{el} C	σ_{ab} A $[\sigma_{inc}-(\sigma_{in}+\sigma_{c.e.})]$	σ_{ab} C	σ_{in} A	σ_{in} C	$\sigma_{c.e.}$ A	$\sigma_{c.e.}$ C	σ_{tot} A	σ_{tot} C
I{ 100	150·8	141·6	276·8	248·3	6·8	2·6	5·2	13·1	439·4	405·6
I{ 200	39·2	45·4	94·7	82·3	18·0	26·9	7·6	8·9	159·7	163·5
I{ 300	14·1	19·4	50·5	43·1	16·4	29·2	4·3	5·3	85·2	97·0
II{ 100	274·6	211·4	214·2	240·9	16·9	25·4	5·3	13·5	510·9	491·6
II{ 200	77·8	82·6	65·1	74·6	58·1	61·2	8·5	9·5	209·5	227·9
II{ 300	29·6	39·5	26·5	34·6	57·8	64·2	4·5	5·2	118·3	143·9

2mb in the K^--Lab. momentum range (100, 300 MeV/c). This is apparently a nice check of the probability conservation of the total flux.

However, Chand calculates all the non-elastic cross-sections, both differential and total, using the closure approximation. This method introduces an infinite number of nucleon-nucleon (or nucleon-hyperon) states in addition to those required by the energy conservation principle. The contribution of these additional states to the fractional cross-sections σ_{in}, $\sigma_{c.e}$ and σ_{ab} can be important at low energies, although the sum $\sigma_{in} + \sigma_{c.e} + \sigma_{ab}$ would not be affected, because the changes thus introduced cancel one another. But the ratios of σ_{in}, $\sigma_{c.e}$ and σ_{ab} to this sum are certainly altered.

A cancellation of this kind (see Table (VI.3)) seems to occur, for K^--Lab. momenta below 200 MeV/c, between the σ_{in} and the $\sigma_{c.e}$ calculated by C, if the values for the same cross-sections obtained by A are accepted as correct. In fact, the differences between the corresponding values of σ_{ab} in A and C are relatively small compared with those for σ_{in} and $\sigma_{c.e}$ and a whole argument can be developed in support of the correctness of the $\sigma_{c.e}$ as given in A.

It takes the following form: the graphs of Figs. (VI.3) and (VI.4) show that, in spite of the wide differences between the two sets of Ross-Humphrey's scattering lengths, $d\sigma_{c.e}/d\Omega$ and $\sigma_{c.e}$ are quite similar for solutions I and II, i.e., the K^-d charge-exchange processes are poorly sensitive to different sets of Dalitz scattering lengths. This observation is confirmed by the former calculations made by Day et al. (1960) who found respectively, for an old set of Dalitz solutions (a^+), (a^-), (b^+) and (b^-) the following values of $\sigma_{c.e.}$ at aK-Lab. momentum of 200 MeV/c: 8·1, 6·4, 5·5, 5·9 mb. These values are not only close to each other but also agree well with the corresponding A and C cross-sections in Table (VI.3). Therefore, one is allowed to consider the result $\sigma_{c.e.} \approx 4$mb at K^--Lab. momentum of 136 MeV/c obtained by Day et al. (1959), which favours the determination of $\sigma_{c.e.}$ made by A. In this paper and for the same K^--Lab momentum, Day et al. state also that the closure approximation overestimates in about 50% the value of $\sigma_{c.e.}$.

Therefore, it seems reasonable to expect that the K^-d charge-exchange cross-sections calculated in this work are more or less correct. It is interesting to note that $d\sigma_{c.e.}/d\Omega$ has a very neat peak at a C.M. scattering angle of about 80^o. ($\sim 67^o$ in the Lab. system).

The smallness of $d\sigma_{c.e}/d\Omega$ for $\theta = 0$ (see Fig. VI.3) confirms the orthogonality of the NN initial and final states wave functions used in this calculation. In fact, the transition matrix element (6.41) for K^-d charge-exchange interactions is nearly proportional to

$$\int \left[\phi^*_{\bar{K}}(\bar{R}) - \phi^*_{\bar{K}}(\bar{R}) \right] \phi_0(R)\, d\bar{R}$$

for the forward scattering, if multiple scattering terms are neglected.

The present work does not include the virtual charge-exchange processes considered by Chand. However, contrary to Chand's work, the two nucleons are supposed to be moving scattering centres, thus allowing for the deuteron's recoil.

It is not possible here to have a check of the probability conservation of the total flux, similar to the one used in Chand's paper, because the independent calculation of σ_{ab} in Chapter V is not sufficiently accurate. However, an indirect proof of this property is obtained in the next chapter. Actually, the treatment of the K^+d interactions given there is analogous to the previous chapters for K^-d collisions. Since the K^+d interactions are free from absorption processes in the kaon-Lab momentum range (100, 300 MeV/c) under inspection, the relation $\sigma_{inc} = \sigma_{in} + \sigma_{c.e}$ should hold in this case.

9. Supplement to Chapter VI: K^-d virtual charge-exchange scattering

It is shown in §1 of Chapter V that only the two isotopic-spin wave functions (5.2) and (5.3) can represent the K^-d system, because they are the only ones, among the eight possible eigenstates of three particles belonging to charge doublets, which have $J = 1/2$ and $J_z = -1/2$. Using an obvious notation, such functions can be written in the form

$$\rho_0 = \frac{1}{\sqrt{2}}(p_1 n_2 - n_1 p_2)K^- \tag{6.63}$$

$$\rho_1 = -\sqrt{\frac{2}{3}}\, n_1 n_2 \bar{K}^0 + \sqrt{\frac{1}{6}}\,(p_1 n_2 + n_1 p_2)K^- \tag{6.64}$$

The channel ρ_1 is a virtual state of the K^-d system, because not only both states, np and nn, are mixed in it, but also the NN functions belonging to ρ_1 have $I = 1$, in contrast with the deuteron's isotopic-spin $I = 0$.

Consider now the exact expansion (4.20) of the transition operator T: if the virtual state ρ_1 is to be included in K^-d collisions, the propagators of the kaon-waves, derived from the Green's operator Go, must have projections on both isotopic-spin channels ρ_0 and ρ_1.

However, in order to obtain a finite expression for T, calculated from (4.20) by Impulse Approximation (see Chapter IV), it is better to replace ρ_0 and ρ_1 by the orthonormal functions:

$$\Lambda_0 = \frac{1}{\sqrt{2}}(\rho_0 + \rho_1)\ ,\ \Lambda_1 = \frac{1}{\sqrt{2}}(\rho_0 - \rho_1) \qquad (6.65)$$

and make the assumption that a kaon-wave in one of these isotopic-spin channels does not interfere with the corresponding wave in the other. The plausibility of this hypothesis is discussed below.

Hence, using the operators t_i (i = 1, 2) and G_0, defined respectively in (4.17') and (4.59), the terms of the T-expansion (4.20) for K^-d elastic collisions are given approximately by (see §§2 and 4 of Chapter IV)

$$T_1 = \langle \rho_0 b | t_1 | a \rho_0 \rangle$$

$$T_{21} = \sum_{j=0}^{1} \underset{r,s}{S} \langle \rho_0 b | t_2 | r \Lambda_j \rangle \langle \Lambda_j r | G_0 | s \Lambda_j \rangle \langle \Lambda_j s | t_1 | a \rho_0 \rangle$$

$$T_{121} = \sum_{j=0}^{1} \underset{r,s,r',s'}{S} \langle \rho_0 b | t_1 | r \Lambda_j \rangle \langle \Lambda_j r | G_0 | s \Lambda_j \rangle \langle \Lambda_j s | t_2 | r' \Lambda_j \rangle \cdot \langle \Lambda_j r' | G_0 | s' \Lambda_j \rangle \cdot \langle \Lambda_j s' | t_1 | a \rho_0 \rangle$$

$$\cdots \qquad (6.66)$$

where a and b stand respectively for the initial (ϕ_a) and final (ϕ_b) K^-d-states and the other indices (r, s, r', s') for the normalised solutions χ_r, χ_s, $\chi_{r'}$ and $\chi_{s'}$ of equation (4.10'). According to the assumption made above no off-diagonal elements of the form

$$<\Lambda_j r \,|t_i|\, r'\Lambda_{j'}> \,,\; i = 0,1 \,,\; j \neq j' \tag{6.67}$$

appear in (6.66)

Supposing that the KN-waves are mainly S-waves and introducing again the hypothesis II (see §5 of Chapter IV) on the behaviour of the off-energy shell $\bar{K}N$-matrix elements, the T-terms T_1, T_{21}, T_{121}, etc., become

$$T_1 = <\rho_0 \bar{p}\,|t_{1,0}|\,\bar{p}\,\rho_0> \int |\phi_0(R)|^2 \, e^{i(\bar{k}_a - \bar{k}_b)\cdot \bar{R}/2} \, d\bar{R}$$

$$T_{21} = \frac{-4\bar{\mu}\gamma^2}{4\pi\hbar^2(1+\gamma^2)} \cdot \sum_{j=0}^{1} <\rho_0 q\,|t_{2,0}|\,q\Lambda_j><\Lambda_j p\,|t_{1,0}|\,p\,\rho_0> \cdot$$

$$\cdot \int |\phi_0(R)|^2 \, e^{-i(\bar{k}_a + \bar{k}_b)\cdot \bar{R}/2} \, P(R)\, d\bar{R} \tag{6.66'}$$

$$T_{121} = \left[\frac{4\bar{\mu}\gamma^2}{4\pi\hbar^2(1+\gamma^2)}\right]^2 \sum_{j=0}^{1} <\rho_0 p\,|t_{1,0}|\,p\Lambda_j><\Lambda_j q\,|t_{2,0}|\,q\Lambda_j> \cdot$$

$$\cdot <\Lambda_j p\,|t_{1,0}|\,p\,\rho_0> \cdot \int |\phi_0(R)|^2 \, e^{-i(\bar{k}_a - \bar{k}_b)\cdot \bar{R}/2} \, P^2(R)\, d\bar{R}.$$

where $\bar{\mu}$ is the kaon reduced mass in the $K^- N$ C.M. referential and P(R) is the propagator i sin(kR)/R.

To be consistent with the existence of the mixed np and nn states of Λ_j ($j = 0, 1$), it is necessary now to allow for $\bar{K}N$ and NN charge-exchange interactions. This is simply achieved by using $\bar{K}N$ potentials of the type defined in (2.3), i.e.,

$$V(r_i) = V^{(1)}(r_i) + V^{(2)}(r_i)P_{3i} \qquad (i = 1, 2) \tag{6.68}$$

where r_i represents the distance between the kaon (particle 3) and the i-nucleon and P_{3i} is the charge-exchange operator between the same particles. No special

form

needs to be assumed for the NN potentials because it does not appear explicitly in the Impulse Approximation formalism.

As it is explained in §1 of Chapter II the potentials (6.68) are subjected to the conditions

$$V^{(1)}(r_i) - V^{(2)}(r_i) = W_0(r_i) \qquad (i = 1, 2) \tag{6.69}$$

$$V^{(1)}(r_i) + V^{(2)}(r_i) = W_1(r_i) \qquad (i = 1, 2) \tag{6.70}$$

where W_0 and W_1 stand respectively for the interactions through the $\bar{K}N$ isotopic-spin channels I = 0 and I = 1. It is also indicated there that the effect of P_{3i} on the $\bar{K}N$-wave functions (2.5), which will be represented here by $\phi_I^{I_Z}(i,3)$, can be summarised as follows:

$$P_{3i}\,\phi_1^{I_z} = \phi_1^{I_z}, \; I_z = 1, 0, -1 \; ; \; P_{3i}\,\phi_0^0 = -\phi_0^0 \tag{6.71}$$

The expressions (6.66') for the T-terms show that one has to calculate three distinct types of $\bar{K}N$-matrix elements: $\langle \rho_0 p | t_{1,0} | p \rho_0 \rangle$, $\langle \rho_0 p | t_{1,0} | p \Lambda_j \rangle$ and $\langle \Lambda_j p | t_{1,0} | p \Lambda_j \rangle$.

To do this, one has to have ρ_0 and Λ_j written as linear combinations of $\phi_I^{I_Z}(i.3)$. For i=1 one gets:

$$\rho_0 = \frac{1}{2}(\phi_0^0 + \phi_1^0)n_2 - \frac{1}{\sqrt{2}}\phi_1^{-1} p_2 \tag{6.72}$$

$$\Lambda_j = \frac{\sqrt{2}}{4}\left[1 + (-1)^j\sqrt{3}\right]\phi_0^0 n_2 + \frac{\sqrt{2}}{4}\left[1 - (-1)^j\frac{\sqrt{3}}{3}\right]\phi_1^0 n_2 - \tag{6.73}$$
$$- \frac{1}{2}\left[1 - (-1)^j\frac{\sqrt{3}}{3}\right]\phi_1^{-1} p_2 \, , \; j = 0, 1$$

Consider now the evaluation of $\langle \rho_0 p | t_{1,0} | p \rho_0 \rangle$ for instance. If $\psi_0^0(r_1)$, $\psi_1^0(r_1)$ and $\psi_1^{-1}(r_1)$ represent respectively the $\bar{K}N$ total waves through the channels ϕ_0^0, ϕ_1^0 and ϕ_1^{-1} one has, by (6.68) → (6.72),

$$t_{1,0} | p \rho_0 \rangle = V | \frac{\eta_2}{2} (\phi_0^0 \psi_0^0 + \phi_1^0 \psi_1^0) - \frac{\xi_2}{\sqrt{2}} \phi_1^{-1} \psi_1^{-1} \rangle ;$$

hence

$$\langle \rho_0 p | t_{1,0} | p \rho_0 \rangle = \frac{1}{4} \langle p | W_0 \psi_0^0 \rangle + \frac{1}{4} \langle p | W_1 \psi_1^0 \rangle + \frac{1}{2} \langle p | W_1 \psi_1^{-1} \rangle \tag{6.74}$$

But $\langle p | W_1 \psi_1^0 \rangle = \langle p | W_1 \psi_1^{-1} \rangle$, because I remains unchanged in ψ_1^0 and ψ_1^{-1} and the interactions involved in K^-N collisions are strong. Thus, if the K^-N scattering amplitudes f^I (I = 0,1) are introduced, one gets

$$-\frac{2\bar{\mu}}{4\pi\hbar^2} \langle \rho_0 p | t_{1,0} | p \rho_0 \rangle = \frac{1}{4} (f^0 + 3 f^1) \tag{6.75}$$

Using the same procedure, one also has

$$-\frac{2\bar{\mu}}{4\pi\hbar^2} \langle \rho_0 p | t_{1,0} | p \Lambda_j \rangle = \frac{\sqrt{2}}{8} \left[f^0 + 3 f^1 - (-1)^j \sqrt{3} (f^1 - f^0) \right] \tag{6.76}$$

and

$$-\frac{2\bar{\mu}}{4\pi\hbar^2} \langle \Lambda_j p | t_{1,0} | p \Lambda_j \rangle = \frac{1}{2} (f^0 + f^1) - (-1)^j \frac{\sqrt{3}}{4} (f^1 - f^0) \tag{6.77}$$

The similar matrix elements, obtained from (6.75), (6.76) and (6.77) by inserting $t_{2,0}$ instead of $t_{1,0}$, or by interchanging ρ_0 and Λ_j, have the same values. Therefore, summing up all T-terms (6.66'), one gets the K^-d

elastic scattering amplitude $f(\theta)$ in the form (4.73), where $f(\theta,\bar{R})$ is now given by

$$f(\theta,\bar{R}) = f^{(1)}(\theta,\bar{R}) + \sum_{j=0}^{1} f_j^{(2)}(\theta,R)\, f_j^{(3)}(\theta,\bar{R}) \tag{6.78}$$

with

$$f^{(1)}(\theta,\bar{R}) = \frac{\gamma}{2}\left(f^{0}+3f^{1}\right)\cos\theta_1 ,$$

$$f_j^{(2)}(\theta,R) = \frac{\gamma^3 \rho(R)}{8(1+\gamma^2)}\left[f^{0}+3f^{1}-(-1)^j\frac{\sqrt{3}}{2}(f^{1}-f^{0})\right]^2 \tag{6.79}$$

and

$$f_j^{(3)}(\theta,\bar{R}) = \frac{\cos\theta_2 + \frac{\gamma^2}{1+\gamma^2}\left[f^{0}+f^{1}-(-1)^j\frac{\sqrt{3}}{2}(f^{1}-f^{0})\right]\rho(R)\cos\theta_1}{1-\left(\frac{\gamma^2}{1+\gamma^2}\right)^2\left[f^{0}+f^{1}-(-1)^j\frac{\sqrt{3}}{2}(f^{1}-f^{0})\right]^2\rho^2(R)}$$

Here θ_1 and θ_2 are respectively equal to $(\bar{k}_a - \bar{k}_b)\bar{R}/2$ and $(\bar{k}_a + \bar{k}_b)|\bar{R}|2$.

As stated above, the off-diagonal terms (6.67) are not included in $f(\theta,\bar{R})$. However, this approximation is reasonable, because the expressions for such terms, calculated at point $\bar{R} = 0$, i.e.

$$-\frac{2\bar{\mu}}{4\pi\hbar^2} < \Lambda j\, p \,|\, t_{i,0} \,|\, p\, \Lambda j' > = \frac{1}{4}(f^{1}-f^{0}),\ j \neq j' \tag{6.67'}$$

show that they are much smaller than the diagonal elements (6.76).

Virtual charge-exchange terms can also be introduced in K^-d inelastic and chargel-exchange scattering. In the calculations of these processes, the potentials (6.68) allow now the use of a more complete np triplet wave function than the one employed in §2 of this chapter - $\phi_{\bar{K}}(\bar{R})$. Such a function can be written in the form

$$\psi_{\bar{K}}(\bar{R}) = \xi_0^{\circ}\,\phi_{0,\bar{K}}(\bar{R}) + \xi_1^{\circ}\,\phi_{1,\bar{K}}(\bar{R}) \tag{6.80}$$

where

$$\phi_{I,\bar{K}}(\bar{R}) = \frac{1}{\sqrt{2}}\left[\phi_{\bar{K}}(\bar{R}) + (-1)^{I}\phi_{\bar{K}}(-\bar{R})\right] \tag{6.81}$$

and

$$\xi_I^{\circ} = \frac{1}{\sqrt{2}}\left[p_1 n_2 - (-1)^{I} n_1 p_2\right] \tag{6.81'}$$

for I = 0, 1.

Consider first the K^-d inelastic collisions. These processes can take place in two different isotopic spin channels, ρ_0 and ρ_1.

The transition matrix element $(\phi_b^0, T\phi_a)$ dealing with the K^-d inelastic scattering in channel ρ_0 can be written in the form

$$(\bar{\Phi}_b^{\circ}, T\bar{\Phi}_a) = \int \phi_{0,\bar{K}}^{*}(\bar{R})\,\phi_0(R) < \rho_0\bar{k}_b | T | \bar{k}_a \rho_0 > d\bar{R}\,, \tag{6.82}$$

$< \rho_0\bar{k}_b | T | \bar{k}_a \rho_0 >$ being given by

$$-\frac{2\mu}{4\pi\hbar^2} < \rho_0\bar{k}_b | T | \bar{k}_a \rho_0 > = f^{(1)}(\theta,\bar{R}) + \sum_{j=0}^{1} f_j^{(2)}(\theta,R)\, f_j^{(3)}(\theta,\bar{R}) \tag{6.83}$$

where $f_j^{(3)}(\theta,\bar{R})$ has the same value as in (6.79), but $f^{(1)}(\theta,\bar{R})$ and $f_j^{(2)}(\theta,R)$ are equal to

$$f^{(1)}(\theta,\bar{R}) = \frac{\gamma}{2}\left(\bar{f}^{\circ} + \bar{f}^{1}\right)\cos\theta_1$$

$$f_j^{(2)}(\theta,R) = \frac{\gamma^3 \rho(R)}{8(1+\gamma^2)}\left[\bar{f}^{\circ} + 3\bar{f}^{1} - (-1)^{j}\sqrt{3}\,(\bar{f}^{1} - \bar{f}^{\circ})\right]. \tag{6.84}$$

$$\cdot\left[f^{\circ} + 3f^{1} - (-1)^{j}\sqrt{3}\,(f^{1} - f^{\circ})\right]$$

Here, $\bar{f}^I$ $(I = 0, 1)$ represents $f^I(I = 0, 1)$ with p_a replaced by p'_a given by (6.7').

The transition matrix element $(\Phi^1_b, T\Phi_a)$ for K^-d collisions in channel ρ_1 is equal to

$$(\bar{\Phi}^1_b, T\,\bar{\Phi}_a) = \int \phi^*_{1,\bar{R}}(\bar{R})\,\phi_0(R) < \rho_1 \bar{k}_b | T | \bar{k}_a \rho_0 > d\bar{R}, \qquad (6.85)$$

where $<\rho_1\bar{k}_b|T|\bar{k}_a\rho_0>$ has a form similar to that of $<\rho_0\bar{k}_b|T|\bar{k}_a\rho_0>$ in (6.83), but with $f^{(1)}(\theta,\bar{R})$ and $f^{(2)}(\theta,R)$ given now by

$$f^{(1)}(\theta,\bar{R}) = -\frac{3\gamma}{2\sqrt{3}}\left(\bar{f}^1 - \bar{f}^0\right)\cos\theta_1$$

and

$$f^{(2)}(\theta,R) = -\frac{3\gamma^3 \rho(R)}{8\sqrt{3}(1+\gamma^2)}\left[\bar{f}^1 - \bar{f}^0 - (-1)^j\sqrt{3}\left(\bar{f}^1 + \tfrac{1}{3}\bar{f}^0\right)\right] \cdot \qquad (6.85)$$

$$\cdot\left[f^0 + 3f^1 - (-1)^j\sqrt{3}\left(f^1 - f^0\right)\right]$$

where $\bar{f}^{(I)}$ $(I = 0, 1)$ has the same meaning as before. The expressions (6.85) are derived from the two-body matrix elements

$$-\frac{2\bar{\mu}}{4\pi\hbar^2} < \rho_1 p | t_{1,0} | p\,\rho_0 > = -\frac{3}{4\sqrt{3}}\left(f^1 - f^0\right)$$

$$-\frac{2\bar{\mu}}{4\pi\hbar^2} < \rho_1 p | t_{1,0} | p\,\Lambda_j > = \frac{3}{4\sqrt{6}}\left[f^1 - f^0 - (-1)^j\sqrt{3}\left(f^0 + \tfrac{1}{3}f^1\right)\right] \qquad (6.86)$$

Therefore, according to the structure of ρ_1 (see (6.64)), one-third of the cross-section calculated from the matrix element $(\Phi^1_b, T\Phi_a)$ measures the amount of K^-d inelastic scattering in channel ρ_1 and two-thirds of the same cross-section is related to the K^-d charge-exchange scattering.

With respect to the formulation of the present theory see a quite recent paper by N.M. Queen (1965).

CHAPTER VII

K^+d Interactions

1 K^+N systems

The differences between K^+N and K^-N interactions are striking: K^+N collisions at low energies never lead to the copious proliferation of $\Sigma\pi$'s and $\Lambda\pi$'s as happens with the K^-p scattering (see equations (1.3) to (1.7) of Chapter I). Such a behaviour of the K^+-mesons, however, is completely explained in terms of the general conservation laws of strangeness and of baryon number.

As it is well-known (see, for instance, Dalitz, 1962, p.2), the strangeness s of any particle belonging to an isotopic-spin multiplet is twice the deviation of the multiplet's average centre of charge from the corresponding centre of a) the mucleon doublet, in the case of the baryons and b) the pion triplet, in the case of the mesons.

This definition assigns a strangeness +1 to the (K^+K^o) doublet, whereas the Λ-, Σ- and Ξ- multiplets have respectively s equal to -1, -1, -2. Therefore, no K^+N strong interaction, leading to $\Lambda\pi$, $\Sigma\pi$ or $\Xi\pi$ production, is possible. Nor is the production of $\bar{\Lambda}\pi$'s and $\bar{\Sigma}\pi$'s, either. Although $\bar{\Lambda}$ and $\bar{\Sigma}$ have s = 1, the conservation law of the baryon number B is violated in these reaction channels. In fact B is equal to +1 for the K^+N system, but $\bar{\Lambda}\pi$ and $\bar{\Sigma}\pi$ have B = -1.

Hence, at low energies the K^+N interactions are free from absorptive processes and should be possible to describe them in terms of two sets of the zero-effective range <u>real</u> parameters (a_I, r_I), related with the S-wave phase shifts δ^I by the equation

$$p \cot \delta^I = \frac{1}{a_I} + \frac{1}{2} r_I p^2 \tag{7.1}$$

where p represents as usual the kaon-wave number in the K^+N C.M. referential and I is the isotopic-spin.

The I_2- invariance (see Chapter I) applied to the K^+p system leads to the only possible total isotopic-spin channel

$$I_2 = 1 \ , \quad I = 1 \qquad \phi_1^1 = |K + p\rangle \tag{7.2}$$

This channel is associated with the reaction

$$K^+ + p \rightarrow K^+ + p$$

and the parameters (a_1, r_1). However, in the case of the K^+n system, the same invariance shows that two total isotopic-spin channels are available:

$$I_2 = 0 \ , \quad I = 0 \ , \quad \phi_0^0 = \frac{1}{\sqrt{2}}(K^+n\rangle - |K^o p\rangle) \tag{7.3}$$

$$I = 1, \quad \phi_1^0 = \frac{1}{\sqrt{2}}(|K^+n\rangle + |K^o p\rangle) \tag{7.4}$$

Obviously, ϕ_o^o is related vith the reaction

$$K^+ + n \rightarrow K^o + p$$

and the set (a_o, r_o), while ϕ_1^o represents the interaction

$$K^+ + n \rightarrow K^+ + n$$

described also by (a_1, r_1).

The numerical values of a_1 and r_1 are thoroughly determined. In fact, the best S-wave zero-effective range χ^2-fit (Goldhuber et al., 1962) to the experimental data on K^+p scattering in the kaon-Lab. momentum interval 140-642 MeV/c gives

$$a_o = -0{\cdot}29 \pm 0{\cdot}015 \text{ fermi}, \quad p_1 = 0{\cdot}5 \pm 0{\cdot}15 \text{ fermi} \qquad (7.5)$$

The same authors use also the zero range formula (I = 1)

$$\cot \delta^I = \frac{1}{a_I} ; \qquad (7.6)$$

the best χ^2-fit of (7.6) to the K^+p experimental data in the Lab momentum range 140-350 MeV/c is now

$$a_1 = -0{\cdot}3 \pm 0{\cdot}01 \text{ fermi} \qquad (7.7)$$

The K^+n potential in channel I = 0 seems to be very small and attractive, in contrast with the K^+p repulsive potential. The corresponding S-wave scattering length is not so well determined as a_1. Robdberg and Thaler (1960) give the following value

$$a_o = 0{\cdot}080 \pm 0{\cdot}068 \text{ fermi} \qquad (7.8)$$

Since the inaccuracy in the determination of a_o is large, the result (7.7) for a_1 will be used in this work as well as the zero-range approximation (7.8) in the I = 0 and I = 1 channels.

2. K^+d cross-sections

The formalism of Impulse Approximation developed in the previous chapters can be used here with minor alterations, if the K^+-Lab. momentum range is the same as in the K^-d problem: 100-300 MeV/c. Obviously, for higher momenta, relativistic effects should be considered, as well as higher phases in the NN wave functions.

a) K^+d elastic and incoherent scattering

The scattering amplitudes for f_{31} (p',p) and f_{32} (q',q) defined respectively in (4.70) and (4.70') are now given by the relations

$$f_{31} = \frac{a_1}{1 - ipa_1} \tag{7.9}$$

and

$$f_{32} = \frac{1}{2}\left(\frac{a_o}{1 - ipa_o} + \frac{a_1}{1 - ipa_1}\right) \tag{7.9'}$$

instead of the expressions (4.87') and (4.87).

Introducing (7.9) and (7.9') in (4.85) the η_L- parameters for K^+d collisions are calculated in the same way as for K^-d scattering.

The correction of Coulomb effects is also brought into the η_L's using the method developed in §7 of Chapter V. However, there is a slight difference: the quantity n appearing in the Coulomb penetration factor (5.77) and in $\Delta\eta_L$ (see (5.79)) has a positive value, because the Coulomb interaction is now repulsive.

The expressions (5.82), (5.85) and (5.87) are used again in the calculation of the K^+d $d\sigma_{el}/d\Omega$, σ_{el} and σ_{inc}, respectively.

b) K^+d inelastic scattering

The coefficients (A_{ij}, A^m_{ij}) and $B_{i,i'}$, $B^m_{i,i'}$) given by equations (6.29), (6.30), (6.36) and (6.37) are calculated in this case with the new expressions (7.9) and (7.9') for f_{3i} (I = 1,2), as well as for f'_{3i} (see §2 of Chapter VI). The Coulomb penetration factor in the expression (6.34) for $d\sigma_{in}/d\Omega$ is calculated considering n positive.

c) K^+d charge-exchange scattering

The alterations pointed out in b) must also be introduced in the evaluation of the K^+d charge-exchange cross-sections. However, a further correction is necessary in this case. Actually, the NN final state for these processes consists in two protons (see (7.3)), instead of two neutrons. Hence, the triplet phase shifts of the pp system (see §6 of Chapter III) are used in the approximate radial functions $U_J^L(KR)$ (see (3.37)) for two nucleons. Also, each $U_J^L(KR)$ should be multiplied by the Coulomb penetration factor Co defined in (3.48). The introduction of Co here is based on the similar correction expressed by formula (5.77).

d) The Δ-correction

Finally, the mass difference between the K^+p and K^on systems, Δ, is handled in the same way as for the K^-p and K^-n sets of particles (see §7 of Chapter VI). Only a slight alteration is needed in the energy conservation principle equation (6.6): Δ must be replaced by

$$\Delta' = \left[m(K^o) - m(K^+) + m(p) - m(n)\right]C^2 \simeq 2{\cdot}2 \text{ MeV} \qquad (7.10)$$

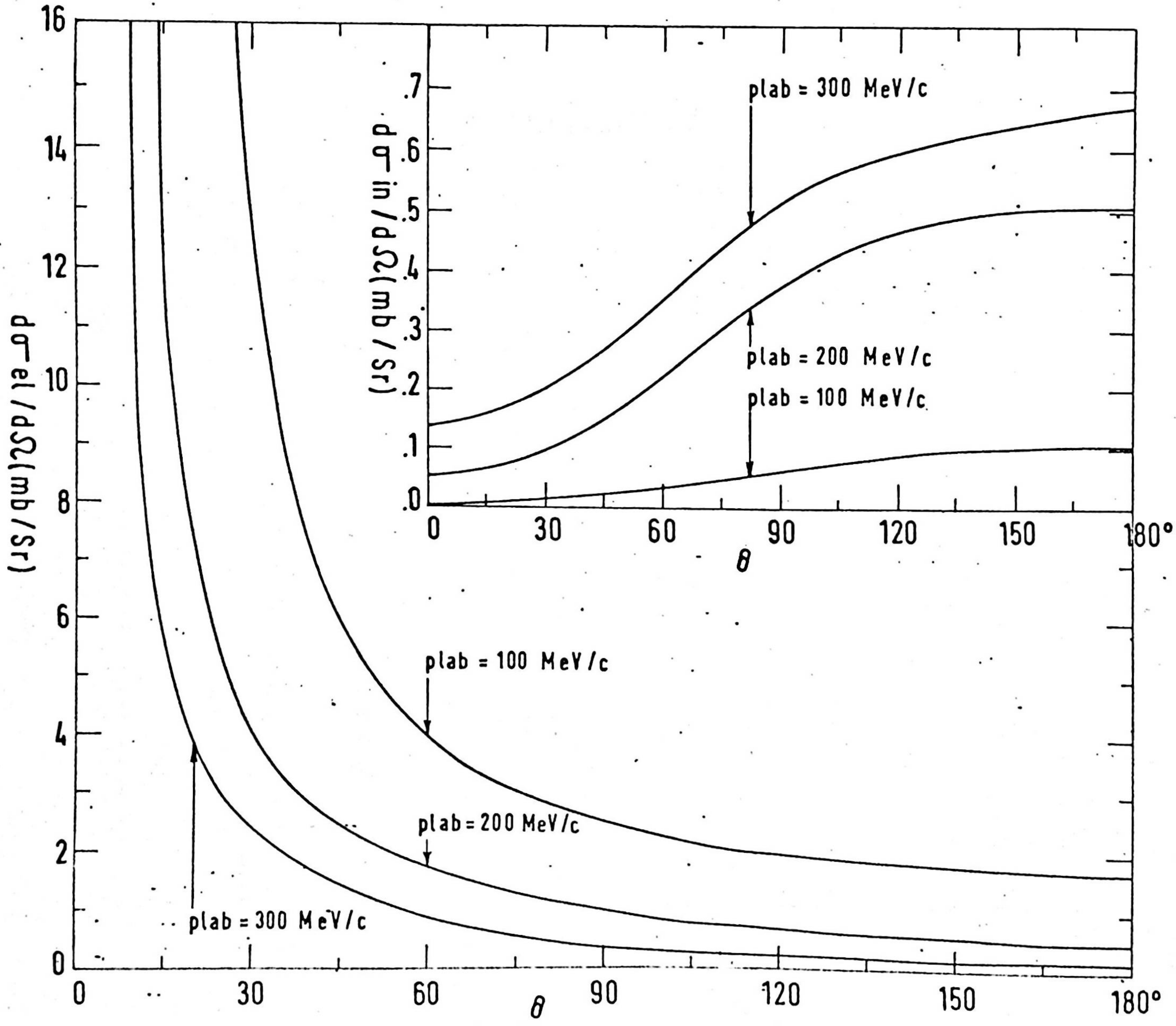
because the NN final state for K^+d charge-exchange scattering is formed by two protons instead of two neutrons as in the corresponding K^-d process.

3. Conclusions

Fig. (VII.1) to (VII.3) show the results of the numerical calculations of σ_{el}, σ_{in} and $\sigma_{c.e.}$, as well as the respective differential cross-sections for K^+d collisions in kaon-lab. momentum range 100-300 MeV/c.

The contribution coming from C.M. scattering angles less than 35^o have been neglected in the calculation of σ_{el}.

The K^+-differential charge-exchange cross-sections (Fig. (VII.2)) show sharp peaks around $\theta_{CM} = 85^o$, similar to those found for K^-d $d\sigma_{c.e.}/d\Omega$. This



dσ el / dΩ (mb/Sr)
dσ in / dΩ (mb/Sr)
plab = 300 MeV/c
plab = 200 MeV/c
plab = 100 MeV/c
plab = 100 MeV/c
plab = 200 MeV/c
plab = 300 MeV/c
θ
0
30
60
90
120
150
180°

FIG. VII.1

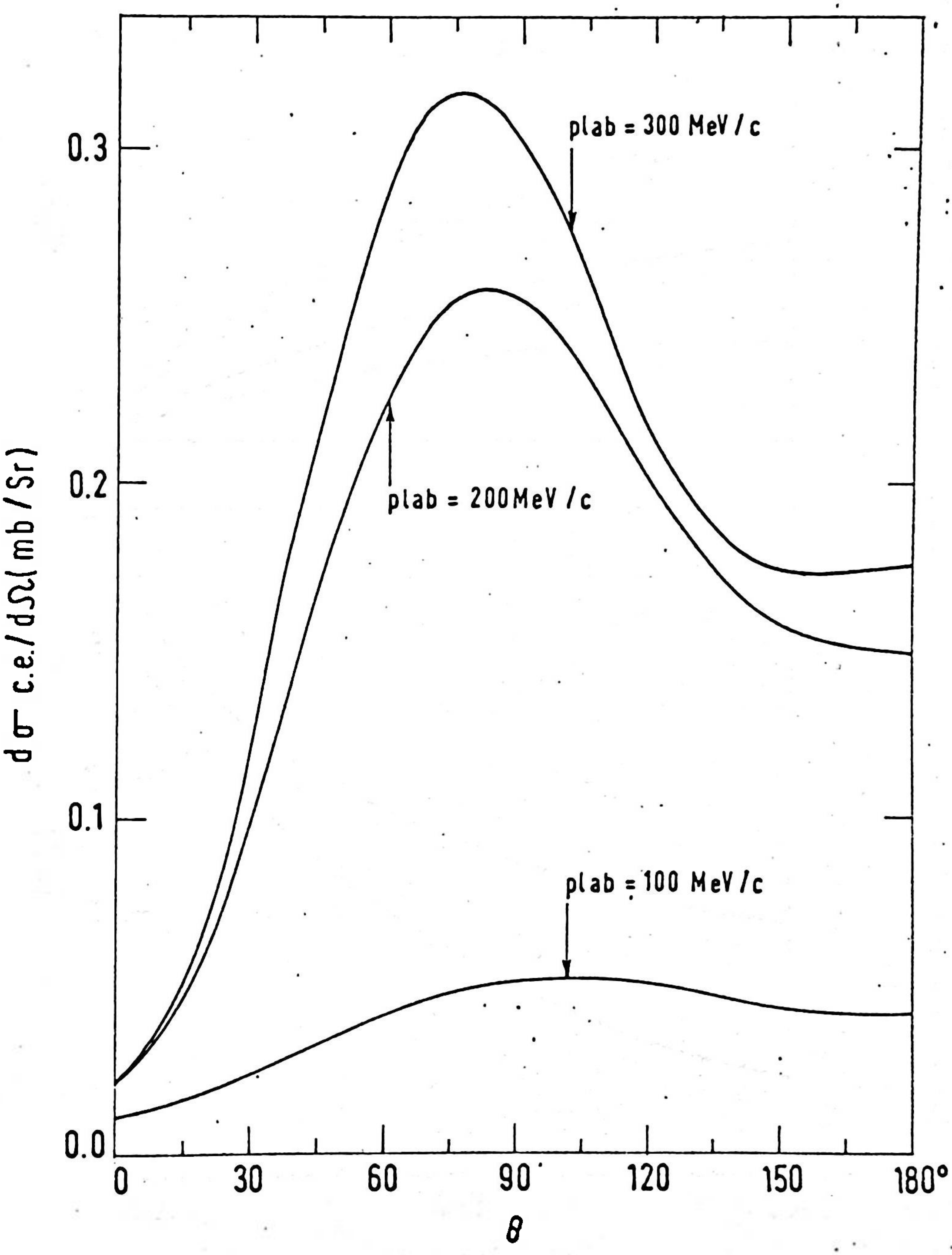

plab = 300 MeV / c
plab = 200MeV / c
plab = 100 MeV / c
0.3
0.2
0.1
0.0
dσ c.e./ dΩ (mb / Sr)
0
30
60
90
120
150
180°
θ

FIG. VII. 2

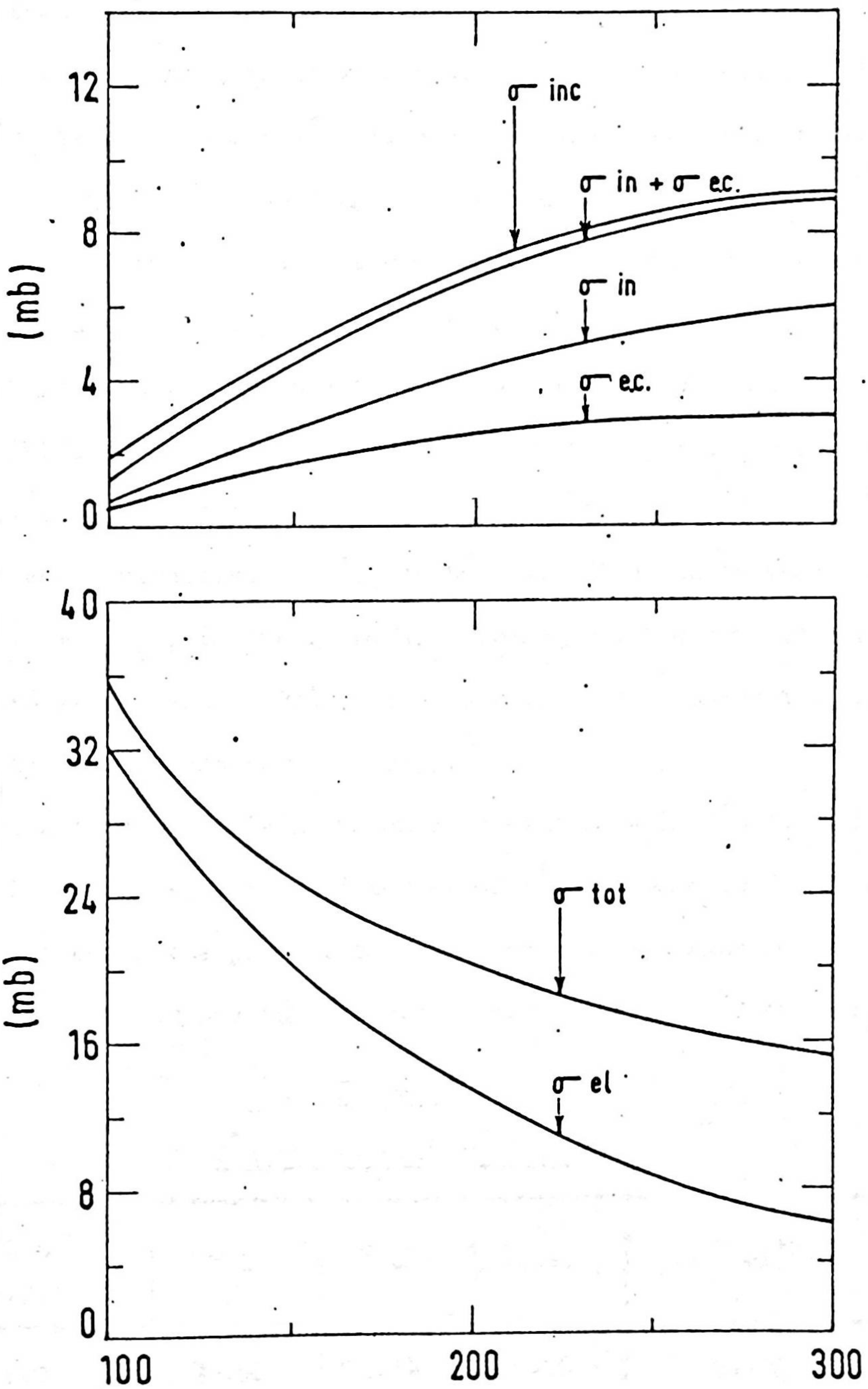

12
8
4
0
(mb)
σ inc
σ in + σ e.c.
σ in
σ e.c.
40
32
24
16
8
0
(mb)
σ tot
σ el
100
200
300
plab (MeV/c)

FIG. VII. 3

indicates that they are due mainly to the properties of the NN wave functions. Such a behaviour is maintained with increasing kaon-energies. The calculations made by Ferreira (1959) for K^+d differential charge-exchange cross-sections at a K^+-Lab energy of 100 MeV show again a peak around the same C.M. scattering angle. Since these calculations are carried out by Impulse Approximation combined with a closure approximation to sum over the pp final states, this result supports the choice made in this work for the pp wave functions.

Fig (VII.3) leads to the conclusiong that $\sigma_{inc} \simeq \sigma_{in} + \sigma_{c.e.}$. This relation is a good test of the probability conservation of total flux in K^+d collisions and indirectly, in K^-d scattering. The gap between the two curves, σ_{inc} and $\sigma_{in} + \sigma_{c.e.}$, is likely to be narrowed, if the damping parameter Z, set equal to 0·8 fermi (see (3.37)), is adjusted, by increasing its value, as discussed by Gourdin and Martin (1949).

To the author's knowledge no experimental data on K^+d collisions is available in the 100-300 MeV/c interval of K^+-Lab momentum. It is not possible, therefore, to check the present calculations with the experiment.

Table VII summarizes the numerical results of the K^+d cross-sections.

Table VII

K^+d cross-sections (in mb)

p_{Lab} (MeV/c)	σ_{el}	σ_{in}	$\sigma_{c.e.}$	$\sigma_{tot}=\sigma_{el}+\sigma_{inc}$
100	39·91	0·74	0·56	35·76
200	12·90	4·26	2·49	20·00
300	5·87	5·91	2·95	14·94

Appendix A

The Order of Magnitude of the Relativistic Effects in K^-N Interactions

A convenient criterion to estimate the relativistic deviation from classical laws in K^-N interactions is the knowledge of the order of magnitude of the relativistic corrections for the centre-of-mass momentum and energy of the K^-N system, calculated at 300 MeV/c, the largest K^--laboratory momentum considered in this work. The reason for choosing such a criterion is that the kinematics as well as the dynamics of the K^-N problem is not appreciably changed, if the relativistic corrections are small.

Consider a negative kaon with mass equal to m_K, moving freely with velocity $\underline{v}$ in an inertial system of reference (the Lab. system). Its relativistic momentum $\bar{p}$ and energy E_1 are given by

$$\bar{p} = \frac{m_K \bar{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad , \quad E_1 = \frac{m_K c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{A.1}$$

where $\underline{c}$ is the velocity of light.

From (A.1) two new expressions are easily derived: one is a scalar,

$$E_1 = p^2 c^2 + m_K^2 c^4 \tag{A.2}$$

and the other, the veotorial equation

$$\bar{p} = \frac{E_1}{c^2} \bar{v} \tag{A.3}$$

Consider now a nucleon with mass equal to m, at rest in the same inertial system. It has a zero momentum and a rest energy

$$E_2 = mC^2$$

From (A.3), the centre-of-mass velocity $\bar{V}$ of the K^-N system is

$$\bar{V} = \frac{C^2 \bar{p}}{E_1 + E_2} \tag{A.4}$$

because the total energy and momentum of this system is the sum of the partial energies and momenta of the two particles. $\bar{V}$ is also equal to the centre-of-mass velocity of the nucleon with negative sign. Thus, the centre-of-mass momentum $\bar{P}$ of this particle is given by

$$\bar{P} = \frac{-m\bar{V}}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{A.5}$$

But the total centre-of-mass momentum for the whole K^-N system vanishes, so that the kaon centre-of-mass momentum is equal to $-\bar{P}$.

Eliminating $\bar{V}$ between (A.4) and (A.5) one has

$$P^2 = \frac{m^2 p^2 c^2}{(E_1 + E_2)^2 - p^2 c^2}$$

or, from (A.2) and the equation $E_2 = mC^2$,

$$P^2 = \frac{p^2}{1 + \left(\frac{m_K}{m}\right)^2 + 2\sqrt{\left(\frac{m_K}{m}\right)^2 + \left(\frac{p}{m_K c}\right)^2}} \tag{A.6}$$

(A.6) gives the relation between the two K^--momenta p (in the Lab system) and $\underline{P}$ (in the C.M. system).

Developing now the square root in (A.6) into powers of C^{-2} one gets

$$\sqrt{\left(\frac{m_K}{m}\right)^2 + \left(\frac{p}{mc}\right)^2} = \frac{m_K}{m}\left[1 + \frac{1}{2}\left(\frac{p}{m_K c}\right)^2 + \dots\right]$$

so that

$$P^2 \simeq \frac{p^2}{\left(1 + \frac{m_K}{m}\right)^2 + \frac{m_K}{m}\left(\frac{p}{m_K c}\right)^2} \tag{A.7}$$

is the approximate expression for $\underline{P}$ up to the second power of C^{-1}.

Remembering that the classical expression equivalent to (A.6) is

$$P_{cl} = \frac{1}{1 + \frac{m_K}{m}} \qquad \text{(A.8)}$$

where P_{cl} represents the classical centre-of-mass K^--momentum, (A.7) gives

$$P \simeq P_{cl} \cdot \frac{1}{\sqrt{1 + \frac{1}{m m_K}\left(\frac{P_{cl}}{c}\right)^2}} \qquad \text{(A.9)}$$

The second term in the expansion of (A.9) into powers of C^{-2} is the relativistic correction $\Delta \underline{P}$ for P_{cl}:

$$\Delta P = -\frac{1}{2} \cdot \frac{P_{cl}}{m m_K}\left(\frac{P_{cl}}{c}\right)^2 \qquad \text{(A.10)}$$

Considering now the relativistic K^-N energy in the C.M. system one has

$$E = \sqrt{m_K^2 c^4 + P^2 c^2} + \sqrt{m^2 c^4 + P^2 c^2} \qquad \text{(A.11)}$$

Developing both square roots in E into powers of C^{-2} and introducing the reduced mass μ ($1/\mu = 1/m_K + 1/m$) of K^-, (A.11) can be written under the form

$$E = m_K c^2 + m c^2 + \frac{1}{2}\frac{P^2}{\mu}\left[1 - \frac{1}{4}\left(\frac{1}{\mu^2} - \frac{3}{m m_K}\right)\left(\frac{P}{c}\right)^2 + \cdots\right]$$

Expressing $\underline{P}$ in terms of P_{cl} by means of (A.9), developed into powers of C^{-2} up to the second order, E gives

$$E = m_K c^2 + m c^2 + \frac{1}{2}\frac{P_{cl}^2}{\mu}\left[1 - \frac{1}{4}\left(\frac{1}{\mu^2} + \frac{1}{m m_K}\right)\left(\frac{P_{cl}}{c}\right)^2\right]$$

The first term in this development represents the K^-N rest energy and, consequently, it is a constant which can be ignored. Therefore the relativistic correction for the K^-N energy in the C.M. system is given by

$$\Delta E = -\frac{1}{8}\frac{P_{cl}^2}{\mu}\left(\frac{1}{\mu^2}+\frac{1}{m m_K}\right)\left(\frac{P_{cl}}{c}\right)^2 \qquad \text{(A.12)}$$

Considering now that $m \simeq 2m_K$ and $m_K \simeq 500$ MeV, from (A.8), (A.10) and (A.12) one has for $p = 300$ MeV/c

$$P_{cl} \simeq 200 \text{ MeV/c}, \qquad \frac{1}{2}\frac{P_{cl}^2}{\mu} \simeq 60 \text{ MeV}$$

$$\Delta P \simeq -200 \times 0{\cdot}04 \simeq 8 \text{ MeV/c}$$

$$\Delta E \simeq -60 \times 0{\cdot}11 \simeq 6{\cdot}6 \text{ MeV}$$

The relativistic corrections $\Delta \underline{P}$ and ΔE amount respectively to 4% and 11% of the classical centre-of-mass momentum and energy of the K^-N system, for the largest K^--Laboratory momentum considered in this work (300 MeV/c).

$\Delta \underline{P}$ is quite small and therefore is negligible. At first sight however ΔE seems to be a larger effect. But if it is considered that

$$P^2 \simeq P_{cl}^2 + 2P_{cl}\Delta \underline{P},$$

so that the relativistic correction for P_{cl}^2 is $2\Delta \underline{P}|P_{cl}$ (= 8%) of P_{cl}^2, the ratio $2\mu\Delta E|P_{cl}$ (= 11%) is in good agreement with the 4% for $\Delta \underline{P}|P_{cl}$. It is then reasonable to ignore the relativistic effects for K^--laboratory momenta below 300 MeV/c.

Appendix B

The integrals of zero-energy variational formula for K^-N potentials

This Appendix is devoted to the calculation of the integrals appearing in Schwinger's variational formula of Chapter II.

The trial function is represented in the same notation as the one used in that Chapter:

$$u(r) = 1 - e^{-br} + \frac{r}{A} \tag{B.1}$$

i) Calculation of the integral

$$\int_0^\infty u(r)\, \frac{e^{-br}}{r}\, dr \int_0^r u(r')\, e^{-br'}\, dr' \tag{B.2}$$

Integrating $\int e^{-\alpha r} r\, dr$ by parts one has

$$\int e^{-\alpha r} r\, dr = -\frac{e^{-\alpha r}}{\alpha} r - \frac{1}{\alpha^2} e^{-\alpha r} \tag{B.3}$$

So that

$$\int_0^\infty e^{-\alpha r} dr = \alpha^{-1} \tag{B.4}$$

$$\int_0^\infty e^{-\alpha r} r\, dr = \alpha^{-2} \tag{B.5}$$

The following integral (See Courant 1947, Volume II, p.240):

$$\int_0^\infty \frac{e^{-\alpha r} - e^{-\beta r}}{r}\, dr = \log\frac{\beta}{\alpha} \quad \text{if } \beta > \alpha > 0, \tag{B.6}$$

is also necessary for the present computations.

To calculate (B.2) one starts with the integral

$$\int_{o}^{r}(1 - e^{-br'} + \frac{r'}{A})e^{-br'}dr'$$

using the integration by parts formula (B.3) its value is quickly obtained

$$\frac{1}{b}\left[\frac{1}{2}(1 - 2e^{-br} + e^{-2br}) + \frac{1}{bA}(1 - e^{-br}(br) - e^{-br})\right] \tag{B.7}$$

Introducing this result in (B.2) and putting $\rho = br$ and $S = bA$, this integral gives

$$\frac{1}{b}\int_{0}^{\infty}\left[\frac{1}{\rho}(e^{-\rho} - e^{-2\rho}) + S^{-1}e^{-\rho}\right]\left[\frac{1}{2}(1 - 2e^{-\rho} + e^{-2\rho}) + S^{-1}(1 - e^{-\rho}\rho - e^{-\rho})\right]d\rho$$

(B.8)

(B.8) can now be reduced to a linear combination of integrals belonging to the forms (B.4), (B.5) or (B.6):

$$\frac{1}{2}\int_{0}^{\infty}\frac{1}{\rho}(e^{-\rho} - e^{-2\rho})(1 - 2e^{-\rho} + e^{-2\rho})\,d\rho = \frac{1}{2}\log\frac{32}{27}$$

$$S^{1}\int_{0}^{\infty}\frac{1}{\rho}(e^{-\rho} - e^{-2\rho})(1 - e^{-\rho}\rho - e^{-\rho})\,d\rho = S^{-1}(\log\frac{4}{3} - \frac{1}{6})$$

$$\frac{S^{-1}}{2}\int_{0}^{\infty}e^{-\rho}(1 - 2e^{-\rho} + e^{-2\rho})\,d\rho = \frac{S^{-1}}{6}$$

$$S^{-2}\int_{0}^{\infty}e^{-\rho}(1 - e^{-\rho}\rho - e^{-\rho})\,d\rho = \frac{S^{-2}}{4}$$

Therefore, one gets for (B.2)

$$\int_0^\infty u(r)\frac{e^{-br}}{r}\,dr \int_0^r u(r')e^{-br'}\,dr' =$$

$$= \frac{1}{b}\left[\log\frac{32}{27} + \frac{1}{\delta}\log\frac{4}{3} + \frac{1}{\delta}\cdot\frac{1}{4}\right] \qquad (B.9)$$

ii) Calculation of the integral

$$\int_0^\infty u^2(r)\,\frac{e^{-br}}{r}\,dr \qquad (B.10)$$

Substitution of (B.1) in this integral gives, putting again ρ = br and δ = bA,

$$\int_0^\infty \frac{1}{\rho}\left[1 - 2e^{-\rho} + e^{-2\rho} + \frac{2}{\delta}(1 - e^{-\rho})\rho + \frac{1}{\delta^2}\rho^2\right]e^{-\rho}d\rho =$$

$$= \log\frac{4}{3} + \frac{1}{\delta} + \frac{1}{\delta^2}$$

Then

$$\int_0^\infty u^2(r)\frac{e^{-br}}{r}\,dr = \log\frac{4}{3} + \frac{1}{\delta} + \frac{1}{\delta^2} \qquad (B.11)$$

iii) Calculation of the integral

$$\int_0^\infty u(r)e^{-br}dr \qquad (B.12)$$

The result is

$$\frac{1}{b}\int_0^\infty (e^{-\rho} - e^{-2\rho} + \frac{e^{-\rho}}{\delta}\rho)\,d\rho = \frac{1}{b}(\frac{1}{2} + \frac{1}{\delta}) \qquad (B.13)$$

Appendix C

Spherical Coulomb and Free Particle Wave Functions

In this appendix some properties of the spherical Coulomb and free particle wave functions are derived from the Radial Schrodinger equations of which such functions are the integrals. The purpose in doing this is to establish by very simple methods expressions that represent these functions in a form suitable for numerical computation.

Consider the radial Schrodinger equation for a Coulomb L-wave function u_L:

$$\frac{d^2u_L}{dr^2} + \left[k^2 - \frac{L(L+1)}{r^2} - \frac{2\mu}{\hbar^2}\frac{\alpha}{r}\right]u_L = 0$$

where α is the product of the electric charges Ze and Z'e of the system under inspection. Then α is positive for a repulsive potential ($\alpha = e^2$ in K^+d system) and negative for an attractive potential ($\alpha = -e^2$ in K^-d system).

Putting

$$n = \frac{\mu\alpha}{\hbar^2 k} \tag{C.1}$$

and making the transformation $\rho = kr$, the radial L-wave Schrodinger equation gives

$$\frac{d^2U_L}{d\rho^2} + \left[1 - \frac{L(L+1)}{\rho^2} - \frac{2n}{\rho}\right]u_L = 0 \tag{C.2}$$

This second order differential equation has a regular singularity at the origin. It is a differential equation of the Fuchsian type and as such it has two linearly independent solutions around the origin: one regular, proportional to ρ^{L+1} for small ρ, the other, irregular, proportional to ρ^{-L} for small ρ. As it is well known (see for instance Albert Messiah, 1961, Volume I, Appendix B) by a convenient linear combination and normalization of these solutions two standard integrals of (C.2), the spherical Coulomb functions $F_L(\rho)$ and $G_L(\rho)$ are defined, with the following asymptotic forms

$$F_L(\rho) \underset{\rho \to \infty}{\sim} \sin(\rho - n \log 2\rho - \frac{\pi}{2}L + Z_L) \tag{C.3}$$

$$G_L(\rho) \underset{\rho \to \infty}{\sim} \cos(\rho - n \log 2\rho - \frac{\pi}{2}L + Z_L) \tag{C.4}$$

where

$$Z_L = \arg \Gamma(L + 1 + in)$$

is the pure Coulomb phase shift.

(C.3) corresponds to the regular solution of (C.2) at the origin and (C.4) to the irregular solution at the same point.

The knowledge of these asymptotic forms is not sufficient to solve a real nuclear scattering problem. Such problem generally leads to the numerical calculation of the regular solution of equation (C.2) with a short range potential added to the coefficient of u_L. Then, if a numerical integration is carried out up to a point ρ_0 in the region where the nuclear potential becomes vanishingly small, the solution of the radial equation beyond ρ_0 is a linear combination of $F_L(\rho)$ and $G_L(\rho)$. To determine accurately the coefficients of this linear combination it is necessary to have a better approximation of the asymptotic behaviour of $F_L(\rho)$ and $G_L(\rho)$ for smaller values of ρ than those for which (C.3) and (C.4) become valid.

Considering that (C.3) and (C.4) are linear combinations of $e^{i\theta}$ and $e^{-i\theta}$

with $\Theta = \rho - n \log 2\rho$, such approximation can be derived from the asymptotic behaviour of the function V_L defined by $u_L = e^{i\Theta}V_L$. Making this transformation in (C.2) and putting $m = L + \frac{1}{2}$ one obtains the differential equation satisfied by V_L:

$$\frac{d^2V_L}{d\rho^2} + 2i(1 - \frac{n}{\rho})\frac{dV_L}{d\rho} - \frac{m^2 - (in + \frac{1}{2})^2}{\rho^2}V_L = 0 \qquad (C.5)$$

This equation has an irregular singular point at infinity. It is possible, then, to translate the asymptotic behaviour of V_L by means of an asymptotic series. Therefore, writing

$$V_L = \sum_{s=0}^{\infty} C_s \rho^{-s}$$

one obtains

$$\frac{dV_L}{d\rho} = -\rho^{-1}\sum C_s s\rho^{-s}, \qquad \frac{d^2V_L}{d\rho^2} = \rho^{-2}\sum C_s s(s+1)\rho^{-s}$$

and substituting these expressions in (C.5) and putting equal to zero the coefficients of the successive powers of $1/\rho$ one gets the recurrence relations for the C_s:

$$2i(s + 1)C_{s+1} + \left[m^2 - (in + s + \tfrac{1}{2})^2\right]C_s = 0 \qquad (C.6)$$

Putting $C_o = 1$, (C.6) leads now to the asymptotic series for V_L (see Whittaker and Watson, 1950, page 342):

$$V_L = 1 + \sum_{s=1}^{\infty} \frac{[m^2 - (in + \frac{1}{2})^2][m^2 - (in + \frac{3}{2})^2] \cdots [m^2 - (in + s - \frac{1}{2})^2]}{s!\,(2i\rho)^s}$$

$$m = L + \tfrac{1}{2} \qquad (C.7)$$

Writing now

$$V_L = C_L + iS_L$$

where C_L = Real $\left[V_L\right]$ and $S_L = I_m \left[V_L\right]$ and using the normalisation constant $e^{i(-L\frac{\pi}{2} + Z_2)}$ for u_L one has, putting $\Theta_L = \rho - n \log 2\rho - L\frac{\pi}{2} + Z_L$,

$$u_L(\rho) = e^{i\Theta_L} V_L$$

or

$$u_L = \left[C_L \cos \Theta_L - S_L \sin \Theta_L\right] + i\left[C_L \sin \Theta_L + S_L \cos \Theta_L\right]$$

so that the improved asymptotic forms of $F_L(\rho)$ and $G_L(\rho)$ are

$$F_L(\rho) \underset{\rho \to \infty}{\sim} C_L \sin \Theta_L + S_L \cos \Theta_L \tag{C.8}$$

$$G_L(\rho) \underset{\rho \to \infty}{\sim} C_L \cos \Theta_L - S_L \sin \Theta_L \tag{C.9}$$

A very important property of $F_L(\rho)$ and $G_L(\rho)$ is the Wronskian relation. Considering that these functions are both solutions of (C.2) one has, putting $g(\rho) = 1 - \frac{L(L+1)}{\rho} - \frac{2n}{\rho}$,

$$\frac{d^2 F_L}{d\rho^2} + g(\rho) F_L = 0$$

$$\frac{d^2 G_L}{d\rho^2} + g(\rho) G_L = 0$$

Multiplying the first equation by G_L and the second by F_L and subtracting the results one gets

$$G_L \frac{d^2 F_L}{d\rho^2} - F_L \frac{d^2 G_L}{d\rho^2} = 0 \tag{C.10}$$

But the first derivative of the function

$$f(\rho) \equiv G_L \frac{dF_L(\rho)}{d\rho} - F_L \frac{dG_L}{d\rho}$$

is equal to (C.10). Then $\frac{df}{d\rho} = 0$ or

$$f(\rho) = \text{const for any value of } \rho$$

Therefore the constant can be calculated from the analytical expressions (C.3) and (C.4) of $F_L(\rho)$ and $G_L(\rho)$ for large values of ρ:

$$F_L(\rho) = \sin\Theta_L \qquad \frac{dF_L}{d\rho} = \cos\Theta_L$$

$$G_L(\rho) = \cos\Theta_L \qquad \frac{dG_L}{d\rho} = -\sin\Theta_L$$

because $\frac{d\Theta}{d\rho} = 1$ when $\rho \to \infty$. Then

$$f(\rho) = \cos^2\Theta_L + \sin^2\Theta_L = 1$$

or

$$G_L \frac{dF_L}{d\rho} - F_L \frac{dG_L}{d\rho} = 1 \tag{C.11}$$

(C.11) is the Wronskian condition.

The evaluation of the pure Coulomb phase shifts Z_L is derived from the Weierstrass's definition of the gamma function (see Courant, 1948, Volume II, page 233):

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{s=1}^{\infty} \left(1 + \frac{z}{s}\right) e^{-\frac{z}{s}} \tag{C.12}$$

where γ is the Euler's constant ($\gamma = 0.5772 \ldots$)

But $Z_o = \arg \Gamma(1 + in)$ or

$$Z_o = -\arg \left[\Gamma(1 + in)\right]^{-1} \tag{C.13}$$

and the arguments of the factors in (C.12) are

$$\arg (1 + in) = \mathrm{Tan}^{-1} n \,, \qquad \arg \left[e^{\gamma(1+in)}\right] = n\gamma$$

$$\arg \left(1 + \frac{1}{\delta} + i\frac{n}{\delta}\right) = \mathrm{Tan}^{-1} \frac{n}{1+\delta} \,, \qquad \arg \left[e^{-(\frac{1}{\delta}+i\frac{n}{\delta})}\right] = -\frac{n}{\delta}$$

Therefore, from (C.12) and (C.13) one gets

$$Z_o = -n\gamma + \sum_{\delta=1}^{\infty} \left(\frac{n}{\delta} - \tan^{-1} \frac{n}{\delta}\right) \tag{C.14}$$

The calculation of the Z_L for $L > 0$ is now worked out by means of a recurrence relation based on the gamma function property $\Gamma(Z+1) = Z\Gamma(Z)$. Then, from

$$Z_L = \arg \Gamma(L + 1 + in)$$

one has

$$Z_L = Z_{L-1} + \mathrm{Tan}^{-1} \frac{n}{L} \tag{C.15}$$

The precedent considerations make the study of the spherical functions for a free particle very easy. In fact such functions are special cases of spherical Coulomb functions for which $n \to 0$.

Therefore, putting $n = 0$ in (C.2) one obtains the differential equations for the spherical free-particle wave functions:

$$\frac{d^2 u_L}{d\rho^2} + \left[1 - \frac{L(L+1)}{\rho^2}\right] u_L = 0 \tag{C.16}$$

The regular and irregular solutions at the origin for this equation are respectively the spherical Bessel function $j_L(\rho)$ multiplied by ρ and the sperical Neumann function $n_L(\rho)$ multiplied by ρ.

The asymptotic forms of $j_L(\rho)$ and $n_L(\rho)$ are obtained directly from (C.3) and (C.4) putting n = 0 (then $Z_L = 0$ from (C.14) and (C.15)):

$$j_L(\rho) \underset{\rho\to\infty}{\sim} \frac{1}{\rho}\sin(\rho - L\frac{\pi}{2}) \tag{C.17}$$

$$n_L(\rho) \underset{\rho\to\infty}{\sim} \frac{1}{\rho}\cos(\rho - L\frac{\pi}{2}) \tag{C.18}$$

If n vanishes in the asymptotic expansion (C.7), the series now terminates and (C.8) and (C.9) represent in this special case the exact $\rho j_L(\rho)$ and $\rho n_L(\rho)$:

$$j_L(\rho) = \frac{1}{\rho}\left\{C_L \sin(\rho - L\frac{\pi}{2}) + S_L \cos(\rho - L\frac{\pi}{2})\right\} \tag{C.19}$$

$$n_L(\rho) = \frac{1}{\rho}\left\{C_L \cos(\rho - L\frac{\pi}{2}) - S_L \sin(\rho - L\frac{\pi}{2})\right\} \tag{C.20}$$

where C_L and $_L$ are, for the first four values of L:

$$C_0 = 1 \qquad S_0 = 0$$

$$C_1 = 1 \qquad S_1 = \frac{1}{\rho}$$

$$C_2 = 1 - \frac{3}{\rho^2} \qquad S_2 = \frac{3}{\rho}$$

$$C_3 = 1 - \frac{15}{\rho^2} \qquad S_3 = \frac{6}{\rho} - \frac{15}{\rho^3}$$

The spherical Bessel and Neumann functions, defined by (C.19) and (C.20), can be also expressed in terms of Bessel functions of half-an-integer exponent:

$$j_l(\rho) = \sqrt{\frac{\pi}{2\rho}}\, J_{l+\frac{1}{2}}(\rho) \tag{C.21}$$

$$n_l(\rho) = (-1)^l \sqrt{\frac{\pi}{2\rho}}\, J_{-(l+\frac{1}{2})}(\rho) \tag{C.22}$$

Many authors use for definition of $n_L(\rho)$ the relation $n_L(\rho) = (-1)^{L+1}\sqrt{\frac{\pi}{2\rho}}\, J_{-(L+\frac{1}{2})}(\rho)$.

In this work, however, (C.22) is used in agreement with Albert Messiah (1961, Appendix B). The same author is followed in the definition of the spherical Hankel functions of first kind ($h_L^{(+)}(\rho)$) and of second kind ($h_L^{(-)}(\rho)$):

$$h_L^{(\pm)}(\rho) \doteq n_L(\rho) \pm i j_L(\rho) \tag{C.23}$$

To finish this brief account on spherical functions, the Wronskian relation for $j_L(\rho)$ and $n_L(\rho)$ is obtained from (C.11):

$$n_L \frac{d\gamma_L}{d\rho} - j_L \frac{dn_L}{d\rho} = \frac{1}{\rho^2} \tag{C.24}$$

Appendix D

The Vanishing Solutions at the Origin of Two Coupled Radial Equations

Consider the system of two second order differential equations

$$\left[\frac{d^2}{dR^2} + K^2 - \frac{(J-1)J}{R^2} + V(R)\right] v = U(R)\, w \qquad \text{(D.1)}$$

$$\left[\frac{d^2}{dR^2} + K^2 - \frac{(J+1)(J+2)}{R^2} + W(R)\right] w = U(R)\, v \qquad \text{(D.2)}$$

or, introducing the linear operators

$$A \equiv \left[\frac{d^2}{dR^2} + K^2 - \frac{(J-1)J}{R^2} + V(R)\right]$$

$$B \equiv \left[\frac{d^2}{dR^2} + K^2 - \frac{(J+1)(J+2)}{R^2} + W(R)\right],$$

$$\mathrm{I} \quad \begin{cases} A v = U w & \text{(D.1')} \\ B w = U v & \text{(D.2')} \end{cases}$$

U(R), V(R) and W(R) are short range (Yukawa) nuclear potentials(*):

$$U = a(J,L)\, e^{-\alpha R}/R\,, \quad V = b(J,L)\, e^{-\beta R}/R\,, \quad W = c(J,L)\, e^{-\gamma R}/R$$

System I can be solved by elimination. Multiplying both sides of (D.1') and (D.2') respectively by $B U^{-1}$ and $A U^{-1}$ one has

$$B U^{-1} A v = U v \qquad \text{(D.1")}$$

$$A U^{-1} B w = U w \qquad \text{(D.2")}$$

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* Actually V(R) and W(R) are linear combinations of Yukawa potentials; but the purpose of this Appendix, which is very general, is not altered if they assume the precedent forms.

Equations (D.1") and (D.2") are fourth order linear differential equations, which can be brought under explicit form. Equation (D.1"), for instance, gives

$$\frac{d^4 v}{dR^4} + \sum_{i=1}^{4} \frac{p_i(R)}{R^i} \cdot \frac{d^{(4-i)} v}{dR^{(4-i)}} = 0 \qquad \text{(D.3)}$$

where the $p_i(R)$, $i = 1,2,3,4$, are regular functions of R not only for $R \geq 0$ but in the whole of the complex plane:

$$p_1(R) = 2 + R\,g_1(R)$$

$$p_2(R) = -[(J-1)J + (J+1)(J+2)] + R\,g_2(R)$$

$$p_3(R) = 2(J-1)J + R\,g_3(R)$$

$$p_4(R) = -2(J-1)J + (J-1)J(J+1)(J+2) + R\,g_4(R)$$

The $g_i(R)$ represent regular functions of R such that $g_i(0) \neq 0$, $i = 1,2,3,4$.

The explicit form (D.3) of (D.1") shows that this equation has a regular singularity at the origin and so it is a linear differential equation of the Fuchsian Type (Goursat, 1924, Vol. II, p. 476). Its four linearly independent solutions around the singularity can be obtained as power series developments at this point. The leading terms for very small values of R of these series are equal to R^{σ_i}, $i = 1,2,3,4$, where the σ_i are the roots of the indicial equation of the differential equation (D.3):

$$\sigma(\sigma-1)(\sigma-2)(\sigma-3) + p_1(0)\sigma(\sigma-1)(\sigma-2) + p_2(0)\sigma(\sigma-1) + p_3(0)\sigma + p_4(0) = 0 \qquad \text{(D.4)}$$

According to the expressions given for the $p_i(R)$, the roots of this algebraic equation are

$$\sigma_1 = J + 3\,,\quad \sigma_2 = J\,,\quad \sigma_3 = -J\,,\quad \sigma_4 = -(J-1)$$

Therefore, only the solutions with $\sigma = J+3\,,\ J$ lead to physically meaningful results because only such solutions vanish at the origin.

Now, the roots of the indicial equation differ from each other by an integer. In such conditions the two solutions of (D.1") with $\sigma = J+3\,,\ J$ can be written in the following way

$$v_1 = R^{J+3} f_1(R)$$

$$v_2 = R^{J} f_2(R) + C\, v_1 \operatorname{Log} R$$

where the $f_i(R)$, $i = 1,2$, are regular functions of R, such that $f_i(0) \neq 0$ and C is a constant (not arbitrary). The $f_i(R)$ are generally given as power series of R. The radius of convergence of such series, centered at the singularity is equal to the minimum of the $p_i(R)$-radii of convergence; therefore $f_1(R)$ and $f_2(R)$ are valid for all values of R, because the $p_i(R)$ are regular functions in the whole of the complex plane.

The foregoing considerations can be repeated for equation (D.2"). In this case the coefficients $p_1(0)$ and $p_2(0)$ of the indicial equation remain unchanged, but $p_3(0)$ and $p_4(0)$ are now

$$p_3(0) = 2(J+1)(J+2)$$

$$p_4(0) = -2(J+1)(J+2) + (J-1)J(J+1)(J+2)$$

Therefore, the roots of (D.3) are

$$\sigma_1 = J+2\,,\quad \sigma_2 = J+1\,,\quad \sigma_3 = -(J-1)\,,\quad \sigma_4 = -(J+1)$$

Here again only two of the four linearly independent solutions vanish at the origin ($\sigma = J+2, J+1$) and their general form is

$$\overline{w}_1 = R^{J+2} f_3(R)$$
$$\overline{w}_2 = R^{J+1} f_4(R) + d\,\overline{w}_1 \operatorname{Log} R$$

where, as before, $f_3(R)$ and $f_4(R)$ are regular functions of R.

However, it is not necessary to consider the solutions of (D.2") to solve the system I. It is sufficient to evaluate the solutions $v_1(R)$ and $v_2(R)$ of (D.1") and, from equation (D.1') written under the form

$$w = U^{-1} A v$$

to derive the corresponding w_1 and w_2:

$$w_1 = U^{-1} A v_1 \qquad \text{(D.5)}$$
$$w_2 = U^{-1} A v_2 \qquad \text{(D.6)}$$

The couple $(v_1 w_1)$ is a solution of system I. In fact, multiplying (D.5) by U one gets

$$U w_1 = A v_1$$

which is equation (D.1'); and multiplying (D.5) by B, one has, according to (D.1")

$$B w_1 = B U^{-1} A v_1 = V v_1$$

The same proof can be carried out for solution (v_2, w_2).

It is also clear that w_1 (or w_2) defined by (D.5) (or (D.6)) is a solution of (D.2"). Actually, the multiplication of w_1 by $A U^{-1} B$ gives, considering (D.1") and (D.2'),

$$A\,U^{-1} B\, w_1 = A\, U^{-1} \underline{B\, U^{-1} A\, v_1} =$$
$$= A\, U^{-1} U\, v_1 = A v_1 =$$
$$= U\, w_1$$

The precedent considerations lead to the effective construction of two linearly independent solutions (v_1,w_1) and (v_2,w_2) of system I, which vanish at the origin. A simple substitution in the equations of system I shows that any linear combination of these solutions

$$v = a_1 v_1 + a_2 v_2$$

$$w = a_1 w_1 + a_2 w_2$$

is again a solution of system I, vanishing at the origin. But no more than two linearly independent solutions with this property can exist.

Solutions (v_1,w_1) and (v_2,w_2) of system I are linked by a generalised Wronskian condition. From (D.1') and (D.2') it is obvious that

$$v_2 A v_1 - v_1 A v_2 = U\,(v_2 w_1 - v_1 w_2)$$
$$w_2 B w_1 - w_1 B w_2 = U\,(w_2 v_1 - w_1 v_2)$$

or, adding the two equations

$$v_2 A v_1 - v_1 A v_2 + w_2 B w_1 - w_1 B w_2 = 0 \quad .$$

Considering now the definition of the linear operators A and B, this expression reduces to

$$\frac{df(R)}{dR} = 0 \rightarrow f(R) = \text{const},$$

where

$$f(R) = v_2\frac{dv_1}{dR} - v_1\frac{dv_2}{dR} + w_2\frac{dw_1}{dR} - w_1\frac{dw_2}{dR}$$

But $v_m(0) = w_m(0) = 0$, $m = 1,2$; therefore the integration constant vanishes and the Wrouskian condition is obtained

$$v_2\frac{dv_1}{dR} - v_1\frac{dv_2}{dR} + w_2\frac{dw_1}{dR} - w_1\frac{dw_2}{dR} = 0 \qquad (D.7)$$

Condition (D.7) can be expressed in terms of the phase shifts of the solutions of system I. For an R sufficiently large the potentials U, V and W vanish and v_m and w_m, $m = 1, 2$, become linear combinations of spherical Bessel and Neumann functions multiplied by KR. Putting $\Theta_J = KR - J.\frac{\pi}{2}$ and using a convenient normalization, the asymptotic behaviour of these solutions is given by the following expressions ($m = 1,2$):

$$v_m = e^{i\delta_m^{J-1}} \sin(\Theta_{J-1} + \delta_m^{J-1})$$
$$w_m = K_m e^{i\delta_m^{J+1}} \sin(\Theta_{J+1} + \delta_m^{J+1})$$

where δ_m^{J-1} and δ_m^{J+1} are the phase shifts and the K_m are coupling constants or mixing parameters. Introducing these functions in (D.7) for large values of R, the new relation is obtained

$$e^{i(\delta_1^{J-1} + \delta_2^{J-1})} \sin(\delta_2^{J-1} - \delta_1^{J-1}) - K_1K_2 e^{i(\delta_1^{J+1} + \delta_2^{J+1})} \sin(\delta_2^{J+1} - \delta_1^{J+1}) = 0 \qquad (D.8)$$

It is now clear that it is possible to seek linear combinations of (v_1,w_1) and (v_2,w_2) (these solutions are linearly independent!), such that the new phase shifts obey the conditions

$$\delta_1^{J-1} = \delta_1^{J+1} = \delta_J^1 \; , \; \delta_2^{J-1} = \delta_2^{J+1} = \delta_J^2 \qquad (D.9)$$

Here δ^1_J and δ^2_J represent the common value of these quantities. For such linear combinations (D.8) reduces to

$$1 + K^1_J K^2_J = 0 \tag{D.10}$$

where K^1_J and K^2_J are the new mixing constants, corresponding to δ^1_J, δ^2_J respectively.

Appendix E

Hulthèn's K^-N Potentials

The dimensionless constants Z_0 and Z_1 were calculated numerically for Ross-Humphrey's sets of solutions in Chapter II by means of Schwinger's variational formula for zero energy. However, in order to achieve greater accuracy, it is convenient to make Z_0 and Z_1 energy-dependent, by adjusting these quantities for each value of p, i.e., the K^--wave number in the K^-N C.M. (p is related to p_{Lab}, the K^--Lab. momentum, by equations (4.57") and (4.90)).

Hulthèn's variational principle (see, for instance, Burhop, 1961) gives the means to perform such programme, if suitable trial functions $U_I(r)$ (I (= 0,1) is the K^-N isotopic-spin quantum number) are used for the S-waves of the K^-N system.

Choosing $U_I(r)$ equal to

$$U_I(r) - \sin pr + (\tan \delta^I + C\, e^{-br})(1 - e^{-br})\cos pr \qquad \text{(E.1)}$$

where $\tan \delta^I$ is linked with the K^-p scattering length A_I by the relation

$$\tan \delta^I = p\, A_I \qquad (2.11)$$

and C is a constant to be determined by the condition

$$\frac{\partial J}{\partial C} = 0\ ;\quad J(U_I) = \int_0^\infty U_I A_I U_I\, dr\ ;\quad A_I \equiv \left[\frac{d^2}{dr^2} + p^2 + Z_I\, e^{-br}/r\right]. \qquad \text{(E.2)}$$

The integrals appearing in Hulthèn's variational formula can be integrated analytically. Furthermore, the boundary conditions

$$U_I(0) = 0; \quad U_I(r) \underset{r \to \infty}{\rightarrow} \sin pr + \tan \delta^I \cos pr$$

are satisfied and the limit of $U_I(r)/p$ when $p \to 0$, i.e.

$$\lim_{p\to 0}[U_I(r)/p] = r + (A_I + C'e^{-br})(1 - e^{-br}),\ C' = \lim_{p\to 0}\frac{C}{p} \tag{E.3}$$

is equal to (except for a normalisation constant and the additive term $C'e^{-br}(1 - e^{-br})$) the trial function (2.53) used in the application of Schwinger's variational principle to the same problem.

Elimination of C between the two conditions translating Hulthèn's variational principle

$$J = 0\ , \qquad \frac{\partial J}{\partial C} = 0 \tag{E.4}$$

leads to an algebraic equation of second degree in Z_I, which can be written under the form

$$4E(A + B\tan\delta^I + D\tan^2\delta^I) = (C + F\tan\delta^I)^2 \tag{E.5}$$

The coefficients appearing in (E.5) depend on p and Z_I as follows ($s = \frac{p}{b}$):

$$A = \frac{Z_I}{4}\log(1 + 4s^2)$$

$$B = Z_I\tan^{-1}\left(\frac{s}{1 + 2s^2}\right) - s$$

$$C = \frac{Z_I}{2}\tan^{-1}\left(\frac{s}{3 + 2s^2}\right)$$

$$D = \frac{Z_I}{2}\left[\log\frac{4}{3} + \frac{1}{2}\log\frac{(4 + 4s^2)^2}{(1 + 4s^2)(9 + 4s^2)}\right] - \frac{1}{4} - \frac{1}{4 + 4s^2}$$

$$E = \frac{Z_I}{2}\left[\log\frac{16}{15} + \frac{1}{2}\log\frac{(16 + 4s^2)^2}{(9 + 4s^2)(25 + 4s^2)}\right] - \frac{1}{12} - \frac{1}{4 + 4s^2} + \frac{3}{9 + 4s^2} - \frac{8}{16 + 4s^2}$$

$$F = Z_I\left[\log\frac{9}{8} + \frac{1}{2}\log\frac{(9+4s^2)^2}{(4+4s^2)(16+4s^2)}\right] - \frac{1}{6} + \frac{2}{4+4s^2} - \frac{6}{9+4s^2}$$

The physically acceptable solution of Z_I is obviously the one that tends to the value obtained by means of Schwinger's variational formula (see Table (II.2)) when p goes to zero.

Table (E) shows the variation of Z_o and Z_1 with the K^--Lab. momentum for Ross-Humphrey's solutions I and II. The values of Z_o and Z_1 corresponding to 0 MeV/c are those given in Table (II.2); the values belonging to the other momenta are calculated by means of (E.5)

Table (E)

	p_{Lab} MeV/c	$Z_o = x_o + iy_o$	$Z_1 = x_1 + iy_1$
I	0	1·6701 + i0·3456	0·3935 + i0·7836
	100	1·7045 + i0·3349	0·4084 + i0·8141
	200	1·8243 + i0·3329	0·4544 + i0·9003
	300	1·9802 + i0·3246	0·5306 + i1·0379
II	0	1·9336 + i1·0853	1·1821 + i0·1679
	100	2·0293 + i1·0019	1·2170 + i0·1695
	200	2·1941 + i0·9425	1·3210 + i0·1776
	300	2.3840 + i0.8507	1.4739 + i0.1879

Bibliography

Akiba, T. and R.H. Capps: Phys. Rev. Letters, 8, 457 (1962)

Alvarez, L.W. in Proceedings of the Ninth Annual International Conference of High-Energy Physics (Kiev, 1959)

Bethe, A.H.: Phys. Rev., 76, 38 (1949)

Blatt, J.M. and L.C. Biedenharn: Phys. Rev. 86, 399 (1952)

Blatt, J.M. and V.F. Weisskopf: Theoretical Nuclear Physics (John Wiley and Sons, New York, 1952)

Breit, G., M.H. Hill, K.E. Lassila and K.D. Pyatt: Phys. Rev., 120, 2227 (1960); 128, 826 (1962)

Bruckner, K.A.: Phys. Rev. 89, 834 (1953)

Buckingham, R.A. and H.S.W. Massey: Proceedings of the Royal Society of London, 179A, 123 (1941)

Burhop, E.H.S.: Theory of Collisions in "Quantum Theory", Volume I, edited by D.R. Bates (Academic Press, New York, 1961)

Burhop, E.H.S. et al.: Physics Letters, 16, 89 (1965)

Capps, R.H. and R.L. Schult: Nuovo Cimento, 23, 416 (1962)

Chand, R.: Annals of Physics, 22, 438 (1963)

Chand, R. and R.M. Dalitz: Annals of Physics, 20, 1 (1962)

Chew, G.F.: Phys. Rev., 80, 196 (1950)

Chew, G.F. and M.L. Goldberger: Phys. Rev., 87, 778 (1952)

Christenson, J.H., J.W. Cronin, V.L. Fitch and R. Turlay: Phys. Rev. Letters, 13, 138 (1964)

Courant, R.: Differential and Integral Calculus, Volumes I and II (Blackie and Son Ltd., London, 1948)

Dahl, O., N. Horwitz, D. Miller, J. Murray and M. Watson: Proceedings of the Annual International Rochester Conference on High-Energy Physics, p. 415 (Interscience Publishers, Inc. 1960)

Dalitz, R.H.: Rev. of Modern Physics, 33, 417 (1961); "Strange Particles and Strong Interactions" (Tate Institute of Fundamental Research, Oxford University Press, 1962)

Dalitz, R.H. and S.F. Tuan: Phys. Rev. Letters, 2, 425 (1959); Annals of Physics, 8, 307 (1960)

Day, T.B., G.A. Snow and J. Sucher: Nuovo Cimento, 13, 614 (1959); 14, 637 (1959a); Phys. Rev., 119, 1100 (1960)

Drell, S.D. and L. Verlet: Phys. Rev. 99, 849 (1955)

Eisenbud, L. and E. Wigner: Proc. Nat. Acad. Sc., 27, 281 (1941)

Ferreira, E.: Phys. Rev. 115, 1727 (1959)

Gammel, J.L. and R.M. Thaler: Phys. Rev., 107, 291 (1957); 1337 (1957a); Prog. in Elementary Particle and Cosmic Ray Physics, 5, 99 (1960)

Goldhaber, R. et al.: Phys. Rev. Letters, 9, 135 (1962)

Gourdin, M. and A. Martin: Nuovo Cimento, 14, 722 (1959)

Goursat, E.: "Cours d'Analyse Mathématique" (Gauthier-Villars et Cie, Paris, 1924, 4th edition)

Hastings, C.: "Approximations for Digital Computers" (Princeton University Press, 1957, 2nd edition)

Hull, M.H., K.E. Lassila, H.M. Ruppel and G. Breit: Phys. Rev., 122, 1606 (1961); 128, 830 (1962)

Jackson, J.D. and J.M. Blatt: Rev. of Modern Physics, 22, 77 (1950)

Jackson, J.D., D.G. Ravenhall and H.W. Wyld: Nuovo Cimento, 9, 834 (1958)

Jost, R. and W. Kohn: Phys. Rev., 87, 977 (1952)

Kim, J.K.: Phys. Rev. Letters, 14, 29 (1965)

Landau, L.D. and E.M. Lifshitz: "Quantum Mechanics, Non-Relativistic Theory" (Pergamon Press, London, 1958)

Lippmann, B.A. and J.S. Schwinger: Phys. Rev. 79, 460 (1950)

Mathews, P.T. and A. Salam: Nuovo Cimento, 13, 381 (1959)

Messiah, A.: "Quantum Mechanics" (North-Holland Publishing Company, Amsterdam; Volume I: 1961; Volume II: 1962)

Noyes, H.P.: Proceedings of the Rutherford Jubilee International Conference at Manchester (Heywood and Company Ltd., London, 1961)

Queen, N.M.: Nuclear Physics, 66, 673 (1965)

Rohrlich, F. and J. Eisenstein: Phys. Rev., 75, 705 (1949)

Roldberg, L.S. and R.M. Thaler: Phys. Rev. Letters, 4, 372 (1960)

Ross, R.: "Elastic and Charge-Exchange Scattering of K^--mesons in Hydrogen" (E. Lawrence Laboratory Report UCRL-9749, 1961)

Sachs, R.: "Nuclear Theory" (Addison-Wesley Pub. Co. Inc., Cambridge, Mass., U.S., 1953)

Sakitt, M.: University of Maryland Technical Report, No.410 (1964)

Schick, L.H.: Rev. of Modern Physics, 33, 603 (1961)

Schult, R.L. and R.H. Capps: Phys. Rev., 122, 1659 (1961)

Thomas, L.H. : Phys. Rev., 51, 202 (1937)

Tripp, R., M. Ferro-Luzzi and M. Watson: Phys. Rev. Letters, 8, 175 (1962)

Wheeler, J.A.: Phys. Rev. 52, 1033; 1107 (1957)

Whittaker, E.T. and G.N. Watson: "A Course of Modern Analysis" (University Press, Cambridge, 1950; 4th edition)

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