Distributed Robust model Predictive Control for Partial Output Consensus of Multi-rate Chain Interconnected Processes

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***Abstract*—In this paper, partial output consensus (POC) based on distributed robust model predictive control (DRMPC) is investigated for multi-rate chain interconnected processes. To accommodate potential differences in sensor sampling characteristics, 'Consensus' and 'Nonconsensus' outputs (i.e. those outputs with and without a consensus target) have different sampling periods. A fusion estimation strategy (FES) is initially designed, which can utilize multi-rate measured outputs to generate state estimates in real time. Using the results of this FES, a DRMPC is then proposed that can simultaneously stabilize all the outputs. The POC cost function and consensus constraint can ensure that all subsystems meet POC requirements. The effectiveness of the proposed approach is shown to be guaranteed theoretically and further demonstrated by simulations and experimental testing.**

***Index Terms*—Partial output consensus, Chain interconnected process, Multi-rate sampling, Fusion estimation strategy, Distributed robust model predictive control**

I. INTRODUCTION

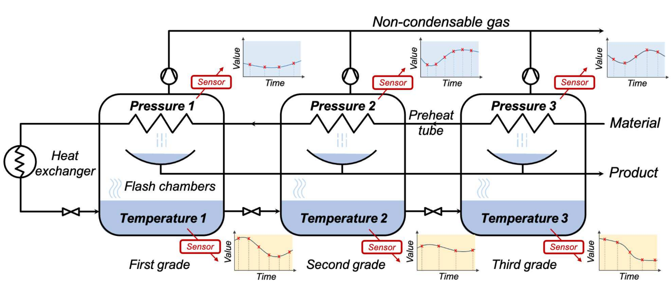

Fig. 1. Multistage flash distillation process for desalination with three grades.

A chain interconnected process is composed of subsystems experiencing chained flow across material, energy and/or information interconnections. These are common in the chemical industry [1] and production dispatching [2]. Optimization and control of such chained, interconnected processes is complex and varied resulting in the problem receiving wide attention in the literature [3], [4]. As shown in Fig.1, the multistage flash distillation process is chained. Its workflow is that materials (sea water) go through multiple chambers (with progressively decreasing pressures), and the product (fresh water) can be obtained through evaporation and condensation of water. Within this process, the pressures should maintain a decreasing relation, which can be modelled using 'Consensus' outputs with a consensus target. The temperatures should reach their own set-points, which must be modelled as 'Nonconsensus' outputs. This problem is called 'partial output consensus'(POC) and it is common in chain interconnected processes.

Regarding the POC problem, there have been many studies presented in [5]–[9]. Previous work [5]–[7] investigated POC control design, but ignored the stability of 'Nonconsensus' outputs and were only suitable for multi-agent systems without interconnections. To address this, the authors of this paper proposed a distributed robust POC control in [8] for chain interconnected systems with uncertainties, which can stabilize all outputs. Further, a distributed optimization method was proposed in [9] to calculate the feasible set-points for 'Consensus' and 'Nonconsensus' outputs. Nevertheless, these methods cannot meet some objectives, such as the satisfaction of state/input constraints and optimal performance. In contrast, distributed robust model predictive control (DRMPC) has excellent robustness and optimality, it has bee extended to consensus control [10], [11] and interconnected system control [12], [13]. DRMPC is a potential method for POC, but it has not been considered for addressing POC until now.

Due to the differences in sensors, subsystems employ multi-rate sampling in practice. In Fig.1, all 'Consensus' and 'Nonconsensus' outputs are sampled asynchronously and there is no measurement information available for the control update at some times. For multi-rate sampling, the conventional method (see for example [14], [15]) is to derive new state-space models with a common rate for the control design. However, the interconnection terms bring the coupling effects into the time series and it can be challenging to find the desired rate. Adopting a different methodology, the authors in [16] proposed a DRMPC control based on distributed Kalman filters for interconnected systems, where the filters can utilize multi-rate outputs to generate the estimated states and DRMPC can provide the stabilized control inputs. Nevertheless, the cross-covariance calculation for the filters is too complex and DRMPC lacks robust theoretical analysis. To reduce the computation complexity, [17] presents distributed set-membership observers that provide performance comparable to the distributed Kalman filters, but this method is only suitable for single-output systems. In summary, the multi-rate sampling control has been widely investigated, but there

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are few available results on interconnected systems, let alone addressing the POC problem.

Motivated by the problems and open questions mentioned above, POC is studied for multi-rate chain interconnected systems in this paper. There are two challenges to be addressed. Firstly, how to handle the influences of multi-rate sampling and utilize measured ‘Consensus’ and ‘Nonconsensus’ outputs to provide reliable real-time information for the control calculation. Secondly, how to design DRMPC controls for POC and guarantee their robustness, feasibility and stability.

A POC control framework is developed using DRMPC and the main contributions are as follows.

- The proposed fusion estimation strategy (FES) can estimate in real-time zonotopes of states. The FES consists of two multi-rate filters and a fusion module. By using the ‘Consensus’ and ‘Nonconsensus’ measured outputs, the filters can synchronously generate two pre-estimated zonotopes of states. The fusion module then integrates them to obtain a more precise estimated zonotope.
- Using the FES results as a basis, an effective DRMPC for POC is proposed by formulating a distributed optimization problem. To handle the chain interconnections, ‘shadow’ variables are defined to represent the dynamics of neighboring subsystems. With the designed POC cost function and consensus constraint, the control inputs can ensure that subsystems meet POC requirements.
- The recursive feasibility and stability of the proposed method are both analyzed.

Compared with [14], [15], multi-rate systems with interconnections are considered in this paper. In contrast to [16], [17], the proposed FES can simultaneously handle two-part outputs with different sampling periods. Although the method in [17] can also solve multi-rate POC, the method in this paper can make full use of ‘Consensus’ and ‘nonconsensus’ measured outputs to improve controller performance and robustness.

The structure of this paper is as follows. Section II formulates POC for a multi-rate chain interconnected process and presents the ‘Consensus’ and ‘Nonconsensus’ filters. Then, FES and DRMPC for POC is developed in Section III, including the design and analysis of the proposed method. The results of numerical simulations and experiments are shown in Section IV. Finally, the conclusion is presented in Section V.

Notation 1: $\mathcal{R}$ and $\mathcal{Z}$ respectively represent the set of real numbers and integers, and $\mathcal{Z}_i^j$ refers to the set $\{i, i+1, \cdots, j\}$ with $i < j \in \mathcal{Z}$. Note P' and $\mathrm{rank}(P)$ denote the transpose and rank of P, respectively. $\emptyset$, $\mathbf{0}_n$ and $\mathbf{0}_{n\times m}$ represent the empty set, n-dimensional zero vector and $n \times m$ dimensional zero matrix. The matrix $\mathrm{diag}[S_i]_N$ denotes the diagonal block matrix composed of $S_1, S_2, \cdots, S_N$. The quadratic norm with respect to a positive definite matrix $P = P'$ is denoted by $\|x\|_P^2 = x'Px$. $\|x\|$ and $\|x\|_\infty$ represent the 2-norm and ∞-norm of x respectively. The eigenvalues of P are denote by $\lambda(P)$. Given two sets $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{R}^n$ and matrix $A \in \mathcal{R}^{m\times n}$, $A\mathcal{X} = \{Ax | x \in \mathcal{X}\}$. The Minkowski set addition is defined by $\mathcal{X} \oplus \mathcal{Y} = \{x + y | x \in \mathcal{X}, y \in \mathcal{Y}\}$ and the Minkowski (Pontryagin) set difference is defined by $\mathcal{X} \ominus \mathcal{Y} = \{z \in \mathcal{R}^n | z \oplus \mathcal{Y} \subseteq \mathcal{X}\}$. A zonotope χ is denoted as $\langle c, E\rangle := \{x \in \mathcal{R}^n | x = c + Eu, \|u\|_\infty \leq 1\}$, where $c \in \mathcal{R}^n$ and $E \in \mathcal{R}^{n\times r}$ are the center and generator matrix of χ.

II. Problem Formulation

In this paper, the considered chain interconnected process is composed of N subsystems, which are shown in Fig.2. Subsystems have chain interconnections which refer to the couplings in mass and energy. The information can be exchanged among subsystems by the communication network. Each subsystem has two types of output, the ‘Consensus’ output $y_{c,i}$ and the ‘Nonconsensus’ output $y_{n,i}$. Due to differences in the sensors, these outputs have different sampling periods. The goal of POC is to make all $y_{c,i}, i \in \mathcal{V}$ achieve consensus and converge to the ‘Consensus’ set-point $y_{d,con}$, while $y_{n,i}$ is stable and converges to the ‘Nonconsensus’ set-point $y_{d,i}$ respectively. Note that, $y_{d,con}$ is only assigned to subsystem 1 and $y_{d,i}$ is assigned to subsystem i.

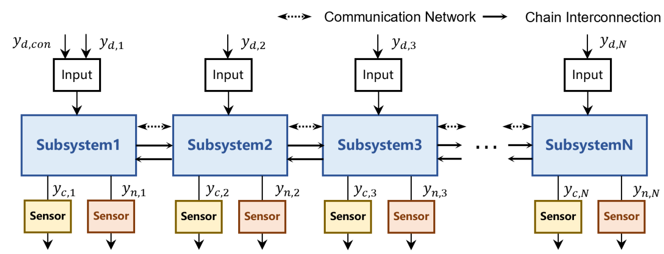

Fig. 2. Multi-rate chain interconnected process.

The subsystems can be formulated as

$$\begin{aligned} x_i(k+1) =& A_{ii}x_i(k) + B_i u_i(k) + \sum_{j\in\mathcal{N}_{c,i}} A_{ij}x_j(k) + w_i(k), \\ y_{c,i}(k) =& C_{c,i}x_i(k) + v_{c,i}(k), \\ y_{n,i}(k) =& C_{n,i}x_i(k) + v_{n,i}(k), \end{aligned} \tag{1}$$

where $i \in \mathcal{V} = \mathcal{Z}_1^N$, $x_i \in \mathcal{X}_i \subseteq \mathcal{R}^n$ and $u_i \in \mathcal{U}_i \subseteq \mathcal{R}^m$ are the state and input, $y_{c,i} \in \mathcal{R}^{p_c}$, $y_{n,i} \in \mathcal{R}^{p_n}$ are the ‘Consensus’ and ‘Nonconsensus’ outputs, $A_{ii}, A_{ij} \in \mathcal{R}^{n\times n}$, $B_i \in \mathcal{R}^{n\times m}$, $C_{c,i} \in \mathcal{R}^{p_c\times n}$ and $C_{n,i} \in \mathcal{R}^{p_n\times n}$ are known matrices. The state and input are constrained, and $\mathcal{X}_i, \mathcal{U}_i$ are convex, compact polytopes whose interiors are not empty. $w_i \in \mathcal{W}_i \subseteq \mathcal{R}^n$ is the disturbance, where $\mathcal{W}_i := \langle \mathbf{0}_n, \eta_{w,i}\mathbf{I}_n\rangle$ and $\eta_{w,i} > 0$. $v_{c,i} \in \mathcal{V}_{c,i} \subseteq \mathcal{R}^{q_c}$, $v_{n,i} \in \mathcal{V}_{n,i} \subseteq \mathcal{R}^{q_n}$ are the measurement noises, where $\mathcal{V}_{c,i} := \langle \mathbf{0}_{q_c}, \eta_{c,i}\mathbf{I}_{q_c}\rangle$, $\mathcal{V}_{n,i} := \langle \mathbf{0}_{q_n}, \eta_{n,i}\mathbf{I}_{q_n}\rangle$, and $\eta_{c,i}, \eta_{n,i} > 0$. $\mathcal{N}_{c,i}$ refers to subsystem i’s neighbor set,

$$\mathcal{N}_{c,i} = \begin{cases} \{2\}, i = 1 \\ \{i-1, i+1\}, i \in \mathcal{Z}_2^{N-1} \\ \{N-1\}, i = N \end{cases},$$

and the chained interconnection satisfies $\sum_{j\in\mathcal{N}_{c,i}} A_{ij}x_j \in \mathcal{V}_{ij}$, where $\mathcal{V}_{ij} = \oplus_{j\in\mathcal{N}_{c,i}} A_{ij}\mathcal{X}_j$.

The following Assumption is required.

Assumption 1: Suppose that the parameters in (1) satisfy

$$\mathrm{rank}\left(\begin{bmatrix} I_{nN} - A & B \\ C_c & \mathbf{0}_{p_cN\times mN} \\ C_n & \mathbf{0}_{p_nN\times mN} \end{bmatrix}\right) = (n + p_c + p_n)N,$$

where $A = [A_{ij}]_{N\times N}$, $B = \mathrm{diag}[B_i]_N$, $C_c = \mathrm{diag}[C_{c,i}]_N$ and $C_n = \mathrm{diag}[C_{n,i}]_N$.

Remark 1: Assumption 1 is made to ensure that all subsystems can attain appropriate steady-states and inputs corresponding to the desired set-points. Similar assumptions are also made in tracking DMPC [18].

Multi-rate sampling is common in process industries. When this occurs, inputs and/or outputs of different subsystems may have different sampling periods. Fig. 3 presents several multi-rate sampling mechanisms. In Fig. 3(a), [15] outputs have a larger sampling period than the inputs, but the subsystems are sampled synchronously. Fig. 3(b) describes the multi-rate sampling in [16], where only the outputs have different sampling periods. Fig. 3(c) describes the sampling scenario considered in this paper, where each subsystem has two output elements which have different sampling periods. Not only that, subsystems are sampled asynchronously and outputs have larger sampling periods than the inputs. Clearly the case considered in this paper is more difficult than the cases conisdered in [15], [16]. Further the control design should consider how best to utilize 'Consensus' and 'Nonconsensus' outputs to provide reliable information for real-time feedback.

For convenience, let $\delta_{c,i}$, $\delta_{n,i}$ represent the sampling periods of $y_{c,i}$, $y_{n,i}$ and δ_u represent the control period. It is assumed that $\delta_{c,i}, \delta_{n,i} \geq \delta_u$. The measured 'Consensus' output $\psi_{c,i}$ and the 'Nonconsensus' output $\psi_{n,i}$ are denoted as $\psi_{c,i}(l_c^i) = y_{c,i}(l_c^i\delta_{c,i})$, $\psi_{n,i}(l_n^i) = y_{n,i}(l_n^i\delta_{n,i})$, where $l_c^i, l_n^i \in \mathcal{Z}_0^\infty$.

The objective of this paper is then to develop a DRMPC-based control method, which can utilize $\psi_{c,i}(l_c^i)$ and $\psi_{n,i}(l_n^i)$ to complete the following POC targets, i.e.

$$\lim_{k\to\infty} \|y_{c,1}(k) - y_{d,con}\| \leq \sigma_{c,1}, \tag{2a}$$

$$\lim_{k\to\infty} \|y_{n,i}(k) - y_{d,i}\| \leq \sigma_{n,i}, \tag{2b}$$

$$\lim_{k\to\infty} \|y_{c,i}(k) - y_{c,j}(k)\| \leq \sigma_{c,i}, j \in \mathcal{N}_{c,i}. \tag{2c}$$

where $\sigma_{c,i}, \sigma_{n,i} > 0$ are constants.

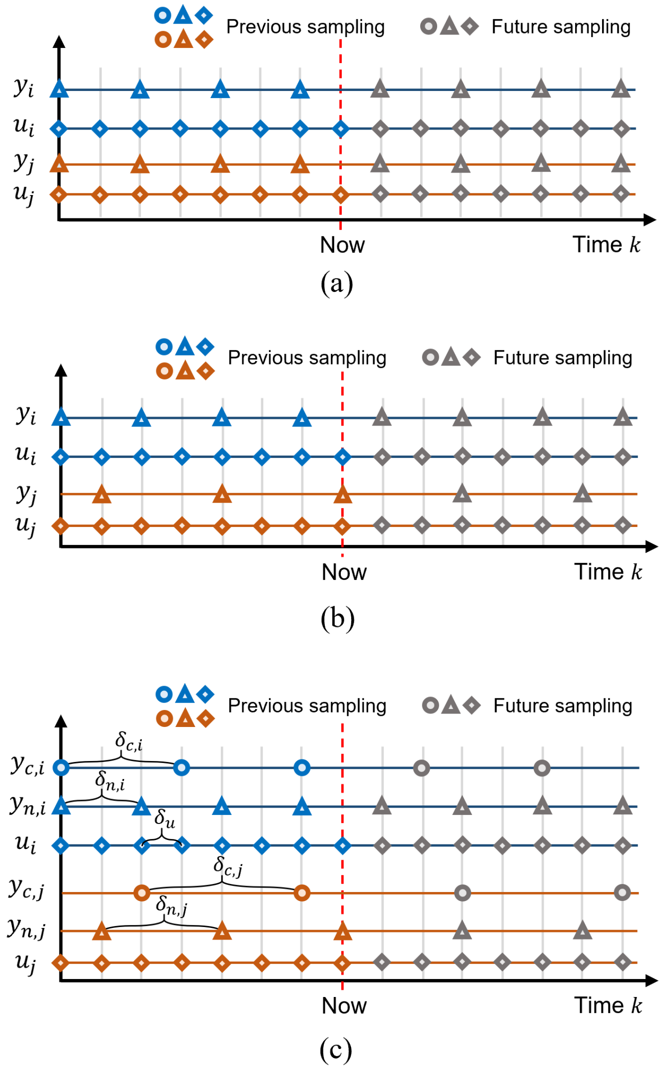

Fig. 3. The common multi-rate sampling mechanisms.

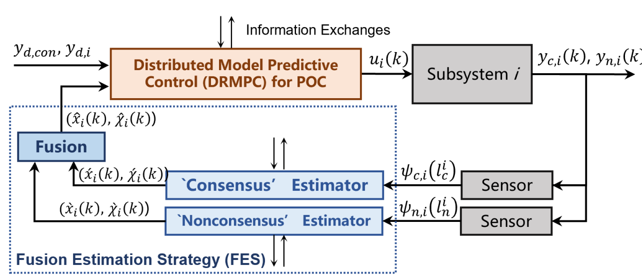

Fig. 4. The POC control framework for multi-rate systems.

III. Main Results

This section presents a POC control framework as shown in Fig. 4, for multi-rate chain interconnected systems. The framework includes both FES and DRMPC where the main principles can be described as follows.

- The FES contains two filters and a fusion module, and can provide the estimated state $\hat{x}_i$ and zonotope $\hat{\chi}_i$ for the control calculation.
- With the results from the FES, DRMPC can provide the control input $u_i(k)$, which can ensure that subsystems meet the POC requirements.

A. FES for Multi-rate Outputs

For each subsystem in (1), the FES is designed as follows

$$\acute{\chi}_i(k) = \begin{cases} \langle \varphi_i(k), \Phi_i(k) \rangle, & \mathrm{mod}(k, \delta_{c,i}) \neq 0, \\ \langle \acute{\xi}_i(l_c^i), \acute{\Xi}_i(l_c^i) \rangle, & k = l_c^i\delta_{c,i}, \end{cases} \tag{3a}$$

$$\grave{\chi}_i(k) = \begin{cases} \langle \varphi_i(k), \Phi_i(k) \rangle, & \mathrm{mod}(k, \delta_{n,i}) \neq 0, \\ \langle \grave{\xi}_i(l_n^i), \grave{\Xi}_i(l_n^i) \rangle, & k = l_n^i\delta_{n,i}. \end{cases} \tag{3b}$$

$$\hat{\chi}_i(k) = \acute{\chi}_i(k) \cap \grave{\chi}_i(k) := \left\langle \hat{x}_i(k), \hat{R}_i(k) \right\rangle, \tag{3c}$$

where $\acute{\chi}_i, \grave{\chi}_i \subseteq \mathcal{R}^n$ are the pre-estimated zonotopes according to 'Consensus' and 'Nonconsensus' measured outputs, respectively. φ_i, $\acute{\xi}_i$, $\grave{\xi}_i$ are the pre-estimated values of $x_i(k)$ at different times, and Φ_i, $\acute{\Xi}_i$, $\grave{\Xi}_i$ are the corresponding generator matrices. $\hat{x}_i \in \mathcal{R}^n$ represents the estimated value of $x_i(k)$ and $\hat{\chi}_i \subseteq \mathcal{R}^n$ is a zonotope with $\hat{x}_i$ as the center and $\hat{R}_i \in \mathcal{R}^{n\times r}$ as the generator matrix. The superscript represents the time index.

The above variables can be updated by the following steps

$$\begin{aligned}\varphi_i(k) =& A_{ii}\hat{x}_i(k-1) + B_i u_i(k-1) \\ &+ \sum_{j\in\mathcal{N}_{c,i}} A_{ij}\hat{x}_j(k-1), \end{aligned} \tag{4a}$$

$$\Phi_i(k) =[A_{ii}\hat{R}_i(k-1), [A_{ij}\hat{R}_j(k-1)]_{j\in\mathcal{N}_{c,i}}, \eta_{w,i}\mathbf{I}], \tag{4b}$$

$$\acute{\xi}_i(l_c^i) =\varphi_i(k) + \acute{K}_i(l_c^i)(\psi_{c,i}(l_c^i) - C_{c,i}\varphi_i(k)), \tag{4c}$$

$$\acute{\Xi}_i(l_c^i) =[(\mathbf{I} - \acute{K}_i(l_c^i)C_{c,i})\Phi_i(k), -\eta_{c,i}\acute{K}_i(l_c^i)], \tag{4d}$$

$$\grave{\xi}_i(l_n^i) =\varphi_i(k) + \grave{K}_i(l_n^i)(\psi_{n,i}(l_n^i) - C_{n,i}\varphi_i(k)), \tag{4e}$$

$$\grave{\Xi}_i(l_n^i) =[(\mathbf{I} - \grave{K}_i(l_n^i)C_{n,i})\Phi_i(k), -\eta_{n,i}\grave{K}_i(l_n^i)], \tag{4f}$$

where $\acute{K}_i(l_c^i) \in \mathcal{R}^{n\times p_c}$ and $\grave{K}_i(l_n^i) \in \mathcal{R}^{n\times p_n}$ are the filtering gains, and $[A_{ij}\hat{R}_j(k-1)]_j \in \mathcal{N}_{c,i}$ is formulated as

$$[A_{ij}\hat{R}_j(k-1)]_{j\in\mathcal{N}_{c,i}} = \begin{cases} A_{12}\hat{R}_2(k-1), i=1, \\ [A_{i,i-1}\hat{R}_{i-1}(k-1), A_{i,i+1}\hat{R}_{i+1}(k-1)], i\in\mathcal{Z}_2^{N-1}, \\ A_{N,N-1}\hat{R}_{N-1}(k-1), i=N. \end{cases}$$

The proposed FES consists of the following parts:

- Equations (3a) and (3b) define 'Consensus' and 'Nonconsensus' filters. Under multi-rate sampling, they can generate the estimated states at every instant. Their input signals are $\psi_{c,i}/\psi_{n,i}$ and the output signals are the pre-estimated states $\acute{x}_i/\grave{x}_i$ and zonotopes $\acute{\chi}_i/\grave{\chi}_i$.
- Equation (3c) defines a fusion module, which is employed to integrate $\acute{\chi}_i$ and $\grave{\chi}_i$ in order to obtain a more precise zonotope $\hat{\chi}_i$ of less size. Its input signals are $\acute{\chi}_i$ and $\grave{\chi}_i$ and its output signal is the estimated state $\hat{\chi}_i$.

Considering minimizing $\|\acute{\Xi}_i(l_c^i)\|_F^2$ and $\|\grave{\Xi}_i(l_n^i)\|_F^2$, the optimal 'Consensus' and 'Nonconsensus' filtering gains are derived as

$$\begin{aligned}\acute{K}_i(l_c^i) =& \Phi_i(k)(C_{c,i}\Phi_i(k))' \\ &(C_{c,i}\Phi_i(k)(C_{c,i}\Phi_i(k))' + \eta_{c,i}^2\mathbf{I})^{-1}, \\ \grave{K}_i(l_n^i) =& \Phi_i(k)(C_{n,i}\Phi_i(k))' \\ &(C_{n,i}\Phi_i(k)(C_{n,i}\Phi_i(k))' + \eta_{n,i}^2\mathbf{I})^{-1}. \end{aligned} \tag{5}$$

Lemma 1: Assume that $x_i(0) \in \hat{\chi}_i(0)$ holds for all $i \in \mathcal{V}$, then $\hat{\chi}_i(k)$ calculated by (3) satisfies $\hat{\chi}_i(k) \neq \emptyset$ and $x_i(k) \in \hat{\chi}_i(k)$ for $k \in \mathcal{Z}_1^{\infty}$.

Proof. According to the principle of induction, this Lemma can be proven by showing that $x_i(k-1) \in \hat{\chi}_i(k-1) \Rightarrow x_i(k) \in \hat{\chi}_i(k)$.

(i) When $\mathrm{mod}(k, \delta_{c,i}) \neq 0$, using the subsystem dynamics in (1), it follows that

$$\begin{aligned} x_i(k) \in& A_{ii}\hat{\chi}_i(k-1) \oplus_{j\in\mathcal{N}_{c,i}} A_{ij}\hat{\chi}_j(k-1) \\ &\oplus B_i\langle u_i(k-1), \mathbf{0}\rangle \oplus \mathcal{W}_i, \end{aligned}$$

so that $x_i(k) \in \langle \varphi_i(k), \Phi_i(k)\rangle$.

(ii) When $k = l_c^i\delta_{c,i}$, consider the dynamics of $\psi_{c,i}(l_c^i)$. It follows that

$$\begin{aligned} x_i(k) =& x_i(k) + \acute{K}_i(l_c^i)(\psi_{c,i}(l_c^i) - C_{c,i}x_i(k) - v_{c,i}(k)) \\ \in& (\mathbf{I} - \acute{K}_i(l_c^i)C_{c,i})\langle\varphi_i(k), \Phi_i(k)\rangle \\ &\oplus (-\acute{K}_i(l_c^i)\mathcal{V}_{c,i}) \oplus \acute{K}_i(l_c^i)\langle\psi_{c,i}(l_c^i), \mathbf{0}\rangle, \end{aligned}$$

so that $x_i(k) \in \langle\acute{\xi}_i(l_c^i), \acute{\Xi}_i(l_c^i)\rangle$.

Based on (i) and (ii), it follows that $x_i(k) \in \acute{\chi}_i(k)$. Similarly, it can be inferred that $x_i(k) \in \grave{\chi}_i(k)$ also holds for all k. According to the convexity of $\acute{\chi}_i(k)$ and $\grave{\chi}_i(k)$, it can be concluded that $\hat{\chi}_i(k)$ in (3c) is non-empty and $x_i(k) \in \hat{\chi}_i(k)$ holds for all k. Then, $x_i(k-1) \in \hat{\chi}_i(k-1) \Rightarrow x_i(k) \in \hat{\chi}_i(k)$ can be proven. Since $x_i(0) \in \hat{\chi}_i(0)$ is satisfied by hypothesis, Lemma 1 can be proven. ■

Lemma 1 demonstrates that the zonotope $\hat{\chi}_i(k)$ can always contain the real state $x_i(k)$. Based on this, $\hat{\chi}_i(k)$ can be utilized as reliable information for the control calculation.

Denote two error sets $\chi_{e,i}(k) := \hat{\chi}_i(k) \ominus \hat{x}_i(k)$ and $\bar{\chi}_{e,i}(k) := A_{ii}\chi_{e,i}(k) \oplus_{j\in\mathcal{N}_{c,i}} A_{ij}\chi_{e,j}(k) \oplus \mathcal{W}_i$. According to (3), $\chi_{e,i}(k) \subseteq \bar{\chi}_{e,i}(k)$ holds for all k. The proposed FES can provide $\hat{x}_i(k)$, $\hat{\chi}_i(k)$, $\chi_{e,i}(k)$ and $\bar{\chi}_{e,i}(k)$ for DRMPC at every instant.

Remark 2: The steps (3c) can be implemented by the zonotope calculation method in [19], or by using the MPT3 toolbox.

B. DRMPC for POC

The prediction horizon is denoted by N_p. For clarity, all the variables in the DRMPC optimization problem are presented:

$x_i(k+l\mid k)$	the predicted value of $x_i(k+l)$ at instant k
$x_{i,j}(k+l\mid k)$	$x_i(k+l\mid k)$ estimated by subsystem j at instant k
$u_i(k+l\mid k)$	the predicted value of $u_i(k+l)$ at instant k
$\boldsymbol{u}_i(k\mid k)$	the sequence of $u_i(k+l\mid k)$, $l \in \mathcal{Z}_0^{N_p}$
$y_{s,c,i}$	the steady-state 'Consensus' output
$y_{s,n,i}$	the steady-state 'Nonconsensus' output
$x_{s,i}$	the steady-state value of x_i
$x_{s,i,j}$	$x_{s,i}$ estimated by subsystem j
$u_{s,i}$	the steady-state value of u_i

Note that $x_{i,j}(k+l|k)$ and $x_{s,i,j}$ are called 'shadow' variables, which can replace $x_i(k)$ and $x_{s,i}$ in subsystem i for decoupling the chain interconnections. $y_{s,c,i}$, $y_{s,n,i}$, $x_{s,i}$, $x_{s,i,j}$ and $u_{s,i}$ are steady-state variables, which satisfy

$$\begin{aligned} x_{s,i} &= A_{ii}x_{s,i} + B_i u_{s,i} + \sum_{j\in\mathcal{N}_{c,i}} A_{ij}x_{s,j,i}, \\ y_{s,c,i} &= C_{c,i}x_{s,i}, \\ y_{s,n,i} &= C_{n,i}x_{s,i}. \end{aligned} \tag{6}$$

Denote $\boldsymbol{d}_i(k) = (x_i(k|k), \boldsymbol{u}_i(k|k), x_{s,i}, u_{s,i})$ as the decision variable for subsystem i, which contains the initial predicted state, the sequence of the predicted inputs and the steady-state outputs. Following [20], a parameterized control input u_i applied to subsystem i is designed as

$$u_i(k) = u_i(k|k) + F_i(\hat{x}_i(k) - x_i(k|k)), \tag{7}$$

where $F_i \in \mathcal{R}^{m\times n}$ is a gain matrix, which satisfies $||A_{s,i}|| = ||A_{ii} + B_iF_i|| < 1$.

According to (2), POC cost function can be designed as

$$J_i(\boldsymbol{d}_i(k)) = V_{p,i} + V_{f,i} + \sum\nolimits_{l=0}^{N_p-1} I_i(l), \tag{8}$$

where $V_{p,i}$ is the POC cost, which is composed of the quadratic-norm differences between steady-state outputs and set-points. $V_{p,i}$ is the terminal cost and I_i^l is the stage cost, which are common in MPC design and can guarantee the stability and optimality of systems. They are formulated as

$$V_{p,i} = \begin{cases} \|y_{s,c,i} - y_{d,con}\|^2_{T_c} + \|y_{s,n,i} - y_{d,i}\|^2_{T_{n,i}}, & i = 1, \\ \|y_{s,n,i} - y_{d,i}\|^2_{T_{n,i}}, & i \neq 1, \end{cases}$$

$$V_{f,i} = \|x_i(k+N_p|k) - x_{s,i}\|^2_{P_{f,i}},$$
$$I_i^l = \|x_i(k+l|k) - x_{s,i}\|^2_{Q_i} + \|u_i(k+l|k) - u_{s,i}\|^2_{R_i},$$

where $P_{f,i} \in \mathcal{R}^{n\times n}$ is the terminal weight matrix, $T_c \in \mathcal{R}^{p_c\times p_c}$, $T_{n,i} \in \mathcal{R}^{p_n\times p_n}$, $Q_i \in \mathcal{R}^{n\times n}$ and $R_i \in \mathcal{R}^{m\times m}$ are weight matrices, which are positive definite symmetric.

Assumption 2: [21]Suppose that there exists a gain matrix $F_i \in \mathcal{R}^{m\times n}$ for the subsystems (1) such that:

- $\mathcal{X}_{f,i} = \left\{a \| \|a\|^2_{P_{f,i}} \le \alpha_i, \alpha_i > 0\right\}$ is a positive invariant set for $x_i - x_{s,i}$, i.e., $x_i(k+1) - x_{s,i} \in \mathcal{X}_{f,i}$, $\forall x_i(k) - x_{s,i} \in \mathcal{X}_{f,i}$, $w_i(k) = 0$.
- $A_o' P_f A_o - P_f \le -Q - K'RK$ holds, where $A_o = A + BF$, $F = \mathrm{diag}[F_i]_N$, $P_f = \mathrm{diag}[P_{f,i}]_N$, $Q = \mathrm{diag}[Q_i]_N$, $R = \mathrm{diag}[R_i]_N$.

Assumption 2 is common in MPC design. Similar assumptions are also adopted in [18] and [22]. Although the inequality is centralized, it can be solved using the distributed method in [8]. Then, F_i can be obtained.

The DRMPC optimization problem can be formulated as

$$\min_{\boldsymbol{d}_i(k)} \quad J_i(\boldsymbol{d}_i(k))$$
$$\text{s.t.} \quad (6) \text{ and}$$
$$x_i(k+l+1|k) = A_{ii}x_i(k+l|k) + B_i u_i(k+l|k) + \sum_{j\in\mathcal{N}_{c,i}} A_{ij} x_{j,i}(k+l|k), l \in \mathcal{Z}_0^{N_p-1}, \tag{9a}$$
$$x_i(k|k) \in \hat{\chi}_i(k), \tag{9b}$$
$$x_i(k+l|k) \in \mathcal{X}_i \ominus S_{x,i}(l|k), \tag{9c}$$
$$u_i(k+l|k) \in \mathcal{U}_i \ominus F_i S_{u,i}(l|k), l \in \mathcal{Z}_0^{N_p-1}, \tag{9d}$$
$$x_i(k+N_p|k) - x_{s,i} \in \mathcal{X}_{f,i}, \tag{9e}$$
$$x_{s,i} \in \mathcal{X}_{s,i}, u_{s,i} \in \mathcal{U}_{s,i}, \tag{9f}$$
$$y_{s,c,i} - y_{s,c,j} = \mathbf{0}, j \in \mathcal{N}_{c,i}. \tag{9g}$$
$$x_i(k+l|k) - x_{i,j}(k+l|k) = \mathbf{0}, l \in \mathcal{Z}_0^{N_p}, \tag{9h}$$
$$x_{s,i} - x_{s,i,j} = \mathbf{0}, j \in \mathcal{N}_{c,i}. \tag{9i}$$

where

$$S_{x,i}(l|k) = \begin{cases} 2\chi_{e,i}(k), & l = 0, \\ A_{ii}S_{x,i}(l-1|k) \\ \oplus_{j\in\mathcal{N}_{c,i}} A_{ij} S_{x,j}(l-1|k), & l > 0. \\ \oplus B_i F_i \chi_{e,i}(k) \oplus \mathcal{W}_i \end{cases}$$

$$S_{u,i}(l|k) = \begin{cases} \chi_{e,i}(k), & l = 0, \\ A_{ii}S_{u,i}(l-1|k) & l > 0. \\ \oplus_{j\in\mathcal{N}_{c,i}} A_{ij} S_{u,j}(l-1|k) \oplus \mathcal{W}_i, \end{cases}$$

$$\mathcal{X}_{s,i} = \mathcal{X}_i \ominus S_{x,i}(N_p|k) \ominus \mathcal{X}_{f,i},$$
$$\mathcal{U}_{s,i} = \mathcal{U}_i \ominus F_i S_{u,i}(N_p-1|k) \ominus \bar{F}_i \mathcal{X}_{f,i}.$$

$S_{x,i}(l|k)$ and $S_{u,i}(l|k)$ are the tube constraints, which can ensure that $x_i(k)$ and $u_i(k)$ always belong to $\mathcal{X}_i$ and $\mathcal{U}_i$. $\mathcal{X}_{f,i} = \left\{a\|\|a\|^2_{P_{f,i}} \le \alpha_i, \alpha_i > 0\right\}$ is the terminal set and $\alpha_i > 0$ is a constant. At every instant, $S_{x,i}(l|k)$ and $S_{u,i}(l|k)$ can be calculated based on $\chi_{e,i}(k)$ which is generated by the FES. After that, (9) can be solved.

The constraints in (9) are explained as follows. (9a) is the prediction equation of subsystem i with $x_i(k+l|k)$ and $x_{j,i}(k+l|k)$. (9b) ensures that $x_i(k|k)$ belongs to $\hat{\chi}_i(k)$ for approximating $x_i(k)$. (9c) and (9d) are the state and input constraints, which can guarantee $x_i(k) \in \mathcal{X}_i$ and $u_i(k) \in \mathcal{U}_i$. (9e) is the terminal constraint and $\mathcal{X}_{f,i}$ is a positive invariant set. $P_{f,i}$ and $\bar{F}_i$ can be obtained by solving the linear matrix inequality in Assumption 2. (6), (9f) and (9g) form the constraints for steady-state variables. Note that, POC cost combined with (9g) can ensure that $y_{s,c,i}$ and $y_{s,n,i}$ meet POC requirements. (9h) and (9i) can ensure the consistency of the 'shadow' variables to guarantee the effectiveness of the solution.

The optimization problem (9) can be directly solved using the distributed optimization methods in [23], [24] and its optimal solution at k is represented by $\boldsymbol{d}_i^*(k) = (x_i^*(k|k), \boldsymbol{u}_i^*(k|k), x_{s,i}^*(k), u_{s,i}^*(k))$.

C. Theoretical Analysis

Algorithm 1: DRMPC for POC of multi-rate chain interconnected process

Input: $K_{max}, N, N_p, \delta_{c,i}, \delta_{n,i}, \eta_{w,i}, \alpha_i, y_{d,con}, y_{d,i}, x_i(0), \hat{x}_i(0), \hat{R}_i(0), Q_i, R_i, T_c, T_{n,i}$.
Output: The responses of $y_{c,i}(k)$ and $y_{n,i}(k)$.
Initialization: calculate $P_{f,i}$, F_i and let $k = 1$.
while $k \le K_{max}$ **do**
  Subsystems exchange $(\hat{x}_i(k-1), \hat{R}_i(k-1), \hat{\chi}_i(k-1))$.
  Calculate $\varphi_i(k), \Phi_i(k)$ by (4a) and (4b).
  if $k = l_c^i \delta_{c,i}$ **then**
    Measure 'Consensus' output and obtain $\psi_{c,i}(l_c^i)$.
    Calculate $\acute{\xi}_i(k), \acute{\Xi}_i(k)$ by (4c) and (4d).
  end
  if $k = l_n^i \delta_{n,i}$ **then**
    Measure 'Nononsensus' output and obtain $\psi_{n,i}(l_n^i)$.
    Calculate $\grave{\xi}_i(k), \grave{\Xi}_i(k)$ by (4e) and (4f).
  end
  Calculate $\hat{x}_i(k)$, $\hat{\chi}_i(k)$ by (3), and calculate $S_{x,i}(l|k)$, $S_{u,i}(l|k)$.
  Solve (9) and obtain the optimal input $u_i^*(k|k)$.
  Calculate $u_i(k)$ in (7) and apply it.
  Let $k = k + 1$.
end

This subsection completes the proof of the recursive feasibility and the stability. Before that, the computation procedures for the proposed method can be summarized by Algorithm 1.

Theorem 1: (Recursive feasibility) Suppose that Assumptions 1 and 2 hold. For each subsystem, provided that $x_i(0) \in \hat{\chi}_i(0)$ holds and (9) has a feasible solution at $k = 1$, if there exists $P_i > 0$, F_i satisfying

$$A'_{s,i} P_i A_{s,i} - A'_{ii} P_i A_{ii} < 0, \tag{10}$$

then Algorithm 1 remains feasible at all times.

The proof of Theorem 1 can be found in Appendix. A.

Theorem 2: (Stability) Suppose that Assumptions 1 and 2 hold and consider the given set-points $y_{d,con}$ and $y_{d,1}, \cdots, y_{d,N}$. For any initial state $x_i(0) \in \mathcal{X}_i$, all subsystems in (1) deploying Algorithm 1 are stable and can achieve POC in (2) with $x_i(k) \in \mathcal{X}_i$ and $u_i(k) \in \mathcal{U}_i$.

The proof of Theorem 2 can be found in Appendix. B.

Remark 3: Compared with the single-filter method in [17], the proposed FES has less conservatism, meaning that it can tighten the boundaries of $\hat{\chi}_i(k)$ to make the size of $\hat{\chi}_i(k)$ as small as possible. Then, the DRMPC can approximate the dynamics of the real process and achieve good performance. This feature will be demonstrated in the following simulation results.

IV. Numerical Simulations and Experiments

A. Numerical Simulations

To validate the effectiveness of the proposed method, the numerical experiments for five subsystems are presented. The parameters of the subsystems are $A_{ii} = [0.8, 0.2; -0.3 + 0.1 * i, 0.9]$, $B_i = [1.0, 0; 0, 1.0 + 0.1 * i]$, $C_{c,i} = [1, 0]$, $C_{n,i} = [-0.1, 0.5]$, $\eta_{w,i} = 0.05$, $\eta_{c,i} = \eta_{n,i} = 0.01$ and the state and input constraints are $\mathcal{X}_i = \{x_i||x_i| \le 10\}, \mathcal{U}_i = \{x_i||u_i| \le 5\}$.

The sampling periods of all outputs are: $\delta_{c,1} = 2s$, $\delta_{c,2} = 3s$, $\delta_{c,3} = 2s$, $\delta_{c,4} = 4s$, $\delta_{c,5} = 3s$, $\delta_{n,1} = 3s$, $\delta_{n,2} = 4s$, $\delta_{n,3} = 3s$, $\delta_{n,4} = 2s$, $\delta_{n,5} = 2s$.

For the 'Consensus' and 'Nonconsensus' filters, the initial values are set as $\hat{x}_1(0) = (0.06, -0.94)'$, $\hat{x}_2(0) = (0.92, -0.45)'$, $\hat{x}_3(0) = (2.08, -0.01)'$, $\hat{x}_4(0) = (2.91, 0.45)'$, $\hat{x}_5(0) = (4.05, 0.91)'$ and $\hat{R}_1(0) = \hat{R}_2(0) = \hat{R}_3(0) = \hat{R}_4(0) = \hat{R}_5(0) = 0.1 * \boldsymbol{I}_2$.

The set-points are piecewise constants, for $k \in [0, 20]$, $y_{d,con} = 2.3$, $y_{d,1} = 1.0$, $y_{d,2} = 1.5$, $y_{d,3} = 2.0$, $y_{d,4} = 2.5$ and $y_{d,5} = 3.0$. For $k \in (21, 40]$, $y_{d,con} = 1.2$, $y_{d,1} = 0$, $y_{d,2} = 0.5$, $y_{d,3} = 1.5$, $y_{d,4} = 1.0$ and $y_{d,5} = 2.0$. For $k \in (41, 60]$, $y_{d,con} = 1.8$, $y_{d,1} = 0.5$, $y_{d,2} = 1.0$, $y_{d,3} = 1.0$, $y_{d,4} = 2.0$ and $y_{d,5} = 2.5$.

In this section, the following two methods are considered: (a) the single-filter method in [17] and (b) the proposed method. The single-filter method estimates the states by $\psi_{c,i}$ and then calculates a DRMPC control. For the sake of impartiality in the results, the parameters for the two methods are set identically. For DRMPC, set $N_p = 10$, $\alpha_i = 0.2$, $Q_i = \boldsymbol{I}_2$, $R_i = \boldsymbol{I}_2$ and $T_c = T_{n,i} = 70$.

The responses of the 'Consensus' outputs with both methods are shown in Fig. 5. All $y_{c,i}$ can achieve consensus and reach a neighborhood of $y_{d,con}$. Note that, with the single-filter method, $y_{c,i}$ in Fig.5(a) can reach $[y_{d,con} - 0.2, y_{d,con} + 0.2]$. In contrast, $y_{c,i}$ in Fig. 5(b) can converge smoothly to a smaller neighborhood of $y_{d,con}$. This difference is more evident in the responses of the 'Nonconsensus' outputs, which are presented in Fig. 6. To quantify the performance, the Integral of Time-weighted Square Errors (ITSEs) are adopted here and are presented in Table I. Obviously, the proposed method has improved performance when compared with the single-filter method.

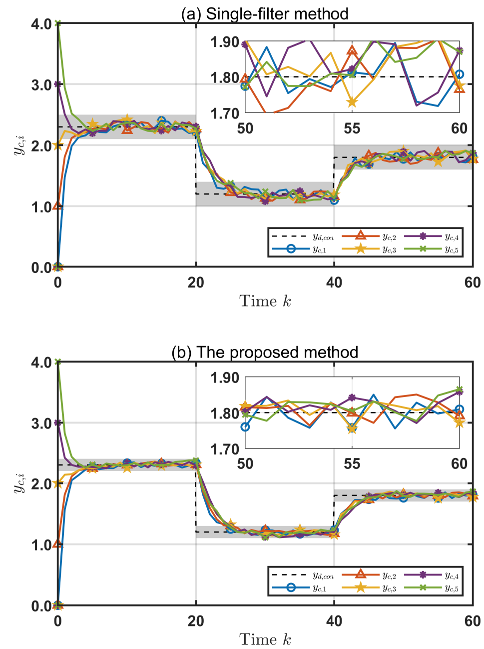

Fig. 5. The responses of 'Consensus' outputs.

TABLE I
THE ITSES OF $y_{c,i}$ AND $y_{n,i}$ UNDER TWO METHODS

	Single-filter method	The proposed method
$y_{c,1}$	841.92	844.25
$y_{n,1}$	606.24	593.16
$y_{c,2}$	719.21	676.40
$y_{n,2}$	652.89	643.36
$y_{c,3}$	705.57	662.68
$y_{n,3}$	1442.01	1427.40
$y_{c,4}$	812.58	808.89
$y_{n,4}$	1529.62	1534.41
$y_{c,5}$	1096.43	1049.50
$y_{n,5}$	720.41	703.36

To illustrate the advantages of FES, Fig. 7 shows $\hat{\chi}_i(k)$ at $k = 2s$ and $6s$, where the black and dark grey areas are $\acute{\chi}_i$ and $\grave{\chi}_i$, the light gray area is $\hat{\chi}_i$, '*' and '+' are the real state and estimated state. Based on (3b), the size of $\hat{\chi}_i(k)$ can be significantly reduced. In this way, DRMPC can approximate the dynamics of real subsystems well and hence achieve good performance.

The proposed method also achieves robustness to the varying sensor sampling periods. To verify this, five groups of

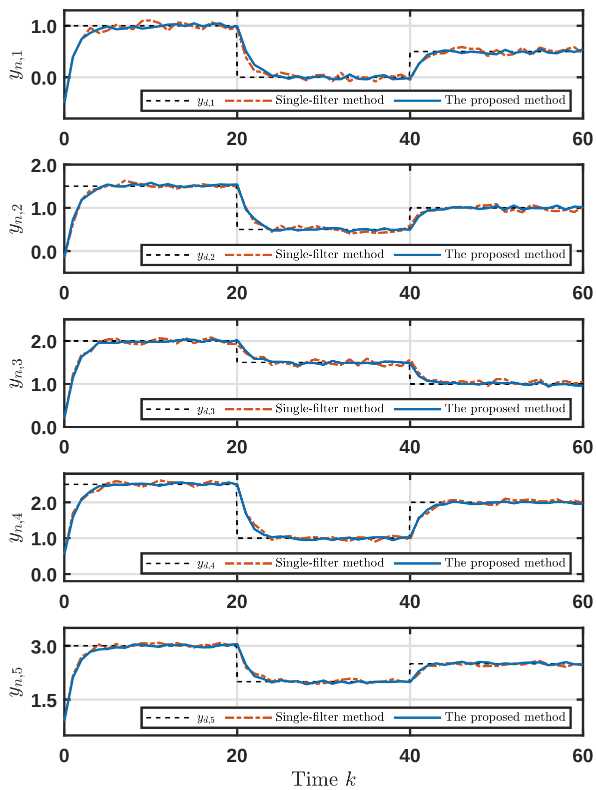

Fig. 6. The responses of 'Nonconsensus' outputs.

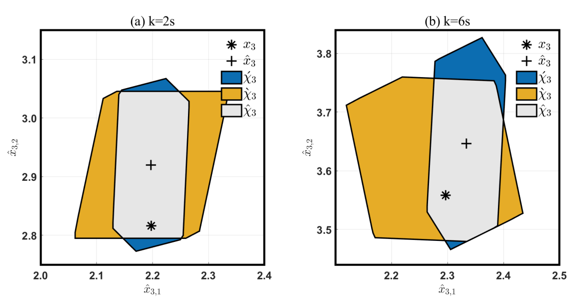

Fig. 7. The zonotope $\hat{\chi}_3(k)$ at $k = 2s$ and $6s$.

$20s$ simulation tests with $\delta_{c,4} = 3s, 4s, 5s, 6s$ and $7s$ are developed. Other sampling periods are the same as before. To avoid randomness, each group is repeated ten times, and the average sum of ITSEs is taken as a reference, which is shown in Table II. The performance of the single-filter method is negatively correlated with $\delta_{c,4}$ and subsystems are divergent when $\delta_{c,4} = 7s$. The proposed method is robust to changes in $\delta_{c,4}$ and the sum of ITSEs remains around $6.75e3 - 6.80e3$. Therefore, the effectiveness and robustness of the proposed method is further demonstrated.

B. Experimental Testing

To further verify the effectiveness of the proposed method, an NaOH solution proportioning experiment is presented here,

TABLE II
THE SUM OF ITSES UNDER DIFFERENT $\delta_{c,4}$

$\delta_{c,4}$	$3s$	$4s$	$5s$	$6s$	$7s$
Single-filter method	$7.19e3$	$7.12e3$	$7.28e3$	$7.26e3$	$7.44e3$
The proposed method	$6.76e3$	$6.76e3$	$6.76e3$	$6.74e3$	$6.79e3$

Fig. 8. The two tanks platform.

which is carried out using the platform in Fig.8. For clarity, the flow chart is shown in Fig.9 and it can be explained by listing the following major elements of equipment. Tank R-101 and R-102 are used for producing the desired NaOH solution. They are connected by a pipe (red line), which can transfer the solution from R-101 to R-102. Therefore, they can be considered as two subsystems with chain interconnections. Tank V-111 and V-112 are the material tanks containing water and NaOH solution, respectively.Tank V-113 is a hot-water tank and the hot water (at about $55°C$) can be transferred to the jackets of R-101 and R-102 for heating.

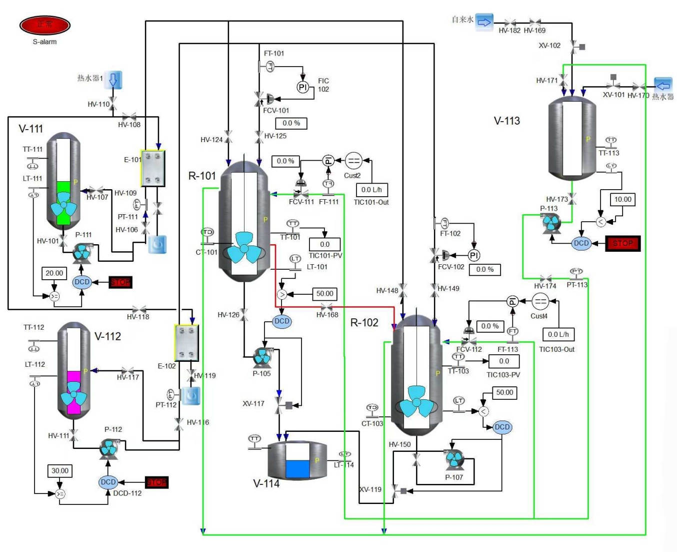

Fig. 9. The flow chart of the solution proportioning experiment.

The objective of this experiment is to ensure that R-101 and R-102 can product the NaOH solution with different temperatures and the same concentration. The temperature and concentration can be respectively modelled as 'Consensus' and 'Nonconsensus' outputs. The partial output consensus problem with multi-rate sampling can be formulated as follows.

- Controlled variables: the temperatures T_1, T_2 and the concentrations C_1, C_2 for R-101 and R-102.
- Manipulated variables: the flow rates $v_{h,1}$, $v_{h,2}$ of hot water and the flow rates $v_{w,1}$, $v_{w,2}$ of water for R-101 and R-102. Their ranges are $0 - 25.0L/h$.
- 'Consensus' and 'Nonconsensus' set-points: the 'Consensus' set-points is set as $C_d = 3.0\ kmol/m^3$, i.e. the concentration set-point. The 'Nonconsensus' set-points are set as $42°C$ and $38°C$, i.e. the temperature set-points.
- Constraints: T_1, T_2 cannot exceed $45°C$ and C_1, C_2 cannot exceed $10.0\ kmol/m^3$.
- Multi-rate sampling: the sampling periods are $\delta_{T_1} = 2s$, $\delta_{T_2} = 3s$, $\delta_{C_1} = 3s$ and $\delta_{C_1} = 4s$.
- Disturbance and measurement noise: they are stochastic and the bounds are $\eta_{w,i} = 0.01, \eta_{c,i} = \eta_{n,i} = 0.005$.

Denote $x_i = col(T_i, C_i)$, the model parameters are $A_{11} = [0.548, 0.006; -0.001, 0.735]$, $B_1 = [0.531, 0.001; 0, 0.465]$, $C_1 = [1, 0]$, $A_{22} = [0.732, -0.004; -0.002, 0.628]$, $B_2 = [0.409, -0.001; -0.001, 0.423]$, $C_2 = [0, 1]$, $A_{12} = [0.001, -0.009; 0, 0.001]$, $A_{21} = [-0.002, 0; 0, -0.001]$.

To highlight the advantages of the proposed method, the single-filter method in [17] is chosen for comparison. Both methods use the same parameters, which are listed in Table III. The solver for the DMPC is CasADi from the MPT3 toolbox. The control inputs are calculated and transmitted to the lower computer via OLE for Process Control (OPC).

TABLE III
THE PARAMETERS OF FES AND DRMPC

Term	Value	Term	Value	Term	Value	Term	Value
N_p	10	R_1	$5 * \boldsymbol{I}_2$	$\hat{T}_1(0)$	37.0	$\hat{R}_1(0)$	$0.05 * \boldsymbol{I}_2$
α	0.6	R_2	$5 * \boldsymbol{I}_2$	$\hat{T}_2(0)$	34.0	$\hat{R}_2(0)$	$0.05 * \boldsymbol{I}_2$
Q_1	$10 * \boldsymbol{I}_2$	T_c	100	$\hat{C}_1(0)$	1.2		
Q_2	$10 * \boldsymbol{I}_2$	T_n	100	$\hat{C}_2(0)$	1.3		

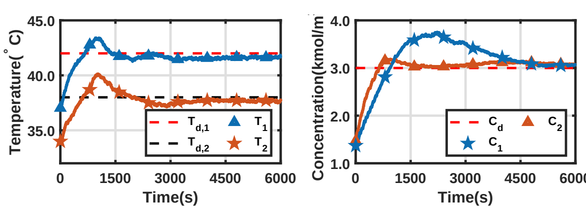

Fig. 10. The results of the single-filter method.

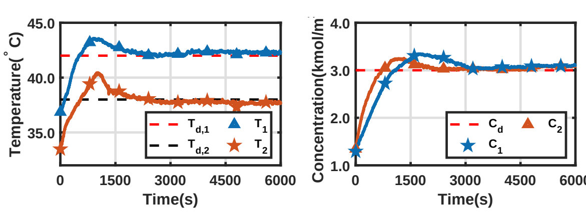

Fig. 11. The results of the proposed method.

The total time is $6000s$, and the results are presented in Fig.10 and Fig.11. After preheating, T_1 and T_2 respectively reached about $37°C$ and $34°C$. The initial values of C_1 and C_2 are 1.2 $kmol/m^3$ and 1.3 $kmol/m^3$. With the two methods, T_1 and T_2 can reach neighborhoods of their own set-points, C_1 and C_2 can achieve consensus at a neighborhood of 3.0 $kmol/m^3$. In TABLE IV, ITSE, overshoot and convergence time are presented. It can be seem that the proposed method has improved performance when compared to the single-filter method. This is particularly notable in the response of C_1 where with the benefit of FES, C_1 exhibits low overshoot and rapid convergence speed. Therefore, the effectiveness of the proposed method is further validated.

TABLE IV
THE SIMULATION RESULTS OF TWO METHODS

Index	Single-filter method	The proposed method
ITSE of T_1	$3.82e6$	$3.43e6$
ITSE of T_2	$4.65e6$	$3.85e6$
ITSE of C_1	$2.01e5$	$1.49e5$
ITSE of C_2	$2.13e6$	$4.96e5$
Overshoot of C_1	23.78%	12.96%
Convergence time of C_1	about 5000s	about 3400s

V. Conclusion

In this paper, a novel POC framework is developed for multi-rate chain interconnected processes. This is composed of FES and DRMPC. The designed FES can generate estimated states in real-time using 'Consensus' and 'Nonconsensus' filters and a fusion module. Based on the output of the FES, a novel DRMPC is developed for POC, which can drive subsystems to meet POC requirements. The recursive feasibility and stability are proven. Compared to previous methods, DRMPC with FES can achieve better performance and robustness to multi-rate sampling. Finally, some numerical simulations and experiments are presented to validate the effectiveness of the proposed method.

VI. Appendix

A. *Proof of Theorem* 1

Based on the solution at k, the feasible solution of (9) for $k+1$ can be formulated as $\bar{\boldsymbol{d}}_i(k+1) = (\bar{x}_i(k+1|k+1), \bar{\boldsymbol{u}}_i(k+1|k+1), x^*_{s,i}(k), u^*_{s,i}(k))$, where $\bar{x}_i(k+1|k+1) = x^*_i(k+1|k)$ and $\bar{\boldsymbol{u}}_i(k+1|k+1) = (u^*_i(k+1|k), \cdots, u^*_i(k+N_p|k), \bar{\kappa}_{f,i})$. Note that, $\bar{\kappa}_{f,i} = u^*_{s,i}(k) + F_i(\bar{x}_i(k+N_p|k+1) - x^*_{s,i}(k))$.

The feasible state trajectory can be derived as

$$\bar{x}_i(k+l|k+1) = \begin{cases} x^*_i(k+l|k), & l \in \mathcal{Z}_1^{N_p}, \\ A_{ii}\bar{x}_i(k+l-1|k+1) + B_i\bar{\kappa}_{f,i} \\ + \sum\limits_{j\in\mathcal{N}_{c,i}} A_{ij}\bar{x}_{j,i}(k+l-1|k+1), & l = N_p+1. \end{cases}$$

According to the feasibility of $\boldsymbol{d}^*_i(k)$, it can be easily concluded that $\bar{\boldsymbol{d}}_i(k+1)$ can satisfy the equality constraints (9a) and (9f)-(9i). Due the complexity of the particular fusion strategy, the feasibility for (9b)-(9e) cannot be directly proven, and the analysis is presented as follows.

(i) The satisfaction of the constraint (9b)**:**

Denote $\acute{x}_i(k)$ and $\grave{x}_i(k)$ as the centers of $\acute{\chi}_i(k)$ and $\grave{\chi}_i(k)$ respectively. Consider the 'Consensus' filter. There are two cases for $\bar{x}_i^{k+1|k+1}$ that need to be considered.

Case 1: while $k \neq l_c^i \delta_{c,i}$, then $\bar{x}_i(k+1|k+1) - \acute{x}_i(k+1) \in \acute{\Gamma}_i(k+1)$, where $\acute{\Gamma}_i(k+1) = A_{s,i}\chi_{e,i}(k) \oplus_{j\in\mathcal{N}_{c,i}} A_{ij}\chi_{e,j}(k)$. The error between $x_i(k+1)$ and $\acute{x}_i(k+1)$ is $x_i(k+1) - \acute{x}_i(k+1) \in \acute{\Pi}_i(k+1)$, where $\acute{\Pi}_i(k+1) = A_{ii}\chi_{e,i}(k) \oplus_{j\in\mathcal{N}_{c,i}} A_{ij}\chi_{e,j}(k) \oplus \mathcal{W}_i$.

Similarly, when $k \neq l_n^i \delta_{n,i}$, it follows that

$$\begin{aligned} \bar{x}_i(k+1|k+1) - \grave{x}_i(k+1) &\in \grave{\Gamma}_i(k+1), \\ x_i(k+1) - \grave{x}_i(k+1) &\in \grave{\Pi}_i(k+1), \end{aligned} \tag{11}$$

where $\grave{\Gamma}_i(k+1) = \acute{\Gamma}_i(k+1)$ and with $\grave{\Pi}_i(k+1) = \acute{\Pi}_i(k+1)$.

Due to $\|A_{s,i}\| < 1$, it can be easily inferred that $\acute{\Gamma}_i(k+1) \subseteq \acute{\Pi}_i(k+1)$ and $\grave{\Gamma}_i(k+1) \subseteq \grave{\Pi}_i(k+1)$.

Case 2: while $k = l_c^i \delta_c$, then $\bar{x}_i(k+1|k+1) - \acute{x}_i(k+1) \in \acute{\Gamma}_i(k+1)$, where $\acute{\Gamma}_i(k+1) = (\acute{K}_{d,i}(l_c^i)A_{ii} + B_iF_i)\chi_{e,i}(k) \oplus_{j\in\mathcal{N}_{c,i}} \acute{K}_{d,i}(l_c^i)A_{ij}\chi_{e,j}(k) \oplus \acute{K}_i(l_c^i)\mathcal{V}_i$ and $\acute{K}_{d,i}(l_c^i) = \boldsymbol{I} - \acute{K}_i(l_c^i)C_{c,i}$.

The error between $x_i(k+1)$ and $\acute{x}_i(k+1)$ satisfies $x_i(k+1) - \acute{x}_i(k+1) \in \acute{\Pi}_i(k+1)$, where $\acute{\Pi}_i(k+1) = \acute{K}_{d,i}(l_c^i)A_{ii}\chi_{e,i}(k) \oplus_{j\in\mathcal{N}_{c,i}} \acute{K}_{d,i}(l_c^i)A_{ij}\chi_{e,j}(k) \oplus \acute{K}_{d,i}\mathcal{W}_i \oplus \acute{K}_i\mathcal{V}_{c,i}$.

Similarly, while $k \neq l_n^i \delta_{n,i}$, then

$$\begin{aligned} \bar{x}_i(k+1|k+1) - \grave{x}_i(k+1) &\in \grave{\Gamma}_i(k+1), \\ x_i(k+1) - \grave{x}_i(k+1) &\in \grave{\Pi}_i(k+1), \end{aligned} \tag{12}$$

where

$$\begin{aligned} \grave{\Gamma}_i(k+1) = {} & (\grave{K}_{d,i}(l_n^i)A_{ii} + B_iF_i)\chi_{e,i}(k) \\ & \oplus_{j\in\mathcal{N}_{c,i}} \grave{K}_{d,i}(l_n^i)A_{ij}\chi_{e,j}(k) \oplus \grave{K}_i(l_n^i)\mathcal{V}_{c,i}, \\ \grave{\Pi}_i(k+1) = {} & \grave{K}_{d,i}(l_n^i)A_{ii}\chi_{e,i}(k) \oplus_{j\in\mathcal{N}_{c,i}} \grave{K}_{d,i}(l_n^i)A_{ij}\chi_{e,j}(k) \\ & \oplus \grave{K}_{d,i}(l_n^i)\mathcal{W}_i \oplus \grave{K}_i(l_n^i)\mathcal{V}_{c,i}. \end{aligned}$$

Based on the above analysis, it follows that

$$\begin{aligned} & \bar{x}_i(k+1|k+1) \in \bar{\chi}_i(k+1) \\ = & (\acute{x}_i(k+1) \oplus \acute{\Gamma}_i(k+1)) \cap (\grave{x}_i(k+1) \oplus \grave{\Gamma}_i(k+1)), \\ & x_i(k+1) \in \hat{\chi}_i(k+1) \\ = & (\acute{x}_i(k+1) \oplus \acute{\Pi}_i(k+1)) \cap (\grave{x}_i(k+1) \oplus \grave{\Pi}_i(k+1)). \end{aligned} \tag{13}$$

If (10) holds, then $\acute{\Gamma}_i(k+1) \subseteq \acute{\Pi}_i(k+1)$ holds for $k = l_c^i \delta_{c,i}$ and $\grave{\Gamma}_i(k+1) \subseteq \grave{\Pi}_i(k+1)$ holds for $k = l_n^i \delta_{n,i}$. Further, the following conditions are satisfied for all time,

$$\begin{aligned} (\acute{x}_i(k+1) \oplus \acute{\Gamma}_i(k+1)) &\subseteq (\acute{x}_i(k+1) \oplus \acute{\Pi}_i(k+1)), \\ (\grave{x}_i(k+1) \oplus \grave{\Gamma}_i(k+1)) &\subseteq (\grave{x}_i(k+1) \oplus \grave{\Pi}_i(k+1)). \end{aligned}$$

According to (13), it can be concluded that $\bar{\chi}_i(k+1) \subseteq \hat{\chi}_i(k+1)$ and $\bar{x}_i(k+1|k+1) \in \hat{\chi}_i(k+1)$. Using Lemma 1, $\hat{\chi}_i(k+1)$ is not empty. Then, it can be proven that $\bar{\boldsymbol{d}}_i(k+1)$ satisfies (9b).

(ii) The satisfaction of the tight constraint (9c)-(9d)**:**

Due to $\bar{x}_i(k+1|k+1), x_i(k+1) \in \hat{\chi}_i(k+1)$ and $x_i(k+1) \in \mathcal{X}_i$, then $\bar{x}_i(k+1|k+1) \in 2\chi_{e,i}(k+1)$ holds. For $\bar{x}_i(k+2|k+1)$, then $\bar{x}_i(k+2|k+1) = x_i^*(k+2|k) \in \mathcal{X}_i \ominus S_{x,i}(2|k)$. According to (3), it follows that $\chi_{e,i}(k+1) \subseteq \bar{\chi}_{e,i}(k+1)$. Then, $S_{x,i}(0|k+1) \subseteq S_{x,i}(1|k)$ holds. By induction, it can be inferred that $S_{x,i}^{l|k+1} \subseteq S_{x,i}(l+1|k)$ holds for $l \in \mathcal{Z}_1^{N_p-1}$. Therefore, $\bar{x}_i(k+l|k+1) \in \mathcal{X}_i \ominus S_{x,i}(l-1|k+1)$ can be proven for $l \in \mathcal{Z}_1^{N_p-2}$. For $l = N_p$, due to $x_{s,i}^*(k) \oplus \mathcal{X}_{f,i} \in \mathcal{X}_i \ominus S_{x,i}(N_p|k)$, it follows that $\bar{x}_i(k+N_p|k) = x_i^*(k+N_p|k) \in x_{s,i}^*(k) \oplus \mathcal{X}_{f,i} \subseteq \mathcal{X}_i \ominus S_{x,i}(N_p-1|k+1)$.

Because $u_i^*(k+l|k) \in \mathcal{U}_i \ominus F_iS_{x,i}(l|k)$ and $\chi_{e,i}(k) \subseteq \bar{\chi}_{e,i}(k)$, it follows that $S_{u,i}(l|k+1) \subseteq S_{u,i}(l+1|k)$. Then, $\bar{u}_i(k+l+1|k+1) \in \mathcal{U}_i \ominus F_iS_{x,i}(l|k+1)$ holds. Therefore, $\bar{d}_i(k+1)$ satisfies the constraint (9c)-(9d).

(iii) The satisfaction of the terminal constraint (9e):

According to Assumption 2, it can be inferred that $\|\bar{x}_i(k+N_p+1|k+1) - x_{s,i}^*(k)\|_{P_{f,i}}^2 \leq \|\bar{x}_i(k+N_p|k+1) - x_{s,i}^*(k)\|_{P_{f,i}}^2$. Due to $\bar{x}_i(k+N_p|k+1) - x_{s,i}^*(k) \in \mathcal{X}_{f,i}$, then $\bar{x}_i(k+N_p+1|k+1) - x_{s,i}^*(k) \in \mathcal{X}_{f,i}$ can be proven.

From the analysis above, $\bar{\boldsymbol{d}}_i(k+1)$ is a feasible solution for (9) at $k+1$. Therefore, it can be deduced that if (9) has a solution at the initial time, then it remains feasible for all time. ■

B. Proof of Theorem 2

For the given set-points $y_{d,con}$ and $y_{d,1}, \cdots, y_{d,N}$, an optimization problem for $y_{s,c,i}$, $y_{s,n,i}$ can be formulated as

$$\begin{aligned} (y_{s,c,i}^\dagger, y_{s,n,i}^\dagger) = \arg\min \sum_{i\in\mathcal{V}} V_{p,i}(y_{s,c,i}, y_{s,n,i}; y_{d,i}) \\ \text{s.t.} \quad (6), (9\text{f}) \text{ and } (9\text{g}). \end{aligned} \tag{14}$$

The optimal solution of (14) satisfies $y_{s,c,1}^\dagger = \cdots = y_{s,c,N}^\dagger = y_{d,con}$ and $y_{s,n,i}^\dagger = y_{d,i}$, $i \in \mathcal{V}$, that is, $(y_{s,c,i}^\dagger, y_{s,n,i}^\dagger)$ meets the requirements of POC. Therefore, if the systems (1) can track $(x_{s,i}^\dagger, u_{s,i}^\dagger)$ corresponding to $y_{s,c,i}^\dagger$, $y_{s,n,i}^\dagger$, then the POC targets are attained. According to Lemma 1 in [25], if Assumption 1 holds, suppose that the optimal solution of (9) is such that $\lim_{k\to\infty} \|x_i(k|k) - x_{s,i}\| = 0$, there is $\lim_{k\to\infty} J(\mathbf{d}_i^*(k)) = V_{p,i}(y_{s,c,i}^\dagger, y_{s,n,i}^\dagger) = 0$, that is $(x_{s,i}^*(k), u_{s,i}^*(k))$ can converge to $(x_{s,i}^\dagger, u_{s,i}^\dagger)$.

For subsystem i, the Lyapunov function is designed as

$$W_i(x_i(k|k)) = J_i(x_i^*(k|k), \boldsymbol{u}_i^*(k|k)) - V_{p,i}(y_{s,c,i}^*, y_{s,n,i}^*), \tag{15}$$

and (15) is the standard form in MPC, which satisfies

$$\alpha_0(|x_i(k|k) - x_{s,i}^\dagger|) \leq W_i(x_i(k|k)) \leq \alpha_W(|x_i(k|k) - x_{s,i}^\dagger|), \tag{16}$$

where $\alpha_0, \alpha_W : \mathcal{R} \to \mathcal{R}$ are both suitable $\mathcal{K}_\infty$ functions [25].

The difference of $W=\sum_{i\in\mathcal{V}}W_i$ is

$$\begin{aligned}\triangle W(k) \leq &\sum_{i\in\mathcal{V}}(J_i(\bar{x}_i(k+1|k+1),\bar{\boldsymbol{u}}_i(k+1|k+1),x^*_{s,i}(k))\\&-J_i(x^*_i(k|k),\boldsymbol{u}^*_i(k|k),x^*_{s,i}(k)))\\=&\sum_{i\in\mathcal{V}}(\|\bar{x}_i(k+N_p+1|k+1)-x^*_{s,i}(k)\|^2_{P_{f,i}}\\&-\|x^*_i(k+N_p|k)-x^*_{s,i}(k)\|^2_{P_{f,i}}\\&+\|\bar{x}_i(k+N_p|k+1)-x^*_{s,i}(k)\|^2_{Q_i}\\&+\|\kappa_{f,i}-u^*_{s,i}(k)\|^2_{R_i}-\|x^*_i(k|k)-x^*_{s,i}(k)\|^2_{Q_i}\\&-\|u^*_i(k|k)-u^*_{s,i}(k)\|^2_{R_i})\\\leq&\sum_{i\in\mathcal{V}}-\|x^*_i(k|k)-x^*_{s,i}(k)\|^2_{Q_i}.\end{aligned}$$

There are $W\geq 0$ and $\triangle W\leq 0$, then it can be obtained that $\lim_{k\to\infty}W(k)=0$. Combined $\lim_{k\to\infty}\|\bar{x}_{s,i}(k+1)-x^*_{s,i}\|=0$ and (16), it follows that $\lim_{k\to\infty}\alpha_0(|x_i(k|k)-x^\dagger_{s,i}|)=0$, i.e. $\lim_{k\to\infty}\|x^*_i(k|k)-x^\dagger_{s,i}\|=0$, $\lim_{k\to\infty}\|u^*_i(k|k)-u^\dagger_{s,i}\|=0$ and $y_{c,i}$, $y_{n,i}$ can reach $y_{d,con}$, $y_{d,i}$ respectively.

According to $x_i(k)-x_i(k|k)\in 2\chi_{e,i}(k)$, the real outputs can achieve the objectives in (2) and reach the neighborhoods of the set points, whose bounds are $\sigma_{c,i}\leq\sqrt{2}\|C_{c,i}\hat{R}_i(k)\|$ and $\sigma_{n,i}\leq\sqrt{2}\|C_{n,i}\hat{R}_i(k)\|$. In addition, because of the satisfaction of (9c) and (9d), $x_i(k)$ and $u_i(k)$ can always satisfy the state and input constraint sets. Theorem 2 is proven. ■

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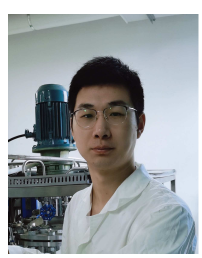

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