On an ecumenical natural deduction with *stoup* - Part I: The propositional case

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Abstract. In 2015 Dag Prawitz proposed a natural deduction ecumenical system, where classical logic and intuitionistic logic are codified in the same system. In his ecumenical system, Prawitz recovers the harmony of rules, but the rules for the classical operators do not satisfy separability. In fact, the classical rules are not pure, in the sense that negation is used in the definition of the introduction and elimination rules for the classical operators. In this work we propose an ecumenical system adapting, to the natural deduction framework, Girard's mechanism of stoup. This allows for the definition of a pure harmonic natural deduction system for the propositional fragment of Prawitz's ecumenical logic. We conclude by presenting an extension proposal to the first-order setting and a different approach to purity based on some ideas proposed by Julien Murzi.

1 Introduction

Natural deduction systems, as proposed by Gentzen [Gen69] and further studied by Prawitz [Pra65], form one of the most well known proof-theoretical frameworks. Part of their success is based on the fact that natural deduction rules provide a simple characterization of logical constants, especially in the case of intuitionistic logic. However, there has been a lot of criticism on extensions of the set of intuitionistic rules in order to deal with classical logic. Indeed, most of such extensions add, to the usual introduction and elimination rules, extra rules governing negation. As a consequence, several meta-logical properties, the most prominent one being *harmony*, are lost.[3]

In [Pra15], Dag Prawitz proposed an *ecumenical* natural deduction system, where classical logic and intuitionistic logic are codified in the same system. In this system, the classical logician and the intuitionistic logician would share the

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[3] A logical connective is called *harmonious* in a certain proof system if there exists a certain balance between the rules defining it. For example, in natural deduction based systems, harmony is ensured when introduction/elimination rules do not contain insufficient/excessive amounts of information [Sch14,DCD21].

universal quantifier, conjunction, negation and the constant for the absurdity $(\forall, \wedge, \neg, \bot)$, but they would each have their own existential quantifier, disjunction and implication, with different meanings $(\exists_j, \vee_j, \rightarrow_j$, where $j \in \{i, c\}$ for the intuitionistic and classical versions, respectively). Prawitz' main idea is that these different meanings are given by a semantical framework that can be accepted by both parties.

In his ecumenical system, Prawitz recovers the harmony of rules, but the rules for the classical operators do not satisfy *separability* [Mur20]. In fact, the classical rules are not *pure*, in the sense that negation is used in the definition of the introduction and elimination rules for the classical operators.

For example, the rules for $\vee_c$ are defined as

$$\dfrac{\begin{array}{c}[\neg A, \neg B]\\ \Pi\\ \bot\end{array}}{A \vee_c B}\ \vee_c\text{-int} \qquad\qquad \dfrac{A \vee_c B \qquad \neg A \qquad \neg B}{\bot}\ \vee_c\text{-elim}$$

The situation is not different in the case of the definition of left and right rules for these classical operators in a *sequent calculus* codification, as presented in [PPdP21]. The rules for $\vee_c$, for example, are defined as

$$\dfrac{\Delta, \neg A, \neg B \Rightarrow \bot}{\Delta \Rightarrow A \vee_c B}\ \vee_c\text{-R} \qquad \dfrac{\Delta_1 \Rightarrow \neg A \qquad \Delta_2 \Rightarrow \neg B}{A \vee_c B, \Delta_1, \Delta_2 \Rightarrow \bot}\ \vee_c\text{-L}$$

where $\Delta, \Delta_1, \Delta_2$ are multisets of formulas.

There are many ways of proposing pure, harmonic natural deduction systems for (propositional) classical logic. Indeed, Murzi [Mur20] proposes a new set of rules for classical logical operators based on absurdity as a punctuation mark, and higher-level rules [Sch14]. D'Agostino [DAg05], on the other hand, brings a totally different approach, presenting a theory of classical natural deduction that makes a distinction between operational rules, governing the use of logical operators, and structural rules dealing with the metaphysical assumptions governing the (classical) notions of truth and falsity, namely the principle of bivalence and the principle of non-contradiction.

Yet another approach appears in [GG05], where Michael and Murdoch Gabbay present the natural deduction version of Dov Gabbay's *Restart* rule

$$\dfrac{A}{B}\ \textit{Restart}$$

with the side-condition that, below every occurrence of *Restart* from A to B, there is (at least) one occurrence of A. The intended meaning is that B is a new start to a line of reasoning concluding A. For example, in the derivation of the Peirce's Law

$$\dfrac{\dfrac{[(A \rightarrow B) \rightarrow A)] \qquad \dfrac{\dfrac{[A]}{B}\ \textit{Restart}^\star}{A \rightarrow B}\ \rightarrow\text{-int}}{A^\dagger}\ \rightarrow\text{-elim}}{((A \rightarrow B) \rightarrow A) \rightarrow A)}\ \rightarrow\text{-int}$$

the restart at $\star$ is justified at $\dagger$.

Similar to Gabbay & Gabbay's approach, Restall's *Alt* rule [Res21]

$$\dfrac{\begin{matrix}\Pi \\ A\end{matrix}}{B}\; Alt, \downarrow A$$

has the following interpretation: having a proof Π of A, one can set A aside and consider some alternative conclusion, B. In the the remaining of the proof, the conclusion A is not ignored, but added to the collection of alternatives current at this point of the proof, which can then be used at some point of the proof.

Peirce's Law with alternatives looks the same as with restart

$$\dfrac{\dfrac{[(A \to B) \to A)] \qquad \dfrac{\dfrac{[A]}{B}\; Alt, \downarrow A}{A \to B}\; \to\text{-int}}{A}\; \to\text{-elim}, \uparrow A}{((A \to B) \to A) \to A)}\; \to\text{-int}$$

It turns out that Gabbay & Gabbay/Restall's approaches can be regarded as particular cases of systems with *stoup* [Par92,Gir91].[4] In such systems, the nodes in derivations have the form $\Delta; \Sigma$, where Δ is a set or multiset of formulas, called the *classical context*, and Σ, the *stoup*, is a multiset containing at most one formula. In this formulation the premise formula of the *Restart* and *Alt* rules moves in/out the classical context via the *dereliction* and *store* rules:

$$\dfrac{\Delta; A}{\Delta, A; B}\; \text{der} \qquad\qquad \dfrac{\Delta, A; \cdot}{\Delta; A}\; \text{store}$$

Considering Δ a set, the proof of Peirce's Law using *stoup* has the following general form

$$\dfrac{\dfrac{\dfrac{\dfrac{[(A \to B) \to A)] \qquad \dfrac{\dfrac{[A]}{A; B}\; \text{der}}{A; A \to B}\; \to\text{-int}}{A; A}\; \to\text{-elim}}{A; \cdot}\; \text{der}}{\cdot; A}\; \text{store}}{\cdot; ((A \to B) \to A) \to A)}\; \to\text{-int}$$

The use of store and dereliction appears hence as a proof theoretic *evidence* of abandoning/making use of the formula A in the *Restart* and *Alt* rules.

[4] It should be noted that this idea appears somewhat hidden in [Res21] for the propositional classical case, where Restall uses sequent style $X \succ A; Y$ (the 'score'), where X represents the undischarged assumptions, A the current conclusion, and Y the *alternatives*.

In this paper, we adapt Parigot and Girard's *stoup* mechanism to the ecumenical setting. This will allow the definition of a pure harmonic natural deduction system $\mathcal{LE}_p$ for the propositional fragment of Prawitz' ecumenical logic.[5]

2 Ecumenical natural deduction system

The language $\mathcal{L}$ used for ecumenical systems is described as follows. We will use a subscript c for the classical meaning and i for the intuitionistic one, dropping such subscripts when formulae/connectives can have either meaning.

Classical and intuitionistic n-ary predicate symbols $(p_c, p_i, \ldots)$ co-exist in $\mathcal{L}$ but have different meanings. The neutral logical connectives $\{\bot, \neg, \wedge, \forall\}$ are common for classical and intuitionistic fragments, while $\{\rightarrow_i, \vee_i, \exists_i\}$ and $\{\rightarrow_c, \vee_c, \exists_c\}$ are restricted to intuitionistic and classical interpretations, respectively. In order to avoid clashes of variables we make use of a denumerable set $a, b, \ldots$ of special variables called *parameters*, which do not appear quantified.

In Fig. 1 we present $\mathcal{NE}$ Prawitz' original ecumenical first-order natural deduction system. In the rules for quantifiers, the notation $A(a/x)$ stands for the substitution of a for every (visible) instance of x in A. In the rules $\exists_i$-elim, $\forall$-int, a is a fresh parameter, *i.e.*, it does not occur free in any assumption that A, B depend on (apart from the assumption discharged by $\exists_i$-elim).

The propositional fragment of the ecumenical natural deduction system proposed by Prawitz (here called $\mathcal{NE}_p$) has been proved normalizing, sound and complete with respect to intuitionistic logic's Kripke semantics in [PR17].

The rules for intuitionistic implication are the traditional ones, while the rules for classical implication make sure that $A \rightarrow_c B$ is treated as $\neg A \vee_c B$, its classical rendering. The surprising facts are that (i) one can have a single constant for absurdity $\bot$ (instead of two, one intuitionistic and one classical, taking that absurd as the unit of disjunction, of which we have two variants) and (ii) that the intuitionistic and classical negations coincide. If negation were simply implication into false (as it is the case for intuitionistic negation) one might expect two negations, one intuitionistic and one classical.

3 Sequent Calculus with *stoup*

LC [Gir91] is a sequent system for classical logic that separates the rules for *positive* and *negative* formulas, being a precursor of the notion of *focusing* in sequent systems [And92]. The polarity of a formula in LC is determined by its outermost connective and the polarity of its subformulas. For example, atoms and unities are always positive and, if P, Q are positive formulas, then $P \wedge Q$ is also positive. The table of polarities can be checked in [Gir91], page 10.

[5] It is important to add, at this point, that there are some other natural deduction systems for combining intuitionistic and classical logics, such as [BD+21]. Moreover, in [LM11] an ecumenical sequent system having LC as a fragment is presented. These systems, however, have little intersection with the present work.

Intuitionistic rules

$$\frac{A \to_i B \quad A}{B}\ \to_i\text{-elim} \qquad \frac{\begin{array}{c}[A]\\ \Pi\\ B\end{array}}{A \to_i B}\ \to_i\text{-int} \qquad \frac{A \vee_i B \quad \begin{array}{c}[A]\\ \Pi_1\\ C\end{array} \quad \begin{array}{c}[B]\\ \Pi_2\\ C\end{array}}{C}\ \vee_i\text{-elim}$$

$$\frac{A_j}{A_1 \vee_i A_2}\ \vee_i\text{-int}_j \qquad \frac{\exists_i x.A \quad \begin{array}{c}[A(a/x)]\\ \Pi\\ B\end{array}}{B}\ \exists_i\text{-elim} \qquad \frac{A(a/x)}{\exists_i x.A}\ \exists_i\text{-int}$$

Classic rules

$$\frac{A \to_c B \quad A \quad \neg B}{\bot}\ \to_c\text{-elim} \qquad \frac{\begin{array}{c}[A, \neg B]\\ \Pi\\ \bot\end{array}}{A \to_c B}\ \to_c\text{-int} \qquad \frac{A \vee_c B \quad \neg A \quad \neg B}{\bot}\ \vee_c\text{-elim}$$

$$\frac{\begin{array}{c}[\neg A, \neg B]\\ \Pi\\ \bot\end{array}}{A \vee_c B}\ \vee_c\text{-int} \qquad \frac{\exists_c x.A \quad \forall x.\neg A}{\bot}\ \exists_c\text{-elim} \qquad \frac{\begin{array}{c}[\forall x.\neg A]\\ \Pi\\ \bot\end{array}}{\exists_c x.A}\ \exists_c\text{-int}$$

$$\frac{p_c \quad \neg p_i}{\bot}\ p_c\text{-elim} \qquad \frac{\begin{array}{c}[\neg p_i]\\ \Pi\\ \bot\end{array}}{p_c}\ p_c\text{-int}$$

Neutral rules

$$\frac{A_1 \wedge A_2}{A_j}\ \wedge\text{-elim}_j \qquad \frac{A \quad B}{A \wedge B}\ \wedge\text{-int} \qquad \frac{A \quad \neg A}{\bot}\ \neg\text{-elim} \qquad \frac{\begin{array}{c}[A]\\ \Pi\\ \bot\end{array}}{\neg A}\ \neg\text{-int}$$

$$\frac{\bot}{A}\ \bot\text{-elim} \qquad \frac{\forall x.A}{A(a/x)}\ \forall\text{-elim} \qquad \frac{A(a/x)}{\forall x.A}\ \forall\text{-int}$$

Fig. 1. Ecumenical natural deduction system $\mathcal{NE}$. In rules $\forall$-int and $\exists_i$-elim, the parameter a is fresh.

In LC, sequents have the form $\Rightarrow \Delta; \Sigma$, where Δ, Σ are multisets of formulas, with Σ, called the *stoup*, containing *at most* one formula. The main idea is that the *stoup* controls the rule applications, in the sense that a positive active formula in the conclusion of a rule is always placed there, while active negative formulas are handled in the classical context. Characteristic examples are the rules for the conjunction of positive/negative formulas

$$\frac{\Rightarrow \Delta_1; P \quad \Rightarrow \Delta_2; Q}{\Rightarrow \Delta_1, \Delta_2; P \wedge Q} \wedge_p \qquad \frac{\Rightarrow \Delta, N; \Sigma \quad \Rightarrow \Delta, M; \Sigma}{\Rightarrow \Delta, M \wedge N; \Sigma} \wedge_n$$

Observe that they also have a multiplicative/additive flavor.

The idea of focusing is also present in LC, with the *dereliction* and *store* rules

$$\frac{\Rightarrow \Delta; P}{\Rightarrow \Delta, P; \cdot} \text{ der} \qquad \frac{\Rightarrow \Delta, N; \cdot}{\Rightarrow \Delta; N} \text{ store}$$

On a bottom-up reading of the rules, while in der positive formulas can be chosen to be focused on, in *store* negative formulas are stored in the classical context. This enables for a *two-phase* proof construction, where the focused formula P is systematically decomposed until reaching a leaf or a negative sub-formula N. In this last case, focusing is lost and N is stored, allowing for the beginning of a new focused phase.

Finally, due to polarities, LC has two admissible cut rules

$$\frac{\Rightarrow \Delta_1; P \quad \Rightarrow \neg P, \Delta_2; \Sigma}{\Rightarrow \Delta_1, \Delta_2; \Sigma} \text{ p-cut} \qquad \frac{\Rightarrow \Delta_1, N; \cdot \quad \Rightarrow \neg N, \Delta_2; \Sigma}{\Rightarrow \Delta_1, \Delta_2; \Sigma} \text{ n-cut}$$

In LC, the sequent $\Rightarrow \Delta; \Sigma$ has an intuitionistic interpretation: $\neg\Delta \Rightarrow \bot$ if Σ is empty and $\neg\Delta \Rightarrow A$ if $\Sigma = A$. Via an implicit double negation elimination, this implies that the context Δ makes the classical information persistent, in the same way as the standard double negation rule in natural deduction systems for classical logic.

This implies that sequents with empty *stoup* have also a classical interpretation, using *e.g.* Gödel's double negation translation. Sequents with non-empty stoup do not have a classical interpretation, as discussed in [Gir91].

Hence one could say that sequents with stoup have a certain ecumenical flavor: formulas with intuitionistic behavior are identified as being *positive*, while formulas with classical behavior are identified as being *negative*. Surprisingly, this is reflected in the composition steps presented in Section 7.1.

In this work, we will carry out a similar idea under the spectrum of Prawitz' ecumenical natural deduction system. While it has some similarities with Girard's original proposal, our system will not consider polarities, and all the conclusion formulas of introduction/elimination rules will be placed in the *stoup*.

4 Natural Deduction with *stoup* - the propositional system $\mathcal{LE}_p$

We will now incorporate the notion of *stoup* to natural deduction in the case of propositional logic, showing its natural connection to the ecumenical setting.

Let the expression $\Delta; \Sigma$ denote a *stoup with a context* (abreviated as *stp-c*), an extension of natural deduction formulas, where Σ is the *stoup* and Δ is its accompanying context (called *alternatives* in [Res21]). As for the case of LC, the *stoup* will carry the intuitionistic (positive) and neutral information, while the context accumulates the classical information related to it.

In the following, we will construct ecumenical introduction and elimination rules over the *stoup* in a step-by-step manner, justifying all our choices.

4.1 Hypothesis formation and dereliction

The hypothesis formation is the usual one and, as in LC, dereliction is needed for guaranteeing the completeness of the system, since ecumenical active formulas are always placed in the *stoup*.

Hypothesis formation

$$\cdot; A$$

Dereliction

$$\dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \Delta; A\end{array}}{\Delta, A; \cdot}\ \text{der}$$

4.2 Structural rules

On choosing the multiplicative version of rules, we need structural rules acting in the classical context, so to transform multisets into sets. As usual in intuitionistic systems, weakening is also allowed in the *stoup*.

Weakening

$$\dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \Delta; \cdot\end{array}}{\Delta; A}\ W_i \qquad\qquad \dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \Delta; C\end{array}}{\Delta, A; C}\ W_c$$

Contraction

$$\dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \Delta, A, A; C\end{array}}{\Delta, A; C}\ C_c$$

4.3 Intuitionistic operators

Implication. The following result, proved in [PPdP21] for the ecumenical sequent calculus system, states that logical consequence in $\mathcal{NE}$ is interpreted intuitionistically.

Theorem 1. *Let Γ be a set of ecumenical formulas. Then B is provable from Γ in ecumenical logic iff $\bigwedge \Gamma \rightarrow_i B$ is provable in $\mathcal{NE}$.*

Hence the natural deduction introduction rule for implication

$$\frac{\begin{array}{c}[A]\\ \Pi\\ B\end{array}}{A \rightarrow B}\ \rightarrow$$

induces the ecumenical intuitionistic version with *stoup*

$$\frac{\begin{array}{c}[\cdot; A] \quad \Gamma\\ \Pi\\ \Delta; B\end{array}}{\Delta; A \rightarrow_i B}\ \rightarrow_i\text{-int}$$

where Γ is a multiset of open assumptions.

For the implication elimination rule, observe that different *stoups* carry different contexts, so we will have the multiplicative version of the rule, combining the classical information

$$\frac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ \Delta_1; A \rightarrow_i B\end{array} \qquad \begin{array}{c}\Gamma_2\\ \Pi_2\\ \Delta_2; A\end{array}}{\Delta_1, \Delta_2; B}\ \rightarrow_i\text{-elim}$$

Disjunction. The rule for introduction is

$$\frac{\begin{array}{c}\Gamma\\ \Pi\\ \Delta; A_i\end{array}}{\Delta; A_1 \vee_i A_2}\ \vee_i\text{-int}$$

while the elimination rule combines all the context information

$$\frac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ \Delta_1; A \vee_i B\end{array} \qquad \begin{array}{c}[\cdot; A] \quad \Gamma_2\\ \Pi_2\\ \Delta_2; C\end{array} \qquad \begin{array}{c}[\cdot; B] \quad \Gamma_3\\ \Pi_3\\ \Delta_3; C\end{array}}{\Delta_1, \Delta_2, \Delta_3; C}\ \vee_i\text{-elim}$$

4.4 Classical operators

Implication. Observe that the negated assumptions in $\mathcal{NE}$ will correspond to the classical counterpart of the stoup, so the consequent of the implication will be stored in this context. The introduction rule

$$\dfrac{\begin{array}{c}[A, \neg B]\\ \Pi\\ \bot\end{array}}{A \to_c B}\ \to_c\text{-int}$$

then becomes

$$\dfrac{\begin{array}{c}[\cdot; A] \qquad \Gamma\\ \Pi\\ \Delta, B; \cdot\end{array}}{\Delta; A \to_c B}\ \to_c\text{-int}$$

For the elimination rule, also the negated formula in the premise become classical, this time with empty stoup.

$$\dfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ \Delta_1; A \to_c B\end{array} \qquad \begin{array}{c}\Gamma_2\\ \Pi_2\\ \Delta_2; A\end{array} \qquad \begin{array}{c}[\cdot; B] \qquad \Gamma_3\\ \Pi_3\\ \Delta_3; \cdot\end{array}}{\Delta_1, \Delta_2, \Delta_3; \cdot}\ \to_c\text{-elim}$$

Disjunction. The same idea applies to disjunction, where negated assumptions become part of the classical context, while positive assumptions are kept as *stoups*

$$\dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \Delta, A, B; \cdot\end{array}}{\Delta; A \vee_c B}\ \vee_c\text{-int}$$

$$\dfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ \Delta_1; A \vee_c B\end{array} \qquad \begin{array}{c}[\cdot; A] \qquad \Gamma_2\\ \Pi_2\\ \Delta_2; \cdot\end{array} \qquad \begin{array}{c}[\cdot; B] \qquad \Gamma_3\\ \Pi_3\\ \Delta_3; \cdot\end{array}}{\Delta_1, \Delta_2, \Delta_3; \cdot}\ \vee_c\text{-elim}$$

4.5 Neutral operators

Negation. Since negation can be defined in classical/intuitionistic logic as "implies bottom", the rules for $\neg$ can be derived from the ones for implication with an empty *stoup*.

$$\dfrac{\begin{array}{c}[\cdot; A] \qquad \Gamma\\ \Pi\\ \Delta; \cdot\end{array}}{\Delta; \neg A}\ \neg\text{-int} \qquad\qquad \dfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ \Delta_1; A\end{array} \qquad \begin{array}{c}\Gamma_2\\ \Pi_2\\ \Delta_2; \neg A\end{array}}{\Delta_1, \Delta_2; \cdot}\ \neg\text{-elim}$$

Conjunction. Since our ecumenical system is essentially intuitionistic (in the terms of Theorem 1), all active formulas are placed in the *stoup*. Hence we will adopt the multiplicative version of Girard's rule for positive conjuncts.

$$\dfrac{\begin{matrix}\Gamma_1\\ \Pi_1\\ \Delta_1; A\end{matrix} \qquad \begin{matrix}\Gamma_2\\ \Pi_2\\ \Delta_2; B\end{matrix}}{\Delta_1, \Delta_2; A \wedge B}\ \wedge\text{-int} \qquad\qquad \dfrac{\begin{matrix}\Gamma\\ \Pi\\ \Delta; A_1 \wedge A_2\end{matrix}}{\Delta; A_j}\ \wedge\text{-elim}_j$$

Derivations are then inductively defined in the usual way.

Definition 1. *We say that the stp-c $\Delta; \Sigma$ is* derivable *from a set Γ of stp-cs in $\mathcal{LE}_p$ (denoted by $\Gamma \vdash_{\mathcal{LE}_p} \Delta; \Sigma$) if and only if there is a derivation of $\Delta; \Sigma$ from Γ. A formula A is a* theorem *of $\mathcal{LE}_p$ if and only if $\vdash_{\mathcal{LE}_p} \cdot; A$.*

As usual, we may also add indices in derivations, for relating a discharged assumption with a specific rule application.

5 Examples

We present below the proofs of some classical tautologies in $\mathcal{LE}_p$.

1. *Peirce's Law*

$$3\ \dfrac{\dfrac{2\ \dfrac{[\cdot; ((A \rightarrow_c B) \rightarrow_c A)]^3 \qquad 1\ \dfrac{\dfrac{\dfrac{[\cdot; A]^1}{A; \cdot}\ \text{der}}{A, B; \cdot}\ W_c}{A; (A \rightarrow_c B)}\ \rightarrow_c\text{-int} \qquad \dfrac{[\cdot; A]^2}{A; \cdot}\ \text{der}}{A, A; \cdot}\ \rightarrow_c\text{-elim}}{A; \cdot}\ C_c}{\cdot; (((A \rightarrow_c B) \rightarrow_c A) \rightarrow_c A)}\ \rightarrow_c\text{-int}$$

   Observe that the only difference of the proof above w.r.t. *Alt* or *Restart* systems lies in the use of structural rules in the classical context.
   More interestingly, note that any sequent of the form $(((A \rightarrow_j B) \rightarrow_k A) \rightarrow_c A)$ with $j, k \in \{i, c\}$ is provable in $\mathcal{LE}_p$. That is, provability is maintained if the outermost implication is classical.
2. *Excluded-middle*

$$\dfrac{\dfrac{1\ \dfrac{\dfrac{[\cdot; A]^1}{A; \cdot}\ \text{der}}{A; \neg A}\ \neg\text{-int}}{A, \neg A; \cdot}\ \text{der}}{\cdot; (A \vee_c \neg A)}\ \vee_c\text{-int}$$

   Of course, $\cdot; A \vee_i \neg A$ is not a theorem in $\mathcal{LE}_p$.
3. *Dummett's linearity axiom*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\cdot; A]^1}{A; \cdot}\,\text{der}}{A, B; \cdot}\,W_c}{A; (A \rightarrow_c B)}\,\rightarrow_c\text{-int}\ (1)}{A, (A \rightarrow_c B); \cdot}\,\text{der}}{(A \rightarrow_c B); (B \rightarrow_c A)}\,\rightarrow_c\text{-int}}{(A \rightarrow_c B), (B \rightarrow_c A)); \cdot}\,\text{der}}{\cdot; ((A \rightarrow_c B) \vee_c (B \rightarrow_c A))}\,\vee_c\text{-int}
$$

This is also an interesting case, where any sequent of the form $((A \rightarrow_j B) \vee_c (B \rightarrow_k A))$ with $j, k \in \{i, c\}$ is provable in $\mathcal{LE}_p$. That is, provability is maintained if the outermost disjunction is classical.

6 Systems equivalence

In the following, we will show that $\mathcal{LE}_p$ is correct and complete w.r.t. $\mathcal{NE}_p$. We will use the following extra notation:

- Given a multiset Δ of formulas, we denote by $\neg\Delta$ the multiset formed by the negation of each formula in Δ.
- If Γ is a set of formulas, we denote by $\Gamma \vdash_{\mathcal{NE}_p} A$ the fact that the formula A depends on the set Γ of assumptions in $\mathcal{NE}_p$.
- If $\Gamma = \bigcup\{\cdot; A_i\}_{0 \leq i \leq n}$ is a multiset of *stoup* with empty contexts in $\mathcal{LE}_p$, we will abuse the notation and also represent by Γ the underlying set of formulas in these *stp-c*, that is, $\Gamma = \bigcup\{A_i\}_{0 \leq i \leq n}$ in $\mathcal{NE}_p$.

Theorem 2. *Let $\Gamma = \bigcup\{\cdot; A_i\}_{0 \leq i \leq n}$ be a set of hypothesis. Then $\Gamma \vdash_{\mathcal{LE}_p} \Delta; \Sigma$ iff $\Gamma, \neg\Delta \vdash_{\mathcal{NE}_p} \Sigma$. In case Σ is empty, we have $\Gamma, \neg\Delta \vdash_{\mathcal{NE}_p} \bot$.*

Proof. By induction on the length of derivations in $\mathcal{LE}_p$ and $\mathcal{NE}_p$. The only interesting cases are the ones involving classical connectives and structural rules.

– Case $\vee_c$-int. Suppose that we have the following derivation in $\mathcal{LE}_p$:

$$
\dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \Delta, A, B; \cdot\end{array}}{\Delta; A \vee_c B}\,\vee_c\text{-int}
$$

By the inductive hypothesis, $\Gamma, \neg A, \neg B, \neg\Delta \vdash_{\mathcal{NE}_p} \bot$. We can then take the desired derivation to be:

$$
n, m\ \dfrac{\begin{array}{c}\Gamma \quad \neg\Delta \quad [\neg A]^n \quad [\neg B]^m\\ \Pi'\\ \bot\end{array}}{A \vee_c B}\,\vee_c\text{-int}
$$

On the other hand, suppose that $\Gamma, \neg\Delta \vdash_{\mathcal{NE}_p} A \vee_c B$ with proof

$$\dfrac{\begin{array}{c} \mathit{\Gamma} \quad \neg\mathit{\Delta} \quad [\neg A] \quad [\neg B] \\ \mathit{\Pi} \\ \bot \end{array}}{A \vee_c B}\ \vee_c\text{-int}$$

By the inductive hypothesis, we have $\mathit{\Gamma} \vdash_{\mathcal{LE}_p} \mathit{\Delta}, A, B; \cdot$. We can then take the desired derivation to be:

$$\dfrac{\begin{array}{c} \mathit{\Gamma} \\ \mathit{\Pi} \\ \mathit{\Delta}, A, B; \cdot \end{array}}{\mathit{\Delta}; A \vee_c B}\ \vee_c\text{-int}$$

– Case $\vee_c$-elim. Suppose that $\mathit{\Gamma}_1, \mathit{\Gamma}_2, \mathit{\Gamma}_3 \vdash_{\mathcal{LE}_p} \mathit{\Delta}_1, \mathit{\Delta}_2, \mathit{\Delta}_3; \cdot$ with proof

$$\dfrac{\begin{array}{c} \mathit{\Gamma}_1 \\ \mathit{\Pi}_1 \\ \mathit{\Delta}_1; A \vee_c B \end{array} \qquad \begin{array}{c} \mathit{\Gamma}_2 \quad [\cdot; A] \\ \mathit{\Pi}_2 \\ \mathit{\Delta}_2; \cdot \end{array} \qquad \begin{array}{c} \mathit{\Gamma}_3 \quad [\cdot; B] \\ \mathit{\Pi}_3 \\ \mathit{\Delta}_3; \cdot \end{array}}{\mathit{\Delta}_1, \mathit{\Delta}_2, \mathit{\Delta}_3; \cdot}\ \vee_c\text{-elim}$$

By the inductive hypothesis we have: $\mathit{\Gamma}_1, \neg\mathit{\Delta}_1 \vdash_{\mathcal{NE}_p} A \vee_c B$, $\mathit{\Gamma}_2, \neg\mathit{\Delta}_2, A \vdash_{\mathcal{NE}_p} \bot$ and $\mathit{\Gamma}_3, \neg\mathit{\Delta}_3, B \vdash_{\mathcal{NE}_p} \bot$. We can the obtain the desired derivation as follows:

$$\dfrac{\begin{array}{c} \mathit{\Gamma}_1 \quad \neg\mathit{\Delta}_1 \\ \mathit{\Pi}'_1 \\ A \vee_c B \end{array} \qquad \begin{array}{c} \mathit{\Gamma}_2 \quad \neg\mathit{\Delta}_2 \quad [A]^n \\ \mathit{\Pi}'_2 \\ n\ \dfrac{\bot}{\neg A}\ \neg\text{-int} \end{array} \qquad \begin{array}{c} \mathit{\Gamma}_3 \quad \neg\mathit{\Delta}_3 \quad [B]^m \\ \mathit{\Pi}'_3 \\ m\ \dfrac{\bot}{\neg B}\ \neg\text{-int} \end{array}}{\bot}\ \vee_c\text{-elim}$$

On the other hand, suppose that $\mathit{\Gamma}_1, \mathit{\Gamma}_2, \mathit{\Gamma}_3, \neg\mathit{\Delta}_1, \neg\mathit{\Delta}_2, \neg\mathit{\Delta}_3 \vdash_{\mathcal{NE}_p} \bot$ with the following derivation:

$$\dfrac{\begin{array}{c} \mathit{\Gamma}_1 \quad \neg\mathit{\Delta}_1 \\ \mathit{\Pi}_1 \\ A \vee_c B \end{array} \qquad \begin{array}{c} \mathit{\Gamma}_2 \quad \neg\mathit{\Delta}_2 \\ \mathit{\Pi}_2 \\ \neg A \end{array} \qquad \begin{array}{c} \mathit{\Gamma}_3 \quad \neg\mathit{\Delta}_3 \\ \mathit{\Pi}_3 \\ \neg B \end{array}}{\bot}\ \vee_c\text{-elim}$$

By the inductive hypothesis: $\mathit{\Gamma}_1, \vdash_{\mathcal{LE}_p} \mathit{\Delta}_1; A \vee_c B$, $\mathit{\Gamma}_2, \vdash_{\mathcal{LE}_p} \mathit{\Delta}_2; \neg A$ and $\mathit{\Gamma}_3 \vdash_{\mathcal{LE}_p} \mathit{\Delta}_3; \neg B$. We can then construct our desired derivation as:

$$\dfrac{\begin{array}{c} \mathit{\Gamma}_1 \\ \mathit{\Pi}'_1 \\ \mathit{\Delta}_1; A \vee_c B \end{array} \qquad \dfrac{\begin{array}{c} \mathit{\Gamma}_2 \\ \mathit{\Pi}'_2 \\ \mathit{\Delta}_2; \neg A \end{array} \quad [\cdot; A]}{\mathit{\Delta}_2; \cdot}\ \neg\text{-int} \qquad \dfrac{\begin{array}{c} \mathit{\Gamma}_3 \\ \mathit{\Pi}'_3 \\ \mathit{\Delta}_3; \neg B \end{array} \quad [\cdot; B]}{\mathit{\Delta}_3; \cdot}\ \neg\text{-int}}{\mathit{\Delta}_1, \mathit{\Delta}_2, \mathit{\Delta}_3; \cdot}\ \vee_c\text{-elim}$$

– Case $\rightarrow_c$-int. Suppose that $\mathit{\Gamma} \vdash_{\mathcal{LE}_p} \mathit{\Delta}; A \rightarrow_c B$ with a derivation as:

$$\dfrac{\begin{array}{c}\mathit{\Gamma} \qquad [\cdot;A] \\ \mathit{\Pi} \\ \mathit{\Delta}, B;\cdot\end{array}}{\mathit{\Delta}; A \to_c B}\ \to_c\text{-int}$$

By the inductive hypothesis, we have $\mathit{\Gamma}, A, \neg B, \neg\mathit{\Delta} \vdash_{\mathcal{NE}_p} \bot$. We can then construct our desired derivation in $\mathcal{NE}_p$ as:

$$\dfrac{\begin{array}{c}\mathit{\Gamma} \qquad [A] \qquad [\neg B] \qquad \neg\mathit{\Delta} \\ \mathit{\Pi}' \\ \bot\end{array}}{A \to_c B}\ \to_c\text{-int}$$

In the other direction, suppose that $\mathit{\Gamma}, \neg\mathit{\Delta} \vdash_{\mathcal{NE}_p} A \to_c B$ with a derivation as:

$$\dfrac{\begin{array}{c}\mathit{\Gamma} \qquad \neg\mathit{\Delta} \qquad [A] \qquad [\neg B] \\ \mathit{\Pi} \\ \bot\end{array}}{A \to_c B}\ \to_c\text{-int}$$

By the inductive hypothesis, we have $\mathit{\Gamma} \cup \{\cdot; A\} \vdash_{\mathcal{LE}_p} \mathit{\Delta}, B;\cdot$. Thus, we can obtain the desired derivation in $\mathcal{LE}_p$ as:

$$\dfrac{\begin{array}{c}\mathit{\Gamma} \qquad [\cdot;A] \\ \mathit{\Pi}' \\ \mathit{\Delta}, B;\cdot\end{array}}{\mathit{\Delta}; A \to_c B}\ \vee_c\text{-elim}$$

– Case $\to_c$-elim. Suppose that $\mathit{\Gamma}_1, \mathit{\Gamma}_2, \mathit{\Gamma}_3 \vdash_{\mathcal{LE}_p} \mathit{\Delta}_1, \mathit{\Delta}_2, \mathit{\Delta}_3;\cdot$ with a derivation as:

$$\dfrac{\begin{array}{c}\mathit{\Gamma}_1 \\ \mathit{\Pi}_1 \\ \mathit{\Delta}_1; A \to_c B\end{array} \qquad \begin{array}{c}\mathit{\Gamma}_2 \\ \mathit{\Pi}_2 \\ \mathit{\Delta}_2; A\end{array} \qquad \begin{array}{c}\mathit{\Gamma}_3 \qquad [\cdot;B] \\ \mathit{\Pi}_3 \\ \mathit{\Delta}_3;\cdot\end{array}}{\mathit{\Delta}_1, \mathit{\Delta}_2, \mathit{\Delta}_3;\cdot}\ \to_c\text{-elim}$$

By inductive hypothesis, we have $\mathit{\Gamma}_1, \neg\mathit{\Delta}_1 \vdash_{\mathcal{NE}_p} A \to_c B$, $\mathit{\Gamma}_2, \neg\mathit{\Delta}_2 \vdash_{\mathcal{NE}_p} A$ and $\mathit{\Gamma}_3, \neg\mathit{\Delta}_3, B \vdash_{\mathcal{NE}_p} \bot$. Then, we can obtain the desired derivation in $\mathcal{NE}_p$ as:

$$\dfrac{\begin{array}{c}\mathit{\Gamma}_1 \qquad \neg\mathit{\Delta}_1 \\ \mathit{\Pi}_1' \\ A \to_c B\end{array} \qquad \begin{array}{c}\mathit{\Gamma}_2 \qquad \neg\mathit{\Delta}_2 \\ \mathit{\Pi}_2' \\ A\end{array} \qquad \dfrac{\begin{array}{c}\mathit{\Gamma}_3 \qquad \neg\mathit{\Delta}_3 \qquad [B] \\ \mathit{\Pi}_3' \\ \bot\end{array}}{\neg B}\ \neg\text{-int}}{\bot}\ \to_c\text{-elim}$$

In the other direction, suppose that $\mathit{\Gamma}_1, \mathit{\Gamma}_2, \mathit{\Gamma}_3, \neg\mathit{\Delta}_1, \neg\mathit{\Delta}_2, \neg\mathit{\Delta}_3 \vdash_{\mathcal{NE}_p} \bot$ with a derivation as:

$$\dfrac{\begin{array}{c}\mathit{\Gamma}_1 \quad \neg\mathit{\Delta}_1\\ \mathit{\Pi}_1\\ A \to_c B\end{array} \qquad \begin{array}{c}\mathit{\Gamma}_2 \quad \neg\mathit{\Delta}_2\\ \mathit{\Pi}_2\\ A\end{array} \qquad \begin{array}{c}\mathit{\Gamma}_3 \quad \neg\mathit{\Delta}_3\\ \mathit{\Pi}_3\\ \neg B\end{array}}{\bot}\ \to_c\text{-elim}$$

By the inductive hypothesis, we have that $\mathit{\Gamma}_1 \vdash_{\mathcal{LE}_p} \mathit{\Delta}_1; A \to_c B$, $\mathit{\Gamma}_2 \vdash_{\mathcal{LE}_p} \mathit{\Delta}_2; A$ and $\mathit{\Gamma}_3 \vdash_{\mathcal{LE}_p} \mathit{\Delta}_3; \neg B$. We can then obtain the desired derivation in $\mathcal{LE}_p$ as:

$$\dfrac{\begin{array}{c}\mathit{\Gamma}_1\\ \mathit{\Pi}'_1\\ \mathit{\Delta}_1; A \to_c B\end{array} \qquad \begin{array}{c}\mathit{\Gamma}_2\\ \mathit{\Pi}'_2\\ \mathit{\Delta}_2; A\end{array} \qquad \dfrac{\begin{array}{c}\mathit{\Gamma}_3\\ \mathit{\Pi}'_3\\ \mathit{\Delta}_3; \neg B\end{array} \qquad [\cdot; B]}{\mathit{\Delta}_2; \cdot}\ \neg\text{-int}}{\mathit{\Delta}_1, \mathit{\Delta}_2, \mathit{\Delta}_3; \cdot}\ \vee_c\text{-elim}$$

– Case der. Consider the derivation

$$\dfrac{\begin{array}{c}\mathit{\Gamma}\\ \mathit{\Pi}\\ \mathit{\Delta}; A\end{array}}{\mathit{\Delta}, A; \cdot}\ \text{der}$$

By the inductive hypothesis, we have $\mathit{\Gamma}, \neg\mathit{\Delta} \vdash_{\mathcal{NE}_p} A$. We can then obtain the desired derivation in $\mathcal{NE}_p$ as:

$$\dfrac{\begin{array}{c}\mathit{\Gamma} \quad \neg\mathit{\Delta}\\ \mathit{\Pi}'\\ A\end{array} \qquad [\neg A]}{\bot}\ \neg\text{-elim}$$

The cases of the other structural rules are trivial.

7 Normalization

We will now describe how normalization works in the natural deduction setting with *stoup*. The idea follows the usual one for natural deduction systems: show how to *compose* derivations, so to eliminate detours. The presence of *stoups*, however, adds an extra case analysis, since the composition may occur in the *stoup* or in the classical context. Both processes will be carefully described in what follows.

7.1 Composition

Before we define reductions and prove the normalization theorem for $\mathcal{LE}_p$, we must guarantee that the process of *composition* of derivations is preserved in $\mathcal{LE}_p$. As quickly mentioned in Sec. 3, sequent systems with *stoup* usually allow for two types of cut: a cut where the cut-formula in the left premise is in the *stoup* and a cut where the cut-formula in the left premise is in the classical region. These two types of cut will correspond to two modes of composition: a composition that occurs in the *stoup* and a composition that occurs in the classical context. We will detail these two forms of compositions below.

Definition 2. *Let Π_1 be a derivation of $\Gamma_1 \vdash_{\mathcal{LE}_p} \Delta_1; A$ and Π_2 be a derivation of $\Gamma_2 \cup \{\cdot; A\} \vdash_{\mathcal{LE}_p} \Delta_2; B$. The* stoup-composition *of Π_1, Π_2 on $\{\cdot; A\}$, denoted by*

$$\begin{array}{ccc} \Gamma_1 & & \\ \Pi_1 & & \\ \Delta_1; [\frac{A}{\cdot;A}] & & \Gamma_2 \\ & \Pi_2 & \\ & \Delta_1, \Delta_2; B & \end{array}$$

is the derivation Π of $\Gamma_1, \Gamma_2 \vdash_{\mathcal{LE}_p} \Delta_1, \Delta_2; B$ defined inductively on Π_2 as follows.

1. *The last rule applied in Π_2 is $\vee_c$-elim with Π_2^1 derivation of $\Gamma_2^1 \cup \{\cdot; A\} \vdash_{\mathcal{LE}_p} \Delta_2^1; C \vee_c D$*

$$\dfrac{\begin{array}{c} \cdot; A \quad \Gamma_2^1 \\ \Pi_2^1 \\ \Delta_2^1; C \vee_c D \end{array} \qquad \begin{array}{c} [\cdot; C] \quad \Gamma_2^2 \\ \Pi_2^2 \\ \Delta_2^2; \cdot \end{array} \qquad \begin{array}{c} [\cdot; D] \quad \Gamma_2^3 \\ \Pi_2^3 \\ \Delta_2^3; \cdot \end{array}}{\Delta_2; \cdot} \vee_c\text{-}elim$$

where Δ_2^i indicates a partition of Δ_2 (the same with Γ_2). The following derivation is given by the inductive step

$$\begin{array}{c} \Gamma_1 \quad \Gamma_2^1 \\ \Pi_1^* \\ \Delta_1, \Delta_2^1; (C \vee_c D) \end{array}$$

The resulting derivation Π is

$$\dfrac{\begin{array}{c} \Gamma_1 \quad \Gamma_2^1 \\ \Pi_1^* \\ \Delta_1, \Delta_2^1; (C \vee_c D) \end{array} \qquad \begin{array}{c} [\cdot; C] \quad \Gamma_2^2 \\ \Pi_2^* \\ \Delta_2^2; \cdot \end{array} \qquad \begin{array}{c} [\cdot; D] \quad \Gamma_2^3 \\ \Pi_3^* \\ \Delta_2^3; \cdot \end{array}}{\Delta_1, \Delta_2; \cdot} \vee_c\text{-}elim$$

The same analysis is done for the cases where $\cdot; A$ appears in Π_2^2, Π_2^3.

2. *The last rule applied in Π_2 is dereliction*

$$\dfrac{\begin{array}{c} \Gamma_2 \quad \cdot; A \\ \Pi_2 \\ \Delta_2; B \end{array}}{\Delta_2, B; \cdot} der$$

The following derivation is given by the inductive step

$$\begin{array}{c} \varGamma_1 \qquad \varGamma_2 \\ \varPi^* \\ \varDelta_1, \varDelta_2; B \end{array}$$

The resulting derivation $\varPi$ is then

$$\dfrac{\begin{array}{c} \varGamma_1 \qquad \varGamma_2 \\ \varPi^* \\ \varDelta_1, \varDelta_2; B \end{array}}{\varDelta_1, \varDelta_2, B; \cdot}\ der$$

All the other cases are similar and/or simpler.

Definition 3. *Let $\varPi_1$ be a derivation of $\varGamma_1 \vdash_{\mathcal{LE}_p} \varDelta_1, A; C$, and $\varPi_2$ be a derivation of $\varGamma_2 \cup \{\cdot; A\} \vdash_{\mathcal{LE}_p} \varDelta_2; B$. The* context-composition *of $\varPi_1, \varPi_2$ on $\{\cdot; A\}$, denoted by*

$$\begin{array}{c} \varGamma_1 \\ \varPi_1 \\ \varDelta_1, [\frac{A}{\cdot;A}]; C \qquad \varGamma_2 \\ \varPi_2 \\ \varDelta_1, \varDelta_2; B \end{array}$$

is the derivation $\varPi$ of $\varGamma_1, \varGamma_2 \vdash_{\mathcal{LE}_p} \varDelta_1, \varDelta_2, B; C$ defined inductively on $\varPi_1$, by replacing the assumption $\cdot; A$ in $\varPi_2$ by the derivation $\varPi_1$ as follows.

1. *The last rule applied in $\varPi_1$ is $\to_i$-elim*

$$\dfrac{\begin{array}{c} [\cdot; D] \qquad \varGamma_1 \\ \varPi_1 \\ \varDelta_1, A; C \end{array}}{\varDelta_1, A; D \to_i C}\ \to_i\text{-}elim$$

   By the inductive step, a derivation $\varPi^*$ of $\varGamma_1, \varGamma_2 \cup \{\cdot; D\} \vdash_{\mathcal{LE}_p} \varDelta_1, \varDelta_2, B; C$ is obtained. The derivation $\varPi$ is defined as

$$\dfrac{\begin{array}{c} [\cdot; D] \qquad \varGamma_1 \qquad \varGamma_2 \\ \varPi^* \\ \varDelta_1, \varDelta_2, \varDelta_2, B; C \end{array}}{\varDelta_1, \varDelta_2, B; D \to_i C}\ \to_i\text{-}elim$$

2. *The last rule applied in $\varPi_1$ is classical weakening*

$$\dfrac{\begin{array}{c} \varGamma_1 \\ \varPi_1 \\ \varDelta_1; C \end{array}}{\varDelta_1, A; C}\ W_c$$

In this case the result of the composition is:

$$\dfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ \Delta_1; C\end{array}}{\Delta_1, \Delta_2, B; C}\; W_c$$

3. The last rule applied in Π_1 is dereliction

$$\dfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ \Delta_1; A\end{array}}{\Delta_1, A; \cdot}\; der$$

By the process described in Definition 2, a derivation Π^* of $\Gamma_1, \Gamma_2 \vdash_{\mathcal{LE}_p} \Delta_1, \Delta_2; B$ is obtained. The derivation Π is defined as

$$\dfrac{\begin{array}{c}\Gamma_1 \qquad \Gamma_2\\ \Pi^*\\ \Delta_1, \Delta_2; B\end{array}}{\Delta_1, \Delta_2, B; \cdot}\; der$$

All the other cases are similar and/or simpler.

In what follows, we will adopt a more concise notation for both compositions:

$$\Pi_1/[\frac{A}{\cdot; A}]/\Pi_2$$

7.2 Reductions

Derivations in $\mathcal{LE}_p$ may contain *detours*. These detours are of two types: we may introduce a formula by an application of an introduction rule to immediately use it as major premiss of an application of an elimination rule; or we may introduce a formula by an application of an introduction rule and use it as major premiss of an application of an elimination rule after several applications of $\vee_i$-elim. The *reductions* defined in this section are intended, as usual, to eliminate *detours* that may occur in a derivation.

Definition 4. *A* segment *in a derivation Π is a sequence $A_1, ..., A_n$ of consecutive formulas in a path in Π such that:*

- *A_1 is not in the stoup of the consequence of an application of $\vee_i$-elim or of an application of C_c;*
- *A_j, for $j < n$, is in the stoup of the minor premiss of an application of $\vee_i$-elim or in the premise of an application of C_c; and*
- *A_n is not in the stoup of the minor premiss of an application of $\vee_i$-elim or of an application of C_c.*

We note the presence of contraction in the last definition. The idea is that contractions move down on reductions, just like in a sequent calculus' cut-elimination process.

Definition 5. *A segment that begins with the consequence of an application of an introduction rule or W_i and ends with an application of an elimination rule is called a* maximal segment. *A maximal segment of length* 1 *is called a* maximum formula.

Definition 6. *Let Π be a derivation in $\mathcal{LE}_p$. The* degree of Π, $d[\Pi]$, *is defined as* $\max\{d[A] : A$ *is the end-formula of a maximal segment in* $\Pi\}$, *where* $d[A]$ *is the* weight *of the formula A, defined inductively by*

$$\begin{array}{ll} d[\bot] = d[p] = 0 & p \textit{ atomic.} \\ d[A \circ B] & = d[A] + d[B] + 1 \textit{ for } \circ \in \{\rightarrow_{i,c}, \vee_{i,c}, \wedge\} \\ d[\neg A] & = d[A] + 1. \end{array}$$

Definition 7. *A derivation Π is called* normal *if and only if* $d[\Pi] = 0$.

We will present next all the reduction steps in $\mathcal{LE}_p$ that will be used in the elimination of maximal segments.

1. $\wedge$-reduction:

   The derivation

$$\dfrac{\dfrac{\begin{matrix}\Gamma_1 \\ \Delta_1; A_1\end{matrix} \qquad \begin{matrix}\Gamma_2 \\ \Delta_2; A_2\end{matrix}}{\Delta_1, \Delta_2; (A_1 \wedge A_2)}\ \wedge\text{-int}}{\Delta_1, \Delta_2; A_j}\ \wedge_j\text{-elim}$$

   Reduces to

$$\begin{matrix}\Gamma_j \\ \dfrac{\Delta_j; A_j}{\Delta_1, \Delta_2; A_j}\ W_c\end{matrix}$$

2. $\rightarrow_i$-reduction:

   The derivation

$$\dfrac{\begin{matrix}\Gamma_1 \\ \Pi_1 \\ \Delta_1; A\end{matrix} \qquad \dfrac{\begin{matrix}[\cdot; A] \quad \Gamma_2 \\ \Pi_2 \\ \Delta_2; B\end{matrix}}{\Delta_2; (A \rightarrow_i B)}\ \rightarrow_i\text{-int}}{\Delta_1, \Delta_2; B}\ \rightarrow_i\text{-elim}$$

   Reduces to:

$$
\begin{array}{ccc}
\Gamma_1 & & \\
\Pi_1 & & \\
\Delta_1; [\frac{A}{\cdot; A}] & & \Gamma_2 \\
& \Pi_2 & \\
& \Delta_1, \Delta_2; B &
\end{array}
$$

Observe that the case for negation is analogous.

3. $\vee_i$-reduction:

The derivation

$$
\dfrac{\dfrac{\begin{array}{c}\Gamma \\ \Pi \\ \Delta; A_j\end{array}}{\Delta; (A_1 \vee_i A_2)}\ \vee_i\text{-int} \qquad \begin{array}{c}\Gamma_1 \quad [\cdot; A_1] \\ \Pi_1 \\ \Delta_1; B\end{array} \qquad \begin{array}{c}\Gamma_2 \quad [\cdot; A_2] \\ \Pi_2 \\ \Delta_2; B\end{array}}{\Delta, \Delta_1, \Delta_2; B}\ \vee_i\text{-elim}
$$

Reduces to:

$$
\begin{array}{ccc}
\Gamma & & \\
\Pi & & \\
\Delta; [\frac{A_j}{\cdot; A_j}] & & \Gamma_j \\
& \Pi_j & \\
& \Delta, \Delta_j; B & \\
\hline\hline
& \Delta, \Delta_1, \Delta_2; B & W_c
\end{array}
$$

4. $\rightarrow_c$-reduction:

The derivation

$$
\dfrac{\dfrac{\begin{array}{c}[\cdot; A] \quad \Gamma_1 \\ \Pi_1 \\ \Delta_2, B; \cdot\end{array}}{\Delta_2; (A \rightarrow_c B)}\ \rightarrow_c\text{-int} \qquad \begin{array}{c}\Gamma_2 \\ \Pi_2 \\ \Delta_2; A\end{array} \qquad \begin{array}{c}[\cdot; B] \quad \Gamma_3 \\ \Pi_3 \\ \Delta_3; \cdot\end{array}}{\Delta_1, \Delta_2, \Delta_3; \cdot}\ \rightarrow_c\text{-elim}
$$

Reduces to:

$$
\begin{array}{cccc}
\Gamma_2 & & & \\
\Pi_2 & & & \\
\Delta_2; [\frac{A}{\cdot; A}] & & \Gamma_1 & \\
& \Pi_1 & & \\
& \Delta_1, \Delta_2, [\frac{B}{\cdot; B}]; \cdot & & \Gamma_3 \\
& & \Pi_3 & \\
& & \Delta_1, \Delta_2, \Delta_3; \cdot &
\end{array}
$$

5. $\vee_c$-reduction:

   The derivation

$$\dfrac{\dfrac{\begin{array}{c}\mathit{\Gamma}_1\\ \mathit{\Pi}_1\\ \mathit{\Delta}_1, A, B; \cdot\end{array}}{\mathit{\Delta}_1; (A \vee_c B)}\vee_c\text{-int} \qquad \begin{array}{c}\mathit{\Gamma}_2 \quad [\cdot; A]\\ \mathit{\Pi}_2\\ \mathit{\Delta}_2; \cdot\end{array} \qquad \begin{array}{c}\mathit{\Gamma}_3 \quad [\cdot; B]\\ \mathit{\Pi}_3\\ \mathit{\Delta}_3; \cdot\end{array}}{\mathit{\Delta}_1, \mathit{\Delta}_2, \mathit{\Delta}_3; \cdot}\vee_c\text{-elim}$$

   Reduces to:

$$\begin{array}{c}\mathit{\Gamma}_1\\ \mathit{\Pi}_1\\ \mathit{\Delta}_1, [\frac{A}{\cdot;A}], [\frac{B}{\cdot;B}]; \cdot \qquad \mathit{\Gamma}_2\\ \mathit{\Pi}_2\\ \mathit{\Delta}_1, \mathit{\Delta}_2, [\frac{B}{\cdot;B}]; \cdot \qquad \mathit{\Gamma}_3\\ \mathit{\Pi}_3\\ \mathit{\Delta}_1, \mathit{\Delta}_2, \mathit{\Delta}_3; \cdot\end{array}$$

6. Permutative reductions:

   (a) The derivation

$$\dfrac{\dfrac{\begin{array}{c}\mathit{\Gamma}_1\\ \mathit{\Pi}_1\\ \mathit{\Delta}_1; (A \vee_i B)\end{array} \quad \begin{array}{c}[\cdot; A] \quad \mathit{\Gamma}_2\\ \mathit{\Pi}_2\\ \mathit{\Delta}_2; C\end{array} \quad \begin{array}{c}[\cdot; B] \quad \mathit{\Gamma}_3\\ \mathit{\Pi}_3\\ \mathit{\Delta}_3; C\end{array}}{\mathit{\Delta}_1, \mathit{\Delta}_2, \mathit{\Delta}_3; C} \quad \begin{array}{c}\mathit{\Omega}_1\\ \mathit{\Theta}_1; \mathit{\Lambda}_1\end{array} \quad \dots \quad \begin{array}{c}\mathit{\Omega}_m\\ \mathit{\Theta}_m; \mathit{\Lambda}_m\end{array}}{\mathit{\Delta}_1, \mathit{\Delta}_2, \mathit{\Delta}_3, \mathit{\Theta}_1, \dots, \mathit{\Theta}_m; \mathit{\Lambda}}$$

   where C is the major premiss of an elimination rule with minor premisses $\mathit{\Theta}_1; \mathit{\Lambda}_1 \dots \mathit{\Theta}_m; \mathit{\Lambda}_m$ (if any), reduces to

$$\dfrac{\dfrac{\begin{array}{c}\mathit{\Gamma}_1\\ \mathit{\Pi}_1\\ \mathit{\Delta}_1; A \vee_i B\end{array} \quad \dfrac{\begin{array}{c}[\cdot; A] \quad \mathit{\Gamma}_2\\ \mathit{\Pi}_2\\ \mathit{\Delta}_2; C\end{array} \quad \begin{array}{c}\mathit{\Omega}_1\\ \mathit{\Theta}_1; \mathit{\Lambda}_1\end{array} \ \dots \ \begin{array}{c}\mathit{\Omega}_m\\ \mathit{\Theta}_m; \mathit{\Lambda}_m\end{array}}{\mathit{\Delta}_2, \mathit{\Theta}_1, \dots, \mathit{\Theta}_m; \mathit{\Lambda}} \quad \dfrac{\begin{array}{c}[\cdot; B] \quad \mathit{\Gamma}_3\\ \mathit{\Pi}_3\\ \mathit{\Delta}_3; C\end{array} \quad \begin{array}{c}\mathit{\Omega}_1\\ \mathit{\Theta}_1; \mathit{\Lambda}_1\end{array} \ \dots \ \begin{array}{c}\mathit{\Omega}_m\\ \mathit{\Theta}_m; \mathit{\Lambda}_m\end{array}}{\mathit{\Delta}_3, \mathit{\Theta}_1, \dots, \mathit{\Theta}_m; \mathit{\Lambda}}}{\mathit{\Delta}_1, \mathit{\Delta}_2, \mathit{\Delta}_3, \mathit{\Theta}_1, \dots, \mathit{\Theta}_m, \mathit{\Theta}_1, \dots, \mathit{\Theta}_m; \mathit{\Lambda}}}{\overline{\mathit{\Delta}_1, \mathit{\Delta}_2, \mathit{\Delta}_3, \mathit{\Theta}_1, \dots, \mathit{\Theta}_m; \mathit{\Lambda}}}\ C_c$$

   (b) The derivation

$$\dfrac{\dfrac{\begin{array}{c}\mathit{\Gamma}_1\\ \mathit{\Pi}_1\\ \mathit{\Delta}_1, A, A; C\end{array}}{\mathit{\Delta}_1, A; C}C_c \quad \begin{array}{c}\mathit{\Omega}_1\\ \mathit{\Theta}_1; \mathit{\Lambda}_1\end{array} \quad \dots \quad \begin{array}{c}\mathit{\Omega}_m\\ \mathit{\Theta}_m; \mathit{\Lambda}_m\end{array}}{\mathit{\Delta}_1, \mathit{\Delta}_2, \mathit{\Delta}_3, \mathit{\Theta}_1, \dots, \mathit{\Theta}_m; \mathit{\Lambda}}$$

where C is the major premiss of an elimination rule with minor premisses $\Theta_1; \Lambda_1 \dots \Theta_m; \Lambda_m$ (if any), reduces to

$$\dfrac{\dfrac{\begin{matrix}\Gamma_1\\ \Pi_1\\ \Delta, A, A : C\end{matrix} \quad \begin{matrix}\Omega_1\\ \Theta_1; \Lambda_1\end{matrix} \quad \dots \quad \begin{matrix}\Omega_m\\ \Theta_m; \Lambda_m\end{matrix}}{\Delta, A, A, \Theta_1, \dots, \Theta_m; \Lambda}}{\Delta, A, \Theta_1, \dots, \Theta_m, A; \Lambda}\, C_c$$

7.3 Normalization

We shall use Pottinger's *critical derivation* strategy [Pot76] to prove the normalization theorem for $\mathcal{LE}_p$. But before the proof of normalization, we need some definitions and preparatory lemmas that relate reductions and composition to the degree of derivations.

First of all, we note that the intuitionistic weakening can be restricted to the atomic case.

Lemma 1. *The following atomic version of the intuitionistic weakening rule is admissible*

$$\dfrac{\begin{matrix}\Gamma\\ \Pi\\ \Delta; \cdot\end{matrix}}{\Delta; p}\, W_i \ \textit{(p atomic)}$$

Proof. The proof is somehow standard, just observing that some intuitionistic weakenings are substituted by classical ones. For example, the following derivation

$$\dfrac{\begin{matrix}\Gamma\\ \Pi\\ \Delta; \cdot\end{matrix}}{\Delta; A \rightarrow_c B}\, W_i$$

can be substituted by

$$\dfrac{\dfrac{\begin{matrix}[\cdot; A] \qquad \Gamma\\ \Pi\\ \Delta; \cdot\end{matrix}}{\Delta, B; \cdot}\, W_c}{\Delta; A \rightarrow_c B}\, \rightarrow_c\text{-int}$$

The proof of the next lemma is obvious.

Lemma 2. *Let Π be $\Pi_1/[\frac{A}{\cdot;A}]/\Pi_2$, the composition of derivations Π_1 with Π_2 at junction point A. Then, $d[\Pi] = max\{d[\Pi_1], d[\Pi_2], d[A]\}$*

Lemma 3. *If Π reduces to Π', then $d[\Pi] \geq d[\Pi']$.*

Proof. Directly from the form of the reductions and Lemma 2.

Definition 8. *A derivation Π is* critical *iff:*

- *Π ends with an elimination rule α;*
- *The major premiss A of α is the end of a maximal segment;*
- *$d[\Pi] = d[A]$; and*
- *For every proper subderivation Π' of Π, $d[\Pi'] < d[\Pi]$.*

Lemma 4. ***(Critical Lemma)**: Let Π be a critical derivation of $\Gamma \vdash_{\mathcal{LE}_p} \Delta; \Sigma$ in $\mathcal{LE}_p$. Then, Π reduces to a derivation Π' of $\Gamma' \vdash_{\mathcal{LE}_p} \Delta; \Sigma$ with $\Gamma' \subseteq \Gamma$, such that $d[\Pi'] < d[\Pi]$.*

Proof. By induction on the length of Π.

- Case 1: The major premiss of the last rule applied in Π is a maximum formula. The result follows directly from the form of the reductions and Lemma 3.
- Case 2: The major premiss of the last rule applied in Π is the end formula of a maximum segment of length >1. There are two sub-cases to be examined:
  1. Π is:

$$\dfrac{\dfrac{\begin{matrix}\Gamma_1\\ \Pi_1\\ \Delta_1;(A \vee_i B)\end{matrix} \quad \begin{matrix}[\cdot;A] \quad \Gamma_2\\ \Pi_2\\ \Delta_2;C\end{matrix} \quad \begin{matrix}[\cdot;B] \quad \Gamma_3\\ \Pi_3\\ \Delta_3;C\end{matrix}}{\Delta_1,\Delta_2,\Delta_3;C} \quad \begin{matrix}\Omega_1\\ \Theta_1;\Lambda_1\end{matrix} \quad \ldots \quad \begin{matrix}\Omega_m\\ \Theta_m;\Lambda_m\end{matrix}}{\Delta_1,\Delta_2,\Delta_3,\Theta_1,\ldots,\Theta_m;\Lambda}$$

By a permutative reduction, Π reduces to the following derivation Π^*:

$$\begin{array}{c}\dfrac{\begin{matrix}\Gamma_1\\ \Pi_1\\ \Delta_1;A \vee_i B\end{matrix} \quad \dfrac{\begin{matrix}[\cdot;A] \quad \Gamma_2\\ \Pi_2\\ \Delta_2;C\end{matrix} \quad \begin{matrix}\Omega_1\\ \Theta_1;\Lambda_1\end{matrix} \quad \ldots \quad \begin{matrix}\Omega_m\\ \Theta_m;\Lambda_m\end{matrix}}{\Delta_2,\Theta_1,\ldots,\Theta_m;\Lambda} \quad \dfrac{\begin{matrix}[\cdot;B] \quad \Gamma_3\\ \Pi_3\\ \Delta_3;C\end{matrix} \quad \begin{matrix}\Omega_1\\ \Theta_1;\Lambda_1\end{matrix} \quad \ldots \quad \begin{matrix}\Omega_m\\ \Theta_m;\Lambda_m\end{matrix}}{\Delta_3,\Theta_1,\ldots,\Theta_m;\Lambda}}{\Delta_1,\Delta_2,\Delta_3,\Theta_1,\ldots,\Theta_m,\Theta_1,\ldots,\Theta_m;\Lambda} \\ \hline\hline \Delta_1,\Delta_2,\Delta_3,\Theta_1,\ldots,\Theta_m;\Lambda\end{array}$$

By Lemma 3, $d[\Pi^*] \leq d[\Pi]$. If $d[\Pi^*] < d[\Pi]$, then we take $\Pi' = \Pi^*$. If $d[\Pi^*] = d[\Pi]$, then at least one of the derivations of the minor premisses has degree $= d[\Pi]$. For the sake of the argument, let's assume that both have degree $= d[\Pi]$. By the induction hypothesis, the derivation

$$\dfrac{\begin{matrix}[\cdot;A] \quad \Gamma_2\\ \Pi_2\\ \Delta_2;C\end{matrix} \quad \begin{matrix}\Omega_1\\ \Theta_1;\Lambda_1\end{matrix} \quad \ldots \quad \begin{matrix}\Omega_m\\ \Theta_m;\Lambda_m\end{matrix}}{\Delta_2,\Theta_1,\ldots,\Theta_m;\Lambda}$$

reduces to a derivation Π_2' of $\Delta_2,\Theta_1,\ldots,\Theta_m;\Lambda$ such that $d[\Pi_2'] < d[\Pi_2]$, and the derivation

$$
\dfrac{\begin{matrix} [\cdot ; B] \quad \Gamma_3 \\ \Pi_3 \\ \Delta_3 ; C \end{matrix} \qquad \begin{matrix} \Omega_1 \\ \Theta_1 ; \Lambda_1 \end{matrix} \quad \ldots \quad \begin{matrix} \Omega_m \\ \Theta_m ; \Lambda_m \end{matrix}}{\Delta_3, \Theta_1, \ldots, \Theta_m ; \Lambda}
$$

Reduces to a derivation Π_3' of $\Delta_3, \Theta_1, \ldots, \Theta_m; \Lambda$ such that $d[\Pi_3'] < d[\Pi_3]$.

Let Π' be:

$$
\dfrac{\begin{matrix} \Gamma_1 \\ \Pi_1 \\ \Delta_1 ; (A \vee_i B) \end{matrix} \qquad \begin{matrix} \Pi_2' \\ \Delta_2, \Theta_1, \ldots, \Theta_m ; \Lambda \end{matrix} \qquad \begin{matrix} \Pi_3' \\ \Delta_3, \Theta_1, \ldots, \Theta_m ; \Lambda \end{matrix}}{\Delta_1, \Delta_2, \Delta_3, \Theta_1, \ldots, \Theta_m ; \Lambda}
$$

We can easily see that Π reduces to Π' and that $d[\Pi'] < d[\Pi]$.

2. Π is

$$
\dfrac{\dfrac{\begin{matrix} \Pi_1 \\ \Delta_1, A, A : C \end{matrix}}{\Delta_1, A : C} C_c \qquad \begin{matrix} \Omega_1 \\ \Theta_1 ; \Lambda_1 \end{matrix} \quad \ldots \quad \begin{matrix} \Omega_m \\ \Theta_m ; \Lambda_m \end{matrix}}{\Delta_1, \Delta_2, \Delta_3, \Theta_1, \ldots, \Theta_m ; \Lambda}
$$

By a permutative reduction, Π reduces to the following derivation Π^*:

$$
\dfrac{\dfrac{\begin{matrix} \Pi_1 \\ \Delta, A, A : C \end{matrix} \qquad \begin{matrix} \Omega_1 \\ \Theta_1 ; \Lambda_1 \end{matrix} \quad \ldots \quad \begin{matrix} \Omega_m \\ \Theta_m ; \Lambda_m \end{matrix}}{\Delta, A, A, \Theta_1, \ldots, \Theta_m ; \Lambda}}{\Delta, A, \Theta_1, \ldots, \Theta_m, A ; \Lambda} C_c
$$

As in the previous case, by Lemma 2, $d[\Pi^*] \leq d[\Pi]$. If $d[\Pi^*] < d[\Pi]$, then we take $\Pi' = \Pi^*$. If $d[\Pi^*] = d[\Pi]$, then by the induction hypothesis, the derivation

$$
\dfrac{\begin{matrix} \Pi_1 \\ \Delta, A, A : C \end{matrix} \qquad \begin{matrix} \Omega_1 \\ \Theta_1 ; \Lambda_1 \end{matrix} \quad \ldots \quad \begin{matrix} \Omega_m \\ \Theta_m ; \Lambda_m \end{matrix}}{\Delta, A, A, \Theta_1, \ldots, \Theta_m ; \Lambda}
$$

reduces to a derivation Π^{**} of $\Gamma^* \subseteq \Gamma \vdash_{\mathcal{LE}_p} \Delta, A, A, \Theta_1, \ldots, \Theta_m; \Lambda$ such that $d[\Pi^{**}] < d[\Pi]$. We can then take the desired derivation Π' as

$$
\dfrac{\begin{matrix} \Pi^{**} \\ \Delta, A, A, \Theta_1, \ldots, \Theta_m ; \Lambda \end{matrix}}{\Delta, A, \Theta_1, \ldots, \Theta_m ; \Lambda}
$$

Lemma 5. *Let Π be a derivation of $\Gamma \vdash_{\mathcal{LE}_p} \Delta; \Sigma$ with $d[\Pi] > 0$. Then, Π reduces to a derivation Π' of $\Gamma' \vdash_{\mathcal{LE}_p} \Delta; \Sigma$ with $\Gamma' \subseteq \Gamma$, such that $d[\Pi'] < d[\Pi]$.*

Proof. By induction on the length of Π.

- Case 1: Π ends with an application of any rule other than an elimination rule. This case follows directly from the induction hypothesis.
- Case 2: Π ends with an application of an elimination rule. The general form of Π is:

$$
\dfrac{\Pi_1 \qquad \Pi_2 \qquad \Pi_3}{\Delta ; \Lambda}
$$

By the induction hypothesis, Π_k reduces to a derivation Π'_k such that $d[\Pi'_k] < d[\Pi_k]$ ($1 \leq k \leq 3$).
Let Π^* be:

$$\frac{\Pi'_1 \qquad \Pi'_2 \qquad \Pi'_3}{\Delta; \Lambda}$$

By Lemma 3, $d[\Pi^*] \leq d[\Pi]$. If $d[\Pi^*] < d[\Pi]$, we take $\Pi^* = \Pi'$. If $d[\Pi^*] = d[\Pi]$, the Π^* is a critical derivation and the result follows from Lemma 4.

Theorem 3. *(**Normalization Theorem** for $\mathcal{LE}_p$) Let Π be a derivation of $\Gamma \vdash_{\mathcal{LE}_p} \Delta; \Sigma$ in $\mathcal{LE}_p$. Then, Π reduces to a normal derivation Π' of $\Gamma' \vdash_{\mathcal{LE}_p} \Delta; \Sigma$ where $\Gamma' \subseteq \Gamma$.*

Proof. Directly from Lemma 5 by induction on $d[\Pi]$.

8 Concluding remarks and future work

There are lots of things to be done in the domain of ecumenical systems and more specifically in connection with pure ecumenical systems. We conclude this paper by mentioning two possible lines of work we are pursuing.

8.1 Pure first-order ecumenical systems

We can easily show that, in Prawitz' ecumenical system, if the main operator of a formula A is classical, if $\Gamma, \neg A \vdash \bot$, then $\Gamma \vdash A$. For example, assume a derivation Π as follows:

$$\begin{array}{cc} \Gamma & \neg(B \vee_c C) \\ & \Pi \\ & \bot \end{array}$$

Then, we can construct the following derivation of $\Gamma \vdash (B \vee_c C)$:

$$\dfrac{\begin{array}{cc} \Gamma & 1\,\dfrac{\dfrac{[(B \vee_c C)]^1 \qquad [\neg B]^2 \qquad [\neg C]^3}{\bot}}{[\neg(B \vee_c C)]} \\ & \Pi \\ & \bot \end{array}}{(B \vee_c C)}\, 2,3$$

The same result holds for the classical existential quantifier. From the derivation

$$\begin{array}{cc} \Gamma & \neg \exists_c x B(x) \\ & \Pi \\ & \bot \end{array}$$

We can easily obtain the derivation:

$$\begin{array}{cc} & \dfrac{[\exists_c x(Bx)]^1 \qquad [\forall x \neg B(x)]^2}{1\;\dfrac{\bot}{[\neg \exists_c c(B(x))]}} \\ \Gamma & \\ & \Pi \\ & 2\;\dfrac{\bot}{\exists_c x B(x)} \end{array}$$

From these results it follows that the classical implication $\to_c$ satisfies *modus ponens* for classical succedents:

Theorem 4. *Let the main operator of the formula B be classical. Then we can prove in Prawitz' system that $\{(A \to_c B), A\} \vdash B$.*

In the propositional system with the *stoup*, we can obtain the same results for the classical propositional operators. For example, given a derivation of

$$\begin{array}{c} \cdot; \neg(B \vee_c C) \qquad \Gamma \\ \Pi \\ \Delta; \cdot \end{array}$$

We can obtain a derivation of

$$\begin{array}{c} \Gamma \\ \Pi \\ \Delta; (B \vee_c C) \end{array}$$

But in the first-order case, a *pure rule* for the classical existential quantifier in the *stoup* format requires an extra attention. Consider the following pure rules with *stoup* for the classical existential quantifier:

$$\begin{array}{c} \Gamma \\ \Pi \\ \dfrac{\Delta, A(t); \cdot}{\Delta; \exists_c x A(x)} \end{array} \qquad\qquad \dfrac{\begin{array}{c} \Gamma_1 \\ \Pi_1 \\ \Delta_1; \exists_c x A(x) \end{array} \qquad \begin{array}{c} \cdot; A(a) \qquad \Gamma_2 \\ \Pi_2 \\ \Delta_2; \cdot \end{array}}{\Delta_1; \Delta_2; \cdot}$$

It is easy to show that the first order system with *stoup* obtained by means of the addition of the intuitionistic rules for $\exists_i$, $\forall$ and these rules for $\exists_c$ is not complete with respect to Prawitz' first-order ecumenical natural deduction. The important relation $\neg \forall x \neg A(x) \vdash \exists_c x A(x)$ between $\forall$ and $\exists_c$ is not derivable in this first-order system with *stoup*. As a future work, we propose to investigate the first-order system obtained by the addition of the rules mentioned above plus a new structural rule, the *store* rule:

$$\dfrac{\Delta, A; \cdot}{\Delta; A}\ \text{store}$$

with the side condition that the main operator of A is classical.

8.2 A different approach to *purity*

A different and interesting approach to pure systems worth exploring is based on some ideas proposed by Julien Murzi in [Mur20]. Murzi proposes a *pure* single-conclusion Natural Deduction system that satisfies the basic inferentialist requirements of harmony and separability. Murzi's proposal combines (in a very interesting way!) Peter Schroeder-Heister's idea of *higher-level rules* with Neil Tennant's idea that the sign $\bot$ for the *absurd* should be conceived as a *punctuation mark*. Using Murzi's idea we can formulate a new *pure* ecumenical natural deduction system for classical and intuitionistic logic.

(1) The *impure* rule for $\vee_c$-Int becomes

$$\begin{array}{c} [\tfrac{A}{\bot}]^j \qquad [\tfrac{B}{\bot}]^k \\ \underbrace{\qquad\qquad} \\ \vdots \\ j,k\ \dfrac{\bot}{(A \vee_c B)} \end{array}$$

(2) $\vee_c$-elim.

$$j,k\ \dfrac{(A \vee_c B) \qquad \begin{array}{c}[A]^j \\ \vdots \\ \bot\end{array} \qquad \begin{array}{c}[B]^k \\ \vdots \\ \bot\end{array}}{\bot}$$

(3) The *impure* rule for $\rightarrow_c$-Int becomes

$$\begin{array}{c} [A]^j \qquad [\tfrac{B}{\bot}]^k \\ \underbrace{\qquad\qquad} \\ \vdots \\ j,k\ \dfrac{\bot}{(A \rightarrow_c B)} \end{array}$$

(4) $\rightarrow_c$-elim.

$$k\ \dfrac{(A \rightarrow_c B) \qquad A \qquad \begin{array}{c}[B]^k \\ \vdots \\ \bot\end{array}}{\bot}$$

It is easy to show that the impure rules can be obtained from the new pure rules. In the case of $\vee_c$, for example, given a derivation Π of $\bot$ from $\neg A$ and $\neg B$, we can construct the following derivation:

$$
\begin{array}{c}
\dfrac{[\frac{A}{\bot}]^j}{[\neg A]} \qquad \dfrac{[\frac{B}{\bot}]^k}{[\neg B]} \\
\Pi \\
j,k\ \dfrac{\bot}{(A \vee_c B)}
\end{array}
$$

In order to prove the other direction, it is convenient to add a new general rule that allows us to conclude rules. In the formulation of the rule we will use (as Murzi does) the expression Δ/A as an alternative to the rule $\frac{\Delta}{A}$.

$$
\begin{array}{c}
[A]^j \\
\Pi \\
j\ \dfrac{B}{(A/B)}
\end{array}
$$

Suppose now that we have a derivation

$$
\begin{array}{c}
\frac{A}{\bot} \qquad \frac{B}{\bot} \\
\Pi \\
\bot
\end{array}
$$

We can then construct the following derivation:

$$
\begin{array}{c}
1\ \dfrac{\dfrac{[A]^1 \qquad [\neg A]^3}{\bot}}{[(A/\bot)]} \qquad 2\ \dfrac{\dfrac{[B]^2 \qquad [\neg B]^4}{\bot}}{[(B/\bot)]} \\
\Pi \\
3,4\ \dfrac{\bot}{(A \vee_c B)}
\end{array}
$$

As a final example, we present again *Peirce's law*

$$
2,3\ \dfrac{\dfrac{((A \to_c B) \to_c A)^2 \qquad 1\ \dfrac{\dfrac{[A]^1 \qquad [\frac{A}{\bot}]^3}{\bot}}{(A \to_c B)} \qquad [\frac{A}{\bot}]^3}{\bot}}{(((A \to_c B) \to_c A) \to_c A)}
$$

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