Separability and harmony in ecumenical systems*

SONIA MARIN, *Department of Computer Science, University of Birmingham, Birmingham B15 2TT, UK.*
E-mail: s.marin@ucl.ac.uk

LUIZ CARLOS PEREIRA, *Department of Philosophy, UERJ, 20550-900 Rio de Janerio, Brazil.*
E-mail: luiz@inf.puc-rio.br

ELAINE PIMENTEL, *Department Computer Science, UCL, London WC1E 6BT, UK.*
E-mail: e.pimentel@ucl.ac.uk

EMERSON SALES, *Gran Sasso Science Institute, 67100 L'Aquila, Italy.*
E-mail: emerson.sales@gssi.it

Abstract

The quest of smoothly combining logics so that connectives from different logics can co-exist in peace has been a fascinating topic of research. In 2015, Dag Prawitz introduced a natural deduction system for an ecumenical first-order logic, unifying classical and intuitionistic logics within a shared language. Building upon this foundation, we introduced, in a series of works, sequent systems for ecumenical logics and modal extensions. In this work we propose a new *pure* sequent calculus version for Prawitz's original system, where each rule features precisely one logical operator. This is achieved by extending sequents with an additional context, called *stoup*, and establishing the ecumenical concept of *polarities*. We smoothly extend these ideas for handling modalities, presenting a new pure labelled system for ecumenical modal logics. Finally, we show how this allows for naturally retrieving the ecumenical modal nested system proposed in a previous work.

Keywords: ecumenical systems, modalities, nested systems, labelled systems, cut-elimination, polarities.

1 Introduction

Ecumenism can be seen as a quest to unity, where diverse thoughts, ideas or points of view coexist in harmony. In logic, ecumenical systems refer, in a broad sense, to proof systems for combining logics.

Designing ecumenical systems that are not a simple disjoint union of their sub-systems can be hard, specially if classical and intuitionistic logics are considered [8]. Indeed, how can one combine, for example, the fact that the excluded middle ($A \vee \neg A$) is valid in classical logic (CL) but not in

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intuitionistic logic and, at the same time, assume a common explosion principle ($(A \wedge \neg A) \rightarrow B$) for both fragments?

According to Prawitz, the key idea is to understand why (and where) classical and intuitionistic logicians (dis)agree. In [38], he proposed an ecumenical natural deduction system codifying both classical and intuitionistic reasoning based on a uniform pattern of meaning explanations. In that system, the conjunction and negation are shared (hence the explosion principle is valid), while the classical and the intuitionistic logicians would have their own disjunction, represented as $\vee_c$ and $\vee_i$, respectively. Quoting Prawitz:

> 'The classical logician is not asserting what the intuitionistic logician denies. For instance, the classical logician asserts $A \vee_c \neg A$ to which the intuitionist does not object; he objects to the universal validity of $A \vee_i \neg A$, which is not asserted by the classical logician.'

For a deeper understanding of the proof-theoretic properties of Prawitz's system a single-conclusion ecumenical sequent calculus corresponding to Prawitz's original system was proposed in [35]. In that work, we showed that the ecumenical entailment is intrinsically intuitionistic, but it turns classical in the presence of classical succedents. We then produced a nested sequent version of the original sequent system and showed all of them sound and complete with respect to (first-order extension of) the ecumenical Kripke semantics [33]. In [24], we lifted this discussion to modal logics, presenting a labelled ecumenical modal system, amenable for modal extensions.

Since the systems presented in [24, 35] are based in Prawitz's original system, they are not *pure* [11] or *separable* [28], in the sense that the introduction rules for some connectives strongly depend on the presence of negation. In [25], we presented a *pure label free* calculus for ecumenical modalities, where every basic object of the calculus can be read as a formula in the language of the logic.

In this work, we will take a complete different path and revisit all these aforementioned systems, starting from a new pure ecumenical first-order system and naturally expand it to the modal case. Such pure systems allow for a clearer notion of the *meaning* for connectives (including modalities), faithfully matching Prawitz's original intention, and the tradition of the *proof-theoretic semantics' school* [19, 40, 41]. For that, we will use a differentiated context called *stoup*, together with an ecumenical notion of *polarities* for formulas.

The notions of stoup and polarities first appeared in [15] on Girard's LC sequent system for CL that separates rules for *positive* and *negative* formulas. LC is a precursor of the notion of *focusing* in sequent systems [1]. The idea is that right rules for positive formulas are applied in the *stoup*—a differentiated context where formulas are focused on. Negative formulas, on the other hand, are stored in a *classical context*, where they can be eagerly decomposed.

Sequents with stoup have the form $\Gamma \Rightarrow \Delta; \Pi$,[1] where Γ, Δ are sets and Π, the stoup, is a multiset containing at most one formula. In LC, the meaning of these contexts is the following:

- Γ is the usual classical left context in well known sequent systems for classical and intuitionistic first order logics, like LK and LJ [45].
- The stoup Π is a differentiated context, where positive formulas are 'worked on'. In a bottom-up read, a *positive formula* can be chosen from the classical right context to populate the stoup

[1] It should be mentioned that, in LC, sequents have a one-sided presentation—the left context is not present. Also, in systems like Girard's LU [16], sequents have two stoups and linear contexts. Here we adopted the simpler possible version for sequents with stoup supporting the intuitionistic setting and avoiding structural rules in classical contexts.

using the dereliction rule

$$\frac{\Gamma \Rightarrow \Delta, P; P}{\Gamma \Rightarrow \Delta, P; \cdot}\ \mathsf{D}$$

– The (classical) right context Δ carries the information of subformulas of *negative formulas* N, in the sense that $N = \neg F$ for some formula F. This means that $N \in \Delta$ can be interpreted as $F \in \Gamma$. Negative formulas are added to the classical context via the store rule

$$\frac{\Gamma \Rightarrow \Delta, N; \cdot}{\Gamma \Rightarrow \Delta; N}\ \mathsf{store}$$

It is interesting to note that, while the sequent $\Gamma \Rightarrow \Delta; \Pi$ is intuitionistically interpreted as $\Gamma, \neg\Delta \Rightarrow \Pi$, it only has a classical interpretation in LC if the stoup Π is *empty*. Moreover, the stoup in LC is *persistent*, in the sense that, after applying dereliction D over a positive formula P in the bottom-up reading, the stoup is emptied only when either P is totally consumed, or a negative subformula is reached—in which case it is stored in the classical right context.

The ecumenical systems we will study in this paper have a quite different behaviour, since they are *intuitionistic* in nature. Hence, the use of stoups will mix some of the characteristics of LC with intuitionistic systems featuring a stoup such as, for example, Herbelin's LJT and LJQ [12, 18]. The base difference is that, in the ecumenical formulation, stoups cannot be persistent since, otherwise, the logic would not be complete (see Example 2.6).

Several approaches have been proposed for combining intuitionistic logics and CLs (see *e.g.* [8, 10, 22, 31, 44]), some of them inspired by Girard's polarised system LU ([16]). Prawitz chose a completely different approach by proposing a natural deduction *ecumenical system* [38]. While it also takes into account meaning-theoretical considerations, it is more focused on investigating the philosophical significance of the fact that CL can be translated into intuitionistic logic.

The careful study of Prawitz's ecumenical system under the view of Girard's original idea of stoup will allow for new pure ecumenical sequent systems that avoid the use of negations in the formulation of rules. In fact, instead of building the modal system over the sequent presentation [35] of Prawitz's ecumenical system [38], we propose a new pure first-order ecumenical system. This not only allows for a better proof theoretic view of Prawitz's original proposal, but it also serves as a solid ground for smoothly accommodating modalities. A new pure labelled system for modalities comes naturally in this approach, and the nested system in [25] is easily proven correct and complete w.r.t. it.

Under this new perspective, we can start some lively discussions about the nature of formulas and systems. For example, in [34] the notion of stoup was brought to the natural deduction setting, while in [29] it is provided not only a medium in which classical and intuitionistic *logics* may coexist, but also one in which classical and intuitionistic notions of *proof* may coexist.

This is an extended journal version of [24, 25]. The rest of the paper is organized as follows: Section 2 introduces the notion of ecumenical systems with stoup (system LCE), and in Section 3 we prove it complete and correct w.r.t. Prawitz's ecumenical system (LE). This involves a non-trivial use of polarities, as well as a non-standard proof of cut-elimination. We show that, is one is not careful, the quest for *purity* ends up in *collapsing*; Section 4 extends the propositional fragment of LCE with modalities, resulting in a new pure labelled ecumenical modal system (labEK); Section 5 brings the nested ecumenical system nEK from [25], which is naturally seen as the label-free counterpart of labEK; Section 6 briefly discusses fragments, axioms and extensions and Section 7 discusses related and future work, and concludes the paper.

INTUITIONISTIC AND NEUTRAL RULES

$$\frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\ \wedge L \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \wedge R \qquad \frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee_i B, \Gamma \Rightarrow C}\ \vee_i L$$

$$\frac{\Gamma \Rightarrow A_j}{\Gamma \Rightarrow A_1 \vee_i A_2}\ \vee_i R_j \qquad \frac{A \rightarrow_i B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{\Gamma, A \rightarrow_i B \Rightarrow C}\ \rightarrow_i L \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow_i B}\ \rightarrow_i R$$

$$\frac{\neg A, \Gamma \Rightarrow A}{\neg A, \Gamma \Rightarrow \bot}\ \neg L \qquad \frac{\Gamma, A \Rightarrow \bot}{\Gamma \Rightarrow \neg A}\ \neg R \qquad \frac{}{\bot, \Gamma \Rightarrow A}\ \bot L$$

$$\frac{A[y/x], \forall x.A, \Gamma \Rightarrow C}{\forall x.A, \Gamma \Rightarrow C}\ \forall L \qquad \frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \forall x.A}\ \forall R \qquad \frac{A[y/x], \Gamma \Rightarrow C}{\exists_i x.A, \Gamma \Rightarrow C}\ \exists_i L \qquad \frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \exists_i x.A}\ \exists_i R$$

CLASSICAL RULES

$$\frac{A, \Gamma \Rightarrow \bot \quad B, \Gamma \Rightarrow \bot}{A \vee_c B, \Gamma \Rightarrow \bot}\ \vee_c L \qquad \frac{\Gamma, \neg A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \vee_c B}\ \vee_c R \qquad \frac{A \rightarrow_c B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow \bot}{A \rightarrow_c B, \Gamma \Rightarrow \bot}\ \rightarrow_c L$$

$$\frac{\Gamma, A, \neg B \Rightarrow \bot}{\Gamma \Rightarrow A \rightarrow_c B}\ \rightarrow_c R \qquad \frac{p_i, \Gamma \Rightarrow \bot}{p_c, \Gamma \Rightarrow \bot}\ L_c \qquad \frac{\Gamma, \neg p_i \Rightarrow \bot}{\Gamma \Rightarrow p_c}\ R_c$$

$$\frac{A[y/x], \Gamma \Rightarrow \bot}{\exists_c x.A, \Gamma \Rightarrow \bot}\ \exists_c L \qquad \frac{\Gamma, \forall x.\neg A \Rightarrow \bot}{\Gamma \Rightarrow \exists_c x.A}\ \exists_c R$$

INITIAL, CUT AND STRUCTURAL RULES

$$\frac{}{p_i, \Gamma \Rightarrow p_i}\ \mathsf{init} \qquad \frac{\Gamma \Rightarrow A \quad A, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}\ \mathsf{cut} \qquad \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow A}\ \mathsf{W}$$

FIGURE 1. Ecumenical sequent system LE. In rules $\forall R, \exists_i L, \exists_c L$, y is fresh; p is atomic.

2 The system LCE

In [38], Dag Prawitz proposed a natural deduction system where classical and intuitionistic logics could coexist in peace. The language $\mathcal{L}$ used for ecumenical systems is described as follows. We will use a subscript c for the classical meaning and i for the intuitionistic one, dropping such subscripts when formulae/connectives can have either meaning.

Classical and intuitionistic n-ary predicate symbols ($p_c, p_i, \ldots$) co-exist in $\mathcal{L}$ but have different meanings. The *neutral* logical connectives $\{\bot, \neg, \wedge, \forall\}$ are common for classical and intuitionistic fragments, while $\{\rightarrow_i, \vee_i, \exists_i\}$ and $\{\rightarrow_c, \vee_c, \exists_c\}$ are restricted to *intuitionistic* and *classical*interpretations, respectively.

The sequent system LE (depicted in Figure 1) was presented in [35][2] as the sequent counterpart of Prawitz's natural deduction system.

LE has very interesting proof theoretical properties including cut-elimination, together with a Kripke semantical interpretation, that allowed the proposal of a variety of ecumenical proof systems, such as multi-conclusion and nested sequent systems, as well as several fragments of such systems [35].

2.1 *Ecumenical consequence and stoup*

Denoting by $\vdash_{\mathsf{S}} A$ the fact that the formula A is a theorem in the proof system S, the following theorems are easily provable in LE:

1. $\vdash_{\mathsf{LE}} (A \vee_c B) \leftrightarrow_i \neg(\neg A \wedge \neg B)$
2. $\vdash_{\mathsf{LE}} (A \rightarrow_c B) \leftrightarrow_i \neg(A \wedge \neg B)$
3. $\vdash_{\mathsf{LE}} (\exists_c x.A) \leftrightarrow_i \neg(\forall x.\neg A)$

[2]The system was called LEci in [35].

There equivalences are of interest since they relate the classical and the neutral operators: the classical connectives can be defined using negation, conjunction and the universal quantifier. On the other hand,

4. $\vdash_{\mathsf{LE}} (\neg\neg A) \rightarrow_c A$ but $\nvdash_{\mathsf{LE}} (\neg\neg A) \rightarrow_i A$ in general
5. $\vdash_{\mathsf{LE}} (A \wedge (A \rightarrow_i B)) \rightarrow_i B$ but $\nvdash_{\mathsf{LE}} (A \wedge (A \rightarrow_c B)) \rightarrow_i B$ in general
6. $\vdash_{\mathsf{LE}} \forall x.A \rightarrow_i \neg\exists_c x.\neg A$ but $\nvdash_{\mathsf{LE}} \neg\exists_c x.\neg A \rightarrow_i \forall x.A$ in general

Observe that (3) and (6) reveal the asymmetry between definability of quantifiers: while the classical existential can be defined from universal quantification, the other way around is not true, in general. This is closely related with the fact that, proving $\forall x.A$ from $\neg\exists_c x.\neg A$ depends on A being a classical formula. We will come back to this in Section 4.

Finally, observe that

7. $\vdash_{\mathsf{LE}} (A \rightarrow_c \bot) \leftrightarrow_i (A \rightarrow_i \bot) \leftrightarrow_i (\neg A)$;

That is, albeit being a neutral connective, the negation has a classical flavour. This fact will be heavily explored throughout this text.

The following result states that logical consequence in LE is intrinsically intuitionistic.

PROPOSITION 2.1 ([35]).
$\Gamma \vdash B$ is provable in LE iff $\vdash_{\mathsf{LE}} \bigwedge \Gamma \rightarrow_i B$.

However, formulas built over neutral and classical operators together with restricted intuitionistic implications have a *classical* behaviour.

DEFINITION 2.2.
Eventually externally classical (*eec* for short) formulas are given by the following grammar:

$$A^{eec} ::= A^c \mid A^{eec} \wedge A^{eec} \mid \forall x.A^{eec} \mid A \rightarrow_i A^{eec}$$

where A is any formula and A^c is an *externally classical* formula given by

$$A^c ::= p_c \mid \bot \mid \neg A \mid A \vee_c A \mid A \rightarrow_c A \mid \exists_c x.A$$

For *eec* formulas, we can prove the following theorems[3]:

1. $\vdash_{\mathsf{LE}} (A \wedge (A \rightarrow_c B^{eec})) \rightarrow_i B^{eec}$
2. $\vdash_{\mathsf{LE}} \neg\neg B^{eec} \rightarrow_i B^{eec}$
3. $\vdash_{\mathsf{LE}} \neg\exists_c x.\neg B^{eec} \rightarrow_i \forall x.B^{eec}$

More generally, notice that all classical right rules as well as the right rules for the neutral connectives in LE are invertible. Since invertible rules can be applied eagerly when proving a sequent, this entails that *eec* formulas can be eagerly decomposed. As a consequence, the ecumenical entailment, when restricted to *eec* succedents (antecedents having an unrestricted form), is classical.

[3]It should be noted that, in [35], these results were proved for externally classical formulas only. However, they are valid over a broader class of formulas, with external neutral operators and a restricted form of the intuitionistic implication.

THEOREM 2.3 ([35], extended to *eec*).
Let A^{eec} be an eventually externally classical formula and Γ be a multiset of ecumenical formulas. Then

$$\vdash_{\mathsf{LE}} \bigwedge \Gamma \rightarrow_c A^{eec} \text{ iff } \vdash_{\mathsf{LE}} \bigwedge \Gamma \rightarrow_i A^{eec}.$$

Proof theoretically, this aligns well with the *ecumenism* of Prawitz's original proposal: consequence relations are intrinsically intuitionistic, but have a classical behaviour when proving a formula that eventually will behave classically.

Moreover, observe that, from a proof π of $\Gamma \Rightarrow A$ in LE, we can derive $\Gamma, \neg A \Rightarrow \bot$:

$$\dfrac{\begin{array}{c}\pi^w\\ \Gamma, \neg A \Rightarrow A\end{array}}{\Gamma, \neg A \Rightarrow \bot}\ \neg L$$

where π^w is the weakened version of π. The other direction does not hold since $\not\vdash_{\mathsf{LE}} \neg\neg A \Rightarrow A$, in general. However, for *eec* formulas, the converse also holds.

PROPOSITION 2.4.
If $\Gamma, \neg A^{eec} \Rightarrow \bot$ is provable in LE so it is $\Gamma \Rightarrow A^{eec}$.

PROOF. Since $\vdash_{\mathsf{LE}} \neg\neg A^{eec} \Rightarrow A^{eec}$ (see 9), then

$$\dfrac{\dfrac{\Gamma, \neg A^{eec} \Rightarrow \bot}{\Gamma \Rightarrow \neg\neg A^{eec}} \quad \neg\neg A^{eec}, \Gamma \Rightarrow A^{eec}}{\Gamma \Rightarrow A^{eec}}\ \mathsf{cut}$$

□

This corroborates the idea that, in an ecumenical system with stoup, formulas in the classical context should hold classical subformulas of *eec* formulas. The stoup, on the other hand, would carry the *intuitionistic* or *neutral* information.

We are now ready to describe the ecumenical system with stoup, where the connections between the 'primitive' sequent calculus in LE and the 'pure' sequent calculus in LCE is established as follows:

- A sequent of the form $\Gamma, \neg\Delta \Rightarrow \Pi$ in LE will be translated as $\Gamma \Rightarrow \Delta; \Pi$ in LCE for some set Δ of negated formulas.
- A sequent of the form $\Gamma \Rightarrow \Delta; \Pi$ in LCE will be translated as $\Gamma, \neg\Delta \Rightarrow \Pi$ in LE.
- The empty stoup will be translated as $\bot$.

As already mentioned, formulas will move over contexts depending on their *polarity*.

DEFINITION 2.5.
An ecumenical formula is called *negative* if its main connective is either classical or the negation, and *positive*, otherwise (we will use N for negative and P for positive formulas).

Figure 2 gives the rules for the ecumenical pure systems with stoup (LCE). Observe that rules in LCE determine positive/negative phases in derivations, and the dynamic of the rules for classical connectives in LCE is as follows: negative formulas in the classical contexts are eagerly decomposed; if a positive formula in the right context is chosen to be worked on, it is placed in the stoup Π, and treated intuitionistically.

INTUITIONISTIC AND NEUTRAL RULES

$$\frac{\Gamma, A, B \Rightarrow \Delta; \Pi}{\Gamma, A \wedge B \Rightarrow \Delta; \Pi}\ \wedge L \qquad \frac{\Gamma \Rightarrow \Delta; A \quad \Gamma \Rightarrow \Delta; B}{\Gamma \Rightarrow \Delta; A \wedge B}\ \wedge R \qquad \frac{\Gamma, A \Rightarrow \Delta; \Pi \quad \Gamma, B \Rightarrow \Delta; \Pi}{\Gamma, A \vee_i B \Rightarrow \Delta; \Pi}\ \vee_i L$$

$$\frac{\Gamma \Rightarrow \Delta; A_j}{\Gamma \Rightarrow \Delta; A_1 \vee_i A_2}\ \vee_i R_j \qquad \frac{\Gamma, A \rightarrow_i B \Rightarrow \Delta; A \quad \Gamma, B \Rightarrow \Delta; \Pi}{\Gamma, A \rightarrow_i B \Rightarrow \Delta; \Pi}\ \rightarrow_i L$$

$$\frac{\Gamma, A \Rightarrow \Delta; B}{\Gamma \Rightarrow \Delta; A \rightarrow_i B}\ \rightarrow_i R \qquad \frac{\Gamma, A[y/x] \Rightarrow \Delta; \Pi}{\Gamma, \exists_i x.A \Rightarrow \Delta; \Pi}\ \exists_i L \qquad \frac{\Gamma \Rightarrow \Delta; A[y/x]}{\Gamma \Rightarrow \Delta; \exists_i x.A}\ \exists_i R$$

$$\frac{\Gamma, \forall x.A, A[y/x] \Rightarrow \Delta; \Pi}{\Gamma, \forall x.A \Rightarrow \Delta; \Pi}\ \forall L \qquad \frac{\Gamma \Rightarrow \Delta; A[y/x]}{\Gamma \Rightarrow \Delta; \forall x.A}\ \forall R$$

CLASSICAL RULES AND NEGATION

$$\frac{\Gamma, A \Rightarrow \Delta; \cdot \quad \Gamma, B \Rightarrow \Delta; \cdot}{\Gamma, A \vee_c B \Rightarrow \Delta; \cdot}\ \vee_c L \qquad \frac{\Gamma \Rightarrow A, B, \Delta; \cdot}{\Gamma \Rightarrow A \vee_c B, \Delta; \cdot}\ \vee_c R \qquad \frac{\Gamma, A \rightarrow_c B \Rightarrow \Delta; A \quad \Gamma, B \Rightarrow \Delta; \cdot}{\Gamma, A \rightarrow_c B \Rightarrow \Delta; \cdot}\ \rightarrow_c L$$

$$\frac{\Gamma, A \Rightarrow B, \Delta; \cdot}{\Gamma \Rightarrow A \rightarrow_c B, \Delta; \cdot}\ \rightarrow_c R \qquad \frac{\Gamma, p_i \Rightarrow \Delta; \cdot}{\Gamma, p_c \Rightarrow \Delta; \cdot}\ L_c \qquad \frac{\Gamma \Rightarrow p_i, \Delta; \cdot}{\Gamma \Rightarrow p_c, \Delta; \cdot}\ R_c$$

$$\frac{\Gamma, A[y/x] \Rightarrow \Delta; \cdot}{\Gamma, \exists_c x.A \Rightarrow \Delta; \cdot}\ \exists_c L \qquad \frac{\Gamma \Rightarrow A[y/x], \exists_c x.A, \Delta; \cdot}{\Gamma \Rightarrow \exists_c x.A, \Delta; \cdot}\ \exists_c R$$

$$\frac{\Gamma, \neg A \Rightarrow \Delta; A}{\Gamma, \neg A \Rightarrow \Delta; \cdot}\ \neg L \qquad \frac{\Gamma, A \Rightarrow \Delta; \cdot}{\Gamma \Rightarrow \Delta, \neg A; \cdot}\ \neg R$$

INITIAL, DECISION AND STRUCTURAL RULES

$$\frac{}{\Gamma, p_i \Rightarrow \Delta; p_i}\ \mathsf{init} \qquad \frac{\Gamma \Rightarrow P, \Delta; P}{\Gamma \Rightarrow P, \Delta; \cdot}\ \mathsf{D} \qquad \frac{\Gamma \Rightarrow N, \Delta; \cdot}{\Gamma \Rightarrow \Delta; N}\ \mathsf{store} \qquad \frac{\Gamma \Rightarrow \Delta; \cdot}{\Gamma \Rightarrow \Delta; A}\ \mathsf{W}$$

CUT RULES

$$\frac{\Gamma \Rightarrow \Delta; P \quad P, \Gamma \Rightarrow \Delta; \Pi}{\Gamma \Rightarrow \Delta; \Pi}\ P-\mathsf{cut} \qquad \frac{\Gamma \Rightarrow \Delta, N; \Pi^* \quad N, \Gamma \Rightarrow \Delta; \Pi}{\Gamma \Rightarrow \Delta; \Pi}\ N-\mathsf{cut}$$

FIGURE 2. Ecumenical pure system LCE. In rules $\forall R$, $\exists_c L$ and $\exists_i L$, y is fresh; N is negative and P is positive; p is atomic; Π^* is either empty or some $P \in \Delta$.

The analogy with focusing [1] stops there, though: explicit weakening in the stoup is needed for completeness, as the next example shows.

EXAMPLE 2.6.
The sequent

$$\neg B, A \rightarrow_c B, A \vdash C$$

is provable in LE, where the succedent C is weakened. This means that, in LCE, the following sequents are provable

$$\neg B, A \rightarrow_c B, A \Rightarrow \cdot; C \qquad \cdot \Rightarrow B, A \wedge \neg B, \neg A; C$$

But the stoup is necessarily erased in the process, *e.g.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow B, A \wedge \neg B, \neg A; A} \qquad \dfrac{\dfrac{\dfrac{}{A, B \Rightarrow B, A \wedge \neg B, \neg A; \cdot}}{A \Rightarrow B, A \wedge \neg B, \neg A, \neg B; \cdot}\ \neg R}{A \Rightarrow B, A \wedge \neg B, \neg A; \neg B}\ \mathsf{store}}{A \Rightarrow B, A \wedge \neg B, \neg A; A \wedge \neg B}\ \wedge}{A \Rightarrow B, A \wedge \neg B, \neg A; \cdot}\ \mathsf{D}}{\cdot \Rightarrow B, A \wedge \neg B, \neg A; \cdot}\ \neg R}{\cdot \Rightarrow B, A \wedge \neg B, \neg A; C}\ \mathsf{W}$$

The double bars correspond to instances of the derivation of $\Gamma, F \Rightarrow \Delta; F$, which is a provable sequent in LCE, for any contexts Γ, Δ and formula F (see Lemma 3.2 in the next section).

Observe that, in Girard's LC [15], sequents with non-empty stoup do not have a classical interpretation. In fact, **none** of the sequents above are provable in LC, if C is a positive formula.

3 Correctness of the systems

We start by stating standard definitions and proof theoretic results.

DEFINITION 3.1.
Let $\mathcal{S}$ be a sequent system. An inference rule

$$\frac{S_1 \quad \cdots \quad S_k}{S}$$

is called:

i. *admissible in* $\mathcal{S}$ if S is derivable in $\mathcal{S}$ whenever $S_1, \ldots, S_k$ are derivable in $\mathcal{S}$;
ii. *invertible* in $\mathcal{S}$ if the rules $\frac{S}{S_1}, \ldots, \frac{S}{S_k}$ are admissible in $\mathcal{S}$.
iii. *totally invertible* in $\mathcal{S}$ if it is invertible and do not have restrictions over contexts.

For example, the rule $\rightarrow_i R$ is totally invertible, but $\rightarrow_c R$ is only invertible since its bottom-up application is restricted to sequents with empty stoup. In general:

LEMMA 3.2.
In LCE:

i. The rules $\vee_c L, \vee_c R, \rightarrow_c L, \rightarrow_c R, \neg L, \neg R, L_c, R_c, \exists_c L, \exists_c R$ and D are invertible.
ii. The rules $\wedge L, \wedge R, \vee_i L, \rightarrow_i R, \exists_i L, \forall L, \forall R$ and store are totally invertible.
iii. Classical weakening and contraction are admissible

$$\frac{\Gamma \Rightarrow \Delta; \Pi}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'; \Pi}\ \mathsf{W}_c \qquad \frac{\Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'; \Pi}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'; \Pi}\ \mathsf{C}_c$$

iv. The general form of initial axioms are admissible

$$\frac{}{\Gamma, A \Rightarrow \Delta; A}\ \mathsf{init_i} \qquad \frac{}{\Gamma, A \Rightarrow \Delta, A; \Pi}\ \mathsf{init_c}$$

PROOF. The proofs are by standard induction on the height of derivations [30, 45]. The proof of admissibility of W_c does not depend on any other result, while the admissibility of C_c depends on the invertibility results and the admissibility of weakening.

The proof of admissibility of the general initial axioms is by mutual induction. Below we show the cases for quantifiers where, by induction hypothesis, instances of the axioms hold for the premises.

$$\dfrac{\dfrac{\dfrac{}{A[y/x], \Gamma \Rightarrow \exists_c x.A, A[x/y], \Delta; \cdot}\ \mathsf{init_c}}{A[y/x], \Gamma \Rightarrow \exists_c x.A, \Delta; \cdot}\ \exists_c R}{\exists_c x.A, \Gamma \Rightarrow \exists_c x.A, \Delta; \cdot}\ \exists_c L \qquad \dfrac{\dfrac{\dfrac{}{\forall x.A, A[y/x], \Gamma \Rightarrow \Delta; A[y/x]}\ \mathsf{init_i}}{\forall x.A, \Gamma \Rightarrow \Delta; A[y/x]}\ \forall L}{\forall x.A, \Gamma \Rightarrow \Delta; \forall x.A}\ \forall R$$

□

The following shows that LCE is correct and complete w.r.t. LE.

THEOREM 3.3.
The sequent $\Gamma \Rightarrow \Delta; \Pi$ is provable in LCE iff $\Gamma, \neg\Delta \Rightarrow \Pi$ is provable in LE.

PROOF. The only interesting cases are the ones involving classical connectives and negation since for the intuitionistic and the other neutral ones the rules in LCE operate in formulas in the stoup, hence faithfully corresponding to the application of rules in the single conclusion system LE.

– Case $\vee_c R$. Suppose that $\Gamma \Rightarrow \Delta, A \vee_c B; \cdot$ is provable in LCE with proof

$$\dfrac{\overset{\pi}{\Gamma \Rightarrow A, B, \Delta; \cdot}}{\Gamma \Rightarrow A \vee_c B, \Delta; \cdot}\ \vee_c R$$

By the inductive hypothesis, $\Gamma, \neg A, \neg B, \neg\Delta \Rightarrow \bot$ has a proof π' in LE. Hence

$$\dfrac{\dfrac{\overset{\pi'}{\Gamma, \neg A, \neg B, \neg\Delta \Rightarrow \bot}}{\Gamma, \neg\Delta \Rightarrow A \vee_c B}\ \vee_c R}{\Gamma, \neg(A \vee_c B), \neg\Delta \Rightarrow \bot}\ \neg L$$

On the other hand, suppose that $\Gamma \Rightarrow A \vee_c B$ is provable in LE with proof

$$\dfrac{\overset{\pi}{\Gamma, \neg A, \neg B \Rightarrow \bot}}{\Gamma \Rightarrow A \vee_c B}\ \vee_c R$$

By the inductive hypothesis, $\Gamma \Rightarrow A, B; \cdot$ has a proof π' in LCE. Thus,

$$\dfrac{\dfrac{\overset{\pi'}{\Gamma \Rightarrow A, B; \cdot}}{\Gamma \Rightarrow A \vee_c B; \cdot}\ \vee_c R}{\Gamma \Rightarrow \cdot; A \vee_c B}\ \mathsf{store}$$

– Case $\rightarrow_c R$. Suppose that $\Gamma \Rightarrow \Delta, A \rightarrow_c B; \cdot$ is provable in LCE with proof

$$\dfrac{\overset{\pi}{\Gamma, A \Rightarrow B, \Delta; \cdot}}{\Gamma \Rightarrow A \rightarrow_c B, \Delta; \cdot}\ \rightarrow_c R$$

By the inductive hypothesis, $\Gamma, A, \neg B, \neg\Delta \Rightarrow \bot$ has a proof π' in LE. Hence

$$\dfrac{\dfrac{\overset{\pi'}{\Gamma, A, \neg B, \neg\Delta \Rightarrow \bot}}{\Gamma, \neg\Delta \Rightarrow A \rightarrow_c B}\ \rightarrow_c R}{\Gamma, \neg(A \rightarrow_c B), \neg\Delta \Rightarrow \bot}\ \neg L$$

On the other hand, suppose that $\Gamma \Rightarrow A \rightarrow_c B$ is provable in LE with proof

$$\dfrac{\overset{\pi}{\Gamma, A, \neg B \Rightarrow \bot}}{\Gamma \Rightarrow A \rightarrow_c B}\ \rightarrow_c R$$

By the inductive hypothesis, $\Gamma, A \Rightarrow B; \cdot$ has a proof π' in LCE. Hence

$$\dfrac{\dfrac{\overset{\pi'}{\Gamma, A \Rightarrow B; \cdot}}{\Gamma \Rightarrow A \rightarrow_c B; \cdot}\ \rightarrow_c R}{\Gamma \Rightarrow \cdot; A \rightarrow_c B}\ \mathsf{store}$$

– Case $\neg R$. Suppose that $\Gamma \Rightarrow \neg A, \Delta; \cdot$ is provable in LCE with proof

$$\dfrac{\overset{\pi}{\Gamma, A \Rightarrow \Delta; \cdot}}{\Gamma \Rightarrow \Delta, \neg A; \cdot}\ \neg R$$

By the inductive hypothesis, $\Gamma, A, \neg\Delta \Rightarrow \bot$ has a proof π' in LE. Hence,

$$\dfrac{\dfrac{\overset{\pi'_w}{\Gamma, \neg\neg A, \neg\Delta, A \Rightarrow \bot}}{\Gamma, \neg\neg A, \neg\Delta \Rightarrow \neg A}\ \neg R}{\Gamma, \neg\neg A, \neg\Delta \Rightarrow \bot}\ \neg L$$

where π'_w represents the translation of the weakened version of π'.
On the other hand, suppose that $\Gamma \Rightarrow \neg A$ is provable in LE with proof

$$\dfrac{\overset{\pi}{\Gamma, A \Rightarrow \bot}}{\Gamma \Rightarrow \neg A}\ \neg R$$

By the inductive hypothesis, $\Gamma, A \Rightarrow \cdot; \cdot$ has a proof π' in LCE. Hence,

$$\dfrac{\dfrac{\overset{\pi'}{\Gamma, A \Rightarrow \cdot; \cdot}}{\Gamma \Rightarrow \neg A; \cdot}\ \neg R}{\Gamma \Rightarrow \cdot; \neg A}\ \mathsf{store}$$

– Case $\exists_c R$. Suppose that $\Gamma \Rightarrow \exists_c x.A, \Delta; \cdot$ is provable in LCE with proof

$$\dfrac{\overset{\pi}{\Gamma \Rightarrow A[y/x], \exists_c x.A, \Delta; \cdot}}{\Gamma \Rightarrow \exists_c x.A, \Delta; \cdot}\ \rightarrow_c R$$

– By the inductive hypothesis, $\Gamma, \neg A[y/x], \neg\exists_c x.A, \neg\Delta \Rightarrow \bot$ has a proof π' in LE. Hence

$$\dfrac{\dfrac{\dfrac{\overset{\pi'}{\Gamma, \neg\exists_c x.A, \neg A[y/x], \forall x.\neg A, \neg\Delta \Rightarrow \bot}}{\Gamma, \neg\exists_c x.A, \forall x.\neg A, \neg\Delta \Rightarrow \bot}\ \forall L}{\Gamma, \neg\exists_c x.A, \neg\Delta \Rightarrow \exists_c x.A}\ \exists_c R}{\Gamma, \neg\exists_c x.A, \neg\Delta \Rightarrow \bot}\ \neg L$$

On the other hand, suppose that $\Gamma \Rightarrow \exists_c x.A$ is provable in LE with proof

$$\dfrac{\overset{\pi}{\Gamma, \forall x.\neg A \Rightarrow \bot}}{\Gamma \Rightarrow \exists_c x.A}\ \exists_c R$$

By the inductive hypothesis, $\Gamma, \forall x.\neg A \Rightarrow \cdot; \cdot$ has a proof π' in LCE. Hence,

$$\dfrac{\dfrac{\dfrac{\overset{\pi'_w}{\Gamma, \forall x.\neg A \Rightarrow \exists_c x.A; \cdot}}{\Gamma \Rightarrow \exists_c x.A, \neg\forall x.\neg A; \cdot}\ \neg R \qquad \overline{\overline{\Gamma, \neg\forall x.\neg A \Rightarrow \exists_c x.A; \cdot}}}{\Gamma \Rightarrow \exists_c x.A; \cdot}\ N-\mathsf{cut}}{\Gamma \Rightarrow \cdot; \exists_c x.A}\ \mathsf{store}$$

where π'_w represents the translation of the weakened version of π' and the double bars in the right branch indicates an adapted proof of 3.

– Case $\to_c L$. Suppose that $\Gamma, A \to_c B \Rightarrow \Delta; \cdot$ is provable in LCE with proof

$$\dfrac{\overset{\pi_1}{\Gamma, A \to_c B \Rightarrow \Delta; A} \quad \overset{\pi_2}{\Gamma, B \Rightarrow \Delta; \cdot}}{\Gamma, A \to_c B \Rightarrow \Delta; \cdot} \to_c L$$

By the inductive hypothesis, $\Gamma, A \to_c B, \neg\Delta \Rightarrow A$ and $\Gamma, B, \neg\Delta \Rightarrow \bot$ have proofs π_1', π_2' in LE, respectively. Hence,

$$\dfrac{\overset{\pi_1'}{\Gamma, A \to_c B, \neg\Delta \Rightarrow A} \quad \overset{\pi_2'}{\Gamma, B, \neg\Delta \Rightarrow \bot}}{\Gamma, A \to_c B, \neg\Delta \Rightarrow \bot} \to_c L$$

and vice-versa. The other left-rule cases are similar.

– Case D. Suppose that $\Gamma \Rightarrow \Delta, P; \cdot$ is provable in LCE with proof

$$\dfrac{\overset{\pi}{\Gamma \Rightarrow \Delta, P; P}}{\Gamma \Rightarrow \Delta, P; \cdot} \, \mathsf{D}$$

By the inductive hypothesis, $\Gamma, \neg P, \neg\Delta \Rightarrow P$ has a proof π' in LE. Hence,

$$\dfrac{\overset{\pi'}{\Gamma, \neg P, \neg\Delta \Rightarrow P}}{\Gamma, \neg P, \neg\Delta \Rightarrow \bot} \neg L$$

– Case store. Suppose that $\Gamma \Rightarrow \Delta; N$ is provable in LCE with proof

$$\dfrac{\overset{\pi}{\Gamma \Rightarrow \Delta, N; \cdot}}{\Gamma \Rightarrow \Delta; N} \, \mathsf{store}$$

By the inductive hypothesis, $\Gamma, \neg\Delta, \neg N, \Rightarrow \bot$ has a proof π' in LE. Hence,

$$\dfrac{\dfrac{\overset{\pi'}{\Gamma, \neg\Delta, \neg N \Rightarrow \bot}}{\Gamma, \neg\Delta \Rightarrow \neg\neg N} \neg R \qquad \overline{\overline{\neg\neg N \Rightarrow N}}}{\Gamma, \neg\Delta \Rightarrow N} \, \mathsf{cut}$$

where the double bar indicates the multiple-steps proof of the fact that, for negative formulas, $N \leftrightarrow_i \neg\neg N$.

– Cases cut. The $P - \mathsf{cut}$ rule in LCE trivially corresponds to cut in LE. Suppose that $\Gamma \Rightarrow \Delta; \Pi$ is provable in LCE with proof

$$\dfrac{\overset{\pi_1}{\Gamma \Rightarrow \Delta, N; \Pi^*} \quad \overset{\pi_2}{N, \Gamma \Rightarrow \Delta; \Pi}}{\Gamma \Rightarrow \Delta; \Pi} \, N - \mathsf{cut}$$

By the inductive hypothesis, $\Gamma, \neg\Delta, \neg N \Rightarrow \Pi^*$ and $N, \Gamma, \neg\Delta \Rightarrow \Pi$ are provable in LE with proofs π_1' and π_2', respectively.
If $\Pi^* = \cdot$, then

$$\dfrac{\dfrac{\overset{\pi_1'}{\Gamma, \neg\Delta, \neg N \Rightarrow \bot}}{\Gamma, \neg\Delta \Rightarrow \neg\neg N} \neg R \qquad \dfrac{\overline{\overline{\neg\neg N, \Gamma, \neg\Delta \Rightarrow N}} \quad \overset{\pi_{2w}'}{\neg\neg N, N, \Gamma, \neg\Delta \Rightarrow \Pi}}{\neg\neg N, \Gamma, \neg\Delta \Rightarrow \Pi} \, \mathsf{cut}}{\Gamma, \neg\Delta \Rightarrow \Pi} \, \mathsf{cut}$$

where π'_{2w} is the weakened version of π_2, and the double bar corresponds to the proof of 9. If $\Pi^* = P \in \Delta$, then the left premise derivation is substituted by

$$\dfrac{\dfrac{\dfrac{\pi'_1}{\Gamma, \neg\Delta, \neg N \Rightarrow P} \quad \overline{\overline{P, \Gamma, \neg\Delta, \neg N \Rightarrow \bot}}\ P \in \Delta}{\Gamma, \neg\Delta, \neg N \Rightarrow \bot}\ \mathsf{cut}}{\Gamma, \neg\Delta \Rightarrow \neg\neg N}\ \neg R$$

Observe that, if $P \in \Delta$ then $\neg P \in \neg\Delta$ and, hence, $P, \Gamma, \neg\Delta, \neg N \Rightarrow \bot$ is trivially provable. □

Observe that, other than the cut rules, store introduces cut in the translation between proofs from LCE to LE, while $\exists_c R$ introduces cuts the other way around. Since LE has the cut-elimination property [35], the translation from LCE to LE is not problematic. However, for proving cut-free completeness from LE to LCE, we need to prove that the later has the cut-elimination property.

3.1 Cut-elimination

A logical connective is called *harmonious* in a certain proof system if there exists a certain balance between the rules defining it. For example, in natural deduction based systems, harmony is ensured when introduction/elimination rules do not contain insufficient/excessive amounts of information [9]. In sequent calculi, this property is often guaranteed by the admissibility of a general initial axiom (*identity-expansion*) and of the cut rule (*cut-elimination*) [27].

In Lemma 3.2, we proved identity-expansion for LCE. In the following, we will complete the proof of harmony for LCE, proving that it enjoys the cut-elimination property. This will also guarantee cut-completeness from LE to LCE, as mentioned above.

Proving admissibility of cut rules in sequent based systems with multiple contexts is often tricky, since the cut formulas can change contexts during cut reductions. This is the case for LCE. The proof is by mutual induction, with inductive measure (n, m) where m is the cut-height, the cumulative height of derivations above the cut, and n is the ecumenical weight of the cut-formula, defined as

$$\begin{array}{ll}
\mathsf{ew}(p_i) = \mathsf{ew}(\bot) = 0 & \mathsf{ew}(A \star B) = \mathsf{ew}(A) + \mathsf{ew}(B) + 1 \text{ if } \star \in \{\wedge, \rightarrow_i, \vee_i\} \\
\mathsf{ew}(p_c) = 4 & \mathsf{ew}(\heartsuit A) = \mathsf{ew}(A) + 1 \text{ if } \heartsuit \in \{\neg, \exists_i x., \forall x.\} \\
\mathsf{ew}(\exists_c x.A) = \mathsf{ew}(A) + 4 & \mathsf{ew}(A \circ B) = \mathsf{ew}(A) + \mathsf{ew}(B) + 4 \text{ if } \circ \in \{\rightarrow_c, \vee_c\}
\end{array}$$

Intuitively, the ecumenical weight measures the amount of extra information needed (the negations added) to define classical connectives from intuitionistic and neutral ones.

THEOREM 3.4.
The rules $N - \mathsf{cut}$ and $P - \mathsf{cut}$ are admissible in LCE.

PROOF. The dynamic of the proof is the following: cut applications either move up in the proof, i.e. the cut-height is reduced or are substituted by simpler cuts of the same kind, i.e. the ecumenical weight is reduced, as in usual cut-elimination reductions. The cut instances alternate between positive and negative (and vice-versa) in the principal cases, where the polarities of the subformulas flip. We will detail the main cut-reductions.

– Base cases. Consider the derivation

$$\dfrac{\dfrac{\pi}{\Gamma \Rightarrow \Delta; p_i} \quad \overline{\Gamma, p_i \Rightarrow \Delta; \Pi}\ init}{\Gamma \Rightarrow \Delta; \Pi}\ P - \mathsf{cut}$$

If p_i is principal, then $\Pi = p_i$ and the derivation reduces to π.
If p_i is not principal, then there is an atom $q_i \in \Gamma \cap \Pi$ and the reduction is a trivial one. Similar analyses hold for $N - \mathsf{cut}$, when the left premise is an instance of init, as well as for the other axioms.

– Non-principal cases. In all the cases where the cut-formula is not principal in one of the premises, the cut moves upwards, for example,

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma, A \Rightarrow \Delta, N; \cdot} \quad \overset{\pi_2}{\Gamma, B \Rightarrow \Delta, N; \cdot}}{\Gamma, A \vee_c B \Rightarrow \Delta, N; \cdot}\ \vee_c L \quad \overset{\pi_3}{N, \Gamma, A \vee_c B \Rightarrow \Delta; \cdot}}{\Gamma, A \vee_c B \Rightarrow \Delta; \cdot}\ N - \mathsf{cut}$$

reduces to

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma, A \Rightarrow \Delta, N; \cdot} \quad \overset{\pi_3^1}{N, \Gamma, A \Rightarrow \Delta; \cdot}}{\Gamma, A \Rightarrow \Delta; \cdot}\ N - \mathsf{cut} \quad \dfrac{\overset{\pi_2}{\Gamma, B \Rightarrow \Delta, N; \cdot} \quad \overset{\pi_3^2}{N, \Gamma, B \Rightarrow \Delta; \cdot}}{\Gamma, B \Rightarrow \Delta; \cdot}\ N - \mathsf{cut}}{\Gamma, A \vee_c B \Rightarrow \Delta; \cdot}\ \vee_c L$$

where π_3^i are the proofs constructed from π_3 using the invertibility of $\vee_c L$.
The only exceptions are when

– dereliction is applied in the left premise

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \Rightarrow \Delta, P, N; P}}{\Gamma \Rightarrow \Delta, P, N; \cdot}\ \mathsf{D} \quad \overset{\pi_2}{N, \Gamma \Rightarrow \Delta, P; \Pi}}{\Gamma \Rightarrow \Delta, P; \Pi}\ N - \mathsf{cut}$$

In this case, we substitute the version of $N - \mathsf{cut}$ for absorbing the dereliction

$$\dfrac{\overset{\pi_1}{\Gamma \Rightarrow \Delta, P, N; P} \quad \overset{\pi_2}{N, \Gamma \Rightarrow \Delta, P; \Pi}}{\Gamma \Rightarrow \Delta, P; \Pi}\ N - \mathsf{cut}$$

– weakening is applied in the left premise

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \Rightarrow \Delta, P, N; \cdot}}{\Gamma \Rightarrow \Delta, P, N; P}\ \mathsf{W} \quad \overset{\pi_2}{N, \Gamma \Rightarrow \Delta, P; \Pi}}{\Gamma \Rightarrow \Delta, P; \Pi}\ N - \mathsf{cut}$$

In this case, we substitute the version of $N - \mathsf{cut}$ for absorbing the weakening

$$\dfrac{\overset{\pi_1}{\Gamma \Rightarrow \Delta, P, N; \cdot} \quad \overset{\pi_2}{N, \Gamma \Rightarrow \Delta, P; \Pi}}{\Gamma \Rightarrow \Delta, P; \Pi}\ N - \mathsf{cut}$$

– Principal cases. If the cut formula is principal in both premises, then we need to be extra-careful with the polarities. We show the two most representative cases.

– $N = P \to_c Q$, with P, Q positive.

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma, P \Rightarrow \Delta, Q; \cdot}}{\Gamma \Rightarrow \Delta, P \to_c Q; \cdot} \to_c R \quad \dfrac{\overset{\pi_2}{\Gamma, P \to_c Q \Rightarrow \Delta; P} \quad \overset{\pi_3}{\Gamma, Q \Rightarrow \Delta; \cdot}}{\Gamma, P \to_c Q \Rightarrow \Delta; \cdot} \to_c L}{\Gamma \Rightarrow \Delta; \cdot} N - \mathsf{cut}_0$$

reduces to

$$\dfrac{\dfrac{\overset{\pi_3}{\Gamma, Q \Rightarrow \Delta; \cdot}}{\Gamma \Rightarrow \Delta, \neg Q; \cdot} \neg R \quad \overset{\pi}{\Gamma, \neg Q \Rightarrow \Delta; \cdot}}{\Gamma \Rightarrow \Delta; \cdot} N - \mathsf{cut}_1$$

where π is

$$\dfrac{\dfrac{\dfrac{\overset{\pi_1^w}{\Gamma, \neg Q, P \Rightarrow \Delta, Q; \cdot}}{\Gamma, \neg Q \Rightarrow \Delta, P \to_c Q; \cdot} \to_c R \quad \overset{\pi_2^w}{\Gamma, \neg Q, P \to_c Q \Rightarrow \Delta; P}}{\Gamma, \neg Q \Rightarrow \Delta; P} N - \mathsf{cut}_2 \quad \overset{\pi_1^{\equiv}}{\Gamma, \neg Q, P \Rightarrow \Delta; \cdot}}{\Gamma, \neg Q \Rightarrow \Delta; \cdot} P - \mathsf{cut}$$

Here, $\pi_1^{\equiv}$ is the same as π_1 where every application of the rule D over Q in the above derivation is substituted by an application of $\neg$ over $\neg Q$. Observe that the cut-formula of $N - \mathsf{cut}_1$ has lower ecumenical weight than $N - \mathsf{cut}_0$, while the cut-height of $N - \mathsf{cut}_2$ is smaller than $N - \mathsf{cut}_0$. Finally, observe that this is a non-trivial cut-reduction: usually, the cut over the implication is replaced by a cut over Q first. Due to polarities, if Q is positive, then $\neg Q$ is negative and cutting over it will add to the left context the classical information Q, hence mimicking the behaviour of formulas in the right input context.

– $N = \exists_c x.P$, with P positive.

$$\dfrac{\dfrac{\overset{\pi_1}{\Gamma \Rightarrow \Delta, \exists_c x.P, P[y/x]; \cdot}}{\Gamma \Rightarrow \Delta, \exists_c x.P; \cdot} \exists_c R \quad \dfrac{\overset{\pi_2}{\Gamma, P[y/x] \Rightarrow \Delta; \cdot}}{\Gamma, \exists_c x.P \Rightarrow \Delta; \cdot} \exists_c L}{\Gamma \Rightarrow \Delta; \cdot} N - \mathsf{cut}_0$$

reduces to

$$\dfrac{\dfrac{\overset{\pi_2}{\Gamma, P[y/x] \Rightarrow \Delta; \cdot}}{\Gamma \Rightarrow \Delta, \neg P[y/x]; \cdot} \neg R \quad \dfrac{\overset{\pi_1^{\equiv}}{\Gamma, \neg P[y/x] \Rightarrow \Delta, \exists_c x.P; \cdot} \quad \dfrac{\overset{\pi_2[z/y]}{\Gamma, \neg P[y/x], P[z/x] \Rightarrow \Delta; \cdot}}{\Gamma, \neg P[y/x], \exists_c x.P \Rightarrow \Delta; \cdot} \exists_c L}{\Gamma, \neg P[y/x] \Rightarrow \Delta; \cdot} N - \mathsf{cut}_2}{\Gamma \Rightarrow \Delta; \cdot} N - \mathsf{cut}_1$$

where the same observations for the above case hold, and $\pi_2[z/y]$ indicates the renaming of fresh variables in the derivation π_2. □

We finish this section noting that polarities play an important role in the cut-elimination process. In fact, without them, adding a general cut rule would collapse the system to CL.

EXAMPLE 3.5.
If the cut rule

$$\dfrac{\Gamma \Rightarrow \Delta, A; \Pi^* \quad A, \Gamma \Rightarrow \Delta; \Pi}{\Gamma \Rightarrow \Delta; \Pi}\ \mathsf{cut}$$

was admissible in LCE for an arbitrary formula A, then $\cdot \Rightarrow \cdot; A \vee_i \neg A$ would have the proof

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$\mathsf{init_i}$}
\UnaryInfC{$A \Rightarrow A \vee_i \neg A; A$}
\RightLabel{$\vee_i$}
\UnaryInfC{$A \Rightarrow A \vee_i \neg A; A \vee_i \neg A$}
\RightLabel{D}
\UnaryInfC{$A \Rightarrow A \vee_i \neg A; \cdot$}
\doubleLine
\RightLabel{$\mathsf{store}, \neg R$}
\UnaryInfC{$\cdot \Rightarrow A \vee_i \neg A; \neg A$}
\RightLabel{$\vee_i$}
\UnaryInfC{$\cdot \Rightarrow A \vee_i \neg A; A \vee_i \neg A$}
\RightLabel{D}
\UnaryInfC{$\cdot \Rightarrow A \vee_i \neg A; \cdot$}
\AxiomC{}
\RightLabel{$\mathsf{init_i}$}
\UnaryInfC{$A \vee_i \neg A, \Gamma \Rightarrow \cdot; A \vee_i \neg A$}
\RightLabel{cut}
\BinaryInfC{$\cdot \Rightarrow \cdot; A \vee_i \neg A$}
\end{prooftree}
$$

4 Ecumenical modalities

We will now extend the propositional fragment of Prawitz ecumenical logic with modalities.

The language of *(propositional, normal) modal formulas* consists of a denumerable set $\mathcal{P}$ of propositional symbols and a set of propositional connectives enhanced with the unary *modal operators* $\Box$ and $\Diamond$ concerning necessity and possibility, respectively [2].

The semantics of modal logics is often determined by means of *Kripke models*. Here, we will follow the approach in [42], where a modal logic is characterized by the respective interpretation of the modal model in the meta-theory (called *meta-logical characterization*).

Formally, given a variable x, we recall the standard translation $[\cdot]_x$ from modal formulas into first-order formulas with at most one free variable, x, as follows: if p is atomic, then $[p]_x = p(x)$; $[\bot]_x = \bot$; for any binary connective $\star$, $[A \star B]_x = [A]_x \star [B]_x$; for the modal connectives

$$[\Box A]_x \quad = \quad \forall y(R(x,y) \rightarrow [A]_y) \qquad [\Diamond A]_x \quad = \quad \exists y(R(x,y) \wedge [A]_y)$$

where $R(x,y)$ is a binary predicate.

Such a translation has, as an underlying justification, the interpretation of alethic modalities in a Kripke model $\mathcal{M} = (W, R, V)$:

$$\begin{array}{lll} \mathcal{M}, w \models \Box A & \text{iff} & \text{for all} v \text{ such that } wRv, \mathcal{M}, v \models A. \\ \mathcal{M}, w \models \Diamond A & \text{iff} & \text{there exists} v \text{ such that } wRv \text{ and } \mathcal{M}, v \models A. \end{array}$$

$R(x,y)$ then represents the *accessibility relation* R in a Kripke frame. This intuition can be made formal based on the one-to-one correspondence between classical/intuitionistic translations and Kripke modal models [42].

The object-modal logic OL is then characterized in the first-order meta-logic ML as

$$\vdash_{OL} A \quad \text{iff} \quad \vdash_{ML} \forall x.[A]_x$$

Hence, if ML is CL, the former definition characterizes the classical modal logic K [2], while if it is intuitionistic logic (IL), then it characterizes the intuitionistic modal logic IK [42].

In this work, we will adopt ecumenical logic as the meta-theory (given by the system LCE), hence characterizing what we will define as the ecumenical modal logic EK.

4.1 An ecumenical view of modalities

The language of *ecumenical modal formulas* consists of a denumerable set $\mathcal{P}$ of (ecumenical) propositional symbols and the set of ecumenical connectives enhanced with unary *ecumenical modal operators*. Unlike for the classical case, there is not a canonical definition of constructive or intuitionistic modal logics. Here we will mostly follow the approach in [42] for justifying our choices for the ecumenical interpretation for *possibility* and *necessity*.

The ecumenical translation $[\cdot]_x^e$ from propositional ecumenical formulas into LCE is defined in the same way as the modal translation $[\cdot]_x$ in the last section. For the case of modal connectives, observe that, due to Proposition 2.1, the interpretation of ecumenical consequence should be essentially *intuitionistic*. This implies that the box modality is a *neutral connective*. The diamond, on the other hand, has two possible interpretations: classical and intuitionistic, since its leading connective is an existential quantifier. Hence, we should have the ecumenical modalities: $\Box, \Diamond_i, \Diamond_c$, determined by the translations

$$\begin{array}{lcllcl} [\Box A]_x^e & = & \forall y(R(x,y) \rightarrow_i [A]_y^e) & & & \\ [\Diamond_i A]_x^e & = & \exists_i y(R(x,y) \wedge [A]_y^e) & \qquad [\Diamond_c A]_x^e & = & \exists_c y(R(x,y) \wedge [A]_y^e) \end{array}$$

We will denote by EK the ecumenical modal logic meta-logically characterized by LCE via $[\cdot]_x^e$, that is,

$$\vdash_{\mathsf{EK}} A \quad \text{iff} \quad \vdash_{\mathsf{LCE}} \forall x.[A]_x$$

Setting $\mathcal{L}_{\mathcal{M}}$ as the ecumenical modal language (that is, built from $\mathcal{L}$ with ecumenical modalities), the translation above naturally induces the labelled language $\mathcal{L}_{\mathcal{L}}$ of *labeled modal formulas*, determined by *labeled formulas* of the form $x : A$ with $A \in \mathcal{L}_{\mathcal{M}}$ and *relational atoms* of the form xRy, where x, y range over a set of variables. *Labeled sequents* have the form $\Gamma \Rightarrow x : A$, where Γ is a multiset containing labelled modal formulas and relational atoms.

In [24], we proposed a non-pure labelled calculus for ecumenical modal logic. We illustrate the system with the rules for the classical diamond

$$\frac{xRy, y : A, \Gamma \Rightarrow x : \bot}{x : \Diamond_c A, \Gamma \Rightarrow x : \bot}\ \Diamond_c L \qquad \frac{x : \Box\neg A, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : \Diamond_c A}\ \Diamond_c R$$

Observe that the $\Diamond_c R$ rule is not pure, since the premise depends on $\neg$ and $\Box$. In the present work, we achieve purity by using polarities and sequents with stoup.

Labeled sequents with stoup have the form $\Gamma \Rightarrow \Delta; x : A$, where Γ is as before and Δ is a multiset containing labelled modal formulas.

Polarities will be extended to the modal case smoothly, that is, formulas with an outermost classical connective or a negation are negative and all the others are positive. Relational atoms are not polarizable.

In Figure 3, we present the pure, labelled ecumenical system labEK. Observe that

$$\vdash_{\mathsf{labEK}} x : \Diamond_c A \leftrightarrow_i \neg\Box\neg A$$

On the other hand, $\Box$ and $\Diamond_i$ are not inter-definable. Finally, if A^{eec} is eventually externally classical, then

$$\vdash_{\mathsf{labEK}} x : \Box A^{eec} \leftrightarrow_i \neg\Diamond_c\neg A^{eec}$$

This means that, when restricted to the classical fragment, $\Box$ and $\Diamond_c$ are duals. This reflects well the ecumenical nature of the defined modalities.

Intuitionistic and neutral rules

$$\frac{\Gamma, x:A, x:B \Rightarrow \Delta;\Pi}{\Gamma, x:A\wedge B \Rightarrow \Delta;\Pi}\ \wedge L \qquad \frac{\Gamma \Rightarrow \Delta; x:A \quad \Gamma \Rightarrow \Delta; x:B}{\Gamma \Rightarrow \Delta; x:A\wedge B}\ \wedge R$$

$$\frac{\Gamma, x:A \Rightarrow \Delta;\Pi \quad \Gamma, x:B \Rightarrow \Delta;\Pi}{\Gamma, x:A\vee_i B \Rightarrow \Delta;\Pi}\ \vee_i L \qquad \frac{\Gamma \Rightarrow \Delta; x:A_j}{\Gamma \Rightarrow \Delta; x:A_1\vee_i A_2}\ \vee_i R_j$$

$$\frac{\Gamma, x:A\rightarrow_i B \Rightarrow \Delta; x:A \quad \Gamma, x:B \Rightarrow \Delta;\Pi}{\Gamma, x:A\rightarrow_i B \Rightarrow \Delta;\Pi}\ \rightarrow_i L \qquad \frac{\Gamma, x:A \Rightarrow \Delta; x:B}{\Gamma \Rightarrow \Delta; x:A\rightarrow_i B}\ \rightarrow_i R$$

Classical rules and negation

$$\frac{\Gamma, x:A\rightarrow_c B \Rightarrow \Delta; x:A \quad \Gamma, x:B \Rightarrow \Delta;\cdot}{\Gamma, x:A\rightarrow_c B \Rightarrow \Delta;\cdot}\ \rightarrow_c L \qquad \frac{\Gamma, x:A \Rightarrow x:B,\Delta;\cdot}{\Gamma \Rightarrow x:A\rightarrow_c B,\Delta;\cdot}\ \rightarrow_c R$$

$$\frac{\Gamma, x:A \Rightarrow \Delta;\cdot \quad \Gamma, x:B \Rightarrow \Delta;\cdot}{\Gamma, x:A\vee_c B \Rightarrow \Delta;\cdot}\ \vee_c L \qquad \frac{\Gamma \Rightarrow x:A, x:B,\Delta;\cdot}{\Gamma \Rightarrow x:A\vee_c B,\Delta;\cdot}\ \vee_c R$$

$$\frac{\Gamma, x:p_i \Rightarrow \Delta;\cdot}{\Gamma, x:p_c \Rightarrow \Delta;\cdot}\ L_c \qquad \frac{\Gamma \Rightarrow x:p_i,\Delta;\cdot}{\Gamma \Rightarrow x:p_c,\Delta;\cdot}\ R_c$$

$$\frac{\Gamma, x:\neg A \Rightarrow \Delta; x:A}{\Gamma, x:\neg A \Rightarrow \Delta;\cdot}\ \neg L \qquad \frac{\Gamma, x:A \Rightarrow \Delta;\cdot}{\Gamma \Rightarrow x:\neg A,\Delta;\cdot}\ \neg R$$

Modal rules

$$\frac{xRy, y:A, x:\Box A,\Gamma \Rightarrow \Delta;\Pi}{xRy, x:\Box A,\Gamma \Rightarrow \Delta;\Pi}\ \Box L \qquad \frac{xRy,\Gamma \Rightarrow \Delta; y:A}{\Gamma \Rightarrow \Delta; x:\Box A}\ \Box R \qquad \frac{xRy, y:A,\Gamma \Rightarrow \Delta;\Pi}{x:\Diamond_i A,\Gamma \Rightarrow \Delta;\Pi}\ \Diamond_i L$$

$$\frac{xRy,\Gamma \Rightarrow \Delta; y:A}{xRy,\Gamma \Rightarrow \Delta; x:\Diamond_i A}\ \Diamond_i R \qquad \frac{xRy, y:A,\Gamma \Rightarrow \Delta;\cdot}{x:\Diamond_c A,\Gamma \Rightarrow \Delta;\cdot}\ \Diamond_c L \qquad \frac{xRy,\Gamma \Rightarrow y:A, x:\Diamond_c A,\Delta;\cdot}{xRy,\Gamma \Rightarrow x:\Diamond_c A,\Delta;\cdot}\ \Diamond_c R$$

Initial, decision and structural rules

$$\frac{}{\Gamma, x:A \Rightarrow \Delta; x:A}\ \mathsf{init}_i \qquad \frac{}{\Gamma, x:A \Rightarrow x:A,\Delta;\Pi}\ \mathsf{init}_c$$

$$\frac{\Gamma \Rightarrow x:P,\Delta; x:P}{\Gamma \Rightarrow x:P,\Delta;\cdot}\ \mathsf{D} \qquad \frac{\Gamma \Rightarrow x:N,\Delta;\cdot}{\Gamma \Rightarrow \Delta; x:N}\ \mathsf{store} \qquad \frac{\Gamma \Rightarrow \Delta;\cdot}{\Gamma \Rightarrow \Delta; x:A}\ \mathsf{W}$$

Cut rules

$$\frac{\Gamma \Rightarrow \Delta; x:P \quad x:P,\Gamma \Rightarrow \Delta;\Pi}{\Gamma \Rightarrow \Delta;\Pi}\ P-\mathsf{cut} \qquad \frac{\Gamma \Rightarrow \Delta, x:N;\Pi^* \quad x:N,\Gamma \Rightarrow \Delta;\Pi}{\Gamma \Rightarrow \Delta;\Pi}\ N-\mathsf{cut}$$

FIGURE 3. Ecumenical pure modal system labEK. In rules $\Box R, \Diamond_i L, \Diamond_c L$, y does not occur free in any formula of the conclusion; N is negative and P is positive; p is atomic; Π^* is either empty or some $z : P \in \Delta$.

4.2 Ecumenical birelational models

The ecumenical birelational Kripke semantics, which is an extension of the proposal in [33] to modalities, was presented in [24].

DEFINITION 4.1.
A *birelational Kripke model* is a quadruple $\mathcal{M} = (W, \leq, R, V)$ where (W, R, V) is a Kripke model such that W is partially ordered with order $\leq$, $R \subset W \times W$ is a binary relation, the satisfaction function $V : \langle W, \leq\rangle \rightarrow \langle 2^{\mathcal{P}}, \subseteq\rangle$ is monotone and

F1. For all worlds w, v, v', if wRv and $v \leq v'$, there is a w' such that $w \leq w'$ and $w'Rv'$;
F2. For all worlds w', w, v, if $w \leq w'$ and wRv, there is a v' such that $w'Rv'$ and $v \leq v'$.

An *ecumenical modal Kripke model* is a birelational Kripke model such that truth of an ecumenical formula at a point w is the smallest relation $\models_{\mathsf{E}}$ satisfying

$$
\begin{array}{lll}
\mathcal{M}, w \models_{\mathsf{E}} p_i & \text{iff} & p_i \in V(w); \\
\mathcal{M}, w \models_{\mathsf{E}} A \wedge B & \text{iff} & \mathcal{M}, w \models_{\mathsf{E}} A \text{ and } \mathcal{M}, w \models_{\mathsf{E}} B; \\
\mathcal{M}, w \models_{\mathsf{E}} A \vee_i B & \text{iff} & \mathcal{M}, w \models_{\mathsf{E}} A \text{ or } \mathcal{M}, w \models_{\mathsf{E}} B; \\
\mathcal{M}, w \models_{\mathsf{E}} A \rightarrow_i B & \text{iff} & \text{for all } v \text{ such that } w \leq v, \mathcal{M}, v \models_{\mathsf{E}} A \text{ implies } \mathcal{M}, v \models_{\mathsf{E}} B; \\
\mathcal{M}, w \models_{\mathsf{E}} \neg A & \text{iff} & \text{for all } v \text{ such that } w \leq v, \mathcal{M}, v \not\models_{\mathsf{E}} A; \\
\mathcal{M}, w \models_{\mathsf{E}} \Box A & \text{iff} & \text{for all } v, w' \text{ such that } w \leq w' \text{ and } w'Rv, \mathcal{M}, v \models_{\mathsf{E}} A. \\
\mathcal{M}, w \models_{\mathsf{E}} \Diamond_i A & \text{iff} & \text{there exists } v \text{ such that } wRv \text{ and } \mathcal{M}, v \models_{\mathsf{E}} A. \\
\mathcal{M}, w \models_{\mathsf{E}} p_c & \text{iff} & \mathcal{M}, w \models_{\mathsf{E}} \neg(\neg p_i); \\
\mathcal{M}, w \models_{\mathsf{E}} A \vee_c B & \text{iff} & \mathcal{M}, w \models_{\mathsf{E}} \neg(\neg A \wedge \neg B); \\
\mathcal{M}, w \models_{\mathsf{E}} A \rightarrow_c B & \text{iff} & \mathcal{M}, w \models_{\mathsf{E}} \neg(A \wedge \neg B). \\
\mathcal{M}, w \models_{\mathsf{E}} \Diamond_c A & \text{iff} & \mathcal{M}, w \models_{\mathsf{E}} \neg\Box\neg A.
\end{array}
$$

We say that a formula A is *valid* in a model $\mathcal{M} = (W, \leq, R, V)$ if for all $w \in W$ we have $w \models_{\mathsf{E}} A$. A formula A is *valid* in a frame $\langle W, \leq, R \rangle$ if, for all valuations V, A is valid in the model $(W, \leq, R, V)$. Finally, we say a formula is *valid*, if it is valid in all frames.

Since, restricted to intuitionistic and neutral connectives, $\models_{\mathsf{E}}$ is the usual birelational interpretation $\models$ for IK [36, 42], and since the classical connectives are interpreted via the neutral ones using the double-negation translation, an ecumenical modal Kripke model coincides with the standard birelational Kripke model for intuitionistic modal logic IK. Hence, the following result easily holds from the similar result for IK.

THEOREM 4.2 ([24]).
The system labEK is sound and complete w.r.t. the ecumenical modal Kripke semantics, that is, $\vdash_{\mathsf{labEK}} x : A$ iff $\models_{\mathsf{E}} A$.

We end this section with a small note on the relationship between the semantics and the dynamics of proofs. On a bottom-up reading of proofs, the store rule is a delay on applying rules over classical connectives. It corresponds to moving the formula up w.r.t. $\leq$ in the birelational semantics. The rule $\Box R$, on the other hand, slides the formula to a fresh new world, related to the former one through the relation R. Finally, rule $\neg R$ moves up the formula w.r.t. $\leq$.

5 A nested system for ecumenical modal logic

The criticism regarding system labEK is that it includes *labels* in the technical machinery, hence allowing one to write sequents that cannot always be interpreted within the ecumenical modal language.

This section is devoted to presentin the pure *label free* calculus for ecumenical modalities, where every basic object of the calculus can be translated as a formula in the language of the logic.

The inspiration comes from Straßburger's *nested system* for IK [43]. The main idea is to add nested layers to sequents, which intuitively corresponds to worlds in a relational structure [4, 5, 14, 20, 37].

The structure of a nested sequent for ecumenical modal logics is a tree whose nodes are multisets of formulas, just like in [43], with the relationship between parent and child in the tree represented by bracketing $[\cdot]$. The difference however is that the ecumenical formulas can be *left inputs* (in the left contexts—marked with a full circle $^\bullet$), *right inputs* (in the classical right contexts—marked with a triangle $^\triangledown$) or a *single right output* (the stoup—marked with a white circle $^\circ$).

DEFINITION 5.1.
Ecumenical nested sequents are defined in terms of a grammar of *input sequents* (written Λ) and *full sequents* (written Γ) where the left/right input formulas are denoted by $A^\bullet$ and $A^\triangledown$, respectively, and A° denotes the output formula. When the distinction between input and full sequents is not essential or cannot be made explicit, we will use Δ to stand for either case.

As usual, we allow sequents to be empty, and we consider sequents to be equal modulo associativity and commutativity of the comma.

We write $\Gamma^{\perp^\circ}$ for the result of replacing an output formula from Γ by $\perp^\circ$, while $\Lambda^{\perp^\circ}$ represents the result of adding the output formula $\perp^\circ$ anywhere in the input context Λ. Finally, Δ^* is the result of erasing an output formula (if any) from Δ.

Observe that full sequents Γ necessarily contain exactly one output-like formula, having the form $\Lambda_1, [\Lambda_2, [\ldots, [\Lambda_n, A^\circ]] \ldots]$.

EXAMPLE 5.2.
The nested sequent $\Diamond_c A^\triangledown, [\neg B^\circ], [C \wedge D^\bullet]$ represents the following tree of sequents:

$$\begin{tikzcd}[column sep=small] & \cdot \Rightarrow \Diamond_c A; \cdot \arrow[dl, dash] \arrow[dr, dash] & \\ \cdot \Rightarrow \cdot; \neg B & & C \wedge D \Rightarrow \cdot; \cdot \end{tikzcd}$$

The next definition (of contexts) allows for identifying subtrees within nested sequents, which is necessary for introducing inference rules in this setting.

DEFINITION 5.3.
An *n-ary context* $\Delta\left\{{}^1\right\}\cdots\left\{{}^n\right\}$ is like a sequent but contains n pairwise distinct numbered *holes* $\{\,\}$, where a formula would otherwise occur. It is a *full* or a *input* context when $\Delta = \Gamma$ or Λ, respectively.

Given n sequents $\Delta_1, \ldots, \Delta_n$, we write $\Delta\{\Delta_1\}\cdots\{\Delta_n\}$ for the sequent where the i-th hole in $\Delta\left\{{}^1\right\}\cdots\left\{{}^n\right\}$ has been replaced by Δ_i (for $1 \leq i \leq n$), assuming that the result is well-formed, i.e., there is at most one output formula. If $\Delta_i = \varnothing$, the hole is removed.

Given two nested contexts $\Gamma^i\{\} = \Delta_1^i, \left[\Delta_2^i, \left[\ldots, \left[\Delta_n^i, \{\}\right]\right] \ldots\right]$, $i \in \{1, 2\}$, their *merge*[4] is

$$\Gamma^1 \otimes \Gamma^2\{\} = \Delta_1^1, \Delta_1^2, \left[\Delta_2^1, \Delta_2^1, \left[\ldots, \left[\Delta_n^1, \Delta_n^2, \{\}\right]\right] \ldots\right]$$

Figure 4 presents the nested sequent system nEK for ecumenical modal logic EK.

[4] As observed in [21, 37], the merge is a 'zipping' of the two nested sequents along the path from the root to the hole. Contexts merging is smoothly extended to nestings without holes, by simply erasing those.

$$\frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}}\wedge^\bullet \qquad \frac{\Lambda\{A^\circ\} \quad \Lambda\{B^\circ\}}{\Lambda\{A \wedge B^\circ\}}\wedge^\circ \qquad \frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee_i B^\bullet\}}\vee_i^\bullet \qquad \frac{\Lambda\left\{A_j^\circ\right\}}{\Lambda\left\{A_1 \vee_i A_2^\circ\right\}}\vee_{i_j}^\circ \qquad \frac{}{\Gamma\{\bot^\bullet\}}\bot^\bullet$$

$$\frac{\Gamma^*\{A \to_i B^\bullet, A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \to_i B^\bullet\}}\to_i^\bullet \qquad \frac{\Lambda\{A^\bullet, B^\circ\}}{\Lambda\{A \to_i B^\circ\}}\to_i^\circ$$

Classical rules and negation

$$\frac{\Gamma^*\{A \to_c B^\bullet, A^\circ\} \quad \Gamma^{\bot^\circ}\{B^\bullet\}}{\Gamma^{\bot^\circ}\{A \to_c B^\bullet\}}\to_c^\bullet \qquad \frac{\Gamma^{\bot^\circ}\{A^\bullet, B^\triangledown\}}{\Gamma^{\bot^\circ}\{A \to_c B^\triangledown\}}\to_c^\triangledown \qquad \frac{\Gamma^{\bot^\circ}\{A^\bullet\} \quad \Gamma^{\bot^\circ}\{B^\bullet\}}{\Gamma^{\bot^\circ}\{A \vee_c B^\bullet\}}\vee_c^\bullet$$

$$\frac{\Gamma^{\bot^\circ}\{A^\triangledown, B^\triangledown\}}{\Gamma^{\bot^\circ}\{A \vee_c B^\triangledown\}}\vee_c^\triangledown \qquad \frac{\Gamma^{\bot^\circ}\left\{p_i^\bullet\right\}}{\Gamma^{\bot^\circ}\{p_c^\bullet\}}p_c^\bullet \qquad \frac{\Gamma^{\bot^\circ}\left\{p_i^\triangledown\right\}}{\Gamma^{\bot^\circ}\{p_c^\triangledown\}}p_c^\triangledown \qquad \frac{\Gamma^*\{\neg A^\bullet, A^\circ\}}{\Gamma^{\bot^\circ}\{\neg A^\bullet\}}\neg^\bullet \qquad \frac{\Gamma^{\bot^\circ}\{A^\bullet\}}{\Gamma^{\bot^\circ}\{\neg A^\triangledown\}}\neg^\triangledown$$

Modal rules

$$\frac{\Delta_1\{\Box A^\bullet, [A^\bullet, \Delta_2]\}}{\Delta_1\{\Box A^\bullet, [\Delta_2]\}}\Box^\bullet \qquad \frac{\Lambda\{[A^\circ]\}}{\Lambda\{\Box A^\circ\}}\Box^\circ \qquad \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond_i A^\bullet\}}\Diamond_i^\bullet \qquad \frac{\Lambda_1\{[A^\circ, \Lambda_2]\}}{\Lambda_1\{\Diamond_i A^\circ, [\Lambda_2]\}}\Diamond_i^\circ$$

$$\frac{\Gamma^{\bot^\circ}\{[A^\bullet]\}}{\Gamma^{\bot^\circ}\{\Diamond_c A^\bullet\}}\Diamond_c^\bullet \qquad \frac{\Delta_1^{\bot^\circ}\left\{\Diamond_c A^\triangledown, \left[A^\triangledown, \Delta_2^{\bot^\circ}\right]\right\}}{\Delta_1^{\bot^\circ}\left\{\Diamond_c A^\triangledown, \left[\Delta_2^{\bot^\circ}\right]\right\}}\Diamond_c^\triangledown$$

Initial, decision and structural rules

$$\frac{}{\Lambda\{A^\bullet, A^\circ\}}\mathsf{init_i} \qquad \frac{}{\Gamma^{\bot^\circ}\{A^\bullet, A^\triangledown\}}\mathsf{init_c} \qquad \frac{\Gamma^*\{P^\triangledown, P^\circ\}}{\Gamma^{\bot^\circ}\{P^\triangledown\}}\mathsf{D} \qquad \frac{\Lambda\{N^\triangledown, \bot^\circ\}}{\Lambda\{N^\circ\}}\mathsf{store} \qquad \frac{\Gamma^{\bot^\circ}}{\Gamma}\mathsf{W}$$

Cut rules

$$\frac{\Gamma^*\{P^\circ\} \quad \Gamma\{P^\bullet\}}{\Gamma\{\varnothing\}}\mathsf{cut}^\circ \qquad \frac{\Gamma^P\{N^\triangledown\} \quad \Gamma\{N^\bullet\}}{\Gamma\{\varnothing\}}\mathsf{cut}^\triangledown$$

FIGURE 4. Nested ecumenical modal system nEK. P is a positive formula, N is a *negative* formula. p is atomic. Γ^P denotes either $\Gamma^{\bot^\circ}$ or $\Gamma^*\{P^\triangledown, P^\circ\}$ for some $P^\triangledown \in \Gamma$.

EXAMPLE 5.4.
Below, left is the proof that $\Diamond_c A$ is a consequence of $\neg\Box\neg A$ for any formula A. Below, right the proof that, if N is negative, then $\Box N$ is a consequence of $\neg\Diamond_i\neg N$. In fact, this holds for and only for eventually externally classical formulas (see Definition 2.2).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\left[A^\bullet, A^\triangledown, \bot^\circ\right]}\mathsf{init_c}}{\left[A^\triangledown, \neg A^\circ\right]}\neg^\circ}{\Diamond_c A^\triangledown, [\neg A^\circ]}\Diamond_c^\triangledown}{\Box\neg A^\circ, \Diamond_c A^\triangledown}\Box^\circ}{\neg\Box\neg A^\bullet, \Diamond_c A^\triangledown, \bot^\circ}\neg^\bullet}{\neg\Box\neg A^\bullet, \Diamond_c A^\circ}\mathsf{store}}{\neg\Box\neg A \to_i \Diamond_c A^\circ}\to_i^\circ \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\left[N^\bullet, N^\triangledown\right]}\mathsf{init_c}}{\left[\neg N^\triangledown, N^\triangledown\right]}\neg^\triangledown}{\left[\neg N^\circ, N^\triangledown\right]}\mathsf{store}}{\Diamond_i\neg N^\circ, \left[N^\triangledown\right]}\Diamond_c^\circ}{\neg\Diamond_i\neg N^\bullet, \left[N^\triangledown, \bot^\circ\right]}\neg^\bullet}{\neg\Diamond_i\neg N^\bullet, [N^\circ]}\mathsf{D}}{\neg\Diamond_i\neg N^\bullet, \Box N^\circ}\Box^\circ}{\neg\Diamond_i\neg N \to_i \Box N^\circ}\to_i^\circ$$

5.1 Proof theoretic properties

As for labEK, the properties of nEK are inherited by the ones in LCE (see Lemma 3.2). We will list them explicitly since the notation is quite different.

THEOREM 5.5.
In nEK:

1. The rules $\vee_c^\bullet, \vee_c^\triangledown, \to_c^\bullet, \to_c^\triangledown, \neg^\bullet, \neg^\circ, p_c^\bullet, p_c{}^\triangledown, \lozenge_c^\bullet, \lozenge_c^\triangledown$ and D are invertible.
2. The rules $\wedge^\bullet, \wedge^\circ, \vee_i^\bullet, \to_i^\circ, \lozenge_i^\bullet, \square^\bullet, \square^\circ$ and store are totally invertible.
3. The following structural rules are admissible:
$$\frac{\Gamma}{\Lambda \otimes \Gamma}\ \mathsf{W}_c \qquad \frac{\Lambda \otimes \Lambda \otimes \Gamma}{\Lambda \otimes \Gamma}\ \mathsf{C}_c$$
4. The rules cut° and $\mathsf{cut}^\triangledown$ are admissible. The ecumenical weight is the following extension of the measure presented in Section 3.1 for propositional connectives.
$$\mathsf{ew}(\heartsuit A) = \mathsf{ew}(A) + 1 \text{ if } \heartsuit \in \{\lozenge_i, \square\} \quad \mathsf{ew}(\lozenge_c A) = \mathsf{ew}(A) + 4$$

The invertible but not totally invertible rules in nEK concern negative formulas; hence, they can only be applied in the presence of empty stoups ($\perp^\circ$). Note also that the rules W, $\vee_i^\circ$, and $\lozenge_i^\circ$ are not invertible, while $\to_i^\bullet$ is invertible only w.r.t. the right premise.

5.2 Soundness and completeness

In this section, we will show that all rules presented in Figure 4 are sound and complete w.r.t. the ecumenical birelational models. The idea is to prove that the rules of the system nEK preserve *validity*, in the sense that if the interpretation of the premises is valid, so is the interpretation of the conclusion.

The first step is to determine the interpretation of ecumenical nested sequents. In this section, we will present the translation of nestings to labelled sequents, hence establishing, at the same time, soundness and completeness of nEK and the relation between this system with labEK.

DEFINITION 5.6.
Let $\Gamma^\bullet, \Delta^\triangledown, \Pi^\circ$ represent that all formulas in the each set/multiset are respectively input left, right or output formulas. The underlying set/multiset will represent in all cases the corresponding multiset of unmarked formulas. The translation $[\cdot]_x$ from nested into labelled sequents is defined recursively by[5]

$$[\Gamma^\bullet, \Delta^\triangledown, \Pi^\circ, [\Sigma_1], \ldots, [\Sigma_n]]_x \quad := \quad (\{xRx_i\}_i, x : \Gamma \Rightarrow x : \Delta; x : \Pi) \otimes \left\{[\Sigma_i]_{x_i}\right\}_i$$

where $1 \leq i \leq n$, x_i are fresh, $\perp$ is translated to the empty set and the merge operation on labelled sequents is defined as

$$(\Gamma_1 \Rightarrow \Delta_1; \Pi_1) \otimes (\Gamma_2 \Rightarrow \Delta_2; \Pi_2) \quad := \quad \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2; \Pi_1, \Pi_2$$

Since full nested sequents have exactly one output formula (which can be $\perp$), the stoup in the labelled setting will have at most one formula, and the merge above is well defined. Given $\mathcal{R}$ a set of relational

[5]We observe that this translation is closely related to those given for classical modal logics in [2] and for intuitionistic modal logics in [4].

formulas, we will denote by xR^*z the fact that there is a path from x to z in $\mathcal{R}$, i.e. there are $y_j \in \mathcal{R}$ for $0 \leq j \leq k$ such that $x = y_0, y_{j-1}Ry_j$ and $y_k = z$.

THEOREM 5.7.
Let Γ be a nested sequent and x be any label. The following are equivalent:

1. Γ is provable in nEK;
2. $[\Gamma]_x$ is provable in labEK.

PROOF. Let $xR^*z \in \mathcal{R}$. Observe that

- $[\Gamma^{\perp^\circ}\{\lozenge_c A^\bullet\}]_x = \mathcal{R}, \Sigma, z : \lozenge_c A \Rightarrow \Delta; \cdot$ iff
  $[\Gamma^{\perp^\circ}\{[A^\bullet]\}]_x = \mathcal{R}, zRy, \Sigma, y : A \Rightarrow \Delta; \cdot$, with y fresh.
- $[\Delta_1^{\perp^\circ}\{\lozenge_c A^\nabla, [\Delta_2^{\perp^\circ}]\}]_x = \mathcal{R}, zRy, \Sigma \Rightarrow \Delta, z : \lozenge_c A; \cdot$ with y the variable related to the nesting of Δ_2 iff
  $[\Delta_1^{\perp^\circ}\{\lozenge_c A^\nabla, [A^\nabla, \Delta_2^{\perp^\circ}]\}]_x = \mathcal{R}, zRy, \Sigma \Rightarrow \Delta, z : \lozenge_c A, y : A; \cdot$.

The other cases are similar. The translation $[\cdot]_x$ is then trivially lifted to rule applications. We will illustrate the $\lozenge_c$ cases.

- Case $\lozenge_c^\bullet$.

$$\frac{\Gamma^{\perp^\circ}\{[A^\bullet]\}}{\Gamma^{\perp^\circ}\{\lozenge_c A^\bullet\}}\ \lozenge_c^\bullet \quad \rightsquigarrow \quad \frac{[\Gamma^{\perp^\circ}\{[A^\bullet]\}]_x}{[\Gamma^{\perp^\circ}\{\lozenge_c A^\bullet\}]_x} \quad = \quad \frac{zRy, y : A, \Sigma \Rightarrow \Delta; \cdot}{z : \lozenge_c A, \Sigma \Rightarrow \Delta; \cdot}\ \lozenge_c L$$

- Case $\lozenge_c^\circ$.

$$\frac{\Delta_1^{\perp^\circ}\{\lozenge_c A^\nabla, [A^\nabla, \Delta_2^{\perp^\circ}]\}}{\Delta_1^{\perp^\circ}\{\lozenge_c A^\nabla, [\Delta_2^{\perp^\circ}]\}}\ \lozenge_c^\nabla \qquad \rightsquigarrow \qquad \frac{[\Delta_1^{\perp^\circ}\{\lozenge_c A^\nabla, [A^\nabla, \Delta_2^{\perp^\circ}]\}]_x}{[\Delta_1^{\perp^\circ}\{\lozenge_c A^\nabla, [\Delta_2^{\perp^\circ}]\}]_x}$$

$$= \quad \frac{zRy, \Sigma \Rightarrow y : A, z : \lozenge_c A, \Delta; \cdot}{zRy, \Sigma \Rightarrow z : \lozenge_c A, \Delta; \cdot}\ \lozenge_c R$$

Given this transformation, (1) $\Leftrightarrow$ (2) is easily proved by induction on a proof of $\Gamma/[\Gamma]_x$ in $\mathsf{nEK}/\mathsf{labEK}$. ☐

Theorems 4.2 and 5.7 immediately imply the following:

COROLLARY 5.8.
Nested system nEK is sound w.r.t. ecumenical birelational semantics.

We observe that, often, passing from labelled to nested sequents is not a simple task, sometimes even impossible. In fact, although the relational atoms of a sequent appearing in labEK *proofs* can be arranged so as to correspond to nestings, as shown here, if the relational context is not tree-like [17], the existence of such a translation is not clear. For instance, how should the sequent $xRy, yRx, x : A \Rightarrow y : B$ be interpreted in modal systems with symmetrical relations?

However, thanks to their tree shape, it is possible to interpret nested sequents as ecumenical modal formulas, and hence prove soundness in the same way as in [43].

We end this section by briefly showing an alternative way of proving of soundness of nEK w.r.t. the ecumenical birelational semantics. Please refer to [25] for a more detailed presentation.

DEFINITION 5.9.
The formula translation $\mathrm{et}(\cdot)$ for ecumenical nested sequents is given by

$$\begin{array}{rcl@{\qquad}rcl} \mathrm{et}(\varnothing) & := & \top & \mathrm{et}(A^\bullet, \Lambda) & := & A \wedge \mathrm{et}(\Lambda) \\ \mathrm{et}(A^\nabla, \Lambda) & := & \neg A \wedge \mathrm{et}(\Lambda) & \mathrm{et}([\Lambda_1], \Lambda_2) & := & \Diamond_i \mathrm{et}(\Lambda_1) \wedge \mathrm{et}(\Lambda_2) \\ \mathrm{et}(\Lambda, A^\circ) & := & \mathrm{et}(\Lambda) \rightarrow_i A & \mathrm{et}(\Lambda, [\Gamma]) & := & \mathrm{et}(\Lambda) \rightarrow_i \Box\, \mathrm{et}(\Gamma) \end{array}$$

where all occurrences of $A \wedge \top$ and $\top \rightarrow_i A$ are simplified to A. We say a sequent is *valid* if its corresponding formula is valid.

The next theorem shows that the rules of nEK preserve validity in ecumenical modal frames w.r.t. the formula interpretation $\mathrm{et}(\cdot)$.

THEOREM 5.10.
Let

$$\frac{\Gamma_1 \quad \dots \quad \Gamma_n}{\Gamma}\ r \qquad n \in \{0, 1, 2\}$$

be an instance of the rule r in the system nEK. Then $\mathrm{et}(\Gamma_1) \wedge \ldots \wedge \mathrm{et}(\Gamma_n) \rightarrow_i \mathrm{et}(\Gamma)$ is valid in the birelational ecumenical semantics.

6 Fragments, axioms and extensions

In this section, we discuss fragments, axioms and extensions of nEK.

6.1 Extracting fragments

For the sake of simplicity, in this sub-section, negation will not be considered a primitive connective, it will rather take its respective intuitionistic or classical form.

DEFINITION 6.1.
An ecumenical modal formula C is *classical* (resp. *intuitionistic*) if it is built from classical (resp. intuitionistic) atomic propositions using only neutral and classical (resp. intuitionistic) connectives but negation, which will be replaced by $A \rightarrow_c \bot$ (resp. $A \rightarrow_i \bot$).

The first thing to observe is that, when only pure fragments are concerned, weakening is admissible (remember that this is not the case for the whole system nEK—see Example 2.6). Also, only positive (resp. eventually externally classical) formulas are present in the intuitionistic (resp. classical) fragment.

Let nEK_i (resp. nEK_c) be the system obtained from $\mathsf{nEK} - \mathsf{W}$ by restricting the rules to the intuitionistic (resp. classical) case—see Figures 5 and 6.

The intuitionistic fragment does not have classical input formulas and it coincides with the system NIK in [43].

Regarding nEK_c, since all the classical/neutral rules are invertible, the following proof strategy is complete:

i. Apply the rules $\wedge^\bullet, \wedge^\circ, \Box^\bullet, \Box^\circ$ and store eagerly, obtaining leaves of the form $\Lambda\{\bot^\circ\}$.
ii. Apply any other rule of nEK_c eagerly, until either finishing the proof with an axiom application or obtaining leaves of the form $\Lambda\{P^\circ\}$, where P is a positive formula in nEK_c, that is, having as main connective $\wedge$ or $\Box$. Start again from step (i).

$$\frac{}{\Lambda\left\{p_i^\bullet, p_i^\circ\right\}}\ \mathsf{init} \qquad \frac{}{\Gamma\{\bot^\bullet\}}\ \bot^\bullet \qquad \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}}\ \wedge^\bullet \qquad \frac{\Lambda\{A^\circ\} \quad \Lambda\{B^\circ\}}{\Lambda\{A \wedge B^\circ\}}\ \wedge^\circ$$

$$\frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee_i B^\bullet\}}\ \vee_i^\bullet \qquad \frac{\Lambda\left\{A_j^\circ\right\}}{\Lambda\left\{A_1 \vee_i A_2^\circ\right\}}\ \vee_{i_j}^\circ \qquad \frac{\Gamma^*\{A \to_i B^\bullet, A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \to_i B^\bullet\}}\ \to_i^\bullet \qquad \frac{\Lambda\{A^\bullet, B^\circ\}}{\Lambda\{A \to_i B^\circ\}}\ \to_i^\circ$$

$$\frac{\Delta_1\{\Box A^\bullet, [A^\bullet, \Delta_2]\}}{\Delta_1\{\Box A^\bullet, [\Delta_2]\}}\ \Box^\bullet \qquad \frac{\Lambda\{[A^\circ]\}}{\Lambda\{\Box A^\circ\}}\ \Box^\circ \qquad \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond_i A^\bullet\}}\ \Diamond_i^\bullet \qquad \frac{\Lambda_1\{[A^\circ, \Lambda_2]\}}{\Lambda_1\{\Diamond_i A^\circ, [\Lambda_2]\}}\ \Diamond_i^\circ$$

FIGURE 5 Intuitionistic fragment nEK_i.

$$\frac{}{\Gamma\{p_c^\bullet, p_c^\triangledown\}}\ \mathsf{init} \qquad \frac{}{\Gamma\{\bot^\bullet\}}\ \bot^\bullet \qquad \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}}\ \wedge^\bullet \qquad \frac{\Lambda\{A^\circ\} \quad \Lambda\{B^\circ\}}{\Lambda\{A \wedge B^\circ\}}\ \wedge^\circ$$

$$\frac{\Gamma^{\bot^\circ}\{A^\bullet\} \quad \Gamma^{\bot^\circ}\{B^\bullet\}}{\Gamma^{\bot^\circ}\{A \vee_c B^\bullet\}}\ \vee_c^\bullet \qquad \frac{\Gamma^{\bot^\circ}\{A^\triangledown, B^\triangledown\}}{\Gamma^{\bot^\circ}\{A \vee_c B^\triangledown\}}\ \vee_c^\triangledown \qquad \frac{\Gamma^*\{A^\circ\} \quad \Gamma^{\bot^\circ}\{B^\bullet\}}{\Gamma^{\bot^\circ}\{A \to_c B^\bullet\}}\ \to_c^\bullet$$

$$\frac{\Gamma^{\bot^\circ}\{A^\bullet, B^\triangledown\}}{\Gamma^{\bot^\circ}\{A \to_c B^\triangledown\}}\ \to_c^\triangledown \qquad \frac{\Delta_1\{\Box A^\bullet, [A^\bullet, \Delta_2]\}}{\Delta_1\{\Box A^\bullet, [\Delta_2]\}}\ \Box^\bullet \qquad \frac{\Lambda\{[A^\circ]\}}{\Lambda\{\Box A^\circ\}}\ \Box^\circ$$

$$\frac{\Gamma^{\bot^\circ}\{[A^\bullet]\}}{\Gamma^{\bot^\circ}\{\Diamond_c A^\bullet\}}\ \Diamond_c^\bullet \qquad \frac{\Delta_1^{\bot^\circ}\left\{\Diamond_c A^\triangledown, \left[A^\triangledown, \Delta_2^{\bot^\circ}\right]\right\}}{\Delta_1^{\bot^\circ}\left\{\Diamond_c A^\triangledown, \left[\Delta_2^{\bot^\circ}\right]\right\}}\ \Diamond_c^\triangledown \qquad \frac{\Gamma^*\{P^\triangledown, P^\circ\}}{\Gamma^{\bot^\circ}\{P^\triangledown\}}\ \mathsf{D} \qquad \frac{\Lambda\{N^\triangledown, \bot^\circ\}}{\Lambda\{N^\circ\}}\ \mathsf{store}$$

FIGURE 6. Classical fragment nEK_c.

This discipline corresponds to the focused strategy for a fragment of the two-sided version of the polarized system defined in [7], exchanging the polarities of diamond and box (which, as observed in the *op.cit.*, is a matter of choice since all rules are invertible).

6.2 About axioms and extensions

The classical modal logic K is defined as propositional CL, extended with the *necessitation rule* (presented in Hilbert style) $A/\Box A$ and the *distributivity axiom* $\mathsf{k} : \ \Box(A \to B) \to (\Box A \to \Box B)$.

There are, however, many variants of the axiom k that induce logics that are classically, but not intuitionistically, equivalent (see [36, 42]). In fact, the following axioms follow from k via the De Morgan laws, but are intuitionistically independent:

$$\begin{array}{ll} \mathsf{k}_1 : \ \Box(A \to B) \to (\Diamond A \to \Diamond B) & \qquad \mathsf{k}_2 : \ \Diamond(A \vee B) \to (\Diamond A \vee \Diamond B) \\ \mathsf{k}_3 : \ (\Diamond A \to \Box B) \to \Box(A \to B) & \qquad \mathsf{k}_4 : \ \Diamond\bot \to \bot \end{array}$$

Combining axiom k with axioms $\mathsf{k}_1 - \mathsf{k}_4$ defines intuitionistic modal logic IK [36].

In the ecumenical setting, this discussion is even more interesting, since there are many more variants of k, depending on the classical or intuitionistic interpretation of implications and diamonds.

It is an easy exercise to show that the intuitionistic versions of $\mathsf{k}_1 - \mathsf{k}_4$ are provable in nEK. One could then ask: what happens if we exchange the intuitionistic versions of the connectives with classical ones?

Consider $\mathsf{k}^{\alpha\beta\gamma} : \Box(A \to_\alpha B) \to_\beta (\Box A \to_\gamma \Box B)$ with $\alpha, \beta, \gamma \in \{i, c\}$. First of all, note that $\mathsf{k}^{c\beta\gamma}$ is not provable, for any β, γ. This is a consequence of the fact that $A \to_c B, A \not\Rightarrow B$ in EK in general (see Equation 5). Moreover, since $C \to_i D \Rightarrow C \to_c D$ in EK, $\mathsf{k}^{\alpha ii} \Rightarrow \mathsf{k}^{\alpha\beta\gamma}$ for any value of β, γ. The same reasoning can be extended to all the other axioms, for example, $\mathsf{k}_3^{\alpha\beta\gamma\delta} = (\Diamond_\alpha A \to_\beta \Box B) \to_\gamma \Box(A \to_\delta B)$ is not provable for $\beta = c$ and k_3^{iiii} implies all the other possible configurations for $\alpha, \beta, \gamma, \delta$.

TABLE 1. Axioms and corresponding first-order conditions on R.

Axiom	**Condition**	**First-order formula**
$\mathsf{t} : \Box A \to A \wedge A \to \Diamond A$	Reflexivity	$\forall x.R(x,x)$
$\mathsf{b} : A \to \Box\Diamond A \wedge \Diamond\Box A \to A$	Symmetry	$\forall x,y.R(x,y) \to R(y,x)$
$\mathsf{4} : \Box A \to \Box\Box A \wedge \Diamond\Diamond A \to \Diamond A$	Transitivity	$\forall x,y,z.(R(x,y) \wedge R(y,z)) \to R(x,z)$
$\mathsf{5} : \Box A \to \Box\Diamond A \wedge \Diamond\Box A \to \Diamond A$	Euclideaness	$\forall x,y,z.(R(x,y) \wedge R(x,z)) \to R(y,z)$

$$\frac{\Gamma\{\Box A^\bullet, A^\bullet\}}{\Gamma\{\Box A^\bullet\}}\ \mathsf{t}^\bullet \quad \frac{\Delta_1\{[\Delta_2, \Box A^\bullet], A^\bullet\}}{\Delta_1\{[\Delta_2, \Box A^\bullet]\}}\ \mathsf{b}^\bullet \quad \frac{\Delta_1\{[\Delta_2, \Box A^\bullet], \Box A^\bullet\}}{\Delta_1\{[\Delta_2], \Box A^\bullet\}}\ \mathsf{4}^\bullet \quad \frac{\Gamma\{[\Box A^\bullet][\Box A^\bullet]\}}{\Gamma\{[\Box A^\bullet][\varnothing]\}}\ \mathsf{5}^\bullet$$

$$\frac{\Lambda\{A^\circ\}}{\Lambda\{\Diamond_i A^\circ\}}\ \mathsf{t}^\circ \quad \frac{\Lambda_1\{[\Lambda_2], A^\circ\}}{\Lambda_1\{[\Lambda_2, \Diamond_i A^\circ]\}}\ \mathsf{b}^\circ \quad \frac{\Lambda_1\{[\Lambda_2, \Diamond_i A^\circ]\}}{\Lambda_1\{[\Lambda_2], \Diamond_i A^\circ\}}\ \mathsf{4}^\circ \quad \frac{\Lambda\{[\varnothing][\Diamond_i A^\circ]\}}{\Lambda\{[\Diamond_i A^\circ][\varnothing]\}}\ \mathsf{5}^\circ$$

$$\frac{\Gamma^{\perp^\circ}\{A^\triangledown\}}{\Gamma^{\perp^\circ}\{\Diamond_c A^\triangledown\}}\ \mathsf{t}^\triangledown \quad \frac{\Delta_1^{\perp^\circ}\left\{\left[\Delta_2^{\perp^\circ}\right], A^\triangledown\right\}}{\Delta_1^{\perp^\circ}\left\{\left[\Delta_2^{\perp^\circ}, \Diamond_c A^\triangledown\right]\right\}}\ \mathsf{b}^\triangledown \quad \frac{\Delta_1^{\perp^\circ}\left\{\left[\Delta_2^{\perp^\circ}, \Diamond_i A^\circ\right]\right\}}{\Delta_1^{\perp^\circ}\left\{\left[\Delta_2^{\perp^\circ}\right], \Diamond_c A^\triangledown\right\}}\ \mathsf{4}^\triangledown \quad \frac{\Gamma^{\perp^\circ}\{[\varnothing][\Diamond_c A^\triangledown]\}}{\Gamma^{\perp^\circ}\{[\Diamond_c A^\triangledown][\varnothing]\}}\ \mathsf{5}^\triangledown$$

FIGURE 7. Ecumenical modal extensions for axioms $\mathsf{t}, \mathsf{b}, \mathsf{4}$ and $\mathsf{5}$.

Hence, the intuitionistic version of the k family of axioms forms their *minimal* version valid in EK. In [25], we proved that nEK was cut-complete w.r.t to EK's Hilbert system based on this set of axioms.

Regarding modal extensions of EK, we can obtain them by restricting the class of frames we consider or, equivalently, by adding axioms over modalities. Many of the restrictions one can be interested in are definable as formulas of first-order logic, where the binary predicate $R(x,y)$ refers to the corresponding accessibility relation. Table 1 summarizes some of the most common logics, the corresponding frame property, together with the modal axiom capturing it [39].

Since the intuitionistic fragment of nEK coincides with NIK, intuitionistic versions for the rules for the axioms $\mathsf{t}, \mathsf{b}, \mathsf{4}$ and $\mathsf{5}$ match the rules ($^\bullet$) and ($^\circ$) presented in [43], and are depicted in Figure 7.

For completing the ecumenical view, the classical ($^\triangledown$) rules for extensions are justified via translations from labelled systems to nEK: We first translate the labelled rules for extensions appearing in [42] to labEK then use the translation on derivations defined in Section 5.2 to justify the rule scheme.

For example, starting with the rule T below left, which is the labelled rule corresponding to the axiom t in [42], the labelled derivation on the middle justifies the classical nested rule in the right.

$$\frac{xRx, \Gamma \vdash z : C}{\Gamma \vdash z : C}\ \mathsf{T} \qquad \cfrac{\cfrac{xRx, \mathcal{R}, \Sigma \Rightarrow \Delta, x : A, x : \Diamond_c A; z : \bot}{xRx, \mathcal{R}, \Sigma \Rightarrow \Delta, x : \Diamond_c A; z : \bot}\ \Diamond_c R}{\mathcal{R}, \Sigma \Rightarrow \Delta, x : \Diamond_c A; z : \bot}\ \mathsf{T} \qquad \frac{\Gamma^{\perp^\circ}\left\{A^\triangledown\right\}}{\Gamma^{\perp^\circ}\left\{\Diamond_c A^\triangledown\right\}}\ \mathsf{t}^\triangledown$$

The rules $\mathsf{b}^\triangledown, \mathsf{4}^\triangledown$ and $\mathsf{5}^\triangledown$, shown in Figure 7, are obtained in the same manner.

Restricted to the fragments described in the last section, by mixing and matching these rules, we obtain ecumenical modal systems for the logics in the $\mathsf{S5}$ modal cube [2] not defined with axiom d.

7 Related and future work

The main idea behind Prawitz's ecumenical system [38] is to build a proof framework in which classical and intuitionistic logics may co-exist in peace. Although one could argue that this is easily done using the well known double-negation translations by Kolmogorov, Gödel, Gentzen and others [13], Prawitz's view matches the idea presented by Liang and Miller in their *PIL* system presented in [22]: not seeing CL as a fragment of intuitionistic logic but rather to determine parts of *reasoning* which are classical or intuitionistic in nature. While double negation acts on *formulas*, the approach in [22] and also followed here concerns *proofs*. For example, we *do not* want to interpret $A \vee \neg A$ as

'it is not the case that A does not hold and it is not the case that it is not the case that A holds'.

Rather, we aim at identifying the points in proofs where the excluded middle is valid and/or necessary.

The similarities between our work and the system presented in [22] ends there, though. Indeed, in the *op.cit.* there are two versions of the constant for absurdum and universal quantifier, and all connectives have a dual version. For example, the intuitionistic implication $\supset$ comes with the intuitionistic dual $\propto$, a form of (non-commutative) conjunction, which has no correspondent in usual classical or intuitionistic systems. Also, these dualized versions have opposite *polarities* (red and green), that do not match Girard's original idea of polarities: They are, instead, defined model-theoretically. In this work, we opt for smoothly extending well known systems and features (like stoup or polarities), which makes LCE and *PIL* incomparable. It would be interesting to investigate, for example, if *PIL* could be smoothly extended to the modal case, as done in this work.

There are other proposals for ecumenical systems in the literature. For example, in [3] the authors present a (type) theory in $\lambda\Pi$-calculus modulo theory, where proofs of several logical systems can be expressed. We are planning to propose type systems related to the systems/fragments described in this paper, and it would be interesting to see the intersection that may appear from the two approaches. It would be also interesting to implement ecumenical provers, as well as to automate the cut-elimination proof in the L-Framework [32].

A completely different approach comes from the school of combining logics [6, 8, 23], where Hilbert-like systems are built from a combination of axiomatic systems. As we trail the exact opposite path, it would be interesting to see if (the propositional fragment of) Prawitz's natural deduction system is axiomatizable.

Finally, the presence of polarization and stoup paves the way for proposing focused ecumenical systems. For getting a complete focused discipline, though, it would be necessary to add polarized versions of conjunction and disjunction, as done *e.g.* in [22] and [7]. This would give a unified focused framework, which could be used, among other things, to automatically extract rules from axioms, as done in [26].

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