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ORIGINAL RESEARCH

An ecumenical view of proof-theoretic semantics

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Abstract

Debates concerning philosophical grounds for the validity of classical and intuitionistic logics often have the very nature of proofs as a point of controversy. The intuitionist advocates for a strictly constructive notion of proof, while the classical logician advocates for a notion which allows the use of non-constructive principles such as *reductio ad absurdum*. In this paper we show how to coherently combine *logical ecumenism* and *proof-theoretic semantics* ($\mathbf{PtS}$) by providing not only a medium in which classical and intuitionistic *logics* coexist, but also one in which their respective *notions of proof* coexist. Intuitionistic proofs receive the standard treatment of $\mathbf{PtS}$, whereas classical proofs are given a semantics based on ideas by David Hilbert. Furthermore, we advance the state of the art in $\mathbf{PtS}$ by introducing a key contribution: treating the absurdity constant $\bot$ as an atomic proposition and requiring all bases to be consistent. This treatment is essential for the obtainment of some ecumenical results, and it can also be used in standard intuitionistic $\mathbf{PtS}$. Additionally, we employ normalization techniques to demonstrate the consistency of simulation bases. These innovations provide fresh technical and conceptual insights into the study of bases in $\mathbf{PtS}$.

Keywords Ecumenical logic · Proof theoretic semantic · Base extension semantic · Intuitionistic logic · Classical logic

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1 Introduction

What is the meaning of a logical connective? This is a very difficult and controversial question, for many reasons. First of all, it depends on the logical setting. For example, asserting that

$A \vee B$ is valid only if it is possible to give a proof of either A or B

clearly does not correctly determine the meaning of the classical disjunction. It turns out, as shown in (Piecha et al., 2015) and further analyzed in (Pym et al., 2025; Gheorghiu & Pym, 2022), that this also does not seem enough for determining meaning in intuitionistic logic, due to the intrinsic non-determinism on choosing between A or B for validating $A \vee B$.

In model-theoretic semantics, mathematical structures help in supporting the notion of validity, which is based on a notion of *truth*. In the case of intuitionistic logic, for example, one could use Kripke structures, where the validation of atomic propositions using the classical notion of truth (e.g., via truth tables) is enough for describing the meaning of the disjunction *in a given world*, where worlds are organized in a pre-order.

Although it became common to specify the meaning of formulas in terms of truth conditions, we agree with Quine's objection to that, quoting Prawitz (Prawitz, 2015):

> Following Tarski, he [Quine] states truth conditions of compound sentences, not as a way to explain the logical constants, but as a first step in a definition of logical truth or logical consequence, which Quine takes to demarcate the logic that he is interested in. He points out that the truth conditions do not explain negation, conjunction, existential quantification and so on, because the conditions are using the corresponding logical constants and are thus presupposing an understanding of the very constants that they would explain. I think that he is essentially right in saying so and that the situation is even worse: when stating truth conditions, one is using an ambiguous natural language expression that is to be taken in a certain specific way, namely in exactly the sense that the truth condition is meant to specify.

Proof-theoretic semantics (Schroeder-Heister, 1991, 2006, 2024) (PtS) provides an alternative perspective for the meaning of logical operators compared to the viewpoint offered by model-theoretic semantics. In PtS, the concept of *truth* is substituted with that of *proof*, emphasizing the fundamental nature of proofs as a means through which we gain demonstrative knowledge, particularly in mathematical contexts. PtS has as philosophical background *inferentialism* (Brandom, 2000), according to which inferences establish the meaning of expressions. This makes PtS a superior approach for comprehending reasoning since it ensures that the meaning of logical operators, such as connectives in logics, is defined based on their usage in inferences.

Base-extension semantics (Sandqvist, 2015) (BeS) is a strand of PtS where proof-theoretic validity is defined relative to a given collection S of inference rules defined over basic formulas of the language.[1] Hence, for example, while satisfiability of an atomic formula p at a state w in a Kripke model $\mathcal{M} = (W, R, V)$ is often given by

$$w \Vdash p \qquad \text{iff} \qquad w \in V(p)$$

in BeS, validity w.r.t. a set S of atomic rules has the general shape

$$\Vdash_S p \qquad \text{iff} \qquad \vdash_S p$$

where $\vdash_S p$ indicates that p is *derivable* in the proof system determined by S. After defining validity for atoms one can also define validity for logical connectives via semantic clauses that express proof conditions (e.g., $A \wedge B$ is provable from S if and only if both A and B are provable from S), which results in a framework that evaluates propositions exclusively in terms of proofs of its constituents.

The switch from truth-functional to proof-functional semantics carries both mathematical and philosophical significance. In Tarskian truth-conditional semantics, as well as in Kripke models, the value of a proposition on a model relies solely on the semantic value assigned to its components. In proof-theoretic semantics, on the other hand, *how* the values of the components of a proposition are assigned is also relevant. For instance, given two atoms p and q, whether an implication $p \rightarrow q$ holds or not in a model $\mathcal{M}$ depends solely on the truth values assigned to p and q. But in BeS and most variants of PtS this implication holds in a base $\mathcal{B}$ only if the base is capable of producing some inferential structure (such as a natural deduction derivation) with premise p and conclusion q (as shown in, e.g. [(Sandqvist, 2015) Theorem 3.1.]). For other variants of PtS it is even possible to show that Kripke models essentially correspond to simplified proof-theoretic structures containing flattened inferential components (Stafford & Nascimento, 2023; Barroso Nascimento, 2024). Those mathematical differences are a direct reflection of the philosophy behind both frameworks: model theory is justified by views giving semantic primacy to the concept of truth, such as Davidson's argument to the effect that by giving sufficient and necessary conditions for the truth of a sentence we provide it with meaning (Davidson, 1967), whereas PtS is justified by views giving primacy to the concept of justification, such as Dummett's arguments to the effect that the meaning of a proposition is given by its assertability conditions (Dummett, 1991). As pointed out by Brandom (Brandom, 1976), such discussions trace back to a longstanding divergence between philosophers concerning whether languages are better understood in terms of the concept of *truth* or the concept of *linguistic use*.

Although the BeS project has been successfully developed for intuitionistic (Sandqvist, 2015) and classical logics (Sandqvist, 2009; Makinson, 2014), it has not yet been systematically developed as a foundation for logical reasoning (Dicher & Paoli, 2021; Kürbis, 2015; Francez, 2016). In this paper, we intend to move on

[1] It should be noted that, in (Sandqvist, 2015), base rules are restricted to formulas in the logic-free fragment only, that is, to *atomic propositions*. Here we will follow (Piecha et al., 2015) and give the unit $\bot$ an "atomic status", allowing it to appear in atomic rules.

with this quest, by proposing a BeS view of *ecumenical logics*, inspired by Prawitz's (2015) proposal of a system combining classical and intuitionistic logics.

In Prawitz's system, the classical logician and the intuitionistic logician would share the universal quantifier, conjunction, negation and the constant for the absurdity, but they would each have their own existential quantifier, disjunction and implication, with different meanings. Prawitz's main motivation was to provide a logical framework that would make possible an inferentialist semantics for the classical logical constants. In this way, inferentialism brought forth a very specific proposal when it emerged in the ecumenical context: to provide acceptable assertability conditions for the operators of a certain logical system in another logical system which does not accept them, thus allowing the acceptance and reinterpretation of the previously rejected operators under the new inferential guise. In the context of conflicting discussions between classical and intuitionistic logicians, this would be comparable to defining assertability conditions for classical operators inside intuitionistic logic, which Prawitz actually does in (Prawitz, 2015). Therefore, the inferentialist's main task is to create ecumenical connectives that, with the assertability conditions exposed in its inferential rules, can represent connectives accepted by one of the logical systems and rejected by the others inside the ecumenical environment.

In this work we *do not* intend to provide a BeS for Prawitz's original system, but rather to proceed with a careful analysis of different aspects of BeS for logical systems where classical and intuitionistic notions of proof coexist in peace (i.e. without collapsing). We define intuitionistic proofs through the usual semantic conditions of BeS, which encapsulate the traditional idea of Brouwer, Heyting and Dummett that mathematical existence of an object can only be guaranteed by means of its construction (Brouwer, 1981; Dummett, 1977; Heyting, 1956). On the other hand, classical proofs are defined by taking into account an idea advanced by David Hilbert to justify non-constructive proof methods: the concept of consistency is conceptually prior to that of truth, and in order to prove the truth of a proposition in a given context it suffices to prove its consistency. In his words (Doherty, 2017; Hilbert et al., 1979; Hilbert, 1900):

> You [Frege] write "From the truth of the axioms it follows that they do not contradict one another". It interested me greatly to read this sentence of yours, because in fact for as long as I have been thinking, writing and lecturing about such things, I have always said the very opposite: if arbitrarily chosen axioms together with everything which follows from them do not contradict one another, then they are true, and the things defined by the axioms exist. For me that is the criterion of truth and existence.

In order to properly represent this idea of classical proof in BeS we must change the semantic treatment given to the absurdity constant $\perp$, but it is shown that this can be done without issues (see Sect. 2.2). As expected of an ecumenical framework, the resulting environment allows both notions of proof to coexist peacefully, retain their independence and fruitfully interact – so we are able, for instance, to analyze the semantic content of a proposition which is in part proved classically and in part proved intuitionistically in terms of interactions between the respective proof notions.

We start by proposing a *weak* version of ecumenical BeS (in Sect. 3). This version relies on the concepts of local and global validity (see e.g., (Cobreros, 2008)), enabling us to examine various aspects of both classical and intuitionistic validities. In particular we demonstrate that, while intuitionistic validity has the property of *monotonicity*, meaning that it remains unchanged under extensions, this characteristic does not hold true for classical formulas. This observation gives rise to the motto:

Classical proof plus monotonicity equals intuitionistic proof of double negation.

In Sect. 4 we will unwrap the full power of ecumenical BeS, by showing a strong notion of validity. In Sect. 5 we define the ecumenical natural deduction system $\mathrm{NE_B}$, and prove its soundness and completeness w.r.t. such (proof-theoretic) semantic. We then conclude with some ideas to push forward the PtS agenda for ecumenical systems.

2 Base extension semantics

2.1 Basic definitions

We will adopt Sandqvist's (2015) terminology, adapted to the ecumenical setting.

The propositional *base language* is assumed to have a set $\mathrm{At} = \{p_1, p_2, \ldots\}$ of countably many atomic propositions, together with the unit $\perp$. The set $\mathrm{At} \cup \{\perp\}$ will be denoted by $\mathrm{At}_\perp$, and its elements will be called *basic sentences*.

We use, as does Sandqvist, systems containing natural deduction rules over basic sentences for the semantical analysis, and we allow inference rules to discharge sets of basic hypotheses. Sets used in the definition of the derivability relation and semantic consequence are always assumed to be finite. Unlike Sandqvist, however, we allow the logical constant $\perp$ to be manipulated by the rules.

Definition 1 (Atomic systems) An *atomic system* (a.k.a. a *base*) S is a (possibly empty) set of atomic rules of the form

$$\frac{\begin{array}{ccc}\Gamma_{\mathrm{At}}\,[P_1] & & \Gamma_{\mathrm{At}}\,[P_n] \\ p_1 & \ldots & p_n\end{array}}{p}$$

where $p_i, p \in \mathrm{At}_\perp$ and $\Gamma_{\mathrm{At}}, P_i$ are (possibly empty) finite sets of basic sentences. The sequence $\langle p_1, \ldots, p_n \rangle$ of premises of the rule can be empty – in this case the rule is called an *atomic axiom*.

Labels will sometimes be written as the superscript of $[P_i]$ and to the right of a rule to denote that P_i was discharged at that rule application.

Definition 2 (Extensions) An atomic system S' is an *extension* of an atomic system S (written $S \subseteq S'$), if S' results from adding a (possibly empty) set of atomic rules to S.

Definition 3 (Deducibility) The *deducibility relation* $\vdash_S$ coincides with the usual notion in the system of natural deduction consisting of just the rules in *S*, that is, $p_1, \ldots, p_n \vdash_S p$ iff there exists a deduction with the rules of *S* whose conclusion is p and whose set of undischarged premises is a subset of $\{p_1, ..., p_n\}$.

Definition 4 (Consistency) An atomic system *S* is called *consistent* if $\nvdash_S \bot$.

Notice that, due to how deducibility is defined, if $p \in \{p_1, ..., p_n\}$ then $p_1, ..., p_n \vdash_S p$ holds regardless of the *S*, as a single occurrence of the assumption *p* already counts as a ruleless deduction of *p* from *p*.

Those basic definitions are usually combined with validity clauses to obtain semantics for intuitionistic logic. For instance, Sandqvist (2015) defines atomic derivability using At instead of $\mathrm{At}_\bot$ and employs the following clauses:

1. $\Vdash_S p$ iff $\vdash_S p$, for $p \in \mathrm{At}$;
2. $\Vdash_S (A \wedge B)$ iff $\Vdash_S A$ and $\Vdash_S B$;
3. $\Vdash_S (A \to B)$ iff $A \Vdash_S B$;
4. $\Vdash_S A \vee B$ iff $\forall S'(S \subseteq S')$ and all $p \in \mathrm{At}$, $A \Vdash_{S'} p$ and $B \Vdash_{S'} p$ implies $\Vdash_{S'} p$;
5. $\Vdash_S \bot$ iff $\Vdash_S p$ for all $p \in \mathrm{At}$
6. For non-empty finite Γ, $\Gamma \Vdash_S A$ iff for all *S′* such that $S \subseteq S'$ it holds that, if $\Vdash_{S'} B$ for all $B \in \Gamma$, then $\Vdash_{S'} A$;
7. $\Gamma \Vdash_{\mathrm{BeS}} A$ iff $\Gamma \Vdash_S A$ for all *S*;

The idea being that BeS validity ($\Vdash_{\mathrm{BeS}}$) is defined in terms of *S*-validity ($\Vdash_S$) and *S*-validity is reducible to derivability in *S* and its extensions, so we obtain a semantics defined exclusively in terms of *proofs* and *proof conditions*. In this sense, BeS not only aims at elucidating the meaning of a logical proof, but also at providing means for its use as a basic concept of semantic analysis.

2.2 On the semantics of $\bot$ in BeS

The semantic conditions for $\bot$ are usually defined in BeS in one of two ways. The first one is to define atomic derivability by using At instead of $\mathrm{At}_\bot$ and to employ the following semantic clause:

$$\Vdash_S \bot \qquad \text{iff} \qquad \Vdash_S p \text{ holds for all } p \in \mathrm{At}$$

Absurdity is treated as a logical constant and cannot figure in atomic bases, hence the switch from $\mathrm{At}_\bot$ to At. This clause, used most notably by Sandqvist (2015), borrows from Dummett (1991) the idea of defining absurdity in terms of logical explosion, but restricts it to just atoms in order to make the definition inductive.

The second one is to consider $\bot$ an atom and require all bases to contain atomic *ex falso* rules concluding *p* from $\bot$ for every $p \in \mathrm{At}_\bot$ (Piecha et al., 2015). If for some *S* we have $\Vdash_S \bot$ this now implies $\vdash_S \bot$; hence, for any $p \in \mathrm{At}_\bot$, the deduction of $\bot$ from empty premises in *S* can be extended by the appropriate *ex falso* rule to a deduction showing $\vdash_S p$ that also shows $\Vdash_S p$. Since $\Vdash_S p$ for all $p \in \mathrm{At}_\bot$ also implies $\Vdash_S \bot$ this means that ($\Vdash_S \bot$ iff $\Vdash_S p$ for all $p \in \mathrm{At}_\bot$) holds in all such *S*, so by restricting ourselves to these "atomically explosive" bases we end up giving the same semantic treatment to $\bot$.

Although technically sound, such definitions are inadequate from a conceptual standpoint because they do not express the intended meaning of $\bot$. The absurdity constant is supposed to represent a statement that never holds. Logical explosion should be a *consequence* of the definition of absurdity, not its definition. Ideally, $\bot \Vdash_S A$ should hold for all *S* because no extension *S*′ of *S* validates $\bot$ and the entailment holds vacuously, not because there are extensions validating $\bot$ that also validate the arbitrary formula *A*.

Unfortunately, we cannot define $\bot$ through the natural semantic clause:

$$\nVdash_S \bot \qquad \text{for all } S.$$

If we define $\neg A$ as $(A \to \bot)$ and adopt the clause above, we can demonstrate that $\Vdash_S \neg\neg p$ holds for every $p \in \mathrm{At}$ in every *S*. This follows because $\Vdash_S \neg p$ would hold only if no extension of *S* validates *p*. However, for every *S* and every $p \in \mathrm{At}$, there is always some extension *S*′ of *S* such that $\Vdash_{S'} p$ (for instance, the extension obtained by adding the atomic axiom with conclusion *p* to *S*). Piecha et al. (2015) observe that, aside from ruling out this definition of $\bot$, the fact that every atom is validated in some extension of every system "*might be considered a fault of validity-based proof-theoretic semantics, since it speaks against the intuitionistic idea of negation $\neg A$ as expressing that A can never be verified*".

As will be shown in this paper, a technically sound and conceptually adequate treatment of $\bot$ has been thus far overlooked by the literature. Even though we cannot define $\bot$ in terms of unsatisfiability through a semantic clause, it is still possible to do it by simply requiring all bases to be consistent:

$$\nvdash_S \bot \qquad \text{for all } S.$$

While at first glance it may seem that the definitions are equivalent, this switch actually allows us to solve the issues with the semantic clause. Moreover, the restriction implements the desiderata of Piecha et al. and allows bases to contain no extensions validating some specific atoms. To see why, consider that if *S* is a consistent base in which $p \vdash_S \bot$ holds then there can be no extension *S*′ of *S* validating *p*, since if $\Vdash_{S'} p$ for some $S \subseteq S'$ we would have a deduction showing $\vdash_{S'} p$ which could be composed with the deduction showing $p \vdash_S \bot$ (which is also a deduction showing $p \vdash_{S'} \bot$, since all rules of *S* are in *S*′ by the definition of extension) to obtain one showing $\vdash_{S'} \bot$, hence *S*′ would be inconsistent. It is easy to show that $p \vdash_S \bot$ now implies both $\Vdash_S \neg p$ and $\nVdash_S \neg\neg p$. As such, by considering $\bot$ an atom but requiring it to always be underivable we allow bases to indirectly restrict their own admissible extensions by conveying information about which formulas will never be validated in their extensions.

The completeness proof presented in Sect. 5 can easily be adapted to standard intuitionistic BeS by simply omitting all steps concerning classical formulas. Since the remaining steps are precisely the constructive ones, this yields a fully constructive proof of completeness for BeS with the consistency constraint and without any semantic clause for $\bot$ w.r.t. intuitionistic propositional logic. It should also be noted that many of the results we will demonstrate for ecumenical logics are essentially dependent on the consistency constraint, so the definition is optional in intuitionistic logic but essential in our ecumenical semantics.

2.3 Ecumenical language

Propositional formulas are built from basic sentences using the binary connectives $\to, \wedge, \vee$. The ecumenical language is defined as follows.

Definition 5 The *ecumenical language* $\mathcal{L}$ is comprised of the following ecumenical formulas:

1. If $p \in \mathrm{At}_{\bot}$, them $p^i, p^c \in \mathcal{L}$;
2. If $A, B \in \mathcal{L}$, then $(A \wedge B)^i$, $(A \vee B)^i$, $(A \to B)^i \in \mathcal{L}$;
3. If $A, B \in \mathcal{L}$, then $(A \wedge B)^c$, $(A \vee B)^c$, $(A \to B)^c \in \mathcal{L}$;

Notation 1 Parenthesis are omitted whenever no confusion ensues. For easing the notation, $\neg A$, $A \to B$, $A \wedge B$ and $A \vee B$ will be abbreviations of $(A \to \bot)^i$, $(A \to B)^i$ $(A \wedge B)^i$, $(A \vee B)^i$, respectively. Finally, we stipulate that if a formula A is used without specification of its superscript, then it may be either *i* or *c*. For instance, $A^i \wedge B^c$ should be read as a placeholder for $(A^i \wedge B^c)^i$, but $A \wedge B$ should be read as a placeholder for $(A^i \wedge B^i)^i$, $(A^c \wedge B^i)^i$, $(A^i \wedge B^c)^i$ and $(A^c \wedge B^c)^i$.

Definition 6 The *complexity* of a formula with shape A^i is the number of logical operators distinct from $\bot$ occurring on it. The *complexity* of a formula with shape A^c is the complexity of A^i plus 1.

Intuitively, an intuitionistic formula A^i holds whenever there exists an *intuitionistic proof* of A, and a classical formula A^c holds whenever there exists a *classical proof* of A. Since every formula of the usual language has both a classical and an intuitionistic version, classical and intuitionistic support in bases is defined for every formula.

In this paper, we will focus on two definitions of semantic ecumenism, called *weak*and *strong* ecumenical semantics, respectively. In both the semantics of classical proofs is given in terms of the consistency of formulas w.r.t. some atomic system, but the notions induce classical behavior in very different ways.

It should be observed that the weak ecumenical semantics proposed next does not have a simple syntactic characterization, and its study is meant for semantic purposes only – the goal is to explore deeply the ecumenical proof-theoretic behavior. In Sect. 5 we present an interesting ecumenical natural deduction system which is sound and complete w.r.t. the strong ecumenical semantics described in Sect. 4.

3 Weak ecumenical semantics

We start by distinguishing between two notions of logical validity for every atomic system S: *local* logical validity (represented by $\Vdash^L_S$) and *global* logical validity (represented by $\Vdash^G_S$). The idea is that, in local validity, we only concern ourselves with what holds in a base and its extensions, but in global validity we also take into account what holds in extensions of a base's extensions. In many contexts both notions collapse, but in some there is reason to make a distinction. Some relationships between both will be studied in more depth later.

A weak ecumenical version of BeS is given by the definition below. As usual, we start by giving semantic conditions for basic sentences in atomic systems, expanding them through semantic clauses.

Definition 7 (Weak Validity) *Weak S-validity* and *weak validity* are defined as follows.

1. $\Vdash^L_S p^i$ iff $\vdash_S p$, for $p \in \mathrm{At}_\bot$;
2. $\Vdash^L_S p^c$ iff $p \nvdash_S \bot$, for $p \in \mathrm{At}_\bot$;
3. $\Vdash^L_S A^c$ iff $A^i \nVdash^L_S \bot^i$, for $A \notin \mathrm{At}_\bot$;
4. $\Vdash^L_S (A \wedge B)^i$ iff $\Vdash^L_S A$ and $\Vdash^L_S B$;
5. $\Vdash^L_S (A \to B)^i$ iff $A \Vdash^G_S B$;
6. $\Vdash^L_S (A \vee B)^i$ iff $\forall S' (S \subseteq S')$ and all $p \in \mathrm{At}_\bot$, $A \Vdash^L_{S'} p^i$ and $B \Vdash^L_{S'} p^i$ implies $\Vdash^L_{S'} p^i$;
7. For non-empty finite Γ, we have that $\Gamma \Vdash^L_S A$ iff for all S' such that $S \subseteq S'$ it holds that, if $\Vdash^L_{S'} B$ for all $B \in \Gamma$, then $\Vdash^L_{S'} A$;
8. For finite Γ, $\Gamma \Vdash^G_S A$ iff for all S' such that $S \subseteq S'$ we have that, if for all S'' such that $S' \subseteq S''$ it holds that $\Vdash^L_{S''} B$ for all $B \in \Gamma$, then for all S'' such that $S' \subseteq S''$ it also holds that $\Vdash^L_{S''} A$;
9. $\Gamma \Vdash A$ iff $\Gamma \Vdash^G_S A$ for all S.

There are important bits of information to unpack in those clauses. First, notice that there is one clause for classical proofs of atoms and one for classical proofs of non-atomic formulas, but both are defined in terms of consistency proofs for the formula's immediate subformula. Second, while Clause 7 is the same as Sandqvist's clause "(Inf)", Clause 8 is slightly more complex; the former is our definition of *local* validity, the later of *global* validity (Cobreros, 2008). This distinction is redundant in usual intuitionistic semantics, but essential in the weak version of ecumenical BeS. Finally, notice that when defining the semantic clause for disjunction we use local entailment instead of global, which is done to show that some desirable semantic properties follows from this weak definition. By using the global notion instead we would obtain an alternative presentation of what we later define as strong ecumenical semantics.

The following result easily follows from Definition 7 and the requirement of atomic systems to be consistent.

Lemma 1 $\nVdash^L_S \bot^i$ *and* $\nVdash^L_S \bot^c$ *for all S.*

Proof For any S, since S must be consistent we have $\nvdash_S \bot$, so $\nVdash^L_S \bot^i$. Moreover, since $\bot \vdash_S \bot$ holds by the definition of deducibility we have $\nVdash^L_S \bot^c$. □

Due to this result, from now on $\bot$ will be used as an abbreviation of $\bot^i$ ($\equiv \bot^c$) in semantic contexts.

Although expected, since intuitionistic provability implies classical provability, the next two results are only possible due to the change from Sandqvist's clause for $\bot$ to the consistency requirement.

Theorem 2 $p^i \Vdash^L_S p^c$ *for any* $p \in At_\bot$.

Proof Assume $\Vdash^L_{S'} p^i$ for some $S \subseteq S'$. Then $\vdash_{S'} p$. Now suppose $p \vdash_{S'} \bot$. Then by composing both deductions we have $\vdash_{S'} \bot$, contradicting the consistency requirement. So $p \nvdash_{S'} \bot$, hence $\Vdash^L_{S'} p^c$. Since S' is an arbitrary extension of S, we have $p^i \Vdash^L_S p^c$ by Clause 7. □

Theorem 3 $A^i \Vdash^L_S A^c$ *for any* $A \notin At_\bot$.

Proof Assume $\Vdash^L_{S'} A^i$ for some $S \subseteq S'$ and suppose that $A^i \Vdash^L_{S'} \bot$. Then by Clause 7 of Definition 7 we have $\Vdash^L_{S'} \bot$, and then $\vdash_{S'} \bot$ by Clause 1, which is a contradiction. Thus, $A^i \nVdash^L_{S'} \bot$, and so $\Vdash^L_{S'} A^c$. Since S' is an arbitrary extension of S, we have $A^i \Vdash^L_S A^c$ by Clause 7. □

If Sandqvist's definition was used, from $\Vdash^L_{S'} A^i$ and $A^i \Vdash^L_{S'} \bot$ we could get $\Vdash^L_{S'} p^i$ for arbitrary $p \in \mathrm{At}_\bot$, but it would not be the case that $A^i \nVdash^L_{S'} \bot$. The same would happen with the proof for atoms if we allowed $\bot$ to occur in atomic bases and required all bases to contain all instances of the atomic *ex falso*.

3.1 Monotonicity

It is well known that BeS validity in intuitionistic logic is *monotonic*, in the sense that it is stable under base extensions. As it turns out, this is not the case in the ecumenical setting, as discussed next.

Definition 8 (Monotonicity) A formula A is called *S-monotonic* with respect to an atomic system S if, for all $S \subseteq S'$, $\Vdash^L_S A$ implies $\Vdash^L_{S'} A$. A is called *monotonic* if it is S-monotonic for any atomic system S.

Some parts of Clause 8 come for free in the presence of monotonicity (as shown next), but they must be explicitly stated on the lack of it. As such, the original notion of logical consequence provides only a weak kind of validation for non-monotonic formulas, and thus would indirectly treat classical and intuitionistic formulas *very differently*.

Theorem 4 *If S-monotonicity holds for A and all formulas in* Γ*, then* $\Gamma \Vdash^L_S A$ *iff* $\Gamma \Vdash^G_S A$.

Proof The result is trivial if $\Gamma = \emptyset$, so we will assume Γ non-empty.

($\Rightarrow$) Suppose $\Gamma \Vdash^L_S A$. Then, by Clause 7, for every S' such that $S \subseteq S'$ we have that, if $\Vdash^L_{S'} B$ for every $B \in \Gamma$ then $\Vdash^L_{S'} A$. Now take any such S' in which, for all S'' such that $S' \subseteq S''$, we have $\Vdash^L_{S''} B$ for all $B \in \Gamma$. Since $S' \subseteq S'$, we have $\Vdash^L_{S'} B$ for every $B \in \Gamma$, so we conclude $\Vdash^L_{S'} A$. But by S-monotonicity we also have that $\Vdash^L_{S'} A$ implies $\Vdash^L_{S''} A$ for all S'' such that $S' \subseteq S''$ and, since S' was an arbitrary extension of S satisfying the antecedent of the second part of Clause 7, we have $\Gamma \Vdash^G_S A$.

($\Leftarrow$) Assume $\Gamma \Vdash^G_S A$. Then, by Clause 7, for every S' such that $S \subseteq S'$ we have that, if $\Vdash^L_{S''} B$ for every $B \in \Gamma$ and for all S'' such that $S' \subseteq S''$ then $\Vdash^L_{S''} A$ for all such S'' as well. Now let S' be any extension of S such that $\Vdash^L_{S'} B$ for all $B \in \Gamma$. By monotonicity, for all formulas $B \in \Gamma$ we have that, if $\Vdash^L_{S'} B$, then $\Vdash^L_{S''} B$ for every S'' such that $S' \subseteq S''$. Taken together with our assumption, this yields $\Vdash^L_{S''} A$ for all such S''. In particular, since $S' \subseteq S'$ we have $\Vdash^L_{S'} A$ and, since S' was an arbitrary extension of S satisfying the antecedent of the second part of Clause 7, we have $\Gamma \Vdash^L_S A$. □

Even though intuitionistic atoms and connectives are monotonic, this is not the case in the classical setting.

Theorem 5 *Every formula containing only intuitionistic subformulas is monotonic. Classical atoms are not monotonic.*

Proof The result for formulas containing only intuitionistic subformulas is easily proven by induction on the complexity of formulas in the same way as in [(Sandqvist, 2015) Lemma 3.2. (a)], where the induction hypothesis is only needed for conjunction (the case for implication holds directly from the definition of general validity).

Regarding classical atoms, for $S = \emptyset$ we have that $p \nvdash_S \bot$ for every $p \in \mathrm{At}$. But if S' is the atomic system containing only the rule obtaining $\bot$ from p, S' is consistent and $p \vdash_{S'} \bot$. Hence $\Vdash^L_S p^c$ and $S \subseteq S'$, but $\nVdash^L_{S'} p^c$. More generally, if p does not occur in the rules of S then $p \nvdash_S \bot$ and by adding a rule obtaining $\bot$ from p to S we have an extension S' guaranteed to be consistent, so whenever p does not appear on the rules of S we have $\Vdash^L_S p^c$but $\nVdash^L_{S'} p^c$ for some $S \subseteq S'$. □

In short, for intuitionistic formulas it is irrelevant whether local or global notions of validity is used. For ecumenical formulas containing classical subformulas, however, this choice makes an enormous difference, as illustrated in Sect. 3.3.

3.2 Basic lemmata

Before proceeding, we briefly present some lemmas that will be useful later. For the sake of readability, some proofs are omitted from the main text, see Appendix A for details.

We start by showing that local validity implies global validity only for non-empty contexts, but global validity implies local validity only when the context is empty. A counter-example for Lemma 7 with non-empty contexts is given in Theorem 18, whereas one for Lemma 6 with empty contexts can be obtained by putting $A = p^c$ and remembering that classical atoms are not always monotonic.

Lemma 6 *For non-empty Γ, $\Gamma \Vdash^L_S A$ implies $\Gamma \Vdash^G_S A$.*

Lemma 7 *$\Vdash^G_S A$ implies $\Vdash^L_S A$.*

Proof Assume $\Vdash^G_S A$. Then for every $S \subseteq S'$, it holds that $\Vdash^L_{S''} A$, for every $S' \subseteq S''$. By putting $S'' = S' = S$ we conclude $\Vdash^L_S A$. $\square$

Lemma 8 *If $\nVdash^L_{S'} A$ for all $S \subseteq S'$ then $\Vdash^L_{S'} \neg A$ for all $S \subseteq S'$.*

Proof Assume $\nVdash^L_{S'} A$ for all $S \subseteq S'$. Take any such extension S'. For any $S' \subseteq S''$ by transitivity of the extension relation we have $S \subseteq S''$, so $\nVdash^L_{S''} A$. But then clearly $A^i \Vdash^G_{S'} \bot$ is satisfied vacuously for any S', so $\Vdash^L_{S'} \neg A$ for all $S \subseteq S'$. $\square$

The following is a form of global *modus ponens*.

Lemma 9 *$\Vdash^G_S A$ and $A \Vdash^G_S B$ implies $\Vdash^G_S B$.*

Proof Assume $\Vdash^G_S A$. Thus for all $S \subseteq S'$ we have that $S' \subseteq S''$ implies $\Vdash^L_{S''} A$. Assume $A \Vdash^G_S B$. Then, for any $S \subseteq S'$, if for all $S' \subseteq S''$ we have $\Vdash^L_{S''} A$, then for all $S' \subseteq S''$ we have $\Vdash^L_{S''} B$. By putting $S = S'$ the antecedent gets satisfied and we immediately get $\Vdash^L_{S''} B$ for all $S \subseteq S''$, hence $\Vdash^G_S B$. $\square$

Finally, the following results show interactions between monotonicity, global validity and negation.

Lemma 10 *If $\Vdash^L_S A$, S-monotonicity holds for A and $A \Vdash^G_S B$, then both $\Vdash^L_S B$ and $\Vdash^G_S B$.*

Proof Since $\Vdash^L_S A$ holds and monotonicity holds for A, for all S' such that $S \subseteq S'$ we have that $\Vdash^L_{S'} A$. Since $A \Vdash^G_S B$ holds and S is an extension of itself, we immediately conclude that $\Vdash^L_{S'} B$ for all S' extending S and all S'' extending any S', and thus $\Vdash^G_S B$. In particular, since S is an extension of itself, we also have $\Vdash^L_S B$. $\square$

Lemma 11 *$(p \vdash_S \bot)$ iff $(p^i \Vdash^L_S \bot)$ iff $(p^i \Vdash^G_S \bot)$ iff $(\Vdash^L_S \neg p^i)$.*

Corollary 12 *$\Vdash^L_S p^c$ iff $p^i \nVdash^L_S \bot$.*

Lemma 13 *$A \nVdash^L_S \bot$ iff there is some $S \subseteq S'$ such that $\Vdash^L_{S'} A$.*

Proof Assume $A \nVdash^L_S \bot$. Suppose there is no $S \subseteq S'$ with $\Vdash^L_{S'} A$. Then $A \Vdash^L_S \bot$ holds vacuously, which is a contradiction. Hence, for some $S \subseteq S^n$ we have $\Vdash^L_{S'} A$. On the other hand, assume that there is some $S \subseteq S'$ such that $\Vdash^L_{S'} A$. Suppose $A \Vdash^L_S \bot$. Then we have $\Vdash^L_{S'} \bot$, yielding a contradiction. Thus, $A \nVdash^L_S \bot$. $\square$

The following is a variation of Makinson's proofs to the effect that every consistent base has a maxiconsistent extension (Makinson, 2014).

Lemma 14 *Every consistent system S has a $\bot$-complete consistent extension $S^{\bot C}$ such that, for every $p \in At_{\bot}$, either $\vdash_{S^{\bot C}} p$ or $p \vdash_{S^{\bot C}} \bot$*

Proof Since $\mathrm{At}_{\bot}$ is countable, assign to each of its elements a unique natural number greater than 0 as a superscript. Define $S = S^0$, and consider the following construction procedure (for $m \geq 1$):

1. If $p^m \nvdash_{S^{m-1}} \bot$, then S^m is obtained by adding an atomic axiom with conclusion p^m to S^{m-1};
2. If $p^m \vdash_{S^{m-1}} \bot$, then $S^m = S^{m-1}$.

We briefly show by a simple induction that every S^m produced this way is consistent. Since $S^0 = S$, S^0 is consistent. Now suppose that S^{m-1} is consistent. If $p^m \vdash_{S^{m-1}} \bot$ then $S^{m-1} = S^m$, so S^m is consistent. If $p^m \nvdash_{S^{m-1}} \bot$, assume for the sake of contradiction that $\vdash_{S^m} \bot$. Then either the proof of $\bot$ in S^m does not use the atomic axiom with conclusion p^m included in S^{m-1}, in which case it is also a deduction showing $\vdash_{S^{m-1}} \bot$, or it does use the atomic axiom, in which case by removing all instances of it from the deduction we obtain a deduction showing $p^m \vdash_{S^{m-1}} \bot$. In the first case we contradict the assumption that S^{m-1} was consistent, and in the second we contradict the assumption that $p^m \nvdash_{S^{m-1}} \bot$, so in any case we obtain a contradiction. Hence, if S^{m-1} is consistent then S^m is consistent, and since S^0 is consistent we have that each S^m is consistent.

Now let $S^{\bot C} = \{R \in S^m | m \geq 0\}$. Clearly, $S \subseteq S^{\bot C}$. To show that $S^{\bot C}$ is also consistent, assume for the sake of contradiction that there is a deduction showing $\vdash_{S^{\bot C}} \bot$. By the definition of deducibility, this deduction can only use finitely many rules. If the deduction does not use any atomic axioms, it is already a deduction in *S*, thus contradicting the fact that *S* is consistent. If it does use atomic axioms, let *m* be the greatest superscript occurring in atomic axioms of the deduction. This deduction only uses axioms with superscript equal to or less than *m*, thus all rules used in it must already occur in S^m (as they could not have been added later in the construction). But then this means that $\vdash_{S^m} \bot$, contradicting our result that each S^m is consistent. In both cases we reach a contradiction, so we conclude that $S^{\bot C}$ is indeed consistent.

Finally, take any $p^m \in \mathrm{At}_{\bot}$. If $p^m \vdash_{S^{m-1}} \bot$ then $p^m \vdash_{S^{\bot C}} \bot$, as by the definition of $S^{\bot C}$ we have $S^{m-1} \subseteq S^{\bot C}$. If $p^m \nvdash_{S^{m-1}} \bot$ then S^m contains an atomic axiom concluding p^m, and since $S^m \subseteq S^{\bot C}$ we conclude $\vdash_{S^{\bot C}} p^m$. Since this holds for every $m \geq 1$ and every atom was assigned such a superscript, we conclude that for every $p \in \mathrm{At}_{\bot}$ either $p \vdash_{S^{\bot C}} \bot$ or $\vdash_{S^{\bot C}} p$, as desired. □

Notice that, unlike Makinson's maxiconsistent extensions, $\bot$-complete extension are not required to be maximal with respect to set inclusion. The two lemmas that follow are also analogues of Makinson's [(2014) Lemma 3.5.], in the sense that they show that $\bot$-complete extensions are classically well-behaved.

Lemma 15 *Let A and B be two formulas and S a system such that, for all $S \subseteq S'$, $\Vdash^L_S A$ iff $\Vdash^L_{S'} A$ and $\Vdash^L_S B$ iff $\Vdash^L_{S'} B$. Then for all $S \subseteq S'$ we have $A \Vdash^L_{S'} B$ iff $A \Vdash^G_{S'} B$ iff either $\nVdash^L_{S'} A$ or $\Vdash^L_{S'} B$.*

Proof Let S be any system A and B be two formulas as specified. Let S' be an arbitrary extension of S. If $\nVdash^L_{S'} A$ then $\nVdash^L_S A$ and for all $S \subseteq S''$ we have $\nVdash^L_{S''} A$. Since $S' \subseteq S'''$ implies $S \subseteq S'''$ we have that $\nVdash^L_{S'''} A$ also holds for all extensions S''' of S', so $A \Vdash^L_{S'} B$ and $A \Vdash^G_{S'} B$ hold vacuously. Likewise, if $\Vdash^L_{S'} B$ then $\Vdash^L_S B$ and so also $\Vdash^L_{S''} B$ for all $S \subseteq S''$, so again for all $S' \subseteq S'''$ we have $\Vdash^L_{S'''} B$, hence by similar reasoning we conclude that both $A \Vdash^L_{S'} B$ and $A \Vdash^G_{S'} B$ hold. Finally, if $\Vdash^L_{S'} A$ and $\nVdash^L_{S'} B$ hold then since $S' \subseteq S'$ clearly $A \nVdash^L_{S'} B$ and $A \nVdash^G_{S'} B$, and since this covers all cases we conclude the desired result.

□

Lemma 16 *Let $S^{\perp C}$ be a $\perp$-complete extension of some system. Then, for every $S^{\perp C} \subseteq S'$ and every A, $\Vdash^L_{S^{\perp C}} A$ iff $\Vdash^L_{S'} A$.*

3.3 Weak ecumenical behavior

This section will be devoted to show some interesting behaviors when monotonicity does not hold for ecumenical formulas. Notice that, due to Corollary 12, classical atoms p^c and classical non-atomic formulas A^c may be treated uniformly in some cases.

Theorem 17 *$A^c \Vdash \neg\neg A^i$ and $\neg\neg A^i \Vdash A^c$.*

Proof Let's first prove that $A^c \Vdash^G_S \neg\neg A^i$ holds for arbitrary S.

Let S be an arbitrary atomic system. Let S' be any extension of S such that, for all $S' \subseteq S''$, we have $\Vdash^L_{S''} A^c$. Then for every $S' \subseteq S''$ we have $A^i \nVdash^L_{S''} \perp$. Suppose, for the sake of contradiction, that there is a $S' \subseteq S''$ such that $\Vdash^L_{S''} \neg A^i$. Then $A^i \Vdash^G_{S''} \perp$. Now let $S^{\perp C}$ be a $\perp$-complete extension of S''. Since $S' \subseteq S^{\perp C}$ we have $A^i \nVdash^L_{S^{\perp C}} \perp$, hence by Lemma 13 there must be a $S^{\perp C} \subseteq S'''$ with $\Vdash^L_{S'''} A$. So by Lemma 16 we conclude $\Vdash^L_{S^{\perp C}} A$ and also $\Vdash^L_{S'''} A$ for arbitrary extensions S''' of $S^{\perp C}$. But since $S'' \subseteq S^{\perp C}$, $A \Vdash^G_{S''} \perp$ and $\Vdash^L_{S'''} A$ for every $S^{\perp C} \subseteq S'''$ we conclude $\Vdash^L_{S'''} \perp$ and $\vdash_{S'''} \perp$ for all $S^{\perp C} \subseteq S'''$, which violates the consistency requirement. Therefore, for all $S' \subseteq S''$ we have $\nVdash^L_{S''} \neg A^i$ and so by Lemma 8 also $\Vdash^L_{S''} \neg\neg A^i$, hence by arbitrariness of S' we have $A^c \Vdash^G_S \neg\neg A^i$.

Now, let's prove $\neg\neg A^i \Vdash A^c$, which amounts to proving $\neg\neg A^i \Vdash^G_S A^c$ for arbitrary S. Let S be an arbitrary atomic system. Let S' be any extension of S such that, for all S'' for which $S' \subseteq S''$, we have that $\Vdash^L_{S''} \neg\neg A^i$ holds. In particular, $\neg A^i \Vdash^G_{S'} \perp$ holds. Now assume for the sake of contradiction that, for some $S' \subseteq S''$, we have $A^i \Vdash^L_{S''} \perp$. Assume there is a $S'' \subseteq S'''$ such that $\Vdash^L_{S'''} A^i$. Then since $A^i \Vdash^L_{S''} \perp$ and

$S'' \subseteq S'''$ we conclude $\Vdash^L_{S'''} \bot$ and obtain a contradiction, so for all $S'' \subseteq S'''$ we have $\nVdash^L_{S'''} A^i$. Since $\nVdash^L_{S'''} A^i$ for all $S'' \subseteq S'''$ we have $\Vdash^L_{S'''} \neg A^i$ for all $S'' \subseteq S'''$ by Lemma 8. But since $\neg A^i \Vdash^G_{S'} \bot$, $S' \subseteq S''$ and $\Vdash^L_{S'''} \neg A^i$ holds for all $S'' \subseteq S'''$ we conclude $\Vdash^L_{S'''} \bot$ for all such S''', leading to a contradiction. Hence we conclude that there can be no extension S'' of S' with $A^i \Vdash^L_{S''} \bot$, so for all $S' \subseteq S''$ we have $A^i \nVdash^L_{S''} \bot$ and thus $\Vdash^L_{S''} A^c$, hence since S' was arbitrary we conclude $\neg\neg A^i \Vdash^G_S A^c$. □

The next two results are interesting, showing that global validity can be preserved locally but that this is not always the case.

Theorem 18 $A^c \Vdash^L_S \neg\neg A^i$ *does not hold for arbitrary S.*

Proof We prove the result for atoms. Consider the atomic system $\emptyset$, which contains no rules. Clearly, since $p \nvdash_\emptyset \bot$, we have $\Vdash^L_\emptyset p^c$. Suppose that $\Vdash^L_\emptyset \neg\neg p^i$. Consider now an extension S of $\emptyset$ containing a rule which concludes $\bot$ from the premise p. Hence $p \vdash_S \bot$ and, due to Lemma 11, $\Vdash^L_S \neg p^i$ holds. Since $\neg\neg p^i$ is intuitionistic, it is monotonic, and thus $\Vdash^L_\emptyset \neg\neg p^i$ implies $\Vdash^L_S \neg\neg p^i$. By the semantic clause for implication we then have $\neg p^i \Vdash^G_S \bot$ and, since $\Vdash^L_S \neg p^i$, by Lemma 10 we have $\Vdash^L_S \bot$, and thus $\vdash_S \bot$. Contradiction. Thus $\nVdash^L_\emptyset \neg\neg p^i$ and, since the empty set is an extension of itself, $p^c \nVdash^L_\emptyset \neg\neg p^i$.

□

Theorem 19 $\neg\neg A^i \Vdash^L_S A^c$ *holds for arbitrary S.*

Proof Let S be any system. Consider any $S \subseteq S'$ such that $\Vdash^L_{S'} \neg\neg A^i$. By the clause for implication, $\neg A^i \Vdash^G_{S'} \bot$. Assume $A^i \Vdash^L_{S'} \bot$ for the sake of contradiction. Then clearly $\nVdash^L_{S''} A^i$ for all $S' \subseteq S''$, so $\Vdash^L_{S''} \neg A^i$ for all $S' \subseteq S''$ by Lemma 8. Since $\neg A^i \Vdash^G_{S'} \bot$ and $\Vdash^L_{S''} \neg A^i$ for all $S' \subseteq S''$ we conclude $\Vdash^L_{S''} \bot$ for all $S' \subseteq S''$, which is a contradiction. Hence $A^i \nVdash^L_{S'} \bot$, so $\Vdash^L_{S'} A^c$, therefore by arbitrariness of S' we have $\neg\neg A^i \Vdash^L_S A^c$.

□

Remark 1 Put together, these results show that classical proof of A is strictly weaker than an intuitionistic proof of $\neg\neg A$, and justify the motto presented in the introduction.

The following results present ecumenical versions of the excluded middle and Peirce's law.

Theorem 20 $\Vdash^L_S A^c \vee \neg A^i$ *holds for arbitrary S.*

Proof Let S be any system. Let S' be any extension of S in which $A^c \Vdash^L_{S'} p^i$ and $\neg A^i \Vdash^L_{S'} p^i$ for some $p \in \mathrm{At}_\bot$. If $A^i \Vdash^L_{S'} \bot$ then by Lemma 6 we have $A^i \Vdash^G_{S'} \bot$ and so $\Vdash^L_{S'} \neg A^i$, hence since $\neg A^i \Vdash^L_{S'} p^i$ we conclude $\Vdash^L_{S'} p^i$. If $A \nVdash^L_{S'} \bot$ then $\Vdash^L_{S'} A^c$, so since $A^c \Vdash^L_{S'} p^i$ we conclude $\Vdash^L_{S'} p^i$. Since either $A \Vdash^L_{S'} \bot$ or $A \nVdash^L_{S'} \bot$ holds this cover all possible cases, hence for all $S \subseteq S'$ we have that $A^c \Vdash^L_{S'} p^i$ and $\neg A^i \Vdash^L_{S'} p^i$ implies $\Vdash^L_{S'} p^i$ for any $p \in \mathrm{At}_\bot$, so we conclude $\Vdash^L_S A^c \vee \neg A^i$.

□

This proof is particularly interesting because it combines our notion of classical proof with the weak definition of disjunction (in terms of local validity) to provide a simple proof of the excluded middle in the language through an application of the excluded middle in the metalanguage. This suggests that the weak disjunction here proposed becomes predominantly classical if combined with our notion of classical proof, and this makes it so that other classical results could also be proved via metalinguistical applications of the excluded middle. This would not be the case if we were to define disjunction through global validity; as will be seen in the strong ecumenical semantics, this strengthened disjunction has much more of an intuitionistic flavor.

Theorem 21 $\Vdash^L_S ((A^i \to B) \to A^i) \to A^c$ *holds for arbitrary S.*

Proof Let S be a system. Let S' be an extension of S with $\Vdash^L_{S''} (A^i \to B) \to A^i$ for all $S' \subseteq S''$. Then, by definition, for all those S'' we have $(A^i \to B) \Vdash^G_{S''} A^i$. Assume, for the sake of contradiction, that for some of those S'' we have $\nVdash^L_{S''} A^c$. Then we have $A^i \Vdash^L_{S''} \bot$ by Clause 3 of Definition 7, and since in any $S'' \subseteq S'''$ with $\Vdash^L_{S'''} A^i$ we could obtain $\Vdash^L_{S'''} \bot$ and thus a contradiction we clearly have $\nVdash^L_{S'''} A^i$ for all $S'' \subseteq S'''$. But notice that, for any such S''', $S''' \subseteq S''''$ implies $S'' \subseteq S''''$ and so $\nVdash^L_{S''''} A^i$, hence we have that $A^i \Vdash^G_{S'''} B$ is vacuously satisfied in all such S''', so we conclude that for all $S'' \subseteq S'''$ we have $\Vdash^L_{S'''} A^i \to B$. Since $(A^i \to B) \Vdash^G_{S''} A^i$ and for all $S'' \subseteq S'''$ it holds that $\Vdash^L_{S'''} A^i \to B$ we conclude that for all $S'' \subseteq S'''$ we have $\Vdash^L_{S'''} A^i$ and, in particular, $\Vdash^L_{S''} A^i$. But we had previously concluded from our assumption for contradiction that $A^i \Vdash^L_{S''} \bot$, so since $\Vdash^L_{S''} A^i$ we have $\Vdash^L_{S''} \bot$, which is indeed a contradiction. Hence we conclude that for no $S' \subseteq S''$ we have $\nVdash^L_{S''} A^c$, so $\Vdash^L_{S''} A^c$ holds for all $S' \subseteq S''$. Since S' is an arbitrary extension of S with $\Vdash^L_{S''} (A^i \to B) \to A^i$ holding for all $S' \subseteq S''$ and we have shown that $\Vdash^L_{S''} A^c$ also holds for all such S'' we conclude $(A^i \to B) \to A^i \Vdash^G_S A^c$, and thus $\Vdash^L_S ((A^i \to B) \to A^i) \to A^c$.

□

There are, however, some drawbacks to our definitions, which are mainly due to the interaction between the clause for disjunction and the definition of local validity. For instance, we lose validities such as the following (proof in Appendix A).

Proposition 22 $(A \vee B), (A \to C), (B \to C) \Vdash C$ *does not hold in general.*

This issue seems to be caused by some interactions between the definition of implication (which uses global validity) and disjunction (which uses local validity). We could, of course, provide a weaker definition of implication that uses local validity, but since logical validity is defined by recourse to global validity by doing we would be giving away the deduction theorem (e.g., $p^c \Vdash^L_\emptyset \neg\neg p^i$ does not hold by Theorem 18 but $p^c \Vdash^G_S \neg\neg p^i$ holds for all S by Theorem 17, so we would have $\nVdash^L_S p^c \to \neg\neg p^i$ and so $\nVdash p^c \to \neg\neg p^i$ even though $p^c \Vdash \neg\neg p^i$).

As seen in the proof of Theorem 20, for all S and A either we have $A^i \Vdash^L_S \bot$ and so $A^i \Vdash^G_S \bot$ by Lemma 6 thus also $\Vdash^L_S \neg A^i$ or $A^i \nVdash^L_S \bot$ and so $\Vdash^L_S \neg A^c$. In other words, the metalinguistic excluded middle is "locally valid" in every S due to our definition of classical proofs. On the other hand, even though from $A^i \Vdash^G_S \bot$ we can conclude $\Vdash^L_S \neg A^i$ it is not the case that from $A^i \nVdash^G_S \bot$ we can conclude $A^i \nVdash^L_S \bot$, which would be necessary for us to conclude $\Vdash^L_S \neg A^c$ (remember that Lemma 7 fails for non-empty contexts), so the metalinguistic excluded middle is not "globally valid". This creates a certain tension between local and global definitions, as local definitions are able to draw on the local excluded middle to validate classical behavior but global definitions are not.

Although the semantic tension and the independent coexistence of classical and intuitionistic features are certainly desirable in the context of ecumenical semantics, the main issue with the definitions we have presented is that, since the usual rule for disjunction elimination is no longer sound, the weak ecumenical semantics is not easily captured in simple syntactic systems. This makes it so that the main motive for studying it lies in the clarification of the ways in which the global and local notions of validity relate to intuitionistic and classical concepts of proof. There might, of course, also be other combinations of local and global definitions which lead to interesting new ecumenical versions of BeS, but we leave the study of any such combinations to future works.

The clarifications provided by the weak semantics on how the notion of classical proof behaves in BeS allow us to formulate a new kind of ecumenical semantics which fixes some of its issues. As such, we propose next an ecumenical BeS with some stronger definitions and very different semantic properties.

4 Strong ecumenical semantics

In the weak semantics we define that a formula has a classical proof in S if and only if it is consistent in S. As a result, classical proofs are not monotonic, so we need to differentiate between local and global validity notions. But there is another possibility: we can define that a formula has a classical proof in S if and only if it is consistent in *S and all its extensions*. This is still faithful to Hilbert's ideas concerning classical proofs and truth, and since we only consider extensions of S it is also faithful to the proposal of the original BeS.

Definition 9 (Strong Validity) *Strong S-validity* and *strong validity* are defined as follows.

1. $\vDash_S p^i$ iff $\vdash_S p$, for $p \in \mathrm{At}_\bot$;
2. $\vDash_S p^c$ iff $\forall S'(S \subseteq S') : p \nvDash_{S'} \bot$, for $p \in \mathrm{At}_\bot$;
3. $\vDash_S A^c$ iff $\forall S'(S \subseteq S') : (A)^i \nvDash_{S'} \bot$, for $A \notin \mathrm{At}_\bot$;
4. $\vDash_S (A \wedge B)^i$ iff $\vDash_S A$ and $\vDash_S B$;
5. $\vDash_S (A \to B)^i$ iff $A \vDash_S B$;
6. $\vDash_S (A \vee B)^i$ iff $\forall S'(S \subseteq S')$ and all $p \in \mathrm{At}_\bot$: $A \vDash_{S'} p^i$ and $B \vDash_{S'} p^i$ implies $\vDash_{S'} p^i$;
7. For non-empty finite Γ, $\Gamma \vDash_S A$ iff for all S' such that $S \subseteq S'$ it holds that, if $\vDash_{S'} B$ for all $B \in \Gamma$, then $\vDash_{S'} A$;
8. $\Gamma \vDash A$ iff $\Gamma \vDash_S A$ for all S.

Weak validity uses a non-monotonic notion, whereas in strong validity classical validities are monotonic by definition. Since by Theorem 4 S-monotonicity induces a collapse between $\Vdash^L_S$ and $\Vdash^G_S$ and all formulas of the strong ecumenical semantics are monotonic, local and global validities are non-distinguishable.

5 An ecumenical proof system for strong ecumenical validity

In this section we will prove soundness and completeness of the natural deduction ecumenical system $\mathrm{NE_B}$ presented in Fig. 1 (which is a version of the system *CIE* presented in (Nascimento, 2018) with a restriction on iterations of the "classicality" operator) w.r.t. the strong ecumenical BeS.

For finite Γ we say that $\Gamma \vdash_{\mathrm{NE_B}} A$ holds if and only if there is a deduction of A from Γ using the rules of $\mathrm{NE_B}$.

5.1 Soundness

Contrary to what happens with completeness, the proof of soundness follows easily from the proof in (Sandqvist, 2015).

Lemma 23 $\vDash_S A^c$ *iff* $\vDash_S \neg\neg A^i$.

Proof ($\Rightarrow$) Suppose $\vDash_S A^c$. Then, for all $S \subseteq S'$, we have $A^i \nvDash_S \bot$. Suppose that, for some of those S', $\vDash_{S'} \neg A^i$ Then we have $A^i \vDash_{S'} \bot$, which yields a contradiction. Thus for all such S' we have $\nvDash_{S'} \neg A^i$, hence $\neg A^i \vDash_S \bot$ is vacuously satisfied and $\vDash_S \neg\neg A^i$ holds.

$$\dfrac{\begin{array}{c}\Gamma\ [A]\\ \Pi\\ B\end{array}}{A \to B}\ \to\text{-int} \qquad \dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2\\ \Pi_1 & \Pi_2\\ A \to B & A\end{array}}{B}\ \to\text{-elim}$$

$$\dfrac{\begin{array}{c}\Gamma\\ \Pi\\ A_j\end{array}}{A_1 \vee A_2}\ \vee_j\text{-int} \qquad \dfrac{\begin{array}{ccc}\Gamma_1 & \Gamma_2\ [A] & \Gamma_3\ [B]\\ \Pi_1 & \Pi_2 & \Pi_3\\ A \vee B & C & C\end{array}}{C}\ \vee\text{-elim}$$

$$\dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2\\ \Pi_1 & \Pi_2\\ A & B\end{array}}{A \wedge B}\ \wedge\text{-int} \qquad \dfrac{\begin{array}{c}\Gamma\\ \Pi\\ A_1 \wedge A_2\end{array}}{A_j}\ \wedge_j\text{-elim}$$

$$\dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \bot\end{array}}{A}\ \bot\text{-elim} \qquad \dfrac{\begin{array}{c}[\neg A^i]\\ \Pi\\ \bot\end{array}}{A^c}\ A^c\text{-int} \qquad \dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2\\ \Pi_1 & \Pi_2\\ A^c & \neg A^i\end{array}}{\bot}\ A^c\text{ - elim}$$

Fig. 1 Ecumenical natural deduction system $\mathrm{NE_B}$

($\Leftarrow$) Suppose $\vDash_S \neg\neg A^i$. Then, for all $S \subseteq S'$, $\neg A^i \vDash_{S'} \bot$. Suppose that for some of those S' it holds that $A^i \vDash_{S'} \bot$. Then we have $\vDash_{S'} \neg A^i$, and thus $\vDash_{S'} \bot$ and $\vdash_{S'} \bot$. Contradiction. Hence $A^i \nvDash_{S'} \bot$ for all $S \subseteq S^n$, and thus $\vDash_S A^c$. □

Theorem 24 *(Soundness) If $\Gamma \vdash_{NE_B} A$ then $\Gamma \vDash A$.*

Proof Due to the collapse between local and global consequence in strong semantics, if we eliminate all clauses for classical formulas and define $A^c = \neg\neg A^i$ we get an equivalent definition. Then, since all the remaining semantic clauses are just Sandqvist's clauses for intuitionistic logic, our proof of soundness follows from his (provided A^c is treated as $\neg\neg A^i$ on induction steps). The only important difference is in the treatment of $\bot - elim$, which is slightly different due to the consistency requirement. The induction hypothesis gives us $\Gamma \vDash \bot$ and thus $\Gamma \vDash_S \bot$ for arbitrary S, from which we conclude that for no $S \subseteq S'$ we have $\vDash_S B$ for all $B \in \Gamma$. But then $\Gamma \vDash_{S'} A$ holds vacuously for all such S', which shows $\Gamma \vDash_S A$ for arbitrary A and arbitrary S and thus $\Gamma \vDash A$. □

5.2 Completeness

We will now prove completeness of the natural deduction system shown in Fig. 1 w.r.t. the strong ecumenical semantics. We use an adaptation of Sandqvist's proof (2015); changes are made only to deal with classical formulas and the consistency constraint.

Lemma 25 *$p \nvdash_S \bot$ iff the system S' obtained by adding a rule which concludes p from empty premises to a consistent S is also consistent.*

Proof This is strengthened syntactic counterpart of Lemma 13.

Assume $p \nvdash_S \bot$. Let S' be the system obtained by adding a rule concluding p from empty premises to S'. Suppose that $\vdash_{S'} \bot$. Then there is a deduction Π in S' showing $\bot$. If it does not use the new rule added to S', Π is also a deduction in S, so S would violate the consistency requirement. If it does use the new rule, by replacing every occurrence of it by an assumption with shape p we get a deduction showing $p \vdash_S \bot$, which contradicts our initial hypothesis. Since a contradiction is obtained in both cases, we conclude $\nvdash_{S'} \bot$

For the other direction, assume the system S' obtained by adding a rule which concludes p from empty premises to S is consistent. Assume $p \vdash_S \bot$. Since $S \subseteq S'$ we have $\vdash_{S'} \bot$, violating the consistency requirement. Thus, $p \nvdash_S \bot$. □

Let Γ^{Sub} be the set of all subformulas of formulas contained in a set Γ. Let $\Delta^c_\Gamma = \{\neg A^i | A^c \in \Gamma\}$. Now, let $\Gamma^\star = ((\Gamma \cup \{A\})^{Sub}) \cup (\Delta^c_{(\Gamma \cup \{A\})^{Sub})})$

We start by producing a mapping α which assigns to each formula A in $\Gamma^\star$ a unique p^A such that:

1. $p^A = q$, if $A = q^i$ (for $q \in \mathrm{At}_\bot$);
2. Else, $p^A \in \mathrm{At}$ and $(p^A)^i \notin \Gamma^\star$.

Notice that, since the assigned atoms are unique, $p^A = p^B$ iff A=B.

Consider now any semantic consequence $\Gamma \vDash A$. Fix any mapping α for $\Gamma^\star$. Notice that, since Γ is finite by definition and there are infinitely many atoms in the language, there are always enough atoms to supply such a mapping. Following Sandqvist's strategy, we start by using the mapping α to build an atomic system $\mathcal{N}$ which is finely tailored for our proof.

We start by defining atomic correspondents of the natural deduction rules (for $i \in \{1, 2\}$):

$$\dfrac{\begin{array}{c}\Gamma\ [p^{A}]\\ \Pi\\ p^{B}\end{array}}{p^{A\rightarrow B}}\ p^{A\rightarrow B}-int \qquad \dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2\\ \Pi_1 & \Pi_2\\ p^{A\rightarrow B} & p^{A}\end{array}}{p^{B}}\ p^{A\rightarrow B}-elim$$

$$\dfrac{\begin{array}{c}\Gamma\\ \Pi\\ p^{A_i}\end{array}}{p^{A_1\vee A_2}}\ p^{A_1\vee A_2}-int \qquad \dfrac{\begin{array}{ccc}\Gamma_1 & \Gamma_2\ [p^{A}] & \Gamma_3\ [p^{B}]\\ \Pi_1 & \Pi_2 & \Pi_3\\ p^{A\vee B} & q & q\end{array}}{q}\ p^{A\vee B}, q-elim$$

$$\dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2\\ \Pi_1 & \Pi_2\\ p^{A} & p^{B}\end{array}}{p^{A\wedge B}}\ p^{A\wedge B}-int \qquad \dfrac{\begin{array}{c}\Gamma\\ \Pi\\ p^{A_1\wedge A_2}\end{array}}{p^{A_i}}\ p^{A_1\wedge_i A_2}-elim$$

$$\dfrac{\begin{array}{c}[p^{\neg A^{i}}]\\ \Pi\\ \bot\end{array}}{p^{A^{c}}}\ p^{A^{c}}-int \qquad \dfrac{\begin{array}{cc}\Gamma_1 & \Gamma_2\\ \Pi_1 & \Pi_2\\ p^{A^{c}} & p^{\neg A^{i}}\end{array}}{\bot}\ p^{A^{c}}-elim \qquad \dfrac{\begin{array}{c}\Gamma\\ \Pi\\ \bot\end{array}}{q}\ \bot, q-elim$$

We now add atomic rules to $\mathcal{N}$ for all formulas $D \in \Gamma^{\star}$, according to the following criteria.

1. For every formula D with shape $A \rightarrow B$ or $A \wedge B$, we add $p^{D}-int$ and $p^{D}-elim$ to $\mathcal{N}$.
2. For every formula D with shape $A \vee B$ we add the rules $p^{D}-int$ to $\mathcal{N}$, and for every D with shape $A \vee B$ and every $q \in \mathrm{At}_{\bot}$ we add $p^{D}, p-elim$ to $\mathcal{N}$.
3. For every formula D with shape A^{c} we add $p^{D}-int$ and $p^{D}-elim$ to $\mathcal{N}$. Notice that, by the definition of Δ^{c}_{Γ} and $\Gamma^{\star}$, if $A^{c} \in (\Gamma \cup \{A\})^{Sub}$ then $\neg A^{i} \in \Gamma^{*}$;
4. For every $q \in \mathrm{At}_{\bot}$ we add $\bot, q-elim$ to $\mathcal{N}$;
5. We also stipulate that $\mathcal{N}$ contains no rules other than those added by this procedure.

Since all atomic systems are now required to be consistent, before using $\mathcal{N}$ in the completeness proof we must prove that it is consistent. One interesting way to do this is by proving *atomic normalization results* for $\mathcal{N}$.

We start by providing some definitions required for the normalization proof.

Definition 10 Rules with shape $p^{A\wedge B}-int$, $p^{A_1\vee A_2}-int$, $p^{A\rightarrow B}-int$ and $p^{A^{c}}-int$ are *introduction rules* of $\mathcal{N}$. Rules with shape $p^{A_1\wedge A_2}-elim$, $p^{A\vee B}$, $q-elim$, $p^{A\rightarrow B}-elim$, $p^{A^{c}}-elim$ and $\bot-elim$ are *elimination rules* of $\mathcal{N}$.

Definition 11 For any rule $p^D - elim$ or $p^D, q - elim$ of $\mathcal{N}$, we say that the atom with shape p^D occurring above it (or the leftmost occurrence if there is more than one)is the rule's *major premise*. The major premise of $\bot, q - elim$ is $\bot$. All other premises are the rule's *minor premises*.

For simplicity, we sometimes refer to the rules of $\mathcal{N}$ simply as $\wedge - int$, $\vee - int$, $\to -int$, $A^c - int$, $\wedge - elim$, $\vee - elim$, $\to -elim$, $A^c - elim$, and $\bot - elim$, omitting further qualifiers where the context makes the meaning clear.

Definition 12 The *length* of a derivation is the number of formula occurrences in it. The *degree* of an atomic formula p^A relative to a previously fixed mapping α of formulas into atoms, denoted by $d[p^A]$, is recursively defined as:

1. $d[p^A] = 0$, if $A = q^i$ for $q \in \mathrm{At}_\bot$;
2. $d[p^{A^c}] = d[p^A] + 2$;
3. $d[p^{\neg A}] = d[p^A] + 1$;
4. $d[p^{A \wedge B}] = d[p^A] + d[p^B] + 1$;
5. $d[p^{A \vee B}] = d[p^A] + d[p^B] + 1$;
6. $d[p^{A \to B}] = d[p^A] + d[p^B] + 1$;

Notice that this is slightly different from Definition 6 because now the degree of classical formulas needs to be the degree of its intuitionistic version plus 2.

Definition 13 A formula occurrence in a derivation Π in $\mathcal{N}$ that is at the same time the conclusion of an application of an introduction rule and the major premise of an elimination rule is said to be a *maximum formula* in Π.

Example 1 The following are examples of maximum formula occurrences:

$$\dfrac{\begin{matrix}\Gamma_1\\ \Pi_1\\ p^A\end{matrix} \qquad \dfrac{\begin{matrix}[p^A]^n \quad \Gamma_2\\ \Pi_2\\ p^B\end{matrix}}{p^{A\to B}}\, p^{A\to B} - int, n}{p^B}\, p^{A\to B} - elim$$

$$\dfrac{\dfrac{\begin{matrix}[p^{\neg A^i}]^n\\ \Pi_1\\ \bot\end{matrix}}{p^{A^c}}\, p^{A^c} - int, n \qquad \begin{matrix}\Pi_2\\ p^{\neg A^i}\end{matrix}}{\bot}\, p^{A^c} - elim$$

Definition 14 A sequence $A_1, A_2, ..., A_n$ of formula occurrences in a deduction is a *thread* if A_1 is a (possibly discharged) assumption, A_m stands immediately below A_{m-1} for every $1 < m \leq n$ and A_n is the conclusion of the deduction.

Definition 15 A sequence $A_1, A_2, ..., A_n$ of consecutive formula occurrences in a thread is a *segment* if and only if it satisfies the following conditions:

1. A_1 is not the consequence of an application of $\vee$-elimination;
2. Each A_m, for $m<n$, is a minor premise of an application of $\vee$-elimination;
3. A_n is not the minor premise of an application of $\vee$-elimination.

The last formula of a segment is called the *vertex* of the segment.

Definition 16 A segment that begins with an application of an introduction rule or the $\perp - elim$ rule and ends with the major premise of an elimination rule is said to be a *maximum segment*.

Observe that maximum formulas are special cases of maximum segments.

Example 2 In the derivation below, the sequence $p^{A_1 \wedge A_2}$, $p^{A_1 \wedge A_2}$ starting with the application of $\wedge$-introduction and ending with an application of $\wedge$-elimination is a *maximum segment*.

$$\dfrac{\begin{matrix}\Gamma_1\\ \Pi_1\\ p^{A\vee B}\end{matrix} \qquad \dfrac{\begin{matrix}[p^A]^n & \Gamma_2\\ \Pi_2 & \\ p^{A_1} & \end{matrix} \qquad \begin{matrix}[p^A]^n & \Gamma_3 \\ \Pi_3 & \\ p^{A_2} & \end{matrix}}{p^{A_1\wedge A_2}}\, p^{A_1\wedge A_2}-int \qquad \begin{matrix}[p^B]^m & \Gamma_4\\ \Pi_4 & \\ p^{A_1\wedge A_2} & \end{matrix}}{\dfrac{p^{A_1\wedge A_2}}{p^{A_1}}\, p^{A_1\wedge A_2}-elim}\, p^{A\vee B}, p^{A_1\wedge A_2}-elim, n, m$$

Definition 17 The *degree* of a derivation Π in $\mathcal{N}$, $d[\Pi]$, is defined as

$$d[\Pi] = \max\{d[p^C] : p^C \text{is a maximum formula or the vertex of a maximum segment in } \Pi\}.$$

We adapt Prawitz's usual proper and permutative reductions for intuitionistic logic (Prawitz, 1965) to the system $\mathcal{N}$. Besides the usual reductions for the operators $\wedge$, $\rightarrow$, $\vee$ and $\neg$, we have a new reduction for maximum formulas of the form A^c:

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}[p^{\neg A^i}]^n\\ \Pi_1\\ \perp\end{matrix}}{p^{A^c}}\, p^{A^c} - int, n \qquad \begin{matrix}\Pi_2\\ p^{\neg A}\end{matrix}}{\perp}\, p^{A^c} - elim \\ \Pi_3\end{matrix} \qquad \text{reduces to} \qquad \begin{matrix}\Pi_2\\ [p^{\neg A^i}]\\ \Pi_1\\ \perp\\ \Pi_3\end{matrix}$$

Definition 18 A derivation Π of $\Gamma \vdash_{\mathcal{N}} A$ is called *critical* if and only if:

1. Π ends with an application β of an elimination rule.
2. The major premise p^B of β is a maximum formula or the vertex of a maximum segment.
3. $d[\Pi] = d[p^B]$.
4. Every other maximum formula or maximum segment in Π has a degree smaller than $d[p^B]$.

The next two lemmas show how derivations can be reduced to lower degrees. The proof is by induction on the length of the derivation (see Appendix A).

Lemma 26 *Let Π be a critical derivation of A from Γ. Then Π reduces to a derivation Π' of A from Γ such that $d[\Pi'] < d[\Pi]$.*

Lemma 27 *Let Π be a derivation of A from Γ in $\mathcal{N}$ such that $d[\Pi] > 0$. Then Π reduces to a derivation Π' of A from Γ in $\mathcal{N}$ such that $d[\Pi'] < d[\Pi]$.*

Proof By induction over the length of Π. We examine two cases depending on the form of the last rule applied in Π.

1. The last rule applied in Π is and introduction rule. The result follows directly from the induction hypothesis.
2. The last rule applied in Π is an elimination rule. Π has the following general form:

$$\frac{\begin{array}{ccc}\Pi_1 & & \Pi_n \\ p_1 & \dots & p_n\end{array}}{p}$$

By the induction hypothesis, each derivation

$$\begin{array}{c}\Pi_i \\ p_i\end{array}$$

$(1 \leq i \leq n)$ reduces to a derivation

$$\begin{array}{c}\Pi_i{}' \\ p_i\end{array}$$

such that $d[\Pi_i'] < d[\Pi_i]$. Let Π^* be:

$$\frac{\begin{array}{ccc}\Pi'{}_1 & & \Pi'{}_n \\ p_1 & \dots & p_n\end{array}}{p}$$

If $d[\Pi^*] < d[\Pi]$, we can take $\Pi' = \Pi^*$. If $d[\Pi^*] = d[\Pi]$, then Π^* is a critical derivation and, by Lemma 26, it reduces to a derivation Π' such that $d[\Pi'] < d[\Pi]$.

□

Theorem 28 *Let Π be a derivation of A from Γ in $\mathcal{N}$. Then Π reduces to a normal derivation Π' of A from Γ in $\mathcal{N}$.*

□

Proof Directly from Lemma 27 by induction on $d[\Pi]$.

Our choice of normalization strategy is incidental—we could just as well have used Prawitz's original strategy or another. The crucial point is that $\mathcal{N}$ satisfies normalization, a property we now leverage to establish its consistency.

Definition 19 Let Π be a derivation in $\mathcal{N}$ and A any formula occurrence in Π. The derivation Π' obtained by removing from Π all formula occurrences except A and those above A is called a *subderivation* of Π.

Lemma 29 *If Π is a normal derivation in $\mathcal{N}$ then all its subderivations are also normal.*

Proof Let Π be a normal derivation and Π' any of its subderivations. It is straightforward to see that if Π' contains a maximal formula or segment then that formula or segment is also maximal in Π, contradicting the assumption that Π was normal. Therefore, no subderivation Π' of Π contains a maximal formula or segment, so every such Π' is normal. □

Lemma 30 *If Π is a normal derivation in $\mathcal{N}$ that does not end with an application of an introduction rule, then Π contains at least one undischarged assumption.*

Proof We prove the result by induction on the length of derivations.

1. Base case: Π has length 1. Then the derivation is just a single occurrence of an assumption p and shows $p \vdash_{\mathcal{N}} p$, so it depends on the undischarged assumption p.
2. Π has length greater than 1 and ends with an application of a elimination rule. Then consider the subderivation Π' of Π which has as its conclusion the major premise of the last rule applied in Π. Since Π' is a subderivation of Π and Π is normal, by Lemma 29 we have that Π' is normal. Notice that, if Π' ended with an application of an introduction rule, since its conclusion is the major premise of an elimination rule there would be a maximum formula in Π, so since Π is normal Π' cannot end with an introduction rule. But then Π' is a deduction with length smaller than that of Π that does not end with an introduction rule, hence by the induction hypothesis it has at least one undischarged assumption. Therefore, since no elimination rule is capable of discharging assumptions occurring above

its major premise, we conclude that the open assumption of Π' are not discharged by the last rule application and so are also open assumptions of Π.

□

Theorem 31 *$\mathcal{N}$ is consistent.*

Proof Assume, for the sake of contradiction, that there is a derivation Π showing $\vdash_{\mathcal{N}} \bot$. By Theorem 28, Π reduces to a normal derivation Π' showing $\vdash_{\mathcal{N}} \bot$. A quick inspection of the shape of introduction rules reveals that no introduction rule can have $p^{\bot^i}(\equiv \bot)$ as its conclusion, hence the last rule of Π' cannot be an introduction rule. But then from Lemma 30 it follows that Π' must have at least one undischarged assumption, hence it cannot be a derivation showing $\vdash_{\mathcal{N}} \bot$. Contradiction. Therefore, $\nvdash_{\mathcal{N}} \bot$. □

We have now established that $\mathcal{N}$ is a valid atomic system, so we can proceed with the completeness proof. Before doing so, we present a lemma traditionally used in BeS completeness proofs—originally proved by Sandqvist in (2015) as Theorem 3.1. In our new framework, this lemma holds in a slightly modified form when considering general bases, but it also retains its original form when restricted to the simulation base.

Lemma 32 *Let $\{p_1, \ldots, p_n\}$ be any set of atoms and S any atomic system. Let q be any atom. Then $\{p_1, \ldots, p_n\} \vDash_S q$ iff either $\{p_1, \ldots, p_n\} \vdash_S q$ or $\{p_1, \ldots, p_n\} \vdash_S \bot$.*

Proof Assume $\{p_1, \ldots, p_n\} \vDash_S q$. Let S' be the set of rules obtained by adding to S a rule with conclusion p_i for each $1 \leq i \leq n$. If S' is inconsistent, there must be a deduction Π showing $\vdash_{S'} \bot$. Notice that, if Π does not use one of the new rules of S', then Π is already a deduction in S and so S is inconsistent, contradicting the assumption that S is an atomic system. Now let Π' be the deduction obtained by replacing every conclusion of one of the rules added to S' with shape p_i by an assumption with shape p_i. Since all new rules added to S to obtain S' are removed by this procedure, we conclude that Π' is a deduction showing $\Gamma \vdash_S \bot$ for some $\Gamma \subseteq \{p_1, \ldots, p_n\}$, which allows us to conclude $\{p_1, \ldots, p_n\} \vdash_S \bot$. If S is consistent, then a single application of each of its new rules yield $\vdash_{S'} p_i$ and thus $\vDash_{S'} p_i$ for all $1 \leq i \leq n$, hence since $\{p_1, \ldots, p_n\} \vDash_S q$ and $S \subseteq S'$ we conclude $\vDash_{S'} q$ and thus $\vdash_{S'} q$. Once again we can replace every new rule of S' with conclusion p_i by an assumption with the same shape and obtain a deduction showing $\Gamma \vdash_S q$ for some $\Gamma \subseteq \{p_1, \ldots, p_n\}$, so it follows that $\{p_1, \ldots, p_n\} \vdash_S q$. Since S' is either consistent or inconsistent, we conclude that either $\{p_1, \ldots, p_n\} \vdash_S q$ or $\{p_1, \ldots, p_n\} \vdash_S \bot$, as desired.

For the converse, assume that either $\{p_1, \ldots, p_n\} \vdash_S q$ or $\{p_1, \ldots, p_n\} \vdash_S \bot$. Take any $S \subseteq S'$ such that $\vDash_{S'} p_i$ for all $1 \leq i \leq n$. Then $\vdash_{S'} p_i$ for all $1 \leq i \leq n$. If $\{p_1, \ldots, p_n\} \vdash_S q$ then by composing the deduction of each p_i from empty premises with the deduction with premises $\{p_1, \ldots, p_n\}$ and conclusion q we obtain a deduction showing $\vdash_{S'} q$, so also $\vDash_{S'} q$ and thus $\{p_1, \ldots, p_n\} \vDash_S q$ by arbitrariness of S'.

If $\{p_1, \ldots, p_n\} \vdash_S \bot$ then by composing the deductions in the same way we obtain a deduction showing $\vdash_{S'} \bot$, meaning that there cannot be any extension S' of S such that $\vDash_{S'} p_i$ for all $1 \leq i \leq n$, so $\{p_1, \ldots, p_n\} \vDash_S q$ holds vacuously.

□

This shows that inclusion of the consistency constraint in the semantics also introduces the possibility of vacuous satisfaction of consequences, which in the case of consequences having sets of atoms in their antecedent boils down to inconsistency of that set with respect to the base. It is also worth remarking that, due to the use of excluded middle in the metalanguage, the proof of this lemma is only constructive when consistency of the extension S' of S is decidable. Although this cannot be guaranteed in general, it can be guaranteed for the particular atomic systems we use at the end step of the completeness proof.

Even though the original version of this result does not hold for bases in general, it holds for simulation bases $\mathcal{N}$ and its extensions:

Lemma 33 *Let $\{p_1, \ldots, p_n\}$ be any set of atoms. Let q be any atom and S any extension of $\mathcal{N}$. Then $\{p_1, \ldots, p_n\} \vDash_S q$ iff $\{p_1, \ldots, p_n\} \vdash_S q$.* □

Proof Assume $\{p_1, \ldots, p_n\} \vDash_S q$. By Lemma 32 we have either $\{p_1, \ldots, p_n\} \vdash_S q$ or $\{p_1, \ldots, p_n\} \vdash_S \bot$. If $\{p_1, \ldots, p_n\} \vdash_S q$ the proof is finished, and if $\{p_1, \ldots, p_n\} \vdash_S \bot$ we can apply $\bot, q - elim$ at the end of the deduction showing this to obtain another deduction showing $\{p_1, \ldots, p_n\} \vdash_S q$, so the proof is finished. The converse is a direct consequence of Lemma 32. □

This means that we are still capable of using the result in its original form in the context of completeness proofs, even though it no longer holds for bases in general.

Our last lemma for completeness is proven by induction on the degree of formulas (see Appendix A).

Lemma 34 *For all $A \in \Gamma^\star$ and all $\mathcal{N} \subseteq S$ it holds that $\vDash_S A$ iff $\vdash_S p^A$.*

Theorem 35 *(Completeness) $\Gamma \vDash A$ implies $\Gamma \vdash_{NE_B} A$.*

Proof Define a mapping α and a system $\mathcal{N}$ for Γ and A as shown earlier. Define a set $\Gamma^{\mathrm{At}_\bot} = \{p^A | A \in \Gamma\}$.

Suppose $\Gamma \vDash A$. By the definition of strong validity, we have $\Gamma \vDash_{\mathcal{N}} A$. Now define $\mathcal{B}$ as the system obtained from $\mathcal{N}$ by adding a rule concluding p^B from empty premises for every $p^B \in \Gamma^{\mathrm{At}_\bot}$.

We split the proof in two cases:

1. $\mathcal{B}$ is consistent. Then it is a valid extension of $\mathcal{N}$. By the definition of $\mathcal{B}$, we have $\vdash_{\mathcal{B}} p^B$ for all $p^B \in \Gamma^{\mathrm{At}_\bot}$. By Lemma 34, for all $B \in \Gamma^\star$ we have that, for any $\mathcal{N} \subseteq S$, $\vdash_S p^B$ iff $\vDash_S B$. Since $\Gamma \subseteq \Gamma^\star$, we conclude $\vDash_{\mathcal{B}} B$ for all $B \in \Gamma$. Since $\Gamma \vDash_{\mathcal{N}} A$ and $\mathcal{N} \subseteq \mathcal{B}$, we also have $\vDash_{\mathcal{B}} A$, and so by another application of

Lemma 34 we conclude $\vdash_{\mathcal{B}} p^A$. Thus, we conclude that there is a deduction Π of p^A in $\mathcal{B}$.

If the deduction does not use any of the rules contained in $\mathcal{B}$ but not in $\mathcal{N}$, Π is a deduction in $\mathcal{N}$, and so $\vDash_{\mathcal{N}} p^A$. If it does use some of the rules, by replacing every new rule concluding p^B by an assumption p^B we obtain a deduction Π' of $\mathcal{N}$ which shows $\Delta \vdash_{\mathcal{N}} p^A$, for some $\Delta \subseteq \Gamma^{\mathrm{At}_\perp}$. Thus, in any case we obtain some deduction showing $\Gamma^{\mathrm{At}} \vdash_{\mathcal{N}} p^A$.

Let Π be the deduction showing $\Gamma^{\mathrm{At}} \vdash_{\mathcal{N}} p^A$ obtained earlier. Define Π' as the deduction obtained by replacing every formula occurrence p^A in Π by A (atoms q occurring on instances of atomic rules for disjunction and $\perp$-elimination which are not mapped to anything by α are not substituted). Since every instance of every atomic rule becomes some instance of a rule in our system of natural deduction, it is straightforward to show by induction on the length of derivations that Π' is a deduction showing $\Gamma \vdash_{\mathrm{NE_B}} A$.

2. $\mathcal{B}$ is inconsistent. Then there is a deduction Π in $\mathcal{B}$ showing $\vdash_{\mathcal{B}} \perp$. If Π does not use any rule contained in $\mathcal{B}$ but not in $\mathcal{N}$, we have $\vdash_{\mathcal{N}} \perp$, contradicting Theorem 31. Then, Π must use some of the new rules. But then we may replace every new rule of $\mathcal{B}$ which concludes p^B by an assumption with shape p^B to obtain a deduction showing $\Delta \vdash_{\mathcal{N}} \perp$ for some $\Delta \subseteq \Gamma^{\mathrm{At}_\perp}$. Define Π' as the deduction obtained by replacing every formula occurrence p^A in Π by A. It is straightforward to show by induction on the length of derivations that Π' is a deduction showing $\Gamma \vdash_{\mathrm{NE_B}} \perp$. As a finishing touch, we apply $\perp - elim$ to obtain a deduction showing $\Gamma \vdash_{\mathrm{NE_B}} A$.

□

Notice that the only non-constructive part of the completeness proof is the step for $A = A^c$ in Lemma 34. This means that by removing A^c from the language we would have a fully constructive proof of completeness for intuitionistic logic.

It is important to remark that our use of the excluded middle in the metalanguage (either $\mathcal{B}$ is consistent or it is inconsistent) is not problematic in this particular instance, in the sense that it does not affect the constructive character of the proof. For the particular extension $\mathcal{B}$ we use during the proof, since $\mathcal{B}$ consists only of logical rules and axioms concluding atoms contained in Γ^{At}, it is straightforward to show that $\vdash_{\mathcal{B}} p^A$ holds if an only if $\Gamma \vdash_{\mathrm{NE_B}} A$, meaning in particular that by putting $A = \perp$ we can show that the consistency of $\mathcal{B}$ is equivalent to the consistency of Γ in $\mathrm{NE_B}$. Naturally, if the same strategy was employed in intuitionistic logic we would use a base $\mathcal{B}$ whose consistency is equivalent to the consistency of Γ in propositional intuitionistic logic, and since the consistency of Γ is decidable for every Γ (especially since Γ is finite) then the consistency of $\mathcal{B}$ is also decidable for every $\mathcal{B}$.

The fact that the inductive steps for constructive proofs only use constructive reasoning but the steps for classical proofs require classical reasoning bears testament to the fact that our definitions indeed capture the meaning of classical and intuitionistic proofs. As such, the ecumenical behavior observed in the metalanguage should be taken as evidence both of the independence between the distinct notions of proof and of their conceptual adequacy.

6 Conclusion

We have proposed a weak and a strong version of BeS for ecumenical systems. While the first helped furthering our understanding concerning the difference between double negations in intuitionistic logic and provability in classical logic, the ecumenical semantics comes into full swing when the strong notion of BeS is provided, since it allows for a new ecumenical natural deduction system which is sound and complete w.r.t. it.

This distinction allows us not only to obtain classical behavior for formulas containing classical atoms and intuitionistic behavior for formulas containing intuitionistic atoms, but also to put on the spotlight basic properties of semantic entailment which are not always evident in traditional semantic analysis. It may also shed light on semantic differences between intuitionistic and classical logics from an even broader perspective.

In the course of this paper we have also shown that it is possible to furnish the absurdity constant $\bot$ with a conceptually adequate and technically sound definition by requiring all systems to be consistent. This can be done in non-ecumenical contexts as well, provided some procedure capable of showing consistency of the syntactic calculus (such as a normalization proof) is available.

There are many ways to further develop this work in the future. First of all, the role of local and global validity in BeS should be better explored, since it opens wide the classical behavior as it appears in other semantic settings for classical logic, e.g., as in Kripke models for classical logic (Ilik et al., 2010). One very interesting step in this direction would be to propose a proof system for our weak version of BeS. Of course, there is the natural question of what would be the BeS proposal for Prawitz's ecumenical system, from which this work took its inspiration but also other ecumenical systems, such as the ones appearing in (Liang & Miller, 2013; Dowek, 2016; Blanqui et al., 2023). Another option would be to investigate new combinations of locally and globally defined connectives for the weak semantics. Finally, it would be interesting to lift this discussion to ecumenical modal logics (Marin et al., 2020).

Appendix A: Detailed proofs of selected results

Proof *of Lemma* 6. Let $\Gamma \Vdash^L_S A$. Then, by the definition of local consequence, for every S' such that $S \subseteq S'$ we have that if $\Vdash^L_{S'} B$ for all $B \in \Gamma$ then $\Vdash^L_{S'} A$. Now, let S' be any extension of S in which for all S'' such that $S' \subseteq S''$, we have $\Vdash^L_{S''} B$ for all $B \in \Gamma$. Consider any such S''. Since $\Gamma \Vdash^L_S A$ holds and S'' is also an extension of S (by transitivity of extension), we have that $\Vdash^L_{S''} B$ for all $B \in \Gamma$ implies $\Vdash^L_{S''} A$. Since S'' is an extension of S', by definition we have $\Vdash^L_{S''} B$ for all $B \in \Gamma$, and thus we have $\Vdash^L_{S''} A$. But this holds for arbitrary S'' extending S', and so for every S'' we have $\Vdash^L_{S''} A$. Since S' is an arbitrary extension of S satisfying the antecedent of Clause 8 of the Definition 7, we conclude $\Gamma \Vdash^G_S A$. □

Proof *of Lemma* 11. It follows from Theorem 5 that $(p^i \Vdash^L_S \bot)$ iff $(p^i \Vdash^G_S \bot)$, since p^i is an intuitionistic formula and $\bot$ is either the intuitionistic $\bot^i$ or the equivalent $\bot^c$.

On the other hand, $(p^i \Vdash^G_S \bot)$ iff $(\Vdash^L_S \neg p^i)$ follows from the clause for implication. Thus we only need to prove that $(p \vdash_S \bot)$ iff $(p^i \Vdash^L_S \bot)$.

($\Rightarrow$) Let $p \vdash_S \bot$. Assume that there is an extension S' of S in which $\Vdash^L_{S'} p^i$. By Clause 1 we have $\vdash_{S'} p$ and, since S' is an extension of S and thus contains all its rules, $p \vdash_{S'} \bot$. By composing both derivations we get $\vdash_{S'} \bot$, which clashes with the consistency requirement. Thus, for all S' extending S we have $\nVdash^L_S p^i$, which together with Clause 4 yields $p^i \Vdash^L_S \bot$.
($\Leftarrow$) Assume $p^i \Vdash^L_S \bot$ and $p \nvdash_S \bot$. Let S' be the system obtained by adding to S only a rule α which concludes p from empty premises. We start by proving that S' is consistent, as thus a valid extension of S in our semantics.

Assume S' is inconsistent, and consider the proof Π of $\bot$ in S'. There are two possibilities:

1. If Π does not use the rule α, then Π is a proof in S. This yields a contradiction, as S must be consistent.
2. If Π uses the rule α, replace each application of α in Π by an assumption p. This immediately yields a derivation showing $p \vdash_S \bot$, contradicting the second initial hypothesis.

We then conclude that S' is consistent. But, given that $\vdash_{S'} p$, we have $\Vdash^L_{S'} p^i$, which can be used together with the assumption $p^i \Vdash^L_S \bot$ to show $\Vdash^L_S \bot$ and thus $\vdash_S \bot$, contradicting the consistency requirement. Thus $p \vdash_S \bot$. □

Proof of Lemma 16 We show the result by induction on the complexity of formulas.

1. $A = p^i$.

   ($\Rightarrow$) If $\Vdash^L_{S^{\bot C}} p^i$ then $\vdash_{S^{\bot C}} p$, and since the deduction showing this is also a deduction in the arbitrary S' we have $\vdash_{S'} p$ and $\Vdash^L_{S'} p^i$.
   ($\Leftarrow$) Assume $\Vdash^L_{S'} p^i$. Then $\vdash_{S'} p$. If we had a deduction showing $p \vdash_{S^{\bot C}} \bot$ then it would also be a deduction showing $p \vdash_{S'} \bot$, which would allow us to compose the deductions to show $\vdash_{S'} \bot$ and obtain a contradiction, hence $p \nvdash_{S^{\bot C}} \bot$. But by the definition of $\bot$-complete extensions we have that $p \nvdash_{S^{\bot C}} \bot$ implies $\vdash_{S^{\bot C}} p$, so we conclude $\Vdash^L_{S^{\bot C}} p^i$.

2. $A = p^c$.

   ($\Rightarrow$) If $\Vdash^L_{S^{\bot C}} p^c$ then $p \nVdash^L_{S^{\bot C}} \bot$, hence definition of $\bot$-complete extensions $\vdash_{S^{\bot C}} p$, so $\Vdash^L_{S'} p^i$ for all $S^{\bot C} \subseteq S'$ and by Theorem 2 and Clause 7 of Definition 7 also $\Vdash^L_{S'} p^c$.
   ($\Leftarrow$) Assume $\Vdash^L_{S'} p^c$ for arbitrary $S^{\bot C} \subseteq S'$. Then $p \nvdash_{S'} \bot$. If $p \vdash_{S^{\bot C}} \bot$ then since $S^{\bot C} \subseteq S'$ we would have $p \vdash_{S'} \bot$ and this would lead to a

contradiction, so $p \nvdash_{S^{\perp C}} \perp$ and by definition of $\perp$-complete extension $\vdash_{S^{\perp C}} p$ and so $\Vdash^{L}_{S^{\perp C}} p^i$, which by Theorem 2 and Clause 7 of Definition 7 yields $\Vdash^{L}_{S^{\perp C}} p^c$.

3. $A = B^C$.

   ($\Rightarrow$) If $\Vdash^{L}_{S^{\perp C}} B^c$ then $B \nVdash^{L}_{S^{\perp C}} \perp$. By Lemma 13 there is a $S^{\perp C} \subseteq S''$ with $\Vdash^{L}_{S''} B^i$. Induction hypothesis: $\Vdash^{L}_{S^{\perp C}} B^i$ and also $\Vdash^{L}_{S'} B^i$ for every $S^{\perp C} \subseteq S'$. Then by Theorem 3 also $\Vdash^{L}_{S'} B^c$ for all $S^{\perp C} \subseteq S'$.
   ($\Leftarrow$) Let $\Vdash^{L}_{S'} B^c$ for arbitrary $S^{\perp C} \subseteq S'$. Then $B^i \nVdash^{L}_{S'} \perp$, hence by Lemma 13 there is a $S' \subseteq S''$ with $\Vdash^{L}_{S''} B^i$. Induction hypothesis: $\Vdash^{L}_{S^{\perp C}} B^i$. Then by Theorem 3 we have $\Vdash^{L}_{S^{\perp C}} B^c$.

4. $A = B \wedge C$.

   ($\Rightarrow$) If $\Vdash^{L}_{S^{\perp C}} B \wedge C$ then $\Vdash^{L}_{S^{\perp C}} B$ and $\Vdash^{L}_{S^{\perp C}} C$. Induction hypothesis: $\Vdash^{L}_{S'} B$ and $\Vdash^{L}_{S'} C$ for all $S^{\perp C} \subseteq S'$. Then $\Vdash^{L}_{S'} B \wedge C$ for arbitrary $S^{\perp C} \subseteq S'$.
   ($\Leftarrow$) Let $\Vdash^{L}_{S'} B \wedge C$ for arbitrary $S^{\perp C} \subseteq S'$. Then $\Vdash^{L}_{S'} B$ and $\Vdash^{L}_{S'} C$. Induction hypothesis: $\Vdash^{L}_{S^{\perp C}} B$ and $\Vdash^{L}_{S^{\perp C}} C$. Then $\Vdash^{L}_{S^{\perp C}} B \wedge C$.

5. $A = B \to C$.

   ($\Rightarrow$) If $\Vdash^{L}_{S^{\perp C}} B \to C$ then $B \Vdash^{G}_{S^{\perp C}} C$. Induction hypothesis: for all $S^{\perp C} \subseteq S'$, $\Vdash^{L}_{S^{\perp C}} B$ iff $\Vdash^{L}_{S'} B$ and $\Vdash^{L}_{S^{\perp C}} C$ iff $\Vdash^{L}_{S'} C$. Then Lemma 15 applies to $S^{\perp C}$ and all its extensions with respect to B and C, so since $B \Vdash^{G}_{S^{\perp C}} C$ we conclude that either $\nVdash^{L}_{S^{\perp C}} B$ or $\Vdash^{L}_{S^{\perp C}} C$. The induction hypothesis then yields $\nVdash^{L}_{S'} B$ or $\Vdash^{L}_{S'} C$ for our chosen S', so by Lemma 15 we have $B \Vdash^{G}_{S'} C$ and thus $\Vdash^{L}_{S'} B \to C$.
   ($\Leftarrow$) Assume $\Vdash^{L}_{S'} B \to C$. Then $B \Vdash^{G}_{S'} C$. Induction hypothesis: for all $S^{\perp C} \subseteq S'$, $\Vdash^{L}_{S^{\perp C}} B$ iff $\Vdash^{L}_{S'} B$ and $\Vdash^{L}_{S^{\perp C}} C$ iff $\Vdash^{L}_{S'} C$. Then again by applying Lemma 15 from $B \Vdash^{G}_{S'} C$ we conclude $\nVdash^{L}_{S'} B$ or $\Vdash^{L}_{S'} C$, from the induction hypothesis we conclude $\nVdash^{L}_{S^{\perp C}} B$ or $\Vdash^{L}_{S^{\perp C}} C$ and by Lemma 15 we conclude $B \Vdash^{G}_{S^{\perp C}} C$, so we conclude $\Vdash^{L}_{S^{\perp C}} B \to C$.

6. $A = B \vee C$.

   ($\Rightarrow$) If $\Vdash^{L}_{S^{\perp C}} B \vee C$ then, for all $S^{\perp C} \subseteq S'$ and all $p \in \mathrm{At}_{\perp}$, if $A \Vdash^{L}_{S'} p^i$ and $B \Vdash^{L}_{S'} p^i$ then $\Vdash^{L}_{S'} p^i$. Now pick any $S^{\perp C} \subseteq S'$, pick any p and let S'' be any extension of S' with $A \Vdash^{L}_{S''} p^i$ and $B \Vdash^{L}_{S''} p^i$. Then since $S' \subseteq S''$ implies $S^{\perp C} \subseteq S''$ we conclude $\Vdash^{L}_{S'} p^i$, so by arbitrariness of S'' we already have $\Vdash^{L}_{S'} B \vee C$.
   ($\Leftarrow$) Assume $\Vdash^{L}_{S'} B \vee C$. Then, for all $p \in \mathrm{At}_{\perp}$, if $A \Vdash^{L}_{S'} p^i$ and $B \Vdash^{L}_{S'} p^i$ then $\Vdash^{L}_{S'} p^i$ (which is a special case of the semantic condition for disjunction). Induction hypothesis: for all $S^{\perp C} \subseteq S''$, $\Vdash^{L}_{S^{\perp C}} B$ iff $\Vdash^{L}_{S''} B$, $\Vdash^{L}_{S^{\perp C}} C$ iff $\Vdash^{L}_{S''} C$ and $\Vdash^{L}_{S^{\perp C}} p^i$ iff $\Vdash^{L}_{S''} p^i$ for all $p \in \mathrm{At}_{\perp}$. Now pick any $S^{\perp C} \subseteq S''$

such that $A \Vdash^L_{S''} p^i$ and $B \Vdash^L_{S''} p^i$ for a particular $p \in \mathrm{At}_\bot$, if any. Then, by the induction hypothesis, since both S' and S'' are extensions of $S^{\bot C}$ we have $\Vdash^L_{S''} B$ iff $\Vdash^L_{S'} B$, $\Vdash^L_{S''} C$ iff $\Vdash^L_{S'} C$ and $\Vdash^L_{S''} p^i$ iff $\Vdash^L_{S'} p^i$, so since $A \Vdash^L_{S''} p^i$ and $B \Vdash^L_{S''} p^i$ we can apply Lemma 15 to conclude based only on the equivalence of B, C and p^i in S'' and S' that $A \Vdash^L_{S'} p^i$ and $B \Vdash^L_{S'} p^i$ also hold, hence since $\Vdash^L_{S'} B \vee C$ we conclude $\Vdash^L_{S'} p^i$. Since $\Vdash^L_{S'} p^i$ by the induction hypothesis we conclude $\Vdash^L_{S''} p^i$, hence for any $S^{\bot C} \subseteq S''$ we have that if $A \Vdash^L_{S''} p^i$ and $B \Vdash^L_{S''} p^i$ for an arbitrary $p \in \mathrm{At}_\bot$ then $\Vdash^L_{S''} p^i$, which yields $\Vdash^L_{S^{\bot C}} B \vee C$.

Proof of Proposition 22 Let $A = (\neg p^i)$, $B = (p^c)$ and $C = (\neg p^i \vee \neg\neg p^i)$. Theorem 20 has already shown that $(A \vee B) = \neg p^i \vee p^c$ is valid in all S, so now we show that this is also the case for $(A \to C) = \neg p \to (\neg p^i \vee \neg\neg p^i)$ and for $(B \to C) = p^c \to (\neg p^i \vee \neg\neg p^i)$.

Let S be any system and S' any extension of it with $\Vdash^L_{S'} \neg p^i$. Since $\neg p^i$ only contains intuitionistic subformulas Theorem 5 shows that it is monotonic, so by Theorem 4 for all $S' \subseteq S''$ we have $\Vdash^L_{S''} \neg p^i$. Then for any $S' \subseteq S''$ and any $q \in \mathrm{At}_\bot$ if both $\neg p^i \Vdash^L_{S''} q^i$ and $\neg\neg p^i \Vdash^L_{S''} q^i$ we can combine $\neg p^i \Vdash^L_{S''} q^i$ with $\Vdash^L_{S''} \neg p^i$ to obtain $\Vdash^L_{S''} q^i$, so $\Vdash^L_{S'} \neg p^i \vee \neg\neg p^i$. Since S' in arbitrary extension of S with $\Vdash^L_{S'} \neg p^i$ we conclude $\neg p^i \Vdash^L_S \neg p^i \vee \neg\neg p^i$ and thus also $\neg p^i \Vdash^G_S \neg p^i \vee \neg\neg p^i$ by Lemma 6, so we conclude $\Vdash^L_S \neg p^i \to (\neg p^i \vee \neg\neg p^i)$.

Now let S be any system and S' any extension of it with $\Vdash^L_{S''} p^c$ for every $S' \subseteq S''$. Then for every such S'' we have $p \nvdash_{S''} \bot$. Assume that for some S'' we have $\Vdash^L_{S''} \neg p^i$. Then by Theorem 11 we have $p \vdash_{S''} \bot$, which yields a contradiction. Hence, for all such S'' we have $\nVdash^L_{S''} \neg p^i$, so by Lemma 8 we get $\Vdash^L_{S''} \neg\neg p^i$. Now pick any such S'' and consider a $S'' \subseteq S'''$ with $\neg p^i \Vdash^L_{S'''} q^i$ and $\neg\neg p^i \Vdash^L_{S'''} q^i$ for some $q \in \mathrm{At}_\bot$. Since $\neg\neg p^i$ only contains intuitionistic subformulas we conclude by Theorem 5 that it is monotonic. Since $S'' \subseteq S'''$ and $\Vdash^L_{S''} \neg\neg p^i$, by Theorem 4 we have $\Vdash^L_{S'''} \neg\neg p^i$, hence $\neg\neg p^i \Vdash^L_{S'''} q^i$ and so $\Vdash^L_{S'''} q^i$. From this we conclude that $\Vdash^L_{S''} \neg p^i \vee \neg\neg p^i$. But notice that S'' is an arbitrary extension of S', so we conclude that if $\Vdash^L_{S''} p^c$ for every $S' \subseteq S''$ then $\Vdash^L_{S''} \neg p^i \vee \neg\neg p^i$ for all $S' \subseteq S''$, and by arbitrariness of S' we conclude $p^c \Vdash^G_S \neg p^i \vee \neg\neg p^i$, and thus $\Vdash^L_S p^c \to (\neg p^i \vee \neg\neg p^i)$.

We have shown that $\Vdash^L_S A \vee B$, $\Vdash^L_S A \to C$ and $\Vdash^L_S B \to C$ hold in any S for our choice of A, B and C. Therefore, it suffices to show for some particular S that $\nVdash^L_S \neg p^i \vee \neg\neg p^i$ (that is, $\nVdash^L_S C$) to prove the desired result. But since both $\neg p^i$ and $\neg\neg p^i$ are purely intuitionistic formulas their semantics is identical to intuitionistic BeS, so we can simply point out that $\neg p^i \vee \neg\neg p^i$ is not an intuitionistic theorem to conclude the desired result. In particular, $\nVdash^L_\emptyset \neg p^i \vee \neg\neg p^i$ (as only intuitionistic theorems hold in the empty system), so we have $(A \vee B), (A \to C), (B \to C) \nVdash^G_\emptyset C$ and also $(A \vee B), (A \to C), (B \to C) \nVdash C$. $\square$

Proof of Lemma 26 Induction over the length of Π. There are two cases to be examined depending on whether $d[\Pi]$ is determined by a maximum formula or by the vertex of a maximum segment.

Case 1: $d[\Pi]$ is determined by a maximum formula. The result follows directly from the application of a reduction to this maximum formula.

1. The critical derivation Π is:

$$\cfrac{\cfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ p^{A_1} & p^{A_2}\end{array}}{p^{A_1\wedge A_2}}\; p^{A_1\wedge A_2}-int}{p^{A_i}}\; p^{A_1\wedge A_2}-elim$$

We know that $d(\Pi)=d[p^{A_1\wedge A_2}]>d[\Pi_i]$ (for $i\in\{1,2\}$). Π reduces to

$$\begin{array}{c}\Pi_i\\ p^{A_i}\end{array}$$

And the degree of this derivation is equal to $d[\Pi_i]$ which is smaller than $d[\Pi]$.

2. The critical derivation Π is:

$$\cfrac{\begin{array}{c}\Gamma_1\\ \Pi_1\\ p^A\end{array} \qquad \cfrac{\begin{array}{c}[p^A]^n \quad \Gamma_2\\ \Pi_2\\ p^B\end{array}}{p^{A\to B}}\; p^{A\to B}-int,n}{p^B}\; p^{A\to B}-elim$$

Π reduces to the following derivation Π':

$$\begin{array}{c}\begin{array}{cc}\begin{array}{c}\Gamma_1\\ \Pi_1\\ [p^A]\end{array} & \Gamma_2\end{array}\\ \Pi_2\\ p^B\end{array}$$

We can easily see that $d[\Pi']\leq max\{d[\Pi_1],d[\Pi_2]\ d[p^A]\}<d[\Pi]=d[p^{A\to B}]$.

3. The critical derivation Π is:

$$\dfrac{\dfrac{\begin{matrix}\Gamma\\ \Pi\\ p^{A_i}\end{matrix}}{p^{A_1 \vee A_2}}\, p^{A_1 \vee A_2} - int \qquad \begin{matrix}[p^{A_1}]^n & \Gamma_1\\ \Pi_1 & \\ q & \end{matrix} \qquad \begin{matrix}[p^{A_2}]^m & \Gamma_2\\ \Pi_2 & \\ q & \end{matrix}}{q}\, p^{A_1 \vee A_2}, q - elim, n, m$$

Π reduces to the following derivation Π':

$$\begin{matrix}\Gamma & \\ \Pi & \\ [p^{A_i}] & \Gamma_i\\ \Pi_i & \\ q & \end{matrix}$$

We can easily see that $d[\Pi'] \leq max\{d[\Pi_1], d[\Pi_2], d[\Pi], d[p^{A_i}]\} < d[\Pi] = d[p^{A_1 \vee A_2}]$.

4. The critical derivation Π is:

$$\dfrac{\dfrac{\begin{matrix}[p^{\neg A^i}]^n\\ \Pi_1\\ \bot\end{matrix}}{p^{A^c}}\, p^{A^c} - int, n \qquad \begin{matrix}\Pi_2\\ p^{\neg A^i}\end{matrix}}{\bot}\, p^{A^c} - elim$$

Π reduces to the following derivation Π':

$$\begin{matrix}\Pi_2\\ [p^{\neg A^i}]\\ \Pi_1\\ \bot\end{matrix}$$

We can easily see that $d[\Pi'] \leq max\{d[\Pi_1], d[\Pi_2], d[p^{\neg A}]\} < d[\Pi] = d[p^{A^c}]$.[2].

5. The critical derivation was obtained through an application of $\bot - elim$. Then:

[2] This is the step in which we cannot use Definition 6.

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \bot\end{matrix}}{p^{A_1\wedge A_2}}\ \bot, p^{A_1\wedge A_2}-elim}{p^{A_i}}\ p^{A_1\wedge A_2}-elim \qquad \text{reduces to} \qquad \dfrac{\begin{matrix}\Pi_1\\ \bot\end{matrix}}{p^{A_i}}\ \bot, p^{A_i}-elim$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \bot\end{matrix}}{p^{A\vee B}}\ \bot, p^{A\vee B}-elim \qquad \begin{matrix}[p^A]^n\\ \Pi_2\\ q\end{matrix} \qquad \begin{matrix}[p^B]^m\\ \Pi_3\\ q\end{matrix}}{q}\ p^{A\vee B}, q-elim \qquad \text{reduces to} \qquad \dfrac{\begin{matrix}\Pi_1\\ \bot\end{matrix}}{q}\ \bot, q-elim$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \bot\end{matrix}}{p^{A\to B}}\ \bot, p^{A\to B}-elim \qquad \begin{matrix}\Pi_2\\ p^A\end{matrix}}{p^B}\ p^{A\to B}-elim \qquad \text{reduces to} \qquad \dfrac{\begin{matrix}\Pi_1\\ \bot\end{matrix}}{p^B}\ \bot, p^B-elim$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \bot\end{matrix}}{p^{A^c}}\ \bot, p^{A^c}-elim \qquad \begin{matrix}\Pi_2\\ p^{\neg A^i}\end{matrix}}{\bot}\ p^{A^c}-elim \qquad \text{reduces to} \qquad \begin{matrix}\Pi_1\\ \bot\end{matrix}$$

In all cases it is straightforward to check that the degree of the derivation is reduced.

Case 2: $d[\Pi]$ is determined by the vertex of a maximal segment. Π is:

$$\dfrac{\dfrac{\begin{matrix}\Gamma_1\\ \Pi_1\\ p^{A\vee B}\end{matrix} \qquad \begin{matrix}[p^A]^n & \Gamma_2\\ \multicolumn{2}{c}{\Pi_2}\\ \multicolumn{2}{c}{q}\end{matrix} \qquad \begin{matrix}[p^B]^m & \Gamma_3\\ \multicolumn{2}{c}{\Pi_3}\\ \multicolumn{2}{c}{q}\end{matrix}}{q}\ p^{A\vee B}, q-elim, n, m \qquad \begin{matrix}\Sigma_1\\ \Delta_1\\ r_1\end{matrix} \quad \dots \quad \begin{matrix}\Sigma_k\\ \Delta_k\\ r_k\end{matrix}}{r}$$

By means of a permutative reduction, Π reduces to the following derivation Π^*:

$$\dfrac{\begin{matrix}\Gamma_1\\ \Pi_1\\ p^{A\vee B}\end{matrix} \qquad \dfrac{\begin{matrix}[p^A]^n & \Gamma_2\\ \multicolumn{2}{c}{\Pi_2}\\ \multicolumn{2}{c}{q}\end{matrix} \quad \begin{matrix}\Sigma_1\\ \Delta_1\\ r_1\end{matrix} \quad \dots \quad \begin{matrix}\Sigma_k\\ \Delta_k\\ r_k\end{matrix}}{r} \qquad \dfrac{\begin{matrix}[p^B]^m & \Gamma_3\\ \multicolumn{2}{c}{\Pi_3}\\ \multicolumn{2}{c}{q}\end{matrix} \quad \begin{matrix}\Sigma_1\\ \Delta_1\\ r_1\end{matrix} \quad \dots \quad \begin{matrix}\Sigma_k\\ \Delta_k\\ r_k\end{matrix}}{r}}{r}\ p^{A\vee B}, r-elim, n, m$$

Without loss of generality, we can assume that the two derivations of the minor premises of the application of $\vee$-elimination are critical. By the induction hypothesis, they reduce to derivations

$$\dfrac{\begin{matrix}\Gamma_1\\ \Pi_1\\ p^{A\vee B}\end{matrix} \qquad \dfrac{\begin{matrix}[p^A]^n & \Gamma_2\\ \multicolumn{2}{c}{\Pi_2}\\ \multicolumn{2}{c}{q}\end{matrix} \quad \begin{matrix}\Sigma_1\\ \Delta_1\\ r_1\end{matrix} \quad \dots \quad \begin{matrix}\Sigma_k\\ \Delta_k\\ r_k\end{matrix}}{r} \qquad \dfrac{\begin{matrix}[p^B]^m & \Gamma_3\\ \multicolumn{2}{c}{\Pi_3}\\ \multicolumn{2}{c}{q}\end{matrix} \quad \begin{matrix}\Sigma_1\\ \Delta_1\\ r_1\end{matrix} \quad \dots \quad \begin{matrix}\Sigma_k\\ \Delta_k\\ r_k\end{matrix}}{r}}{r}\ p^{A\vee B}, r-elim, n, m$$

such that $d[\Pi_2'] < d[\Pi]$ and $d[\Pi_3'] < d[\Pi]$. We can then take Π' to be:

$$\dfrac{\begin{matrix}\Gamma_1\\ \Pi_1\\ p^{A\vee B}\end{matrix} \qquad \begin{matrix}[p^A]^n & \Gamma_2'\\ \multicolumn{2}{c}{\Pi_2'}\\ \multicolumn{2}{c}{r}\end{matrix} \qquad \begin{matrix}[p^B]^m & \Gamma_3'\\ \multicolumn{2}{c}{\Pi_3'}\\ \multicolumn{2}{c}{r}\end{matrix}}{r}\ p^{A\vee B}, r-elim, n, m$$

□

Proof of Lemma 34 We show the result by induction on the degree of formulas.

1. $A = p^i$, for some $p \in \mathrm{At}_\bot$. Then $p^A = A$, and the result follows immediately from Clause 1 of strong validity;
2. $A = A \wedge B$.

   (⇒) Assume $\vDash_S A \wedge B$. Then $\vDash_S A$ and $\vDash_S B$. Induction hypothesis: $\vdash_S p^A$ and $\vdash_S p^B$. By $p^{A\wedge B} - int$, we obtain $\vdash_S p^{A\wedge B}$.
   (⇐) Assume $\vdash_S p^{A\wedge B}$. Then, by $p^{A\wedge B} - elim$ we get both $\vdash_S p^A$ and $\vdash_S p^B$. Induction hypothesis: $\vDash_S A$ and $\vDash_S B$. Then, by the semantic clause for conjunction, $\vDash_S A \wedge B$.

3. $A = A \vee B$.

   (⇒) Assume $\vDash_S A \vee B$. Let S' be any extension of S with $\vDash_{S'} A$. The induction hypothesis yields $\vdash_{S'} p^A$, and by applying one of the rules for $p^{A\vee B} - int$ we get $\vdash_{S'} p^{A\vee B}$ and so $\vDash_{S'} p^{A\vee B}$. By arbitrariness of S' we conclude $A \vDash_S p^{A\vee B}$. An analogous argument establishes $B \vDash_S p^{A\vee B}$. Since $\vDash_{S'} A \vee B$, $B \vDash_S p^{A\vee B}$ and $B \vDash_S p^{A\vee B}$ we obtain $\vDash_S p^{A\vee B}$ by the clause for disjunction, so also $\vdash_S p^{A\vee B}$.
   (⇐) Assume $\vdash_S p^{A\vee B}$. Let S' be any extension of S with both $A \vDash_{S'} q^i$ and $B \vDash_{S'} q^i$ for some $q \in \mathrm{At}_\bot$. Let S'' be an extension of S' with $\vDash_{S''} p^A$. Then $\vdash_{S''} p^A$, so the induction hypothesis yields $\vDash_{S''} A$. Since $A \vDash_{S'} q^i$ and $\vDash_{S''} A$ we conclude $\vDash_{S''} q^i$, hence by arbitrariness of S'' also $p^A \vDash_{S'} q^i$. An analogous argument establishes $p^B \vDash_{S'} q^i$. Lemma 33 yields $p^A \vdash_{S'} q$ and $p^B \vdash_{S'} q$. Since $\vdash_S p^{A\vee B}$ we also have $\vdash_{S'} p^{A\vee B}$, so using the deduction of this atom together with the deductions showing $p^A \vdash_{S'} q$ and $p^B \vdash_{S'} q$ we can obtain a deduction showing $\vdash_{S'} q$ by applying $p^{A\vee B}, q - elim$, hence $\vDash_{S'} q^i$. Since S' was an arbitrary extension of S with $A \vDash_{S'} q^i$ and $B \vDash_{S'} q^i$ for arbitrary $q \in \mathrm{At}_\bot$ and we have shown $\vDash_{S'} q^i$ we conclude $\vDash_S A \vee B$ by the clause for disjunction.

4. $A = A \to B$.

   (⇒) Assume $\vDash_S A \to B$. Then, for any $S \subseteq S'$, $A \vDash_{S'} B$. Let S' be an extension of S with $\vDash_{S'} p^A$. Then $\vdash_{S'} p^A$, so the induction hypothesis yields $\vDash_{S'} A$, hence $\vDash_{S'} B$ and thus $\vdash_{S'} p^B$ by another application of the induction hypothesis, whence $\vDash_{S'} p^B$. By arbitrariness of S' we conclude $p^A \vDash_S p^B$, so Lemma 33 yields $p^A \vdash_S p^B$. We can then use the rule $p^{A\to B} - int$ to conclude $\vdash_S p^{A\to B}$.
   (⇐) Assume $\vdash_S p^{A\to B}$. Let S' be any extension of S with $\vDash_{S'} A$. The induction hypothesis yields $\vdash_{S'} p^A$. Since $\vdash_S p^{A\to B}$ we also have $\vdash_{S'} p^{A\to B}$, and since $\vdash_{S'} p^A$ we can use both deductions to obtain a deduction showing

$\vdash_{S'} p^B$ through an application of $p^{A \to B} - elim$. The induction hypothesis yields $\vDash_{S'} B$, hence by arbitrariness of S' we conclude $A \vDash_S B$ and so $\vDash_S A \to B$.

5. $A = A^c$.

   ($\Rightarrow$) Assume $\vDash_S A^c$.
   Then, for any $S \subseteq S'$, $A^i \nvDash_{S'} \bot$. For the sake of contradiciton, assume that for some $S \subseteq S'$ we have $p^{A^i} \vdash \bot$. Let S'' be an extension of S' such that $\vDash_{S''} A^i$. The induction hypothesis yields $\vdash_{S''} p^{A^i}$, so by composition of deductions we obtain $\vdash_{S''} \bot$ and thus $\vDash_{S''} \bot$, whence by arbitrariness of S'' also $A^i \vDash_{S'} \bot$. This contradicts the fact that $A^i \nvDash_{S'} \bot$, so we conclude $p^{A^i} \nvdash_{S'} \bot$ for all $S \subseteq S'$, hence it also follows by Lemma 25 that, for every $S \subseteq S'$, the system obtained by adding a rule concluding p^{A^i} from empty premises to S' is consistent.
   Assume, for the sake of contradiction, that $p^{\neg A^i} \nvdash_S \bot$. Then, by Lemma 25, the system S' obtained by adding a rule concluding $p^{\neg A^i}$ from empty premises to S is consistent. But by the previous result we also have that the system S'' obtained by adding a rule concluding p^{A^i} from empty premises to S' must be consistent. However, since $\vdash_{S''} p^{A^i}$ and $\vdash_{S''} p^{\neg A^i}$, we can apply the $p^{\neg A^i} - elim$ rule to show $\vdash_{S''} p^{\bot}$, and thus $\vdash_{S''} \bot$ due to the properties of the mapping α. Contradiction. Thus, $p^{\neg A^i} \vdash_S \bot$, and so $\vdash_S p^{A^c}$ can by obtained trough an application of $p^{A^c} - int$.
   ($\Leftarrow$) Assume $\vdash_S p^{A^c}$. Suppose there is an $S \subseteq S'$ such that $p^A \vdash_{S'} p^{\bot}$. Then, by $p^{\neg A^i} - int$ we conclude $\vdash_{S'} p^{\neg A^i}$ and, since $S \subseteq S'$ and thus $\vdash_{S'} p^{A^c}$, we conclude $\vdash_{S'} p^{\bot}$ through an application of $p^{A^c} - elim$, and thus $\vdash_{S'} \bot$. Contradiction. Hence, for all $S \subseteq S'$ we have $p^{A^i} \nvdash_{S'} \bot$. Induction hypothesis: for all $S \subseteq S'$ it holds that $A^i \nvDash_{S'} \bot$, which by the clauses for classical formulas yield $\vDash_S A^c$. □

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Declarations

Competing interests The authors declare no conflicts of interest.

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