Invariant Sets and Stability in Dynamical Systems Applied to Theoretical Ecology and Population Genetics

by

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Declaration

I, Hamid Naderi Yeganeh confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

Dynamical systems play a crucial role in studying various aspects of ecology and population genetics. Among discrete dynamical systems, different instances of Kolmogorov maps have been widely used to model populations of organisms and predict their long-term behavior. Two well-known Kolmogorov maps in theoretical ecology are the Leslie-Gower model and the Ricker model, while the Stein-Ulam spiral map has been applied in population genetics. In this thesis, I establish a theoretical framework that resolves a conjecture stating that the local stability of the interior fixed point of the Ricker map implies the global stability of the system, within a specific range of parameters. As a second topic, I prove the convexity of carrying simplices in the logarithmically scaled Leslie-Gower and Ricker models for a range of parameters. Finally, I develop a method that provides deeper insight into the behavior of the Stein-Ulam spiral map near its boundary.

Impact Statement

This thesis makes theoretical contributions to the study of dynamical systems applied to ecology and population genetics by advancing our understanding of invariant sets, global stability, and complex behaviors through discrete models. By rigorously analyzing and extending properties of Kolmogorov maps—particularly the Leslie-Gower and Ricker models—the research enhances mathematical insight in a range of models ecologists and biologists deal with in their research.

One of the central achievements of this work is a contribution to resolve a conjecture regarding the Ricker model. Specifically, the thesis demonstrates that under certain conditions, the local stability of the interior fixed point of the planar Ricker map implies global stability. By developing a novel framework based on forward-invariant sets that don't intersect problematic regions, the research overcomes previous obstructions and broadens the known parameter ranges where global stability can be guaranteed.

In another development, the thesis proves the convexity of carrying simplices in the logarithmic coordinates of the Leslie-Gower and Ricker models for a range of parameters. Carrying simplices are essential invariant sets that encapsulate long-term population behavior. Establishing their convexity strengthens our understanding of the geometry of ecological phase spaces.

A third major contribution is the derivation of an approximation formula for the Stein-Ulam spiral map, which arises in population genetics. By analyzing its boundary behavior in depth, the thesis opens a path for better understandings and more accurate predictions in genetic dynamics.

Overall, this work provides applicable mathematical tools that could support future research in areas such as ecology, population genetics and possibly other fields that apply similar models.

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Chapter 1

Introduction

1.1 Introduction to Dynamical Systems

Dynamical systems theory is a fundamental area of mathematics that studies systems which evolve over time according to specific rules. These systems arise naturally in a wide range of disciplines, including physics, biology, economics, engineering, and computer science. At its core, a dynamical system describes how a point in a given space moves with time, where the evolution is determined by a fixed rule.

The behavior of such systems can vary greatly, from very predictable and stable to highly complex and chaotic. Analyzing the long-term behavior of these systems provides profound insights into the structure and nature of temporal processes, stability, periodicity, bifurcations and chaos.

Broadly speaking, dynamical systems can be classified into two major types based on the nature of time: **continuous dynamical systems** and **discrete dynamical systems**.

1.1.1 Continuous Dynamical Systems

In continuous dynamical systems, the evolution of the system occurs continuously over time. The state of the system is typically described by a set of ordinary differential equations (ODEs) or, in more complex settings, by partial differential equations (PDEs). The solutions to these equations define the trajectories (also called orbits) of the system in a continuous time framework.

Mathematically, an autonomous continuous dynamical system on a set X is defined as a function $\phi : \mathbb{R} \times X \to X$ such that for all $x \in X$ and $t, s \in \mathbb{R}$:

$$\phi(0, x) = x \quad \text{and} \quad \phi(t + s, x) = \phi(t, \phi(s, x)).$$

Such systems are particularly useful in modeling physical systems where time changes smoothly, such as planetary motion and fluid dynamics.

1.1.2 Discrete Dynamical Systems

In contrast, discrete dynamical systems evolve in discrete steps, typically indexed by the integers $\mathbb{Z}$ or natural numbers $\mathbb{N}$. Instead of differential equations, such systems are governed by recurrence relations. A discrete dynamical system is characterized by the repeated application of a function $f : X \to X$ to an initial point x_0, producing a sequence:

$$\begin{aligned} &x_0 \\ &x_1 = f(x_0) \\ &x_2 = f(f(x_0)) \\ &\quad\vdots \\ &x_n = f^n(x_0) \\ &\quad\vdots \end{aligned}$$

Discrete systems naturally arise in areas where time progresses in steps, such as population models in ecology, economic systems measured over fiscal periods, computational algorithms, and digital signal processing.

One of the key interests in studying discrete dynamical systems is understanding the asymptotic behavior of orbits: whether they converge to fixed points, enter periodic cycles, or exhibit chaotic dynamics.

A key distinction between autonomous discrete and continuous dynamical systems lies in the nature of their evolution rules. In an autonomous continuous-time system, defined by a flow $\phi : \mathbb{R} \times X \to X$, one can derive an infinite family of discrete systems by fixing a time step h and defining the map $f_h : X \to X$ via $f_h(x) = \phi(h, x)$. However, the reverse is not necessarily possible: not every discrete dynamical system arises as the time-h map of a continuous-time flow. A necessary condition for this to hold is that the discrete system be invertible.

This distinction underscores the richness of discrete systems. While an infinite number of discrete systems can be extracted from a single continuous-time system, discrete systems—despite their seemingly simpler formulation—can encode and express dynamics of far greater complexity.

1.1.3 Focus of the Thesis

The focus of this thesis is on **discrete dynamical systems**, with sole attention to a class of maps known as **Kolmogorov maps**. These maps arise naturally in the modeling of population dynamics within ecological systems, where the evolution of species populations is described in discrete time steps. Kolmogorov maps provide a powerful framework for studying how populations interact, stabilize, fluctuate, or collapse over time under various biological and environmental conditions. In this thesis, we explore some different aspects of the theory of Kolmogorov maps. This includes an analysis of stability criteria, the existence and nature of invariant sets and some other important properties.

1.1.4 Kolmogorov Maps

A special class of discrete dynamical systems that plays an important role in both theoretical and applied contexts is that of **Kolmogorov maps**. These maps are named after Andrey Kolmogorov (1903 – 1987), who introduced such models in the context of theoretical population dynamics. Formally, a map $K : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is called a **Kolmogorov map** if it is of the form:

$$K(x) = (x_1 f_1(x),\, x_2 f_2(x),\, \ldots,\, x_n f_n(x)),$$

where each $f_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ is a continuously differentiable function that satisfies $f_i(x) > 0$ for all $x \in \mathbb{R}^n_+$. Therefore, the orbit of the discrete dynamical system defined by this map with the initial condition $\mathbf{x_0} \in \mathbb{R}^n_+$ is as follows

$$\mathbf{x_{n+1}} = K(\mathbf{x_n}), \qquad n = 0, 1, 2, ...$$

In this context, $\mathbb{R}_+$ denotes the set of all non-negative real numbers. Each component x_i typically represents the population of the i-th species, and $f_i(x)$ is the growth rate of the species.

Kolmogorov maps possess the important property that the origin is always a fixed point, i.e. $K(0) = 0$. The long-term dynamics are governed by the interaction of growth functions f_i, leading to phenomena such as existence of positive equilibria, periodic orbits, and the emergence of invariant sets known as carrying simplices.

A famous and simple example of a Kolmogorov Map is the Lotka–Volterra predator–prey model one of whose versions is given in [44]:

$$\begin{cases} x_{n+1} = x_n(a - by_n) \\ y_{n+1} = y_n(c + dx_n) \end{cases}$$

where a, b, c, d are constants describing the interaction of the two species. The Kolmogorov maps we will discuss in this thesis are the Leslie-Gower map, the Ricker map and the Stein-Ulam spiral map.

1.2 Theoretical framework versus Numerical Methods in Dynamical Systems

The study of dynamical systems often requires a delicate balance between rigorous theoretical analysis and computational numerical methods. In contrast to the analysis of real functions $f : \mathbb{R} \to \mathbb{R}$ defined by basic analytic expressions—where both theoretical and numerical tools are so well-developed that few problems fall outside their scope—the field of dynamical systems sometimes presents challenges that remain beyond the reach of both current theory and computation.

Theoretical and numerical approaches each offer distinct insights and capabilities. Their relative advantages depend on the specific context, the nature of the system under investigation, and the objectives of the researcher. This section briefly discusses the complementary roles, strengths, and limitations of these two approaches in the study of both discrete and continuous dynamical systems.

While the primary focus of this thesis is on theoretical analysis, it is worth noting that the author's master's dissertation concentrated on numerical methods in dynamical systems. Furthermore, Chapter 4 presents an effort to bridge the two methodologies. For these reasons, a brief comparison of the two approaches is both relevant and timely.

1.2.1 The Role and Strength of Theoretical Analysis

Theoretical analysis in dynamical systems focuses on the derivation of precise results that are universally valid under clearly defined assumptions, particularly for long-term behaviors.

It offers deep qualitative insights into the nature of long-term behavior, such as convergence of trajectories, the existence and stability of equilibria, periodic orbits, bifurcations, and invariant structures like attractors, carrying simplices and ω-limit sets. The key strength of theory lies in its ability to provide general proofs and guarantees, including:

- **Exact statements and universality:** Theorems can establish the existence and other properties of fixed points, periodic orbits, bifurcations, invariant sets, etc. for broad classes of systems and wide range of parameter values rigorously.

- **Global insights:** Concepts such as topological properties, Lyapunov functions and invariant manifolds enable the understanding of the whole phase-space rather than only short-term behavior of individual trajectories. Theoretical framework is much stronger in the establishment of these studies than the numerical methods.

- **Dealing with the cases when it is impossible to use numerical methods:** When a system is extremely sensitive to its initial conditions and the errors of computational methods grow extraordinarily, theoretical analysis is the only practical tool to study such cases.

Despite its power, theoretical analysis is often limited by tractability. Many realistic models, particularly nonlinear and high-dimensional ones, resist comprehensive theoretical analysis and leave a lot of open problems to researchers. This is where numerical methods play a crucial complementary role.

1.2.2 The Power and Flexibility of Numerical Methods

Numerical methods provide a practical and flexible way to explore dynamical systems' short-term behavior and also long-term properties with the help of theoretical frameworks. These methods rely on discretizing time or space and approximating the evolution of the system using algorithms implemented in software. Key benefits include:

- **Practicality in the industry and other real-life computational uses:** While it may take time to develop rigorous theories on applied models, the industry can use computational evaluations derived from numerical methods as an already available tool, though those methods' ability might be limited due to sensitivity of the models.
- **Exploration of complex dynamics:** Numerical simulations allow for the experimental visualization and investigation of chaotic behavior, strange attractors, and fractal boundaries that are analytically elusive. This is especially useful to develop theoretical conjectures.
- **High-dimensional systems:** Numerical tools can handle systems with a lot of dimensions, which are common in real-world applications like weather prediction.

However, numerical methods are often developed with support from theoretical analysis to some extent. Estimating the errors of numerical methods requires theoretical insights, and more specialized methods may rely on properties proven within theoretical frameworks. For example, if a numerical method is designed under the assumption that a certain set is convex, then theorems identifying the structure of the system and the valid range of parameters become necessary.

1.2.3 Key Differences

While theoretical and numerical methods serve different purposes, they are not mutually exclusive. Rather, they reinforce each other. Their major differences and complementarities can be summarized as follows:

Table 1.1: Comparison of theoretical analysis and numerical methods in dynamical systems

Aspect	**Theoretical Analysis**	**Numerical Methods**
Scope	General properties across parameter ranges particularly for long-term behaviors	Specific behaviors for selected initial conditions and parameters particularly for short-term behaviors
Nature of Output	Abstract results, qualitative theorems	Quantitative simulations and approximations
Limitations	Intractable for many nonlinear systems, leaving a lot of open problems.	Sensitive to discretization error and may be computationally expensive. Sometimes, even relies on the development of a theoretical framework.
Typical Use	Providing foundational insight to mathematicians and researchers of other fields by studying global structure and long-term behaviors in a precise way, validation of numerical observations, supporting development of numerical methods.	Industrial and other real-life computational uses, visualization, developing and testing hypotheses.

1.2.4 The Bridges Between the Two Approaches

In many modern research programs, particularly in applied dynamical systems, the workflow begins with numerical simulations to suggest conjectures or phenomena, followed by theoretical work to confirm and generalize the observed behavior. Alternatively, theoretical results may guide the design of new numerical methods for special and even general purposes.

In the chapter of this thesis that develops a theory for the Stein-Ulam spiral map, first, we develop an approximation formula with specified errors. Then we will use that approximation formula to prove a topological property of the system. This use of an approximation formula in a theoretical analysis, reveals a deeper connection between the theoretical and numerical frameworks.

1.3 Long-Term Behavior in Discrete Dynamical Systems

The long-term behavior of iterates in a dynamical system is of central interest in both theoretical and applied settings. In discrete dynamical systems, understanding how orbits evolve under repeated application of a map sheds light on the system's global structure, stability properties, and potential for complex dynamics such as periodicity or chaos.

Let $f : X \to X$ be a map on a metric space X, and let $x_0 \in X$ be an initial point. The sequence $\{x_n\}_{n=0}^{\infty}$ defined recursively by $x_{n+1} = f(x_n)$ for all $n \in \mathbb{N}$ is called the **orbit** of x_0.

The primary goal of dynamical systems theory is to describe the qualitative and geometric properties of these orbits, particularly as $n \to \infty$.

1.3.1 Fixed Points and Stability

A point $x^* \in X$ is a **fixed point** of f if $f(x^*) = x^*$. Fixed points serve as potential forward or backward long-term destinations of orbits and are often associated with equilibrium states in models of physical, biological, or economic systems.

A fixed point x^* is said to be **locally stable** if all nearby points stay close to x^* under iteration. More formally, x^* is **Lyapunov stable** if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\|x_0 - x^*\| < \delta$, then $\|f^n(x_0) - x^*\| < \varepsilon$ for all $n \geq 0$.

A fixed point is **asymptotically stable** if it is Lyapunov stable and attracts nearby points; that is, there exists $\delta > 0$ such that if $\|x_0 - x^*\| < \delta$, then $\lim_{n\to\infty} f^n(x_0) = x^*$.

1.3.2 Periodic Orbits

A point $x \in X$ is **periodic** of period $p \in \mathbb{N}$ if $f^p(x) = x$ and $f^k(x) \neq x$ for all $0 < k < p$. The set $\{x, f(x), f^2(x), \ldots, f^{p-1}(x)\}$ is called a *periodic orbit*. Periodic orbits generalize fixed points (which are simply period-1 orbits) and are foundational in the classification of dynamical behavior.

1.3.3 ω-Limit Sets

To capture the asymptotic behavior of orbits that may not settle into fixed or periodic states, we define the **ω-limit set** of a point $x \in X$ as:

$$\omega_f(x) := \left\{ y \in X \mid \exists \{n_k\}_{k=1}^{\infty} \text{ with } n_k \to \infty \text{ and } f^{n_k}(x) \to y \right\}.$$

That is, $\omega_f(x)$ consists of all accumulation points of the forward orbit $\{x_n\}_{n=0}^{\infty}$. ω-limit sets generalize the notion of convergence and include fixed points, periodic orbits, or more exotic invariant sets such as strange attractors.

1.3.4 Invariant Sets

A subset $A \subset X$ is said to be **forward-invariant** under f if $f(A) \subset A$. If $f(A) = A$, we say that A is **strictly invariant** or simply we call it **invariant**. Invariant sets are central objects in the structural analysis of dynamical systems. Examples include stable manifolds, attractors, and carrying simplices.

In particular, when A is compact and invariant, and every orbit starting nearby remains close to A for all time, A is often called an **attractor**. If every point in a neighborhood of A is eventually attracted to it, then A is a **global attractor** for that neighborhood.

1.3.5 Basins of Attraction

Given an asymptotically stable fixed point x^*, the **basin of attraction** of x^* is the set:

$$\mathcal{B}(x^*) := \left\{ x \in X \;\middle|\; \lim_{n\to\infty} f^n(x) = x^* \right\}.$$

Basins of attraction may have smooth or fractal boundaries. Their structure is crucial in understanding the global phase portrait of dynamical systems.

1.3.6 Interpretations in Applied Discrete Models

In discrete models such as those in population dynamics, epidemiology, or economic cycles, these concepts enable predictions of qualitative outcomes. For instance, whether a population stabilizes, oscillates, or collapses depends on whether the corresponding state is a stable fixed point, a periodic orbit, or part of a chaotic attractor. For example, if a population or economic model is globally stable, with orbits converging to a fixed point, it means the fate of the population or economy is a permanent fixed balance among the components of the model.

Furthermore, ω-limit sets provide a way to analyze systems where exact convergence does not occur, but the behavior remains confined. While the system is not entirely predictable in such scenarios, it may still be quite useful to know which set of points the values are limited to.

1.4 Historical and Biological Motivation

The origins of mathematical ecology can be traced to the early 20th century, when Alfred J. Lotka [36] and Vito Volterra [55] independently formulated continuous-time predator–prey systems. Their work laid the groundwork for the usage of both continuous and discrete time dynamical systems in ecological models.

To determine whether a continuous or discrete dynamical system is more appropriate, scientists consider properties such as whether a species has overlapping or non-overlapping breeding patterns. An overlapping generation is a mating system in which individuals of different generations do breed simultaneously. Non-overlapping generation usually happens in species that can only experience one breeding season. For example some fish and bee species do have such a property. In ecological systems with non-overlapping generations it is more natural to consider discrete-time models.

In the 1940s, P. Leslie [25] [26] introduced age-classified projection matrices to model populations of species such as flour beetle and brown rat. In this method, the population was divided into age groups by using a matrix. This then led to the introduction of the Leslie–Gower map in a work [27] co-authored with P. H. Gower.

The Ricker model as another famous population model proposed by William E. Ricker in 1954 [46], emerged from the need to predict stock and recruitment in fisheries. As of 2025, The Ricker model is still a source of many challenging open problems.

In 1950s at Los Alamos National Laboratory, Stanisław Ulam, Paul Stein and Mary Tsingou Menzel [54] conducted pioneering research on quadratic maps applied to population genetics. Among the 97 maps studied, one exhibited remarkable distinct dynamics, which later termed the Stein–Ulam Spiral Map. This discrete dynamical system models interactions among three competing populations. While early studies conjectured that all non-trivial interior trajectories' ω-limit sets are the same and equal to the whole boundary, later work by Barański and Misiurewicz [8] disproved this, showing that any admissible subset of the boundary could arise as an ω-limit set.

Through these historical developments, two mathematical properties have had pivotal roles:

1. **Stability.** While local stability analyses rely on eigenvalues of the Jacobian at fixed points, global stability seeks to guarantee convergence of all concerned initial conditions. Bridging the gap between local linear criteria and global attractivity has remained a central challenge.

2. **Carrying simplices.** A carrying simplex is a compact, attracting, invariant manifold that is homeomorphic to the standard simplex. The presence of such a structure is considered highly advantageous, as it greatly simplifies the analysis of a system's global dynamics.

Beyond these notions rooted in dynamical systems theory, studying topological properties of sets of interest is also important. For example, proving the local connectedness of the Mandelbrot set—a problem known as the MLC conjecture—is considered one of the most significant open problems in complex dynamics. Thus, exploring topological properties remains a key area of interest.

1.5 Structure and Contributions of the Thesis

This thesis is organized into five chapters, including the current introduction chapter. Below is a summary of the subsequent chapters and their key contents.

- **Chapter 2** addresses a conjecture regarding the planar Ricker model. By introducing novel forward-invariant sets, this chapter extends the known region in the parameter space for which local stability implies global stability.

- **Chapter 3** establishes the convexity of non-compact carrying simplices in logarithmic coordinates for the Leslie-Gower and Ricker models. This result offers new geometric insights into the phase space of competitive ecological systems and strengthens our theoretical understanding of invariant manifolds.
- **Chapter 4** focuses on the Stein-Ulam spiral map, a three-dimensional system arising in population genetics. The chapter presents an approximation formula that nicely estimates the boundary behavior of orbits. This formula facilitates deeper topological analysis of the system and lays the groundwork for future studies.
- **Chapter 5** concludes the thesis by summarizing the main results, discussing their implications in broader scientific contexts, and proposing directions for future research.

Collectively, the chapters demonstrate that Kolmogorov-type maps—when analyzed using invariant sets, Lyapunov functions, and transformation techniques—can yield rigorous results about global stability, long-term behavior, and topological features of interest. The tools developed in this thesis can be applied beyond ecological and genetic systems, potentially impacting fields such as epidemiology, mathematical neuroscience, and theoretical economics.

Chapter 2

New Global Stability Results for the Planar Ricker Model

The planar Ricker map, $(x, y) \mapsto \left(xe^{r-x-\alpha y}, ye^{s-y-\beta x}\right)$, is a non-injective map on the first quadrant arising in the study of population dynamics. It has been conjectured that if a positive fixed point of the planar Ricker map is locally asymptotically stable, then it must also be globally asymptotically stable. In a significant development, Baigent et al. [6] introduced a new Lyapunov function and demonstrated that when the parameters satisfy $0 \leq r, s \leq 2$, every orbit of the Ricker map converges to a fixed point. However, beyond this parameter range, the same Lyapunov function fails to satisfy the conditions needed for a direct proof of global stability. In this chapter, we develop a theory that couples the use of certain forward-invariant sets with the same Lyapunov function for the purpose of proving global asymptotic stability of the planar Ricker map for a range of positive parameters outside of $[0, 2]^2$.[1]

2.1 Introduction

In this chapter, we consider the following form of the planar Ricker map:

$$F(x, y) := (F_1(x, y), F_2(x, y)) := \left(xe^{r-x-\alpha y}, ye^{s-y-\beta x}\right), \quad (x, y) \in \mathbb{R}_+^2 \tag{2.1}$$

where $r, s, \alpha, \beta > 0$ and $\mathbb{R}_+ := [0, +\infty)$. For convenience, we adopt the following notation from [6]:

$$X := r - x - \alpha y,$$

[1] The content of this chapter has been submitted as a manuscript by Hamid Naderi Yeganeh and Steve Baigent, entitled *"New Global Stability Results for the Planar Ricker Model"*, to the *Journal of Difference Equations and Applications.*

$$Y := s - y - \beta x.$$

So the map can also be expressed as $F(x, y) = \big(xe^X, ye^Y\big)$.

Until recently, global stability results for the planar Ricker model were limited to some special cases (e.g. [33, 38, 7, 34]), but then global stability for all $\alpha, \beta > 0$ and $0 \le r, s \le 2$ was established in 2023 by way of a Lyapunov function [6]. While a significant improvement on previous results, the Lyapunov function approach of [6] still leaves open the important question:

> *Does local asymptotic stability of a positive fixed point of the planar Ricker model guarantee its global asymptotic stability?*

(It is known, and we show here, that there are locally stable positive fixed points outside of $0 \le r, s \le 2$). The key problem when studying (2.1) outside of the parameter range $0 \le r, s \le 2$ is that on the axis the dynamics is no longer globally stable, but exhibits cycles or chaos [41]. Thus we do not expect the Lyapunov function of [6] to work on all of $\mathbb{R}^2_+$. Here we use the same Lyapunov function of [6] but limit its application to a selection of forward-invariant subsets of the whole plane. Perhaps surprisingly, we find that a positive fixed point may be globally asymptotically stable (on the interior of the first quadrant) even though the boundary behaviour may be periodic.

2.2 Background on the Ricker map

The Ricker map was introduced by William Ricker in 1954 to model the expected number of individuals in the next generation as a unimodal function of the current generation. It is an example of a model of growth of a pioneer species [11] in which small populations increase and sufficiently large populations decrease.

A standard form of the Ricker model for a single species with population size x_t at generation t is given by specifying the population x_{t+1} at generation $t + 1$ via

$$x^{t+1} = x^t e^{r-x^t}, \quad t = 1, 2, \dots, \quad x_0 \text{ given}, \tag{2.2}$$

where $r > 0$. The graph of $x \mapsto e^{r-x}$ is strictly decreasing and $x \to xe^{r-x}$ is unimodal with value 0 where $x = 0$ and $x \to \infty$ with a single maximum at $x = 1$. Setting $F(x) = xe^{r-x}$ we may understand the Ricker model as the result of repeated composition of the map F: $x_t = F^t(x_0)$ where F^t means the $t-$fold composition of F.

Despite the apparently simple functional form of F, the dynamics of (2.2) can be highly complex, as demonstrated by Robert May [40, 41] who showed that as r increases the positive fixed point loses stability at $r = 2$ and thereafter the model embarks on a period-doubling route to chaos.

The functional form of the single-species Ricker map can also be found within models involving more species (for example [1, 32, 30]). The n species Ricker model takes the standard form

$$x_i^{t+1} = x_i^t e^{r_i - A_i \cdot x}, \ i = 1, \dots, n, \ t = 0, 1, \dots \tag{2.3}$$

where $x = (x_1, \dots, x_n)$ is the $n-$vector of species populations, $r_i > 0$ and each A_i is a positive $n-$vector with $A_{ii} = 1$. Notice that when all but one species, say species i, is absent, the dynamics is given by the single-species Ricker equation for species i: $x_i^{t+1} = x_i^t e^{r_i - x_i^t}$. Hence we expect periodic solutions and chaos in Ricker models with multiple species. The n species Ricker model has attracted much attention in the study of global asymptotic stability [5], exclusion principles [1] and permanence [32, 29], but the picture is not complete even for 2 and 3 species models [48]. One complicating feature of the Ricker map is that it is not injective, and is only competitive (or rather retrotone [49, 43] in the sense of monotone systems theory) in a bounded set that is determined by parameter values. When the map is competitive on a compact global attractor the existence of a carrying simplex can help classify global dynamics [28, 30], especially in 3 species models [28].

Ricker-type interactions have also been studied within hybrid models that include multiple functional forms for species growth. One model in particular is the model studied by Franke and Yakubu [15, 14] and more recently by Cheng et al. in [9] in which one species functional form is Ricker-like, whereas the other functional form is of type Beverton-Holt.

The planar Ricker model that we focus on here has been studied by many authors, including cases where parameters are time-dependent. Early investigations were by Hassell and Comins [31], May [41] and later [37], [38, 50, 51, 7] and most recently [34, 6]. As mentioned above, to date [6] has the most complete dynamical picture for the planar Ricker model and our purpose here is to expand the region of parameter values where the planar Ricker model can be proved to possess a globally attracting positive fixed point.

2.3 Narrowing down parameter ranges for asymptotic stability

We use the following notation for the second iterate of the map (2.1):

$$F(F(x,y)) := F^2(x,y) := (F_1^2(x,y), F_2^2(x,y)). \tag{2.4}$$

and define the following sets which are subsets of images of $\mathbb{R}_+^2$ under F and F^2

$$F(\mathbb{R}_+^2) := \{F(x,y)|\ (x,y) \in \mathbb{R}_+^2\},$$

$$A_x^{(1)} := \left\{(u,v) \in \mathbb{R}_+^2 | F_1(u,v) \geq u\right\}, \qquad A_y^{(1)} := \left\{(u,v) \in \mathbb{R}_+^2 | F_2(u,v) \geq v\right\},$$

$$A_x^{(2)} := \left\{(u,v) \in \mathbb{R}_+^2 | F_1^2(u,v) \geq u\right\}, \qquad A_y^{(2)} := \left\{(u,v) \in \mathbb{R}_+^2 | F_2^2(u,v) \geq v\right\}.$$

See Figure 2.1 for illustrative examples of these sets.

In addition, we define the sets $A_{x+}^{(1)}, A_{y+}^{(1)}, A_{x+}^{(2)}, A_{y+}^{(2)}$ by replacing "$\geq$" with "$>$" in the definitions of $A_x^{(1)}, A_y^{(1)}, A_x^{(2)}, A_y^{(2)}$ respectively and similarly define $A_{x-}^{(1)}, A_{y-}^{(1)}, A_{x-}^{(2)}, A_{y-}^{(2)}$ by replacing "$\geq$" with "$<$".

Each of the sets we have just defined are bounded. That $F(\mathbb{R}_+^2)$ is bounded, and in fact compact, has been established by several authors [7, 6]. $A_y^{(1)}$ consists of points in $\mathbb{R}_+^2$ lying on or below the line $Y = 0$. As the line $Y = 0$ has negative slope, the region below it has bounded intersection with $\mathbb{R}_+^2$. For $A_y^{(2)}$, we have

$$(x,y) \in A_y^{(2)} \iff y \leq F_2^2(x,y) = ye^{2s-y(e^Y+1)-\beta x(e^X+1)}$$

$$\iff 2s - y(e^Y+1) - \beta x(e^X+1) \leq 0$$

This inequality corresponds to the condition that the midpoint between (x, y) and $F(x, y)$,

$$\left(\frac{x+xe^X}{2}, \frac{y+ye^Y}{2}\right),$$

lies below the line $Y = 0$. As $F(x, y) \in F(\mathbb{R}_+^2)$, which is compact, for only a bounded set of points from $\mathbb{R}_+^2$, the middle point between the point and its first iteration can lie below the line $Y = 0$. Thus $A_y^{(2)}$ is bounded. $F(\mathbb{R}_+^2)$ is actually the region on or below the graph of a decreasing function [7, 6]. That function, which may be convex or concave (see Figure 2.1 (a), (b)), is the image of a curve on which the Jacobian of (2.1) vanishes. See [7] for the formulas for these curves. See also [45] for discussion on the curves in logarithmic coordinates and their convexity and invariance.

Here we focus on the map (2.1) for parameter values where it has exactly four fixed points

$$\begin{aligned} \mathbf{c}_1 &:= (0,0), \\ \mathbf{c}_2 &:= (r,0), \\ \mathbf{c}_3 &:= (0,s), \\ \mathbf{c}_4 &:= \left(\tfrac{r-s\alpha}{1-\alpha\beta}, \tfrac{s-r\beta}{1-\alpha\beta}\right), \end{aligned} \tag{2.5}$$

with $\mathbf{c}_4$ always assumed to be asymptotically stable and located in the interior of $\mathbb{R}_+^2$.

Among these fixed points, $\mathbf{c}_4$ is the only one that guarantees existence of both species. Hence it is called the coexistence fixed point. In cases where exactly four fixed points exist in $\mathbb{R}_+^2$, the following inequalities have been provided by [38] and [6] as necessary and sufficient conditions

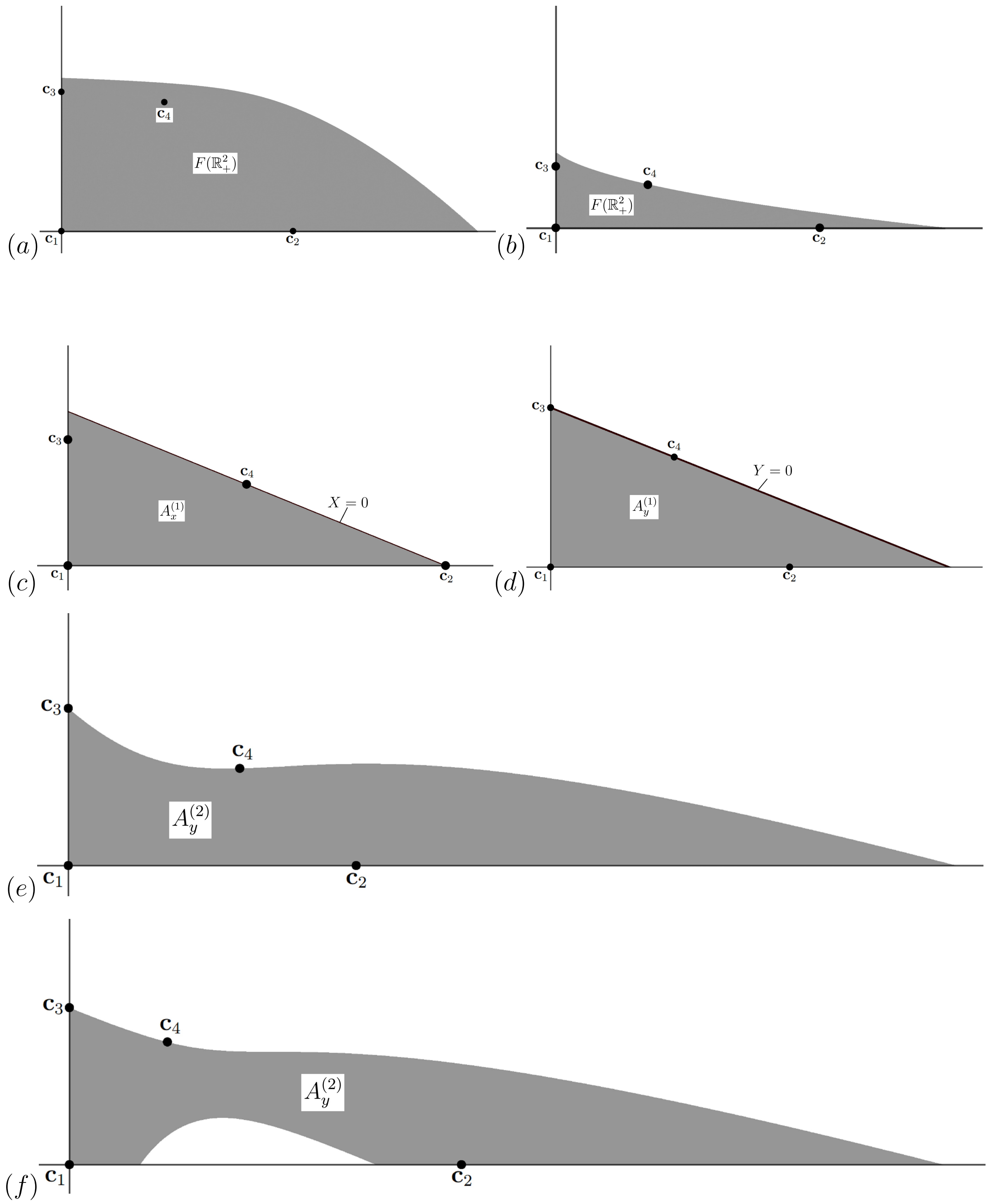

Figure 2.1: (a) $F(\mathbb{R}^2_+)$ for $r = 2.5, s = 1.5, \alpha = 1, \beta = 0.1$, with the graph of a concave function as boundary (b) $F(\mathbb{R}^2_+)$ for $r = 2.15, s = 0.5, \alpha = 4, \beta = 0.2$, with the graph of a convex function as boundary (c) $A_x^{(1)}$ for $r = 3, s = 1, \alpha = 2.45, \beta = 0.25$, (d) $A_y^{(1)}$ for $r = 2.4, s = 1.6, \alpha = 1.05, \beta = 0.4$, (e) $A_y^{(2)}$ for $r = 2.2, s = 1.2, \alpha = 1.2, \beta = 0.35$, (f) $A_y^{(2)}$ with a cavity between $\mathbf{c}_1$ and $\mathbf{c}_2$ for $r = 3, s = 1.2, \alpha = 2.4, \beta = 0.35$.

for the asymptotic stability of $\mathbf{c}_4$:

$$4(\alpha\beta - 1) + 2(r - \alpha s) + 2(s - \beta r) < (r - \alpha s)(s - \beta r) < (r - \alpha s) + (s - \beta r). \tag{2.6}$$

It is also known (e.g. [38]) that if $\mathbf{c}_4$ is asymptotically stable, then $\alpha\beta < 1$. This fact can be easily observed by the above inequalities: Since $\mathbf{c}_4$ is located in the interior of $\mathbb{R}^2_+$, we have

$$\frac{r - \alpha s}{1 - \alpha\beta} > 0, \quad \frac{s - \beta r}{1 - \alpha\beta} > 0. \tag{2.7}$$

Therefore, $r - \alpha s$ and $s - \beta r$ must have the same sign, which means

$$(r - \alpha s)(s - \beta r) > 0.$$

From (2.6) we then have $0 < (r - \alpha s) + (s - \beta r)$. As $(r - \alpha s) + (s - \beta r) < 2(r - \alpha s) + 2(s - \beta r)$ and using (2.6) again, we deduce $4(\alpha\beta - 1) < 0$. This implies $\alpha\beta < 1$.

It has been shown by Baigent et al. [6] that if $0 < r, s < 2$, (and $\alpha, \beta > 0$) then the asymptotic stability of $\mathbf{c}_4$ implies its global asymptotic stability. A natural question is whether this equivalence between local and global stability of $\mathbf{c}_4$ holds for all $\alpha, \beta, r, s > 0$ for which $\mathbf{c}_4$ is asymptotically stable. This was, in fact, conjectured as an open problem in [13].

Our first result shows that there is a significant restriction on the values of r and s when we expect the map to have an asymptotically stable interior fixed point.

Lemma 2.1. *Suppose that for given $\alpha, \beta, r, s > 0$ the Ricker map (2.1) has a unique fixed point $\mathbf{c}_4$ in the interior of $\mathbb{R}^2_+$ which is asymptotically stable. Then r and s cannot both be greater than or equal to 2.*

Proof. From $1 - \alpha\beta > 0$ and (2.7) we can also deduce

$$r - \alpha s > 0, \quad s - \beta r > 0.$$

Now let us put $x = r - \alpha s,\ y = s - \beta r,\ p = \frac{y}{x}$. Then we can rewrite the first inequality in (2.6) as

$$4(\alpha\beta - 1) + (2 + 2p)x < px^2,$$

which is equivalent to $px^2 - (2+2p)x + 4(1-\alpha\beta) > 0$. The polynomial $px^2 - (2+2p)x + 4(1-\alpha\beta)$ has the following roots:

$$\frac{2 + 2p \pm \sqrt{(2 + 2p)^2 - 16p(1 - \alpha\beta)}}{2p}.$$

We have

$$(2+2p)^2 - 16p(1-\alpha\beta) > (2+2p)^2 - 16p = 4 - 8p + 4p^2 = (2-2p)^2 \geq 0.$$

Therefore, the polynomial has exactly two distinct real and positive roots. Now because the polynomial begins with px^2 and $p > 0$, it is negative if and only if x lies between these roots. So since the polynomial must have a positive value two cases are possible:

($\star$) $\frac{2+2p+\sqrt{(2+2p)^2-16p(1-\alpha\beta)}}{2p} < x$,

($\star\star$) $x < \frac{2+2p-\sqrt{(2+2p)^2-16p(1-\alpha\beta)}}{2p}$.

Consider ($\star$). From the second inequality in (2.6) we have $px^2 < x + px$, which implies $x < \frac{1+p}{p}$. But we also have

$$\frac{1+p}{p} < \frac{2+2p+\sqrt{(2+2p)^2-16p(1-\alpha\beta)}}{2p} < x, \tag{2.8}$$

which is a contradiction. Hence ($\star$) is impossible.

Now let us consider ($\star\star$). We have

$$\begin{aligned} & x < \frac{1+p-\sqrt{(1+p)^2-4p(1-\alpha\beta)}}{p} \\ \Longrightarrow\ & \frac{x(1+p+\sqrt{(1+p)^2-4p(1-\alpha\beta)})}{1-\alpha\beta} < \frac{(1+p)^2-(1+p)^2+4p(1-\alpha\beta)}{p(1-\alpha\beta)} = 4. \end{aligned} \tag{2.9}$$

Now suppose for the sake of contradiction that r and s are both greater than or equal to 2. Note that we have

$$r = \frac{x+\alpha y}{1-\alpha\beta}, \quad s = \frac{y+\beta x}{1-\alpha\beta}. \tag{2.10}$$

So $r, s \geq 2$ implies

$$x \geq \frac{2(1-\alpha\beta)}{1+\alpha p}, \quad x \geq \frac{2(1-\alpha\beta)}{p+\beta}. \tag{2.11}$$

From the two inequalities in (2.11) we have

$$\frac{x(1+p+\sqrt{(1+p)^2-4p(1-\alpha\beta)})}{1-\alpha\beta} \geq \frac{2(1+p+\sqrt{(1+p)^2-4p(1-\alpha\beta)})}{1+\alpha p} \tag{2.12}$$

and

$$\frac{x(1+p+\sqrt{(1+p)^2-4p(1-\alpha\beta)})}{1-\alpha\beta} \geq \frac{2(1+p+\sqrt{(1+p)^2-4p(1-\alpha\beta)})}{p+\beta}. \tag{2.13}$$

Combining with (2.9) and simplifying yields

$$\sqrt{(1+p)^2-4p(1-\alpha\beta)}-(1+(2\alpha-1)p)<0, \tag{2.14}$$
$$\sqrt{(1+p)^2-4p(1-\alpha\beta)}-(p+2\beta-1)<0. \tag{2.15}$$

If we denote

$$a:=\sqrt{(1+p)^2-4p(1-\alpha\beta)}, \quad b_1:=1+(2\alpha-1)p, \quad b_2:=p+2\beta-1,$$

then we can rewrite the above inequalities as follows

$$a-b_1<0, \qquad a-b_2<0.$$

However, we have

$$\begin{aligned}(a-b_1)(a+b_1)&=(1+p)^2-4p(1-\alpha\beta)-(1+(2\alpha-1)p)^2\\&=(1-p)^2+4p\alpha\beta-((1-p)+2\alpha p)^2,\\&=4\alpha p(\beta-1+p(1-\alpha)),\end{aligned}$$

and

$$\begin{aligned}(a-b_2)(a+b_2)&=(1+p)^2-4p(1-\alpha\beta)-(p+2\beta-1)^2,\\&=(1-p)^2+4p\alpha\beta-(p-1+2\beta)^2\\&=-4\beta(\beta-1+p(1-\alpha)).\end{aligned}$$

If $\beta-1+p(1-\alpha)\neq 0$, then at least one of the expressions $4\alpha p(\beta-1+p(1-\alpha))$ or $-4\beta(\beta-1+p(1-\alpha))$ is positive. Hence $(a-b_1)(a+b_1)>0$ or $(a-b_2)(a+b_2)>0$. Without loss of generality we can suppose $(a-b_1)(a+b_1)>0$. Hence $a-b_1$ and $a+b_1$ are both positive or negative. But they cannot be both negative since $a>0$. Therefore $a-b_1$ and $a+b_1$ are both positive. But it contradicts $a-b_1<0$ and therefore, $\beta-1+p(1-\alpha)\neq 0$ is impossible.

Now consider $\beta-1+p(1-\alpha)=0$. Note that the set of points (x,y) satisfying (2.6) is open, so if it is nonempty, then it cannot be just a subset of the line $\beta-1+p(1-\alpha)=\beta-1+\frac{x}{y}(1-\alpha)=0$. Therefore, the set is empty. □

2.4 Some useful forward-invariant sets

In this section, we identify two forward-invariant sets which attract all interior orbits. If such sets lie completely inside the region where $\Delta V(\mathbf{x}):=V(F(\mathbf{x}))-V(\mathbf{x})$ is negative for a Lyapunov function V, then we later prove that the region where ΔV vanishes attracts all positive orbits.

Lemma 2.2. *The following set is forward-invariant under the map F:*

$$B_1 := \left\{(x,y) \in F(\mathbb{R}_+^2) |\ y \geq \gamma_1\right\} \tag{2.16}$$

where

$$\gamma_1 := \inf\left\{y \in \mathbb{R}_+ |\ \exists x \in \mathbb{R}_+ \ \ s.t.\ \ (x,y) \in F\left(F(\mathbb{R}_+^2) - A_y^{(1)}\right)\right\}. \tag{2.17}$$

Proof. It is clear that $F(B_1) \subset F(\mathbb{R}_+^2)$. So to prove $F(B_1) \subset B_1$ it is sufficient to prove that for every $(x,y) \in F(B_1)$ we have $y \geq \gamma_1$. Suppose for the sake of contradiction that $(x,y) \in F(B_1)$ but $y < \gamma_1$. Based on the definition of γ_1, (x,y) cannot be a member of $F\left(F(\mathbb{R}_+^2) - A_y^{(1)}\right)$. So if for $(x_0,y_0) \in B_1$ we have $F(x_0,y_0) = (x,y)$, then $(x_0,y_0) \notin F(\mathbb{R}_+^2) - A_y^{(1)}$. This along with $(x_0,y_0) \in F(\mathbb{R}_+^2)$ implies $(x_0,y_0) \in A_y^{(1)}$ and means that $y = F_2(x,y) \geq y_0$. Now since $(x_0,y_0) \in B_1$ we have $y_0 \geq \gamma_1$. Hence $y \geq \gamma_1$ which is a contradiction. $\square$

Now we prove that in addition to being forward-invariant, B_1 also attracts all positive orbits.

Lemma 2.3. *Suppose that $\{\mathbf{x}_n\}$ is an interior orbit of (2.1). Then for every $\varepsilon > 0$ there exists an $N \geq 0$ such that for every $n > N$ we have $\mathbf{B}_\varepsilon(\mathbf{x}_n) \cap B_1 \neq \emptyset$, where $\mathbf{B}_\varepsilon(\mathbf{x}_n)$ is the open ball of radius ε and center $\mathbf{x}_n$.*

Proof. For $n = 1, 2, ...$ we have $\mathbf{x}_n = F(\mathbf{x}_{n-1}) \in F(\mathbb{R}_+^2)$. So only two cases are possible:

(i) There exists an $m \geq 1$ such that $\mathbf{x}_m \in F(\mathbb{R}_+^2) - A_y^{(1)}$.

(ii) For every $n = 1, 2, ...$ we have $\mathbf{x}_n \in F(\mathbb{R}_+^2) \cap A_y^{(1)}$.

If (i) is true, then by definition, $\mathbf{x}_{m+1} = F(\mathbf{x}_m) \in B_1$. So $\mathbf{x}_{m+1}$ is a point in the forward-invariant set B_1 which means that for every $n \geq m+1$ we have $\mathbf{x}_n \in B_1$. Therefore, if for $\varepsilon > 0$ we choose $N = m+1$, then for every $n \geq N$ we have $\emptyset \neq \{\mathbf{x}_n\} \subset \mathbf{B}_\varepsilon(\mathbf{x}_n) \cap B_1$.

For (ii), since $(x_n, y_n) := \mathbf{x}_n \in F(\mathbb{R}_+^2) \cap A_y^{(1)}$ for every $n = 1, 2, ...$, the definition of $A_y^{(1)}$ implies that the bounded sequence $\{y_n\}$ is increasing, and so converges to a y^*. It can be easily proven that $y^* \geq \gamma_1$.

Let $\varepsilon > 0$. We prove that there exists $\rho_\varepsilon > 0$ such that if $(x,y) \in F(\mathbb{R}_+^2)$ and $y > \gamma_1 - \rho_\varepsilon$, then $(x,y) \in \bigcup_{\mathbf{x}' \in B_1} \mathbf{B}_\varepsilon(\mathbf{x}')$.

Assume for the sake of contradiction that for every $\rho > 0$ there exists $(x_\rho, y_\rho) \in F(\mathbb{R}_+^2)$ with $y_\rho > \gamma_1 - \rho$ and $(x_\rho, y_\rho) \notin \bigcup_{\mathbf{x}' \in B_1} \mathbf{B}_\varepsilon(\mathbf{x}')$. The bounded sequence $\{(x_{\frac{1}{n}}, y_{\frac{1}{n}})\}$ has a limit point (x^{**}, y^{**}). Since $F(\mathbb{R}_+^2)$ is compact, $(x^{**}, y^{**}) \in F(\mathbb{R}_+^2)$ and it is clear that $y^{**} \geq \gamma_1$. So $(x^{**}, y^{**}) \in B_1$. This along with $(x_\rho, y_\rho) \notin \bigcup_{\mathbf{x}' \in B_1} \mathbf{B}_\varepsilon(\mathbf{x}')$ implies $|(x^{**}, y^{**}) - (x_{\frac{1}{n}}, y_{\frac{1}{n}})| \geq \varepsilon$ for every n, which contradicts that (x^{**}, y^{**}) is a limit point of the sequence $\{(x_{\frac{1}{n}}, y_{\frac{1}{n}})\}$.

Now since $\{y_n\}$ converges to a y^* and we have $y^* \geq \gamma_1$, there exists N such that for every $n \geq N$ we have $y_n > \gamma_1 - \rho_\varepsilon$. So $(x_n, y_n) \in \bigcup_{\mathbf{x}' \in B_1} \mathbf{B}_\varepsilon(\mathbf{x}')$ for every $n \geq N$. Thus $\mathbf{B}_\varepsilon(\mathbf{x}_n) \cap B_1 \neq \emptyset$.

□

We can also construct another forward-invariant set by using F^2 instead of F. Using a similar argument to Lemma 2.2 we can prove the following lemma.

Lemma 2.4. *The following set is forward-invariant under the map F^2:*

$$B_2 := \left\{(x, y) \in F(\mathbb{R}_+^2) |\ y \geq \gamma_2\right\} \tag{2.18}$$

where

$$\gamma_2 := \inf \left\{y \in \mathbb{R}_+ |\ \ \exists x \in \mathbb{R}_+ \ \ s.t.\ \ (x, y) \in F^2 \left(F(\mathbb{R}_+^2) - A_y^{(2)}\right)\right\}. \tag{2.19}$$

For our approach, we need a set that at least satisfies the condition of forward-invariance under F. While being forward-invariant under F^2, B_2 is unfortunately not necessarily forward-invariant under F. We therefore need a method to construct a set that is forward-invariant under F by using B_2 in order to proceed.

Lemma 2.5. *The set $B_2 \cup F(B_2)$ is forward-invariant under both F and F^2.*

Proof. We have

$$F(B_2 \cup F(B_2)) = F(B_2) \cup F(F(B_2)) = F(B_2) \cup F^2(B_2). \tag{2.20}$$

According to Lemma 2.4 we have $F^2(B_2) \subset B_2$. So clearly we have $F(B_2) \cup F^2(B_2) \subset F(B_2) \cup B_2$. □

Lemma 2.6. *Suppose that $\{\mathbf{x}_n\}$ is an interior orbit of (2.1). Then for every $\varepsilon > 0$ there exists an $N \geq 0$ such that for every $n > N$ we have $\mathbf{B}_\varepsilon(\mathbf{x}_n) \cap (B_2 \cup F(B_2)) \neq \emptyset$, where $\mathbf{B}_\varepsilon(\mathbf{x}_n)$ is the open ball of radius ε and centre $\mathbf{x}_n$.*

Proof. For $n = 1, 2, ...$ we have $\mathbf{x}_n = F(\mathbf{x}_{n-1}) \in F(\mathbb{R}_+^2)$. So only two cases are possible:

(i) There exists an $m \geq 1$ such that $\mathbf{x}_m \in F(\mathbb{R}_+^2) - A_y^{(2)}$.

(ii) For every $n = 1, 2, ...$ we have $\mathbf{x}_n \in F(\mathbb{R}_+^2) \cap A_y^{(2)}$.

If (i) is true, then by definition, $\mathbf{x}_{m+2} = F^2(\mathbf{x}_m) \in B_2$. So $\mathbf{x}_{m+2}$ is a point in the forward-invariant set $B_2 \cup F(B_2)$. It means that for every $n \geq m + 2$ we have $\mathbf{x}_n \in B_2 \cup F(B_2)$. Therefore, if for $\varepsilon > 0$ we choose $N = m + 2$, then for every $n \geq N$ we have $\emptyset \neq \{\mathbf{x}_n\} \subset \mathbf{B}_\varepsilon(\mathbf{x}_n) \cap (B_2 \cup F(B_2))$.

For (ii), first we observe that $F(\mathbb{R}^2_+)$ is compact. Hence F is uniformly continuous on $F(\mathbb{R}^2_+)$. So for $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{x}, \mathbf{x}' \in F(\mathbb{R}^2_+)$ and $|\mathbf{x}-\mathbf{x}'| < \delta$, then $|F(\mathbf{x})-F(\mathbf{x}')| < \varepsilon$. So if $\mathbf{x} \in \bigcup_{\mathbf{x}'\in B_2} \mathbf{B}_\delta(\mathbf{x}')$ then $F(\mathbf{x}) \in \bigcup_{\mathbf{x}'\in F(B)} \mathbf{B}_\varepsilon(\mathbf{x}')$, which implies $\mathbf{B}_\varepsilon(F(\mathbf{x}))\cap(B_2\cup F(B_2)) \neq \emptyset$.

Since $(x_n, y_n) := \mathbf{x}_n \in F(\mathbb{R}^2_+)\cap A^{(2)}_y$ for every $n = 1, 2, ...$, the definition of $A^{(2)}_y$ implies that the bounded sequence $\{y_{2n+1}\}$ is increasing. So it converges to a y^*. It can be easily proven that $y^* \geq \gamma_2$.

Let $\mu := \min\{\delta, \varepsilon\}$. We prove that there exists $\rho_\mu > 0$ such that if $(x, y) \in F(\mathbb{R}^2_+)$ and $y > \gamma_2 - \rho_\mu$, then $(x, y) \in \bigcup_{\mathbf{x}'\in B_2} \mathbf{B}_\mu(\mathbf{x}')$. Assume for the sake of contradiction that for every $\rho > 0$ there exists $(x_\rho, y_\rho) \in F(\mathbb{R}^2_+)$ with $y_\rho > \gamma_2 - \rho$ and $(x_\rho, y_\rho) \notin \bigcup_{\mathbf{x}'\in B_2} \mathbf{B}_\mu(\mathbf{x}')$. The bounded sequence $\{(x_{\frac{1}{n}}, y_{\frac{1}{n}})\}$ has at least a limit point (x^{**}, y^{**}). Since $F(\mathbb{R}^2_+)$ is compact, $(x^{**}, y^{**}) \in F(\mathbb{R}^2_+)$ and it is clear that $y^{**} \geq \gamma_2$. So $(x^{**}, y^{**}) \in B_2$. This along with $(x_\rho, y_\rho) \notin \bigcup_{\mathbf{x}'\in B_2} \mathbf{B}_\mu(\mathbf{x}')$ implies $|(x^{**}, y^{**}) - (x_{\frac{1}{n}}, y_{\frac{1}{n}})| \geq \mu$ for every n, which contradicts the fact that (x^{**}, y^{**}) is a limit point of the sequence $\{(x_{\frac{1}{n}}, y_{\frac{1}{n}})\}$.

Now since $\{y_{2n+1}\}$ converges to a y^* and we have $y^* \geq \gamma_2$, there exists N such that for every $2n + 1 \geq N$ we have $y_{2n+1} > \gamma_2 - \rho_\mu$. So $(x_{2n+1}, y_{2n+1}) \in \bigcup_{\mathbf{x}'\in B_2} \mathbf{B}_\mu(\mathbf{x}')$ for every $2n + 1 \geq N$. From $\mu \leq \varepsilon$ it is clear that $(x_{2n+1}, y_{2n+1}) \in \bigcup_{\mathbf{x}'\in B_2} \mathbf{B}_\varepsilon(\mathbf{x}')$. Thus $\mathbf{B}_\varepsilon(\mathbf{x}_{2n+1})\cap(B_2\cup F(B_2)) \neq \emptyset$.

Now from $\mu \leq \delta$ it is clear that $(x_{2n+1}, y_{2n+1}) \in \bigcup_{\mathbf{x}'\in B_2} \mathbf{B}_\delta(\mathbf{x}')$. So based on what was proven earlier we have $\mathbf{B}_\varepsilon(F(x_{2n+1}, y_{2n+1}))\cap(B_2\cup F(B_2)) \neq \emptyset$, thus $\mathbf{B}_\varepsilon(x_{2n+2}, y_{2n+2})\cap(B_2\cup F(B_2)) \neq \emptyset$. Now combining the above arguments proves the lemma for case (ii).

□

Theorem 2.7. *Suppose that $V : \mathbb{R}^2_+ \to \mathbb{R}$ is a Lyapunov function for (2.1) that satisfies all the needed conditions to establish asymptotic stability of the interior fixed point: V is continuous, $V(\mathbf{c_4}) = 0$, V is positive at any other interior point and ΔV is negative at any point in a deleted neighbourhood of $\mathbf{c_4}$.*

If B_1 or $B_2\cup F(B_2)$ has a positive distance from the set $\Delta V_+ := \{\mathbf{x} \in \mathring{\mathbb{R}}^2_+|\Delta V(\mathbf{x}) \geq 0\} - \{\mathbf{c}_4\}$, then every interior orbit converges to the interior fixed point. In other words, if there exists $\varepsilon > 0$ such that

$$\Delta V_+ \cap \bigcup_{\mathbf{x}\in B_1} \mathbf{B}_\varepsilon(\mathbf{x}) = \emptyset, \tag{2.21}$$

or

$$\Delta V_+ \cap \bigcup_{\mathbf{x}\in B_2\cup F(B_2)} \mathbf{B}_\varepsilon(\mathbf{x}) = \emptyset, \tag{2.22}$$

then (2.1) is globally asymptotically stable in $\mathring{\mathbb{R}}^2_+$.

Proof. Suppose that $\{\mathbf{x}_n\}$ is an interior orbit of (2.1). By Lemma 2.3, there exists an $N_1 \geq 0$ such that for every $n > N_1$ we have $\mathbf{B}_{\frac{\varepsilon}{2}}(\mathbf{x}_n) \cap B_1 \neq \emptyset$ and by Lemma 2.6, there exists an $N_2 \geq 0$ such that for every $n > N_2$ we have $\mathbf{B}_{\frac{\varepsilon}{2}}(\mathbf{x}_n) \cap (B_2 \cup F(B_2)) \neq \emptyset$. Therefore, for $n > N_1$

$$\mathbf{x}_n \in \bigcup_{\mathbf{x} \in B_1} \mathbf{B}_{\frac{\varepsilon}{2}}(\mathbf{x}) \subset \bigcup_{\mathbf{x} \in B_1} \mathbf{B}_{\varepsilon}(\mathbf{x}),$$

and for $n > N_2$

$$\mathbf{x}_n \in \bigcup_{\mathbf{x} \in B_2 \cup F(B_2)} \mathbf{B}_{\frac{\varepsilon}{2}}(\mathbf{x}) \subset \bigcup_{\mathbf{x} \in B_2 \cup F(B_2)} \mathbf{B}_{\varepsilon}(\mathbf{x}).$$

Hence, $\Delta V(\mathbf{x_n}) < 0$ for every $n > N_1$ or $n > N_2$. Thus $V(\mathbf{x_n})$ is convergent to 0 since $\mathbf{x_n}$ is bounded. $V(\mathbf{x_n})$ vanishes only at the fixed points. So the orbit approaches the fixed points. The fixed points that the orbit can approach must be in $\bigcup_{\mathbf{x} \in B_1} \mathbf{B}_{\varepsilon}(\mathbf{x})$ and $\bigcup_{\mathbf{x} \in B_2 \cup F(B_2)} \mathbf{B}_{\varepsilon}(\mathbf{x})$ for every $\varepsilon > 0$. Only $\mathbf{c}_4$ and $\mathbf{c}_2$ satisfy this condition. The orbit cannot oscillate between the regions near $\mathbf{c}_4$ and $\mathbf{c}_2$. Because $\mathbf{c}_4$ is locally asymptotically stable it attracts every orbit close enough to it. So only two cases are possible: (i) the orbit converges to $\mathbf{c}_4$ (ii) the orbit converges to $\mathbf{c}_2$. The area near $\mathbf{c}_2$ is located inside the region $X > 0$, meaning that the first component of every orbit close to $\mathbf{c}_2$ only grows as long as it remains inside the region $X > 0$. Thus such an orbit can never approach $\mathbf{c}_2$ and only (i) is possible. □

2.5 The Lyapunov Function

It is well-known that for a continuous map f and its fixed point a, if a continuous function V (called Lyapunov function) is found such that it vanishes at only a, V is positive at any other point and ΔV is negative at any point in a deleted neighbourhood of a, then any orbit entering the neighborhood and staying in it converges to a. Here we use the Lyapunov function introduced by Baigent et al. [6]:

$$V(x, y) := \beta x^2 + \alpha y^2 + 2\alpha\beta xy - 2r\beta x - 2s\alpha y. \tag{2.23}$$

Here is the expression provided in [6] for ΔV:

$$\begin{aligned}\Delta V =& \beta x^2(e^X - 1)^2 + 2\alpha\beta xy(e^X - 1)(e^Y - 1) + \alpha y^2(e^Y - 1)^2 \\ &- 2\beta x X(e^X - 1) - 2\alpha y Y(e^Y - 1).\end{aligned}$$

Then [6] provides a decoupled expression which is greater than ΔV by using the well-known inequality $2ab \leq a^2 + b^2$. Since here we deal with parameter values beyond the range discussed in [6], we need to make an improvement to that decoupled expression (See (8) in [6]). We use

$2ab \leq ca^2 + c^{-1}b^2$, $(c > 0)$ instead of $2ab \leq a^2 + b^2$. Using this inequality, and by taking a and b as $e^X - 1$ and $e^Y - 1$, we obtain

$$\begin{aligned}\Delta V \leq & \beta x^2(e^X-1)^2 + \alpha\beta xy(c(e^X-1)^2 + c^{-1}(e^Y-1)^2) + \alpha y^2(e^Y-1)^2 \\ & \qquad - 2\beta xX(e^X-1) - 2\alpha yY(e^Y-1) \\ = & -\beta x(e^X-1)((X-r^*)(e^X-1)+2X) - \alpha y(e^Y-1)((Y-s^*)(e^Y-1)+2Y) \\ =: & \ \Delta^*,\end{aligned}$$

where

$$r^* := \alpha y(c-1) + r,$$

$$s^* := \beta x\left(\frac{1}{c} - 1\right) + s.$$

In [6] it was shown that if $0 < t \leq 2$, then

$$\begin{cases} (e^u-1)((u-t)(e^u-1)+2u > 0 & \text{if} \quad u \in \mathbb{R} - \{0\} \\ (e^u-1)((u-t)(e^u-1)+2u = 0 & \text{if} \quad u = 0. \end{cases}$$

Therefore, we can deduce that if $0 < r^*, s^* \leq 2$ then

$$\begin{cases} \Delta^* < 0 & \text{if} \quad X, Y \in \mathbb{R} - \{0\} \\ \Delta^* = 0 & \text{if} \quad X = Y = 0. \end{cases}$$

Now we can state the following theorem which identifies a number of regions in which ΔV is negative.

Theorem 2.8. *For every $r, s, \alpha, \beta > 0$ that satisfies (2.6), ΔV is negative at every point in each of the following sets except at the fixed points (where it vanishes):*

(a) $$\begin{cases} \left\{(x,y) \in \mathbb{R}^2_+ \mid y > \frac{\beta(r-2)}{\alpha(2-s)}x + \frac{r-2}{\alpha}\right\} & \text{if} \quad s < 2 \text{ and } r > 2 \\ \left\{(x,y) \in \mathbb{R}^2_+ \mid x > \frac{\alpha(s-2)}{\beta(2-r)}y + \frac{s-2}{\beta}\right\} & \text{if} \quad r < 2 \text{ and } s > 2 \end{cases}$$

(b) $$\begin{cases} \{X = 0\} := \{(x,y) \in \mathbb{R}^2_+ | r - x - \alpha y = 0\} & \text{if} \quad s \leq 2 \\ \{Y = 0\} := \{(x,y) \in \mathbb{R}^2_+ | s - y - \beta x = 0\} & \text{if} \quad r \leq 2 \end{cases}$$

(c) $$\left(A^{(1)}_{x+} \cap A^{(2)}_{x+} \cap A^{(1)}_{y+} \cap A^{(2)}_{y+}\right) \cup \left(A^{(1)}_{x-} \cap A^{(2)}_{x-} \cap A^{(1)}_{y+} \cap A^{(2)}_{y+}\right) \cup \left(A^{(1)}_{x+} \cap A^{(2)}_{x+} \cap A^{(1)}_{y-} \cap A^{(2)}_{y-}\right) \cup \left(A^{(1)}_{x-} \cap A^{(2)}_{x-} \cap A^{(1)}_{y-} \cap A^{(2)}_{y-}\right)$$

Proof. (a) If $s < 2$ and $r > 2$ and for $x, y \geq 0$ we have $y > \frac{\beta(r-2)}{\alpha(2-s)}x + \frac{r-2}{\alpha}$, then it is clear that

$y > \frac{r-2}{\alpha}$. Hence $0 < 2 - r + \alpha y < \alpha y$ and by choosing $c = \frac{2-r+\alpha y}{\alpha y}$ we have $0 < c < 1$. Thus

$$s^* = \beta x \left(\frac{1}{c} - 1\right) + s > 0.$$

We also have

$$\begin{aligned} r^* &= \alpha y \left(\frac{2-r+\alpha y}{\alpha y} - 1\right) + r = 2 - r + r = 2, \\ s^* &= \beta x \left(\frac{\alpha y}{2-r+\alpha y} - 1\right) + s = \beta x \left(\frac{r-2}{2-r+\alpha y}\right) + s. \end{aligned}$$

Now since $y > \frac{\beta(r-2)}{\alpha(2-s)}x + \frac{r-2}{\alpha}$, we have $x < \frac{\alpha(2-s)}{\beta(r-2)}\left(y - \frac{r-2}{\alpha}\right)$ and

$$\begin{aligned} s^* &< \beta \frac{\alpha(2-s)}{\beta(r-2)} \left(y - \frac{r-2}{\alpha}\right) \left(\frac{r-2}{2-r+\alpha y}\right) + s \\ &= \beta \frac{\alpha(2-s)}{\beta(r-2)} \left(\frac{2-r+\alpha y}{\alpha}\right) \left(\frac{r-2}{2-r+\alpha y}\right) + s = 2. \end{aligned}$$

Therefore, $0 < r^*, s^* \leq 2$ and based on the theory we mentioned before from [6], we deduce that for this (x, y) we have $\Delta V < 0$. The proof for the other case is similar.

(b) For every point on $\{X = 0\}$ we have

$$\begin{aligned} \Delta^* = &- \beta x(e^X - 1)((X - r^*)(e^X - 1) + 2X) - \alpha y(e^Y - 1)((Y - s^*)(e^Y - 1) + 2Y) \\ &= -\alpha y(e^Y - 1)((Y - s^*)(e^Y - 1) + 2Y). \end{aligned}$$

Now with $c = 1$, we have $s^* = s$. So when $s \leq 2$, we have $\Delta V \leq \Delta^* < 0$. The proof is similar for the other case.

(c) We prove that ΔV is negative at every point in $\left(A_{x+}^{(1)} \cap A_{x+}^{(2)} \cap A_{y+}^{(1)} \cap A_{y+}^{(2)}\right) - \{\mathbf{c}_4\}$. The proof for the other sets in the union is quite similar. We have

$$\begin{aligned} \Delta V &= \beta x^2(e^X - 1)^2 + 2\alpha\beta xy(e^X - 1)(e^Y - 1) + \alpha y^2(e^Y - 1)^2 \\ &- 2\beta x X(e^X - 1) - 2\alpha y Y(e^Y - 1) \\ &= \beta x(e^X - 1)\left(x(e^X - 1) + \alpha y(e^Y - 1) - 2X\right) \\ &+ \alpha y(e^Y - 1)\left(y(e^Y - 1) + \beta x(e^X - 1) - 2Y\right) \\ &= \beta x(e^X - 1)\left(x(e^X + 1) + \alpha y(e^Y + 1) - 2r\right) \\ &+ \alpha y(e^Y - 1)\left(y(e^Y + 1) + \beta x(e^X + 1) - 2s\right). \end{aligned}$$

It is easily observed that

$$\begin{aligned} A_x^{(1)} &= \left\{(x,y) \in \mathbb{R}_+^2 |\ e^X - 1 \geq 0\right\} \\ A_y^{(1)} &= \left\{(x,y) \in \mathbb{R}_+^2 |\ e^Y - 1 \geq 0\right\} \\ A_x^{(2)} &= \left\{(x,y) \in \mathbb{R}_+^2 |\ x(e^X + 1) + \alpha y(e^Y + 1) - 2r \leq 0\right\} \\ A_y^{(2)} &= \left\{(x,y) \in \mathbb{R}_+^2 |\ y(e^Y + 1) + \beta x(e^X + 1) - 2s \leq 0\right\}. \end{aligned}$$

Therefore, when $(x,y) \in A_{x+}^{(1)} \cap A_{x+}^{(2)} \cap A_{y+}^{(1)} \cap A_{y+}^{(2)}$ we have

$$\begin{aligned} \beta x(e^X - 1)\left(x(e^X + 1) + \alpha y(e^Y + 1) - 2r\right) \leq 0 \\ \alpha y(e^Y - 1)\left(y(e^Y + 1) + \beta x(e^X + 1) - 2s\right) \leq 0. \end{aligned}$$

The previous two inequalities vanish simultaneously only at $\mathbf{c}_4$. Hence, as a direct result of these inequalities and the expression we obtained for ΔV, ΔV is negative at every point in $\left(A_{x+}^{(1)} \cap A_{x+}^{(2)} \cap A_{y+}^{(1)} \cap A_{y+}^{(2)}\right) - \{\mathbf{c}_4\}$. □

As a direct result of Theorem 2.7 and Theorem 2.8, we obtain the following corollary, which offers a more easily verifiable sufficient condition for the global stability of the Ricker model. Parts (a) and (b) follow directly from the statements of Theorem 2.7 and Theorem 2.8. Parts (c) and (d) can be established using a similar line of reasoning, by interchanging the roles of (r,α) and (s,β) in the arguments developed thus far.

Corollary 2.9. *Suppose that $r, s, \alpha, \beta > 0$ and the positive fixed point $\mathbf{c}_4$ exists. If one of the following holds then every positive orbit of the Ricker model converges to $\mathbf{c}_4$:*
(a) $s < 2$ and $B_1 \subset \left\{(x,y)|\ y > \frac{\beta(r-2)}{\alpha(2-s)}x + \frac{r-2}{\alpha}\right\}$.
(b) $s < 2$ and $(B_2 \cup F(B_2)) \subset \left\{(x,y)|\ y > \frac{\beta(r-2)}{\alpha(2-s)}x + \frac{r-2}{\alpha}\right\}$.
(c) $r < 2$ and $B_1^\star \subset \left\{(x,y)|\ x > \frac{\alpha(s-2)}{\beta(2-r)}y + \frac{s-2}{\beta}\right\}$, where

$$\begin{aligned} B_1^\star &:= \left\{(x,y) \in F(\mathbb{R}_+^2)|\ x \geq \gamma_1^\star\right\} \\ \gamma_1^\star &:= \inf\left\{x \in \mathbb{R}_+ |\ \exists y \in \mathbb{R}_+ \ \ s.t.\ \ (x,y) \in F\left(F(\mathbb{R}_+^2) - A_x^{(1)}\right)\right\}. \end{aligned}$$

(d) $r < 2$ and $(B_2^\star \cup F(B_2^\star)) \subset \left\{(x,y)|\ x > \frac{\alpha(s-2)}{\beta(2-r)}y + \frac{s-2}{\beta}\right\}$, where

$$\begin{aligned} B_2^\star &:= \left\{(x,y) \in F(\mathbb{R}_+^2)|\ x \geq \gamma_2^\star\right\} \\ \gamma_2^\star &:= \inf\left\{x \in \mathbb{R}_+ |\ \exists y \in \mathbb{R}_+ \ \ s.t.\ \ (x,y) \in F^2\left(F(\mathbb{R}_+^2) - A_x^{(2)}\right)\right\}. \end{aligned}$$

We can now prove an intriguing property of the boundaries of the sets mentioned in part (a)

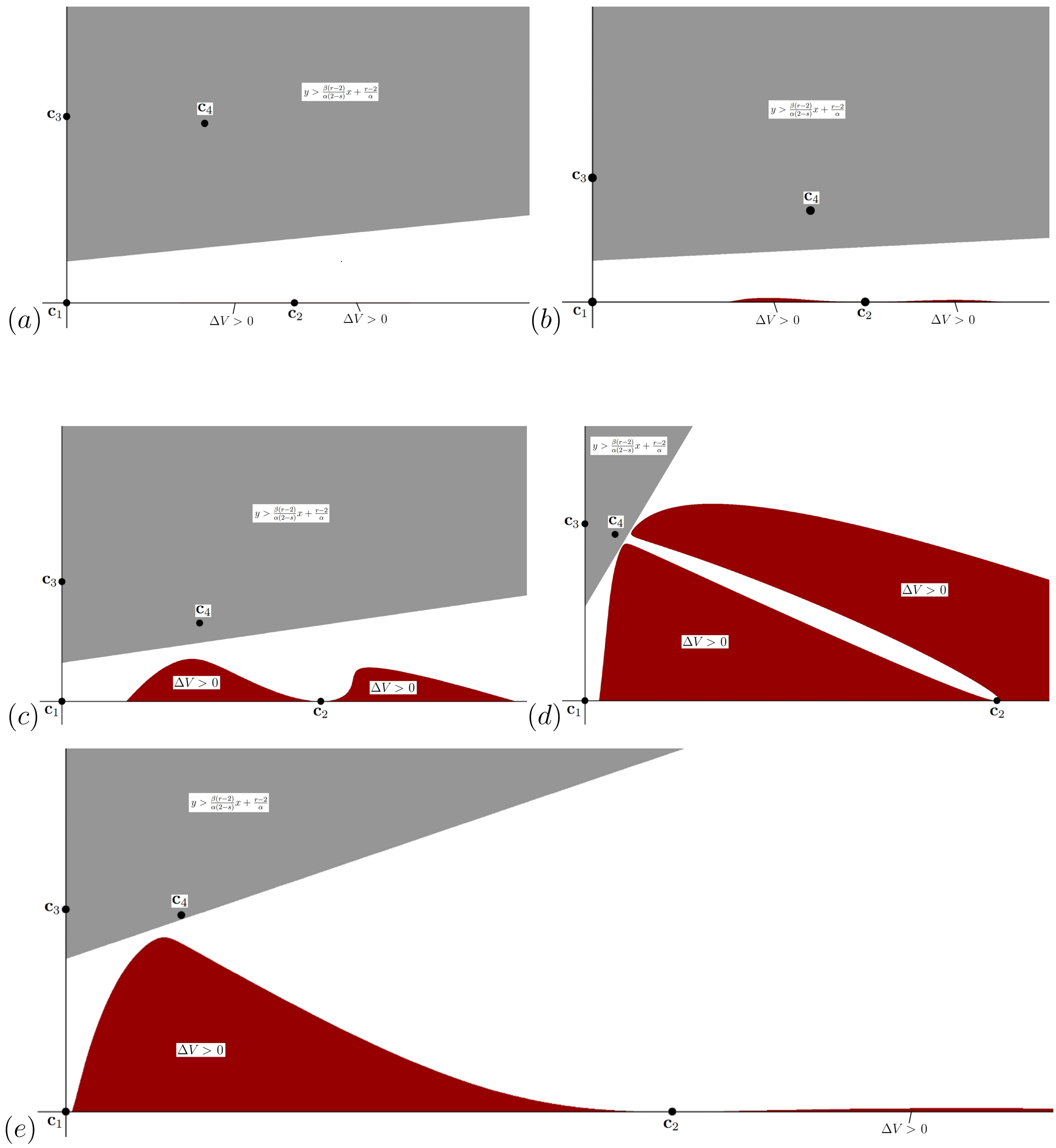

Figure 2.2: Examples of the sets $\{\Delta V > 0\}$ and their placement relative to the region $y > \frac{\beta(r-2)}{\alpha(2-s)}x + \frac{r-2}{\alpha}$. The parameter values are: (a) $r = 2.2, s = 1.8, \alpha = 0.5, \beta = 0.05$, (b) $r = 2.2, s = 1, \alpha = 0.6, \beta = 0.15$, (c) $r = 2.6, s = 1.2, \alpha = 1.55, \beta = 0.3$, (d) $r = 4.2, s = 1.8, \alpha = 2.3, \beta = 0.35$, (e) $r = 5.4, s = 1.8, \alpha = 2.5, \beta = 0.05$.

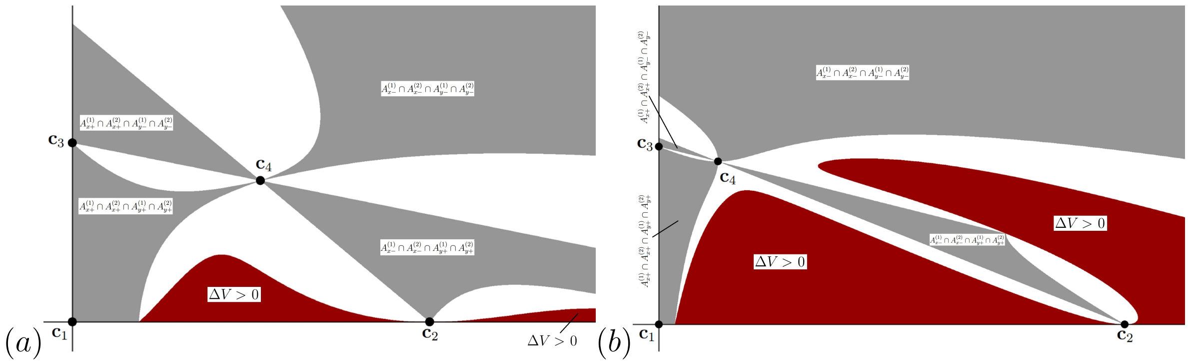

Figure 2.3: Examples of the set stated in Theorem 2.8 (c) and its placement relative to the region ΔV_+. The parameter values are: (a) $r = 2.8, s = 1.4, \alpha = 1.2, \beta = 0.2$, (b) $r = 4.2, s = 1.6, \alpha = 2.5, \beta = 0.25$.

of Theorem 2.8 as delimited by the two lines given by

$$y = \frac{\beta(r-2)}{\alpha(2-s)}x + \frac{r-2}{\alpha}$$

and

$$x = \frac{\alpha(s-2)}{\beta(2-r)}y + \frac{s-2}{\beta}.$$

First we note that these are actually two equations for the same line. As we show below the line not only defines a region where ΔV is negative but also leads to necessary and sufficient conditions for the local stability of the interior fixed point. This interesting connection between the Lyapunov function and the stability of the fixed point may suggest a deeper theoretical role for this line in the study of the Ricker model.

Theorem 2.10. *Assume $\alpha, \beta, r, s > 0$ are such that the Ricker map (2.1) has a unique fixed point in the interior of $\mathbb{R}^2_+$. Then*

(a) if $s < 2$, $\mathbf{c}_4$ is asymptotically stable if and only if it is located above the line $y = \frac{\beta(r-2)}{\alpha(2-s)}x + \frac{r-2}{\alpha}$

(b) if $r < 2$, $\mathbf{c}_4$ is asymptotically stable if and only if it is located on the right-hand side of the line $x = \frac{\alpha(s-2)}{\beta(2-r)}y + \frac{s-2}{\beta}$

(c) if $r, s \geq 2$, $\mathbf{c}_4$ is not asymptotically stable.

Proof. (c) is already proven in Lemma 2.1. We only provide proof for (a). The proof for (b) is quite similar by interchange of parameters.

Assume that $s < 2$ and $\mathbf{c}_4$ is located above the line $y = \frac{\beta(r-2)}{\alpha(2-s)}x + \frac{r-2}{\alpha}$. By Theorem 2.8 there exists a neighborhood N_1 of $\mathbf{c}_4$ in which ΔV is negative everywhere except at $\mathbf{c}_4$. As the contour lines of V are ellipses centered at $\mathbf{c}_4$ (recall that we are considering the locally asymptotically stable case where V is convex) and with the value of V decreasing as it gets closer to $\mathbf{c}_4$, $\varepsilon > 0$ can be found such that the set $\{(x, y)|\ V(x, y) < \varepsilon\}$ is a subset of N_1. Now suppose that $\mathbf{x}_0 \neq \mathbf{c}_4$ is a point on $\{(x, y)|\ V(x, y) < \varepsilon\}$. ΔV is negative at $\mathbf{x}_0$ since it is a point in N_1. Therefore, $V(\mathbf{x}_1) < V(\mathbf{x}_0) < \varepsilon$ and $V(\mathbf{x}_1)$ is a point in N_1 as well (hence $\Delta V(\mathbf{x}_1) < 0$). Now by using induction and a similar argument at each step, we have $\cdots < V(\mathbf{x}_2) < V(\mathbf{x}_1) < V(\mathbf{x}_0) < \varepsilon$. Hence, $\mathbf{x}_n \to \mathbf{c}_4$. As every orbit starting from $\{(x, y)|\ V(x, y) < \varepsilon\}$ converges to $\mathbf{c}_4$, the fixed point is asymptotically stable.

Now suppose that $s < 2$ and $\mathbf{c}_4$ is asymptotically stable. We prove that (2.6) implies $\frac{s-\beta r}{1-\alpha\beta} > \frac{\beta(r-2)}{\alpha(2-s)}\frac{r-\alpha s}{1-\alpha\beta} + \frac{r-2}{\alpha}$. From (2.6) we have

$$\begin{aligned}
&4(\alpha\beta - 1) + 2(r - \alpha s) + 2(s - \beta r) < (r - \alpha s)(s - \beta r) \\
\Longrightarrow\ &4\alpha\beta < (r - \alpha s)(s - \beta r) - 2(r - \alpha s) - 2(s - \beta r) + 4 \\
&= (r - 2 - \alpha s)(s - 2 - \beta r) \\
\Longrightarrow\ &- 4\alpha\beta - \alpha s(s - 2) - \beta r(r - 2) + \alpha\beta rs > (r - 2)(2 - s).
\end{aligned}$$

Now by using the previous inequality we have

$$\begin{aligned}
&\alpha(s - 2)(s - \beta r) + \beta(r - 2)(r - \alpha s) + (r - 2)(2 - s)(1 - \alpha\beta) \\
&= \alpha(s - 2)(s - \beta r) + \beta(r - 2)(r - \alpha s) + (r - 2)(2 - s) - \alpha\beta(r - 2)(2 - s) \\
&< \alpha(s - 2)(s - \beta r) + \beta(r - 2)(r - \alpha s) - 4\alpha\beta - \alpha s(s - 2) - \beta r(r - 2) + \alpha\beta rs \\
&\quad - \alpha\beta(r - 2)(2 - s) \\
&= \alpha s(s - 2) - \alpha(s - 2)\beta r + \beta(r - 2)(r - \alpha s) - 4\alpha\beta - \alpha s(s - 2) - \beta r(r - 2) \\
&\quad + \alpha\beta rs - \alpha\beta(r - 2)(2 - s) \\
&= -\alpha(s - 2)\beta r + \beta(r - 2)(r - \alpha s) - 4\alpha\beta - \beta r(r - 2) + \alpha\beta rs - \alpha\beta(r - 2)(2 - s) \\
&= -\alpha\beta(s - 2)r + \beta(r - 2)(r - \alpha s) - 4\alpha\beta - \beta r(r - 2) + \alpha\beta rs - \alpha\beta(2 - s)r + 2\alpha\beta(2 - s) \\
&= \beta(r - 2)(r - \alpha s) - 4\alpha\beta - \beta r(r - 2) + \alpha\beta rs + 2\alpha\beta(2 - s) \\
&= \beta(r - 2)(r - \alpha s) - 4\alpha\beta - \beta r(r - 2) + \alpha\beta rs + 4\alpha\beta - 2s\alpha\beta \\
&= \beta(r - 2)(r - \alpha s) - \beta r(r - 2) + \alpha\beta rs - 2s\alpha\beta \\
&= \beta r(r - 2) - \beta\alpha s(r - 2) - \beta r(r - 2) + \alpha\beta rs - 2s\alpha\beta \\
&= -\beta\alpha s(r - 2) + \alpha\beta rs - 2s\alpha\beta = -\beta\alpha s(r - 2) + \beta\alpha s(r - 2) = 0.
\end{aligned}$$

From $s < 2$ and Lemma (2.1), we obtain the inequality $\alpha(2 - s)(1 - \alpha\beta) > 0$. Now dividing both sides of the above inequality by $\alpha(2 - s)(1 - \alpha\beta)$ yields

$$-\frac{s - \beta r}{1 - \alpha\beta} + \frac{\beta(r - 2)}{\alpha(2 - s)}\frac{r - \alpha s}{1 - \alpha\beta} + \frac{r - 2}{\alpha} < 0$$

which is equivalent to the desired inequality. □

2.6 Estimation of the Forward-Invariant Sets

We have introduced a formulation that simplifies the application of the Lyapunov function within our theoretical framework, particularly in terms of computational effort. Since it is also challenging to directly apply the exact equations that define our forward-invariant sets (e.g. because of the presence of exponentials), we must resort to approximations that can meaningfully reduce the complexity of our calculations. For a fixed set of parameters, numerical methods can verify the sufficient conditions for global stability. However, when these conditions must be assessed over a range of parameter sets, the problem becomes significantly more complex, making estimations essential.

The following lemma provides a rectangle that contains $F(\mathbb{R}^2_+)$ [52, 6].

Lemma 2.11. *$F(\mathbb{R}^2_+) \subset [0, e^{r-1}] \times [0, e^{s-1}]$. Additionally, the points $(e^{r-1}, 0)$ and $(0, e^{s-1})$ are both part of $F(\mathbb{R}^2_+)$.*

Proof. As the expression xe^{r-x} takes its maximum at $x = 1$, for every $(x, y) \in \mathbb{R}^2_+$ we have

$$F_1(x, y) = xe^{r-x-\alpha y} \leq xe^{r-x} \leq e^{r-1}.$$

Hence $F(\mathbb{R}^2_+) \subset [0, e^{r-1}] \times \mathbb{R}_+$. Similarly, we have $F(\mathbb{R}^2_+) \subset \mathbb{R}_+ \times [0, e^{s-1}]$. Now the proof is completed by observing $F(1, 0) = (e^{r-1}, 0)$ and $F(0, 1) = (0, e^{s-1})$. □

If $\gamma_1 = 0$ then $B_1 = F(\mathbb{R}^2_+)$. We also have $B_2 \cup F(B_2) = F(\mathbb{R}^2_+)$ when $\gamma_2 = 0$. This means that in the cases where $\gamma_1 = 0$ or $\gamma_2 = 0$, the corresponding forward-invariant sets are the same as $F(\mathbb{R}^2_+)$. So we cannot expect an empty intersection between ΔV_+ and those forward-invariant sets in these cases. So $\gamma_1 > 0$ or $\gamma_2 > 0$ are necessary conditions for our theory to work. The following lemma provides some conditions for $\gamma_1 > 0$ and $\gamma_2 > 0$.

Lemma 2.12. *For $r, s, \alpha, \beta > 0$,*

(a) If $\frac{s}{\beta} > e^{r-1}$ then $\gamma_1 > 0$.

(b) If $\frac{s}{\beta} < e^{r-1}$ then $\gamma_1 = 0$.

(c) If e^{r-1} is less than the smallest solution of $\beta x(e^{r-x} + 1) - 2s = 0$ then $\gamma_2 > 0$.

(d) $\gamma_1 > 0$ implies $\gamma_2 > 0$.

Proof. (a) We prove when $\gamma_1 = 0$ then $\frac{s}{\beta} \leq e^{r-1}$. Suppose that $\gamma_1 = 0$. This means that for every $\varepsilon > 0$, there exists $(x, y) \in \mathbb{R}^2_+$ such that $y < \varepsilon$ and $(x, y) \in F\left(F(\mathbb{R}^2_+) - A^{(1)}_y\right)$.

Since $(x, y) \notin A_y^{(1)}$, we have $Y > 0$ for that point. Therefore, every neighborhood of the set $[0, e^{r-1}] \times \{0\}$ has non-empty intersection with the region $Y > 0$ and, as part of such a neighborhood also has non-empty intersection with the region $Y < 0$, the set $[0, e^{r-1}] \times \{0\}$ must contain a point on the line $Y = 0$. The only possible point for that scenario is $(s/\beta, 0)$, and this is the point where $Y = 0$ and $y = 0$ cross each other. Hence, from $(s/\beta, 0) \in [0, e^{r-1}] \times \{0\}$ we deduce $\frac{s}{\beta} \leq e^{r-1}$.

(b) If $\frac{s}{\beta} < e^{r-1}$ then by Lemma 2.11, $(s/\beta, 0) \in F(\mathbb{R}_+^2)$. Taking (x, y) as $(s/\beta, 0)$ gives

$$Y = s - y - \beta x = s - 0 - \beta \frac{s}{\beta} = 0.$$

Hence, as $(s/\beta, 0)$ is located on the line $Y = 0$, it is located on the boundary of the set $F(\mathbb{R}_+^2) - A_y^{(1)}$ as well. Therefore, for every $\varepsilon > 0$, there exists $(x_\varepsilon, y_\varepsilon) \in \mathbb{R}_+^2$ such that $y_\varepsilon < \varepsilon$ and $(x_\varepsilon, y_\varepsilon) \in F(\mathbb{R}_+^2) - A_y^{(1)}$. As $\varepsilon \to 0$, $F_2(x_\varepsilon, y_\varepsilon) \to 0$. Now since $F_2(x_\varepsilon, y_\varepsilon) \geq \gamma_1$, we deduce $\gamma_1 = 0$.

(c) We have

$$\begin{aligned} A_y^{(2)} &= \left\{(x, y) \in \mathbb{R}_+^2 |\ \ y > 0 \ \ \textit{and} \ \ F_2^2(x, y) = y e^{s-y-\beta x} e^{s - y e^{s-y-\beta x} - \beta x e^{r-x-\alpha y}} \geq y \right\} \\ &= \left\{(x, y) \in \mathbb{R}_+^2 |\ \ y > 0 \ \ \textit{and} \ \ y e^{2s - y(1+e^{s-y-\beta x}) - \beta x(1+e^{r-x-\alpha y})} \geq y \right\} \\ &= \left\{(x, y) \in \mathbb{R}_+^2 |\ \ y > 0 \ \ \textit{and} \ \ 2s - y(1+e^{s-y-\beta x}) - \beta x(1+e^{r-x-\alpha y}) \geq y \right\}. \end{aligned}$$

When $y = 0$, for (x, y) we have

$$2s - y(1+e^{s-y-\beta x}) - \beta x(1+e^{r-x-\alpha y}) = 2s - \beta x(1+e^{r-x}),$$

thus for $(x, y) = (0, 0)$

$$2s - y(1+e^{s-y-\beta x}) - \beta x(1+e^{r-x-\alpha y}) = 2s > 0.$$

Hence, since $2s - y(1+e^Y) - \beta x(1+e^X)$ is positive at $(x, y) = (0, 0)$, it is positive for any x less than the smallest solution of

$$2s - y(1+e^{s-y-\beta x}) - \beta x(1+e^{r-x-\alpha y}) = \beta x(e^{r-x} + 1) - 2s = 0.$$

So if e^{r-1} is less than the smallest solution of $\beta x(e^{r-x}+1) - 2s = 0$, then $2s - y(1+e^Y) - \beta x(1+e^X)$ is positive for any $(x, y) \in [0, e^{r-1}] \times \{0\}$. This implies the existence of a neighborhood P of $[0, e^{r-1}] \times \{0\}$ such that for every $(x, y) \in P \cap \mathring{\mathbb{R}}_+^2$ we have

$$F_2^2(x, y) = y e^{2s - y(1+e^{s-y-\beta x}) - \beta x(1+e^{r-x-\alpha y})} > y.$$

Therefore, $P \cap \mathring{\mathbb{R}}_+^2 \subset A_y^{(2)}$. Since P is a neighborhood of the compact set $[0, e^{r-1}] \times \{0\}$, $\varepsilon > 0$

can be found such that $[0, e^{r-1}] \times [0, \varepsilon] \subset P \cap \mathring{\mathbb{R}}^2_+ \subset A^{(2)}_y$. Thus

$$\left([0, e^{r-1}] \times [0, \varepsilon]\right) \cap \left(F(\mathbb{R}^2_+) - A^{(2)}_y\right) = \emptyset$$

implies the existence of $\varepsilon' > 0$ such that $([0, e^{r-1}] \times [0, \varepsilon']) \cap F^2 \left(F(\mathbb{R}^2_+) - A^{(2)}_y\right) = \emptyset$. Hence $\gamma_2 > 0$.

(d) Assume that $\gamma_1 > 0$. There must be $\varepsilon > 0$ such that

$$(\mathbb{R}_+ \times [0, \varepsilon]) \cap F\left(F(\mathbb{R}^2_+) - A^{(1)}_y\right) = \emptyset,$$

which implies the existence of $\varepsilon' > 0$ such that $(\mathbb{R}_+ \times [0, \varepsilon']) \cap \left(F(\mathbb{R}^2_+) - A^{(1)}_y\right) = \emptyset$. Hence $\left((\mathbb{R}_+ \times [0, \varepsilon']) \cap F(\mathbb{R}^2_+)\right) \subset A^{(1)}_y$. So for every $(x, y) \in F(\mathbb{R}^2_+)$ with $y \leq \varepsilon'$ we have $F_2(x, y) < y$. $0 < \varepsilon'' < \varepsilon'$ can be found such that for every $(x, y) \in F(\mathbb{R}^2_+)$ with $y \leq \varepsilon''$, $F_2(x, y) < \varepsilon'$. Thus both (x, y) and $(F_1(x, y), F_2(x, y))$ are located in $(\mathbb{R}_+ \times [0, \varepsilon']) \cap F(\mathbb{R}^2_+)$ and $y < F_2(x, y) < F_2^2(x, y)$, which implies $(x, y) \subset A^{(2)}_y$. Now since

$$(\mathbb{R}_+ \times [0, \varepsilon'']) \cap \left(F(\mathbb{R}^2_+) - A^{(2)}_y\right) = \emptyset,$$

$\delta > 0$ exists such that for every $(x, y) \in \left(F(\mathbb{R}^2_+) - A^{(2)}_y\right)$ we have $F_2^2(x, y) > \delta$, which implies $\gamma_2 > \delta > 0$. □

As a direct result of Lemma 2.12 (d), we may expect that a theory based on $\gamma_2 > 0$ and F^2 spans a wider range of parameter values than a similar theory based on $\gamma_1 > 0$ and F. Some examples of cases for which global stability can be proven by using $B_2 \cap F(B_2)$, whereas $B_1 = F(\mathbb{R}^2_+)$ cannot be used are shown in Figure 2.4.

Theorem 2.13. *For $r, s, \alpha, \beta > 0$,*

$$\gamma_1 \geq (s - \beta e^{r-1}) e^{s - e^{s-1} - \beta e^{r-1}} =: \rho,$$

hence, $B_1 \cap \{(x, y) |\, y < \rho\} = \emptyset$.

Proof. If $s - \beta e^{r-1} \leq 0$ then the inequality is obvious as $\gamma_1 \geq 0$. Assume that $s - \beta e^{r-1} > 0$. The point (e^{r-1}, e^{s-1}) is the most distant point to the line $Y = 0$ which is located above that line and on $[0, e^{r-1}] \times [0, e^{s-1}]$. Therefore, Y takes its minimum on $[0, e^{r-1}] \times [0, e^{s-1}]$ at that point. As $F(\mathbb{R}^2_+) \subset [0, e^{r-1}] \times [0, e^{s-1}]$, for every $(x, y) \in F(\mathbb{R}^2_+)$ we have

$$y e^Y \geq y e^{s - e^{s-1} - \beta e^{r-1}}.$$

The line $Y = 0$ crosses the line $x = e^{r-1}$ at $(e^{r-1}, s - \beta e^{r-1})$. As $Y = 0$ is decreasing, the point $(e^{r-1}, s - \beta e^{r-1})$ has the lowest second component in the region $\{Y \geq 0\} \times [0, e^{r-1}]$. Therefore,

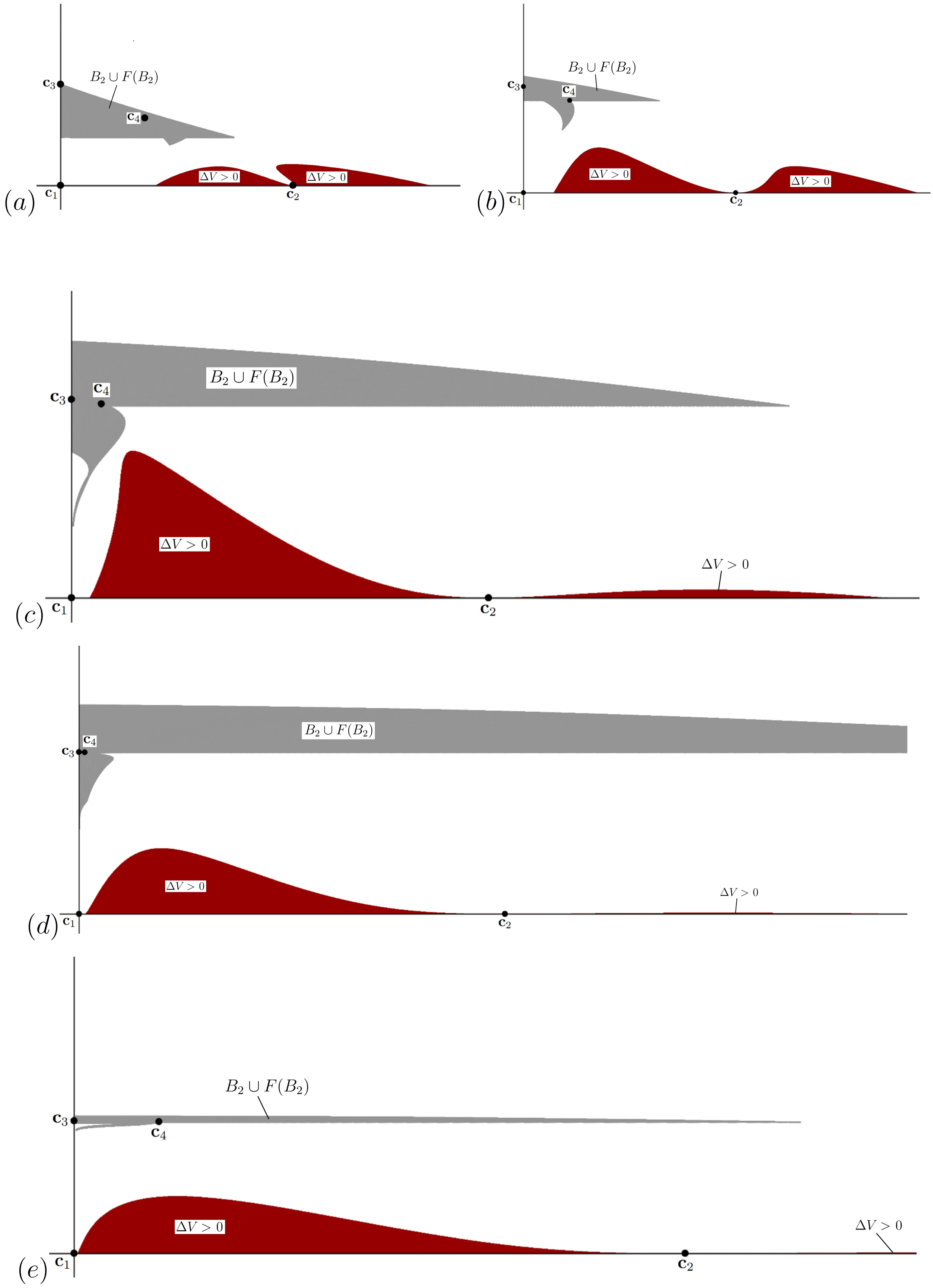

Figure 2.4: Examples of parameter sets with $\gamma_1 = 0$ but with empty intersection between $B_2 \cap F(B_2)$ and ΔV_+: (a) $r = 2.3, s = 1, \alpha = 2.2, \beta = 0.4$, (b) $r = 3, s = 1.5, \alpha = 1.8, \beta = 0.3$, (c) $r = 4, s = 1.9, \alpha = 2, \beta = 0.15$, (d) $r = 5, s = 1.9, \alpha = 2.6, \beta = 0.04$, (e) $r = 6, s = 1.3, \alpha = 4, \beta = 0.01$.

for every $(x,y) \in F(\mathbb{R}^2_+) - A^{(1)}_y$, we have

$$y \geq s - \beta e^{r-1}.$$

Now combining both inequalities gives

$$ye^Y \geq (s - \beta e^{r-1}) e^{s - e^{s-1} - \beta e^{r-1}}, \qquad (x,y) \in F(\mathbb{R}^2_+) - A^{(1)}_y.$$

This implies

$$F(F(\mathbb{R}^2_+) - A^{(1)}_y) \cap \left(\mathbb{R} \times \left(-\infty, (s - \beta e^{r-1}) e^{s - e^{s-1} - \beta e^{r-1}}\right)\right) = \emptyset.$$

Therefore, by definition $\gamma_1 \geq \rho$.

The second component of each point on B_1 is greater than γ_1. Thus,

$$B_1 \cap \{(x,y) |\ y < \rho\} = \emptyset.$$

□

Theorem 2.14. *For $r \geq 2$ and $s, \alpha, \beta > 0$, if $\mathbf{c}_4$ is asymptotically stable, and*

$$(s - \beta e^{r-1}) e^{s - e^{s-1} - \beta e^{r-1}} > \frac{r-2}{\alpha} \left(\frac{\beta}{2-s} e^{r-1} + 1 \right), \tag{2.24}$$

then every orbit starting from the interior of $\mathbb{R}^2_+$ converges to $\mathbf{c}_4$.

Proof. We prove that the inequality implies empty intersection between B_1 and the region below the line $y = \frac{\beta(r-2)}{\alpha(2-s)} x + \frac{r-2}{\alpha}$. As the line is increasing, the second component of the points on $[0, e^{r-1}] \times [0, e^{s-1}]$ and below the line $y = \frac{\beta(r-2)}{\alpha(2-s)} x + \frac{r-2}{\alpha}$ takes its maximum at $x = e^{r-1}$ and that maximum value is

$$\frac{\beta(r-2)}{\alpha(2-s)} e^{r-1} + \frac{r-2}{\alpha} = \frac{r-2}{\alpha} \left(\frac{\beta}{2-s} e^{r-1} + 1 \right).$$

By Theorem 2.16, we have $B_1 \cap \{(x,y) |\ y < \rho\} = \emptyset$. Now the proof is complete by observing the fact that the inequality implies

$$\left\{ (x,y) |\ y < \frac{\beta(r-2)}{\alpha(2-s)} x + \frac{r-2}{\alpha} \right\} \subset \{(x,y) |\ y < \rho\}.$$

□

By applying a similar line of reasoning and interchanging the roles of (r, α) and (s, β) in our arguments, we also obtain the following corollary:

Corollary 2.15. *For $s \geq 2$ and $r, \alpha, \beta > 0$, if $\mathbf{c}_4$ is asymptotically stable, and*

$$(r - \alpha e^{s-1}) e^{r - e^{r-1} - \alpha e^{s-1}} > \frac{s-2}{\beta}\left(\frac{\alpha}{2-r} e^{s-1} + 1\right), \tag{2.25}$$

then every orbit starting from the interior of $\mathbb{R}^2_+$ converges to $\mathbf{c}_4$.

Now we give an example of a box of parameters at which, every set of parameters satisfies the inequality stated in the above theorem. Consider the ranges $2 \leq r \leq 2.5$, $\beta < \frac{1}{20}$, $\alpha > 1.5$ and $1 < s < 1.25$. Then clearly $0 < \frac{r-2}{\alpha} < \frac{1}{3}$ and $0 < \frac{\beta}{2-s} e^{r-1} + 1 < \frac{4}{3}$. Thus

$$\frac{r-2}{\alpha}\left(\frac{\beta}{2-s} e^{r-1} + 1\right) < \frac{4}{9},$$

we also have $s - \beta e^{r-1} > 0.75$ and $e^{s - e^{s-1} - \beta e^{r-1}} > 0.6$. Hence

$$(s - \beta e^{r-1}) e^{s - e^{s-1} - \beta e^{r-1}} > \frac{9}{20},$$

Therefore, as $\frac{4}{9} < \frac{9}{20}$, for every (r, s, α, β) in the following box and with locally stable $\mathbf{c}_4$, the Ricker map is globally stable:

$$\left[2, \frac{5}{2}\right] \times \left[1, \frac{5}{4}\right] \times \left[\frac{3}{2}, +\infty\right] \times \left[0, \frac{1}{20}\right].$$

Similarly, by applying Corollary 2.15, we also obtain the following box:

$$\left[1, \frac{5}{4}\right] \times \left[2, \frac{5}{2}\right] \times \left[0, \frac{1}{20}\right] \times \left[\frac{3}{2}, +\infty\right].$$

Now we prove that for every $r > 0$, the set of all $s, \alpha, \beta > 0$ for which the conditions of Theorem 2.14 are satisfied is non-empty. Therefore, Theorem 2.14 proves global stability for an unbounded set of parameters.

Theorem 2.16. *For every $r > 0$, there exists $s, \alpha, \beta > 0$ for which the conditions of Theorem 2.14 are satisfied.*

Proof. Let $r > 0$ be fixed and $s = 1$. Then the inequality (2.25) becomes equivalent to the following inequality:

$$\frac{r-2}{\alpha} < \frac{1 - \beta e^{r-1}}{1 + \beta e^{r-1}} e^{-\beta e^{r-1}}. \tag{2.26}$$

It is clear that

$$\lim_{\beta\to 0}\frac{1-\beta e^{r-1}}{1+\beta e^{r-1}}e^{-\beta e^{r-1}}=1.$$

If we choose any $\alpha\in(r-2,r)$, then $\frac{r-2}{\alpha}<1$. So there exists a small enough $\beta_\alpha>0$ such that for every $\beta\in(0,\beta_\alpha)$ the inequality (2.26) is satisfied.

In addition to (2.25), we must have $\mathbf{c}_4\in\mathbb{R}^2_+$ and $\mathbf{c}_4$ must be asymptotically stable.
Let $r>0$, $s=1$ and $\alpha\in(r-2,r)$. We have

$$\mathbf{c}_4=\left(\frac{r-\alpha}{1-\alpha\beta},\frac{1-r\beta}{1-\alpha\beta}\right),$$

now since

$$\lim_{\beta\to 0}\frac{r-\alpha}{1-\alpha\beta}=r-\alpha>0,\qquad \lim_{\beta\to 0}\frac{1-r\beta}{1-\alpha\beta}=1,$$

there exists a small enough $\beta'>0$ such that for every $\beta\in(0,\beta')$, $\mathbf{c}_4\in\mathbb{R}^2_+$.

For the local stability condition, we again let $r>0$, $s=1$ and $\alpha\in(r-2,r)$. Since $s=1$, by Theorem 2.10 (a), $\mathbf{c}_4$ is asymptotically stable if and only if $\frac{1-\beta r}{1-\alpha\beta}>\frac{\beta(r-2)}{\alpha}\frac{r-\alpha}{1-\alpha\beta}+\frac{r-2}{\alpha}$. As we have

$$\lim_{\beta\to 0}\frac{1-\beta r}{1-\alpha\beta}=1,\qquad \lim_{\beta\to 0}\frac{\beta(r-2)}{\alpha}\frac{r-\alpha}{1-\alpha\beta}=0,$$

there exists a small enough $\beta''>0$ such that for every $\beta\in(0,\beta'')$, the desired inequality is satisfied.

Now by letting $\beta^\star=\min\{\beta_\alpha,\beta',\beta''\}$, for $r>0$, $s=1$, $\alpha\in(r-2,r)$ and $\beta\in(0,\beta^\star)$ the conditions of Theorem 2.14 are satisfied. $\square$

Chapter 3

Convexity of Non-Compact Carrying Simplices in Logarithmic Coordinates

We study a non-compact version of the carrying simplex for the planar Leslie-Gower and planar Ricker maps when they are written in logarithmic variables. We show that for both of these models there is a convex (unbounded) invariant set X_∞, and all orbits are attracted to X_∞. For the Leslie-Gower map, which is injective, the boundary of X_∞ globally attracts all orbits and we identify it with a non-compact carrying simplex. As the Ricker map is not invertible, the boundary of X_∞ may not be invariant. We establish conditions on the parameters of the Ricker map which guarantee that there is a convex non-compact carrying simplex when $r, s < 1$ which maps into a compact carrying simplex in the standard untransformed coordinates.[1]

3.1 Introduction

Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_{++} = (0, \infty)$ and $F : \mathbb{R}_+^2 \to \mathbb{R}_+^2$ be a continuous function. Consider the following planar difference equation:

$$\mathbf{x}_{n+1} := F(\mathbf{x}_n) = F^{n+1}(\mathbf{x}_0), \quad n \in \mathbb{N} := \{0, 1, 2, ...\}, \quad \mathbf{x}_0 \in \mathbb{R}_+^2. \tag{3.1}$$

In this chapter we are interested in a special invariant curve of (3.1) known as the *carrying simplex.* Hirsch's definition [17] of a carrying simplex, when applied to the above system, is as follows.

Definition 3.1. We call $\Sigma \subset \mathbb{R}_{++}^2$ a carrying simplex if
(CS1) Σ is compact and invariant.
(CS2) For any $\mathbf{x} \in \mathbb{R}_+^2 - \{\mathbf{0}\}$ there exists $\mathbf{y} \in \Sigma$ such that $\lim_{n\to\infty} \mid F^n(\mathbf{x}) - F^n(\mathbf{y}) \mid = 0$. (asymptotic completeness)

[1]The content of this chapter has been published in the Journal of Difference Equations and Applications as a paper by Hamid Naderi Yeganeh and Steve Baigent, titled "Convexity of Non-Compact Carrying Simplices in Logarithmic Coordinates" [45].

(CS3) Σ is unordered. (i.e. for $(x_1, y_1), (x_2, y_2) \in \Sigma$, if $x_1 < x_2$, then $y_2 < y_1$, and if $y_1 < y_2$, then $x_2 < x_1$)

When it exists, the carrying simplex Σ is thus a compact and invariant manifold for (3.1) that attracts $\mathbb{R}^2_+ - \{\mathbf{0}\}$ and that has the special property that Σ is the graph of a decreasing and continuous function. To date, and to the best of the authors' knowledge, planar carrying simplices for discrete dynamics have been studied exclusively in the context of retrotone systems (e.g. [17, 49, 19, 43]).

This chapter explores the pros and cons of working in alternative coordinates where compactness of the carrying simplex is lost.

Definition 3.2. A map $F = (F_1, F_2) : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is retrotone (e.g. [17, 49]) in a subset $D \subset \mathbb{R}^2_+$ if for $(x_1, x_2), (y_1, y_2) \in D$ such that $F_1(x_1, x_2) \geq F_1(y_1, y_2)$ and $F_2(x_1, x_2) \geq F_2(y_1, y_2)$ but $F(x_1, x_2) \neq F(y_1, y_2)$ we have $x_1 > y_1$ provided $y_1 > 0$ and $x_2 > y_2$ provided $y_2 > 0$.

A retrotone map is sometimes also called a competitive map (see, for example, [52]). In the planar case a map satisfying Definition 3.2 has the special property that it maps the graph of a decreasing function on D to the graph of a new decreasing function on D [4, 6].

The Leslie-Gower map from ecology [35] that we study in Section 3.3.1 is retrotone for all biologically realistic parameter values, and it is well-known that it has a unique carrying simplex [20]. On the other hand the Ricker map is not retrotone everywhere in $\mathbb{R}^2_+$ (see for example [17, 49, 19]), and so existence of a carrying simplex in the standard coordinates of population densities, by means of retronicity, is only known for a limited set of parameter values.

Here we will extend the notion of the carrying simplex applied to planar systems to allow it to be non-compact, and we will call a set $\Sigma \subset \mathbb{R}^2$ a *non-compact carrying simplex* if it satisfies (CS1) without compactness and (CS3), but (CS2) is replaced by the lesser requirement that Σ globally attracts $\mathbb{R}^2$. The issue of asymptotic completeness will be addressed elsewhere.

In working with non-compact carrying simplices we may work in alternative coordinate systems for which the systems (3.1) that we consider here have at most one (finite) fixed point, but in so doing we lose compactness of the global attractor and asymptotic completeness. We have found that by using logarithmically transformed coordinates, we are sometimes able to obtain stronger geometrical properties for the non-compact carrying simplex, namely that it is the graph of a concave decreasing function. While the corresponding compact carrying simplices are also known to be graphs of decreasing functions, whether or not those functions are convex or concave is not generally known (for results on convexity of carrying simplices see [4] [3] [56] [2]). Here, we will also discuss the convexity of the boundary of the basin of repulsion of infinity in the logarithmically scaled Leslie-Gower and Ricker models. When all the parameters are positive, then the maps in the logarithmically scaled versions of both models are concave (i.e. each component of the map is a concave function [23]). We take advantage of this fact to prove that the basin of repulsion of infinity is an invariant convex set. We establish a relationship between the convexity and the strict decreasingness of the members of a sequence of sets that converges to the boundary of the basin of repulsion of infinity. Then, it becomes straightforward to show that this boundary satisfies (CS3).

3.2 Preliminary results

In this section, we prove three lemmas that play pivotal roles. The first lemma will enable us to prove that the boundary of each of the sets we are discussing is the graph of a continuous strictly decreasing function. The second lemma shows that for given a set in a certain class of subsets of $\mathbb{R}^2$ whose members have boundary that is the graph of a continuous strictly decreasing function, that set must be convex.

Lemma 3.3. *Let $X \subset \mathbb{R}^2$ and $a, b \in \mathbb{R}$ be given. Suppose there exist two continuous functions $A : (-\infty, a) \to \mathbb{R}$ and $B : (-\infty, b) \to \mathbb{R}$ such that*

$$\{x \mid (x, c) \in X\} = \begin{cases} (-\infty, A(c)], & c \in (-\infty, a) \\ \emptyset, & \textit{otherwise} \end{cases} \tag{3.2}$$

$$\{y \mid (d, y) \in X\} = \begin{cases} (-\infty, B(d)], & d \in (-\infty, b) \\ \emptyset, & \textit{otherwise.} \end{cases} \tag{3.3}$$

Then $X \subset (-\infty, b) \times (-\infty, a)$, both A and B are strictly decreasing functions, and

$$\partial X = \Big\{(A(c), c) \mid \ c \in (-\infty, a)\Big\} \tag{3.4}$$

$$= \Big\{(d, B(d)) \mid \ d \in (-\infty, b)\Big\}. \tag{3.5}$$

In other words, the boundary of X is the graph of a strictly decreasing function and X is the set of all points on or under the graph of that function.

Proof. It is clear that we have

$$\Big\{(A(c), c) \mid \ c \in (-\infty, a)\Big\} \subseteq \partial X$$
$$\Big\{(d, B(d)) \mid \ d \in (-\infty, b)\Big\} \subseteq \partial X.$$

To prove (3.4), we observe that for each $(x, y) \in \partial X$ there exist $\{(x_n, y_n)\}_{n=1}^{\infty} \subseteq X$ and $\{(x'_n, y'_n)\}_{n=1}^{\infty} \subseteq (-\infty, b) \times (-\infty, a) - X$ such that

$$\lim_{n\to\infty} (x_n, y_n) = \lim_{n\to\infty} (x'_n, y'_n) = (x, y).$$

For each $n \in \mathbb{N}$ we have $x_n \leq A(y_n)$ and $x'_n > A(y'_n)$. It follows that since A is continuous we have

$$x = \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} A(y_n) = A(y) = \lim_{n\to\infty} A(y'_n) \leq \lim_{n\to\infty} x'_n = x.$$

Hence, $x = A(y)$ and $(x, y) \in \Big\{(A(c), c) \mid \ c \in (-\infty, a)\Big\}$. This proves (3.4). Proving (3.5) is similar. It is clear that (3.4) and (3.5) imply that A and B are inverse of each other and they are both bijective. Hence, by using the fact that they are continuous functions, we deduce that A and B are strictly decreasing functions (These functions cannot be strictly increasing since $X \subset (-\infty, b) \times (-\infty, a)$ and the boundary of X is equal to each of the graphs of A and B). $\square$

Before stating Lemma 3.4, we have to define the relation "$\ll$" between some members of $\mathbb{R}^2$. Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we write $(x_1, y_1) \ll (x_2, y_2)$ if $x_1 < x_2$ and $y_1 < y_2$.

Lemma 3.4. *Let $X \subset (-\infty, b) \times (-\infty, a)$ be the set of points on or under the graph of the continuous strictly decreasing function $B : (-\infty, b) \to (-\infty, a)$. Assume that for every $\mathbf{x}, \mathbf{y} \in X$ and $0 < \lambda < 1$ there exists at least one $\mathbf{z} \in X$ such that*

$$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \ll \mathbf{z}. \tag{3.6}$$

Then X is convex.

Proof. Assume that X is not convex. Then there exist $\mathbf{x}, \mathbf{y} \in X$ and $0 < \lambda < 1$ such that

$$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \notin X.$$

Assume that $\mathbf{z}$ is as stated in the theorem. Since $z_1 > \lambda x_1 + (1 - \lambda)y_1$ and B is strictly decreasing, we have

$$B(z_1) < B(\lambda x_1 + (1 - \lambda)y_1)$$

and since $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \notin X$ and $\mathbf{z} \in X$, we have

$$z_2 \leq B(z_1)$$

$$B(\lambda x_1 + (1 - \lambda)y_1) < \lambda x_2 + (1 - \lambda)y_2.$$

Hence,

$$z_2 < \lambda x_2 + (1 - \lambda)y_2$$

which contradicts (3.6). $\square$

Lemma 3.5. *Let $x_0 \in \mathbb{R}$ be given and suppose that $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $K : (-\infty, x_0) \to \mathbb{R}$ are continuous functions and p satisfies*

$$\lim_{x \to -\infty} \sup\{p\,(x, y) \mid y \in \mathbb{R}\} = -\infty. \tag{3.7}$$

Suppose also that there exists $G : (-\infty, y^\star) \to \mathbb{R}$ defined by

$$G(c) := \sup p\left(\{(x, q(x, c)) \mid x \in (-\infty, K(c)]\}\right).$$

Then G is continuous and

$$\Omega_c := p\left(\{(x, q(x, c)) \mid x \in (-\infty, K(c)]\}\right) = (-\infty, G(c)], \quad c \in (-\infty, y^\star).$$

Proof. Fix $c \in (-\infty, y^\star)$. Since the continuous image of a connected set is connected, we deduce that Ω_c is connected. Equation (3.7) implies that

$$\lim_{x \to -\infty} p(x, q(x, c)) = -\infty,$$

and hence Ω_c is unbounded below and there exists $L \in (-\infty, K(c))$ such that for every $x < L$ we have $p(x, q(x, c)) < G(c) - 1$. Therefore, by compactness of $Y_c := [L, K(c)]$ we deduce that

$$G(c) = \sup \Omega_c \in p\left(\{(x, q(x, c)) \mid x \in Y_c\}\right) \subset \Omega_c. \tag{3.8}$$

Connectedness of Ω_c along with the fact that it is unbounded below and $G(c) = \sup \Omega_c \in \Omega_c$ proves $\Omega_c = (-\infty, G(c)]$.

We now prove continuity of G by contradiction. Suppose that G is not continuous at some $c_0 \in (-\infty, x_0)$. Then there exist a sequence $\{a_n\}$ which converges to c_0 and $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ we have $|G(c_0) - G(a_n)| \geq \varepsilon$. By (3.8) we know that $G(c_0) \in \Omega_{c_0}$. Hence there exists $w_{c_0} \in Y_{c_0}$ such that $p(w_{c_0}, q(w_{c_0}, c_0)) = G(c_0)$. Similarly, for every $n \in \mathbb{N}$ there exists $w_{a_n} \in Y_{a_n}$ such that $p(w_{a_n}, q(w_{a_n}, a_n)) = G(a_n)$. Since K is continuous, we can find a sequence $\{v_n\}$ which converges to w_{c_0} and for every $n \in \mathbb{N}$ we have $v_n \in (-\infty, K(a_n)]$. By continuity of p and q we have

$$\lim_{n \to \infty} p(v_n, q(v_n, a_n)) = p(w_{c_0}, q(w_{c_0}, c_0)) = G(c_0).$$

For every $n \in \mathbb{N}$ we have

$$p(v_n, q(v_n, a_n)) \leq \sup p\left(\{(x, q(x, a_n)) \mid x \in (-\infty, K(a_n)]\}\right) = G(a_n).$$

Thus

$$G(c_0) \leq \liminf_{n \to \infty} G(a_n). \tag{3.9}$$

Inequality (3.9) along with $|G(c_0) - G(a_n)| \geq \varepsilon$ implies that there exists $M > 0$ such that for every $n > M$ we have $G(c_0) + \frac{\varepsilon}{2} < G(a_n) = p(w_{a_n}, q(w_{a_n}, a_n))$. From (3.7), there exists $x_1 \in \mathbb{R}$ such that for every $x < x_1$ and $y \in \mathbb{R}$ we have $p(x, y) < G(c_0) + \frac{\varepsilon}{2}$. Hence for every $n > M$ we have $x_1 \leq w_{a_n} \leq K(a_n)$. This along with the fact that K is continuous, implies that there exists $M' > 0$ such that for every $n > M'$ we have $x_1 \leq w_{a_n} \leq K(c_0) + 1$. Thus $\{w_{a_n}\}$ is bounded and has a convergent subsequence $\{w_{a_{m_n}}\}$. By the continuity of p and q we have

$$G(c_0) + \frac{\varepsilon}{2} \leq \lim_{n\to\infty} G(a_{m_n}) = \lim_{n\to\infty} p(w_{a_{m_n}}, q(w_{a_{m_n}}, a_{m_n})) = p(b, q(b, c_0)), \tag{3.10}$$

where $b = \lim_{n\to\infty} w_{a_{m_n}}$. But it is clear that we also have

$$p(b, q(b, c_0)) \leq \sup p\left(\{(x, q(x, c_0)) \mid x \in (-\infty, K(c_0)]\}\right) = G(c_0)$$

which contradicts (3.10). Therefore, G is continuous at c_0. Since $c_0 \in (-\infty, x_0)$ is arbitrary we see that G is continuous on $(-\infty, x_0)$.

□

We will now combine Lemmas 3.3, 3.4 and 3.5 to show that two well-known maps from theoretical ecology have globally attracting and invariant 1-dimensional manifolds, and also determine when they are the invariant boundary of an invariant convex set.

3.3 Applications to ecological models

In this section, we use the above theory to prove the convexity of a unique non-compact carrying simplex in logarithmically scaled versions of the Leslie–Gower Model and Ricker models from theoretical ecology. Before detailing the theory, first we need to state an important notation followed by a lemma that will be applied to both maps. Suppose that $D \subset \mathbb{R}^2$ is a closed set and $g : D \to \mathbb{R}^2$. We define

$$X_0 := D,$$

$$X_n := \overline{g(X_{n-1})}, \quad n = 1, 2, 3, ...$$

and finally

$$X_\infty := \bigcap_{n=0}^{\infty} X_n. \tag{3.11}$$

Now we have the following lemma.

Lemma 3.6. *Suppose that $D \subset \mathbb{R}^2$ is a closed set and $g : D \to \mathbb{R}^2$ is proper (i.e. the inverse* *image of any compact subset of D is compact) and continuous and $g(D) \subset D$. Then X_∞ is invariant under the map g.*

Proof. If $\mathbf{x} \in g(X_\infty)$ then $\mathbf{x} \in g(X_n) \subset \overline{g(X_n)} = X_{n+1}$ for $n \in \mathbb{N}$. Hence $\mathbf{x} \in \bigcap_{n=1}^{\infty} X_n$ and since $X_1 = \overline{g(D)} \subset \overline{D} = D = X_0$, we have $\mathbf{x} \in \bigcap_{n=1}^{\infty} X_n = \bigcap_{n=0}^{\infty} X_n = X_\infty$. This proves $g(X_\infty) \subset X_\infty$.

We prove that $\overline{g(X_n)} = g(X_n)$ for $n \in \mathbb{N}$. It is sufficient to show that $g(X_n)$ is closed. Suppose that $\mathbf{x}^*$ is a limit point of $g(X_n)$. There exists a sequence $\{\mathbf{y}_n\} \subset X_n$ such that $\lim_{n\to\infty} g(\mathbf{y}_n) = \mathbf{x}^*$. $\{\mathbf{y}_n\}$ is bounded since, being proper, g maps unbounded subsets of X_0 into unbounded sets whereas $\{g(\mathbf{y}_n)\}$ is convergent, and hence bounded. Boundedness of $\{\mathbf{y}_n\} \subset X_n$ and the fact that X_n is closed, imply that $\{\mathbf{y}_n\}$ has a limit point $\mathbf{y}^* \in X_n$. Continuity of g implies $g(\mathbf{y}^*) = \mathbf{x}^*$. Thus $\mathbf{x}^* \in g(X_n)$, which proves that $g(X_n)$ is closed and hence $\overline{g(X_n)} = g(X_n)$.

Now if $\mathbf{x} \in X_\infty$, then $\mathbf{x} \in X_{n+1} = \overline{g(X_n)} = g(X_n)$ for $n \in \mathbb{N}$. Hence for $n \in \mathbb{N}$ there exists $\mathbf{z}_n \in X_n$ such that $g(\mathbf{z}_n) = \mathbf{x}$. Since g is proper, $g^{-1}(\{\mathbf{x}\})$ is compact. Therefore, since $\{\mathbf{z}_n\} \subset g^{-1}(\{\mathbf{x}\})$, $\{\mathbf{z}_n\}$ is bounded and has at least one limit point. Let $\mathbf{z}^*$ be such a limit point. $\mathbf{z}^*$ is also a limit point of $g^{-1}(\{\mathbf{x}\})$, and since $g^{-1}(\{\mathbf{x}\})$ is closed, we have $\mathbf{z}^* \in g^{-1}(\mathbf{x})$. Thus $g(\mathbf{z}^*) = \mathbf{x}$ and since $\mathbf{z}^*$ is a limit point of $\{\mathbf{z}_n\}$, and for $n \in \mathbb{N}$ we have $\mathbf{z}_n \in X_n$ where $\{X_n\}$ is a decreasing sequence of closed sets, we conclude that $\mathbf{z}^* \in X_\infty$. This proves $X_\infty \subset g(X_\infty)$. □

3.3.1 The Leslie–Gower Model

The planar Leslie-Gower model [35, 10] is defined by the Leslie-Gower map

$$F(u,v) := \left(\frac{ru}{1+u+\alpha v}, \frac{sv}{1+v+\beta u} \right). \tag{3.12}$$

When $r, s < 1$ and $\alpha, \beta > 0$, then $(0,0)$ is globally asymptotically stable on $\mathbb{R}^2_+$ (see [10]). Hence, the system has no carrying simplex when $r, s < 1$ and $\alpha, \beta > 0$ since no $\Sigma \subset \mathbb{R}^2_+ - \{\mathbf{0}\}$ can satisfy (CS2).

When $r, s > 1$ the Leslie-Gower map has fixed points

$$(0,0),\ (r-1,0),\ (0,s-1),\ \text{and}\ \left(\frac{\alpha(s-1)-r+1}{\alpha\beta-1}, \frac{\beta(r-1)-s+1}{\alpha\beta-1} \right) \text{if positive.} \tag{3.13}$$

A number of authors [17, 21, 4, 19] have shown that for $r, s > 1$ and $\alpha, \beta > 0$, the model (3.12) has a unique carrying simplex. In our approach, we use an alternative set of coordinates to those in (3.12): We scale (3.12) as follows

$$u = e^x, \quad v = e^y, \tag{3.14}$$

to obtain the following log-scaled version of the model:

$$f(x,y) := \big(\ln(r) + x - \ln(1 + e^x + \alpha e^y),\ \ln(s) + y - \ln(1 + e^y + \beta e^x)\big). \tag{3.15}$$

The only finite fixed point of the log-scale Leslie-Gower map is

$$\left(\log\left(\frac{\alpha(s-1) - r + 1}{\alpha\beta - 1}\right), \log\left(\frac{\beta(r-1) - s + 1}{\alpha\beta - 1}\right)\right), \tag{3.16}$$

when the expressions are real.

We wish to study the invariant sets of the map $f : \mathbb{R}^2 \to \mathbb{R}^2$, and to this end we consider X_∞ defined by (3.11) with $D = \overline{f(\mathbb{R}^2)}$ and $f = g$ at every point on D. It can be easily observed that this $g : D \to \mathbb{R}^2$ is proper and continuous and $g(D) \subset D$. Therefore, X_∞ satisfies the conditions of Lemma 3.6 and as a result, it's invariant.

The first lemma is needed because standard theorems on nonempty intersections of decreasing sequences of compact sets cannot be applied here as our sets X_n are not compact.

Lemma 3.7. *When $r, s > 1$ we have $X_\infty \neq \emptyset$.*

Proof. Suppose that $\zeta_{x,y}$ and $\eta_{x,y}$ are defined as follows:

$$\zeta_{x,y} := \frac{1 - \frac{\alpha y}{y-s}}{r - x + \frac{\alpha\beta xy}{y-s}},$$
$$\eta_{x,y} := \frac{1 - \frac{\beta x}{x-r}}{s - y + \frac{\alpha\beta xy}{x-r}}.$$

Since $\lim_{(x,y)\to(0,0)}(\zeta_{x,y}, \eta_{x,y}) = \left(\frac{1}{r}, \frac{1}{s}\right) \ll (1,1)$, there exists $x', y' \in \mathbb{R}_{++}$ such that for every $(0,0) \leq (x,y) \leq (x',y')$ we have $\zeta_{x,y} < 1$, $\eta_{x,y} < 1$. Hence if $S = (0, x'] \times (0, y']$, for every $(x,y) \leq (x',y')$ we have $(x\zeta_{x,y}, y\eta_{x,y}) \in S$. It can also be easily verified that $F(x\zeta_{x,y}, y\eta_{x,y}) = (x,y)$. Thus $S \subset F(S)$ and if we define $S^* := \{(\ln(x), \ln(y)) | (x,y) \in S\}$, then $S^* \subset f(S^*)$. It means that for $n = 0, 1, 2, ...$ we have $S^* \subset X_n$. Therefore we have $\emptyset \neq S^* \subset X_\infty$.

□

In the following, we rely strongly on the fact that f in (3.12) is invertible.

Lemma 3.8. *The set ∂X_∞ is invariant.*

Proof. As f defines a diffeomorphism from X_∞ onto X_∞, both f and f^{-1} map the interior of X_∞ into itself. Hence the interior of X_∞ is invariant. As X_∞ is also invariant, ∂X_∞ must be invariant.

□

Lemma 3.9. *For any $r, s, \alpha, \beta > 0$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $\mathbf{x} \neq \mathbf{y}$ and $0 < \lambda < 1$ we have*

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \gg \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}). \tag{3.17}$$

Proof. Define $\xi_1 : [0,1] \to \mathbb{R}$ as follows

$$\xi_1(\lambda) := f_1(\lambda \mathbf{x} + (1-\lambda)\mathbf{y})$$

If $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, then we have

$$\begin{aligned}\xi_1''(\lambda) = &-\frac{(x_1-y_1)^2 e^{\lambda x_1 + (1-\lambda)y_1} + \alpha(x_2-y_2)^2 e^{\lambda x_2+(1-\lambda)y_2}}{(1+e^{\lambda x_1+(1-\lambda)y_1} + \alpha e^{\lambda x_2+(1-\lambda)y_2})^2} \\ &-\frac{\alpha(x_1-y_1-x_2+y_2)^2 e^{\lambda x_1+(1-\lambda)y_1} e^{\lambda x_2+(1-\lambda)y_2}}{(1+e^{\lambda x_1+(1-\lambda)y_1} + \alpha e^{\lambda x_2+(1-\lambda)y_2})^2} < 0.\end{aligned}$$

Hence ξ_1 is strictly concave. Similarly $\xi_2 : [0,1] \to \mathbb{R}$ defined by $\xi_2(\lambda) := f_2(\lambda \mathbf{x} + (1-\lambda)\mathbf{y})$ is also strictly concave. The inequality (3.17) is now a direct result of the strict concavity of ξ_1 and ξ_2 and the following facts:

$$\xi_i(0) = f_i(\mathbf{y}), \quad \xi_i(1) = f_i(\mathbf{x}), \quad i = 1, 2.$$

$\square$

Lemma 3.10. *(a) Let $X \subset \mathbb{R}^2$ be the set of all points on or under the graph of a continuous strictly decreasing function $B : (-\infty, b) \to \mathbb{R}$. Then for the log-scaled Leslie-Gower map $f = (f_1, f_2)$ in (3.15) we have*

$$\{x \mid (x,c) \in f(X)\} = \begin{cases} f_1\left(\{(x, g(x) + h(c)) \mid x \in (-\infty, K(c))\}\right), & c \in (-\infty, y^\star) \\ \emptyset, & \textit{otherwise} \end{cases}$$

where $y^\star = \sup\{f_2(x,y) \mid (x,y) \in f(X)\}$, and

$$g(x) = \ln(1 + \beta e^x)$$

$$h(c) = c - \ln(s - e^c)$$

$$K(c) = H^{-1}(c - \ln(s - e^c))$$

and H is the invertible continuous function defined by

$$H(x) = B(x) - \ln(1 + \beta e^x).$$

(b) We have

$$\{x \mid (x,c) \in f(\mathbb{R}^2)\} = \begin{cases} \left(-\infty, \ln\left(\frac{r}{1+\frac{e^c\alpha\beta}{s-e^c}}\right)\right), & c \in (-\infty, \ln(s)) \\ \emptyset, & \textit{otherwise} \end{cases}$$

where h and g are as defined in part (a).

Proof. (a) For $c \in (-\infty, y^\star)$ we have

$$\{x \mid (x,c) \in f(X)\} = f_1\left(X \cap \left\{(x,y) \in \mathbb{R}^2 \mid f_2(x,y) = c\right\}\right)$$

$$= f_1\left(X \cap \left\{(x,y) \in \mathbb{R}^2 \mid y = c - \ln(s - e^c) + \ln(1 + \beta e^x)\right\}\right)$$

$$= f_1\left(\left\{(x,y) \in \mathbb{R}^2 \mid y = c - \ln(s - e^c) + \ln(1 + \beta e^x), y \leq B(x)\right\}\right)$$

$$= f_1\left(\{(x, \ln(1 + \beta e^x) + c - \ln(s - e^c)) \mid c - \ln(s - e^c) + \ln(1 + \beta e^x) \leq B(x)\}\right)$$

$$= f_1\left(\{(x, g(x) + h(c)) \mid c - \ln(s - e^c) \leq B(x) - \ln(1 + \beta e^x)\}\right)$$

$$= f_1\left(\{(x, g(x) + h(c)) \mid c - \ln(s - e^c) \leq H(x)\}\right).$$

Since B is strictly decreasing, H is strictly decreasing and invertible. Hence,

$$\{x \mid (x,c) \in f(X)\} = f_1\left(\{(x, g(x) + h(c)) \mid x \in (-\infty, H^{-1}(c - \ln(s - e^c))]\}\right).$$

(b) For $c \in (-\infty, \ln(s))$ we have

$$\{x \mid (x,c) \in f(\mathbb{R}^2)\} = f_1\left(\left\{(x,y) \in \mathbb{R}^2 \mid f_2(x,y) = c\right\}\right)$$

$$= f_1\left(\left\{(x,y) \in \mathbb{R}^2 \mid y = c - \ln(s - e^c) + \ln(1 + \beta e^x)\right\}\right)$$

$$= f_1\left(\{(x, \ln(1 + \beta e^x) + c - \ln(s - e^c)) \mid x \in \mathbb{R}\}\right)$$

$$= f_1\left(\{(x, g(x) + h(c)) \mid x \in \mathbb{R}\}\right),$$

$$= \{f_1(x, g(x) + h(c)) \mid x \in \mathbb{R}\},$$

$$= \{\ln(r) + x - \ln(1 + e^x + \alpha e^{g(x)+h(c)}) \mid x \in \mathbb{R}\},$$

$$= \{\ln(r) + x - \ln(1 + e^x + \alpha e^{\ln(1+\beta e^x)+c-\ln(s-e^c)}) \mid x \in \mathbb{R}\},$$

$$= \left\{\ln(r) + x - \ln\left(1 + \frac{\alpha e^c}{s - e^c} + e^x\left(1 + \frac{e^c \alpha\beta}{s - e^c}\right)\right) \mid x \in \mathbb{R}\right\}.$$

Since $c \in (-\infty, \ln(s))$, $s - e^c$ is positive. Hence, $1 + \frac{\alpha e^c}{s-e^c} > 0$ and $1 + \frac{e^c\alpha\beta}{s-e^c} > 0$ which implies that the derivative of the function $w(x) = \ln(r) + x - \ln\left(1 + \frac{\alpha e^c}{s-e^c} + e^x\left(1 + \frac{e^c\alpha\beta}{s-e^c}\right)\right)$ is always positive. Thus

$$\left\{\ln(r) + x - \ln\left(1 + \frac{\alpha e^c}{s - e^c} + e^x\left(1 + \frac{e^c \alpha\beta}{s - e^c}\right)\right) \mid x \in \mathbb{R}\right\} = \left(-\infty, \lim_{x\to+\infty} w(x)\right)$$

$$= \left(-\infty, \ln\left(\frac{r}{1 + \frac{e^c\alpha\beta}{s-e^c}}\right)\right).$$

□

Combining the previous lemmas together we obtain.

Theorem 3.11. *For any $r, s > 1$ and $\alpha, \beta > 0$, for the log-scaled Leslie-Gower map (3.15), the set X_∞ defined by (3.11) is convex and invariant. Moreover, ∂X_∞ is invariant and attracts $\mathbb{R}^2$.*

Proof. It is clear that X_0 is convex. We use induction to prove that for $n = 1, 2, ...,$ X_n is convex, from which it follows that their intersection X_∞ is convex. To prove convexity of X_1, first we observe that by Lemma 3.10 (b) for every $c \in (-\infty, \ln(c))$ we have

$$\{x \mid (x, c) \in X_1\} = \overline{\{x \mid (x, c) \in f(\mathbb{R}^2)\}} = \left(-\infty, \ln\left(\frac{r}{1 + \frac{e^c\alpha\beta}{s-e^c}}\right)\right].$$

Now $A(c) := \ln\left(\frac{r}{1+\frac{e^c\alpha\beta}{s-e^c}}\right)$ and $X = X_1$ satisfy the conditions stated for A in Lemma 3.3.

So far, we have proven the existence of A which satisfies the conditions of Lemma 3.3 for $X = X_1$. But to apply Lemma 3.3 we also need to prove the existence of the second function B of that lemma. Indeed it is easy check that B is given by $B(c) = A^{-1}(c) = \ln\left(\frac{s}{1+\frac{e^c\alpha\beta}{r-e^c}}\right)$. Hence by Lemma 3.3, X_1 is the set of all points on or under the graph of a continuous strictly decreasing function. It is obvious that for every $\mathbf{x}, \mathbf{y} \in X_1$ and $0 < \lambda < 1$ we have $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \in$

$f(\mathbb{R}^2) = X_1$. Moreover by Lemma 3.9 we have $\lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}) \ll f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y})$. Therefore, for every $\mathbf{x}, \mathbf{y} \in X_1$ and $0 < \lambda < 1$ there exists $\mathbf{z} = f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \in X_1$ such that $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} < \mathbf{z}$. Now since X_1 satisfies the conditions of Lemma 3.4, we deduce that X_1 is convex.

Assume that for $n \geq 1$, X_n is convex and that it is the set of all points on or under the graph of a continuous strictly decreasing function $B : (-\infty, b) \to \mathbb{R}$. By Lemma 3.10 (a), for every $c \in (-\infty, y^\star)$ we have

$$\{x \mid (x,c) \in X_{n+1}\} = \{x \mid (x,c) \in f(X_n)\} = f_1\left(\{(x, g(x) + h(c)) \mid x \in (-\infty, K(c))\}\right).$$

where g, h and K are as defined in Lemma 3.10. The functions $p(x,y) := f_1(x,y)$, $q(x,y) := g(x) + h(y)$ and K satisfy the conditions of Lemma 3.5, so that for every $c \in (-\infty, y^\star)$ we have $\Omega_c = (-\infty, G(c)]$, where $G : (-\infty, y^\star) \to \mathbb{R}$ is continuous. Thus

$$X_{n+1} = \bigcup_{c\in(-\infty,y^\star)} \Omega_c = \bigcup_{c\in(-\infty,y^\star)} (-\infty, G(c)].$$

$A := G$ and $X := X_{n+1}$ satisfy the conditions stated for A in Lemma 3.3. Owing to the symmetric structure of the definition of the log-scaled Leslie-Gower map we can prove the existence of B which satisfies the conditions of Lemma 3.3 for $X = X_{n+1}$. Therefore, Lemma 3.3 shows that X_{n+1} is the set of all points on or under the graph of a continuous strictly decreasing function.

Suppose that $\mathbf{x}, \mathbf{y} \in X_{n+1}$. Since the sequence $\{X_n\}$ is decreasing, we have $\mathbf{x}, \mathbf{y} \in X_n$. Now since X_n is convex, for every $0 < \lambda < 1$ we have $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in X_n$, thus $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \in f(X_n) = X_{n+1}$. By Lemma 6 we have $\lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}) < f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y})$. Therefore, for every $\mathbf{x}, \mathbf{y} \in X_{n+1}$ and $0 < \lambda < 1$ there exists $\mathbf{z} = f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \in X_{n+1}$ such that $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} < \mathbf{z}$. Since X_{n+1} satisfies the conditions of Lemma 2, we deduce that X_{n+1} is convex. We conclude that X_∞ is convex.

∂X_∞ is invariant by Lemma 3.8. We know that ∂X_∞ is the graph of a concave and strictly decreasing function A. By invariance, $f_1(A^{-1}(y), y) = A^{-1}(f_2(A^{-1}(y), y))$, i.e. $A^{-1}(y) + \log r - \log(1 + \alpha e^{A^{-1}(y)} + e^y) = A^{-1}(y + \log s - \log(1 + e^y + \beta e^{A^{-1}(y)}))$. As A^{-1} is strictly decreasing and bounded above by zero, $\lim_{y\to-\infty} A^{-1}(y) = x^*$ and invariance yields $x^* + \log r - \log(1 + e^{x^*}) = x^*$, so $x^* = \log(r-1)$. A similar argument shows that $\lim_{x\to-\infty} A(x) = \log(s-1)$.

To show that ∂X_∞ attracts $\mathbb{R}^2$, first we show that any finite fixed point of (3.15) must belong to ∂X_∞. As there can be at most one finite fixed point, if $P = (P_1, P_2)$ is a finite fixed point not in ∂X_∞ then ∂X_∞ contains no finite fixed point and dynamics on ∂X_∞ is monotone. On ∂X_∞ we may consider the one-dimensional dynamics $x_{n+1} = f_1(x_n, A^{-1}(x_n))$ or $y_{n+1} = f_2(A(y_n), y_n)$. Suppose $x_n \to -\infty$ when $n \to \infty$:

$$0 > x_{n+1} - x_n = \log r - \log(1 + e^{x_n} + \alpha e^{A^{-1}(x_n)}) \to \log r - \log(1 + \alpha(s-1))$$

so we need $r-1<\alpha(s-1)$. On the other hand

$$0<y_{n+1}-y_n=\log s-\log(1+e^{y_n}+\beta e^{A(y_n)})\to\log s-\log(1+\beta(r-1))$$

so we also need $s-1>\beta(r-1)$. The pair of conditions $r-1<\alpha(s-1)$ and $s-1>\beta(r-1)$ are incompatible with existence of a finite fixed point (3.16) of (3.15). A similar contradiction is obtained when the dynamics is monotone increasing in x_n. Hence we conclude that whenever a finite fixed point exists for (3.15) it must belong to ∂X_∞.

Next we recall (e.g. [10]) that all non-trivial dynamics for the unscaled Leslie-Gower map converge to a fixed point which can be $(r-1,0)$, $(0,s-1)$, or $\left(\frac{\alpha(s-1)-r+1}{\alpha\beta-1},\frac{\beta(r-1)-s+1}{\alpha\beta-1}\right)$ when it is positive. Hence any orbit of (3.12) not convergent to the positive fixed point must converge to $(r-1,0)$ or $(0,s-1)$.

Now consider an orbit (x_n,y_n) of (3.15) that does not converge to a finite fixed point. Then by convergence of Leslie-Gower orbits, (e^{x_n},e^{y_n}) tends to $(r-1,0)$ or $(0,s-1)$ as $n\to\infty$. Suppose (e^{x_n},e^{y_n}) tends to $(r-1,0)$ as $n\to\infty$. Then $x_n\to\log(r-1)$, $y_n\to-\infty$ as $n\to\infty$. On the other hand $A^{-1}(y_n)\to\log(r-1)$ as $n\to\infty$. Thus $\|(x_n,y_n)-(A^{-1}(y_n),y_n)\|=|x_n-A^{-1}(y_n)|\to 0$ as $n\to\infty$. Hence in this case we have $(x_n,y_n)\to\partial X_\infty$ as $n\to\infty$. The case (e^{x_n},e^{y_n}) tends to $(0,s-1)$ is similar.

Finally for the case that an orbit (x_n,y_n) of (3.15) converges a positive fixed point P, as we showed in the previous paragraph $P\in\partial X_\infty$.

Thus we conclude that ∂X_∞ is attracting.

□

3.3.2 The Ricker Model

The planar Ricker model is defined by the non-invertible map

$$F(u,v):=\left(ue^{r-u-\alpha v},ve^{s-v-\beta u}\right),\quad (u,v)\in\mathbb{R}^2_+ \tag{3.18}$$

where $\alpha,\beta,r,s>0$.

With the coordinates stated in (3.14), we have the following log-scaled version of the model:

$$f(x,y):=(x+r-e^x-\alpha e^y,y+s-e^y-\beta e^x)\,. \tag{3.19}$$

For the time being we work with the Ricker map in the standard coordinates u,v to see when we can expect a carrying simplex to be unique when it exists. Log coordinates will be introduced later. The following points are always fixed points of F:

$$\mathbf{c}_1:=(0,0),\ \ \mathbf{c}_2:=(r,0),\ \ \mathbf{c}_3:=(0,s). \tag{3.20}$$

When $\alpha\beta \neq 1$ and $\mathbf{c}_4$ defined in (3.21) below is a member of $\mathbb{R}^2_+$, then F has exactly four fixed points and the fourth fixed point is:

$$\mathbf{c}_4 := \left(\frac{s\alpha - r}{\alpha\beta - 1}, \frac{r\beta - s}{\alpha\beta - 1} \right). \tag{3.21}$$

We will find it more convenient to use the log-scaled version (3.19).

In this section we consider X_∞ defined by (3.11) with $D = Q_3$, where $Q_3 := \{(x, y) : x \leq 0 \ \textit{and} \ y \leq 0\}$, and $f = g$ at every point on D. It can be easily observed that g is proper and continuous and if $0 < r, s < 1$, we have $g(D) \subset D$ since by Lemma 2.11, we can deduce that $g(D) \subset f(\mathbb{R}^2) \subset (-\infty, r - 1] \times (-\infty, s - 1] \subset Q_3$.

We also define Y_∞ and Y_n as another X_∞ and X_n defined by (3.11) with $D = \mathbb{R}^2$.

Lemma 3.12. *If $s, r < 1$ then $X_\infty = Y_\infty$.*

Proof. $X_\infty \subset Y_\infty$ is obvious. As mentioned above, we have $f(\mathbb{R}^2) \subset Q_3 = X_0$, hence $Y_0 = \overline{f(\mathbb{R}^2)} \subset X_0$. Therefore, by using induction $Y_n \subset X_n$ for every $n = 1, 2, 3, ...$ and $Y_\infty \subset X_\infty$. □

Lemma 3.13. *When $r, s > 0$ we have $X_\infty \neq \emptyset$.*

Proof. $x', y' < 0$ can be found such that for every $(x_1, y_1) \ll (x', y')$ the following equations have at least one solution with $(x, y) \ll (x_1, y_1)$:

$$x + r - e^x - \alpha e^y = x_1,$$
$$y + s - e^y - \beta e^x = y_1.$$

Hence if $S = (-\infty, x'] \times (-\infty, y']$ then $S \subset f(S)$. Therefore for $n = 0, 1, 2, ...$ we have $S \subset X_n$ and $\emptyset \neq S \subset X_\infty$. □

Now we will show that X_∞ is invariant. Since the log-scaled Ricker map f is not invertible, to show invariance of X_∞ we need a property weaker than invertibility. We use the fact that the log-scaled Ricker map f is a proper map (i.e. for every compact set $X \subset \mathbb{R}^2$, $f^{-1}(X)$ is compact). To see this, note that if $\{\mathbf{x}_n\}$ is a sequence such that $|\mathbf{x}_n| \to \infty$, then, according to the terms in (3.19), $|f(\mathbf{x}_n)| \to \infty$. Since f is continuous, we conclude that for every closed and bounded set X, $f^{-1}(X)$ is closed and bounded. Therefore, for every compact set X, $f^{-1}(X)$ is compact and we conclude that the log-scaled Ricker map f is proper.

Lemma 3.14. *When $r, s < 1$, X_∞ is invariant.*

Proof. As mentioned, $g : D \to \mathbb{R}^2$ is proper and continuous for $D = Q_3$ and if $0 < r, s < 1$, we have $g(D) \subset D$. So X_∞ is invariant by Lemma 3.6. □

Lemma 3.15. *Lemma 3.9 also holds for the log-scaled Ricker map f defined in (3.19).*

Proof. We define $\chi_1 : [0,1] \to \mathbb{R}$ and $\chi_2 : [0,1] \to \mathbb{R}$ as follows

$$\chi_1(\lambda) := f_1(\lambda \mathbf{x} + (1-\lambda)\mathbf{y})$$

$$\chi_2(\lambda) := f_2(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}).$$

It is easy to show that both χ_1 and χ_2 are strictly concave (being the sum of linear minus exponential terms), and the rest of the proof is straightforward. □

Lemma 3.16. *(a) Let $b \in \mathbb{R}$ and $X \subset Q_3$ be the set of all points on or under the graph of a continuous strictly decreasing function $B : (-\infty, b) \to \mathbb{R}$. Let $f = (f_1, f_2)$ denote the log-scaled Ricker map. Then we have*

$$\{x \mid (x,c) \in f(X)\} = \begin{cases} f_1\left(\{(x, P(c - s + \beta e^x)) \mid x \in (-\infty, K(c)]\}\right), & c \in (-\infty, y^\star) \\ \emptyset, & \textit{otherwise} \end{cases}$$

where $y^\star = \sup\{f_2(x,y) | \, (x,y) \in f(X)\}$,

$$P(x) := x - W_0(-e^x),$$

W_0 is the principal branch of the Lambert W function (see, for example, [42]),

$$K(c) := G^{-1}(c - s)$$

and G is the continuous invertible function defined by

$$G(x) := B(x) - e^{B(x)} - \beta e^x.$$

(b) We have

$$\{x \mid (x,c) \in f(Q_3)\} = \begin{cases} f_1\left(\{(x, P(c - s + \beta e^x)) \mid x \in (\infty, K(c)]\}\right), & c \in (-\infty, y^\star) \\ \emptyset, & \textit{otherwise} \end{cases}$$

where P is the function defined in part (a) and

$$y^\star = \sup\{f_2(x,y) | \, (x,y) \in f(Q_3)\},$$

$$K(c) = \min\left\{0, \ln\left(\frac{-1 - c + s}{\beta}\right)\right\}.$$

Proof. (a) For $c \in (-\infty, y^\star)$ we have

$$\{x \mid (x,c) \in f(X)\} = f_1\left(X \cap \{(x,y) \in Q_3 \mid f_2(x,y) = c\}\right)$$

$$= f_1\left(X \cap \{(x,y) \in Q_3 \mid y + s - e^y - \beta e^x = c\}\right)$$

$$= f_1\left(X \cap \{(x,y) \in Q_3 \mid y - e^y = c - s + \beta e^x\}\right)$$

When $t < 0$, $t - e^t$ is strictly increasing. Hence, for every $l \in \mathbb{R}$, $t - e^t = l$ has at most one solution for $t \leq 0$. Since $\{t - e^t \mid t \leq 0\} = (-\infty, -1]$, $t - e^t = l$ has a unique solution for $t \leq 0$ and $l \in (-\infty, -1]$ and no solution when $t \leq 0$ and $l > -1$. We claim that $t = P(l)$ is the unique solution for $t - e^t = l$ when $t \leq 0$ and $l \in (-\infty, -1]$. To prove that, we have

$$P(l) - e^{P(l)} = l - W_0(-e^l) - e^{l - W_0(-e^l)}. \tag{3.22}$$

By the properties of the Lambert W function W_0 we have $W_0(-e^l)e^{W_0(-e^l)} = -e^l$. Hence $-e^{l-W_0(-e^l)} = W_0(-e^l)$. This along with (3.22) implies $P(l) - e^{P(l)} = l$ (since $t = P(l)$ must satisfy $t < 0$, only the principal branch of the Lambert W function can be used to provide a solution). Now with $l = c - s + \beta e^x$ and $t = y$ we have

$$f_1\left(X \cap \{(x,y) \in Q_3 \mid y - e^y = c - s + \beta e^x\}\right) = f_1\left(X \cap \{(x,y) \in Q_3 \mid y = P(c - s + \beta e^x)\}\right)$$

$$= f_1\left(\{(x,y) \in Q_3 \mid y = P(c - s + \beta e^x), y \leq B(x)\}\right)$$

$$= f_1\left(\{(x, P(c - s + \beta e^x)) \mid P(c - s + \beta e^x) \leq B(x)\}\right)$$

Since $P(c - s + \beta e^x) \leq 0$, $B(x) \leq 0$ in the above sets, and since when $t < 0$, $t - e^t$ is strictly increasing, we have

$$P(c - s + \beta e^x) \leq B(x) \implies c - s + \beta e^x = P(c - s + \beta e^x) - e^{P(c-s+\beta e^x)} \leq B(x) - e^{B(x)},$$

Thus

$$f_1\left(\{(x, P(c - s + \beta e^x)) \mid P(c - s + \beta e^x) \leq B(x)\}\right)$$

$$= f_1\left(\{(x, P(c - s + \beta e^x)) \mid c - s + \beta e^x \leq B(x) - e^{B(x)}\}\right)$$

$$= f_1\left(\{(x, P(c - s + \beta e^x)) \mid c - s \leq B(x) - e^{B(x)} - \beta e^x\}\right)$$

$$= f_1\left(\{(x, P(c - s + \beta e^x)) \mid c - s \leq G(x)\}\right)$$

(b) For $c \in (-\infty, s-1)$ we have

$$\{x \mid (x, c) \in f(Q_3)\} = f_1\left(\{(x, y) \in Q_3 \mid f_2(x, y) = c\}\right)$$

$$= f_1\left(\{(x, y) \in Q_3 \mid y + s - e^y - \beta e^x = c\}\right)$$

$$= f_1\left(\{(x, y) \in Q_3 \mid y - e^y = c - s + \beta e^x\}\right)$$

$$= f_1\left(\{(x, y) \in Q_3 \mid y = P(c - s + \beta e^x)\}\right)$$

$$= f_1\left(\{(x, P(c - s + \beta e^x)) \mid x \in (-\infty, 0], P(c - s + \beta e^x) \in (-\infty, 0]\}\right)$$

$$= f_1\left(\{(x, P(c - s + \beta e^x)) \mid x \in (-\infty, 0], P(c - s + \beta e^x) - e^{P(c-s+\beta e^x)} \leq -1\}\right)$$

$$= f_1\left(\{(x, P(c - s + \beta e^x)) \mid x \leq 0, \;\; c - s + \beta e^x \leq -1\}\right)$$

$$= f_1\left(\left\{(x, P(c - s + \beta e^x)) \mid x \leq 0, \;\; x \leq \ln\left(\frac{-1 - c + s}{\beta}\right)\right\}\right)$$

$$= f_1\left(\left\{(x, P(c - s + \beta e^x)) \mid x \leq \min\left\{0, \ln\left(\frac{-1 - c + s}{\beta}\right)\right\}\right\}\right).$$

□

Lemma 3.17. *For any $0 < r, s < 1$ and $\alpha, \beta > 0$ and f is the log-scaled Ricker map, X_∞ is invariant and convex.*

Notice that Lemma 3.17 is not a complete analogue of Theorem 3.11 since we are not claiming that ∂X_∞ is necessarily invariant. We will address this after proving Lemma 3.17.

Proof. (Lemma 3.17) Convexity of X_∞ can be proved with a similar argument to that used for the scaled Leslie-Gower model. So we explain the argument more briefly. It is obvious that X_0 is convex, and by using induction we can prove convexity of X_n for $n = 1, 2, ...$, and that implies convexity of X_∞.

To prove convexity of X_1, by Lemma 3.16 (b) for $c \in (-\infty, y^\star)$ we have

$$\{x \mid (x,c) \in f(X_1)\} = \{x \mid (x,c) \in f(Q_3)\} = f_1\left(\{(x, P(c - s + \beta e^x)) \mid x \in (\infty, K(c)]\}\right)$$

where P, K and $y^\star$ are defined in that part of the lemma. Now it is easy to verify that $p = f_1$ and $q : \mathbb{R}^2 \to \mathbb{R}$ defined by $q(x,y) = P(y - s + \beta e^x)$ and K satisfy the conditions of Lemma 3.5. By Lemma 3.5 for every $c \in (-\infty, y^\star)$ we have $\Omega_c = (-\infty, G(c)]$, where $G : (-\infty, y^\star) \to \mathbb{R}$ is continuous. Thus

$$X_1 = \bigcup_{c \in (-\infty, y^\star)} \Omega_c = \bigcup_{c \in (-\infty, y^\star)} (-\infty, G(c)].$$

Now $A := G, X := X_1$ satisfy the conditions stated for A in Lemma 3.3.

Owing to the symmetric structure of the definition of the log-scaled Ricker map, we can state quite similar lemmas to prove that there exists B such that it satisfies the conditions of Lemma 3.3 for $X = X_1$. Now since, by Lemma 3.3, X_1 is the set of all points on or under the graph of a continuous strictly decreasing function, we can use Lemma 3.4 and Lemma 3.15 with a similar argument to that used in Theorem 1 to prove that X_1 is convex.

Assume that for $n \geq 1$, X_n is convex and it is the set of all points on or under the graph of a continuous strictly decreasing function $B : (-\infty, b) \to \mathbb{R}$. By Lemma 3.16 (a) for every $c \in (-\infty, y^\star)$ we have

$$\{x \mid (x,c) \in X_{n+1}\} = \{x \mid (x,c) \in f(X_n)\} = f_1\left(\{(x, P(c - s + \beta e^x)) \mid x \in (-\infty, K(c))\}\right).$$

Then $p := f_1$ and $q : \mathbb{R}^2 \to \mathbb{R}$ defined by $q(x,y) := P(y - s + \beta e^x)$ satisfy the conditions of Lemma 3 and G defined in that lemma is continuous. So $A := G$ and $X := X_{n+1}$ satisfy the conditions stated for A in Lemma 3.3. Again, owing to the symmetric structure of the definition of the log-scaled Ricker map we can prove the existence of B which satisfies the conditions of Lemma 1 for $X = X_{n+1}$. Therefore, by that lemma, X_{n+1} is the set of all points on or under the graph of a continuous strictly decreasing function. Now we can use Lemma 3.4 and Lemma 3.15 and a similar argument to that used in Theorem 1 to prove that X_{n+1} is convex.

According to Lemma 3.14 X_∞ is invariant, and as the intersection of convex sets it is convex. $\square$

As the log-scaled Ricker map f is not invertible we cannot conclude that ∂X_∞ is also invariant. In order to prove that $\partial X_\infty \subset X_1$ is invariant, it is sufficient to show that the restriction $f|_{X_1 \to f(X_1)}$ is invertible. If the Jacobian of f is nonvanishing throughout X_1 then f is locally invertible at any point of X_1. But as is well known, locally invertibility does not always imply global invertibility. Ho [18] proved that a local homeomorphism between a pathwise connected Hausdorff space and a simply connected Hausdorff space is a global homeomorphism if and only if that map is proper. We have already established that the log-scaled Ricker map f is proper (in the paragraph preceding Lemma 3.14). Hence, if we prove that the Jacobian of f does not vanish anywhere in X_1 for a given range of parameters, then we can deduce that $\partial X_\infty \subset X_1$ is invariant for that same range of parameters.

Thus now we consider where the Jacobian vanishes.

Using [7] (which studies the unscaled Ricker map (3.18)) the Jacobian of the log-scaled Ricker map f only vanishes on LC_{-1} defined by

$$LC_{-1} := \{(x,y) \in Q_3 : 1 - e^x = e^y(1-(1-\alpha\beta)e^x)\}. \tag{3.23}$$

Set

$$q(t) = \frac{1-e^t}{(\alpha\beta-1)e^t+1}.$$

When $\alpha\beta \geq 1$, then $q(t) > 0$ if and only if $t < 0$. If $\alpha\beta < 1$, then $q(t) > 0$ if and only if $t \in (-\infty, 0) \cup (-\ln(1-\alpha\beta), +\infty)$. In this case, LC_{-1} is the union of two curves. By Lemma 3.12, if $r, s < 1$, then $X_\infty = Y_\infty \subset Q_3$. So in this case we only need

$$LC^1_{-1} := \{(\ln(q(t)), t) \mid t < 0\}.$$

to see whether or not the Jacobian vanishes at some points on X_∞.

According to [7], $Y_1 := f(\mathbb{R}^2)$ is bounded by the space on or under LC^1_0 defined by

$$LC^1_0 := \left\{\left(\ln\left(q(t)\right) + r - q(t) - \alpha e^t, t + s - e^t - \beta q(t)\right) \mid t < 0\right\}. \tag{3.24}$$

Since $X_\infty \subset Y_1$, X_∞ is a subset of the set of points on or under LC^1_0. Hence, if $r, s < 1$ and LC^1_{-1} does not intersect that space, then the Jacobian of f does not vanish anywhere in X_∞ since $LC_{-1} \cap X_\infty = \emptyset$.

Lemma 3.18. *If $r, s < 1$ then LC^1_{-1} does not intersect the set of points on or under LC^1_0. Consequently, if $r, s < 1$ then ∂X_∞ is invariant.*

Proof. It is sufficient to show that if $(x_{-1}, y) \in LC^1_{-1}$ and $(x_0, y) \in LC^1_0$, then $x_{-1} > x_0$. Since $(x_0, y) \in LC^1_0$, for some $t < 0$ we have $(x_0, y) = (\ln\left(q(t)\right) + r - q(t) - \alpha e^t, t + s - e^t - \beta q(t))$. Since $x_{-1} = \ln(q(y))$ and $y = t + s - e^t - \beta q(t)$, we have

$$x_{-1} = \ln(q(t + s - e^t - \beta q(t))).$$

We define $R(t) := x_{-1} - x_0 = \ln(q(t + s - e^t - \beta q(t))) - (\ln\left(q(t)\right) + r - q(t) - \alpha e^t)$. We have

$$q(t)\left(e^{R(t)} - 1\right) = q(t + s - e^t - \beta q(t))e^{-r+q(t)+\alpha e^t} - q(t). \tag{3.25}$$

It is easy to show that when $h < 0$, then $q(h) > 0$. We use this fact multiple times in this proof.

From $t < 0$ we have $t - e^t < -1$. This along with $s < 1$ and $q(t) > 0$ implies $t + s - e^t - \beta q(t) < 0$. Thus $q(t + s - e^t - \beta q(t)) > 0$. Now since $e^{-r+q(t)+\alpha e^t} \geq 1 - r + q(t) + \alpha e^t$, we have

$$q(t + s - e^t - \beta q(t))e^{-r+q(t)+\alpha e^t} \geq q(t + s - e^t - \beta q(t))(1 - r + q(t) + \alpha e^t). \qquad (3.26)$$

Now (3.25) and (3.26) imply

$$q(t)\left(e^{R(t)} - 1\right) \geq q\left(t + s - e^t - \beta q(t)\right)(1 - r + q(t) + \alpha e^t) - q(t). \qquad (3.27)$$

For the sake of expressing equations in a simpler way, let $T := e^t$. We have $0 < T < 1$, $q(t) = q(\ln(T)) = \frac{1-T}{(\alpha\beta-1)T+1}$ and we can rewrite (3.27) as follows

$$q(t)\left(e^{R(t)} - 1\right) \geq q\left(\ln(T) + s - T - \beta q(\ln(T))\right)\left(1 - r + q(\ln(T)) + \alpha T\right) - q(\ln(T)). \qquad (3.28)$$

We have

$$q\left(\ln(T) + s - T - \beta q(\ln(T))\right) = \frac{1 - e^{\ln(T)+s-T-\beta q(\ln(T))}}{(\alpha\beta - 1)e^{\ln(T)+s-T-\beta q(\ln(T))} + 1}$$

$$= \frac{1 - Te^{s-T-\beta\frac{1-T}{(\alpha\beta-1)T+1}}}{(\alpha\beta - 1)Te^{s-T-\beta\frac{1-T}{(\alpha\beta-1)T+1}} + 1} = 1 - \frac{\alpha\beta Te^{s-T-\beta\frac{1-T}{(\alpha\beta-1)T+1}}}{(\alpha\beta - 1)Te^{s-T-\beta\frac{1-T}{(\alpha\beta-1)T+1}} + 1}$$

$$= 1 - \frac{\alpha\beta T}{(\alpha\beta - 1)T + e^{-s+T+\beta\frac{1-T}{(\alpha\beta-1)T+1}}}. \qquad (3.29)$$

As we mentioned before, $\ln(T) + s - T - \beta q(\ln(T)) = t + s - e^t - \beta q(t) < 0$. Hence, from $\alpha\beta - 1 > -1$, we deduce

$$(\alpha\beta - 1)Te^{s-T-\beta\frac{1-T}{(\alpha\beta-1)T+1}} + 1 = (\alpha\beta - 1)e^{\ln(T)+s-T-\beta q(\ln(T))} + 1 > 0,$$

thus

$$(\alpha\beta - 1)T + e^{-s+T+\beta\frac{1-T}{(\alpha\beta-1)T+1}} = e^{-s+T+\beta\frac{1-T}{(\alpha\beta-1)T+1}}\left((\alpha\beta - 1)Te^{s-T-\beta\frac{1-T}{(\alpha\beta-1)T+1}} + 1\right) > 0. \quad (3.30)$$

From $T > 0$, $s < 1$ and $q(t) > 0$ we have

$$(\alpha\beta - 1)T + 1 + \left(-s + T + \beta\frac{1 - T}{(\alpha\beta - 1)T + 1}\right) = \alpha\beta T + 1 - s + \beta q(t) > 0. \qquad (3.31)$$

Since $e^h > 1 + h$ for $h \in \mathbb{R}$, we have

$$(\alpha\beta - 1)T + e^{-s+T+\beta\frac{1-T}{(\alpha\beta-1)T+1}} \geq (\alpha\beta - 1)T + 1 + \left(-s + T + \beta\frac{1 - T}{(\alpha\beta - 1)T + 1}\right). \qquad (3.32)$$

Now (3.30), (3.31) and (3.32) imply

$$1-\frac{\alpha\beta T}{(\alpha\beta-1)T+e^{-s+T+\beta\frac{1-T}{(\alpha\beta-1)T+1}}}\geq 1-\frac{\alpha\beta T}{\alpha\beta T+1-s+\beta\frac{1-T}{(\alpha\beta-1)T+1}}.$$

This along with (3.28) and (3.29) implies

$$q(t)\left(e^{R(t)}-1\right)\geq\left(1-\frac{\alpha\beta T}{\alpha\beta T+1-s+\beta\frac{1-T}{(\alpha\beta-1)T+1}}\right)(1-r+q(\ln(T))+\alpha T)-q(\ln(T))$$

$$=\left(1-\frac{\alpha\beta T((\alpha\beta-1)T+1)}{(\alpha\beta T+1-s)((\alpha\beta-1)T+1)+\beta(1-T)}\right)\left(1-r+\frac{1-T}{(\alpha\beta-1)T+1}+\alpha T\right)$$
$$-\frac{1-T}{(\alpha\beta-1)T+1}$$

$$=\frac{(1-s)((\alpha\beta-1)T+1)+\beta(1-T)}{(\alpha\beta T+1-s)((\alpha\beta-1)T+1)+\beta(1-T)}\left(1-r+\frac{1-T}{(\alpha\beta-1)T+1}+\alpha T\right)-\frac{1-T}{(\alpha\beta-1)T+1}$$

$$=\frac{E(T)}{(\alpha\beta-1)T+1},$$

where

$$E(T):=\frac{(1-s)((\alpha\beta-1)T+1)+\beta(1-T)}{(\alpha\beta T+1-s)((\alpha\beta-1)T+1)+\beta(1-T)}((1-r+\alpha T)((\alpha\beta-1)T+1)+1-T)$$
$$-(1-T)$$

$$=\frac{(1-s)((\alpha\beta-1)T+1)+\beta(1-T)}{(\alpha\beta T+1-s)((\alpha\beta-1)T+1)+\beta(1-T)}((1-r+\alpha T)((\alpha\beta-1)T+1)+1-T)$$
$$-\frac{(\alpha\beta T+1-s)((\alpha\beta-1)T+1)(1-T)+\beta(1-T)^2}{(\alpha\beta T+1-s)((\alpha\beta-1)T+1)+\beta(1-T)}$$

$$=\frac{(1-s)((\alpha\beta-1)T+1)(1-r+\alpha T)((\alpha\beta-1)T+1)+\beta(1-T)(1-r+\alpha T)((\alpha\beta-1)T+1)}{(\alpha\beta T+1-s)((\alpha\beta-1)T+1)+\beta(1-T)}$$
$$+\frac{(1-s)((\alpha\beta-1)T+1)(1-T)-(\alpha\beta T+1-s)((\alpha\beta-1)T+1)(1-T)}{(\alpha\beta T+1-s)((\alpha\beta-1)T+1)+\beta(1-T)}.$$

Now by the above inequalities we have

$$q(t)\left(e^{R(t)}-1\right)\geq\frac{(1-s)(1-r+\alpha T)((\alpha\beta-1)T+1)+\beta(1-T)(1-r+\alpha T)}{(\alpha\beta T+1-s)((\alpha\beta-1)T+1)+\beta(1-T)}$$
$$+\frac{(1-s)(1-T)-(\alpha\beta T+1-s)(1-T)}{(\alpha\beta T+1-s)((\alpha\beta-1)T+1)+\beta(1-T)}$$

$$= \frac{(1-s)(1-r+\alpha T)((\alpha\beta-1)T+1)+\beta(1-T)(1-r)}{(\alpha\beta T+1-s)((\alpha\beta-1)T+1)+\beta(1-T)}.$$

From $s<1$, $r<1$, $\alpha\beta-1>-1$ and $0<T<1$ we can deduce that

$$\frac{(1-s)(1-r+\alpha T)((\alpha\beta-1)T+1)+\beta(1-T)(1-r)}{(\alpha\beta T+1-s)((\alpha\beta-1)T+1)+\beta(1-T)} > 0.$$

Therefore, $q(t)\left(e^{R(t)}-1\right)>0$. Now since $q(t)>0$, we deduce that $e^{R(t)}-1>0$, which implies $x_{-1}-x_0=R(t)>0$. This proves $x_{-1}>x_0$. Now since X_∞ is a subset of the set of points on or under LC_0^1, LC_{-1}^1 does not intersect X_∞ and by the argument we stated before, we deduce that ∂X_∞ is invariant. □

We may now put together Lemma 3.17 and Lemma 3.18 to obtain the analogue of Theorem 3.11 for the Ricker map:

Theorem 3.19. *For any $0<r,s<1$ and $\alpha,\beta>0$ and f is the log-scaled Ricker map, X_∞ is invariant and convex, ∂X_∞ is invariant and attracts $\mathbb{R}^2$.*

Proof. All that is left to do is show that ∂X_∞ is attracting. It is proven that if $r,s<2$, then every non-trivial orbit converges to one of the non-zero fixed points (see [6]). The possible non-zero fixed points on the x or y axis are the same as for the Leslie-Gower model with $r-1$ replaced by r and $s-1$ replaced by s. So we may use the same method as used for the log-scaled Leslie-Gower map to show attraction to ∂X_∞. □

From this we obtain the following improvement on the known conditions

$$r+s<1+rs(1-\alpha\beta)<2,$$

(e.g [17, 49, 16, 43]) for the existence of a carrying simplex for the Ricker model. These inequalities fail for some $\alpha,\beta>0$, when $r,s<1$, namely those α,β that satisfy $r,s<1$ and $\alpha\beta<(1-1/r)(1-1/s)$. However, Lemma 3.18 also shows that the unscaled Ricker map is retrotone on $[0,r]\times[0,s]$ when $r,s<1$ so we obtain

Corollary 3.20. *When $r,s<1$ and $\alpha,\beta>0$ the Ricker map $(x,y)\mapsto(xe^{r-x-\alpha y},ye^{s-y-\beta x})$ defined on $\mathbb{R}^2_+$ has a (compact) carrying simplex.*

Proof. In the absence of asymptotic completeness in Theorem 3.19, we apply standard results on retrotone systems (e.g. [17, 49, 19, 43]). □

Chapter 4

An Approximation Formula for the Stein-Ulam Spiral Map

The Stein–Ulam Spiral Map, introduced in a 1959 Los Alamos report by S.M. Ulam, P.R. Stein, and M.T. Menzel, models competitive dynamics among large populations of three types of individuals via a discrete dynamical system. Due to the intricate structure of the map, early investigations speculated that the ω-limit set of every non-trivial interior trajectory coincides with the entire boundary of the domain. This conjecture was later refuted by Barański and Misiurewicz, who demonstrated that every admissible subset of the boundary arises as the ω-limit set of some trajectory. In this chapter, we develop an approximation formula for the iterates of the Stein–Ulam Spiral Map, with the error vanishing as trajectories approach the boundary. This tool enables us to probe the topological properties of the basins of attraction associated with ω-limit sets—an area traditionally fraught with difficulties, especially in proving connectivity properties akin to the Mandelbrot set's MLC conjecture. Using our approximation technique, we establish that for every ω-limit set basin, there exist two-dimensional simply connected sets whose intersection with the basin fails to be path connected. This result underscores both the complexity of the system and the strength of the approximation method introduced herein, opening new avenues for studying connectivity in discrete dynamical systems.

4.1 Introduction

4.1.1 Binary Reaction System

The term binary reaction system was introduced by S.M. Ulam, P. R. Stein and M. T. Menzel [54] to describe a special case of the maps they used to model the competition of large populations of a number of types of individuals with conditions such as the size of the total population remaining constant and the current generation being totally replaced by the next generation at each iteration. The general case of their system was expressed as follows:

$$x_i' = \sum_{k,l=1}^{N} \gamma_i^{kl} x_k x_l, \quad (i = 1, ..., N) \tag{4.1}$$

where N is the number of types of individuals and for $i = 1, ..., N$, x_i and x_i' are the fractions of individuals of type i in the present and next generation respectively. Since the size of the population remains constant it's assumed that

$$\sum_{k=1}^{N} x_k = \sum_{k=1}^{N} x_k' = 1. \tag{4.2}$$

As $0 \leq x_k \leq 1$ for every $k = 1, ...N$, the system is defined on Δ_N defined as follows

$$\Delta_N = \left\{ (x_1, x_2, ..., x_N) \in [0,1]^N \mid \sum_{k=1}^{N} x_k = 1 \right\}. \tag{4.3}$$

(4.2) implies $\sum_{i=1}^{N} \gamma_i^{kl} = 1$ for every $1 \leq k, l \leq N$. Ulam and Stein defined binary reaction systems as those with $\gamma_i^{kl} \in \{0, 1\}$ for every $1 \leq k, l \leq N$ and $x_i' \neq 0$ for every $1 \leq i \leq N$. If we consider equivalent systems as the same when counting the number of binary reaction systems, there are 97 systems for $N = 3$. Out of these 97 systems, only one map lacks an attracting periodic orbit. That map which this chapter discusses, is called Stein-Ulam Spiral map.

4.1.2 Stein-Ulam Spiral Map

The Stein-Ulam spiral map is defined on the simplex

$$\Delta := \{(u, v, w) | u, v, w \geq 0, u + v + w = 1\} \tag{4.4}$$

by the recurrence relation

$$\begin{cases} u_{n+1} = u_n(u_n + 2v_n), \\ v_{n+1} = v_n(v_n + 2w_n), \\ w_{n+1} = w_n(w_n + 2u_n). \end{cases} \tag{4.5}$$

This map models the interaction of three large populations of individuals. The variables u_n, v_n, w_n represent the fraction of the total population belonging to each type in the n-th generation, under the assumption that each generation completely replaces the previous one. Since the phase space is constrained to the triangle Δ, where the sum of the components is always one, the interpretation of these variables as population fractions is consistent and natural.

The map admits four fixed points:

$$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\quad (1,0,0),\quad (0,1,0),\quad (0,0,1). \tag{4.6}$$

The central fixed point $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ is repelling, and every trajectory, except the trivial orbit at this point, either lies entirely on the boundary of Δ or asymptotically approaches it. Kesten [22] proved this fact by showing that $V(u, v.w) = uvw$ works as a Lyapunov function for the map.

Since we always have $u + v + w = 1$, one variable can be considered dependent on the other two. Therefore, the system is equivalent to each of the following three planar systems:

$$\begin{cases} u_{n+1} = u_n(u_n + 2v_n), \\ v_{n+1} = v_n(2 - v_n - 2u_n). \end{cases} \qquad (u_0, v_0) \in \Delta^* \tag{4.7}$$

$$\begin{cases} v_{n+1} = v_n(v_n + 2w_n), \\ w_{n+1} = w_n(2 - w_n - 2v_n). \end{cases} \qquad (v_0, w_0) \in \Delta^* \tag{4.8}$$

$$\begin{cases} w_{n+1} = w_n(w_n + 2u_n), \\ u_{n+1} = u_n(2 - u_n - 2w_n). \end{cases} \qquad (w_0, u_0) \in \Delta^* \tag{4.9}$$

Where $\Delta^* = \{(x, y)|\ x, y \geq 0,\ x + y \leq 1\}$.

The triangular shape of the phase space makes it challenging to analyze the precise behavior of nontrivial trajectories as they approach the boundary of Δ. This difficulty arises because trajectories are not only attracted to the boundary —which is a non-smooth union of line segments— but also experience significant deceleration near the corners of the triangle. The motion slows down as trajectories approach a vertex and speeds up again as they move away toward another corner. This cycle of deceleration and acceleration repeats infinitely, making the long-term behavior of the system extremely sensitive to initial conditions.

The authors of the Los Alamos report [54] appear to have attempted to partly address these difficulties by suggesting a circular transformation that maps the boundary of Δ to a circle. Here we find it very useful to study the logarithmically scaled versions of these maps. Not only we can transform products into sums with this approach, but also we can have a very better picture of what's going on near the boundary of Δ. So letting[1] $x = \ln(u)$, $y = \ln(v)$, $z = \ln(w)$ and $\Delta^{**} = \{(x, y)|\ x, y < 0,\ e^x + e^y \leq 1\}$ gives us the following log-scaled versions

[1] x, y, z defined here are not the same as those in the previous section.

of (4.7), (4.8) and (4.9):

$$\begin{cases} x_{n+1} = x_n + \ln(e^{x_n} + 2e^{y_n}), \\ y_{n+1} = y_n + \ln(2 - e^{y_n} - 2e^{x_n}). \end{cases} \qquad (x_0, y_0) \in \Delta^{**} \tag{4.10}$$

We define $z_n := \ln(1 - e^{x_n} - e^{y_n})$ for this system.

$$\begin{cases} y_{n+1} = y_n + \ln(e^{y_n} + 2e^{z_n}), \\ z_{n+1} = z_n + \ln(2 - e^{z_n} - 2e^{y_n}). \end{cases} \qquad (y_0, z_0) \in \Delta^{**} \tag{4.11}$$

We define $x_n := \ln(1 - e^{y_n} - e^{z_n})$ for this system.

$$\begin{cases} z_{n+1} = z_n + \ln(e^{z_n} + 2e^{x_n}), \\ x_{n+1} = x_n + \ln(2 - e^{x_n} - 2e^{z_n}). \end{cases} \qquad (z_0, x_0) \in \Delta^{**} \tag{4.12}$$

We define $y_n := \ln(1 - e^{z_n} - e^{x_n})$ for this system.

Note 4.1. *In each of the above three systems we have* $e^{x_n} + e^{y_n} + e^{z_n} = 1$.

Here, we develop a theory for (4.10), but this theory is only usable when an orbit is traveling alongside the side $(1, 0, 0) - (0, 0, 1)$. So we also need completely similar theories for (4.11) and (4.12) in order to have similar theories for when the orbit is traveling alongside the other sides. In fact, for the final result, once we prove the existence of a set with certain properties in (4.10), we will use the transformed version of that set in (4.12) and again we prove the existence of a set with certain properties in (4.12). Then we will use the transformed version of that set in (4.11) and again we prove the existence of a set with certain properties in (4.11). This time we will use the transformed version of that set in (4.10). Now that we are in (4.10) again, we repeat this process. So we can do this process for infinite times. So we will have a process with the following order:

$$(4.10) \longrightarrow (4.12) \longrightarrow (4.11) \longrightarrow (4.10) \longrightarrow (4.12) \longrightarrow (4.11) \longrightarrow ... \tag{4.13}$$

Every lemma or theorem that is proven here for (4.10), has completely similar versions for (4.12) and (4.11) as well. Meaning that if $P(x, y, z)$ is a statement true for (4.10), then $P(z, x, y)$ is true for (4.12) and $P(y, z, x)$ is true for (4.11). For example, the following are the $(z - x)$ and $(y - z)$ versions of Lemma 4.4 (a) respectively:

Suppose that for a non-negative integer n we have $x_n \leq -2$ and

$$2y_n < x_n < y_n - M,$$

then

$$-2e^{\frac{x_n}{2}} < z_n < -e^{x_n+M}.$$

Suppose that for a non-negative integer n we have $z_n \leq -2$ and

$$2x_n < z_n < x_n - M,$$

then

$$-2e^{\frac{z_n}{2}} < y_n < -e^{z_n+M}.$$

Definition 4.2. Suppose that $X \subset \partial\Delta$. We say that X is an admissible set if

(a) X contains all the three fixed points of (4.5) on $\partial\Delta$.

(b) $X - \{(1,0,0),(0,1,0),(0,0,1)\}$ has nonempty intersection with each side of $\partial\Delta$.

(c) X is invariant under the map (4.5).

Since any orbit starting from an interior point of Δ, other than the central fixed point, approaches $\partial\Delta$, the ω-limit set of such a point is a subset of $\partial\Delta$. It is also well known that only admissible sets can arise as ω-limit sets of nontrivial interior trajectories.

An admissible set is a closed subset of $\partial\Delta$ containing the boundary fixed points, and consists of fully invariant trajectories of each of the forms:

$$\left\{\left(u_0^{2^n}, 0, 1 - u_0^{2^n}\right) \mid\ n \in \mathbb{Z}\right\}$$

or

$$\left\{\left(1 - v_0^{2^n}, v_0^{2^n}, 0\right) \mid\ n \in \mathbb{Z}\right\}$$

or

$$\left\{\left(0, 1 - w_0^{2^n}, w_0^{2^n}\right) \mid\ n \in \mathbb{Z}\right\},$$

where $0 < u_0, v_0, w_0 < 1$. Thus, an admissible set is of the form:

$$\overline{\bigcup_{u_0\in U}\{(u_0^{2^n},0,1-u_0^{2^n})\mid\ n\in\mathbb{Z}\}}\cup\overline{\bigcup_{v_0\in V}\{(1-v_0^{2^n},v_0^{2^n},0)\mid\ n\in\mathbb{Z}\}}$$

$$\cup\overline{\bigcup_{w_0\in W}\{(0,1-w_0^{2^n},w_0^{2^n})\mid\ n\in\mathbb{Z}\}},$$

for some non-empty subsets $U, V, W \subset (0, 1)$.

Definition 4.3. Suppose that $X \subset \partial\Delta$. We define $\Lambda(X)$ to be the set of all points in $\mathring{\Delta} - \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$ whose ω-limit set is X.

It was initially conjectured that $\partial\Delta$ is the ω-limit set for every non-trivial interior trajectory. However, Barański and Misiurewicz [8] disproved this by showing that for any set $X \subset \partial\Delta$, the set $\Lambda(X)$ is nonempty if and only if X is admissible. They further demonstrated that $\Lambda(\partial\Delta)$ is residual—that is, its complement is a countable union of nowhere dense sets—implying that, in the topological sense, $\Lambda(\partial\Delta)$ is significantly larger than $\Lambda(X)$ for any other admissible set X. Moreover, they proved that for any admissible set X, the Hausdorff dimension of the intersection of X with any open subset of Δ is at least 1.

Proving properties such as connectedness, path connectedness, local connectedness, or local path connectedness has long posed a significant challenge for sets generated by iterated function systems (IFSs). For instance, consider the most famous fractal: the Mandelbrot set. While Douady and Hubbard [12] proved its connectedness in the 1980s—contradicting the initial conjecture that it was disconnected—the question of its local connectedness remains a prominent open problem. This problem is so well-known that it has earned its own abbreviation in the literature: MLC, short for "Mandelbrot Locally Connected." It is widely regarded as the most important unresolved question concerning the Mandelbrot set [39].

In this work, we develop an approximation theory and demonstrate how this tool is powerful enough to at least partially address the path connectedness of basins of ω-limit sets. These sets, which arise from even more complex iterations than the Mandelbrot set, present significant challenges of their own. The authors believe that this tool has the potential to yield further exciting results and may even hint at a more general approach for studying the topological properties of fractals.

4.2 The Approximation Formula

Every orbit of (4.5) originating from any point in $\mathring{\Delta} - \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$ approaches the boundary of Δ in a spiral fashion. The behavior of this spiral is as follows: after reaching the area near

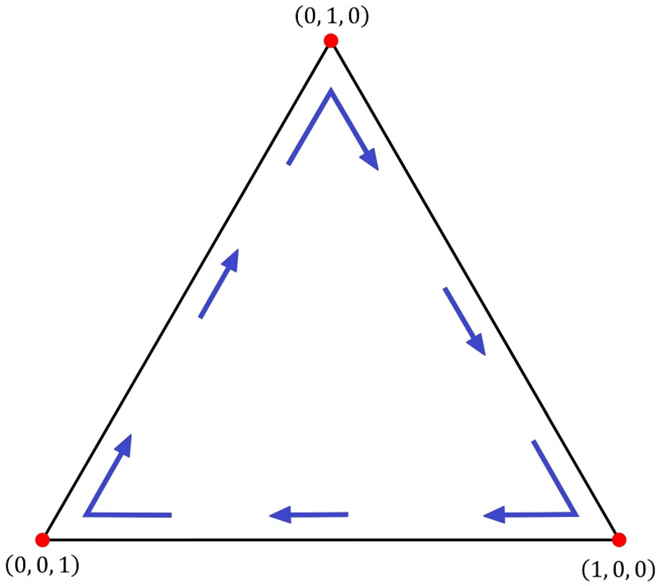

Figure 4.1: The direction of a non-trivial interior trajectory as it approaches the boundary.

one corner of Δ, the orbit departs that area and travels along a side of Δ toward the area near another corner. In this way, the orbit spirals around the boundary region infinitely, with each revolution following a specific order illustrated in Figure 4.1.

When an orbit travels along the side $(1, 0, 0) - (0, 0, 1)$, the second component of the point is small. A similar phenomenon occurs for the first component when the orbit moves along the side $(0, 0, 1) - (0, 1, 0)$, and for the third component when the orbit is near the side $(0, 1, 0) - (1, 0, 0)$.

The core idea of the approximation theory is to disregard these small component values in order to derive a simplified formula. This formula approximates the orbit as it moves from the vicinity of one corner to that of the next. In other words, given the value of an iterate near one corner, the formula provides an approximation of subsequent iterates until the orbit reaches the region near the next corner.

We will see that sometimes ignoring small values is useful. However, we can't always do so. In fact, although a small value can be ignored when approximating other components, we still need to approximate the value of the component itself, even if it is small. The system is sensitive to a component's value when it is close to zero. For that reason, we need to scale the system appropriately in order to accurately approximate small values.

The logarithmic transformation of the system not only allows us to approximate small values of components, but also converts certain products into sums, thereby simplifying some calculations. The following table specifies which log-scaled system the approximation theory uses when an orbit travels along each side of Δ:

The processes of the approximation theory in the three cases mentioned in the above table are

Table 4.1: The log-scaled system the approximation theory uses when an orbit travels along each side of Δ.

Side	**System**	**Component expected to be small**
$(1,0,0)-(0,0,1)$	(4.10)	2nd component (y_n)
$(0,0,1)-(0,1,0)$	(4.11)	1st component (x_n)
$(0,1,0)-(1,0,0)$	(4.12)	3rd component (z_n)

extremely similar to each other. So here we only express the arguments only for the first case of the table (ie. when the orbit is traveling alongside the side $(1,0,0)-(0,0,1)$ and we use (4.10) in the approximation theory).

Let's put our focus on the first case. Suppose that $(u_0,v_0,w_0) \in \mathring{\Delta} - \left\{\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\right\}$. Let's denote $(x_k,y_k,z_k) = (\ln(u_k),\ln(v_k),\ln(w_k))$ for $k \in \mathbb{Z}$. (x_k,y_k) is the scaled version of (u_k,v_k,w_k) in (4.10). Suppose that for a non-negative integer n, (u_n,v_n,w_n) is located in the area close to $(1,0,0)$. So after (u_n,v_n,w_n), the orbit starts traveling alongside the side $(1,0,0)-(0,0,1)$, meaning that the second component remains close to zero until it reaches the area close to $(0,0,1)$. Suppose that the orbit remains close to the side $(1,0,0)-(0,0,1)$ until its k-th iteration after (u_n,v_n,w_n). So $e^{y_n}, e^{y_{n+1}}, ..., e^{y_{n+k}}$ are all small. If we ignore these values when calculating each iteration and the error remains small in each step, then we may have:

$$\begin{aligned}
x_{n+1} &= x_n + \ln(e^{x_n} + 2e^{y_n}) \approx x_n + \ln(e^{x_n}) = 2x_n,\\
x_{n+2} &= x_{n+1} + \ln(e^{x_{n+1}} + 2e^{y_{n+1}}) \approx x_{n+1} + \ln(e^{x_{n+1}}) = 2x_{n+1} \approx 4x_n,\\
&\cdots \quad \cdots \quad \cdots\\
x_{n+k} &= x_{n+k-1} + \ln(e^{x_{n+k-1}} + 2e^{y_{n+k-1}}) \approx x_{n+k-1} + \ln(e^{x_{n+k-1}}) = 2x_{n+k-1} \approx 2^k x_n.
\end{aligned}$$

So the idea is to approximate x_{n+r} with $2^r x_n$ for $r = 1,2,...,k$. Although we ignored each e^{y_r}, y_r's are big negative numbers and we need to approximate each of them to get an approximation of the log-scaled orbit. If we ignore each e^{y_r} again and replace x_{n+r} by $2^r x_n$ when calculating y_r's, we may get:

$$\begin{aligned}
y_{n+1} &= y_n + \ln(2 - e^{y_n} - 2e^{x_n}) \approx y_n + \ln(2 - 2e^{x_n}),\\
y_{n+2} &= y_{n+1} + \ln(2 - e^{y_{n+1}} - 2e^{x_{n+1}}) \approx y_n + \ln(2 - 2e^{x_n}) + \ln(2 - 2e^{2x_n}),\\
&\cdots \quad \cdots \quad \cdots\\
y_{n+k} &= y_{n+k-1} + \ln(2 - e^{y_{n+k-1}} - 2e^{x_{n+k-1}}) \approx y_n + \textstyle\sum_{p=1}^{k} \ln(2 - 2e^{2^{p-1}x_n}).
\end{aligned}$$

So we are hopeful to approximate y_{n+r} with $y_n + \sum_{p=1}^{r} \ln(2 - 2e^{2^{p-1}x_n})$ for $r = 1,2,...,k$.

Now suppose that $(u_{n-1},v_{n-1},w_{n-1})$ is located outside the area colored gray in Figure 4.2 and (u_n,v_n,w_n) is located inside that area. So the iterations after (u_n,v_n,w_n) travel alongside the side $(1,0,0)-(0,0,1)$ until the orbit reaches the area near $(0,0,1)$ and finally it gets out of the gray area. Suppose that m is the smallest natural number such that $(u_{n+m-1},v_{n+m-1},w_{n+m-1})$ is located inside the gray area but $(u_{n+m},v_{n+m},w_{n+m})$ is located outside of it (see Figure 4.3).

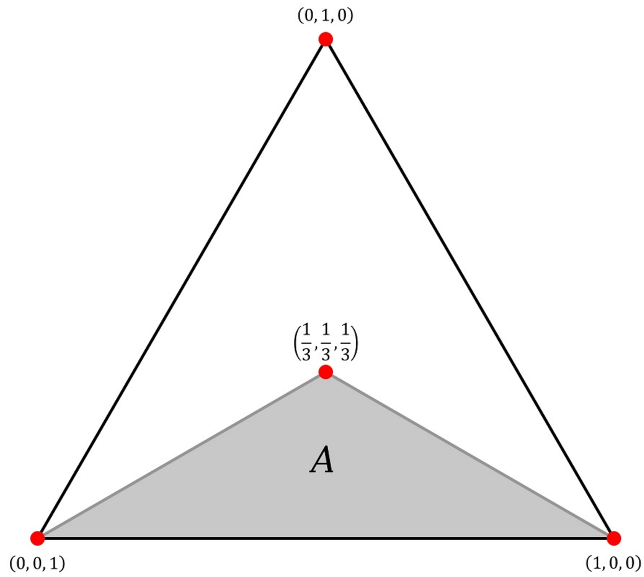

Figure 4.2: The placement of the set A inside Δ.

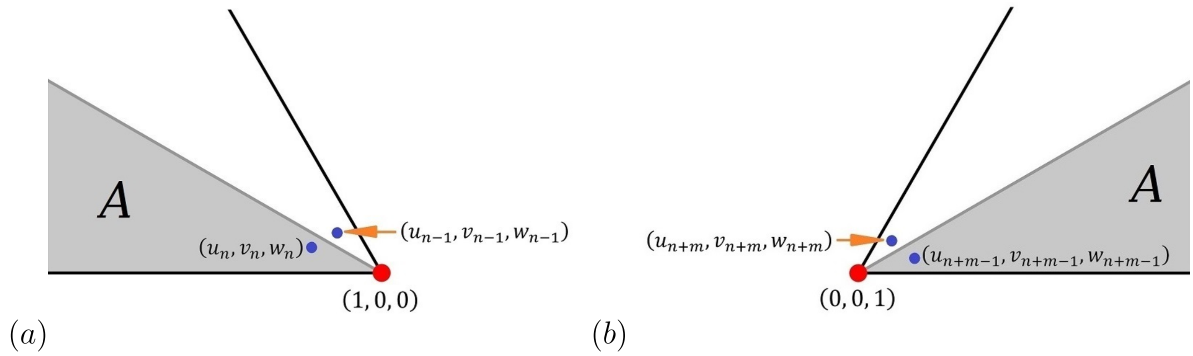

Figure 4.3: (a) The Locations of $(u_{n-1}, v_{n-1}, w_{n-1})$ and (u_n, v_n, w_n) when n is an A-enterer. (b) The Locations of $(u_{n+m-1}, v_{n+m-1}, w_{n+m-1})$ and $(u_{n+m}, v_{n+m}, w_{n+m})$ as the orbit exits A.

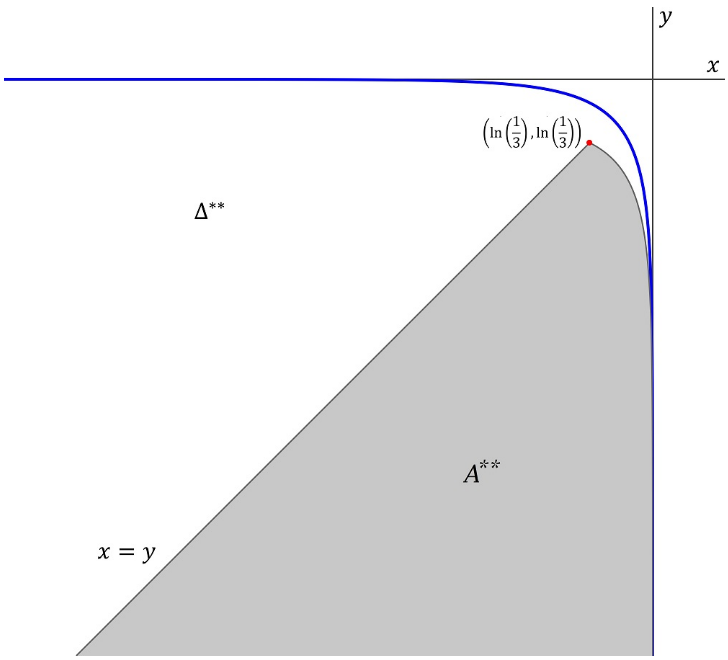

Figure 4.4: The placement of the set A^{**} inside Δ^{**}.

Let's denote that m by $\gamma(n)$ and the area colored gray by A. We also call such an n an A-enterer.

$$A = \{(u, v, w) \in \Delta |\; v < u \; and \; v < w\}.$$

Here is a question: how big is the error if we approximate $x_{n+\gamma(n)}$ with $2^{\gamma(n)}x_n$ and $y_{n+\gamma(n)}$ with $y_n + \sum_{p=1}^{\gamma(n)} \ln(2 - 2e^{2^{p-1}x_n})$. As the main result of the approximation theory, we will prove that under certain conditions not only the error is small but it also tends to 0 as the orbit approaches the boundary of Δ.

Figure 4.4 shows Δ^{**} (bounded by a curve colored blue) as the space in which (4.10) is defined. The area colored gray in Figure 4.4 is the transformed version of A in (4.10). We denote that gray area by A^{**}.

$$A^{**} = \{(x, y) \in \Delta^{**} |\; e^y < e^x \; and \; e^y < 1 - e^x - e^y\}.$$

In the following lemmas and theorems, which aim to develop the theory behind the approximation formula, we assume that (x_0, y_0) is a non-trivial point in the interior of Δ^{**}, and for every $n = 1, 2, 3, ...$, the point (x_n, y_n) is defined by the map (4.10). Moreover, for every non-negative integer n, we define $z_n = \ln(1 - e^{x_n} - e^{y_n})$.

The following lemma establishes inequalities that will be used repeatedly throughout our arguments. This lemma introduces the constant M. The constants M_1 and M_2 will also appear in some other lemmas. These constants play an important role in controlling the error of the approximation formula, as the error is inversely related to them. M in the following lemma ensures a minimum separation between y_n and z_n, which is necessary for the approximation theory to proceed. Importantly, these conditions does not exclude any non-trivial orbit from being eligible for the application of the approximation theory. Even if the condition fails to hold for a given iteration and a not-too-small value of M (or M_1, M_2), it may still hold at a subsequent iteration (or within a few further iterations), depending on the value of M.

Lemma 4.4. *(a) Suppose that for a non-negative integer n and $M > 0$ we have $y_n \leq -2$ and*

$$2z_n < y_n < z_n - M,$$

then

$$-2e^{\frac{y_n}{2}} < x_n < -e^{y_n+M}.$$

(b) Suppose that for a non-negative integer n we have $y_n \leq -2$ and

$$\beta z_n \leq y_n \leq \alpha z_n,$$

where $1 \leq \alpha < \beta \leq 2$, then

$$-4e^{\frac{y_n}{\beta}} < x_n < -e^{\frac{y_n}{\alpha}}.$$

Proof. (a) We have

$$x_n = \ln(1 - e^{y_n} - e^{z_n}) < \ln(1 - e^{y_n} - e^{y_n+M}) = \ln(1 - e^{y_n}(1 + e^M)) < -e^{y_n}(1 + e^M) < -e^{y_n+M}.$$

Since $y_n \leq -2$ and $2z_n < y_n$, we have

$$-2e^{\frac{y_n}{2}} < \ln(1 - e^{\frac{y_n}{2}}(1 + e^{\frac{y_n}{2}})) = \ln(1 - e^{y_n} - e^{\frac{y_n}{2}}) < \ln(1 - e^{y_n} - e^{z_n}) = x_n.$$

(b) We have

$$x_n = \ln(1 - e^{y_n} - e^{z_n}) \leq \ln(1 - e^{y_n} - e^{\frac{y_n}{\alpha}}) < -e^{y_n} - e^{\frac{y_n}{\alpha}} < -e^{\frac{y_n}{\alpha}}.$$

Since $y_n \leq -2$ and $\beta z_n < y_n$, we have

$$x_n = \ln(1 - e^{y_n} - e^{z_n}) \geq \ln\left(1 - e^{y_n} - e^{\frac{y_n}{\beta}}\right) = \ln\left(1 - e^{\frac{y_n}{\beta}}\left(1 + e^{\frac{\beta-1}{\beta}y_n}\right)\right) > \ln\left(1 - 2e^{\frac{y_n}{\beta}}\right)$$

$$= -\sum_{k=1}^{\infty} \frac{\left(2e^{\frac{y_n}{\beta}}\right)^k}{k} = -2e^{\frac{y_n}{\beta}} \sum_{k=1}^{\infty} \frac{\left(2e^{\frac{y_n}{\beta}}\right)^{k-1}}{k} \geq -2e^{\frac{y_n}{\beta}} \sum_{k=1}^{\infty} \frac{(2e^{\frac{y_n}{2}})^{k-1}}{k} \geq -2e^{\frac{y_n}{\beta}} \sum_{k=1}^{\infty} \frac{(2e^{-1})^{k-1}}{k}$$

$$> -4e^{\frac{y_n}{\beta}}.$$

□

Suppose that n is an A-enterer and that the point (u_n, v_n, w_n) lies close to the boundary. As mentioned earlier, we intend to ignore the terms $e^{y_{n+r}}$ for $r = 0, 1, ..., \gamma(n)$ in our approximation formula. Therefore, the accuracy of this approximation depends on the magnitude of each $e^{y_{n+r}}$. We already know that e^{y_n} is small when (u_n, v_n, w_n) is near the boundary, but we must also ensure that $e^{y_{n+r}}$ remains uniformly small for all $r = 0, 1, ..., \gamma(n)$.

Lemmas 4.5 and 4.7 provide general upper bounds for y_{n+k}, valid for all non-negative integers n. However, these bounds do not directly control the size of $e^{y_{n+r}}$ over the range $r = 0, 1, ..., \gamma(n)$ when n is an A-enterer. Nevertheless, by applying these lemmas, we will derive Corollary 4.8, which asserts that under a certain condition, we have

$$y_{n+r} < y_n \quad \text{for all} \quad r = 1, ..., \left\lfloor -\frac{2\ln(-x_n)}{\ln(2)} \right\rfloor - 1.$$

We will later show that if n is an A-enterer and (u_n, v_n, w_n) is sufficiently close to the boundary, then

$$\gamma(n) < \left\lfloor -\frac{2\ln(-x_n)}{\ln(2)} \right\rfloor - 1.$$

In this case, it follows that $e^{y_{n+r}} \leq e^{y_n}$ for all $r = 0, 1, ..., \gamma(n)$, implying that throughout the portion of the orbit inside A, the second component remains at most as large as $v_n = e^{y_n}$.

Lemma 4.5. *For every $n, k = 0, 1, 2, ...$ we have*

$$y_{n+k} \leq y_n + k\ln(2).$$

Proof. We have

$$y_{n+1} = y_n + \ln(2 - 2e^{x_n} - e^{y_n}) = y_n + \ln(2) + \ln(1 - e^{x_n} - \frac{1}{2}e^{y_n}) < y_n + \ln(2).$$

Now the lemma follows by using induction. □

Lemma 4.6 establishes three key inequalities that will be essential in our arguments.

Lemma 4.6. *For every $n = 0, 1, 2, ...$ we have*

$$y_{n+1} < y_n + \ln(2) + \ln(-x_n). \tag{4.14}$$

$$2x_n < x_{n+1}. \tag{4.15}$$

$$y_{n+2} < y_n + 3\ln(2) + 2\ln(-x_n). \tag{4.16}$$

Proof. (4.14). We have

$$y_{n+1} = y_n + \ln(2) + \ln(1 - e^{x_n} - \frac{1}{2}e^{y_n}).$$

Since $x_n < 0$, we have $1 - e^{x_n} < -x_n$. Thus

$$y_{n+1} < y_n + \ln(2) + \ln(-x_n - \frac{1}{2}e^{y_n}) < y_n + \ln(2) + \ln(-x_n).$$

(4.15). We have

$$x_{n+1} = x_n + \ln(2e^{y_n} + e^{x_n}) > x_n + \ln(e^{x_n}) = 2x_n.$$

(4.16). By (4.14) we have

$$y_{n+1} < y_n + \ln(2) + \ln(-x_n),$$
$$y_{n+2} < y_{n+1} + \ln(2) + \ln(-x_{n+1}).$$

Hence we have

$$y_{n+2} < y_n + \ln(2) + \ln(-x_n) + \ln(2) + \ln(-x_{n+1}) = y_n + 2\ln(2) + \ln(-x_n) + \ln(-x_{n+1}).$$

By (4.15) we have $\ln(-2x_n) > \ln(-x_{n+1})$. Thus

$$y_{n+2} < y_n + 2\ln(2) + \ln(-x_n) + \ln(-2x_n) = y_n + 3\ln(2) + 2\ln(-x_n).$$

□

As a direct result of Lemma 4.5, for every $n, k = 0, 1, 2, ...$ we have

$$y_{n+2+k} \leq y_{n+2} + k\ln(2).$$

This along with Lemma 4.6 implies

$$y_{n+2+k} < y_n + 3\ln(2) + 2\ln(-x_n) + k\ln(2) = y_n + (k+3)\ln(2) + 2\ln(-x_n).$$

Therefore, we have the following lemma:

Lemma 4.7. *For every $n = 0, 1, 2, \ldots$ and $k = 2, 3, 4, \ldots$ we have*

$$y_{n+k} < y_n + (k+1)\ln(2) + 2\ln(-x_n). \tag{4.17}$$

Now as result of Lemma 4.6 and Lemma 4.7 we prove the following corollary.

Corollary 4.8. *If for a non-negative integer n we have $-\frac{1}{2} \leq x_n$, then for every $0 \leq k \leq \left\lfloor -\frac{2\ln(-x_n)}{\ln(2)} \right\rfloor - 1$ we have*

$$y_{n+k} \leq y_n. \tag{4.18}$$

Proof. Since $-\frac{1}{2} \leq x_n$, we have $-\frac{2\ln(-x_n)}{\ln(2)} > 1$ and $\left\lfloor -\frac{2\ln(-x_n)}{\ln(2)} \right\rfloor - 1 \geq 0$. If $1 = k \leq \left\lfloor -\frac{2\ln(-x_n)}{\ln(2)} \right\rfloor - 1$, then by Lemma 4.6 we have

$$y_{n+k} = y_{n+1} < y_n + \ln(2) + \ln(-x_n) \leq y_n + \ln(2) - \ln(2) = y_n.$$

Suppose that $2 \leq k \leq \left\lfloor -\frac{2\ln(-x_n)}{\ln(2)} \right\rfloor - 1$. By Lemma 4.7 we have

$$y_{n+k} < y_n + (k+1)\ln(2) + 2\ln(-x_n) \leq y_n + \left\lfloor -\frac{2\ln(-x_n)}{\ln(2)} \right\rfloor \ln(2) + 2\ln(-x_n).$$

$$\leq y_n - \frac{2\ln(-x_n)}{\ln(2)} \ln(2) + 2\ln(-x_n) = y_n.$$

□

Lemma 4.9 provides an upper bound for the error when we consider $2^k x_n$ as an approximation of x_{n+k}.

Lemma 4.9. *Suppose that for a non-negative integer n and every $k = 0, 1, \ldots, m$ we have $y_{n+k} \leq x_{n+k} - M$, where $M > 0$. For $k = 0, 1, \ldots, m+1$ we have*

$$0 \leq x_{n+k} - 2^k x_n < 2^{k+1} e^{-M}. \tag{4.19}$$

Proof. By using induction, we prove that for every $k = 0, 1, \ldots, m+1$ we have

$$0 \leq x_{n+k} - 2^k x_n \leq (2^{k+1} - 2)e^{-M}. \tag{4.20}$$

Then (4.19) is a direct result of (4.20). (4.20) is obvious for $k = 0$. Now assume that (4.20) is true for a $0 \leq k \leq m$. We have

$$x_{n+k+1} = x_{n+k} + \ln(2e^{y_{n+k}} + e^{x_{n+k}}) \leq x_{n+k} + \ln(2e^{x_{n+k}-M} + e^{x_{n+k}})$$
$$= x_{n+k} + \ln(e^{x_{n+k}}) + \ln(2e^{-M} + 1) = 2x_{n+k} + \ln(2e^{-M} + 1) < 2x_{n+k} + 2e^{-M}$$
$$\leq 2(2^k x_n + (2^{k+1} - 2)e^{-M}) + 2e^{-M} = 2^{k+1}x_n + (2^{k+2} - 2)e^{-M}.$$

Thus $x_{n+k+1} - 2^{k+1}x_n \leq (2^{k+2} - 2)e^{-M}$. By (4.15) and the induction assumption we have

$$2^{k+1}x_n \leq 2x_{n+k} < x_{n+k+1}$$

Thus $0 \leq x_{n+k+1} - 2^{k+1}x_n$. □

Suppose that n is an A-enterer with (u_n, v_n, w_n) close to the boundary. So v_n is small and u_n is close to 1. This implies a big difference between y_n and x_n. But as the orbit approaches the line $y = x$, the difference between y_{n+k} and x_{n+k} gets smaller and it doesn't allow us to find a big enough M which satisfies the conditions of Lemma 4.9. On the other hand, we will see that for most of the $k = 0, 1, ..., \gamma(n)$, u_{n+k} is close to 1, keeping the difference between y_{n+k} and x_{n+k} big, and for just a small portion of these k's we have smaller differences between y_{n+k} and x_{n+k}. So in order to get a better error, we will need to divide the k's into two parts, those with big differences between y_{n+k} and x_{n+k} and those with smaller ones. That's why we need to improve Lemma 4.9 by stating Lemma 4.10.

Lemma 4.10. *Suppose that for a non-negative integer n and every $k = 0, 1, \ldots, m_1$ we have $y_{n+k} < x_{n+k} - M_1$ and for every $k = m_1 + 1, m_1 + 2, \ldots, m_2$ we have $y_{n+k} < x_{n+k} - M_2$, where $M_1, M_2 > 0$. For $k = m_1 + 1, m_1 + 2, \ldots, m_2 + 1$ we have*

$$0 \leq x_{n+k} - 2^k x_n < 2^{k+1}e^{-M_1} + 2^{k-m_1}e^{-M_2}. \tag{4.21}$$

Proof. By using induction, we prove that for every $k = m_1 + 1, m_1 + 2, \ldots, m_2$ we have

$$0 \leq x_{n+k} - 2^k x_n < 2^{k+1}e^{-M_1} + (2^{k-m_1} - 2)e^{-M_2}. \tag{4.22}$$

Then (4.21) is a direct result of (4.22). (4.22) is true for $k = m_1 + 1$ according to Lemma 4.9. Assume that (4.22) is true for a $m_1 + 1 \leq k \leq m_2$. We have

$$x_{n+k+1} = x_{n+k} + \ln(2e^{y_{n+k}} + e^{x_{n+k}}) < x_{n+k} + \ln(2e^{x_{n+k}-M_2} + e^{x_{n+k}})$$
$$= x_{n+k} + \ln(e^{x_{n+k}}) + \ln(2e^{-M_2} + 1) = 2x_{n+k} + \ln(2e^{-M_2} + 1) < 2x_{n+k} + 2e^{-M_2}$$
$$\leq 2(2^k x_n + 2^{k+1}e^{-M_1} + (2^{k-m_1} - 2)e^{-M_2}) + 2e^{-M_2} = 2^{k+1}x_n + 2^{k+2}e^{-M_1} + (2^{k+1-m_1} - 2)e^{-M_2}.$$

Thus $x_{n+k+1} - 2^{k+1}x_n < 2^{k+2}e^{-M_1} + (2^{k+1-m_1} - 2)e^{-M_2}$. With a quite similar argument to what we had in Lemma (4.9) we can deduce that $0 \leq x_{n+k} - 2^k x_n$ implies $0 \leq x_{n+k+1} - 2^{k+1}x_n$. □

Lemma 4.11 gives an upper bound for the error of approximating y_{n+k+1} with $y_{n+k} + \ln(2 - 2e^{x_{n+k}})$ under certain conditions. Then by using that upper bound, we will obtain an upper bound for the error of approximating y_{n+k} with $y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1 - e^{x_{n+r}})$ in Lemma 4.12. The formula which Lemma 4.12 provides as the approximation of y_{n+k}, uses the values of x_{n+r} for $r = 0, 1, ..., k-1$. This is where we may think about the idea of using $2^r x_n$ instead of x_{n+r} in that formula. Lemmas 4.9 and 4.10 provide upper bounds for the error of approximating x_{n+r} with $2^r x_n$. So these lemmas will enable us to get the error of replacing x_{n+r} by $2^r x_n$ in the formula. Lemma 4.13 uses the result which Lemma 4.12 gives to obtain an upper bound for the error of approximating y_{n+k} with $y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1 - e^{2^r x_n})$.

Lemma 4.11. *Suppose that for a $n \in \{0, 1, 2, ...\}$ and a $k \in \{0, 1, 2, ...\}$ we have*

$$x_{n+k} < -e^{y_n + M}, \qquad y_{n+k} \le y_n,$$

where $1 < M < -y_n$. Then

$$y_{n+k} + \ln(2 - 2e^{x_{n+k}}) - 2e^{-M} < y_{n+k+1} < y_{n+k} + \ln(2 - 2e^{x_{n+k}}).$$

Proof. We have

$$y_{n+k+1} = y_{n+k} + \ln(2 - 2e^{x_{n+k}} - e^{y_{n+k}}) < y_{n+k} + \ln(2 - 2e^{x_{n+k}}).$$

To prove the other inequality, we have

$$y_{n+k+1} = y_{n+k} + \ln(2 - 2e^{x_{n+k}} - e^{y_{n+k}}) = y_{n+k} + \ln(2 - 2e^{x_{n+k}}) + \ln\left(1 - \frac{1}{2}\frac{e^{y_{n+k}}}{1 - e^{x_{n+k}}}\right). \tag{4.23}$$

Now we consider two cases: $-1 \le x_{n+k} < 0$ and $x_{n+k} < -1$. When $-1 \le x_{n+k} < 0$ we have

$$1 - e^{x_{n+k}} = -x_{n+k} - \frac{x_{n+k}^2}{2} - \frac{x_{n+k}^3}{6} - ... > -x_{n+k} - \frac{x_{n+k}^2}{2} = -\frac{x_{n+k}}{2} - \frac{1}{2}\left(x_{n+k} + x_{n+k}^2\right) \ge -\frac{x_{n+k}}{2}$$

$$\Longrightarrow \quad \frac{e^{y_{n+k}}}{1 - e^{x_{n+k}}} < \frac{e^{y_{n+k}}}{-\frac{x_{n+k}}{2}}$$

$$\Longrightarrow \quad 1 - \frac{1}{2}\frac{e^{y_{n+k}}}{1 - e^{x_{n+k}}} > 1 - \frac{1}{2}\frac{e^{y_{n+k}}}{-\frac{x_{n+k}}{2}} = 1 + \frac{e^{y_{n+k}}}{x_{n+k}}$$

$$\Longrightarrow \quad \ln\left(1 - \frac{1}{2}\frac{e^{y_{n+k}}}{1 - e^{x_{n+k}}}\right) > \ln\left(1 + \frac{e^{y_{n+k}}}{x_{n+k}}\right). \tag{4.24}$$

Now (4.23) and (4.24) imply

$$y_{n+k+1} > y_{n+k} + \ln(2 - 2e^{x_{n+k}}) + \ln\left(1 + \frac{e^{y_{n+k}}}{x_{n+k}}\right). \tag{4.25}$$

Since $x_{n+k} < -e^{y_n+M}$ and $y_{n+k} \leq y_n$ we have

$$1 + \frac{e^{y_{n+k}}}{x_{n+k}} > 1 - \frac{e^{y_{n+k}}}{e^{y_n+M}} \geq 1 - \frac{e^{y_n}}{e^{y_n+M}} = 1 - e^{-M}.$$

$$\implies \quad \ln\left(1 + \frac{e^{y_{n+k}}}{x_{n+k}}\right) > \ln\left(1 - e^{-M}\right)$$

Since $M > 1$ we have $\ln\left(1 - e^{-M}\right) > -2e^{-M}$. Thus

$$\ln\left(1 + \frac{e^{y_{n+k}}}{x_{n+k}}\right) > -2e^{-M}$$

$$\implies \quad y_{n+k+1} > y_{n+k} + \ln(2 - 2e^{x_{n+k}}) - 2e^{-M}.$$

When $x_{n+k} < -1$ we have

$$1 - e^{x_{n+k}} > \frac{1}{2}$$

$$\implies \quad 1 - \frac{1}{2}\frac{e^{y_{n+k}}}{1 - e^{x_{n+k}}} > 1 - \frac{1}{2}\frac{e^{y_{n+k}}}{\frac{1}{2}} = 1 - e^{y_{n+k}} \geq 1 - e^{y_n}$$

$$\implies \quad \ln\left(1 - \frac{1}{2}\frac{e^{y_{n+k}}}{1 - e^{x_{n+k}}}\right) > \ln\left(1 - e^{y_n}\right).$$

Since $y_n < -M < -1$ we have $\ln\left(1 - e^{y_n}\right) > -2e^{y_n} > -2e^{-M}$. Thus

$$\ln\left(1 - \frac{1}{2}\frac{e^{y_{n+k}}}{1 - e^{x_{n+k}}}\right) > -2e^{-M}.$$

This along with (4.23) implies

$$y_{n+k+1} > y_{n+k} + \ln(2 - 2e^{x_{n+k}}) - 2e^{-M}.$$

□

Lemma 4.12. *Suppose that for a non-negative integer n and for every $0 \leq k \leq m$ we have*

$$x_{n+k} < -e^{y_n+M}, \qquad y_{n+k} \leq y_n,$$

where $1 < M < -y_n$. Then for every $1 \leq k \leq m+1$ we have

$$y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1-e^{x_{n+r}}) - 2ke^{-M} < y_{n+k} < y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1-e^{x_{n+r}}). \tag{4.26}$$

Proof. By Lemma 4.11, (4.26) is true for $k = 1$. Assume for the sake of induction that (4.26) is true for a $0 \leq k \leq m$. By Lemma 4.11 and (4.26) we have

$$\begin{aligned} y_{n+k+1} < y_{n+k} + \ln(2) + \ln(1-e^{x_{n+k}}) < y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1-e^{x_{n+r}}) + \ln(2) + \ln(1-e^{x_{n+k}}) \\ = y_n + k\ln(2) + \sum_{r=0}^{k} \ln(1-e^{x_{n+r}}). \end{aligned}$$

Again by Lemma 4.11 and (4.26) we have

$$\begin{aligned} & y_n + (k+1)\ln(2) + \sum_{r=0}^{k} \ln(1-e^{x_{n+r}}) - 2(k+1)e^{-M} \\ = {} & y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1-e^{x_{n+r}}) - 2ke^{-M} + \ln(2) + \ln(1-e^{x_{n+k}}) - 2e^{-M} \\ & < y_{n+k} + \ln(2) + \ln(1-e^{x_{n+k}}) - 2e^{-M} < y_{n+k+1}. \end{aligned}$$

□

Lemma 4.13. *Suppose that for a non-negative integer n and for every $0 \leq k \leq m$ we have*

$$x_{n+k} < -e^{y_n+M}, \qquad y_{n+k} \leq y_n, \qquad |x_{n+k} - 2^k x_n| < \varepsilon_{n+k} < |2^{k-1}x_n|,$$

where $1 < M < -y_n$. Then for every $1 \leq k \leq m+1$ we have

$$\left| y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1-e^{2^r x_n}) - y_{n+k} \right| < 2ke^{-M} + \sum_{r=0}^{k-1} \frac{\varepsilon_{n+r}}{|2^{r-1}x_n|}. \tag{4.27}$$

Proof. Suppose that $0 \leq r \leq k-1$. By the mean value theorem there exists $2^r x_n \leq \xi \leq x_{n+r}$ such that

$$\frac{\ln(1-e^{x_{n+r}}) - \ln(1-e^{2^r x_n})}{x_{n+r} - 2^r x_n} = \frac{-e^{\xi}}{1-e^{\xi}}.$$

Since $\left|\frac{-e^{\xi}}{1-e^{\xi}}\right|$ is strictly increasing and $\xi \leq x_{n+r} < 2^r x_n + \varepsilon_{n+r}$, we have

$$\left|\frac{-e^{\xi}}{1-e^{\xi}}\right| < \left|\frac{-e^{2^r x_n+\varepsilon_{n+r}}}{1-e^{2^r x_n+\varepsilon_{n+r}}}\right|$$

$$\Longrightarrow \quad \frac{|\ln(1-e^{x_{n+r}}) - \ln(1-e^{2^r x_n})|}{|x_{n+r} - 2^r x_n|} \leq \left|\frac{-e^{2^r x_n+\varepsilon_{n+r}}}{1-e^{2^r x_n+\varepsilon_{n+r}}}\right|$$

$$\Longrightarrow \quad |\ln(1-e^{x_{n+r}}) - \ln(1-e^{2^r x_n})| \leq \left|\frac{-e^{2^r x_n+\varepsilon_{n+r}}}{1-e^{2^r x_n+\varepsilon_{n+r}}}\right| |x_{n+r} - 2^r x_n|$$

$$\Longrightarrow \quad |\ln(1-e^{x_{n+r}}) - \ln(1-e^{2^r x_n})| \leq \left|\frac{-e^{2^r x_n+\varepsilon_{n+r}}}{1-e^{2^r x_n+\varepsilon_{n+r}}}\right| \varepsilon_{n+r}. \tag{4.28}$$

We also have

$$\left|\frac{-e^{2^r x_n+\varepsilon_{n+r}}}{1-e^{2^r x_n+\varepsilon_{n+r}}}\right| = \frac{1}{e^{-(2^r x_n+\varepsilon_{n+r})} - 1} = \frac{1}{\sum_{s=1}^{\infty} \frac{(-(2^r x_n+\varepsilon_{n+r}))^s}{s!}} < \frac{1}{-(2^r x_n + \varepsilon_{n+r})}, \tag{4.29}$$

since $\varepsilon_{n+r} < |2^{r-1} x_n|$, we have

$$0 < \frac{\varepsilon_{n+r}}{-(2^r x_n + \varepsilon_{n+r})} < \frac{\varepsilon_{n+r}}{-(2^{r-1} x_n)}, \tag{4.30}$$

(4.28), (4.29) and (4.30) imply

$$|\ln(1-e^{x_{n+r}}) - \ln(1-e^{2^r x_n})| < \frac{\varepsilon_{n+r}}{-(2^{r-1} x_n)}.$$

Thus

$$\ln(1-e^{2^r x_n}) - \frac{\varepsilon_{n+r}}{-(2^{r-1} x_n)} < \ln(1-e^{x_{n+r}}) < \ln(1-e^{2^r x_n}) + \frac{\varepsilon_{n+r}}{-(2^{r-1} x_n)} \tag{4.31}$$

$$\Longrightarrow \quad \sum_{r=0}^{k-1} \ln(1-e^{2^r x_n}) - \frac{\varepsilon_{n+r}}{-(2^{r-1} x_n)} < \sum_{r=0}^{k-1} \ln(1-e^{x_{n+r}}) < \sum_{r=0}^{k-1} \ln(1-e^{2^r x_n}) + \frac{\varepsilon_{n+r}}{-(2^{r-1} x_n)},$$

$$\Longrightarrow \quad \left|\sum_{r=0}^{k-1} \ln(1-e^{2^r x_n}) - \sum_{r=0}^{k-1} \ln(1-e^{x_{n+r}})\right| < \sum_{r=0}^{k-1} \frac{\varepsilon_{n+r}}{|2^{r-1} x_n|}.$$

Now (4.27) is a direct result of the above inequalities and Lemma 4.12.

□

Suppose that n is an A-enterer with (u_n, v_n, w_n) close to the boundary. We will become able to observe that if $0 \leq m \leq \gamma(n)$ is the smallest non-negative integer such that $x_{n+r} < -1$ for $r = m+1, ..., \gamma(n)$, then $\frac{m}{\gamma(n)} \to 1$ as $n \to \infty$. It means that for most of $r = 0, 1, ..., \gamma(n)$ we have $-1 < x_{n+r}$. Since we expect to have $x_{n+r} \approx 2^r x_n$ for $r = 0, 1, ..., \gamma(n)$, we could estimate that m to be approximately equal to $\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$. As we still don't have an estimation of $\gamma(n)$, It's challenging to estimate the errors of the approximation formulas for every $r = 0, 1, ..., \gamma(n)$ at the same time. So we use this strategy: First we develop the theory for $r = 0, 1, ..., \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$, then we will become able to get an estimation of $\gamma(n)$ to complete the theory for the rest of r's.

Lemmas 4.14 and 4.15 give sufficient conditions for n to statisfy the conditions of Lemma 4.13 for $k = 0, 1, ..., \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$.

Lemma 4.14. *Suppose that for a non-negative integer n we have $-\frac{1}{2} \leq x_n$ and $y_n < -1$. Then for every $k = 0, 1, ..., \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1$ we have*

$$y_{n+k} \leq x_{n+k} + y_n + 1, \tag{4.32}$$

$$0 \leq x_{n+k} - 2^k x_n < 2^{k+1} e^{y_n+1}. \tag{4.33}$$

Proof. Since $-\frac{1}{2} \leq x_n$, we have $\frac{-\ln(-x_n)}{\ln(2)} \geq 1$ and $1 \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor \leq \left\lfloor \frac{-2\ln(-x_n)}{\ln(2)} \right\rfloor - 1$. Hence by Corollary 4.8, for every $k = 0, 1, ..., \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$ we have $y_{n+k} \leq y_n$. We also have

$$-1 = 2^{\frac{-\ln(-x_n)}{\ln(2)}} x_n \leq 2^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} x_n \leq 2^k x_n \leq x_{n+k}.$$

Now since $y_{n+k} \leq y_n$ and $-1 \leq x_{n+k}$, we have

$$y_{n+k} \leq -1 + (y_n + 1) \leq x_{n+k} + (y_n + 1) = x_{n+k} - M, \tag{4.34}$$

where $M = -(y_n + 1) > 0$. Now (4.33) is a direct result of Lemma 4.9 and (4.32). □

Lemma 4.15. *Suppose that for a natural number n we have $y_n \leq -3$ and $2z_n < y_n < z_n - M$, where $M \geq 3$. Then for every $k = 0, 1, ..., \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$ we have*

$$|x_{n+k} - 2^k x_n| < 2^{k+1} e^{y_n+1} < |2^{k-1} x_n|. \tag{4.35}$$

$$x_{n+k} < -e^{y_n+M}. \tag{4.36}$$

Proof. By Lemma 4.4 we have $-2e^{\frac{y_n}{2}} < x_n$ and this along with $y_n \leq -3$ implies $-\frac{1}{2} < x_n$. Therefore, x_n and y_n satisfy the conditions of Lemma 4.14. Suppose that $0 \leq k \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$, by Lemma 4.4 we have

$$|2^{k-1}x_n| > 2^{k-1}e^{y_n+M} = 2^{k+1}e^{y_n+1}\frac{e^{M-1}}{4},$$

Now since $M \geq 3$, we have $\frac{e^{M-1}}{4} > 1$. Thus $|2^{k-1}x_n| > 2^{k+1}e^{y_n+1}$.

To prove (4.36), first we observe that by Lemma 4.4 we have $x_n < -e^{y_n+M}$. So (4.36) is true for $k = 0$. Now suppose that $0 < k \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$, (4.35) implies

$$x_{n+k} < 2^k x_n + |2^{k-1}x_n| = 2^k x_n - 2^{k-1}x_n = 2^{k-1}x_n \leq x_n < -e^{y_n+M}.$$

□

Now we can obtain an upper bound for the error of approximating y_{n+k} by $y_n + k\ln(2) + \sum_{r=0}^{k-1}\ln(1-e^{2^r x_n})$ for $k = 1, 2, ..., \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1$.

Lemma 4.16. *Suppose that for a natural number n we have $y_n \leq -3$ and $2z_n < y_n < z_n - M$, where $3 \leq M < -y_n$. Then for every $k = 1, 2, ..., \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1$ we have*

$$\left| y_n + k\ln(2) + \sum_{r=0}^{k-1}\ln(1-e^{2^r x_n}) - y_{n+k} \right| < (2+4e)ke^{-M}. \tag{4.37}$$

Proof. It's clear that n satisfies the conditions of Lemma 4.15. Hence, for every $0 \leq k \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$, (4.35) and (4.36) are true. By Lemma 4.4 we have $-2e^{\frac{y_n}{2}} < x_n$, and this along with $y_n \leq -3$ implies $-\frac{1}{2} < x_n$. Hence by Corollary 4.8, (4.18) holds for $0 \leq k \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$. Now from (4.35), (4.36) and (4.18) we deduce that n satisfies the conditions of Lemma 4.13 for $m = \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$ and $\varepsilon_{n+k} = 2^{k+1}e^{y_n+1}$. Now by (4.27) and (4.36) for every $1 \leq k \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1$ we have

$$\left| y_n + k\ln(2) + \sum_{r=0}^{k-1}\ln(1-e^{2^r x_n}) - y_{n+k} \right| < 2ke^{-M} + \sum_{r=0}^{k-1}\frac{2^{r+1}e^{y_n+1}}{|2^{r-1}x_n|} < 2ke^{-M} + \sum_{r=0}^{k-1}\frac{2^{r+1}e^{y_n+1}}{2^{r-1}e^{y_n+M}}$$

$$= 2ke^{-M} + \sum_{r=0}^{k-1} 4e \cdot e^{-M} = (2+4e)ke^{-M}.$$

□

As mentioned before, first we need to develop the approximation theory for $k = 1, 2, ..., \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1$ and then we will also become able to develop the theory for the whole period the orbit is traveling through A. Now that Lemma 4.16 provides the error for those values of k, we especially need an estimation of the error of approximating $y_{n+\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor}$ in order to get an estimation of $\gamma(n)$ and to estimate the approximation error of x_{n+k} and y_{n+k} for $k = \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 2, ..., \gamma(n)$.

So we express Corollary 4.17.

Corollary 4.17. *Suppose that for a natural number n we have $2z_n < y_n < z_n - M$, where $3 \leq M < -y_n$. Then*

$$\left| y_n + \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor \ln(2) + \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} \ln(1 - e^{2^r x_n}) - y_{n+\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} \right| < -19 y_n e^{-M}. \quad (4.38)$$

Proof. Lemma 4.16 and (4.36) imply that for $k = \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$ we have

$$\left| y_n + \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor \ln(2) + \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} \ln(1 - e^{2^r x_n}) - y_{n+\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} \right| < (2+4e) \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor e^{-M}$$

$$\leq (2+4e)\frac{-\ln(-x_n)}{\ln(2)} e^{-M} < (2+4e)\frac{-\ln(e^{y_n+M})}{\ln(2)} e^{-M} = (2+4e)\frac{-(y_n+M)}{\ln(2)} e^{-M}$$

$$< -\frac{2+4e}{\ln(2)} y_n e^{-M} < -19 y_n e^{-M}.$$

□

Now it's time to develop the theory for $k > \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$. In Proposition 4.18, we put some conditions on a natural number m. As it can be seen, when n is an A-enterer, $m = \gamma(n) - 1$, may satisfy the conditions of this proposition (because the condition $y_{n+k} < x_{n+k} - M_2$ guarantees $(u_{n+k}, v_{n+k}, w_{n+k}) \in A$ and if M_2 is chosen appropriately, we would have $(u_{n+m+1}, v_{n+m+1}, w_{n+m+1}) \notin A$). This exactly what we intend to do with the value of m mentioned in Proposition 4.18.

Proposition 4.18. *Suppose that for a natural number n we have $2z_n < y_n < z_n - M_1$, where $4 \leq M_1 < -y_n$ and for an $\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor < m \leq \left\lfloor \frac{-2\ln(-x_n)}{\ln(2)} \right\rfloor - 1$ and every $\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor < k \leq m$ we have $y_{n+k} < x_{n+k} - M_2$ where $M_2 > 3$. Then for $\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1 \leq k \leq m+1$ we have*

$$0 \leq x_{n+k} - 2^k x_n < 2^{k+1} e^{y_n+1} + 2^{k - \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} e^{-M_2} < |2^{k-1} x_n|, \quad (4.39)$$

$$\left| y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1 - e^{2^r x_n}) - y_{n+k} \right| < -38 y_n e^{-M_1} - 12 y_n e^{-M_2}. \quad (4.40)$$

Proof. Lemma 4.4 and $4 < -y_n$ imply $-\frac{1}{2} \leq x_n$. Hence n satisfies the conditions of Lemma 4.14. So by (4.32), for every $k = 0, 1, ..., \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$ we have

$$y_{n+k} \leq x_{n+k} - (-y_n - 1).$$

This along with the fact that for every $\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor < k \leq m$ we have $y_{n+k} < x_{n+k} - M_2$ and Lemma 4.10 implies (4.39).

Now by (4.39) for every $\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1 \leq k \leq m+1$ we have

$$|x_{n+k} - 2^k x_n| < 2^{k+1} e^{y_n+1} + 2^{k-\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} e^{-M_2} = 2^{k-1}\left(4e^{y_n+1} + 2^{1-\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} e^{-M_2}\right)$$
$$< 2^{k-1}\left(4e^{y_n+1} + 2^{2-\frac{-\ln(-x_n)}{\ln(2)}} e^{-M_2}\right) = 2^{k-1}\left(4e^{y_n+1} - 4x_n e^{-M_2}\right).$$

Since by Lemma 4.4, $|x_n| > e^{y_n+M_1}$ and $M_1 \geq 4$, we have $4e^{y_n+1} < |\frac{x_n}{2}|$ and $M_2 > 3$ implies $4e^{-M_2} < \frac{1}{2}$. Thus for $\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1 \leq k \leq m+1$,

$$2^{k+1} e^{y_n+1} + 2^{k-\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} e^{-M_2} < 2^{k-1}\left(\left|\frac{x_n}{2}\right| - \frac{x_n}{2}\right) = |2^{k-1} x_n|. \tag{4.41}$$

n satisfies the conditions of Lemma 4.15 for $M = M_1$. Hence for every $0 \leq k \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor$

$$|x_{n+k} - 2^k x_n| < 2^{k+1} e^{y_n+1} < |2^{k-1} x_n|. \tag{4.42}$$

Since $m \leq \left\lfloor \frac{-2\ln(-x_n)}{\ln(2)} \right\rfloor - 1$ and $-\frac{1}{2} \leq x_n$, by Corollary 4.8 we have $y_{n+k} \leq y_n$ for every $k = 0, 1, 2, ..., m$. Now since by Lemma 4.4 we have $x_n < -e^{y_n+M_1}$, for $k = 0, 1, ..., m$, $|x_{n+k} - 2^k x_n| < |2^{k-1} x_n|$ implies

$$x_{n+k} < 2^k x_n + |2^{k-1} x_n| = 2^{k-1} x_n \leq x_n < -e^{y_n+M_1}.$$

So m and n satisfy the conditions of Lemma 4.13 with $M = M_1$ and ε_{n+k} defined as follows

$$\varepsilon_{n+k} = \begin{cases} 2^{k+1} e^{y_n+1}, & 0 \leq k \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor \\ 2^{k+1} e^{y_n+1} + 2^{k-\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} e^{-M_2}, & \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1 \leq k \leq m+1 \end{cases}$$

hence for $k = \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 2, ..., m+1$

$$\left| y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1 - e^{2^r x_n}) - y_{n+k} \right| < 2ke^{-M} + \sum_{r=0}^{k-1} \frac{\varepsilon_{n+r}}{|2^{r-1} x_n|}$$

$$= 2ke^{-M_1} + \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} \frac{2^{r+1} e^{y_n+1}}{|2^{r-1} x_n|} + \sum_{r=\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1}^{k-1} \frac{2^{r+1} e^{y_n+1} + 2^{r-\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} e^{-M_2}}{|2^{r-1} x_n|}$$

$$= 2ke^{-M_1} + \sum_{r=0}^{k-1} \frac{2^{r+1}e^{y_n+1}}{|2^{r-1}x_n|} + \sum_{r=\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor+1}^{k-1} \frac{2^{r-\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} e^{-M_2}}{|2^{r-1}x_n|}$$

$$= 2ke^{-M_1} + \sum_{r=0}^{k-1} \frac{4e^{y_n+1}}{|x_n|} + \sum_{r=\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor+1}^{k-1} \frac{2^{-\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} e^{-M_2}}{|2^{-1}x_n|}$$

$$< 2ke^{-M_1} + \sum_{r=0}^{k-1} \frac{4e^{y_n+1}}{e^{y_n+M_1}} + \sum_{r=\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor+1}^{k-1} \frac{2^{1-\frac{-\ln(-x_n)}{\ln(2)}} e^{-M_2}}{|2^{-1}x_n|}$$

$$= 2ke^{-M_1} + 4ke^{1-M_1} + 4\left(k - 1 - \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor\right) e^{-M_2}$$

$$< 2ke^{-M_1} + 4ke^{1-M_1} + 4ke^{-M_2} = (2+4e)ke^{-M_1} + 4ke^{-M_2}$$

$$\leq (2+4e)\left\lfloor \frac{-2\ln(-x_n)}{\ln(2)} \right\rfloor e^{-M_1} + 4\left\lfloor \frac{-2\ln(-x_n)}{\ln(2)} \right\rfloor e^{-M_2}$$

$$\leq (2+4e)\frac{-2\ln(-x_n)}{\ln(2)} e^{-M_1} + 4\frac{-2\ln(-x_n)}{\ln(2)} e^{-M_2}$$

$$\leq (2+4e)\frac{-2\ln(e^{y_n+M_1})}{\ln(2)} e^{-M_1} + 4\frac{-2\ln(e^{y_n+M_1})}{\ln(2)} e^{-M_2} = \frac{y_n+M_1}{\ln(2)}(-(4+8e)e^{-M_1} - 8e^{-M_2})$$

$$< \frac{y_n}{\ln(2)}(-(4+8e)e^{-M_1} - 8e^{-M_2}) < -38y_ne^{-M_1} - 12y_ne^{-M_2}.$$

□

Proposition 4.18 provides an upper bound for the error of approximating y_{n+k} for $\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1 \leq k \leq m+1$, and as we mentioned before, our candidate for the value of m is $\gamma(n) - 1$. Now we need to examine under which conditions we could get similar errors for a similar m when approximating x_{n+k} for $k \leq m+1$. Then combining these conditions with the conditions of Proposition 4.18 will give us the conditions under which we can estimate the approximation error of an orbit while it travels through A.

Proposition 4.19. *Suppose that for a natural number n we have $2z_n < y_n < z_n - M_1$, where $28 \leq M_1 < -y_n$ and for an $\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor < m \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 5\ln(-y_n)$ and every $\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor <$*

$k \le m$ we have $y_{n+k} < x_{n+k} - M_2$ where $M_2 > 3$. Then for $\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 2 \le k \le m+1$ we have

$$0 \le x_{n+k} - 2^k x_n < 6y_n^4 e^{-M_1} + y_n^4 e^{-M_2}. \tag{4.43}$$

Proof. By Lemma 4.4 we have $-2e^{\frac{y_n}{2}} < x_n$, hence

$$\left\lfloor \frac{-2\ln(-x_n)}{\ln(2)} \right\rfloor - 1 > \frac{-2\ln(-x_n)}{\ln(2)} - 2 > \frac{-\ln(-x_n)}{\ln(2)} + \frac{-\ln(2e^{\frac{y_n}{2}})}{\ln(2)} - 2$$
$$\ge \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + \frac{-\ln(2e^{\frac{y_n}{2}})}{\ln(2)} - 2 = \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 3 - \frac{y_n}{2\ln(2)}.$$

Since $y_n < -28$ we have $-3 - \frac{y_n}{2\ln(2)} > 5\ln(-y_n)$. Thus $\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 5\ln(-y_n) < \left\lfloor \frac{-2\ln(-x_n)}{\ln(2)} \right\rfloor - 1$ and m satisfies the conditions of Proposition 4.18. Hence (4.39) implies

$$0 \le x_{n+k} - 2^k x_n < 2^{k+1} e^{y_n+1} + 2^{k - \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} e^{-M_2}$$
$$< 2^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 5\ln(-y_n)+1} e^{y_n+1} + 2^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 5\ln(-y_n) - \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} e^{-M_2}$$

$$\le 2^{\frac{-\ln(-x_n)}{\ln(2)} + 5\ln(-y_n)+1} e^{y_n+1} + 2^{5\ln(-y_n)} e^{-M_2} = \frac{2(-y_n)^{5\ln(2)}}{-x_n} e^{y_n+1} + (-y_n)^{5\ln(2)} e^{-M_2}$$

$$< \frac{2y_n^4}{-x_n} e^{y_n+1} + y_n^4 e^{-M_2} < \frac{2y_n^4}{e^{y_n+M_1}} e^{y_n+1} + y_n^4 e^{-M_2}$$

$$= 2e y_n^4 e^{-M_1} + y_n^4 e^{-M_2} < 6y_n^4 e^{-M_1} + y_n^4 e^{-M_2}.$$

□

Now we combine Propositions 4.18 and 4.19 to obtain conditions under which we can use the upper bounds for the approximation errors stated in both propositions. It's easy to observe that m and n in Proposition 4.19 also satisfy the conditions of Proposition 4.18. So we only need to suppose that m and n satisfy the conditions of Proposition 4.19 to combine Propositions 4.18 and 4.19.

Theorem 4.20. *Suppose that m and n satisfy the conditions of Proposition 4.19. Then for every $0 \le k \le m$ we have*

$$0 \le x_{n+k} - 2^k x_n < 6y_n^4 e^{-M_1} + y_n^4 e^{-M_2}, \tag{4.44}$$

$$\left| y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1-e^{2^r x_n}) - y_{n+k} \right| < -38 y_n e^{-M_1} - 12 y_n e^{-M_2}. \tag{4.45}$$

Lemma 4.21. *Suppose that for a natural number n we have* $y_n < -2$ *and* $2z_n < y_n < z_n - M$*, where* $\ln(2) < M$*. Then for every* $k = 1, 2, ...$ *we have*

$$y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1-e^{2^r x_n}) > -\frac{1}{\ln(2)}(y_n)^2 + y_n - 1. \tag{4.46}$$

Proof. First we prove that for every $1 \le k \le \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1$

$$\sum_{r=0}^{k-1} \ln(1-e^{2^r x_n}) > k\ln(|x_n|) - k\ln(2).$$

Suppose that $0 \le k \le \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1$. We have

$$\sum_{r=0}^{k-1} \ln(1-e^{2^r x_n}) = \sum_{r=0}^{k-1} \ln\left(1 - 1 + |2^r x_n| - \frac{|2^r x_n|^2}{2} + \frac{|2^r x_n|^3}{6} - \frac{|2^r x_n|^4}{24} + ...\right)$$

$$= \sum_{r=0}^{k-1} \ln\left(|2^r x_n| - \frac{|2^r x_n|^2}{2} + \frac{|2^r x_n|^3}{6} - \frac{|2^r x_n|^4}{24} + ...\right).$$

For every $0 \le r \le k-1$

$$0 < |2^r x_n| \le |2^{k-1} x_n| \le \left| 2^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} x_n \right| \le \left| 2^{\frac{-\ln(-x_n)}{\ln(2)}} x_n \right| = 1.$$

Therefore $\ln\left(|2^r x_n| - \frac{|2^r x_n|^2}{2} + \frac{|2^r x_n|^3}{6} - \frac{|2^r x_n|^4}{24} + ...\right) > \ln\left(|2^r x_n| - \frac{|2^r x_n|^2}{2}\right)$. Thus

$$\sum_{r=0}^{k-1} \ln(1-e^{2^r x_n}) > \sum_{r=0}^{k-1} \ln\left(|2^r x_n| - \frac{|2^r x_n|^2}{2}\right) = \sum_{r=0}^{k-1} \ln\left(|2^r x_n|\right) + \sum_{r=0}^{k-1} \ln\left(1 - \frac{|2^r x_n|}{2}\right)$$

$$> \sum_{r=0}^{k-1} \ln\left(|x_n|\right) + \sum_{r=0}^{k-1} \ln\left(1 - \frac{|2^r x_n|}{2}\right) = k\ln(|x_n|) + \sum_{r=0}^{k-1} \ln\left(1 - \frac{|2^r x_n|}{2}\right)$$

Now since $|2^r x_n| \le 1$ for every $0 \le r \le k-1$, we have $\ln\left(1 - \frac{|2^r x_n|}{2}\right) \ge -\ln(2)$ and

$$\sum_{r=0}^{k-1} \ln(1-e^{2^r x_n}) > k\ln(|x_n|) + \sum_{r=0}^{k-1} \ln\left(1 - \frac{|2^r x_n|}{2}\right) > k\ln(|x_n|) - k\ln(2).$$

Now for every $0 \leq k \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1$ we have

$$y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1 - e^{2^r x_n}) > y_n + k\ln(2) + k\ln(|x_n|) - k\ln(2) = y_n + k\ln(|x_n|) \quad (4.47)$$

By Lemma 4.4, $e^{y_n} < e^{y_n + M} < |x_n|$. Hence $\ln(|x_n|) > y_n + M > y_n$. Moreover,

$$0 \leq k \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1 \leq \frac{-\ln(|x_n|)}{\ln(2)} + 1 < \frac{-y_n}{\ln(2)} - \frac{M}{\ln(2)} + 1$$

So from $\ln(2) < M$ we have $k < \frac{-y_n}{\ln(2)}$. This along with (4.47) and $y_n < \ln(|x_n|) < 0$ implies

$$y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1 - e^{2^r x_n}) > y_n + k\ln(|x_n|) > y_n + \frac{-y_n}{\ln(2)} \ln(|x_n|)$$

$$> y_n + \frac{-y_n}{\ln(2)} y_n > -\frac{(y_n)^2}{\ln(2)} + y_n - 1.$$

To prove (4.46) for $k > \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1$, we have

$$y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1 - e^{2^r x_n}) = y_n + \left(\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1 \right) \ln(2) + \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} \ln(1 - e^{2^r x_n})$$

$$+ \left(k - \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1 \right) \ln(2) + \sum_{r=\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1}^{k-1} \ln(1 - e^{2^r x_n}),$$

From the argument we had for $1 \leq k \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1$, we have

$$y_n + \left(\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1 \right) \ln(2) + \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor} \ln(1 - e^{2^r x_n}) > -\frac{(y_n)^2}{\ln(2)} + y_n,$$

this along with $\left(k - \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1 \right) \ln(2) > 0$ implies

$$y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1 - e^{2^r x_n}) = -\frac{(y_n)^2}{\ln(2)} + y_n + \sum_{r=\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 1}^{k-1} \ln(1 - e^{2^r x_n})$$

$$> -\frac{(y_n)^2}{\ln(2)} + y_n + \sum_{r=\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor+1}^{\infty} \ln(1 - e^{2^r x_n}) = -\frac{(y_n)^2}{\ln(2)} + y_n + \sum_{r=0}^{\infty} \ln\left(1 - e^{2^r 2^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor+1} x_n}\right)$$

$$\geq -\frac{(y_n)^2}{\ln(2)} + y_n + \sum_{r=0}^{\infty} \ln\left(1 - e^{2^r 2^{\frac{-\ln(-x_n)}{\ln(2)}} x_n}\right) = -\frac{(y_n)^2}{\ln(2)} + y_n + \sum_{r=0}^{\infty} \ln\left(1 - e^{-2^r}\right)$$

$$> -\frac{(y_n)^2}{\ln(2)} + y_n - 1.$$

□

The following theorem provides an estimate of $\gamma(n)$, which plays an important role in our subsequent discussion. This estimate allows us to conclude that when applying the approximation formula to a segment of an orbit that enters the set A in the beginning and exits it in the end, the value of $\gamma(n)$ is sufficiently small to prevent the error from becoming significant.

Theorem 4.22. *Suppose that for a non-negative integer n we have $2z_n < y_n < z_n - M_1$, where $60 \leq M_1 < -y_n$ and*

$$6y_n^4 e^{-M_1} < 1,$$

$$-38 y_n e^{-M_1} < 1.$$

Then for some $\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor < m \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 5\ln(-y_n)$ we have $x_{n+m} < y_{n+m}$ and $y_{n+m-1} < x_{n+m-1}$. In other words,

$$\gamma(n) \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 5\ln(-y_n).$$

Proof. Let m' be the largest natural number such that for $0 \leq k \leq m'$ we have $y_{n+k} < x_{n+k} - 4\ln(-y_n)$. First we prove that $m' \leq m^* := \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 5\ln(-y_n) - 2$. Suppose for the sake of contradiction that $m^* < m'$.

Since $y_n < -60 < -2$ and $\ln(2) < 60 \leq M_1$, By lemma 4.21 for every $k = 1, 2, ...$ we have

$$y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1 - e^{2^r x_n}) > -\frac{1}{\ln(2)}(y_n)^2 + y_n - 1.$$

By Theorem 4.20 and the assumed inequalities and letting $M_2 = 4\ln(-y_n)$ for every $0 \leq k \leq$

$\lfloor m^* \rfloor$ we have

$$0 \leq x_{n+k} - 2^k x_n < 6y_n^4 e^{-M_1} + y_n^4 e^{-M_2} < 2,$$

$$\left| y_n + k\ln(2) + \sum_{r=0}^{k-1} \ln(1 - e^{2^r x_n}) - y_{n+k} \right| < -38y_n e^{-M_1} - 12y_n e^{-M_2} < 2.$$

hence

$$x_{n+k} \leq 2^k x_n + 1,$$

$$-\frac{1}{\ln(2)}(y_n)^2 + y_n - 3 < y_{n+k}.$$

We also have

$$2^{m^*} x_n + 1 = 2^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 5\ln(-y_n) - 2} x_n + 1 \leq 2^{5\ln(-y_n)-2} \frac{x_n}{-2x_n} + 1 = -\frac{1}{8} 2^{5\ln(-y_n)} + 1$$

$$= -\frac{1}{8} e^{5\ln(2)\ln(-y_n)} + 1 < -\frac{1}{8} e^{3\ln(-y_n)} + 1 = -\frac{1}{8}|y_n|^3 + 1.$$

Now since $y_n < 60$

$$-\frac{1}{8}|y_n|^3 + 1 < -\frac{1}{\ln(2)}(y_n)^2 + y_n,$$

and combining the above inequalities gives

$$2^{m^*} x_n + 1 < -\frac{1}{\ln(2)}(y_n)^2 + y_n.$$

Thus

$$x_{n+\lfloor m^* \rfloor} \leq 2^{\lfloor m^* \rfloor} x_n + 1 \leq 2^{m^*} x_n + 1 < -\frac{1}{\ln(2)}(y_n)^2 + y_n < y_{n+\lfloor m^* \rfloor}.$$

Now $x_{n+\lfloor m^* \rfloor} < y_{n+\lfloor m^* \rfloor}$ contradicts $m^* < m'$ because for $0 \leq k \leq m'$ we have $y_{n+k} < x_{n+k} - 4\ln(-y_n)$.

Now that we have shown that $m' \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 5\ln(-y_n)$, by Theorem 4.20 and letting $M_2 = 4\ln(-y_n)$ we have

$$0 \leq x_{n+m'+1} - 2^{m'+1} x_n < 6y_n^4 e^{-M_1} + y_n^4 e^{-M_2} < 2,$$

$$\left| y_n + (m'+1)\ln(2) + \sum_{r=0}^{m'} \ln(1 - e^{2^r x_n}) - y_{n+m'+1} \right| < -38 y_n e^{-M_1} - 12 y_n e^{-M_2} < 2.$$

We also have

$$(m'+1)\ln(2) + \sum_{r=0}^{m'} \ln(1 - e^{2^r x_n}) = \sum_{r=0}^{m'} \ln(2 - 2e^{2^r x_n}).$$

Observe that $\ln(2 - 2e^{2^r x_n}) > 0$ if and only if $2^r x_n < -\ln(2)$, and since $m' \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor + 5\ln(-y_n)$, $\ln(2 - 2e^{2^r x_n})$ is positive for a number of times which is less than $5\ln(-y_n) + 2$. Moreover, we always have $\ln(2 - 2e^{2^r x_n}) < \ln(2)$. Thus

$$(m'+1)\ln(2) + \sum_{r=0}^{m'} \ln(1 - e^{2^r x_n}) < (5\ln(-y_n) + 2)\ln(2).$$

Therefore

$$y_{n+m'+1} < y_n + (5\ln(-y_n) + 2)\ln(2) + 2 < y_n + 4\ln(-y_n) + 4.$$

Two cases are possible:

(i) $x_{n+m'+1} < y_{n+m'+1}$.
(ii) $y_{n+m'+1} \leq x_{n+m'+1}$.

If (i) is true, then the theorem is true by letting $m = m' + 1$.

If (ii) is true, by the definition of m' we have $x_{m'+1} - 4\ln(-y_n) \leq y_{n+m'+1} \leq x_{n+m'+1}$. Now we have

$$x_{n+m'+2} = x_{n+m'+1} + \ln(e^{x_{n+m'+1}} + 2e^{y_{n+m'+1}}) \leq x_{n+m'+1} + \ln(e^{x_{n+m'+1}} + 2e^{x_{n+m'+1}})$$

$$= x_{n+m'+1} + \ln(3e^{x_{n+m'+1}}) = 2x_{n+m'+1} + \ln(3) \leq 2y_{n+m'+1} + 8\ln(-y_n) + \ln(3)$$

$$< y_{n+m'+1} + 8\ln(-y_n) + \ln(3) + y_n + 4\ln(-y_n) + 4 < y_{n+m'+1} + 12\ln(-y_n) + 6 + y_n.$$

Since $y_n < -60$, we have $12\ln(-y_n) + 6 + y_n < 0$, thus

$$x_{n+m'+2} < y_{n+m'+1}.$$

We also have

$$y_{n+m'+2} = y_{n+m'+1} + \ln(2 - e^{y_{n+m'+1}} - 2e^{x_{n+m'+1}})$$

$$> y_{n+m'+1} + \ln(2 - e^{y_{n+m'+1}} - 2e^{y_{n+m'+1}+4\ln(-y_n)})$$

$$> y_{n+m'+1} + \ln(2 - (y_n)^4 e^{y_n+4} - 2(y_n)^8 e^{y_n+4}) > y_{n+m'+1}.$$

Now combining the inequalities gives $x_{n+m'+2} < y_{n+m'+2}$, which means we can put $m = m' + 2$ in this case.

□

Suppose that n is an A-enterer. Using the estimate for $\gamma(n)$ provided by Theorem 4.22, we see that the error in the approximation formula given by Theorem 4.20 remains small for $0 \le k \le \gamma(n)$. Thus, we have achieved the primary goal outlined earlier in developing this approximation theory. To demonstrate an application of the theory, we now prove a sequence of theorems establishing a topological property of the basins of attraction of the ω-limit sets of the system.

Lemma 4.23. *Suppose that for a natural number n we have $y_n < -5$ and $\beta z_n \le y_n \le \alpha z_n$, where $1 \le \alpha < \beta \le 2$. If $0 \le h \le |x_n|$ then for every $1 \le k \le \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1$ we have*

$$0 \le \sum_{r=0}^{k-1} \ln(1 - e^{2^r(x_n-h)}) - \sum_{r=0}^{k-1} \ln(1 - e^{2^r x_n}) < \frac{2|y_n|}{\ln(2)} \frac{h}{|x_n|}, \tag{4.48}$$

and for every $k > \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1$

$$\left(\frac{|y_n|}{\beta\ln(2)} - 6\right) \frac{h}{|x_n - h|} < \sum_{r=0}^{k-1} \ln(1 - e^{2^r(x_n-h)}) - \sum_{r=0}^{k-1} \ln(1 - e^{2^r x_n}) < 2 + \frac{2|y_n|}{\ln(2)} \frac{h}{|x_n|}. \tag{4.49}$$

Proof. By the mean value theorem, for every $r = 0, 1, 2, ...$ we have

$$\ln(1 - e^{2^r(x_n-h)}) - \ln(1 - e^{2^r x_n}) = \frac{-e^{c_r}}{1 - e^{c_r}}(2^r(x_n - h) - 2^r x_n),$$

where $2^r(x_n - h) \le c_r \le 2^r x_n$. Hence

$$\sum_{r=0}^{k-1} \ln(1 - e^{2^r(x_n-h)}) - \sum_{r=0}^{k-1} \ln(1 - e^{2^r x_n}) = \sum_{r=0}^{k-1} \ln(1 - e^{2^r(x_n-h)}) - \ln(1 - e^{2^r x_n})$$

$$= \sum_{r=0}^{k-1} -2^r h \frac{-e^{c_r}}{1-e^{c_r}}. \tag{4.50}$$

Since $\frac{d}{dx}\frac{-e^x}{1-e^x} = \frac{-e^x}{(1-e^x)^2} < 0$, $\frac{-e^x}{1-e^x}$ is strictly decreasing. So $2^r(x_n - h) \le c_r \le 2^r x_n$ implies

$$-2^r h \frac{-e^{2^r(x_n-h)}}{1-e^{2^r(x_n-h)}} \le -2^r h \frac{-e^{c_r}}{1-e^{c_r}} \le -2^r h \frac{-e^{2^r x_n}}{1-e^{2^r x_n}}.$$

$$\implies \quad \sum_{r=0}^{k-1} 2^r h \frac{e^{2^r(x_n-h)}}{1-e^{2^r(x_n-h)}} \le \sum_{r=0}^{k-1} -2^r h \frac{-e^{c_r}}{1-e^{c_r}} \le \sum_{r=0}^{k-1} 2^r h \frac{e^{2^r x_n}}{1-e^{2^r x_n}}. \tag{4.51}$$

By Lemma 4.4 and $y_n < -5$ we have $-\frac{1}{2} < -2e^{\frac{y_n}{2}} < x_n$. Then for every $0 \le r \le \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1$, we have $-\frac{1}{2} \le 2^r x_n$. Moreover, $0 \le h \le -x_n$ implies $2x_n \le x_n - h$. Hence, $-1 \le 2^r(x_n - h)$ for every $0 \le r \le \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1$. So for these values of r, since $|2^r(x_n - h)| \le 1$ and $|2^r x_n| \le 1$ we have

$$0 < 1 - e^{2^r(x_n-h)} < |2^r(x_n-h)| - \frac{|2^r(x_n-h)|^2}{2} + \frac{|2^r(x_n-h)|^3}{6},$$

$$1 - e^{2^r x_n} > |2^r x_n| - \frac{|2^r x_n|^2}{2} > 0.$$

This along with (4.51) implies

$$\sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} 2^r h \frac{e^{2^r(x_n-h)}}{|2^r(x_n-h)| - \frac{|2^r(x_n-h)|^2}{2} + \frac{|2^r(x_n-h)|^3}{6}} \le \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} -2^r h \frac{-e^{c_r}}{1-e^{c_r}}$$

$$\le \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} 2^r h \frac{e^{2^r x_n}}{|2^r x_n| - \frac{|2^r x_n|^2}{2}}. \tag{4.52}$$

Again since $|2^r(x_n - h)| \le 1$ for $0 \le r \le \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1$, we have

$$\sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} 2^r h \frac{e^{2^r(x_n-h)}}{|2^r(x_n-h)| - \frac{|2^r(x_n-h)|^2}{2} + \frac{|2^r(x_n-h)|^3}{6}}$$

$$> \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} 2^r h \frac{1 - |2^r(x_n-h)|}{|2^r(x_n-h)| - \frac{|2^r(x_n-h)|^2}{2} + \frac{|2^r(x_n-h)|^3}{6}}$$

$$= \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} \frac{h}{|x_n - h|} \frac{1 - |2^r(x_n - h)|}{1 - \frac{|2^r(x_n-h)|}{2} + \frac{|2^r(x_n-h)|^2}{6}}$$

$$= \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} \frac{h}{|x_n - h|} - \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} \frac{h}{|x_n - h|} \frac{\frac{|2^r(x_n-h)|}{2} + \frac{|2^r(x_n-h)|^2}{6}}{1 - \frac{|2^r(x_n-h)|}{2} + \frac{|2^r(x_n-h)|^2}{6}}$$

$$= \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} \frac{h}{|x_n - h|} - \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} 2^r h \frac{\frac{1}{2} + \frac{|2^r(x_n-h)|}{6}}{1 - \frac{|2^r(x_n-h)|}{2} + \frac{|2^r(x_n-h)|^2}{6}}. \tag{4.53}$$

Since $|2^r(x_n - h)| \leq 1$ for $0 \leq r \leq \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1$ we have

$$\frac{1}{2} + \frac{|2^r(x_n - h)|}{6} < \frac{2}{3}, \qquad 1 - \frac{|2^r(x_n - h)|}{2} + \frac{|2^r(x_n - h)|^2}{6} > \frac{1}{2}.$$

$$\implies \qquad \frac{\frac{1}{2} + \frac{|2^r(x_n-h)|}{6}}{1 - \frac{|2^r(x_n-h)|}{2} + \frac{|2^r(x_n-h)|^2}{6}} < \frac{4}{3}.$$

Hence

$$\sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} 2^r h \frac{\frac{1}{2} + \frac{|2^r(x_n-h)|}{6}}{1 - \frac{|2^r(x_n-h)|}{2} + \frac{|2^r(x_n-h)|^2}{6}} < \frac{4h}{3} \sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} 2^r < \frac{4h}{3} 2^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor}.$$

This along with (4.53) and $0 \leq h \leq |x_n|$ implies

$$\sum_{r=0}^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor - 1} 2^r h \frac{e^{2^r(x_n-h)}}{|2^r(x_n - h)| - \frac{|2^r(x_n-h)|^2}{2} + \frac{|2^r(x_n-h)|^3}{6}} > \left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor \frac{h}{|x_n - h|} - \frac{4h}{3} 2^{\left\lfloor \frac{-\ln(-x_n)}{\ln(2)} \right\rfloor}.$$

$$> \left(\frac{-\ln(-x_n)}{\ln(2)} - 1 \right) \frac{h}{|x_n - h|} - \frac{4h}{3} 2^{\frac{-\ln(-x_n)}{\ln(2)}} = \left(\frac{-\ln(-x_n)}{\ln(2)} - 1 \right) \frac{h}{|x_n - h|} - \frac{4h}{3} \frac{1}{|x_n|}.$$

By Lemma 4.4 we have $|x_n| < 4e^{\frac{y_n}{\beta}}$, thus

$$\left(\frac{-\ln(-x_n)}{\ln(2)} - 1 \right) \frac{h}{|x_n - h|} - \frac{4h}{3} \frac{1}{|x_n|} > \left(\frac{-\ln(4e^{\frac{y_n}{\beta}})}{\ln(2)} - 1 \right) \frac{h}{|x_n - h|} - \frac{4h}{3} \frac{1}{|x_n|}.$$

$$= \left(-\frac{y_n}{\beta \ln(2)} - 3 \right) \frac{h}{|x_n - h|} - \frac{4h}{3} \frac{1}{|x_n|} \geq \left(-\frac{y_n}{\beta \ln(2)} - 3 \right) \frac{h}{|x_n - h|} - \frac{8h}{3} \frac{1}{|x_n - h|}.$$

$$\geq \left(-\frac{y_n}{\beta\ln(2)} - 6\right)\frac{h}{|x_n - h|}. \tag{4.54}$$

We also have

$$\sum_{r=0}^{\left\lfloor\frac{-\ln(-x_n)}{\ln(2)}\right\rfloor-2} 2^r h \frac{e^{2^r x_n}}{|2^r x_n| - \frac{|2^r x_n|^2}{2}} = \sum_{r=0}^{\left\lfloor\frac{-\ln(-x_n)}{\ln(2)}\right\rfloor-2} \frac{h}{|x_n|}\frac{e^{2^r x_n}}{1 - 2^{r-1}|x_n|}$$

$$\leq \sum_{r=0}^{\left\lfloor\frac{-\ln(-x_n)}{\ln(2)}\right\rfloor-2} \frac{h}{|x_n|}\frac{1}{1 - 2^{r-1}|x_n|} \leq \sum_{r=0}^{\left\lfloor\frac{-\ln(-x_n)}{\ln(2)}\right\rfloor-2} \frac{h}{|x_n|}\frac{1}{1-\frac{1}{2}} = \frac{2h}{|x_n|}\left(\left\lfloor\frac{-\ln(-x_n)}{\ln(2)}\right\rfloor - 1\right)$$

$$\leq \frac{2h}{|x_n|}\frac{-\ln(-x_n)}{\ln(2)} < \frac{2h}{|x_n|}\frac{-\ln(e^{y_n+M})}{\ln(2)} = \frac{2h}{|x_n|}\frac{-(y_n+M)}{\ln(2)} < \frac{2h}{|x_n|}\frac{|y_n|}{\ln(2)}. \tag{4.55}$$

Now if $1 \leq k \leq \left\lfloor\frac{-\ln(-x_n)}{\ln(2)}\right\rfloor - 1$ then it's clear that by (4.50), (4.52) and (4.55) we have

$$0 < \sum_{r=0}^{k-1}\ln(1 - e^{2^r(x_n-h)}) - \sum_{r=0}^{k-1}\ln(1 - e^{2^r x_n}) \leq \sum_{r=0}^{k-1} 2^r h \frac{e^{2^r x_n}}{|2^r x_n| - \frac{|2^r x_n|^2}{2}}$$

$$\leq \sum_{r=0}^{\left\lfloor\frac{-\ln(-x_n)}{\ln(2)}\right\rfloor-2} 2^r h \frac{e^{2^r x_n}}{|2^r x_n| - \frac{|2^r x_n|^2}{2}} < \frac{2|y_n|}{\ln(2)}\frac{h}{|x_n|}.$$

If $k \geq \left\lfloor\frac{-\ln(-x_n)}{\ln(2)}\right\rfloor$, then by (4.50), (4.52), (4.54) and (4.55) and since $2^{\left\lfloor\frac{-\ln(-x_n)}{\ln(2)}\right\rfloor-1}x_n \leq \frac{1}{2}$, we have

$$\left(\frac{|y_n|}{\beta\ln(2)} - 6\right)\frac{h}{|x_n - h|} < \sum_{r=0}^{\left\lfloor\frac{-\ln(-x_n)}{\ln(2)}\right\rfloor-1}\ln(1 - e^{2^r(x_n-h)}) - \sum_{r=0}^{\left\lfloor\frac{-\ln(-x_n)}{\ln(2)}\right\rfloor-1}\ln(1 - e^{2^r x_n})$$

$$\leq \sum_{r=0}^{k-1}\ln(1 - e^{2^r(x_n-h)}) - \sum_{r=0}^{k-1}\ln(1 - e^{2^r x_n}),$$

and

$$\sum_{r=0}^{k-1}\ln(1 - e^{2^r(x_n-h)}) - \sum_{r=0}^{k-1}\ln(1 - e^{2^r x_n})$$

$$= \sum_{r=\lfloor \frac{-\ln(-x_n)}{\ln(2)} \rfloor -1}^{k-1} \ln(1-e^{2^r(x_n-h)}) - \ln(1-e^{2^r x_n}) + \sum_{r=0}^{\lfloor \frac{-\ln(-x_n)}{\ln(2)} \rfloor -2} \ln(1-e^{2^r(x_n-h)}) - \ln(1-e^{2^r x_n})$$

$$\leq \sum_{r=\lfloor \frac{-\ln(-x_n)}{\ln(2)} \rfloor -1}^{k-1} -\ln(1-e^{2^r x_n}) + \sum_{r=0}^{\lfloor \frac{-\ln(-x_n)}{\ln(2)} \rfloor -2} \ln(1-e^{2^r(x_n-h)}) - \ln(1-e^{2^r x_n})$$

$$= \sum_{r=0}^{k-\lfloor \frac{-\ln(-x_n)}{\ln(2)} \rfloor +1} -\ln(1-e^{2^r 2^{\lfloor \frac{-\ln(-x_n)}{\ln(2)} \rfloor -1} x_n}) + \sum_{r=0}^{\lfloor \frac{-\ln(-x_n)}{\ln(2)} \rfloor -2} \ln(1-e^{2^r(x_n-h)}) - \ln(1-e^{2^r x_n})$$

$$\leq \sum_{r=0}^{k-\lfloor \frac{-\ln(-x_n)}{\ln(2)} \rfloor +1} -\ln(1-e^{-2^r \frac{1}{2}}) + \sum_{r=0}^{\lfloor \frac{-\ln(-x_n)}{\ln(2)} \rfloor -2} \ln(1-e^{2^r(x_n-h)}) - \ln(1-e^{2^r x_n})$$

$$\leq \sum_{r=0}^{\infty} -\ln(1-e^{-2^{r-1}}) + \sum_{r=0}^{\lfloor \frac{-\ln(-x_n)}{\ln(2)} \rfloor -2} \ln(1-e^{2^r(x_n-h)}) - \ln(1-e^{2^r x_n})$$

$$< 2 + \frac{2|y_n|}{\ln(2)} \frac{h}{|x_n|}.$$

□

Lemma 4.24. *Suppose that $-1 < a$ and $[a,b] \times [c,d] \subset \Delta^{**}$ is a set in (4.12) and the transformed version of $[a,b] \times [c,d]$ in (4.10), which is denoted by S, is completely located above the line $y = x$. Then*

(a)

$$S = \bigcup_{y \in [c,d]} \{y\} \times [\ln(1-e^b-e^y), \ln(1-e^a-e^y)].$$

In other words, S is a closed simply connected set with the following boundary:

$$\partial S = \{c\} \times [\ln(1-e^b-e^c), \ln(1-e^a-e^c)] \cup \{d\} \times [\ln(1-e^b-e^d), \ln(1-e^a-e^d)] \cup D_1 \cup D_2,$$

where D_1 and D_2 are two curves defined as follows:

$$D_1 = \{(y, \ln(1-e^a-e^y)) |\, y \in [c,d]\},$$

$$D_2 = \{(y, \ln(1-e^b-e^y))|\ y \in [c,d]\}.$$

(b)

$$S \subset [c,d] \times [\ln(|b|) - 2\ln(2), \ln(|a|)]\,.$$

(c) If $C \subset [c,d] \times \mathbb{R}$ is a continuous curve with one endpoint on or above the line $y = \ln(|a|)$ and the other endpoint on or below the line $y = \ln(|b|) - 2\ln(2)$, then there exists continuous curve $C' \subset C$ such that the transformed version of C' in (4.12), which is denoted by C^*, is a continuous curve with one endpoint on the line $x = a$ and the other endpoint on the line $x = b$ and we have $C^* \subset [a,b] \times [c,d]$.

Proof. (a) Part (a) is a direct result of the fact that we have

$$S = \{(y, \ln(1-e^x-e^y))|\ (x,y) \in [a,b] \times [c,d]\}\,.$$

(b) It's sufficient to show that for every $(x,y) \in [a,b] \times [c,d]$

$$\ln(|b|) - 2\ln(2) \leq \ln(1-e^x-e^y) \leq \ln(|a|).$$

We have

$$1 - e^x - e^y < 1 - e^x \leq -x \leq |a|,$$

hence $\ln(1-e^x-e^y) \leq \ln(|a|)$.

Since S is completely located above the line $y = x$, we have $\ln(1-e^x-e^y) > y$, hence

$$1 - e^x - e^y > e^y$$

$$\Longrightarrow \quad 1 - e^x > 2e^y$$

$$\Longrightarrow \quad 1 - e^x - e^y > 1 - e^x - \frac{1}{2} + \frac{1}{2}e^x = \frac{1}{2} - \frac{1}{2}e^x$$

$$\Longrightarrow \quad \ln(1-e^x-e^y) > \ln(1-e^x) - \ln(2) > \ln\left(-x - \frac{x^2}{2}\right) - \ln(2)$$

$$= \ln(|x|) + \ln\left(1 - \frac{x}{2}\right) - \ln(2) > \ln(|x|) - 2\ln(2) > \ln(|b|) - 2\ln(2).$$

(c) Each of the curves D_1 and D_2 divides $[c, d] \times \mathbb{R}$ into two parts: (1) the set of points in $[c, d] \times \mathbb{R}$ that are above D_i. (2) the set of points in $[c, d] \times \mathbb{R}$ that are below D_i. Since one endpoint of C is above the line $y = \ln(|a|)$, that endpoint is a member of the first set and Since one endpoint of C is below the line $y = \ln(|b|) - 2\ln(2)$, that endpoint is a member of the second set. So continuity of C implies that this curve intersects each of the curves D_1 and D_2. Therefore, there exists at least one continuous curve $C' \subset C$ such that C' has one endpoint on D_1 and the other endpoint on D_2 and C' is completely located in the space between D_1 and D_2 (i.e. $C' \subset S$). So $C^* \subset [a, b] \times [c, d]$ and one of the endpoints of C^* is located on the transformed version of D_1 in (4.12), which is a subset of the line $x = a$ and the other endpoint of C^* is located on the transformed version of D_2 in (4.12), which is a subset of the line $x = b$. □

Theorem 4.25. *Suppose that $0 < \alpha_2 < \alpha_1 < 1$ and $C \subset \Delta^{**}$ is a continuous curve with endpoints (a_1, b_1) and (a_2, b_2), where $\frac{a_2}{a_1} = 1 + \alpha_2$ and for a $y^* \in \Delta^{**}$ we have $y^* < -5$ and*

$$C \subset [a_2, a_1] \times [y^*(1 + \alpha_1), y^*] =: S,$$

and there exist $1 \le \alpha < \beta \le 2$ such that every $(x_n, y_n) \in [a_2, a_1] \times [y^*(1 + \alpha_1), y^*]$ satisfies the conditions of Lemma 4.23, and we have

$$|y^*| \left(\frac{\alpha_2}{\beta \ln(2)(1 + \alpha_2)} - \alpha_1 \right) - 2 - \frac{6\alpha_2}{1 + \alpha_2} > \ln(20), \tag{4.56}$$

$$-2^m a_1 > -\frac{1 + \alpha_1}{\alpha_2 - \alpha_1}, \tag{4.57}$$

then there exists a natural number m and continuous curve $C' \subset C$ such that the transformed version of the m-th iteration of C' in (4.12), which is denoted by C^*, is a continuous curve with endpoints (a_1^*, b_1^*) and (a_2^*, b_2^*), where $\frac{a_2^*}{a_1^*} = 5$ and for a $y^{**} \in \Delta^{**}$ we have

$$C^* \subset [a_2^*, a_1^*] \times [y^{**}(1 + \alpha_1), y^{**}].$$

Proof. Let m be the natural number such that the m-th iteration of S is completely above the line $y = x$ and its $m - 1$-th iteration is completely below the line $y = x$. Let's denote the m-th iteration of S by S_m. According to Theorem 4.22, for every $(x_0, y_0) \in S$ we have

$$0 \le x_m - 2^m x_0 < 1, \tag{4.58}$$

$$\left| y_0 + m\ln(2) + \sum_{r=0}^{m-1} \ln(1 - e^{2^r x_0}) - y_m \right| < 1. \tag{4.59}$$

From (4.58) we deduce that we have

$$S_m \subset [2^m a_2, 2^m a_1 + 1] \times \mathbb{R} \tag{4.60}$$

and if we denote the m-th iterations of (a_1, b_1) and (a_2, b_2) by $\left(a_1^{(m)}, b_1^{(m)}\right)$ and $\left(a_2^{(m)}, b_2^{(m)}\right)$ respectively, (4.59) implies

$$b_2^{(m)} > b_2 + m\ln(2) - 1 + \sum_{r=0}^{m-1} \ln(1 - e^{2^r a_2}), \tag{4.61}$$

$$b_1^{(m)} < b_1 + m\ln(2) + 1 + \sum_{r=0}^{m-1} \ln(1 - e^{2^r a_1}), \tag{4.62}$$

hence

$$b_2^{(m)} - b_1^{(m)} > b_2 - b_1 - 2 + \sum_{r=0}^{m-1} \ln(1 - e^{2^r a_2}) - \sum_{r=0}^{m-1} \ln(1 - e^{2^r a_1}). \tag{4.63}$$

Now we have

$$|b_2 - b_1| \leq |y^* \alpha_1| \tag{4.64}$$

and by Lemma 4.23 we have

$$\sum_{r=0}^{m-1} \ln(1 - e^{2^r a_2}) - \sum_{r=0}^{m-1} \ln(1 - e^{2^r a_1}) > \left(\frac{|y^*|}{\beta \ln(2)} - 6\right) \frac{a_1 - a_2}{|a_2|}. \tag{4.65}$$

The above inequalities imply

$$b_2^{(m)} - b_1^{(m)} > -|y^*|\alpha_1 - 2 + \left(\frac{|y^*|}{\beta \ln(2)} - 6\right) \frac{a_1 - a_2}{|a_2|}. \tag{4.66}$$

$$= -|y^*|\alpha_1 - 2 + \left(\frac{|y^*|}{\beta \ln(2)} - 6\right) \frac{\alpha_2}{1 + \alpha_2}. \tag{4.67}$$

$$= |y^*| \left(\frac{\alpha_2}{\beta \ln(2)(1 + \alpha_2)} - \alpha_1\right) - 2 - \frac{6\alpha_2}{1 + \alpha_2} > \ln(20). \tag{4.68}$$

Now let's define $y^{**} = 2^m a_1 + 1$, $a_2^* = -e^{b_2^{(m)}}$ and $a_1^* = -\frac{1}{5} e^{b_2^{(m)}}$. We may use Lemma 4.24 for the following set in (4.12):

$$[a,b]\times[c,d]=\left[-e^{b_2^{(m)}},-\frac{1}{5}e^{b_2^{(m)}}\right]\times[y^{**}(1+\alpha_1),y^{**}]. \tag{4.69}$$

By Lemma 4.24, if S^* is the transformed version of $[a,b]\times[c,d]$ in (4.10), then

$$S^*\subset[c,d]\times[\ln(|b|)-2\ln(2),\ln(|a|)]=[c,d]\times\left[\ln\left(\left|-\frac{1}{5}e^{b_2^{(m)}}\right|\right)-2\ln(2),\ln\left(\left|-e^{b_2^{(m)}}\right|\right)\right]$$

$$\subset[c,d]\times[\ln(|b|)-2\ln(2),\ln(|a|)]=[c,d]\times\left[b_2^{(m)}-\ln(20),b_2^{(m)}\right].$$

If we denote the m-th iteration of C by C_m, then C_m is a continuous curve with endpoints $\left(a_1^{(m)},b_1^{(m)}\right)$ and $\left(a_2^{(m)},b_2^{(m)}\right)$ and we have $C_m\subset S_m\subset[2^m a_2,2^m a_1+1]\times\mathbb{R}$. Now by (4.57) we have

$$2^m a_2=2^m a_1(1+\alpha_2)=2^m a_1(1+\alpha_1)+2^m a_1(\alpha_2-\alpha_1)$$

$$>2^m a_1(1+\alpha_1)+1+\alpha_1=(2^m a_1+1)(1+\alpha_1)=y^{**}(1+\alpha_1)=c.$$

It means that $[2^m a_2,2^m a_1+1]\subset[y^{**}(1+\alpha_1),y^{**}]$ and $C_m\subset[c,d]\times\mathbb{R}$. C_m has one endpoint on the line $y=b_2^{(m)}$ and one on $y=b_1^{(m)}$. Now since $b_1^{(m)}<b_2^{(m)}-\ln(20)$, C_m has one endpoint on or above the line $y=\ln(|a|)$ and the other endpoint on or below the line $y=\ln(|b|)-2\ln(2)$, so by Lemma 4.24, there exists continuous curve $C'_m\subset C_m$ such that the transformed version of C'_m in (4.12), which we denote by C^*, is a continuous curve with one endpoint on the line $x=a_2^*$ and the other endpoint on the line $x=a_1^*$ and we have $C^*\subset[a_2^*,a_1^*]\times[y^{**}(1+\alpha_1),y^{**}]$. □

Theorem 4.26. *Suppose that $0<\alpha_2<\alpha_1<1$ and $1<\beta\le 2$ and we have*

$$\frac{\alpha_2}{\beta\ln(2)(1+\alpha_2)}-\alpha_1>0.$$

Suppose in addition that H and H' are two admissible sets such that $H'\subset H$ and there exist $K,L<0$ and $w>0$ such that if $S=[5a,a]\times[y^*(1+\alpha_1),y^*]$ is a set with $K<5a$ and $y^*<L$, then

(i) There exists $a'\in[5a,a]$ and a natural number m such that

$$\left[a'(1+\alpha_2)(1+w),\frac{a'}{(1+w)}\right]\times[y^*(1+\alpha_1),y^*]\subset S-(H'\times\mathbb{R})$$

and if we denote the m-th iteration of $S^*:=[a'(1+\alpha_2),a']\times[y^*(1+\alpha_1),y^*]$ by S_m^* then for every $(x,y)\in S_m^*$ we have $\beta y<x<\alpha y$.

(ii) We have

$$|y^*|\left(\frac{\alpha_2}{\beta\ln(2)(1+\alpha_2)}-\alpha_1\right)-2-\frac{6\alpha_2}{1+\alpha_2}>\ln(20), \tag{4.70}$$

$$-2^m a > -\frac{1+\alpha_1}{\alpha_2-\alpha_1}. \tag{4.71}$$

Then there exists simply connected set P such that $\Lambda(H)\cap P$ is not path connected.

Proof. Suppose that $S=[5a,a]\times[y^*(1+\alpha_1),y^*]$ with $K<5a$ and $y^*<L$. Let C_0 be a continuous curve with endpoints $(5a,b_1)$ and (a,b_2) where $b_1,b_2\in[y^*(1+\alpha_1),y^*]$ and we have $C_0\subset S$. Let a', m and S^* be those mentioned in (i). Continuity of C_0 implies that there exists a continuous curve $C\subset C_0$ such that $C\subset S^*$ and one of the endpoints of C is on the line $x=a'(1+\alpha_2)$ and the other one is on the line $x=a'$.

(i) and (ii) imply that C satisfies the conditions of Theorem 4.25. Therefore, according to that theorem, there exists continuous curve C^* with endpoints (a_1^*,b_1^*) and (a_2^*,b_2^*), where $\frac{a_2^*}{a_1^*}=5$ and for a $y^{**}\in\Delta^{**}$ we have $C^*\subset[5a_1^*,a_1^*]\times[y^{**}(1+\alpha_1),y^{**}]=:S^*$ and C^* is the transformed version of C' in (4.12), where C' is a continuous curve and $C'\subset C$. We define $C_1=C^*$, $C_1'=C'$ and $S_1=S^*$. We can easily see that C_1 is a once transformed version of an iteration of C_1' and C_1' is continuous curve and a subset of C_0.

We can repeat the above argument for S_1 and C_1 instead of S and C_0 and as a result, we get another C' now in (4.12) and another S^* and C^* now in (4.11). This C' is a continuous curve which is a subset of C_1. Since C_1 itself is a once transformed version of an iteration of C_1', there exists a continuous curve $C_2'\subset C_1'$ such that C^* is a once transformed version of an iteration of C_2'. We also define $C_2=C^*$ and $S_2=S^*$.

We continue the process according to (4.13). Now suppose that for a natural number k, we have repeated that argument k times and we already have defined C_{k+1}, C_{k+1}' and S_{k+1}. The continuous curve C_{k+1} is a $k+1$ times transformed version of an iteration of the continuous curve C_{k+1}' and we have

$$C_{k+1}'\subset C_k'\subset\ldots\subset C_2'\subset C_1'\subset C_0. \tag{4.72}$$

If we repeat the argument for S_{k+1} and C_{k+1} instead of S and C_0 we get another C' in the system in which C_{k+1} is defined and another S^* and C^* in the next system according to (4.13). This C' is a continuous curve which is a subset of C_{k+1}. Since C_{k+1} itself is a $k+1$ times transformed version of an iteration of C_{k+1}', there exists a continuous curve $C_{k+2}'\subset C_{k+1}'$ such that C^* is a $k+1$ times transformed version of an iteration of C_{k+2}'. We also define $C_{k+2}=C^*$ and $S_{k+2}=S^*$.

Now we have defined a sequence $\{C'_k\}$ of continuous curves with the following property:

$$... \subset C'_3 \subset C'_2 \subset C'_1 \subset C_0. \tag{4.73}$$

Hence, since every C'_k is a closed set, $\bigcap_{k=1}^{\infty} C'_k$ is nonempty and has at least one point $\mathbf{c}$. According to the way we have defined $\{C'_k\}$ and since we have used (ii) when defining every C'_k, for every natural number k and every $(x, y) \in C_k$ and every $u \in H'$ we have $x < u(1 + w)$ or $u < x(1 + w)$. By using the approximation theory we established earlier in this chapter, we can prove that for every $t_1 < t_2 < 0$ and every $\varepsilon > 0$, there exists $M > 0$ such that for every $k > M$, if $(x', y') \in [t_1, t_2] \times \mathbb{R}$ is an iteration of a point in C_k before passing the line $x = y$, then we have $x' < u(1 + w) + \varepsilon$ or $u < x'(1 + w) - \varepsilon$ for every $u \in H'$. If for $t_1 < t_2 < 0$, ε is chosen small enough, $x' < u(1 + w) + \varepsilon$ or $u < x'(1 + w) - \varepsilon$ for every mentioned (x', y') guarantees the existence of $v > 0$ such that for every mentioned (x', y') and for every $u \in H'$ we have $|x' - u| > v$. It means that for every $t_1 < t_2 < 0$, there exist $v > 0$ and $M > 0$ such that for every $k > M$, if any iteration of any point in C_k before passing the line $x = y$ is a point on $[t_1, t_2] \times \mathbb{R}$, then the first component of that iteration has a distance of at least v with any point on H'. So this fact is true for the point of C_k which is k times transformed version of an iteration of $\mathbf{c}$. It means that the trajectory of $\mathbf{c}$ finally gets distance from H' when the first component of its transformed version is between t_1 and t_2. We can choose t_1 and t_2 in a way that (t_1, t_2) has nonempty intersection with H'. In that situation, the omega limit set of $\mathbf{c}$ cannot have any member of $H' \cap (t_1, t_2)$, which is possible only if $\omega(\mathbf{c}) \cap H' = \emptyset$. From $H' \subset H$ we deduce that $\omega(\mathbf{c}) \neq H$. Hence $\mathbf{c} \notin \Lambda(H)$. It means that C_0 is not a subset of $\Lambda(H)$.

Now let's choose $\varepsilon' > 0$ in a way that if we define $P = [5a - \varepsilon', a + \varepsilon'] \times [y^*(1 + \alpha_1), y^*]$, then $P \subset \Delta^{**}$. We know that

$$P'_1 := [5a - \varepsilon', 5a] \times [y^*(1 + \alpha_1), y^*] \cap \Lambda(H) \neq \emptyset \tag{4.74}$$

$$P'_2 := [a, a + \varepsilon'] \times [y^*(1 + \alpha_1), y^*] \cap \Lambda(H) \neq \emptyset \tag{4.75}$$

So if $P \cap \Lambda(H)$ is path connected, for any $\mathbf{p}_1 \in P'_1$ and $\mathbf{p}_2 \in P'_2$ there must be a continuous curve $C'' \subset P \cap \Lambda(H)$ with endpoints $\mathbf{p}_1$ and $\mathbf{p}_2$. This implies the existence of a continuous curve $C_0 \subset C'' \cap [5a, a] \times [y^*(1 + \alpha_1), y^*] \subset \Lambda(H)$ and with one endpoint on the line $x = 5a$ and the other one on the line $x = a$. But we have proven that such a C_0 cannot be a subset of $\Lambda(H)$, which is a contradiction. Therefore, $P \cap \Lambda(H)$ is not path connected. □

Definition 4.27. Let H be an admissible set. We define the corresponding admissible sets of H in (4.10), (4.12) and (4.11), denoted by H_1, H_2, H_3 respectively, as follows:

$$H_1 := \{\ln(x) \mid (x, y, z) \in H, \quad y = 0, \quad x \neq 0, \quad z \neq 0\}, \tag{4.76}$$

$$H_2 := \{\ln(z) \mid (x, y, z) \in H, \;\; x = 0, \;\; z \neq 0, \;\; y \neq 0\}, \tag{4.77}$$

$$H_3 := \{\ln(y) \mid (x, y, z) \in H, \;\; z = 0, \;\; y \neq 0, \;\; x \neq 0\}. \tag{4.78}$$

Now as an application of the approximation formula we developed, we can now prove a topological property of $\Lambda(H)$ for every admissible set H.

Theorem 4.28. *For any nonempty admissible set H there exists a two-dimensional simply connected set P (transformed as a filled rectangle in one of the log coordinates) such that $\Lambda(H) \cap P$ is not path connected.*

Proof. Let H_1, H_2 and H_3 be the corresponding admissible sets of H in (4.10), (4.12) and (4.11) respectively. Each of these corresponding admissible sets is nonempty. We choose points v_1, v_2 and v_3 from H_1, H_2 and H_3 respectively. Let's define

$$H_i' = \{2^n v_i \mid n \in \mathbb{Z}\}. \qquad (i = 1, 2, 3) \tag{4.79}$$

Now there exists a unique admissible set H' whose corresponding admissible sets in (4.10), (4.12) and (4.11) are H_1', H_2' and H_3' respectively. Let's choose $\alpha_1 = 0.01$, $\alpha_2 = 0.0095$, $w = 0.01$, $\alpha = 1.01$ and $\beta = 1.2$. Then we have

$$\alpha_1 > \alpha_2 > 0,$$

$$\frac{\alpha_2}{\beta \ln(2)(1 + \alpha_2)} - \alpha_1 = 0.00131... > 0.$$

Now let's focus on (4.10). For any $y^* < 0$ we define

$$S^{(y^*)} = [5a, a] \times [y^*(1 + \alpha_1), y^*],$$

$$k^{(y^*)} = \{5a\} \times [y^*(1 + \alpha_1), y^*],$$

$$l^{(y^*)} = \{a\} \times [y^*(1 + \alpha_1), y^*],$$

and for every natural number n we denote the n-th iteration of $S^{(y^*)}$, $k^{(y^*)}$ and $l^{(y^*)}$ by $S_n^{(y^*)}$, $k_n^{(y^*)}$ and $l_n^{(y^*)}$ respectively.

By using the approximation theory, $L_1 < 0$ can be found such that for every $y^* < L_1$, if m is the largest natural number such that $l_m^{(y^*)}$ is completely below the line $y = x$, then $k_m^{(y^*)}$ is completely above the line $2y = x$.

Again by using the approximation theory, $L_2 \leq L_1$ can be found such that for every $y^* < L_1$

(a) There exists $g < 0$ such that

$$S_m^{(y^*)} \subset [6g, \frac{1}{3}g] \times [1.001g, g],$$

(b) For every $(x_0, y_0) \in S^{(y^*)}$ we have

$$|x_m - 2^m x_0| < -\frac{x_m}{1000}.$$

Now suppose that $(x_0, y_0) \in S^{(y^*)}$ and $\frac{1.19}{2^m} g \leq x_0 \leq \frac{1.02}{2^m} g$. We have

$$2^m x_0 + \frac{x_m}{1000} < x_m < 2^m x_0 - \frac{x_m}{1000}$$

$$\implies \quad 2^m \left(\frac{1.19}{2^m} g\right) + \frac{6g}{1000} < x_m < 2^m \left(\frac{1.02}{2^m} g\right) - \frac{g}{6000}$$

$$\implies \quad 1.196g < x_m < 1.019g$$

Now since $y_m \in [1.001g, g]$ we have $1.196y_m < 1.196g$ and $1.019g < 1.017y_m$. Thus

$$1.196y_m < x_m < 1.017y_m$$

$$\implies \quad \beta y_m < x_m < \alpha y_m$$

The set $[\frac{1.19}{2^m} g, \frac{1.02}{2^m} g] \cap H_1'$ has at most one member. So since we have

$$\left[\frac{1.15}{2^m} g(1 + \alpha_2)(1 + w), \frac{\frac{1.15}{2^m} g}{(1 + w)}\right] \subset \left[\frac{1.19}{2^m} g, \frac{1.02}{2^m} g\right],$$

$$\left[\frac{1.05}{2^m} g(1 + \alpha_2)(1 + w), \frac{\frac{1.05}{2^m} g}{(1 + w)}\right] \subset \left[\frac{1.19}{2^m} g, \frac{1.02}{2^m} g\right],$$

$$\left[\frac{1.05}{2^m} g(1 + \alpha_2)(1 + w), \frac{\frac{1.05}{2^m} g}{(1 + w)}\right] \cap \left[\frac{1.15}{2^m} g(1 + \alpha_2)(1 + w), \frac{\frac{1.15}{2^m} g}{(1 + w)}\right] = \emptyset,$$

for at least one of the choices $\alpha' = \frac{1.05}{2^m}g$ or $\alpha' = \frac{1.15}{2^m}g$ we have

$$\left[a'(1+\alpha_2)(1+w), \frac{a'}{(1+w)}\right] \times [y^*(1+\alpha_1), y^*] \subset S^{(y^*)} - (H' \times \mathbb{R}).$$

Therefore, any $L < L_2$ and $K = -1$ satisfy the condition (i) of Theorem 4.26.

It can be easily shown that since $\frac{\alpha_2}{\beta \ln(2)(1+\alpha_2)} - \alpha_1 > 0$, there exists $L_3 < 0$ such that every $y^* < L_3$ satisfies (4.56).

If $y^* < L_1$ and $(x_0, y_0) = (a, y^*)$, then as mentioned before $|x_m - 2^m a| < -\frac{x_m}{1000}$. Hence $-2^m a > 0.999|x_m|$. Since it's obvious that $x_m \to -\infty$ as $y^* \to -\infty$, there exists $L_4 < 0$ such that for every $y^* < L_4$, we have $-2^m a > 0.999|x_m| > -\frac{1+\alpha_1}{\alpha_2-\alpha_1}$. Therefore, any $L < L_4$ satisfies the condition (ii) of Theorem 4.26.

So we conclude that $L = \min\{L_1, L_2, L_3, L_4,\}$ and $K = -1$ satisfy the conditions of Theorem 4.26, which implies the existence of P with the mentioned property.

□

In conclusion, the main achievement of this chapter lies in the development of an approximation theory for the Stein–Ulam spiral map. While the final result can be viewed as a portfolio of potential applications, including a novel improvement toward proving connectivity properties similar to the Mandelbrot set's MLC conjecture, the deeper significance is in the approximation framework itself. By allowing precise control over the errors as orbits approach the boundary, this theory provides a powerful new insight through which the intricate topology of basins of attraction can be studied. This highlights the promising potential of the approximation approach, not only for the Stein–Ulam spiral map but also as a general tool for addressing long-standing questions in discrete dynamical systems and beyond.

Chapter 5

Conclusion

This thesis has investigated deep mathematical structures within the field of discrete dynamical systems, focusing particularly on their applications to theoretical ecology and population genetics. The central framework is built upon Kolmogorov maps—a class of maps capturing the multiplicative growth and interactions among species in discrete time steps. Through rigorous analytical methods, this work has made meaningful contributions to the understanding of invariant sets, local and global stability, and the geometry of population models.

5.1 Summary of Key Contributions

The first major achievement of this thesis is the development of a new framework to address a conjecture concerning the global stability of the planar Ricker model. Prior to this work, it was known that if both parameters r and s in the planar Ricker system were constrained to values less than or equal to 2, then the local stability of the interior fixed point implies its global stability. However, extending this result to broader parameter ranges had remained an open challenge. By introducing carefully constructed forward-invariant sets that avoid the problematic region ΔV^+ (where the Lyapunov function fails to decrease), this thesis extends the result to additional regions of the parameter space—specifically, when only one of r or s exceeds 2.

These forward-invariant sets, denoted B_1 and $B_2 \cup F(B_2)$, serve as globally attracting regions. Proving that orbits are eventually trapped in these sets enabled the establishment of global convergence toward the interior fixed point. The arguments involve a delicate interplay of dynamical system theory, inequalities, and geometric intuition.

Secondly, the thesis establishes the convexity of non-compact carrying simplices in logarithmic coordinates for both the Leslie-Gower and Ricker models. Carrying simplices, as invariant attracting sets, characterize the long-term coexistence states of species in competitive ecological models. The result on convexity is especially meaningful in the context of dynamical systems because it provides geometric clarity and simplifies the global analysis of trajectories. It also has potential implications for the stability and monotonicity properties of the system by simplifying the analysis of the system.

This convexity was proven by employing a logarithmic transformation and demonstrating that under certain conditions, the transformed carrying simplex does not develop concavities that would complicate the dynamics. This work refines and extends earlier studies in the literature that had primarily considered compact settings.

A third original contribution is made through the analysis of the Stein-Ulam spiral map, a three-dimensional system arising in population genetics and connected to binary reaction systems. While previous research had noted its unusual spiral structure, its precise behavior near the boundary of the simplex had remained analytically inaccessible. This thesis develops an explicit approximation formula that describes the evolution of orbits as they approach the boundary. The derivation includes error estimation, offering a novel computational tool for further studies.

The approximation strategy not only enables a more tractable analysis of this map but also paves the way for addressing subtle topological questions, such as whether the basin of attraction of possible ω-limit sets are connected in some simply connected regions. The broader prospect is the demonstration that even highly complicated systems can be investigated by a combination of an approximation theory and a rigorous analysis.

5.2 Broader Context and Implications

Beyond the direct mathematical results, this thesis contributes to a broader understanding of how ecological and genetic systems behave over long periods under deterministic dynamics. The tools developed here, particularly those concerning forward-invariant sets and Lyapunov-based stability analysis, may be applicable in diverse domains such as mathematical epidemiology, economic competition models, or the analysis of neural population dynamics.

The use of Kolmogorov maps serves as a unifying theme across the different chapters. These maps encapsulate the essential feature of many biological systems: growth rates that depend nonlinearly on current population levels and their interactions. The thesis shows that through thoughtful transformations, these complex systems can be more easily analyzed.

5.3 Future Work

Despite the progress made, there are natural limitations that suggest directions for future research:

- **Higher-dimensional generalizations:** The methods developed for the planar Ricker model could potentially be extended to higher-dimensional systems, where more than two species interact. However, such extensions may require more geometric tools and more sophisticated invariant set constructions.

- **Topology of the sets in question:** In the Stein-Ulam spiral map, questions remain about the detailed topological properties of the basins of attraction. Are they globally connected? What is the fractal dimension of each? These are subtle but foundational questions in dynamical systems and topology.

- **Bridging numerical and theoretical methods:** While this thesis leans heavily on rigorous theoretical analysis, some constructions suggest strategies for numerical implementation. In the third theory, it is also demonstrated that the relationship is bidirectional: not only can theoretical analysis support numerical methods, but numerical approaches can also aid theoretical developments.

5.4 Concluding Remarks

In summary, this thesis advances our understanding of discrete-time dynamical systems applied to ecology and genetics by exploring diverse problem, developing new methods, and illuminating the geometric and topological nature of the concerning sets. It applies rigorous theory to the modeling needs of biological systems, making it not only a mathematical investigation but also a contribution to applied science. The ideas, methods, and results herein provide fertile ground for further theoretical development and interdisciplinary collaboration in the years to come.

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