KNÖRRER PERIODICITY AS A DEFORMATION OF MATRIX FACTORISATION CATEGORIES

by

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Declaration

I, Edwin Hollands confirm that the work presented in my thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

Knörrer periodicity is an equivalence of matrix factorisations categories which has been proven to hold under certain fairly restrictive conditions. We show that the phenomenon can be understood as a deformation of categories even in the cases where there is no equivalence. The class of this deformation in the Hochschild cohomology of the category can be thought of as a local obstruction to Knörrer periodicity.

Furthermore, we study this deformation and show there is a strong argument that if we assume the equivalence to hold and the triviality of the deformation up to first order, then the conditions mentioned above are necessary. This presents an approximate converse to Knörrer periodicity.

Impact Statement

This thesis contributes to the understanding of relationships between categories of matrix factorisations, one of the key classes of category involved in conjectures around homological mirror symmetry. This is a broad and very active area of research within academia: to systematically understand mirror symmetry at a categorical level. The more general versions of a theorem are still very open. Mirror symmetry itself is a highly studied conjecture which came about as a result of predictions made by physicists working on string theoretical models for space-time, and a mathematical understanding of the phenomenon could lead to interesting conclusions for this theory.

This research fits into this body of work by presenting a new framework for understanding the most well-known equivalence of matrix factorisation categories, allowing an exploration of the limitations of the equivalence as well as to what extent it can be salvaged when it does fail. This will be useful for other researchers within the field both as a source of intuition about the relationships of these categories and as a source of tests to apply when figuring out if two categories are equivalent. There are a few low-hanging generalisations, for example there should be little work in making the same statements outside of the affine context, but there are more interesting questions raised as well. For instance: what corresponds to the non-trivial deformations described below 'through the mirror' of mirror symmetry - when studying the symplectic geometry of the mirror space in terms of Fukaya categories?

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Contents

1 Introduction

Matrix Factorisations were introduced by Eisenbud in [4] in order to study maximal Cohen-Macaulay modules. In particular an equivalence is established, using the cokernel functor, between the matrix factorisation category of a regular local ring paired with a distinguished non-unit element and the stable category of maximal Cohen-Macaulay modules of the ring obtained from the quotient with the aforementioned element.

This was then followed by an interesting result from Knörrer [10] which, along with its many descendants, became known as Knörrer periodicity. The original purpose of this result was for applications in algebra, but it expresses a fundamental property of categories of matrix factorisations themselves. Namely, that extending the ring by new independent variables and adding a quadratic term in these variables to the superpotential constitutes no change to the actual category: there is an equivalence of categories given by tensoring with a particular bimodule.

Outside of abstract algebra, matrix factorisations also find application in the homological mirror symmetry conjecture. This comes from there being two topological twists of the field theory in topological string theory: the A model and the B model. The relationship between the spaces in these models is what is known as mirror symmetry - they are referred to as 'mirrors' of each other. The endpoints of open strings are required to lie on a subspace called a D brane. Mathematically, we think of D branes as the objects of category, and states of open strings spanning between them as morphisms. For a Calabi-Yau target space, the category of D branes for the B model (called B branes) is the derived category of the Calabi-Yau. For other natural classes of examples such as Fano varieties, we are led to mirrors called Landau-Ginzburg models. The B branes of Landau-Ginzburg models are categories of matrix factorisations.

These categories of matrix factorisations were studied within the world of derived algebraic geometry by Orlov [15] via an equivalence with the 'singularity category' of the associated hypersurface cut out by the vanishing of the superpotential. Orlov proves a more general version of Knörrer periodicity in [16] where the previously quadratic function is replaced with the product of a fibre coordinate of a bundle and a regular function on the base. The functor that gives the equivalence is also realised geometrically as the composition of a pull up and a push forward. This method suggests that the phenomenon is closely tied to the presence of this 'linearity' given by the fibres which provides a direction to pull up along. As a consequence we will use the shorthand 'semilinear' to refer to this case. Several different proofs and extensions of the result have since been developed by Isik [9], Hirano [7], Shipman [20] and Teleman [21]. In particular, if we consider regular functions $f, g \in \mathcal{O}_X$ on an affine scheme X, these results roughly speaking compare the derived category of $Y = V(f, g)$ with matrix factorisations of $W = fg$ on X. Orlov's result says that there is an equivalence when one of f and g is linear and the other is invariant in this linear direction. From Teleman we learn that when Y is smooth, $Y = Crit(fg)$ and there is an equivalence up to some global obstructions such as from the super-Brauer group. In a formal neighbourhood of a point, however, things look like the previous case where Knörrer periodicity holds since ∇f and ∇g are non-zero and linearly independent. We shall explore what happens when Y is singular, focusing on local obstructions.

In general, there are relatively few equivalences known between different matrix factorisation categories, with examples coming either from different kinds of flops or from versions of Knörrer's result. The goal of the work presented here is to understand the result as a special case of a certain deformation, which exists more generally, and to use this to present something of a converse argument: that if the deformation is trivial to first order then we only have Knörrer periodicity in the 'semilinear' case.

Our main theorem shows that given a regular pair of functions we can construct a close relationship between categories of matrix factorisations on the vanishing locus Y of the functions and on a formal neighbourhood of Y where the product of the functions is added to the superpotential. The semilinearity assumptions would usually require that one of these functions is linear in the sense of being a fibre coordinate function for some vector bundle on Y.

Theorem 1.1. *Let $f, g, h \in \mathcal{O}_X$ be a regular sequence of regular functions on a smooth affine scheme X and $\widehat{X}_{f,g}$ be the completion of X with respect to f and g. Let*

$$\mathcal{K} = \mathcal{O}_X \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{O}_X \in \mathrm{MF}(X, fg),$$

then there is a differential graded algebra $\mathcal{R}_1$ which gives an equivalence

$$\widehat{\mathrm{MF}}(\mathcal{R}_1, h) \underset{-\otimes_X \mathcal{K}}{\overset{Hom_{\mathcal{R}_1}(\mathcal{K}, -)}{\underset{\cong}{\rightleftarrows}}} \widehat{\mathrm{MF}}(\widehat{X}_{f,g}, h + fg) .$$

to the category of completed matrix factorisations on $\widehat{X}_{f,g}$. Furthermore, $\mathcal{R}_1$ is a deformation of $\mathcal{R}_0$, the Kozsul resolution $Y = V(f, g) \subset X$.

The DG algebra $\mathcal{R}_1$ is the endomorphism algebra of the object K. We can see it as the fibre over 1 of $\mathcal{R} \to \mathbb{A}^1_t$ where $\mathcal{R}$ is the $\mathbb{Z}/2\mathbb{Z}$-graded DG algebra:

$$\mathcal{O}_X[t]\langle \epsilon_1, \epsilon_2 \rangle \Big/ ([\epsilon_1, \epsilon_2] = t, \epsilon_1^2 = \epsilon_2^2 = 0, \partial\epsilon_1 = f, \partial\epsilon_2 = g) \text{ where } |\epsilon_1| = |\epsilon_2| = 1.$$

$\mathcal{R}_0$, the fibre over 0, is quasi-isomorphic to $\mathcal{O}_Y$ and in fact this induces a quasi-equivalence of the derived categories of these two objects. However, our matrix factorisation categories are instead analagous to the absolute derived categories of Positselski [18] for which it is not necessarily true that a quasi-equivalence of DG-algebras induces an equivalence of matrix factorisation categories. For our specific case of a complete intersection Y and its two-step Koszul resolution we expect and conjecture the equivalence $\widehat{\mathrm{MF}}(\mathcal{R}_0, h) \cong \widehat{\mathrm{MF}}(Y, h)$. So matrix factorisations of h on Y are a deformation of (completed) matrix factorisations of $h + fg$ on X (completed with respect to f and g).

The next question of particular interest is to know when this deformation is trivial, since this suggests an equivalence of categories much like Knörrer periodicity. However, the deformation is non-formal and therefore most of our tools are inadequate to study it directly. We can, however, study it up to first order by investigating the associated class in the Hochschild cohomology of the category of matrix factorisations on the DG-algebra $\mathcal{R}_0$. When this class vanishes it is suggestive that we should have an equivalence. The concept seems closely tied to Knörrer periodicity, however for that we need full triviality of the deformation, not just infinitesimally or to first order. As for the converse, strictly speaking it is possible to hypothesise pathological situations where the deformation is non-trivial to first order yet the categories at $\mathcal{R}_0$ and $\mathcal{R}_1$ happen to be equivalent, but we expect these not to exist in any meaningful way.

To find the class of this deformation in Hochschild cohomology, we will use a version of the Hochchild-Konstant-Rosenberg theorem. It turns out, since we are deforming the relation $[\epsilon_1, \epsilon_2] = 0$ to $[\epsilon_1, \epsilon_2] = 1$, that this class is represented by the degree 2 differential operator $\partial_{\epsilon_1}\partial_{\epsilon_2}$. When this class is nonzero, it can be thought of as a local obstruction to Knörrer periodicity. This is in contrast to previous obstructions, such as in Teleman's work [21], which are due to global phenomena.

Since this study is local in nature, we can pass to a formal neighbourhood of a point in X, which is the same as looking at a formal neighbourhood of the origin in affine space, since X is a smooth affine scheme.

Our final study is a heuristic argument that is close to a converse to the previous statements of Knörrer periodicity. We argue that when such an equivalence holds we are only a trivial deformation away from the semilinear case. This is done by first assuming our deformation representing Knörrer periodicity is trivial which allows us to change coordinates so that one of our functions f or g is just the first coordinate function. Without loss of generality suppose this is f. From here we can set up a commuting square of deformations, where our Knoerrer periodicity equivalence is obtained by deforming from a semilinear case where we take the parts of g and the superpotential h which are independent of the coordinate direction associated to f. Further yet, if the deformation from the semilinear case is trivial then the classes of the deformation are everywhere zero in Hochschild cohomology, in particular vanishing in the Jacobi ring meaning it is locally a coordinate change. If we can integrate these coordinate changes up to the non-formal deformation then we really were in the semilinear case to begin with.

1.1 Summary

In section 2 we will cover some background topics and definitions. Starting with definitions of matrix factorisation categories first in the smooth affine case and then expanding to include singular, non-affine and then a definition for factorisations over differential graded algebras too. We then cover a few more topics, such as relations to categories of maximal Cohen-Macaulay modules and derived singularity categories, and then some necessary technicalities on completions, limits and Hochschild cohomology.

Section 3 will be a recap on the origins and various approaches and different statements of the Knörrer periodicity theorem to date.

In section 4, we set up functors between $\widehat{\mathrm{MF}}(X, fg + h)$ and $\widehat{\mathrm{MF}}(\mathcal{R}_1, h)$ and prove Theorem 1.1, setting up an equivalence after completion of X with respect to f and g.

Section 5 then addresses the deformation between $\widehat{\mathrm{MF}}(\mathcal{R}_0, h)$ and $\widehat{\mathrm{MF}}(\mathcal{R}_1, h)$ using Hochschild cohomology and passing to a formal neighbourhood of a point.

We explore the implications when the second cohomology class associated to the deformation vanishes and make a heuristic converse argument to the Knörrer periodicity theorem.

2 Background

2.1 Matrix Factorisations

Consider W, an element of a ring A (often a k-algebra where k is a field of characteristic zero). A factorisation is a pair of other elements f, g such that $fg = gf = W$. However, we can find more factorisations if we instead consider a pair of matrices $F, G \in \mathrm{Mat}_{n\times n}(A)$ such that $F \cdot G = G \cdot F = W \cdot \mathrm{Id}_n$.

As an illustration, factorisations in terms of matrices are one way of arriving at the concept of complex numbers. Let $A = \mathbb{Z}$ and $W = -1$. The only way to factorise is $1 \cdot -1 = -1 \cdot 1 = -1$. However, if we allow matrices, there is an interesting factorisation

$$\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = 1 \cdot \mathrm{Id}_2$$

If we identify $\mathbb{Z}$ with an isomorphic subring $\mathbb{Z} \cdot \mathrm{Id}_2$ of $\mathrm{Mat}_{n\times n}(\mathbb{Z})$, then the extension by this 'matrix factorisation' of -1 gives us a subring $\mathbb{Z}\left[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right]$ which is isomorphic to the Gaussian integers $\mathbb{Z}[i]$. Similarly with $\mathbb{R}$ replacing $\mathbb{Z}$ we can discover $\mathbb{C}$ as a subring of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$.

Remark. This example is just an analogy to illustrate the concept. The actual category we will define would make this factorisation trivial.

The examples of interest to us come from polynomials. Take $wx - yz$ in $\mathbb{C}[w, x, y, z]$. This does not factorise in terms of other polynomials, however it does in terms of matrices of polynomials.

$$\left(\begin{smallmatrix} w & -y \\ -z & x \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) = (wx - yz) \cdot \mathrm{Id}_2. = \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} w & -y \\ -z & x \end{smallmatrix}\right).$$

2.1.1 The smooth affine case

To study matrix factorisations we must define an appropriate category. Let A be a regular commutative algebra over a field of characteristic zero and $W \in A$ a regular element, which we call the 'superpotential'. We will define a category of matrix factorisations associated to the pair (A, W). We will model our approach on the definition of the bounded derived category of coherent sheaves so that we can hope to retain many of its useful properties.

Let us first then look at how the derived category behaves in the simple case of a smooth affine scheme $X = Spec(A)$. The differential graded bounded derived category of finitely generated A-modules $\mathcal{D}^b(A)$ has a differential graded subcategory $Perf(A)$ of perfect complexes. Perfect complexes are those quasi-isomorphic to bounded complexes of finite projective A-modules. For all our purposes A will be Noetherian, so we can treat projective and locally free interchangeably. Under our smoothness condition, there is an equivalence of triangulated categories $D^b(A) \cong [Perf(A)]$ (where the square brackets indicate passing from the differential graded category to the homotopy category and D^b represents the usual triangulated bounded derived category). This is because all complexes in $D^b(A)$ have finite resolutions by finite locally free A-modules. In fact, if we take the smaller category $\mathcal{V}$ of these bounded complexes of finite locally free modules, all quasi-isomorphisms of such complexes come from genuine homotopy equivalences. So in fact $[\mathcal{V}] \cong D^b(A)$ without needing to invert quasi-isomorphisms (equivalently there is no need to quotient by acyclic objects since they are already contractible).

Remark. To see why smoothness is critical here, consider the singular point in the scheme $Spec(\mathbb{C}[x]/x^2)$. Locally free resolutions of this point are necessarily infinite, the obvious one being

$$\cdots \xrightarrow{x} \mathbb{C}[x]/x^2 \xrightarrow{x} \mathbb{C}[x]/x^2 \xrightarrow{x} \mathbb{C}[x]/x^2 \longrightarrow \mathbb{C}.$$

We shall model our approach to matrix factorisations on these properties of the derived category, at least in the smooth affine case. A matrix factorisation shall be a pair of finite locally free modules with morphisms between them:

$$\mathcal{M}_0 \underset{d_1}{\overset{d_0}{\rightleftarrows}} \mathcal{M}_1$$

such that both compositions of these morphisms are just the action of W on the respective modules, i.e. $d_1 d_0 = W\mathrm{Id}_{\mathcal{M}_0}$ and $d_0 d_1 = W\mathrm{Id}_{\mathcal{M}_1}$. We shall often instead think of this pair as a $\mathbb{Z}/2\mathbb{Z}$-graded module with differential d of odd degree, which squares to give the action of W. We can then consider two matrix factorisations (M, d_M) and (N, d_N), and the chain complex of homomorphisms given by

$$\mathrm{Hom}_{\mathrm{MF}_{\mathrm{LF}}(A,W)}(M, N) = \bigoplus_{i,j\in\{0,1\}} \mathrm{Hom}_A(M_i, N_j)$$

with grading $(i-j)\mathrm{mod}2$ and induced differential defined on homogeneous elements $h \in \mathrm{Hom}_A(M_i, N_j)$ as $dh = d_N h - (-1)^{i-j} h d_M$ and extended linearly.

Definition 2.1. Take a regular commutative algebra A, over a field of characteristic zero, and a regular element W. The DG category of matrix factorisations $\mathrm{MF}_{\mathrm{LF}}(A, W)$ has as its objects the set of $\mathbb{Z}/2\mathbb{Z}$-graded finite locally free modules equipped with differentials d of odd degree such that $d^2 = W$. The morphisms are the chain complexes

$$\mathrm{Hom}_{\mathrm{MF}_{\mathrm{LF}}(A,W)}(M, N) = \Big(\bigoplus_{i,j\in\{0,1\}} \mathrm{Hom}_A(M_i, N_j),\ d \Big)$$

with differential 'd' as above.

The homotopy category of matrix factorisations $[\mathrm{MF}_{LF}(A, W)]$ has the same objects and its morphisms are

$$\mathrm{Hom}_{[\mathrm{MF}_{\mathrm{LF}}(A,W)]}(M, N) = H^0(\mathrm{Hom}_{\mathrm{MF}_{\mathrm{LF}}(A,W)}(M, N)).$$

Proposition 2.2. $\mathrm{MF}_{LF}(A,W)$ *can be given the structure of a pre-triangulated category which also makes* $[\mathrm{MF}_{LF}(A,W)]$ *a triangulated category.*

We shall not give a proof here, since this is essentially a matter of straightforward checks once the structure is specified. A thourough treatment is given in [17]. So we shall define that structure: namely the shift functor and a class of distinguished triangles. The shift functor simply switches degrees and the sign of the differential, so an object

$$M = \mathcal{M}_0 \underset{d_1}{\overset{d_0}{\rightleftarrows}} \mathcal{M}_1$$

becomes

$$M[1] = \mathcal{M}_1 \underset{-d_0}{\overset{-d_1}{\rightleftarrows}} \mathcal{M}_0 \ .$$

and a morphism (F_0, F_1) becomes $(F_0, F_1)[1] = (F_1, F_0)$.

To define distinguished triangles, we first take the cone of a morphism F

$$\begin{array}{ccc} \mathcal{M}_0 & \underset{d_1^M}{\overset{d_0^M}{\rightleftarrows}} & \mathcal{M}_1 \\ \downarrow F_0 & & \downarrow F_1 \\ \mathcal{N}_0 & \underset{d_1^N}{\overset{d_0^N}{\rightleftarrows}} & \mathcal{N}_1 \end{array}$$

and then take the totalisation when the diagram above is viewed as a $\mathbb{Z}/2\mathbb{Z}$-graded double complex:

$$\mathcal{N}_1 \oplus \mathcal{M}_0 \underset{\begin{pmatrix} d_0^N & F_1 \\ 0 & -d_1^M \end{pmatrix}}{\overset{\begin{pmatrix} d_1^N & F_0 \\ 0 & -d_0^M \end{pmatrix}}{\rightleftarrows}} \mathcal{N}_0 \oplus \mathcal{M}_1 \ .$$

We define the standard triangles as those coming from cones as above:

$$M \xrightarrow{F} N \xrightarrow{(id,0)} Cone(F) \xrightarrow{(0,-id)} M[1]$$

in the usual manner and distinguished (or 'exact') triangles as those isomorphic to these standard ones in the homotopy category.

2.1.2 Including singularity

Now, to generalise to the singular case we should heed a warning indicated by our comparison with the derived category. For a singular algebra A, $[Perf(A)]$ is a strict subcategory of $D^b(Coh(A))$. So we should consider more objects, either complexes of coherent sheaves or even all quasi-coherent sheaves. This also forces us to address the need to now invert quasi-isomorphisms, since these will no longer simply be homotopy equivalences as in the smooth case (or equivalently to quotient by acyclic objects since they are no longer necessarily contractible).

We will use the naïve generalisation as an intermediate category. Let $\mathcal{MF}_{\mathrm{Coh}}(A, W)$ be defined as $\mathrm{MF}_{\mathrm{LF}}(A, W)$ was above, only with A not necessarily smooth and with objects being pairs of any finitely generated modules, not just locally free ones. We can define our 'acyclic' matrix factorisations by taking all exact sequences in $\mathcal{MF}_{\mathrm{Coh}}(A, W)$ and fold them up by degrees into a single object, T, called the totalisation.

Definition 2.3. Let

$$\mathcal{M}^n \xrightarrow{f_n} \mathcal{M}^{n+1} \xrightarrow{f_{n+1}} \dots \xrightarrow{f_m} \mathcal{M}^m$$

be a complex in $\mathcal{MF}_{\mathrm{Coh}}(A, W)$, then the totalisation $T(\mathcal{M})$ is an object in $\mathcal{MF}_{\mathrm{Coh}}(A, W)$ given by

$$T(\mathcal{M})_i = \bigoplus_{n+j=i \ mod(2)} \mathcal{M}_j^n$$

with differential $d_i^T = \sum_{n+j=i \ mod(2)} (f_j^n + (-1)^n d_j^n)$ where d^n is the internal differential on $\mathcal{M}^n$.

Now we denote by $T(A, W)$ the smallest full thick DG subcategory containing all such totalisations of exact sequences.

For this and the next section, we follow the definitions set out by Orlov in [17].

Definition 2.4. For an algebra A and a regular element W, the pre-triangulated DG category of matrix factorisations $\mathrm{MF}_{\mathrm{Coh}}(A, W)$ is given by the DG quotient of $\mathcal{MF}_{\mathrm{Coh}}(A, W)$ by the full thick DG subcategory $T(A, W)$. The associated triangulated homotopy category is $[\mathrm{MF}_{\mathrm{Coh}}(A, W)]$.

Remark. We are justified in calling this 'the' category of matrix factorisations since when A is smooth, $[\mathrm{MF}_{\mathrm{Coh}}(A, W)] \cong [\mathrm{MF}_{\mathrm{LF}}(A, W)]$, just like how $[\mathcal{V}] \cong Perf(A) = D^b(Coh(A))$. This is because coherent sheaves on smooth schemes have finite bounded resolutions in terms of locally free modules and short exact sequences of projective modules are split. This means the totalisations of exact sequences are homotopically trivial, so our 'acylic' objects in $\mathrm{MF}_{\mathrm{LF}}(A, W)$ are already zero in the homotopy category.

2.1.3 Non-affine matrix factorisations

We can also consider matrix factorisations on a non-affine scheme X, paired with $W \in \mathcal{O}_X$ a regular function (see [17]).

Definition 2.5. Let X be a scheme over k and $W \in \mathcal{O}_X$ a regular function, then $\mathrm{MF}_{\mathrm{Coh}}(X, W)$ is defined just as in Definition 2.4, where instead of $\mathbb{Z}/2\mathbb{Z}$-graded finite A modules we instead consider $\mathbb{Z}/2\mathbb{Z}$-graded coherent sheaves on X.

Furthermore, our calculations will present us with objects most naturally understood as quasi-coherent, so we will expand our definition to include this case.

Definition 2.6. Let X be a scheme and $W \in \mathcal{O}_X$ a regular function, then $\mathrm{MF}_{\mathrm{QCoh}}(X, W)$ is defined just as in Definition 2.4, where instead of $\mathbb{Z}/2\mathbb{Z}$-graded finite A modules we instead consider $\mathbb{Z}/2\mathbb{Z}$-graded quasi-coherent sheaves on X.

Remark. We have defined our notion of 'acyclic' well, since just like with the derived category, the quotient neatly handles issues presented by both singularities and non-affineness.

2.1.4 Curved differential graded algebras

We shall need one further generalisation of the affine case: a category of matrix factorisations over a differential graded algebra (DGA). In order to define this, we must first take a detour into the curvy world of curved differential graded (CDG) modules over curved differential graded algebras.

Definition 2.7. A curved differential ($\mathbb{Z}/2\mathbb{Z}$-)graded algebra over a field k is a triple (A, d, h) consisting of a (not necessarily commutative) $\mathbb{Z}/2\mathbb{Z}$-graded k-algebra A, a graded derivation d (i.e. satisfying the graded Leibniz rule) of odd degree, and a 'curvature' element h of even degree satisfying the equations $d^2 a = [h, a]$ for all $a \in A$ and $dh = 0$.

Remark. This is a specialisation of the definition of a curved A_∞ algebra where the higher maps vanish. These are in turn special cases of A_∞ categories, just as k-algebras are k-linear categories with one object.

Definition 2.8. A CDG module of a CDG algebra (A, d, h) is a pair (M, d_M) consisting of a graded A-module M and an odd derivation d_M of degree 1 satisfying the equation $d_M^2 m = hm$ for all $m \in M$. The fact that the derivation is odd means it satisfies the graded Leibniz rule $d_M(am) = (da)m + (-1)^{\deg(a)} a(d_M m)$ for all $a \in A$ and $m \in M$.

We will often just write (A, h) for the CDG algebra or M instead of (M, d_M) leaving the differential implicit.

Definition 2.9. Let M and N be CDG modules over a CDG algebra, then the complex of homomorphisms

$$\mathrm{Hom}_{A,h}(M, N) = (\bigoplus_{i,j \in \mathbb{Z}} \mathrm{Hom}_A(M_i, N_j), d_{M,N})$$

where the grading is given by $(j-i)$ and the differential is defined on homogeneous elements $f \in \mathrm{Hom}_A(M_i, N_j)$ by $d_{M,N}(f)(m) = d_N(f(m) - (-1)^{\mathrm{deg(f)}} f(d_M(m)))$ for all $m \in M$ (and extended linearly).

We start once again with an intermediate definition which does not have the 'derived' properties that we want.

Definition 2.10. Let (A, d, h) be a CDG algebra. The DG category of (finitely generated) CDG modules $CDG(A, d, h)$ over A has as objects the CDG modules (M, d_M) defined above for which M is a finitely generated A-module. The complexes of morphisms are $\mathrm{Hom}_{A,h}(M, N)$.

Now we must again quotient out by some 'acyclic object' equivalent. Similarly to before, we let $T(A, d, h)$ be the DG subcategory of totalisations of exact sequences in $CDG(A, d, h)$.

Definition 2.11. Let (A, d, h) be a CDG algebra. The Derived DG category of (finitely generated) CDG modules $\mathcal{D}_{\mathrm{CDG}}(A, d, h)$ over A is the DG quotient of $CDG(A, d, h)$ by the full thick subcategory $T(A, d, h)$.

Now, with these many definitions out of the way, we will fix a notation. When we refer to $\mathrm{MF}(X, W)$, if X is a scheme, we shall mean $\mathrm{MF}_{\mathrm{Coh}}(X, W)$, though we shall sometimes need to implicitly use the fact that this is a subcategory of $\mathrm{MF}_{\mathrm{QCoh}}(X, W)$. When X is smooth and affine, $\mathrm{MF}(X, W)$ is quasi-equivalent to $\mathrm{MF}_{\mathrm{LF}}(X, W)$.

When instead we have a central element h of a DG algebra (A, d), we will write $\mathrm{MF}(A, h)$ to mean the DG category $\mathcal{D}_{\mathrm{CDG}}(A, d, h)$ of (finitely generated) CDG modules over the CDG algebra (A, d, h).

Remark. Since h is taken to be central, $[h, -] \equiv 0$, so since $d^2 = 0$ in a DG algebra, (A, d, h) is indeed a CDG algebra. We can see that this is actually a generalisation of the definition of matrix factorisations since we can always consider an algebra and superpotential (A, h) as a CDG algebra with trivial grading and differential.

2.2 Relationships to Other Categories

Starting with a ring R and an element W which is not a zero divisor, we shall explore two ways in which $\mathrm{MF}_{\mathrm{LF}}(R, W)$ is related to the hypersurface ring $S = R/W$.

2.2.1 Maximal Cohen-Macaulay Modules

Matrix factorisations were introduced by Eisenbud [4] as a means of studying (the 'stable category' of) maximal Cohen-Macaulay modules.

To define this class of modules we will need the notion of 'depth'.

Definition 2.12. Let M be a finitely generated R module for a local ring (R, m). The depth of M is the maximal length of a sequence $f_1, ..., f_n \in m$ such that f_1 is not a zero divisor in M and for all $1 \leq i \leq n-1$, f_{i+1} is not a zero divisor in $M/(f_1, ..., f_i)$.

Definition 2.13. A maximal Cohen-Macaulay module M over a local ring (R, m) is one that is finitely generated such that $\mathrm{depth}(M) = \dim_{\mathrm{Krull}}(R)$.

Remark. If R is regular, the Auslander-Buchsbaum formula tells us that the projective dimension of M is the difference between the Krull dimension of R and the depth of M, so maximal Cohen-Macaulay modules are just finitely generated free modules.

A maximal Cohen-Macaulay module M over the hypersurface ring $S = R/W$ is also an R module, and since the Krull dimension of R is one greater than that of S, by the Auslander-Buchsbaum formula the projective dimension of M over R

is 1. Therefore M has a length 1 free resolution.

$$0 \longrightarrow \mathcal{M}_1 \xrightarrow{d_0} \mathcal{M}_0 \longrightarrow M \longrightarrow 0$$

Since W acts trivially on M, there is a contracting homotopy $d_1 : \mathcal{M}_0 \to \mathcal{M}_1$ such that $W \cdot \mathrm{Id}_{\mathcal{M}_0 \oplus \mathcal{M}_1} = d(d_1) = d_0 d_1 + d_1 d_0$. Put another way, we have

$$\mathcal{M}_0 \underset{d_1}{\overset{d_0}{\rightleftarrows}} \mathcal{M}_1$$

such that $d_0 d_1 = W \cdot \mathrm{Id}_{\mathcal{M}_1}$ and $d_1 d_0 = W \cdot \mathrm{Id}_{\mathcal{M}_0}$. So maximal Cohen-Macaulay modules on $S = R/W$ allow us to construct elements of $\mathrm{MF}_{\mathrm{LF}}(R, W)$. We can also recover M as $coker(d_0)$. In fact, starting with the matrix factorisation we can construct a maximal Cohen-Macaulay module in this way, since $W\mathcal{M}_1 \in \mathrm{Im}(d_0)$ so W annihilates $coker(d_0)$, which can therefore be thought of as an S module. It has projective dimension 1 over R and therefore is maximal Cohen-Macaulay over S.

So we have a correspondence between matrix factorisations of (R, W) and maximal Cohen-Macaulay S modules. To compare morphisms, we take the S module M and find a free resolution. We can obtain one by starting with the resolution over R and modding out W.

$$\ldots \xrightarrow{d_0} \mathcal{M}_1/W\mathcal{M}_1 \xrightarrow{d_1} \mathcal{M}_0/W\mathcal{M}_0 \xrightarrow{d_0} \mathcal{M}_1/W\mathcal{M}_1 \xrightarrow{d_1} \mathcal{M}_0/W\mathcal{M}_0 \longrightarrow M$$

We observe that the resulting resolution is 2 periodic. We want to consider the morphisms which themselves become 2 periodic along this resolution, so we quotient out by F, the set of S module homomorphisms which factor through a free module.

Definition 2.14. The stable category of maximal Cohen-Macaulay modules of S has as objects the maximal Cohen-Macaulay modules of S and as morphisms sets $\mathrm{Hom}_{SMCM(S)}(M, N) = \mathrm{Hom}_S(M, N)/F$.

Which finally leads to the context in which matrix factorisations were studied by Eisenbud, Knörrer and their contemporaries.

Theorem 2.15. *(Eisenbud [4])*
There is a functor coker : $[\mathrm{MF}_{LF}(R, W)] \to SMCM(S)$ *which takes a matrix factorisation*

$$\mathcal{M}_0 \underset{d_1}{\overset{d_0}{\rightleftarrows}} \mathcal{M}_1$$

and sends it to a maximal Cohen-Macaulay module coker(d_0)*. This functor induces an equivalence of categories.*

Remark. Although we defined matrix factorisations for k-algebras only due to our geometric motivation, it works exactly the same for a ring R and an element W which is not a zero divisor.

2.2.2 Derived Singularity Categories

For a ring R, as was peviously mentioned, when R is smooth we have that $Perf(R) = \mathcal{D}^b(Coh(R))$ (where we are now staying at the DG level). For singular R, $Perf(R)$ is a strict subcategory of $\mathcal{D}^b(Coh(R))$. Therefore it is sensible to think of the difference between these two categories as an obstruction to smoothness, or a measure of singularity.

Definition 2.16. The Derived Category of Singularities of a ring R is the homotopy category of the DG quotient

$$D_{sg}(R) := [\mathcal{D}^b(Coh(R))/Perf(R)].$$

Take an R module, M. We can consider a (potentially infinite) locally free resolution of M. Since we are quotienting by perfect complexes, we can essentially ignore any finite chunk of this resolution, so that in the singularity category it seems we only care about the behaviour off towards infinity.

Theorem 2.17. *(Eisenbud [4])*
Let (R, m) be a local ring, $W \in R$ and $S = R/W$, then every finitely generated S module has an eventually 2 periodic free resolution.

In fact, we can drop the local condition on the ring as long as we instead consider locally free resolutions. Note that we now have something of a generalisation of the maximal Cohen-Macaulay case.

Theorem 2.18. *(Buchweitz [1])*
Let (R, m) be a regular local ring and $W \in R$ such that $S = R/W$ is a singular hypersurface. The obvious embedding

$$SMCM(S) \to D_{sg}(S)$$

is an equivalence of categories.

Therefore, combining Theorems 2.15 and 2.18 there is also an equivalence of categories

$$[\mathrm{MF}_{\mathrm{LF}}(R, W)] \cong D_{sg}(S)$$

which also happens to be an equivalence of triangulated categories. Thus we should think of the study of matrix factorisations as also the study of singularities of hypersurfaces.

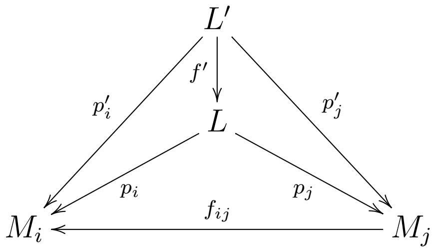
2.3 Completions

We will eventually need to work on formal neighbourhoods, so we define completed matrix factorisations and describe some technical results on limits and completions.

2.3.1 Completed matrix factorisations

Definition 2.19. An inverse system is a partially ordered set $(I, \leq)$, a family of objects $(M_i, f_{ij})_{i,j \in I}$ and morphisms $f_{ij} : M_i \leftarrow M_j$ in a category $\mathcal{C}$ such that $f_{ii} = \mathrm{Id}_{M_i}$ and $f_{ik} = f_{ij} f_{jk}$ for all $i \leq j \leq k$ in I.

Definition 2.20. The inverse limit of an inverse system $(M_i, f_{ij})_{i,j\in I}$ in the category $\mathcal{C}$ is the universal object L with morphisms $p_i : L \to M_i$ satisfying $p_i = f_{ij}p_j$ for all $i \leq j$ in I such that for any other such object L' with morphisms p'_i there is a unique morphism $f' : L' \to L$ which makes the following diagram commutes for all $i \leq j$ in I.



We write

$$L = \varprojlim M_i.$$

Definition 2.21. A filtered abelian group is an abelian group G with a descending filtration $G = F^0G \supset F^1G \supset F^2G...$ of subgroups. This forms an inverse system in the category of abelian groups with $I = \mathbb{N}$ and the morphisms $f_{ij} : G/F^jG \leftarrow G/F^iG$ for $i \geq j$ are just given by the quotient

$$\left(G/_{F^iG} \right)/_{F^jG} \cong G/_{F^jG}$$

since $F^iG \subset F^jG$.

The completion of a filtered abelian group G is the inverse limit

$$\widehat{G} = \varprojlim (G/(F^nG)).$$

Remark. We can construct the completion as the subgroup of the direct product given by

$$\widehat{G} = \left\{ (\overline{m_n})_{n\geq 0} \in \prod_{n\geq 0} (G/(F^nG)) \;\middle|\; \text{for all } i \leq j,\ a_i = a_j \pmod{F^nG} \right\}.$$

where essentially we take a class from every quotient and ensure they are compatible in the sense that they map to each other under the quotient maps.

Remark. We define a filtered ring in the same was as for a filtered abelian group, and similarly for a filtered curved differential graded algebra. The completion of a filtered ring will have an induced ring structure.

Now let M be a matrix factorisation in $\mathrm{MF}(R, W)$ for some filtered ring R. Suppose $\widehat{R}$ is the completion of R. Then we can consider a 'completed matrix factorisation' $\widehat{M}$ which is simply the completion of the filtered R-module $M = MF^0R \supset MF^1R \supset MF^2R \supset ....$ This is now a matrix factorisation over $\widehat{R}$. Completion then gives a functor from $\mathrm{MF}(R, W)$ to $\mathrm{MF}(\widehat{R}, W)$.

Remark. There is, of course, a forgetful functor in the opposite direction but this is not an inverse since there are $\widehat{R}$ modules that are not equal to their completions.

Example 2.22. Consider the $\mathbb{C}[[x, y]]$ module $\mathbb{C}[[x]][[y]]$. The completion of this module is the limit of $\mathbb{C}[x, y]/(x, y)^n$ which is just $\mathbb{C}[[x, y]]$ itself.

Example 2.23. Similarly the completion of the $\mathbb{C}[[x]]$ module $\mathbb{C}[[x]][x^{-1}]$ is the limit of $\mathbb{C}[x, x^{-1}]/x^n = 0$ which is just 0.

Definition 2.24. Given a filtered, curved differential graded algebra (A, d, h) where $A = F^0A \supset F^1A \supset F^2A \supset ...$, we obtain an inverse system of quotient rings $f_{ij} : A/F^jA \leftarrow A/F^iA$ for $i \geq j$ whose morphisms induce functors which give us an inverse system of categories

$$F_{ij} : \mathrm{MF}(A/F^iA, h) \leftarrow \mathrm{MF}(A/F^jA, h)$$

We take the limit of this inverse system of categories to obtain a category of completed matrix factorisations which we denote $\widehat{\mathrm{MF}}(\widehat{A}, h)$. The objects of the category are completions of matrix factorisations and the morphisms are limits of morphisms in the inverse system of categories.

2.3.2 Limits of cohomology and tensor products

We will need to use a notion of product '$\cdot$' which validates such statements as $\mathbb{C}[[x]] \cdot \mathbb{C}[[y]] = \mathbb{C}[[x, y]]$. The usual tensor product does not satisfy this condition, however we can use a 'completed tensor product' to realise it.

Definition 2.25. Consider two filtered modules $M = F^0 M \supset F^1 M \supset F^2 M ...$ and $N = G^0 N \supset G^1 N \supset G^2 N ...$ with completions $\widehat{M}$ and $\widehat{N}$ respectively. The completed tensor product is then the inverse limit

$$\widehat{M} \widehat{\otimes} \widehat{N} = \lim_{\leftarrow} (M/(F^m M) \otimes (N/G^n N)).$$

This is particularly useful for us in combination with the following condition.

Definition 2.26. We call an inverse system $(M_i, f_{ij})_{i,j \in I}$ Mittag-Leffler if for each $N \in I$ there exists $M \geq N$ such that for all $n \geq M$,

$$Im(f_{Nn}) = Im(f_{NM}).$$

In other words the image at any given degree eventually stabilises.

Proposition 2.27. *The inverse limit functor is exact on filtered chain complexes and therefore commutes with taking homology of these complexes.*

Any filtered group G is Mittag-Leffler since all the maps are surjections between quotients. Therefore, using C.11.1 from [12], the first derived functor of the inverse limit vanishes $lim^1(G/F^n G) = 0$. In other words the inverse limit functor is exact on filtered groups, of which filtered chain complexes are a special case. A functor being exact is the precise condition needed to commute with the homology functor.

This, along with the completed tensor product, will help us to separate cohomology calculations into constituent parts.

Proposition 2.28. *Suppose we have two bounded below cochain complexes of R-modules $A^\bullet$ and $B^\bullet$, one of which is flat, with filtrations given by the truncations $F^n = \tau_{\geq n}$. Then*

$$H^*(\widehat{A^\bullet}\widehat{\otimes}\widehat{B^\bullet}) = H^*(\widehat{A^\bullet})\widehat{\otimes}H^*(\widehat{B^\bullet}).$$

The completions $\widehat{A^\bullet}$ and $\widehat{B^\bullet}$ can be thought of as direct product complexes. The claim is that cohomology distributes over the completed tensor product.

Proof.

$$H^*(\widehat{A^\bullet}\widehat{\otimes}\widehat{B^\bullet}) = H^*\Big(\lim_{\leftarrow}\big({A^\bullet}/{(\tau_{\geq m}A^\bullet)} \otimes {B^\bullet}/{(\tau_{\geq n}B^\bullet)}\big)\Big)$$

by the definition of the completed tensor product. Since we trivially have Mittag-Leffler, $\lim_{\leftarrow}$ is exact and therefore

$$= \lim_{\leftarrow}\Big(H^*\big({A^\bullet}/{(\tau_{\geq m}A^\bullet)}\otimes{B^\bullet}/{(\tau_{\geq n}B^\bullet)}\big)\Big) = \lim_{\leftarrow}\Big(H^*\big({A^\bullet}/{(\tau_{\geq m}A^\bullet)}\big)\otimes H^*\big({B^\bullet}/{(\tau_{\geq n}B^\bullet)}\big)\Big)$$

where homology distributes over the tensor product by our flatness assumption. Now we apply the definition of the completed tensor product again

$$= \lim_{\leftarrow}\Big(H^*\big({A^\bullet}/{(\tau_{\geq m}A^\bullet)}\big)\Big)\widehat{\otimes}\lim_{\leftarrow}\Big(H^*\big({B^\bullet}/{(\tau_{\geq n}B^\bullet)}\big)\Big)$$

and we can once again use the previous proposition since these systems are Mittag-Leffler to conclude that

$$= H^*\Big(\lim_{\leftarrow}\big({A^\bullet}/{(\tau_{\geq m}A^\bullet)}\big)\Big)\widehat{\otimes}H^*\Big(\lim_{\leftarrow}\big({B^\bullet}/{(\tau_{\geq n}B^\bullet)}\big)\Big) = H^*(\widehat{A^\bullet})\widehat{\otimes}H^*(\widehat{B^\bullet}).$$

qed

2.4 Deformation Theory

2.4.1 Hochschild Cohomology

Let k be an algebraically closed field of characteristic zero. We will study the deformation theory of A, an associative k-algebra.

Definition 2.29. The bar complex $C_*^{bar}(A)$ is the chain complex of A-bimodules (or $A \otimes_k A^{op}$ modules)

$$\cdots \xrightarrow{d_{n+1}} C_n^{bar}(A) = A^{\otimes_k n+2} \xrightarrow{d_n} \cdots \xrightarrow{d_2} A^{\otimes_k 3} \xrightarrow{d_1} A \otimes_k A \longrightarrow 0$$

with differentials

$$d_n(a_0 \otimes_k a_1 \otimes_k ... \otimes_k a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes_k ... \otimes_k a_i a_{i+1} \otimes_k ... \otimes_k a_{n+1}.$$

Definition 2.30. The Hochschild chain complex of an A bimodule M is $C_*(A, M) = M \otimes_{A \otimes_k A^{op}} C_*^{bar}(A)$.

The point is to resolve A as a bimodule, essentially to then compute $Tor(A, M)$. For our purposes we will only need $C_*(A) := C_*(A, A)$, the chain complex of A-bimodules

$$\cdots \xrightarrow{d_{n+1}} C_n(A) = A^{\otimes_k n+1} \xrightarrow{d_n} \cdots \xrightarrow{d_3} A^{\otimes_k 3} \xrightarrow{d_2} A \otimes_k A \xrightarrow{d_1} A \longrightarrow 0$$

with differentials

$$d_n(a_0 \otimes_k a_1 \otimes_k ... \otimes_k a_n) = (-1)^n a_n a_0 \otimes_k a_1 \otimes_k ... \otimes_k a_{n-1} + \sum_{i=0}^{n-1} (-1)^i a_0 \otimes_k ... \otimes_k a_i a_{i+1} \otimes_k ... \otimes_k a_n.$$

Definition 2.31. The Hochschild cohomology of A is the cohomology of the cochain complex given by $\mathrm{Hom}_{A \otimes_k A^{op}}(C_*^{bar}(A), A)$ which computes $Ext_{A \otimes_k A^{op}}(A, A)$.

Example 2.32. Let us look at the affine space $A = k[x_1, ..., x_n]$. We resolve A over the enveloping algebra $A \otimes_k A^{op} \cong k[x_1, ..., x_n, y_1, ..., y_n]$. We have a map $A \otimes_k A^{op} \to A$ with kernel $I = (x_1 - y_1, ..., x_n - y_n)$. We can extend this to a Koszul resolution of A over $A \otimes_k A^{op}$ to see that A is quasi-isomorphic to the chain complex

$$\wedge^n I \to \cdots \to \wedge^2 I \to I \to A \otimes_k A^{op}.$$

Now we must tensor with A considered as an $A \otimes_k A^{op}$ module. In this context $A \cong A \otimes_k A^{op}/I$, so we get the complex

$$\wedge^n I/I^2 \to \cdots \to \wedge^2 I/I^2 \to I/I^2 \to A$$

with zero differential. Dualising gives us

$$\mathrm{Hom}_A(A, A) \to \mathrm{Hom}_A(I/I^2, A) \to \cdots \to \mathrm{Hom}_A(\wedge^n I/I^2, A).$$

It can be shown that $\mathrm{Hom}_A(I/I^2, A) = Der_k(A)$ and that dualising commutes with the wedge product here so that

$$HH^*(A) \cong \wedge^* Der_k(A).$$

$HH^*(A)$ can be given the structure of a differential graded Lie algebra using the cup product and the Gerstenhaber bracket [5].

The algebra in the example above, $\wedge^* Der_k(A)$, is the algebra of polyvector fields on A and it has its own bracket called the Schouten bracket.

Our observation in the example can famously be generalised.

Theorem 2.33. *(Hochschild-Konstant-Rosenberg [8])*
If A is smooth as a k-algebra, there is an isomorphism $HH^*(A) \cong \wedge^* Der_k(A)$ of graded k-algebras which furthermore sends the Gerstenhaber bracket to the Schouten bracket.

There are two further generalisations of interest to us which can be seen as stepping stones to dealing with curved differential algebras.

Suppose first that we now consider A to be a differential graded algebra with differential d. On the left hand side we simply include this differential and grading when computing the homology $HH^*(A, d)$ while on the right we must find a corresponding action on polyvector fields. To do so we use the correspondence between derivations and vector fields $Der_k(A) \cong T_X$ where $X = Spec(A)$ to define a vector field Q_d corresponding to the differential d. The natural, and indeed correct, thing to do then is to let d act via the Lie derivative $\mathcal{L}_{Q_d}$ and take cohomology.

$$HH^*(A) \cong H^*(\wedge T_X, \mathcal{L}_{Q_d}).$$

We are mainly interested in the Hochschild cohomology of categories, not just algebras. Fortunately, we have a correspondence between functors in derived categories and objects which are the Fourier-Mukai kernels of those functors. In particular, the Hochschild cohomology of the derived category can be computed as the extensions of the identity functor, whose Fourier-Mukai kernel is the diagonal whose extensions compute the Hochschild cohomology of the algebra.

$$Ext_{D^b(A)}(\mathbb{1}) \cong Ext_{X\times X}(\mathcal{O}_\Delta)$$

This argument is due to Toën [22] who then establishes the equivalence

$$HH^*(D^b(A)) \cong HH^*(A)$$

By an adaptation of the same argument, if (A, h) is a curved algebra then

$$HH^*(\mathrm{MF}(A, h)) \cong HH_c^{II}(A, h).$$

where the right hand side represents the Hochschild cohomology of the second kind (see [18] and [6]), which is essentially arrived at by swapping direct products and direct sums with each other throughout the theory.

What is the effect of the curvature element h on our polyvector fields? Since it is just an element of A, the commutator $[h, -]$ is a derivation on A and therefore gives a vector field Q_h and acts via the lie derivative as before (see [3] [14] [18] [19])

$$HH^*(\mathrm{MF}(A, h)) \cong H^*(\wedge T_X, \mathcal{L}_{Q_h}).$$

2.4.2 First Order Deformations

Of particular interest to us is the connection between the second Hochschild cohomology group and first order deformations of an algebra.

Definition 2.34. A first order deformation of a k-algebra A is an algebra structure on $A[t]/t^2$ which extends the usual product on A and is linear over t.

We consider two deformations equivalent if there is an isomorphism between the algebras that restricts to the identity on A. Let μ_0 be multiplication on A and μ_t be our new product. We observe that by the linearity and distribution properties of the product and the vanishing of t^2,

$$\mu_t(a + ct, b + dt) = \mu_0(a, b) + (\mu_t(a, d) + \mu_t(c, b))t.$$

So our deformations are defined by how the product acts on elements of A. We set $\mu_t(a, b) = \mu_0(a, b) + \mu_1(a, b)t$, the the deformation is in fact determined by the Hochschild 2-cocycle μ_1. It is not hard to check that the equivalence relation on these deformed products corresponds to that of the cohomology.

Proposition 2.35. *There is a one to one correspondence between cohomology classes in $HH^2(A)$ and classes of first order deformations of A which matches the trivial deformation with the trivial cohomology class.*

3 Knörrer Periodicity, A Recap

We start with some heuristic motivation. Since our motivation comes from algebraic geometry, we start with affine space. One of the first non-trivial matrix factorisation categories we can write down is $\mathrm{MF}(\mathbb{C}[u, v], uv)$ with the matrix factorisation

$$\mathbb{C}[u,v] \underset{v}{\overset{u}{\rightleftarrows}} \mathbb{C}[u,v] \ .$$

This turns out to be the only object in the category up to quasi-isomorphism and it has endomorphism group $\mathbb{C}$.

Another way to see this is since $\mathrm{MF}(\mathbb{C}[u,v], uv) \cong D_{sg}(\mathbb{C}[u,v]/uv)$, we are really examining the singularity category of the ordinary double point or node. It should not be too surprising that this category looks like the derived category of a point, since this is exactly the critical locus. It is not quite true that $\mathrm{MF}(\mathbb{C}[u,v], uv) \cong D^b(\mathbb{C})$ however, since the former is $\mathbb{Z}/2\mathbb{Z}$-graded and the latter is $\mathbb{Z}$-graded.

Remark. In general, for a k-scheme Y the $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{MF}(Y, W)$ can be upgraded to a $\mathbb{Z}$-grading, in the presence of a k^* action (internal $\mathbb{Z}$-grading). If, in our construction of the singularity category, we quotient the k^*-equivariant derived category by the full subcategory of k^*-equivarient perfect complexes, we obtain $D_{sg}^{k^*}(Y)$ which has a $\mathbb{Z}$-grading induced by the k^*-action.

In our case this allows us to say that $\mathrm{MF}(\mathbb{C}[u,v], \mathbb{C}^*, uv) \cong D^b(\mathbb{C})$ with the standard $\mathbb{C}^*$ action that makes u, v degrees 0 and 2 respectively.

We might sensibly wonder how to generalise this. Suppose we have $Y = \mathbb{A}_k^n$ and an element $h \in k[x_1, ..., x_n]$. We can also consider $h + uv \in k[x_1, ..., x_n, u, v]$. It is generally true that matrix factorisations are supported on the critical locus of the superpotential. We can see this since points away from the critical locus are perfect using a Koszul resolution. Therefore they don't support the singularity category of the hypersurface cut out by the superpotential, which as we have seen is equivalent to it's matrix factorisation category. Because of this, we should find the fact that $Crit(h) = Crit(h + uv)$ rather suggestive.

Remark. It is natural to ask exactly what the relationship between matrix factorisations and the derived category of the critical locus of the superpotential. It is tempting to try to generalise the fact above to claim that it is just some kind of

2-periodisation of the critical locus, but this is not true in general. For Morse functions it is true up to a potential Clifford algebra factor (in odd dimensions) and if the superpotential is only Morse-Bott then we must also include a deformation to get from one to the other. These global obstructions are shown in the paper of Teleman [21].

This is alright, however, because there are known cases of Knörrer periodicity where the critical loci of the superpotentials on either side are not the same. This is part of the motivation for this project.

3.1 Knörrer

Take a commutative ring R and an element $h \in R$. The story starts with Knörrer, who originally studied a functor

$$[\mathrm{MF}(R,h)] \xrightarrow{H} [\mathrm{MF}(R[u,v], h+uv)]$$

$$\mathcal{F} \longmapsto \begin{array}{c} (\mathcal{F}_0 \otimes_R R[u,v]_{even}) \oplus (\mathcal{F}_1 \otimes_R R[u,v]_{odd}) \\ \begin{pmatrix} v & f_0 \\ f_1 & -u \end{pmatrix} \Big\uparrow \Big\downarrow \begin{pmatrix} u & f_1 \\ f_0 & -v \end{pmatrix} \\ (\mathcal{F}_0 \otimes_R R[u,v]_{odd}) \oplus (\mathcal{F}_1 \otimes_R R[u,v]_{even}). \end{array}$$

where $1, u^2, uv, v^2, ... \in R[u,v]_{even}$ and $u, v, u^3, u^2v, ... \in R[u,v]_{odd}$ are obtained by considering a grading where R is in degree zero and u, v are elements of degree 1. One can think of this functor as tensoring with the element

$$k[u,v] \underset{v}{\overset{u}{\rightleftarrows}} k[u,v]$$

of $\mathrm{MF}(k[u,v], uv)$. He showed that this functor induced an equivalence of categories and then applied this to show the equivalence for maximal Cohen-Macaulay modules.

Theorem 3.1. *(Knörrer, see [10])*
The functor H induces an equivalence of triangulated categories

$$[\mathrm{MF}(R, h)] \cong [\mathrm{MF}(R[u, v], h + uv)]$$

and equivalently

$$\mathrm{SMCM}(R/(h)) \cong \mathrm{SMCM}(R[u, v]/(h + uv)).$$

3.2 Orlov

Orlov's approach [15] uses the language of derived categories of singularities $D_{sg}(Z)$ defined as the quotient of $D^b(Z)$, the bounded derived category of coherent sheaves on Z, by the full triangulated subcategory of perfect complexes $\mathrm{Perf}(Z)$.

Orlov then proves [16] a version of Knörrer Periodicity; we will first focus on a local picture where $X = Z' \times \mathbb{A}^1_x$ for some smooth scheme Z' and h is a regular function on Z'. This will provide a more direct comparison to our theorem later. Let Y be a smooth hypersurface in Z', cut out by a regular function g, for which $h|_Y$ is non-constant.

$$\begin{array}{ccccc} V(h|_Y) \times \mathbb{A}^1_x & \overset{i}{\hookrightarrow} & V(h + xg) & \hookrightarrow & X \\ \downarrow p & & & \searrow & \downarrow \\ V(h|_Y) & \hookrightarrow & Y & \hookrightarrow & Z' \end{array}$$

Theorem 3.2. *(Orlov [16])*
There is an equivalence of triangulated categories $D_{sg}(V(h|_Y)) \cong D_{sg}(V(h + xg))$ induced by the functor Ri_*p*. Therefore also $\mathrm{MF}(Y, h|_Y) \cong \mathrm{MF}(X, h + xg)$.

In fact, more generally, Orlov sets up with a regular section $s \in H^0(S, \mathcal{E})$ of a vector bundle $\mathcal{E}$ on a noetherian seperated regular scheme S of finite Krull dimension. Then Y is the zero subscheme of s, and Z is the zero subscheme in $\mathcal{E}$ of the section on the canonical line bundle induced by s.

$$\begin{array}{ccccc} \mathcal{N}_{Z/S} & \xhookrightarrow{i} & Z & \hookrightarrow & Tot(\mathcal{E}^\vee) \\ \downarrow p & & & \searrow & \downarrow \\ Y & & \hookrightarrow & & S \end{array}$$

Theorem 3.3. *(Orlov [16])*
There is an equivalence of triangulated categories $D_{sg}(Y) \cong D_{sg}(Z)$ induced by the functor $Ri_* p*$.

3.3 Post-Orlov

Following Orlov there have been several reimaginings of the result in different contexts and perspectives.

The case of $\mathrm{MF}(Y, 0)$ is interesting, because it is very close to simply being the derived category of Y, only we need a k^* action to induce a $\mathbb{Z}$-grading. Once we have that, the previous statements can be adapted to say $D^b(Y) \cong \mathrm{MF}(Y, k^*, 0) \cong \mathrm{MF}(X, k^*, xg) \cong D^{k^*}_{sg}(Z)$ where $Z = V(xg) \subset X$.

Isik sets this up in his paper [9] as follows. Take a vector bundle $\mathcal{E}$ on a smooth variety S. Just like in Orlov's set up we have a section $s \in H^0(S, \mathcal{E})$ which cuts out the zero subscheme Y in S. We also have Z, the zero subscheme of the induced section on X, the total space of the dual vector bundle $\mathcal{E}^\vee$.

Theorem 3.4. *(Isik [9]) Under the conditions above,*

$$D^b(Y) \cong D^{k^*}_{sg}(Z)$$

is an equivalence both of triangulated categories and of their DG enhancements.

The equivalence here is given by resolving the projection of S onto Z, applying a form of (linear) Koszul duality. It is then shown that perfect objects on the Z side correspond to objects supported on S. Taking quotents by these subcategories then gives $D^{k^*}_{sg}(Z)$ on the one hand and the complement of the zero section on the other, which is also a shift of the Kozsul resolution of Y, leaving a category equivalent to $D^b(Y)$.

There is another proof by Shipman [20] of a similar result which sets up the situation much like in Orlov's version. He uses Γ, the double of the k^* action, with character χ. He constructs functors using a distinguished matrix factorisation $\mathcal{K}$, which plays essentially the same role as the $\mathcal{K}$ we will use later.

If X is the total space of the vector bundle $\mathcal{E}^\vee$ and $p : X \to S$ is the projection, then Shipman notes that $p^*\mathcal{E}$ has a canonical section s_S which cuts out the $S \subset X$. There is also a cosection from the pullback of our section s.

$$\mathcal{O}_X \xrightarrow{s_S} p^*\mathcal{E} \xrightarrow{s} \mathcal{O}_X$$

Then the matrix factorisation $\mathcal{K}$ is $(\bigwedge p^*\mathcal{E}^\vee(\chi^{-1}), d)$ with differential $d(-) = s_S \wedge (-) + s \vee (-)$. Importantly, the endomorphisms of $\mathcal{K}$ are quasiisomorphic to $\mathcal{O}_Y$. In fact $\mathcal{K}$ is isomorphic to $i_*\mathcal{O}_{q^{-1}Y}$ as matrix factorisations.

$$\begin{array}{ccc} q^{-1}Y & \xhookrightarrow{i} & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ Y & \hookrightarrow & S \end{array}$$

We also need the regular function Q_s on X which is induced by the section s.

Theorem 3.5. *(Shipman [20])*
The functor i_*p^* induces an equivalence of triangulated categories $D^b(Y) \cong \mathrm{MF}(X, \Gamma, Q_s)$.

A natural further extension then is to replace k^* with more general group actions. This is what Hirano does in the paper [7]. Let $\chi : G \to \mathbb{G}_m$ be a character of a reductive affine algebraic group acting on a regular scheme S, and let h be a χ semi-invariant regular function. Then we can define $\mathrm{MF}(S, \chi, h)$ by considering factorisations F where F_0, F_1 are equivariant, f_0, f_1 are invariant and $f_1 : F_1 \to F_0(\chi)$.

$$Hom(F,G)^{2m} := Hom(F_1, G_1(\chi^m)) \oplus Hom(F_0, G_0(\chi^m)) \quad (1)$$

$$Hom(F,G)^{2m+1} := Hom(F_1, G_0(\chi^m)) \oplus Hom(F_0, G_1(\chi^{m+1})) \quad (2)$$

If we take a G-invariant section s of a G-equivariant, finite rank locally free sheaf $\mathcal{E}$ and Y the zero locus of s, there is an induced action of G on the vector bundle $V(\mathcal{E}(\chi))$, as well as an induced χ semi invariant regular function Q_s.

$$\begin{array}{ccc} V(\mathcal{E}(\chi))|_Y & \overset{i}{\hookrightarrow} & V(\mathcal{E}(\chi)) \\ \downarrow p & & \downarrow q \\ Y & \hookrightarrow & S \end{array}$$

Theorem 3.6. *(Hirano [7])*
For h a χ semi invariant regular function on Z, such that $h|_Y$ is flat, then i_*p^* induces an equivalence of triangulated categories

$$\mathrm{MF}(Y, \chi, h|_Y) \cong \mathrm{MF}(V(\mathcal{E}(\chi)), \chi, q^*h + Q_s).$$

The case we shall focus on is when the total space of the bundle is $X = S \times \mathbb{A}^1_x$, g is a regular function that gives a section of this bundle and $Y = V(g) \subset S$. The theorem then tells us the following.

Theorem 3.7. *(Hirano)*

$$\mathrm{MF}(Y, \chi, h) \cong \mathrm{MF}(X, \chi, h + xg)$$

Subsequent work by Teleman [21] examines global obstructions to Knörrer periodicity. Take a Morse-Bott function W on X, so the critical locus Y is smooth (the function is locally quadratic), then there is n element τ of the super-Brauer group formed from the first two Stiefel-Whitney classes.

Theorem 3.8. *(Teleman [21])* $\mathrm{MF}(X, W)$ *is equivalent to a deformation of the* τ*-twisted differential super-category* $DS^\tau(Y)$*.*

The deformation is trivial when the first neighbourhood is split. The obstruction to this comes about when a tubular neighbourhood of Y in X is not equal to the normal bundle $\mathcal{N}_{Y/X}$.

Remark. Both τ and the deformation here are global obstructions to Knörrer periodicity which are locally trivial. In Section 5 we will find a local obstruction.

4 Comparing Categories

4.1 Set Up

We fix X a regular affine scheme, (f, g) a regular sequence of regular functions. We shall explore an expansion of Knörrer periodicity, where we search for a relationship between the category of matrix factorisations $\mathrm{MF}(X, fg)$ and the derived category of the vanishing locus of the functions $Y = V(f, g) \subset X$. More generally, between $\mathrm{MF}(X, fg + h)$ and $\mathrm{MF}(Y, h)$ under some conditions on h. Essentially we are asking if we can relax the condition maintained in the previous versions that one of these functions should be linear.

We will, of course, study the matrix factorisation

$$\mathcal{K} = \mathcal{O}_X \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{O}_X \in \mathrm{MF}(X, fg),$$

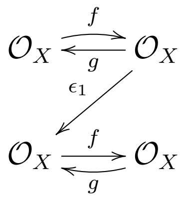
since this is just about the only interesting object we can write down. The first natural question is what the endomorphisms of this obect are.

Proposition 4.1. *Let* $\mathcal{R}_1$ *be the* $\mathbb{Z}/2\mathbb{Z}$*-graded DGA:*

$$\mathcal{O}_X\langle\epsilon_1, \epsilon_2\rangle \Big/ ([\epsilon_1, \epsilon_2] = 1, \epsilon_1^2 = \epsilon_2^2 = 0, \partial\epsilon_1 = f, \partial\epsilon_2 = g)$$

where $|\epsilon_1| = |\epsilon_2| = 1$*.* $\mathcal{R}_1$ *is the endomorphism algebra of* $\mathcal{K}$*.*

Proof. We can see two endomorphisms in degree 1 which identify copies of $\mathcal{O}_X$ along the two diagonals:



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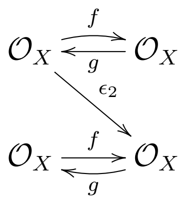
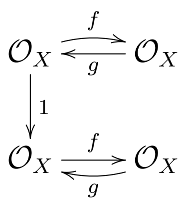
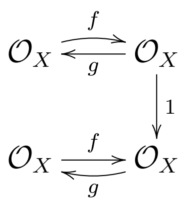


whose differentials are f and g respectively, and whose compositions $\epsilon_1\epsilon_2$ and $\epsilon_2\epsilon_1$ are



and



respectively. So we see the relation $\epsilon_1\epsilon_2 + \epsilon_2\epsilon_1 = [\epsilon_1, \epsilon_2] = 1$.

Now we can look at the endomorphism algebra as given by the object $\mathcal{K}^\vee \otimes_X \mathcal{K}$. Note that $\mathcal{K}^\vee$ is the dual matrix factorisation

$$\mathcal{O}_X \underset{f}{\overset{-g}{\rightleftarrows}} \mathcal{O}_X \ \in \mathrm{MF}(X, -fg).$$

and that we can tensor these CDG modules over the commutative algebra $\mathcal{O}_X$ to obtain a new curved module whose curvature is the sum of the others. We label the generators (as an $\mathcal{O}_X$ module) for $\mathcal{K}$ as a, b so that $\partial a = fb$ and $\partial b = ga$.

Similarly we can give $\mathcal{K}^\vee$ corresponding generators b', a' so that $\partial b' = -ga'$ and $\partial a' = fb'$. Now we carefully chose a generating set for $\mathcal{K}^\vee \otimes \mathcal{K}$: $(b' \otimes a - a' \otimes b, a' \otimes a, b' \otimes b, a' \otimes b)$, which if we identify with $(1, \epsilon_1, \epsilon_2, \epsilon_1\epsilon_2)$ gives an $\mathcal{R}_1$ module isomorphism $\mathcal{K}^\vee \otimes_X \mathcal{K} \cong \mathcal{R}_1$. qed

Remark. Note that $H^0(End(\mathcal{K})) = \mathcal{O}_Y$, whence the interest in this object for building our desired connection. However, $\mathcal{O}_Y$ is not in general quasi-equivalent to $\mathcal{R}_1$, but it is equivalent to a closely related DGA.

Let $\mathcal{R}$ be the $\mathbb{Z}/2\mathbb{Z}$-graded DGA:

$$\mathcal{O}_X[\epsilon_1, \epsilon_2, t] \Big/ ([\epsilon_1, \epsilon_2] = t, \partial\epsilon_1 = f, \partial\epsilon_2 = g)$$

where $|\epsilon_1| = |\epsilon_2| = 1$. From the obvious projection map $\mathcal{R} \to k[t]$ we see that $\mathcal{R}_1$ is a deformation of the fibre $\mathcal{R}_0$. All these fibres when considered as $\mathcal{O}_X$ modules are just the Koszul resolution of Y in X, but the fibre $\mathcal{R}_0$ is the one with the correct product structure to be quasi-equivalent to $\mathcal{O}_Y$ as a DGA.

As $\mathcal{K}$ is both an $\mathcal{R}_1$ module and a matrix factorisation of fg on X, we can use it to construct functors:

$$\mathrm{MF}(\mathcal{R}_1, h) \underset{-\otimes_X \mathcal{K}}{\overset{\mathrm{Hom}_{\mathcal{R}_1}(\mathcal{K}, -)}{\rightleftarrows}} \mathrm{MF}(X, h + fg)$$

.

We would like some kind of equivalence here, so that we can then work our way through the deformation given by $\mathcal{R} \to \mathbb{A}^1_t$ to $\mathcal{R}_0$ and therefore Y.

4.2 The endofunctors

Proposition 4.2. *The composition of these functors $Hom_{\mathcal{R}_1}(\mathcal{K}, -) \otimes_X \mathcal{K}$ is the identity endofunctor on* $\mathrm{MF}(\mathcal{R}_1, h)$.

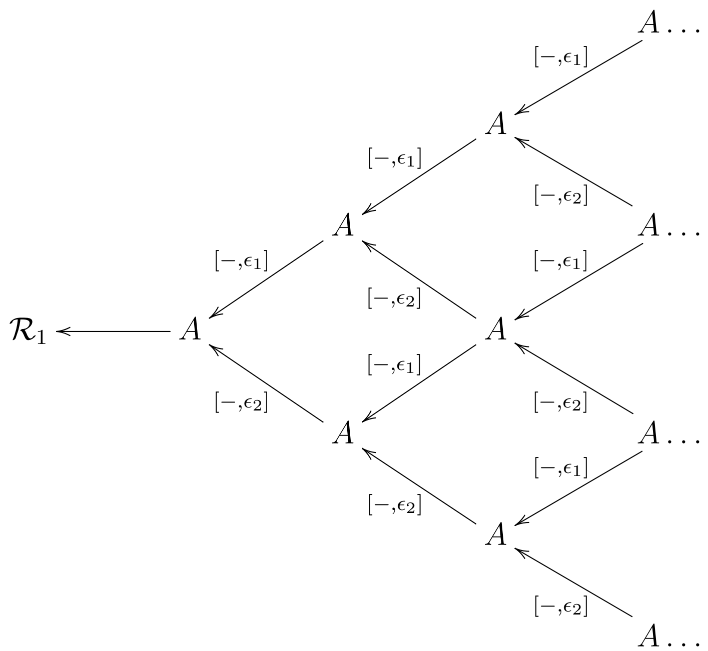
Proof. Take a matrix factorisation M belonging to $\mathrm{MF}(\mathcal{R}_1, h)$. If we apply both functors we are left with the object $\mathrm{Hom}_{\mathcal{R}_1}(\mathcal{K}, M) \otimes_X \mathcal{K}$. Which we can rewrite using hom-tensor adjunction as

$$\mathrm{Hom}_X(\mathcal{K}^\vee, \mathrm{Hom}_{\mathcal{R}_1}(\mathcal{K}, M)) = \mathrm{Hom}_{\mathcal{R}_1}(\mathcal{K}^\vee \otimes_X \mathcal{K}, M) = \mathrm{Hom}_{\mathcal{R}_1}(\mathcal{R}_1, M) = M.$$

Thus, the composition functor is given by homomorphisms from the module $\mathcal{R}_1$ so must be the identity functor on $\mathrm{MF}(\mathcal{R}_1, h)$.

qed

Now, the other composition is significantly more complicated. To understand the functor, we will need to find an $\mathcal{R}_1$ free resolution of the module $\mathcal{K}$, for which we shall use the following complex over the DG algebra $A = \mathcal{R}_1 \otimes_X \mathcal{R}_1$.



Proposition 4.3. *This is a chain complex, is exact, and is therefore a free resolution of $\mathcal{R}_1$ as an A module.*

Proof. First of all, quotienting $A = \mathcal{R}_1 \otimes_X \mathcal{R}_1$ by the image of $[-, \epsilon_1]$ and $[-, \epsilon_2]$ corresponds to identifying the right and left copies of $\mathcal{R}_1$ via the $\mathcal{O}_X$-linear $\mathcal{R}_1$-

module isomorphism from $\mathcal{R}_1$ considered as a left module to a right module. It reverses the order of multiplication and changes signs in odd degree:

$$1, \epsilon_1, \epsilon_2, \epsilon_1\epsilon_2 \mapsto 1, -\epsilon_1, -\epsilon_2, \epsilon_2\epsilon_1.$$

So taking A modulo the images of $[-, \epsilon_1]$ and $[-, \epsilon_2]$ just leaves a single copy of the bimodule $\mathcal{R}_1$. Now for the rest of the complex.

Let $\alpha \in A$, then we can use the Jacobi Identity to check the compositions

$$\begin{aligned}
0 &= (-1)^{|\alpha||\epsilon_2|}[[\alpha, \epsilon_1], \epsilon_2] + (-1)^{|\epsilon_1||\alpha|}[[\epsilon_1, \epsilon_2], \alpha] + (-1)^{|\epsilon_2||\epsilon_1|}[[\epsilon_2, \alpha], \epsilon_1] \\
&= (-1)^{|\alpha|}[[\alpha, \epsilon_1], \epsilon_2] - [[\epsilon_2, \alpha], \epsilon_1] \\
&= (-1)^{|\alpha|}[[\alpha, \epsilon_1], \epsilon_2] - [-(-1)^{|\alpha||\epsilon_2|}[\epsilon_2, \alpha], \epsilon_1] \\
&= (-1)^{|\alpha|}([[\alpha, \epsilon_1], \epsilon_2] + [[\alpha, \epsilon_2], \epsilon_1])
\end{aligned}$$

where we used that $|\epsilon_1| = |\epsilon_2| = 1$, the fact that $[\epsilon_1, \epsilon_2] = 1$ by the definition of $\mathcal{R}_1$ and therefore $[[\epsilon_1, \epsilon_2], \alpha] = 0$ and the super-skew symmetry rule. We see the composition $[-, \epsilon_1] \circ [-, \epsilon_2] + [-, \epsilon_2] \circ [-, \epsilon_1]$ is zero. Using the same calculations for the other compositions shows that

$$0 = [[\alpha, \epsilon_1], \epsilon_1] + [[\alpha, \epsilon_1], \epsilon_1]$$

and

$$0 = [[\alpha, \epsilon_2], \epsilon_2] + [[\alpha, \epsilon_2], \epsilon_2]$$

so that $[-, \epsilon_1] \circ [-, \epsilon_1] = 0$ and $[-, \epsilon_2] \circ [-, \epsilon_2] = 0$, therefore we have a chain complex.

To see exactness, we can view our complex as the tensor product (over A) of the two complexes

$$A \xleftarrow{[-,\epsilon_1]} A \xleftarrow{[-,\epsilon_1]} A \xleftarrow{[-,\epsilon_1]} A \xleftarrow{[-,\epsilon_1]} A \xleftarrow{[-,\epsilon_1]} \ldots$$

and

$$A \xleftarrow{[-,\epsilon_2]} A \xleftarrow{[-,\epsilon_2]} A \xleftarrow{[-,\epsilon_2]} A \xleftarrow{[-,\epsilon_2]} A \xleftarrow{[-,\epsilon_2]} \ldots$$

which are exact. We can see this simply by observing that A is free over $\mathcal{O}_X$ and can be split into 16 summands such that, for each of $[-, \epsilon_1]$ and $[-, \epsilon_2]$, 8 are in the kernel and the same 8 are the image.

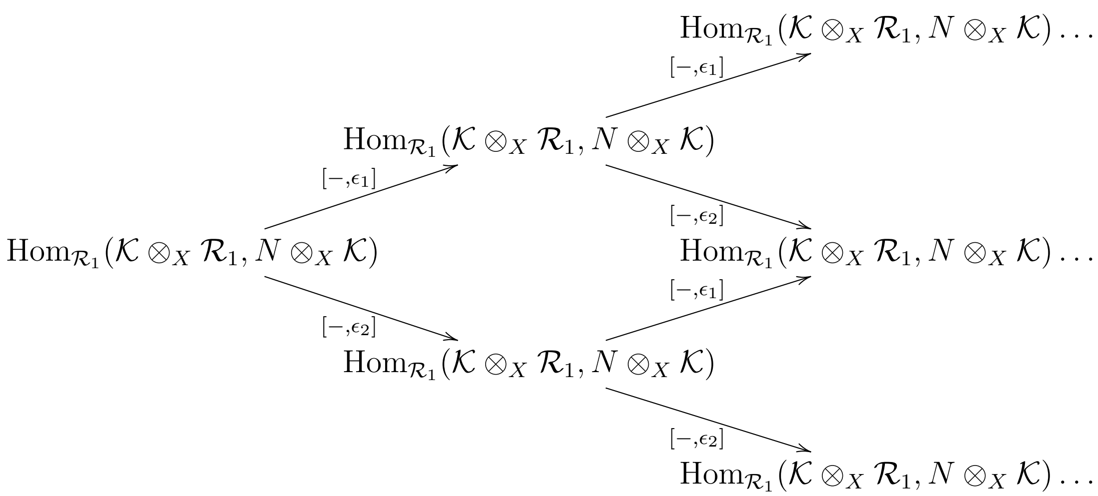
The tensor product functor preserves exactness here since the complexes are made up of free A modules which are therefore flat. So the entire complex above is indeed a A free resolution of $\mathcal{R}_1$.

qed

Proposition 4.4. *The composition of functors* $Hom_{\mathcal{R}_1}(\mathcal{K}, - \otimes_X \mathcal{K})$ *that gives an endofunctor on* $\mathrm{MF}(X, h + fg)$ *is completion with respect to* f *and* g.

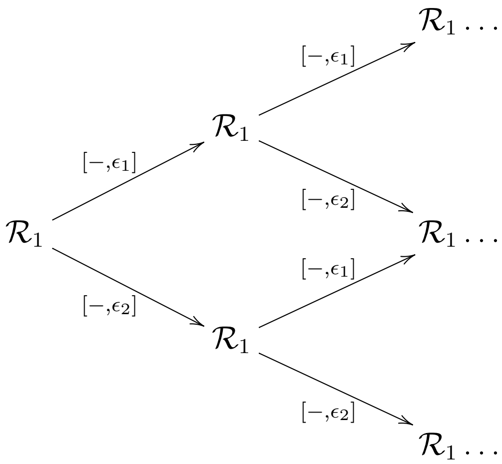
Proof. Take some $N \in \mathrm{MF}(X, h + fg)$, and apply both functors to get the object $\mathrm{Hom}_{\mathcal{R}_1}(\mathcal{K}, N \otimes_X \mathcal{K})$.

We can now apply $\mathcal{K} \otimes_{\mathcal{R}_1} -$ to the complex in proposition 4.3 to obtain an $\mathcal{R}_1$-free resolution of $\mathcal{K}$. This allows us to calculate $\mathrm{Hom}_{\mathcal{R}_1}(\mathcal{K}, N \otimes_X \mathcal{K})$ using the following chain complex which we consider as an inverse limit of the inverse system of finite quotients:

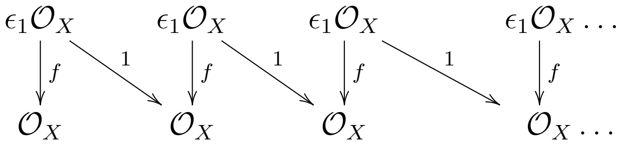


where each term is $\mathrm{Hom}_{\mathcal{R}_1}(\mathcal{K} \otimes_X \mathcal{R}_1, N \otimes_X \mathcal{K})$, which, using Hom-Tensor adjunction, is $\mathrm{Hom}_X(\mathcal{K}, \mathrm{Hom}_{\mathcal{R}_1}(\mathcal{R}_1, N \otimes_X \mathcal{K})) = \mathrm{Hom}_X(\mathcal{K}, N \otimes_X \mathcal{K}) = N \otimes_X \mathcal{K} \otimes_X$

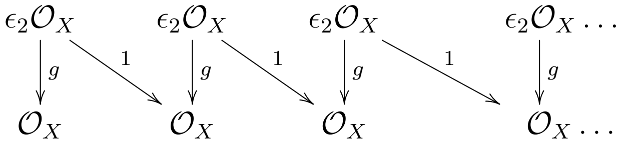
$\mathcal{K}^\vee = N \otimes_X \mathcal{R}_1$. So our composition of functors is completed tensor product with the following element of $D^b(X)$.



Since we are interested in this object only as a complex of $\mathcal{O}_X$ modules, we can divide it into the completed tensor product of two complexes



and



which have cohomology $\mathcal{O}_X[[f]]$ and $\mathcal{O}_X[[g]]$ respectively. Since in the inverse system of quotients each map is actually a surjection, we trivially have the Mittag-Leffler condition meaning the inverse limit functor is exact and therefore commutes with taking cohomology. Using Proposition 2.28 since these complexes are free and therefore flat over X, the cohomology of the direct product complex above is then the completed tensor product of the cohomologies of the two component complexes $\mathcal{O}_X[[f]]\hat{\otimes}\mathcal{O}_X[[g]] = \mathcal{O}_X[[f,g]]$ (all complexes here are inverse limits after applying $\mathrm{Hom}_X(\mathcal{K} \otimes_{\mathcal{R}_1} -, N \otimes_X \mathcal{K})$, so we can write the finite quotients of the larger complex as the tensor product of quotients of the smaller complexes and then pass to the limit). Therefore our object N is actually mapped to $N \otimes_X \mathcal{O}_X[[f,g]]$.

qed

4.3 An Equivalence

Denote by $\widehat{X}_{f,g}$ the completion of X with respect to f and g, i.e. $Spec(\mathcal{O}_X[[f,g]])$. We will complete $\mathcal{R}_1$ similarly to $\widehat{\mathcal{R}_1}$, and consider the completed categories of matrix factorisations which inherit the functors $\mathrm{Hom}_{\mathcal{R}_1}(\mathcal{K}, -)$ and $- \otimes_X \mathcal{K}$ as limits of analagous functors involving finite truncations of $\mathcal{K}$.

Theorem 4.5. *There is an equivalence of categories induced by the two functors.*

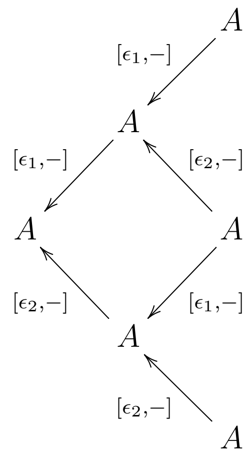
$$\widehat{\mathrm{MF}}(\widehat{\mathcal{R}_1}, h) \underset{-\otimes_X \mathcal{K}}{\overset{Hom_{\mathcal{R}_1}(\mathcal{K},-)}{\rightleftarrows}} \widehat{\mathrm{MF}}(\widehat{X}_{f,g}, h + fg) \quad (\cong)$$

Proof. Using propositions 4.2 and 4.4 we can establish a commuting triangle of functors.

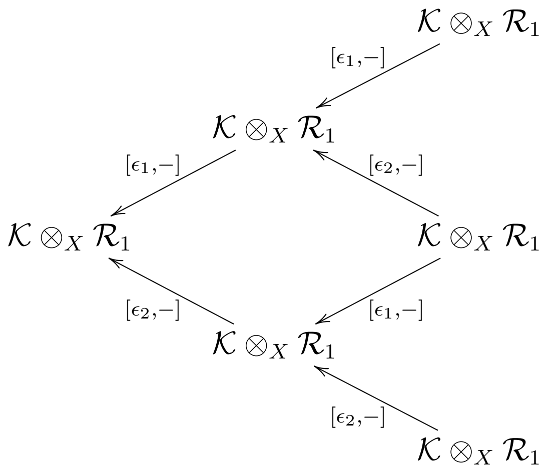
$$\begin{array}{ccc}
\mathrm{MF}(\mathcal{R}_1, h) & \xleftarrow{-\otimes_X \mathcal{K}} & \mathrm{MF}(X, h+fg) \\
\downarrow \rho & \searrow^{\mathrm{Hom}_{\mathcal{R}_1}(\mathcal{K},-)} & \downarrow \rho \\
\widehat{\mathrm{MF}}(\widehat{\mathcal{R}_1}, h) & & \widehat{\mathrm{MF}}(\widehat{X}_{f,g}, h+fg)
\end{array}$$

In order to construct functors between the completed categories, we use truncations the resolution of $\mathcal{R}_1$ from 4.3 and tensor these with $\mathcal{K}$ to obtain truncations of a resolution of $\mathcal{K}$. For example,

$\mathcal{R}_1^3 =$



$\mathcal{K}^3 =$



Now, let I be the ideal $(f, g) \subset \mathcal{O}_X$ (or by abuse of notation in $\mathcal{R}_1$ too). We use these truncations to construct functors on the inverse system

$$\mathrm{MF}(\mathcal{R}_1/I^n, h) \underset{-\otimes_X \mathcal{K}^n}{\overset{\mathrm{Hom}_{\mathcal{R}_1}(\mathcal{K}^n, -)}{\rightleftarrows}} \mathrm{MF}(\mathcal{O}_X/I^n, fg + h)$$

Since we use finite truncations, the functors send finitely generated matrix factorisations to finitely generated CDG modules and vice-versa. We can now take the limit of this entire diagram along the inverse system to induce functors on the completed categories for which the proofs in the previous section still apply, so that both compositions $- \otimes \widehat{\mathcal{R}_1}$ and $- \otimes \mathcal{O}_X[[f,g]]$ are now the identity. This gives an equivalence of categories. qed

Remark. Although we deal only with the affine case here, reading Orlov's paper [16] should make clear how we can relax this restriction by considering sections of bundles instead of regular functions.

5 Converse to Knörrer Periodicity

So we have a Knörrer periodicity type equivalence (after completion) between matrix factorisations (or derived categories of CDG modules) on $(\mathcal{O}_X, h + fg)$ and $(\mathcal{R}_1, h)$. However, in order to compare to previous work and to obtain more geometric relevance we need to pass via the deformation to $\mathcal{R}_1$ from $\mathcal{R}_0$ and then the quasi-equivalence to $\mathcal{O}_Y$ (all over the completion of X w.r.t. f and g).

Conjecture 5.1. *There is an equivalence of categories* $\widehat{\mathrm{MF}}(\widehat{\mathcal{R}_0}, h) \cong \widehat{\mathrm{MF}}(Y, h)$.

$\widehat{\mathcal{R}_0}$ is just the Koszul resolution of the subspace $Y = V(f,g) \subset \widehat{X}_{f,g}$, and so there is a homomorphism of DG algebras to $\mathcal{O}_Y$ which gives a quasi-isomorphism since it does not affect the cohomology $H^n(\widehat{\mathcal{R}_0}) \cong H^n(Y)$. It is well known that such quasi-isomorphisms induce equivalences of derived categories. The argument is that for any $\widehat{\mathcal{R}_0}$ DG module M, $H^n(M) \cong H^n(M \otimes^L_{\widehat{\mathcal{R}_0}} \mathcal{O}_Y)$ and therefore that the endofunctor $- \otimes^L_{\widehat{\mathcal{R}_0}} \mathcal{O}_Y : D^b(\widehat{\mathcal{R}_0}) \to D^b(\widehat{\mathcal{R}_0})$ is fully faithful and clearly has no kernel. This functor is really the composition of $- \otimes^L_{\widehat{\mathcal{R}_0}} \mathcal{O}_Y : D^b(\widehat{\mathcal{R}_0}) \to D^b(Y)$ with the functor induced by the resolution map from $\widehat{\mathcal{R}_0} \to \mathcal{O}_Y$. Similarly, we can compose the functors the other way to obtain an endofunctor on $D^b(Y)$ which is fully faithful and has no kernel. This proves that these functors are both equivalences.

Regarding our completed categories, we can also take a different approach. Let I be the ideal (f,g) in $\mathcal{R}_0$, then our completed category $\widehat{\mathrm{MF}}(\widehat{\mathcal{R}_0}, 0)$ is the limit of the categories $\mathrm{MF}(\mathcal{R}_0/I^n, 0)$. Each of these categories contains only finitely generated CDG modules.

$$\mathrm{MF}(\mathcal{R}_0/I^n, h) \underset{-\otimes_{\mathcal{R}_0/I^n}\mathcal{O}_Y}{\overset{\mathrm{Hom}_{\mathcal{R}_0/I^n}(\mathcal{O}_Y,-)}{\rightleftarrows}} \mathrm{MF}(Y, h)$$

Taking the limit over the invese system gives us functors

$$\widehat{\mathrm{MF}}(\widehat{\mathcal{R}_0}, h) \underset{-\otimes_{\widehat{\mathcal{R}_0}}\mathcal{O}_Y}{\overset{\mathrm{Hom}_{\widehat{\mathcal{R}_0}}(\mathcal{O}_Y,-)}{\rightleftarrows}} \mathrm{MF}(Y, h)$$

Investigating the kernel of the compositions of these functors and take the limit over the inverse system gives the diagonal object, meaning we should get an equivalence.

Remark. These arguments work for ordinary derived categories, however they do not generalise to absolute derived categories, where the acyclic subcategory is defined similarly to how we defined it for matrix factorisations. For example, the matrix factorisation $\mathbb{C}[x] \underset{x}{\overset{x}{\rightleftarrows}} \mathbb{C}[x]$ in $\mathrm{MF}(\mathbb{C}[x]/x^2, 0)$ is acyclic in the usual derived category and so quasi-isomorphic to the zero object but it is non-zero in our definition of matrix factorisations which coincides with the absolute derived category instead. Unfortunately when dealing with curved modules there is no analogue to the usual derived category since that version of 'acyclicity' breaks down.

The author is currently unaware of a general result that applies to this specific case in the curved setting, however we conjecture that for the two step Koszul resolution of a complete intersection that the desired equivalence of matrix factorisation categories does indeed hold: $\widehat{\mathrm{MF}}(\widehat{\mathcal{R}_0}, h) \cong \mathrm{MF}(Y, h)$. In any case it is undoubtable that these categories will share a very close relationship.

5.1 Deformation

Suppose that we have some case where Knörrer periodicity holds: a smooth scheme X and a regular sequence $f, g, h \in \mathcal{O}_X$ such that if $Y = V(f, g) \subset X$ then $\mathrm{MF}(Y, h) \cong \mathrm{MF}(X, fg + h)$. We choose a point in X that lies along Y and pass to a formal neighbourhood of that point. We replace X, and $\mathcal{R}_t$ with their completions (omitting the notation) and also complete all our matrix factorisation categories. This assumption will carry for the rest of this chapter.

Around this point, since X is smooth, the formal neighbourhood has an isomorphism to $Spec(k[[x_1, ..., x_n]])$ where n is the dimension of X. We then abuse our notation and consider $f, g, h \in k[[x_1, ..., x_n]]$ and $Y = V(f, g) \subset Spec(k[[x_1, ..., x_n]])$. It follows that $\widehat{\mathrm{MF}}(Y, h) \cong \widehat{\mathrm{MF}}(k[[x_1, ..., x_n]], fg + h)$. Note that $k[[x_1, ..., x_n]]$ is already complete with respect to f and g.

Then, since we assume $\widehat{\mathrm{MF}}(Y, h) \cong \widehat{\mathrm{MF}}(\mathcal{R}_0, h)$, using Theorem 1.1, it follows for

$$\mathcal{R}_t = k[[x_1, ..., x_n]][\epsilon_1, \epsilon_2] \Big/ ([\epsilon_1, \epsilon_2] = t, \partial\epsilon_1 = f, \partial\epsilon_2 = g)$$

that $\widehat{\mathrm{MF}}(\mathcal{R}_0, h) \cong \widehat{\mathrm{MF}}(\mathcal{R}_1, h)$.

So we have a non-formal deformation $\widehat{\mathrm{MF}}(\mathcal{R}_0, h) \rightsquigarrow \widehat{\mathrm{MF}}(\mathcal{R}_1, h)$ for which the endpoints are equal when Knörrer periodicity holds. In this case we should expect the deformation to be trivial, however it is possible to think of rare cases where a nontrivial deformation produces isomorphic fibres over 0 and 1. Pathological situations aside though, it behoves us to study this deformation further. Our main tool here is Hochschild cohomology, however this tells us about formal deformations, so we can only really study our deformation locally and to finite order.

We use a version of the Hochschild-Kostant-Rosenberg theorem 2.33 for curved differential graded algebras. [1]

[1] While we can only currently find references for the differential graded algebra and curved algebra cases separately, we can see no reason why the arguments should not work for the

Conjecture 5.2. *Let (A, d, h) be a smooth curved differential graded algebra, $X = Spec(A)$ with associated vector fields Q_d and Q_h. Then we can compute the Hochschild cohomology of the matrix factorisation category as polyvector fields:*

$$HH^*(\mathrm{MF}(A, d, h)) \cong H^*(\wedge T_X, \mathcal{L}_{Q_d+Q_h}).$$

In our case this should imply that

$$HH^*(\widehat{\mathrm{MF}}(\mathcal{R}_0, h)) \cong H^*(X, (\bigwedge \mathcal{T}_X[[\partial_{\epsilon_1}, \partial_{\epsilon_2}]][\epsilon_1, \epsilon_2], \mathcal{L}_Q))$$

where $Q = f\partial_{\epsilon_1} + g\partial_{\epsilon_2} + h$ is the vector field coming from our differential and $\mathcal{L}_Q$ is the Lie derivative. For our purposes we are specifically just taking an $\mathcal{O}_X$-free CDGA (whose Hochschild cohomology will give the polyvector field) and deforming the superpotential and differential.

Example 5.3. (Isolated Singularities) Suppose we want to compute the Hochschild cohomology of $\mathrm{MF}(A, h)$ with h cutting out an isolated hypersurface singularity. Then we have a Koszul resolution of the critical locus given by the partial derivatives of h. This means when computing the cohomology using this resolution that in degree 1 we have the cokernel of these functions (and nothing in higher degrees) which is the Jacobian algebra (see [2]). In this case we can construct a converse to Knörrer periodicity using the Mather-Yau theorem [13]. If $\mathrm{MF}(A, h) \cong \mathrm{MF}(A, h')$ then $HH^*(\mathrm{MF}(A, h)) \cong HH^*(\mathrm{MF}(A, h'))$, therefore $Jac(h) \cong Jac(h')$ and so since the Jacobian algebra dictates the singularity type for isolated hypersurface singularities, this dictates h', given h, up to a nondegenerate quadratic form.

Since our deformation is taking the relation $[\epsilon_1, \epsilon_2] = 0$ in $\mathcal{R}_0$ and making it nonzero: $[\epsilon_1, \epsilon_2] = 1$, the class of polyvector fields representing the infinitesimal first order deformation in this direction is $[\partial_{\epsilon_1}\partial_{\epsilon_2}]$.

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combination. Proof of this is beyond our scope, however, but the reader can see similar results in Efimov [3], Guan-Holstein-Lazarev [6], Lin-Pomerleano [11], also section 3.3 in Nordstrom [14] as well as Polishchuk-Positselski [18], Preygel [19] and Toen [23].

The differential on our polyvector fields acts as follows:

$$\partial_{x_i} \longmapsto f_{x_i}\partial_{\epsilon_1} + g_{x_i}\partial_{\epsilon_2} + h_{x_i} \qquad \epsilon_1 \longmapsto f \qquad \epsilon_2 \longmapsto g$$

and extends linearly over $\mathcal{O}_X[\partial_{\epsilon_1}, \partial_{\epsilon_2}]$.

It is interesting to consider when the Hochschild cohomology class corresponding to our deformation is trivial to first order, i.e. when $[\partial_{\epsilon_1}\partial_{\epsilon_2}] = 0$, since we can expect this to be closely related to when $\widehat{\mathrm{MF}}(\mathcal{R}_0, h) \cong \widehat{\mathrm{MF}}(\mathcal{R}_1, h)$. However, neither direction is a direct implication.

There is one case where the class vanishes for trivial reasons, that is when h does not have a critical point at the origin. We can see that in this case all categories $\widehat{\mathrm{MF}}(\mathcal{R}_t, h)$ are trivial. Let us see what happens in the Hochschild cohomology.

Proposition 5.4. *If the origin is not a critical point for the function h, then $[\partial_{\epsilon_1}\partial_{\epsilon_2}] = 0$.*

Proof. If we are not a critical point, then there is some partial derivative h_{x_i} which does not vanish at the origin. In this case

$$\sum_{m=0}^{\inf}\sum_{j=0}^{m} \frac{\partial_{x_i}}{h_{x_i}^{m+1}} f_{x_i}^k g_{x_i}^{m-j+1} \partial_{\epsilon_1}^{j+1} \partial_{\epsilon_2}^{m-j+1} \mapsto \partial_{\epsilon_1}\partial_{\epsilon_2}.$$

So the class $[\partial_{\epsilon_1}\partial_{\epsilon_2}]$ is trivial. qed

Remark. Note that this is not necessarily a finite sum, so it is important that we are considering direct product complexes rather than direct sums.

Now in the more interesting case where our categories are not trivial, what can we glean from the vanishing of this cohomology class?

Proposition 5.5. *If we are at a critical point of h and $[\partial_{\epsilon_1}\partial_{\epsilon_2}] = 0$ then either $\nabla f \neq 0$ or $\nabla g \neq 0$.*

Proof. There is a bi-grading that we can track simply by counting '∂'s and 'ϵ's, with the former coming from the differential structure and the latter from the grading on $\mathcal{R}_0$. For example the degree of $\epsilon_1\partial_{\epsilon_2}$ is $(1,2)$. We can see that at the origin, because the h_{x_i} terms all vanish, the Lie/Hochschild differential preserves the first grading (i.e. the number of '∂'s is fixed).
Therefore we consider the classes with the same ∂-degree as the class $[\partial_{\epsilon_1}\partial_{\epsilon_2}]$, wich are $\mathcal{O}_X$-linear combinations of:

$$\begin{aligned}
\epsilon_1\partial_{\epsilon_1}\partial_{\epsilon_2} &\mapsto f\partial_{\epsilon_1}\partial_{\epsilon_2} \\
\epsilon_2\partial_{\epsilon_1}\partial_{\epsilon_2} &\mapsto g\partial_{\epsilon_1}\partial_{\epsilon_2} \\
\partial_{x_i}\partial_{\epsilon_1} &\mapsto f_{x_i}(\partial_{\epsilon_1})^2 + g_{x_i}\partial_{\epsilon_1}\partial_{\epsilon_2} + h_{x_i}\partial_{\epsilon_1} \\
\partial_{x_i}\partial_{\epsilon_2} &\mapsto -f_{x_i}\partial_{\epsilon_1}\partial_{\epsilon_2} + g_{x_i}(\partial_{\epsilon_2})^2 + h_{x_i}\partial_{\epsilon_2}.
\end{aligned}$$

Our class is trivial if and only if it is in the image of the differential. We take a general object in the stratum of interest. At the origin f, g and all of the partial derivatives of h are 0. Restricting to the origin then, for any $\bar{\alpha}, \bar{\beta} \in k[[x_1, ..., x_n]]^n$,

$$\bar{\alpha} \cdot \partial\bar{x}\partial_{\epsilon_1} + \bar{\beta} \cdot \partial\bar{x}\partial_{\epsilon_2} \mapsto \bar{\alpha} \cdot \nabla f(\partial_{\epsilon_1})^2 + (\bar{\alpha} \cdot \nabla g + \bar{\beta} \cdot \nabla f)\partial_{\epsilon_1}\partial_{\epsilon_2} + \bar{\beta} \cdot \nabla g(\partial_{\epsilon_2})^2.$$

Therefore $[\partial_{\epsilon_1}\partial_{\epsilon_2}] = 0$ can happen only if one of the $\partial_{\epsilon_1}\partial_{\epsilon_2}$ coefficients is non-zero, meaning either ∇f or ∇g is non-zero. qed

Example 5.6. Let $f = x + y + y^2 + yz^2$, $g = yz - z^4$ and $h = y^2 - z^2$. We can see that our Hochschild cohomology class is the image of $\partial_x\partial_{\epsilon_2}$, so $[\partial_{\epsilon_1}\partial_{\epsilon_2}] = 0$.

Example 5.7. Conversely, supposing f and g no terms of degree 1, then their partial derivatives will vanish, so as long as we are at a critical point of h, it follows from Proposition 5.5 that $[\partial_{\epsilon_1}\partial_{\epsilon_2}] \neq 0$.

Remark. The contrapositive of the proposition yields a useful heuristic: if $\nabla f = \nabla g = 0$ at a point then $[\partial_{\epsilon_1}\partial_{\epsilon_2}] \neq 0$ so we expect Knörrer periodicity not to hold.

Remark. We will always assume without loss of generality that it is f whose gradient is nonzero. This means then that there is a coordinate change of $k[[x_1, ..., x_n]]$ such that $f = x_1$. If we could somehow argue now that $g, h \in k[[x_2, ..., x_n]]$ then we would be back in the case where Knörrer periodicity is already proven by Orlov. We shall use the moniker 'semilinear' as a shorthand to refer to this case.

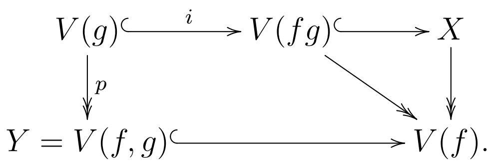
Definition 5.8. We shall call a triple of functions $(f, g, h) \in k[[x_1, x_2, ..., x_n]]$ 'semilinear' if we can choose the co-ordinates x_i such that $f = x_1$ and $g, h \in k[[x_2, ..., x_n]]$ are regular functions.

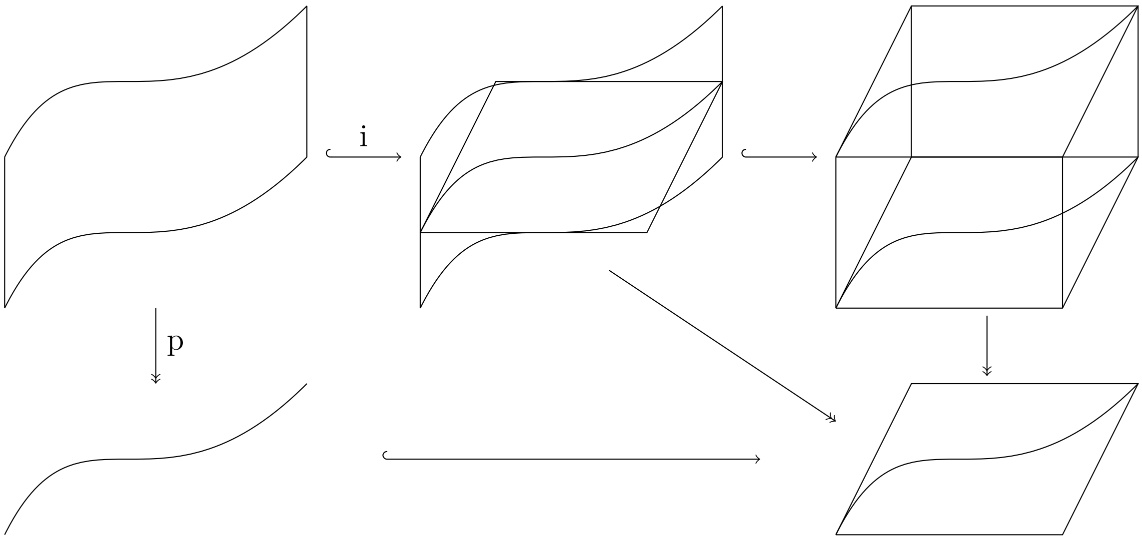
Remark. Note that if $Y = V(f, g)$ is smooth, then ∇f and ∇g are non-zero and linearly independent, $Y = Crit(f, g)$, and we can choose coordinates on our formal neighbourhood such that $f = x_1$ and $g = x_2$. In this case we are clearly semilinear so Knörrer periodicity holds. This means in the case of smooth Y that obstructions happen only at the global level, see [21]. We will be interested in the case where Y is singular, where obstructions to the theorem can arise locally.

5.2 The Semi-Linear Case

Now let us recast Orlov's argument in the terminology used here.

Consider the matrix factorisation category $\mathrm{MF}(X, fg + h)$. Suppose $Y = V(f, g) \subset X = Spec(k[[x_1, ..., x_n]])$ and (f, g, h) is semilinear. Then Orlov's geometric picture is the following commuting diagram:





He proves that the functor Ri_*p^* induces an equivalence between $\mathrm{MF}(X, fg+h) \cong \mathrm{MF}(Y, h)$.

If we look at the algebraic side of this picture,

$$\begin{array}{ccccc} k[[f]]\hat{\otimes}_k\mathcal{O}_Y \cong \mathcal{O}_{V(g)} & \twoheadleftarrow & \mathcal{O}_{V(fg)} & \twoheadleftarrow & \mathcal{O}_X \\ \uparrow & & \nwarrow & & \uparrow \\ \mathcal{O}_Y & \twoheadleftarrow & & & \mathcal{O}_{V(f)} \end{array}$$

we can see that the key point that allows us to set up an appropriate functor is that the linearity of f lets us create inclusion functors where there would otherwise only be quotient functors in the opposite direction.

In our language then, the functors in the semilinear case (let us now assume we make some local coordinate change so that $f = x_1$) will come from the following picture

$$\begin{array}{ccc} & & \mathcal{R}_1 \\ & & \uparrow \\ k[[x_2, ..., x_n]]/g & \twoheadleftarrow & S \end{array}$$

where the horizontal arrow is just a quasi-isomorphism between Y and its one step free resolution in $k[[x_2, ..., x_n]]$, S considered as a DG algebra, namely

$$S = k[[x_2, ..., x_n]] \xleftarrow{g} \epsilon_2 k[[x_2, ..., x_n]]$$

or more geometrically

$$\left[\mathcal{O}_{\widehat{\mathbb{A}}^{n-1}} \xrightarrow{g} \mathcal{O}_{\widehat{\mathbb{A}}^{n-1}}\right] \cong \mathcal{O}_Y.$$

This induces a quasi-equivalence of the matrix factorisation categories with the superpotential h, given by the derived pushforward functor. The vertical arrow is given by the inclusion into $\mathcal{R}_1$, which also gives a quasi-equivalence since the cohomology of $\mathcal{R}_1$ considered as a chain complex over X

$$\mathcal{O}_{\widehat{\mathbb{A}}^n} \xrightarrow{(x_1, -g)} \mathcal{O}_{\widehat{\mathbb{A}}^n}^{\oplus 2} \xrightarrow{\binom{g}{x_1}} \mathcal{O}_{\widehat{\mathbb{A}}^n}$$

is just $\mathcal{O}_Y$.

Together these two functors give an equivalence of categories $\widehat{\mathrm{MF}}(Y, h) \cong \widehat{\mathrm{MF}}(\mathcal{R}_1, h)$ which by Theorem 1.1 is equivalent to $\widehat{\mathrm{MF}}(X, fg + h)$, giving our Knörrer periodicity statement.

In fact, since the cohomology of $\mathcal{R}_t$ as a chain complex is independent of t, all fibres of this deformation are trivial so the deformation itself is trivial in the semi-linear case.

Theorem 5.9. *(Knörrer Periodicity) Let $f, g, h \in k[[x_1, ..., x_n]]$ be a semilinear triple, then $\widehat{\mathrm{MF}}(Y, h) \cong \widehat{\mathrm{MF}}(X, fg + h)$. Moreover the deformation from $\mathcal{R}_0$ to $\mathcal{R}_1$ is trivial.*

Remark. The argument presented here is not quite the same as what Orlov does, since we are effectively going around the square the other way and we are composing functors which are themselves already equivalences.

5.3 Approaching a Converse

We know that in the 'semilinear' case that Knörrer periodicity holds. We also know that there is a deformation between the two categories of interest whether or not they are equivalent. We will now make a heuristic argument for why Knörrer periodicity does not have generalisations beyond this semilinear case except in some exceptional examples.

We work in a formal neighbourhood of a point as in the previous section. Suppose first not only that $\widehat{\mathrm{MF}}(\mathcal{R}_0, h) \cong \widehat{\mathrm{MF}}(\mathcal{R}_1, h)$ but also that the deformation $\widehat{\mathrm{MF}}(\mathcal{R}_0, h) \rightsquigarrow \widehat{\mathrm{MF}}(\mathcal{R}_1, h)$ is trivial. It follows that in particular it is trivial to first order and therefore, given the generalisation of HKR which we mentioned previously, the cohomology class $[\partial_{\epsilon_1}\partial_{\epsilon_2}] = 0$. By Proposition 5.5 and the following remark, we can choose coordinates such that $f = x_1$ in $k[[x_1, ..., x_n]]$.

This allows us to consider another deformation. We can write $g_s = g_0 + sx_1g'$ and $h_s = h_0 + sx_1h'$ where $g_0, h_0 \in k[[x_2, ..., x_n]]$ and $g', h' \in k[[x_1, ..., x_n]]$ such that $g = g_1$ and $h = h_1$. The point being that (f, g_0, h_0) is 'semilinear', so our situation is a deformation of the semilinear case.

We combine both deformations in a square given by

$$\mathcal{R}_{t,s} := \mathcal{O}_X[t, s, \epsilon_1, \epsilon_2] \Big/ [\epsilon_1, \epsilon_2] = t, \partial_{\epsilon_1} = x_1, \partial_{\epsilon_2} = g_s$$

so that

$$\begin{array}{ccc} \widehat{\mathrm{MF}}(R_{0,0}, h_0) & \overset{A}{\rightsquigarrow} & \widehat{\mathrm{MF}}(R_{1,0}, h_0) \\ \wr\downarrow B & & \wr\downarrow C \\ \widehat{\mathrm{MF}}(R_{0,1}, h_1) & \overset{D}{\rightsquigarrow} & \widehat{\mathrm{MF}}(R_{1,1}, h_1). \end{array}$$

The deformations going left to right are the ones coming from Knörrer periodicity and the vertical arrows are deforming from the semilinear case to our given one. Consider the deformation on the left. Assuming 5.1 allows us to think of B as passing from $\widehat{\mathrm{MF}}(V(x_1, g_0), h_0)$ to $\widehat{\mathrm{MF}}(V(x_1, g_1), h_1)$. Since we annihilate x_1, $h =$

$h_1 = h_s = h_0$ and $g = g_1 = g_s = g_0$. This implies that B is actually constant.
The deformation at the top, A, is simply Knörrer periodicity for the semilinear case since with 5.1 and Theorem 1.1, we can see it as the deformation from $\mathrm{MF}(Y, h_0)$ to $\widehat{\mathrm{MF}}(X, fg_0 + h_0)$. which we know to be trivial also by our previous discussion.
C is given by deforming the superpotential by $x_1h' + x_1^2g'$. It has a class $[C]$ in the Jacobi ring of the curved differential graded algebra $(\mathcal{R}_{0,1}, h_1)$. The Jacobi ring classifies these deformations to first order up to change of coordinates, and it embeds into $HH^2(\widehat{\mathrm{MF}}(\mathcal{R}_{0,1}, h_1))$.
D is the deformation given to first order by $[\partial_{\epsilon_1}\partial_{\epsilon_2}]$ which we assumed to be trivial in $HH^2(\widehat{\mathrm{MF}}(\mathcal{R}_{0,1}, h_1))$.

Therefore we have a square of deformations where three sides are trivial. This does not imply that the final side, C, is trivial also. However, were it not then we would once again have a deformation between categories which is non-trivial yet circles back to an equivalent category.

So what can we conclude? If we have a version of Knörrer periodicity, at least locally:

$$\widehat{\mathrm{MF}}(Y, h) \cong \widehat{\mathrm{MF}}(X, h + fg)$$

then our equivalence comes from a deformation $\mathcal{R}_t$ which, barring the potential for strange coincidences and exceptions will be itself trivial. In this case we are in a deformation of a semilinear case where Knörrer periodicity is already known. This deformation too we would expect to be trivial. If so, then in particular it is trivial everywhere to first order, so at each value of s the class $[C]$ of the deformation in the Hochschild cohomology is zero. Since the Jacobi ring injects into this cohomology, that means the class of our deformation is zero here also. The Jacobi ring categorises first order deformations up to coordinate changes, so if we could integrate together all these coordinate changes at every point to get from $s = 0$ to $s = 1$ then we could show that we actually are in the semilinear case already.

In summary we need three assumptions: that we don't have a coincidence of non-trivial deformations over the unit interval with equivalent endpoints, the integrability of the coordinate change, and that the quasi-isomorphism from $\mathcal{R}_0$ to Y induces an equivalence of categories. While this is not on the nose a converse to Knörrer periodicity, it is a strongly implied bound on the possible generalisations and should be understood as a warning that cases of the equivalence outside the semilinear we expect at best to be rare and unusual, and may very well not exist at all.

6 Examples

So let us see what we can now say about comparing the categories $\mathrm{MF}(X, fg+h)$ and $\mathrm{MF}(Y, h)$ where $Y = V(f, g) \subset X$ and $f, g \in \mathcal{O}_X$ are a regular sequence of functions on a smooth (affine) scheme and previous assumptions such as Conjecture 5.1 hold.

If Y is smooth, then we can use Teleman's result [21] since $Y = Crit(fg)$ to show that Knörrer periodicity holds up to some potential global obstructions.

Otherwise, we pass to the formal neighbourhood of a point on $Y \subset X$. We choose a critical point of h, otherwise our obstruction vanishes and the categories are trivially equivalent since they both vanish.

Now, if $\nabla f = \nabla g = 0$ at the origin then, by Proposition 5.5, $[\partial_{\epsilon_1}\partial_{\epsilon_2}] \neq 0$ so $\widehat{\mathrm{MF}}(X, fg+h)$ and $\mathrm{MF}(Y, h)$ are related by a non-trivial deformation and we therefore expect them not to be equivalent.

Example 6.1. Let $f = x^2$ and $g = y^2$ with $h = -z^2$. Then our obstruction is nonzero, so we expect not to find an equivalence. Indeed, $\mathrm{MF}(Y, h) \cong D_{sg}(\mathbb{C}[x, y, z]/(x^2, y^2, z^2))$ while $\mathrm{MF}(X, fg+h) \cong D_{sg}(\mathbb{C}[x, y, z]/(x^2y^2 - z^2))$ which we can also see from the critical loci cannot be equivalent (only one is an isolated singularity).

Now, without loss of generality, say $\nabla f \neq 0$ at the origin, then the deformation of $\mathrm{MF}(Y, h)$ to $\widehat{\mathrm{MF}}(X, fg+h)$ is itself deformed from a trivial deformation coming from a related semilinear case.

If there exists a fixed vector $\beta \in k^n$ such that $\beta \cdot \nabla f \neq 0$ but $\beta \cdot \nabla g = 0$, then we can choose coordinates $x_1, ..., x_n$ such that $f = x_1$. We can then transform by $y_1 = \frac{1}{\beta \cdot \nabla f} x_1 = \frac{1}{\beta_1} x_1$ and for $i \geq 2$, $y_i = x_i - \frac{\beta_i}{\beta_1} x_1$ so that

$$\begin{aligned} \frac{\partial g}{\partial y_1} &= \sum_{i=1}^{n} \frac{\partial x_i}{\partial y_1} \frac{\partial g}{\partial x_i} \\ &= \sum_{i=1}^{n} \beta_i g_{x_i} \\ &= \beta \cdot \nabla g \\ &= 0 \end{aligned}$$

Hence this coordinate change puts us in the semilinear case where Knörrer periodicity is known to hold. Otherwise, things are more complicated.

Example 6.2. If $f = x_1$ and $g = 1 - 3x_1 - 9x_2 + x_1^2 + 6x_1x_2 + 9x_2^2$, then we can see that $g_{x_2} = 3g_{x_1}$ so $\beta = (3, -1)$ and we use the coordinate change $y_1 = \frac{1}{3}x_1$ and $y_2 = x_2 - \frac{1}{3}x_1$ which gives us $f = 3y_1$ and $g = 1 - 9y_2 + 9y_2^2$.

Example 6.3. Let $f = x$, $g = xy + y^3$ and $h = yz^2$. In this case we compare $\mathrm{MF}(Y, h) \cong D_{sg}(\mathbb{C}[y, z]/(y^3, yz))$ with $\mathrm{MF}(X, fg + h) \cong D_{sg}(\mathbb{C}[x, y, z]/(x^2y + xy^3 + yz))$. We can look to see if our obstruction vanishes:

$$-\partial_x \partial_{\epsilon_2} \mapsto \partial_{\epsilon_1} \partial_{\epsilon_2} - y\partial_{\epsilon_2}^2$$

We need to see if we can eliminate the y term somehow. The terms we could use in the image of the differential are multiples of f, g and the partial derivatives of g and h, none of which produce y, so our obstruction does not vanish and the deformation is non-trivial. We therefore expect that Knörrer periodicity fails and the categories are not equivalent.

Example 6.4. Let $f = x$, $g = xy + y^2$ and $h = 0$. We compare $D_{sg}(\mathbb{C}[y]/(y^2))$ with $D_{sg}(\mathbb{C}[x,y]/(x^2y + xy^2))$. This time our obstruction vanishes since we have

$$\partial_y \partial_{\epsilon_2} \mapsto 2y\partial_{\epsilon_2}^2$$

so $\partial_{\epsilon_1}\partial_{\epsilon_2}$ is the image of $\frac{1}{2}\partial_y\partial_{\epsilon_2} - \partial_x\partial_{\epsilon_2}$. So the deformation from $\mathcal{R}_0$ to $\mathcal{R}_1$ is trivial to first order at least. The semilinear case which we deform from is $f = x$, $g = y^2$. Let $W_s = x(sxy + y^2)$, then $W_0 = xy^2$ and $W_1 = xy(x + y)$ which clearly gives a non-trivial deformation $\mathcal{R}_{1,s}$. It is trivial to first order, however, since it is represented by x^2y which is zero in the Jacobi ring of W_0 since $x^2y = \frac{x}{2}\partial_y W_0$. In fact it is nontrivial already at second order, so the deformation from the semilinear case is not a coordinate change and we also see that Knörrer periodicity does not hold here.

Example 6.5. Similarly if $f = x$ and $g = x^2 + y^2$ then since

$$-\partial_x\partial_{\epsilon_2} + 2\epsilon_1\partial_{\epsilon_2}^2 \mapsto \partial_{\epsilon_1}\partial_{\epsilon_2},$$

the obstruction in Hochschild cohomology vanishes, Knörrer periodicity again doesnt hold but this time the deformation from the semilinear case is already nontrivial at first order.

Of course, in general if the deformation from the semilinear case is trivial then Knörrer Periodicity does hold since

$$\mathrm{MF}(Y,h) \cong \widehat{\mathrm{MF}}(\mathcal{R}_{0,1},h) \cong \widehat{\mathrm{MF}}(\mathcal{R}_{0,0},h_0) = \widehat{\mathrm{MF}}(\mathcal{R}_{1,0},h_0) \cong \widehat{\mathrm{MF}}(\mathcal{R}_{1,1},h) \cong \widehat{\mathrm{MF}}(X, fg+h).$$

That just leaves the possibility of exceptions where the deformation $\widehat{\mathrm{MF}}(\mathcal{R}_{1,0},h_0) \rightsquigarrow \widehat{\mathrm{MF}}(\mathcal{R}_{1,1},h_1)$ is nontrivial but the categories are equal. These examples are unlikely to exist, so we argue is only reasonable to expect Knörrer periodicity in the semilinear case.

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