Quantum matter backreacting on classical spacetime

By

Andrea Russo

A thesis submitted to

University College London

for the degree of

Doctor of Philosophy

Department of Physics and Astronomy

University College London

September 2024

I, Andrea Russo, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Signed .... Date ....

Quantum matter backreacting on classical spacetime

Andrea Russo

Doctor of Philosophy of Physics

University College London

Prof. Jonathan Oppenheim, Supervisor

Abstract

This work explores the weak gravitational regime and the cosmological implications of a recently proposed unification framework known as classical-quantum (CQ) theory. CQ dynamics attempts to coherently unify quantum physics with classical general relativity by stochastically coupling the two theories.

After an overview of the background material, which includes three different formulations of the CQ framework, this work explores the Newtonian limit of classical general relativity coupled with quantum matter. Due to the backreaction generated by the coupling, we find that the gravitational field diffuses around Poisson's equation while quantum matter decoheres into mass eigenstates. We study the bounds in the coefficients regulating diffusion and decoherence and compare and contrast with previous works on the interaction between a classical Newtonian potential and quantum matter.

Secondly, we focus on deriving the Newtonian limit of classical-quantum Nordström's theory of gravitation to show that a self-consistent scalar theory of gravity leads to the same results obtained from its general relativistic counterpart. This model reinforces the previous results and highlights the consistency of the Newtonian limit with a diffeomorphism-invariant CQ theory of gravity.

Lastly, we delve deeper into the gravitational sector of CQ theories of gravity to explore the consequences of stochasticity on the rotational curves of galaxies. We find evidence that the presence of stochastic noise could act to explain the phenomenology usually attributed to dark matter. A deviation from the expected general relativistic behaviour appears at low accelerations and connects it with cosmological parameters. A dark energy-like effect is expected even if the starting theory does not have a bare cosmological constant. We propose an explanation

and compare our results with tabletop experiments to understand how large-scale diffusion and local noise might relate.

We expect these results to be relevant to future theoretical and experimental tests of the quantum nature of gravity.

Impact Statement

In recent years, attempts to reconcile gravity and quantum theory have broadened to include hybrid theories in which spacetime remains classical while matter and energy are quantised. While known since the nineties, hybrid approaches have gained momentum after the development of the classical-quantum (CQ) formalism of [1]. This program presents itself as a plausible alternative to quantum gravity, proposing a resolution to the long-standing incompatibility of general relativity and quantum physics without the need to quantise gravity. The framework preserves the classical nature of gravity and the quantum nature of matter fields while coupling them coherently. Their interaction results in the classical system undergoing a noisy diffusion process while the quantum system decoheres as if it is being weakly measured. While the theoretical foundations of the framework have been already laid out in [1, 2, 3, 4, 5], including formulations of the dynamics in terms of master equation, stochastic differential equations and path integrals, experimental predictions have been primarily limited to tabletop experiments obtained by directly considering a Newtonian gravitational field [3]. Therefore, results did not originate from taking the limit of the full general-relativistic CQ theory.

This work takes on the task of deriving such a limit from the complete theory and providing testable experimental predictions. Given that, within good approximation, we live in weak gravitational regimes, understanding this context makes it easier to derive and test predicted behaviours. Moreover, the analysis of such regimes can be extended to study the rotational velocity of edge stars in galaxies.

The importance of diffeomorphism invariance when working with theories of gravity is paramount. Hence, this work also derives the weak-field limit from the self-contained manifestly diffeomorphism invariant theory of Nordström gravity, providing evidence that the Newtonian limit is compatible with a diffeomorphism invariant framework. Once the limit has been established, this work explores the effects on galaxies' rotational curves resulting from the underlying noise generated by the hybrid coupling of classical and quantum degrees of freedom. The CQ framework's effects on gravity lead to a deviation from the rotational velocities expected from Einstein's theory. We find evidence that this could reproduce phenomenology usually attributed to dark matter. We use the results to propose an explanation for anomalies in the rotational velocities of stars away from the galactic core.

This work provides the tools and foundations for future studies of the interaction between quantum matter and stochastic spacetimes, especially in weak field regimes. We also hope these techniques will be of broad interest in exploring explanations alternative to dark matter for galactic rotation velocity curves.

List of Publications and Preprints

The work presented in this thesis contains material from the following publications and preprints:

1. Isaac Layton, Jonathan Oppenheim, Andrea Russo and Zachary Weller-Davies. The weak field limit of quantum matter back-reacting on classical spacetime. JHEP 08 (2023) 163 [6]

2. Jonathan Oppenheim, Andrea Russo, Zachary Weller-Davies. Diffeomorphism invariant classical-quantum path integrals for Nordström gravity. Phys.Rev.D 110.024007 [7]

3. Jonathan Oppenheim, Andrea Russo. Anomalous contribution to galactic rotation curves due to stochastic spacetime. arXiv:2402.19459 [8]

Other publications and preprints by the author are:

4. Andrzej Grudka, Jonathan Oppenheim, Andrea Russo, Muhammad Sajjad. Renormalisation of postquantum-classical gravity. arXiv:2402.17844 [9].

5. Muhammad Sajjad, Andrea Russo, Maite Arcos, Andrzej Grudka, Jonathan Oppenheim. A quantum oscillator interacting with a classical oscillator. arXiv:2403.07479 [10].

6. Philipp A. Hoenn, Andrea Russo, Alexander R. H. Smith. Matter relative to quantum hypersurfaces. Phys.Rev.D 109.105011 [11].

7. Juan F. Pedraza, Andrea Russo, Andrew Svesko and Zachary Weller-Davies. Lorentzian threads as 'gatelines' and holographic complexity. Phys.Rev.Lett. 127 (2021) 27, 271602. [12]

8. Juan F. Pedraza, Andrea Russo, Andrew Svesko and Zachary Weller-Davies. Sewing spacetime with Lorentzian threads: complexity and the emergence of time in quantum gravity. JHEP 02 (2022) 093. [13].

9. Juan F. Pedraza, Andrea Russo, Andrew Svesko and Zachary Weller-Davies. Computing spacetime. Honourable mention for the 2022 Essay Competition of the Gravity Research Foundation. Int.J.Mod.Phys.D 31 (2022) 14, 2242010 [14].

Acknowledgments

None of the work in this thesis could have been possible without the incredible number of great people I have been lucky to encounter in my life.

First, I would like to thank my supervisor, Jonathan. Since joining, I have always been treated like a colleague rather than a student. While stepping up to the role was not always straightforward, it taught me much about thinking and acting like a physicist. From him, I learned how to fight and believe in ideas that dare to be radically different, even when most people might strongly disagree. I also learned a great deal about not getting distracted by unnecessary details and staying focused on the core of a concept to really pave the way forward. I will be forever grateful for all the (many hours long) discussions and the funny moments we have experienced together with the rest of the group. We should also remember all the fantastic lunches we had around UCL.

As much as my supervisor, I owe this accomplishment to my academic family and their support:

To Zach, for being like an older brother when I joined the group. Thank you for being so patient in answering hundreds of questions while I was getting to grips with the PhD life and for being a great friend (and an even greater researcher) even when you were on a different continent.

To Andy, for being a great mentor, friend and my favourite academic uncle. I am constantly inspired by your work ethic, paired with your humbleness and ability to help anyone believe in themselves. You have been a second supervisor to all of us; we will be forever grateful for this. When you will have your own PhD students, I hope they will realise how lucky they have been.

To Isaac, for sharing my passion for Japanese animation and being one of the kindest and most lovable friends I ever met in my life. Your insights on CQ and your knowledge of everything quantum were more than appreciated in these years. I know that amazing years are waiting for you, and rest assured I will come to visit you as much as possible.

To Maite, for all the hours of mutual improvised therapy mixed with gossiping. I am so proud of you for everything you have accomplished in these difficult times, and I know that in years to come, only the good memories will remain. Thanks for being a great friend.

To Emanuele, thank you for helping me bring some Italian culture into this foreign research group and for all the coffee breaks. You are one of the best physicists I have met in recent years. Thanks for being an amazing friend and letting me bring out your nerdiest side.

It is difficult to convey how having great friends and colleagues can improve life.

Thanks to Alexander R. H. Smith, Philipp A Höhn, and Juan Pedraza for allowing me to conduct great research on fascinating topics and being so helpful and kind while I was still learning.

Thanks to Henrique Gomes for being a great secondary supervisor and for his precious advice on work, life and this thesis.

Many thanks to Lenny Susskind for mentoring me during my time at Stanford, pushing me to do better than I could have ever imagined, and changing the course of my life.

Un grazie immenso ad Enrico, Giulia, Margherita, Mattia, Nicoló e Oliviero. Questi anni lontano da casa non sarebbero stati minimamente cosí incredibili se non fosse stato per tutte le avventure, serate, risate e feste indimenticabili passate insieme.

Grazie a Dalí, Dino, Elisabetta e Matteo per starmi vicino da una vita ed essere stati come una famiglia per me. Siete una parte di me e non riesco ad immaginare come farei senza di voi.

Grazie ad Arianna, Federico, Giulia, Ilaria, Sara e Vanessa. Anche se a migliaia di chilometri di distanza, siamo riusciti ad avvicinarci invece di allontanarci. Pochi possono vantare amici del genere.

Grazie a Luca e Alessandro per avermi fatto appassionare alla fisica ed essere sempre disponibili a discutere dei miei nuovi progetti.

Infine, i ringraziamenti piú importanti.

A Valeria, per avermi amato ed essermi stata accanto in questi anni, per avermi sempre saputo risollevare il morale, per le nostre mille avventure, per le fantastiche serate e per le milioni di risate che mi hai regalato. Averti al mio fianco mi rende una persona migliore.

Alla mia famiglia, é difficile esprimere la gratitudine immensa che provo per l'amore ed il supporto che mi avete dato in tutti questi anni. Non sarei qua se non fosse per voi. Vi voglio un bene immenso.

Contents

Chapter 1

Introduction

"Insanity is doing the same thing over and over
and expecting different results"
∼ Albert Einstein

On September 21, 1908, a few years after Einstein's first work on special relativity, Hermann Minkowski addressed the audience at the 80th Assembly of German Natural Scientists and Physicians in Cologne [15]. His address began with a statement that perfectly embodied the radical change of view brought by the theory of relativity:

"The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth, space by itself and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

The old ways of the observer-independent "Newtonian" absolute space and time were being abandoned and replaced with the notion of *spacetime*, a new object born from the union of the two [15, 16]. This object, technically a manifold, serves as the theatre where the events of physics occur. Each event described by the laws of nature is associated with a well-defined point in this four-dimensional background, and different reference frames measure intervals of space and time differently depending on their relative motion, the only constant being the speed of light. Furthermore, this new view had a solid tie to experimental physics. Spacetime turned out to also be ideal for describing gravity [17]. The background where physics unfolds is not a static object but a player itself. It can bend and stretch, and it is through these deformations that matter and energy feel the influence of gravity. In 1915, Einstein explained gravity and its effects on matter as a consequence of spacetime deformations in his theory of General Relativity, expressed beautifully through Einstein's field equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{1.1}$$

On the left side of the equality, Einstein's tensor $G_{\mu\nu}$ summarises how spacetime is curved and how this curvature tells matter and energy how to move. On the right side, the stress-energy tensor $T_{\mu\nu}$ summarises the distribution of matter and energy, and it explains how this distribution bends and deforms spacetime.

However, the 20th century did not only bring the revolution of relativity and spacetime but also an entirely new framework of quantum mechanics. Originating from Planck's hypothesis of the absorption and emission of light as *quanta* [18], then proved by Einstein through the photoelectric effect [19], quantum mechanics forced physicists to rethink the world as composed of discrete entities. Energy is transmitted in quanta, but atoms are also composed of subatomic particles in bound states with discrete energy levels [20]. The laws of physics had to be revisited in this new light, where the deterministic evolution of systems was replaced by probabilistic evolution and uncertainty [21], matter and energy exhibit both particles and wave nature [22] and physical states, known as *quantum states*, are best described by a probability amplitude known as *wavefunction* [23].

However, the framework of quantum mechanics and the theory of relativity had to be reconciled. Quantum mechanics did not allow for the transformation between mass and energy and did not respect Lorentz invariance. Eventually, special relativity and quantum mechanics were accommodated through a series of works, among which the seminal contributions of Fock, Dirac, Pauli and Heisenberg [24, 25, 26, 27], that developed the first quantum field theory of the electron. The decades after Dirac's 1928 paper that laid the foundation of quantum field theory and predicted antimatter [25] saw an explosion of determining contributions as all the known forms of energy and matter were explained in terms of relativistic quantum field theories and brought into the framework of quantum physics. Sin-Itiro Tomonaga, Julian Schwinger, and Richard Feynman independently developed QED [28, 29, 30], the first successful interacting quantum field theory describing the interaction of light and matter. Yang and Mills introduced non-abelian gauge theories [31], which later became fundamental in developing the Standard Model, particularly for strong and weak nuclear interactions. Soon after, Sheldon Glashow, Steven Weinberg, and Abdus Salam formulated the electroweak theory [32, 33, 34], unifying the electromagnetic and weak nuclear forces. Murray Gell-Mann and George Zweig independently proposed the quark model [35, 36], describing protons, neutrons, and other hadrons as composed of quarks, fundamental particles, leading to quantum chromodynamics. Together with Higgs, who proposed the mechanism of spontaneous symmetry breaking, crucial for explaining the mass of fundamental particles and the Higgs boson [37], the relativistic quantum field theory of matter composes the *Standard Model* of particle physics.

While progress was made towards the creation of the Standard Model, it soon became clear that the role of gravity as the only known fundamental interaction that was not yet completely formulated in the language of quantum theory was going to become the centre of the theoretical physics debate in the coming decades. Matvei Bronstein's paper on the quantum theory of weak gravitational fields [38] was a significant early attempt to address the quantization of gravity. Originally working in the Soviet Union in the 1930s, Bronstein's insights were far ahead of his time. One of Bronstein's significant contributions was recognizing that a quantum theory of gravity would likely need to address the non-linear nature of Einstein's field equations. This foresight pointed to the complexities that later researchers would encounter, especially the challenge of dealing with the feedback loop where gravity affects the quantum fields, which, in turn, affect the gravitational field. However, at the time, the community did not appreciate this task's significance entirely until DeWitt published his essay in 1953 [39]. There are a couple of notable points of interest in the essay. Firstly, the essay paints the picture of a community mostly uninterested in the quantisation of gravity, with DeWitt himself leaving the door open to the possibility that gravity could be a *fundamentally classical* field, different from all the others present in Nature:

"At this point, one may well ask to know the reasons for attempting quantization of the gravitational field in the first place. As a matter of fact, the overwhelming weight of opinion of physicists is opposed to the attempt. The prime reason for this is the experimental fact that gravitation has never been observed to take part in physical events on a quantum level, and where there is no evidence, it is bad form to speculate. Even if the covariance failure

mentioned above could be regarded as definite negative evidence, it would cause no upheaval in physics. It may actually be that the gravitational field is the one and only field which is not quantized in Nature. The gravitational field, with its attendant phenomena, could, under these circumstances, constitute the ultimate classical level which must be postulated, even in the quantum theory, in order to have a consistent "quantum theory of measurement."

However, while discussing this possibility, DeWitt highlights that if the gravitational field is classical, it could not be sourced by the quantum stress-energy tensor but by its expectation value, leading to the *semiclassical* Einstein's equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4}\langle \hat{T}_{\mu\nu}\rangle. \tag{1.2}$$

Consequently, he points out that if Ψ is taken to be the quantum state of matter and energy, then its evolution will be given by a covariant Schroedinger equation

$$i\hbar c\,\frac{\partial \Psi}{\partial x^\mu} = \mathcal{H}_\mu(g_{\mu\nu})\Psi \tag{1.3}$$

with $\mathcal{H}_\mu$ being the gravitational Hamiltonian. Given that the metric would depend on the average stress-energy tensor, it would also be dependent on both Ψ and Ψ^*, making the evolution of the quantum state *non-linear* and hence violating one of the fundamental tenets of quantum mechanics, namely the principle of *superposition*. We will return to this point later. Secondly, DeWitt seems to suggest that the strongest argument in favour of the quantisation of gravity is the aesthetic appeal of finding a unified quantum theory able to encompass all fundamental interactions. If such a task were to be accomplished, the boundary of what we mean by gravity might become blurred, as in a unified theory, different interactions can be indistinguishable from one another.

The impact of DeWitt's essay was one of the catalysts that led to the famous conference on "The Role of Gravitation in Physics" held at the University of North Carolina, Chapel Hill, from January 18 to January 23, 1957 [40]. It was planned as a working session to discuss problems in the theory of gravitation, which had recently received attention. The transcript was taken down by Cécile DeWitt and several other "reporters" as part of a conference funding agreement. As the conference progressed, it became increasingly apparent that a report of the discussions would be of scientific interest, partly due to an increasing number of requests for a report from physicists unable to attend the conference and partly because of the nature of the discussions. Thomas Gold pointed out that electromagnetism required quantisation because a range of phenomena closely related to it (like the ultraviolet catastrophe) required explanation. When applying the same line of reasoning with gravity, one would realise that, at the time, there were no such instances in need of clarification and that the only reasonable motive for gravity quantisation was if it were to contradict the principles of quantum mechanics. As an answer to Gold, Feynman outlined an experiment to show the incompatibility of general relativity and quantum theory. Here, we present an improvement on the original argument due to Aharonov [41]. Imagine a double-slit experiment where a massive particle forms an interference pattern as its superposed wavefunction from passing through the slits interferes with itself as shown in Figure 1.1. Imagine now that the gravitational field was classical and that we were in possession of a device capable of detecting changes in the gravitational field with arbitrary precision, which would be possible due to the classical nature of spacetime. If the mass in superposition was deterministically coupled to the gravitational field, one could monitor the gravitational signature of the mass going through the slits and discern which slit it passed through without measuring its position directly, which would destroy the interference pattern. Even worse, the correlation between the quantum system and the classical gravitational field would prevent the quantum state from being in a pure superposition of two locations. This would lead to a contradiction which appears only to be resolved by the quantisation of gravity. If the gravitational field were quantum, it would get into an entangled superposition with the possible particle states. Leaving it be would preserve the pure state, and its measurement would decohere the entire quantum system, as expected.

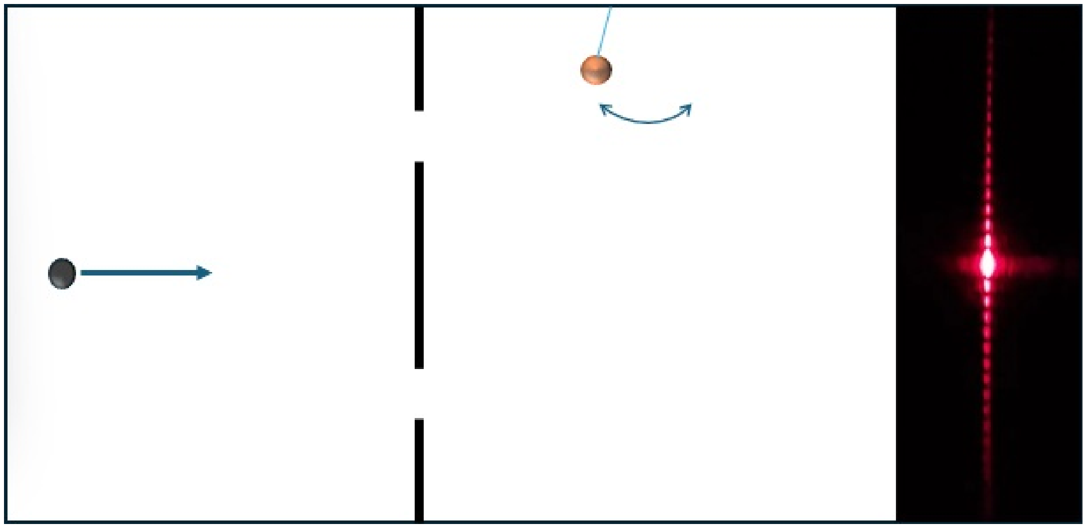


Figure 1.1: Aharonov's thought experiment [41]. The interference pattern should be destroyed if the gravitational field can be measured to obtain arbitrarily precise information about the quantum state through their coupling. Interference image retrieved from [42].

It has been more than sixty years since the Chapel Hill conference, and the community's opinion seems to have unilaterally shifted towards the necessity of a quantised theory of gravity. Multiple no-go theorems forbidding the coupling of quantum matter and classical spacetime were developed [43, 44, 45, 46, 47, 48, 49, 50, 51, 52] with some opposition from [53, 54, 55]. At the same time, research programs aiming at a fully quantised theory of gravity have sprouted and developed for decades in seemingly radically different directions, albeit all aimed at the same goal of a unified theory of quantum gravity and matter. We have reached a point where the quantum nature of gravity is often taken for granted, despite the absence of direct empirical evidence, and students entering theoretical physics typically encounter only debates between competing quantum gravity theories.

When discussing the approaches to quantum gravity, Isham's classification [56, 57] comes in handy. He divides these theories into "primary" and "secondary" theories of quantum gravity. Primary theories start with a given classical theory to which a heuristic quantisation scheme is applied, hoping to replicate the success obtained with other classical theories like electromagnetism, which became QED. The starting point is then chosen to be general relativity, and the resulting theories are referred to as "quantum geometrodynamics". The clear strength of this approach is that the starting point is given and well-known, while the trade-off is that an eventual positive outcome would be a quantum theory of gravity and not a unified quantum theory of all fundamental interactions. Furthermore, primary theories can be divided into "canonical" and "covariant" approaches, where the former treats space and time as separate entities in a 3+1 split, while the latter preserves a full 4-dimensional spacetime picture. The canonical program originated in 1958 from Dirac [58] with the Hamiltonian formulation of general relativity and was shortly followed by the 1967 development of the Wheeler-DeWitt equation for the wavefunction of the universe [59]. However, the Hamiltonian of general relativity seemed to vanish on physical states, leading to the so-called *problem of time* (see [60] for a great review), where time in the theory appears to be completely frozen, its flow forever interrupted the instant the theory is quantised. Significant progress slowed until 1986, when Abhay Ashtekar introduced a new set of variables for canonical quantum gravity that greatly simplified the mathematical formulation of the theory [61]. The breakthrough was identifying the fundamental gravitational degrees of freedom with a $SU(2)$ connection and its complementary variable. It is through the use of these "Ashtekar variables" that Lee Smolin and Carlo Rovelli developed loop quantum gravity [62] (see [63] for a modern introduction to the approach). Originally a canonical theory, LQG became covariant with the introduction of spinfoams. These networks evolve over time, providing a way to describe the quantum spacetime fabric in terms of a sum-over-histories (path integral) approach [64, 65, 66]. Loop quantum gravity provides an explicitly background-independent theory capable of notable feats like explaining the Black-Hole entropy formula [67, 68]. However, the quantum states in LQG are based on spin networks that provide a granular spacetime structure. While this achieves UV-finiteness because the spin-foam network degrees of freedom live on a lattice, which introduces a cut-off related to its spacing, bridging the gap between this discrete structure and the smooth spacetime of general relativity is non-trivial. Moreover, developing semiclassical states that are well-behaved and that approximate classical spacetime over large distances remains a significant challenge.

On the other hand, by far the most well-known example of a secondary quantum gravity theory is *string theory*. While primary approaches start from the assumption that gravity can be quantised separately, the starting point of secondary theories is the belief that only a unified quantum framework of all interactions would be possible. The key revolution of string theory lies in replacing quantised classical fields with one-dimensional 'strings' as its fundamental objects. This shift represents a clean break from the traditional concept of local fields defined at specific points in spacetime. While string theory stemmed from an attempt at explaining the spectrum of hadrons, Scherk, Schwartz and Yoneya realised in 1974 that a massless spin-2 particle was present in the spectrum of these objects [69, 70], which necessarily leads to the emergence of general relativity in the low-energy limit. This insight is paired with the fact that the string spectrum is capable of generating gauge bosons and, with the implementation of supersymmetry, fermions [71, 72, 73]. The ultimate goal would be the recovery of the entire Standard Model of Physics integrated with gravity, but the program is still far from it. However, even if in the 1980s there was a strong sense that string theory was going to be the ultimate "Theory of Everything", the last 20 years have seen the program suffer from much criticism [74, 75, 76]. Without delving into details, the biggest points are the lack of experimental evidence to support string theory's predictions, the presence of a vast number of possible solutions (often referred to as the "landscape problem" [77]), which challenges the predictive power of the theory, and the theory's reliance on extra dimensions and supersymmetry, which remain unobserved. Additionally, the theory's requirement of very high energy scales for direct testing puts it beyond the reach of current and near-future experimental technology. This limitation complicates efforts to validate or falsify string theory within a reasonable timeframe. Critics also point to internal theoretical challenges, such as the problem of moduli stabilization [78], where the theory does not uniquely predict the compactification parameters of the extra dimensions, leading to potentially innumerable physically distinct universes. As a reminder, string theories predict 11-spacetime dimensions (26 for bosonic strings), but large extra dimensions have been excluded by cosmological observations [79], while compactified extra dimensions, which are expected to be accessible by modern-day collider experiments like CERN, have not been observed [80]. Furthermore, the absence of a clear mechanism within string theory to naturally generate the observed cosmological constant or explain the universe's dark matter and dark energy components compounds these difficulties [76]. Lastly, there's a philosophical critique regarding whether a theory that encompasses such a broad range of possibilities can be considered truly predictive or explanatory in the traditional sense used in physical sciences [74].

While the importance of the aforementioned quantum gravity programs in better understanding the relation between gravity and quantum theories is indisputable, they have yet to achieve the goal they set out to complete. It has now been more than 50 years, and it is only fair to wonder if the quantisation of gravity truly is the *only* way forward. At first glance, it would seem that Feynman's argument and the following no-go theorems truly forbid the coupling of classical theories of spacetime with quantised matter fields. However, realising the existence of a common subtle assumption in all these arguments can allow one to sidestep them without violating the no-go theorems.

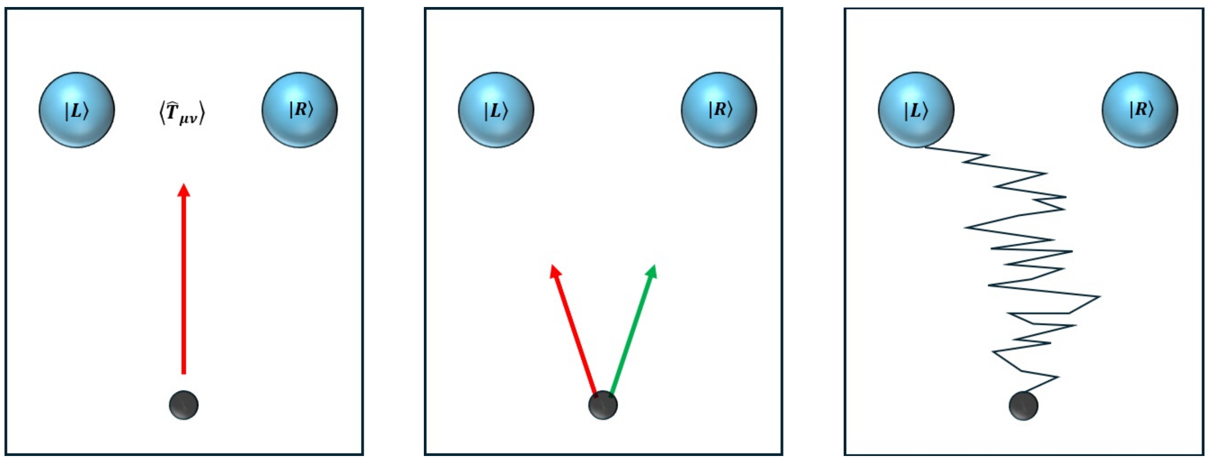
The assumption is based on identifying the concepts of *classicality* and *determinism*. While a deterministic theory is necessarily classical, the converse is not true. We identify classical theories with theories where observables are commutative, meaning the order of measurements does not affect the outcomes, where no concept of entanglement exists, meaning that objects are independent unless physically connected, and where specific values of measurable properties like position and momentum usually describe the state of a system. On the other hand, in quantum theories, states are described by density matrices in a Hilbert space. Moreover, quantum states can be in superpositions, where a system simultaneously exists in multiple states until a measurement is made. It is imprecise to associate the properties of classicality with determinism. A deterministic theory is simply one where the future behaviour of a system can be precisely predicted from its initial conditions using the laws of physics. Classical gravitational theories, as formulated by Newton and Einstein, are a prime example where the equations of motion provide a definite prediction of future states as the gravitational field is deterministically produced by matter. On the other hand, quantum theories are *intrinsically* probabilistic. Probability amplitudes lead to uncertainty of measurement results, and the theory provides probabilities of different outcomes rather than definite predictions. We are familiar with the notion that quantum mechanics cannot predict exactly where an electron will be found when measured, but it can predict the probability of finding it at different locations. This uncertainty is inherent in the description of nature and cannot be removed by better measurements or clever tricks.

When considering probabilities in a classical theory, it is often assumed that any probabilistic behaviour can be attributed to a lack of knowledge about an intrinsically deterministic state. Consider some gas molecules of a perfume bottle left open in a room. If one wonders where these molecules could be, one might take a guess using the probability distribution of their positions, but one could have had deterministic knowledge of their exact locations *if only* it was possible to know their initial position and precisely track their movements. The probabilistic outcome directly results from ignorance about some parts of the system or a constraint of the measurements' precision. What if, instead, a classical system had some inherent probabilistic behaviour, such as some fundamental intrinsic noise process in its evolution laws? In such a scenario, the classical system would still not allow for superposition or entanglement, but the theory would also not be deterministic.

When Feynman and the authors of the no-go theorems coupled classical gravity with quantum theories of matter to show their incompatibility, they implicitly assumed that their coupling had to be *deterministic*. In other words, a specific quantum state is associated with each measurement outcome of the classical gravitational field. This will inevitably collapse any superposition, as the classical deterministic state can be measured with arbitrary accuracy and, through it, the quantum state itself. The gravitational field will act as a strong measurement apparatus, destroying quantum coherence without the need for an external observer.

However, if the coupling between the classical and the quantum state were noisy, a single measurement of the classical system would not be sufficient to determine the quantum state, and only repeated measurements would extract enough knowledge to collapse any present superposition into a defined eigenstate. Therefore, if the coupling between the classical and the quantum system were *stochastic* (had some intrinsic random noise), the coherence properties of the quantum state would be preserved, at least for some time.

This discussion is easier to understand if one considers an ideal superposition of the planet Prospero in a left and right state $|P\rangle = \frac{1}{\sqrt{2}}|L\rangle + \frac{1}{\sqrt{2}}|R\rangle$. Let us treat the gravitational field classically and compare the semiclassical Einstein's equation (1.2) with deterministic and noisy coupling. As one can notice in the leftmost panel of Figure 1.2, if one utilises the semiclassical Einstein's equations, the test mass (which can be taken to represent measurements of the gravitational field) will head towards the centre. This pathological motion towards a point where no mass is present can be attributed to the fact that taking the expectation value of the stress-energy tensor discards any correlation between the classical and quantum systems. Even worse, the same pathological effect can be observed with a simple statistical mixture of the planet on the left and on the right. What would be the correct behaviour? Given that we expect to find the planet Prospero on the left 50% of the time and on the right 50% of the time, we would expect the test mass to move either to the left or to the right with the same probability. However, looking at the central panel, we notice how a deterministic coupling, which preserves the correlation and results in the test mass moving left or right 50% of the time, will also result in an immediate collapse of the planet's superposition. In this deterministic case, as soon as the test mass starts moving, we will immediately know which side the planet is on, destroying the quantum coherence of the state. This is the scenario forbidden by Feynman's example and the aforementioned no-go theorems.



(a) Semiclassical Gravity (b) Deterministic Coupling (c) Stochastic Coupling

Figure 1.2: Comparative view of three coupling models for classical gravity and a quantum superposition of the planet Prospero. Panel (a) shows the effect of semiclassical gravity. The test mass heads to the centre, where nothing is present. Panel (b) shows the deterministic coupling referred to by Feynman and forbidden by experiments. The quantum state collapses as soon as we observe the test mass move. Panel (c) represents the stochastic coupling between classical and quantum systems. We must wait to collapse the quantum state, but the correlation between the planet's location and the test mass direction is preserved.

We can observe a stochastic coupling of classical gravity and the quantum state in the rightmost panel. As the test mass starts moving, it receives noisy kicks, resulting in a jittery motion that will contribute to hiding the planet's location. Only by waiting long enough do we learn whether Prospero is on the right or the left, eventually collapsing the quantum state.

Moreover, the higher the noise, the longer the coherence of the quantum state will be preserved, but we will see more of this later. Having developed an intuition on how we can sidestep the issues regarding the coupling of classical gravity and quantum matter, the rest of this work will be concerned with introducing the most general linear, positive and probability-preserving realisation of this *hybrid* framework, which we refer to as "CQ theory", and its application to the weak gravitational regimes. Previous attempts at a hybrid approach to gravity and quantum matter exist, which deserve credit, and we discuss them more in detail in Chapter 2.

Before getting into the mathematical formulation of this work, one last reflection should be directed towards the implications of a hybrid coexistence of a classical theory of gravity and a quantum theory of matter. Physicists agree that the semiclassical Einstein's equations have a relatively small degree of validity restricted to physical states for which quantum fluctuations of the stress-energy tensor are small with respect to its expected value; relying on them outside this regime often results in pathological scenarios. Moreover, they cannot correctly account for the backreaction of quantum fields on spacetime. They should not be considered a viable general model and have been cleverly ruled out in this regard in [81]. A better model is therefore necessary for regimes where gravity can be treated classically and matter can be treated as quantum theory. It may be that the CQ framework (and hybrid theories in general) should be considered the correct model only for this regime, able to handle correlations between the two sides of the hybrid system and model the correct dynamics. The results would be insightful on their own, as we would still observe the intrinsic probabilism of quantum theory trade-off with the determinism of classical gravitational theory. The quantum side becomes "more deterministic" as it is slowly measured, and the classical side becomes "more probabilistic" as it inherits a form of intrinsic noise. Moreover, phenomena like Hawking's black hole radiation [82, 83] or CMB fluctuation during inflationary cosmology [84, 85, 86, 87] have been derived in a semiclassical regime where the backreaction of quantum fields on spacetime (and vice versa) is crucial. A correct theory of classical-quantum backreaction could lead to a better understanding of these phenomena, with possible new insights.

However, it is much more interesting to consider the possibility that the gravitational field is *fundamentally* classical. If this were to be true, there would be two major consequences on top of everything mentioned above. As an outcome of the classical system inheriting some *intrinsic non-determinism* from its quantum partner, there could be a fundamental process of *information destruction*. A famous no-go theorem by Coleman [88] states that even if it were possible that a purely quantum theory could allow for the destruction of information, it would simply result in false decoherence corresponding to unknown coupling constants. On the other hand, Gross, Banks, Peskin and Susskind [89, 90] argued that fundamental information destruction must be tied to violation of locality and anomalous heating. However, none of the theories they considered correctly handled the backreaction of matter on spacetime. This is not enough to completely disregard their arguments, even if they might not directly apply to CQ theory, and the heating question has not been resolved yet. As we will see in Chapter 2, the general formulation of the framework does contain a few choices of couplings and correlations, even if these are greatly reduced in its continuous version. It might be that some regularisation at the Planck scale is necessary or that symmetry principles, like diffeomorphism invariance, will step up to help with anomalous heating. Regardless of the drawbacks, the upside of entertaining the idea of fundamental information destruction is that it may bring about a clean resolution of the black-hole information problem highlighted first in [91, 92] and then refined into the "AMPS" paradox in [93, 38]. There would be no need for Planckian black hole remnants [94] or for complex holographic dynamics necessary for the information to escape the black hole [95, 96]. However, we would then have to deal with the consequences of information destruction.

On the other hand, the quantum system would inherit some *intrinsic determinism* from the fundamentally classical theory of gravity. The biggest consequence would be an (at least partial) resolution of the *measurement problem* [97]. Any theory attempting to quantise gravity will inevitably have to explain the transition to the semiclassical regime, invoking some form of the measurement postulate or a restriction of observables [98, 99, 100]. Generalising this train of thought, if the nature of reality is described by a quantum theory undergoing unitary evolution, then the emergence of classicality will have to be explained either through a mechanism for wavefunction collapse, which is external to the unitary dynamics of quantum theories of spacetime and matter, or through some ontological stance akin to the many worlds interpretation. It has been proposed that the measurement problem could be resolved through gravitational collapse [101, 102, 103] or theories of stochastic collapse [104, 105, 106, 107, 108, 109]. However, in gravitational attempts like Diosi's original collapse model (his later models did not suffer from this issue) or Penrose's, the decoherence of the quantum state is due to nonlinear interactions. In the former, this happens through the non-linear Schroedinger-Newton equation, while in the latter, the timelike killing vectors in two different branches of the universe necessarily interact. However, we will see that the hybrid CQ theory presented here is completely linear in the quantum state, such that different branches of a superposition evolve independently. On the other hand, theories of stochastic collapse need to insert an ad-hoc field capable of inducing decoherence. Here, we do not need to do that. Gravity acts as the measurement apparatus and performs a stochastic collapse of the wave function. Moreover, this is an objective collapse, not just a wavefunction decoherence. CQ theory cannot be reduced to a linear Markovian collapse model, as the theory is linear only when both quantum and classical degrees of freedom are considered. As we will explore in Chapter 2, when conditioned on the quantum degrees of freedom, the quantum state remains pure [5] since, in such conditions, an ontological classical trajectory in configuration/phase space is associated with the evolution of the quantum states.

Thesis contributions

We now outline the main contributions of this work.

Newtonian limit of quantum matter backreacting on classical spacetime

In Chapter 4, we present the first complete derivation of the Newtonian limit of the gravitational CQ framework, also known as Post Quantum Gravity [1]. The limit is derived in all three formulations of the framework: the master equation formalism, the unravelling and the CQ path integral. The results are in agreement with each other; the Newtonian potential undergoes a diffusion process away from the usual Poisson equation while the quantum state decoheres into mass eigenstates. The diffusion is lower bounded by the amount of decoherence, resulting in completely positive dynamics. The bounds can be experimentally tested to validate or reject the theory. We recover the result of [110] and compare and contrast it with other hybrid models of Newtonian gravity coupled with quantum matter.

This work was done in collaboration with Jonathan Oppenheim, Zach Weller-Davies, and Isaac Layton [6].

Diffeomorphism invariant classical-quantum path integrals for Nordström gravity

When deriving the Newtonian limit of CQ theories, dealing with the Hamiltonian constraint requires carefully treating the conjugate momenta to the gravitational variable and introducing a shift-vector. This might contribute to the impression that the Newtonian limit might not be compatible with a diffeomorphism invariant CQ theory of gravity. In Chapter 5, we show that a fully self-contained and diffeomorphism invariant theory of gravity is compatible with the CQ framework by utilising Nordström gravitational theory. The Newtonian limit is derived, and it matches the one obtained directly from general relativity in the CQ framework.

This work has been done in collaboration with Jonathan Oppenheim and Zach Weller-Davies [7]

Anomalous contribution to the rotation of galaxies

In Chapter 6, we apply gravity in the context of the CQ framework to the rotational curve of galaxies. We see that even in a simple spherically symmetric vacuum solution, new gravitational contributions that could not emerge from Poisson's equation appear. We use these contributions to argue that we may be able to account for anomalies in galaxies' rotational velocity curves without invoking dark matter. We find a connection between the cosmological constant and a key acceleration parameter of the rotational curves.

This work has been done in collaboration with Jonathan Oppenheim [8].

Structure of the thesis

The thesis is composed of the main body and appendices. The main body is split into two parts: the first introduces the background of the CQ framework in all its formulations, and the second contains the framework's applications to the weak field limit of gravity. The second part is based on research work carried out and published entirely during my PhD in collaboration with other researchers from the group. The rest of the research papers listed above are also the result of collaborations carried out and published with internal and external researchers during my PhD, but they are not included in this thesis.

In Chapter 2, we explain the foundations of the CQ framework. We define the classical-quantum state, the fundamental object of CQ theories, and explain its dynamics as formulated in the master equation and unravelling formalism. We also introduce the tradeoff between the diffusion of the classical system and the decoherence of the quantum degrees of freedom.

In Chapter 3, we focus on the path integral formulation of the CQ framework. We dedicate an entire chapter to it because it will be the most used formalism in the second part of the thesis. The CQ path integral is introduced by directly comparing it to the purely quantum and classical-stochastic path integrals. The covariant formulation and the manifestation of the decoherence-diffusion tradeoff in this formulation of the CQ framework are also explained.

Chapter 4 is the first chapter of the second part of the thesis. Here, we apply the machinery of the CQ framework to derive and study the weak field Newtonian limit of classical gravity coupled with quantum matter. We consider all three formulations of the framework and present the Newtonian limit in the master equation, unravelling, and path integral version. We compare and contrast with other versions of hybrid Newtonian gravity.

In Chapter 5, we use the CQ path integral to derive the Newtonian limit of Nordström gravity and check that it matches with the result of Chapter 4. While Nordström gravity is not a description of gravity in our universe, it is nonetheless a self-consistent and diffeomorphism invariant theory of gravity. Therefore, this short chapter supports the previous chapter by confirming that a fully diffeomorphic CQ theory of gravity reduces to the same Newtonian limit.

In Chapter 6, we conclude our work by applying the CQ gravitational framework to study its impact on the rotational curve of galaxies. In particular, applying the CQ framework to gravity in this context allows for solutions that, at low accelerations, meaningfully deviate from the usual Newtonian weak field behaviour. As a result, we suggest that we can explain the observed deviation from the expected rotational velocity without invoking dark matter. Moreover, we directly connect the relevant acceleration to the observed cosmological constant, which suggests a resolution for the so-far unexplained connection observed between the relevant MOND acceleration a_0 and the observed cosmological constant Λ.

Chapter 7 presents a few closing remarks and the last part of the thesis contains the appendices. Here, we present detailed calculations that, if inserted in the main body, would have hindered the clarity of the narration. Appendix F has been done in collaboration with Andrzej Grudka, Tim Morris, Jonathan Oppenheim and Muhammad Sajjad [9].

Part I

Background

Chapter 2

Classical-Quantum framework

"In this group, we do not say the word quantisation"

∼ Jonathan Oppenheim

Having discussed at length why a consistent theory able to couple quantum and classical degrees of freedom consistently is relevant, and with our sights set on its applications to gravity, it is now the moment to explain how such a framework is constructed and how its dynamic is defined. In this chapter, we describe the formalism used to arrive at the general form of consistent coupling between classical and quantum degrees of freedom [1, 2]. We start by discussing and justifying the state of CQ as a framework rather than a theory. While doing so, we explain the assumptions of this framework and how it defines states and their evolution. In particular, in this chapter, we focus on the master equation and unravelling formalisms of CQ dynamics, while the path integral is covered thoroughly in Chapter 3.

2.1 Classical-quantum framework

When discussing the nature of the classical-quantum approach, one should keep in mind the difference between theory and framework. A good definition of what a theory is is reported in [111]: "A theory is a set of interrelated constructs (concepts), definitions, and propositions that present a systematic view of phenomena by specifying relations among variables, with the purpose of explaining and predicting the phenomena". While this definition does not pertain only to physical theories, we can readily apply it in the context of physical descriptions of nature. For example, the *theory* of Newtonian gravity presents a systematic description of the gravitational interaction between masses, relating in a precise and mathematical way the concepts of mass, distance and force through the formula $F_g = -\frac{Gm_1m_2}{r^2}$. However, we can place Newtonian gravity in the *framework* of classical mechanics. By this, we mean that underlying the specific relations used to describe the gravitational force lies a plethora of assumptions shared among all the theories in the framework of classical Newtonian physics. Among these, some of the most important are determinism (in the absence of random behaviour), absolute space and time, continuity and differentiability of the physical quantities describing the system, the principle of superposition of forces, conservation laws of quantities such as energy, momentum, and angular momentum, the ability to make sensible material point approximations, the restriction to non-relativistic speeds and the independence between the existence of the physical properties of a system and the observer's measurements.

As new physical phenomena are discovered, physicists attempt to formulate new theories based on known frameworks in order to incorporate new experimental evidence. However, this may be insufficient, and the need for a new framework arises. Sometimes, it is enough to swap or update some of the assumptions of an already-known framework. This was the case for general relativity, for which the new assumptions of the principle of relativity, constancy of the speed of light, equivalence principle and general covariance define a wider framework where Newtonian mechanics is nested as the low-speed, low-energy limit. However, general relativity still maintains some of the core assumptions already present in the Newtonian framework, among which are the ideas of determinism and the description of states through real-valued variables. On the other hand, a different framework had to be developed when quantum mechanics and quantum field theory were developed to describe atomic and sub-atomic physics. In the *quantum framework*, many fundamental assumptions of classical physics need to be scrapped to create the tools necessary for describing quantum phenomena. Among them, we see that the perfect predictability of future outcomes implied by classical determinism is forgotten in favour of a probabilistic interpretation of matter, behaviour and experimental outcomes. Objective reality is not a given either, and the observer plays a crucial role in determining the properties of a physical system. While it is simple enough to pinpoint the transition from the modern general relativistic framework to older frameworks of classical mechanics, it is not as easy to define the transition between a quantum framework and any classical model of reality, even if we know that decoherence plays a crucial role [112, 113, 114, 115]. However, it is often the case when the need to describe the interaction between a quantum and a classical system arises. The most relevant example to this work would be the coupling of quantum fields with classical spacetime, which we have explored in Chapter 1 to be, at best, incomplete due to its inability to account for the correlations between quantum and classical systems. In this setting, CQ finds its role as a hybrid *Classical-Quantum framework*, giving us the tools to describe the coupling of a classical and a quantum theory. While this introduction might seem quite general, it is intentionally so. As we will now see in detail, using CQ, we are able to make a choice of classical and quantum systems and couple them such that they stay true to the rules of their respective frameworks (with some adjustments, as we will see) presenting a coherent picture of the behaviour of the whole system.

When describing a framework, one can distinguish between the assumptions going into the state and those that concern the dynamics. The former gives us the tools to consistently describe the physical state under consideration, while the latter sets the rules that the evolution of those states must obey. For example, in non-relativistic quantum mechanics, the state is assumed to be a semi-definite operator in a Hilbert space known as a density matrix, and the assumptions also define how the state of composite systems is defined and how measurement outcomes are obtained. At the same time, the dynamics is assumed to be unitary, implying the conservation of probability and information and following the Schrodinger equation.

2.2 Classical-Quantum states

In the Classical-Quantum framework, the state of the entire physical system takes the form of a hybrid classical-quantum object. The state is defined such that it describes both quantum and classical degrees of freedom. Classical degrees of freedom live in a classical configuration space $\mathcal{M}$ and are denoted by z. There are no restrictions on what the classical degrees of freedom can represent. The standard example is for them to be the position and momentum of a particle, in which case $\mathcal{M} = (\mathbb{R}^2)$, but they could be representing a collection of localised masses or a classical field. At the same time, quantum degrees of freedom live in a Hilbert space $\mathcal{H}$. We denote the set of positive semi-definite operators living in this Hilbert space as $S(\mathcal{H})$. Therefore, the total space is a trivial fibre bundle (*product space*) with base space $\mathcal{M}$ and fibre $S(\mathcal{H})$ such that $E = \mathcal{M} \times S(\mathcal{H})$. Hence, we define the CQ state of the system to be described at any time by the section

$$\begin{aligned} \varrho &: \mathcal{M} \to S(\mathcal{H}), \\ \varrho &: (t, z) \to \varrho(t, z). \end{aligned} \tag{2.1}$$

To assign physical meaning to the CQ state, we interpret the probability density over the classical degrees of freedom to be its trace over the Hilbert space

$$p(t, z) = Tr_{\mathcal{H}}[\varrho(t, z)], \tag{2.2}$$

and the density matrix of the quantum degrees of freedom to be the normalised positive semi-definite operator

$$\hat{\sigma}(t, z) = \frac{\varrho(t, z)}{Tr_{\mathcal{H}}[\varrho(t, z)]}. \tag{2.3}$$

As a direct consequence, classical-quantum states always admit a decomposition $\varrho(t, z) = p(t, z)\hat{\sigma}(t, z)$ where $\hat{\sigma}(t, z)$ is a normalised quantum state. Intuitively, $\hat{\sigma}(t, z)$ can be understood as the quantum state one assigns to the system, given the classical state z is observed. Since the density matrix has a statistical foundation, $p(z, t)$ is then understood as the probability (density) of being in the classical state z. There should always be a separation of classical and quantum degrees of freedom, as we do not expect classical degrees of freedom to become quantised and vice versa. However, we do expect to be able to interpret the state and its decomposition probabilistically. Therefore, the CQ framework requires that $\varrho(t, z)$ is at every instant subject to a normalisation constraint

$$\int_{\mathcal{M}} \mathrm{Tr}_{\mathcal{H}}[\varrho(t, z)]\, dz = 1. \tag{2.4}$$

In other words, we associate to each classical degree of freedom an un-normalised density operator $\varrho(t, z)$ such that $\mathrm{Tr}_{\mathcal{H}}[\varrho(t, z)] = p(t, z) \geq 0$ is a normalised probability distribution over the classical degrees of freedom and $\int_{\mathcal{M}} \varrho(t, z)\, dz$ is a normalised density operator on $\mathcal{H}$.

An example of such a CQ-state is the *CQ qubit*, where we take a 2-dimensional Hilbert space and couple it to classical position, and momenta [1, 116]. If we treat classical degrees of freedom from a phase space approach, we have that $z = (q, p)$. The quantum density matrix will instead be that of a 2-level quantum system. Then, the CQ state takes the form of a 2×2 matrix over phase space

$$\varrho(t, q, p) = \begin{pmatrix} u_0(t, q, p) & c(t, q, p) \\ c^\star(t, q, p) & u_1(t, q, p) \end{pmatrix}. \tag{2.5}$$

If one desires to treat gravity coupled to matter fields in the CQ framework, a natural choice for the classical degree of freedom is the metric, while the matter fields are quantised. This allows for proper treatment of semiclassical gravity, where correlations between the gravitational field and matter are not ignored as in the semiclassical Einstein equations.

Having defined the CQ state, we are left with the task of specifying the dynamics. Equations of motion for hybrid states have been defined in the past, especially in the context of coupling classical gravitational degrees of freedom and quantum matter. An early attempt was that of [117] where the equation

$$\frac{\partial \varrho(t, z)}{\partial t} = -i[\hat{H}(z), \varrho] + \frac{1}{2}\left(\{\hat{H}(z), \varrho(t, z)\} - \{\varrho(t, z), \hat{H}(z)\}\right) \tag{2.6}$$

evolves the semiclassical state with the help of Alexandrov brackets [118, 119]. One can see the attempt at having consistent hybrid dynamics in which the quantum Hamiltonian $\hat{H}$ depends on the classical degrees of freedom. Unfortunately, this form of time evolution does not preserve probabilities [98, 117]. Let us illustrate this with a simple example. Consider the CQ qubit described by Eq. (2.5). We can imagine the quantum state corresponding to its spin and the hybrid state describing its coupling to the classical position and momentum of the particle during a Stern-Gerlach experiment. The minimally coupled Hamiltonian describing the interaction of spin and position can be written as:

$$\hat{H}(p, q) = \frac{p^2}{2m}\hat{\mathbb{I}} + \lambda q \hat{\sigma}_z \tag{2.7}$$

where m is the particle's mass, λ controls the strength of the interaction, and we have chosen to couple the position to the Pauli z operator. Substituting in Eq. (2.6) and simplifying, we arrive at

$$\frac{\partial \varrho}{\partial t} = -i\lambda q[\hat{\sigma}_z, \varrho] - \frac{\partial}{\partial q}\left(\frac{p}{m}\varrho\right) + \frac{\lambda}{2}\frac{\partial}{\partial p}\{\hat{\sigma}_z, \varrho\}_+ \tag{2.8}$$

with $\{\cdot,\cdot\}_+$ being the anticommutator. By choosing the starting state to be, for example, the $|+\rangle$ state and looking at the differential equations for the components of the CQ state, one can see that while the probability distributions of the diagonal components drift in opposite phase space directions and increase, the off-diagonal ones can only acquire a phase due to the action of the commutator and hence cannot decrease to counterbalance the diagonal terms. Over time, this leads to the loss of the probabilistic interpretation.

In the context of gravity, an attempt was made with the Schroedinger-Newton equation [120, 121, 122], which instead failed to achieve dynamics linear in the density matrix, inviting faster than light signalling and a breakdown of the probabilistic interpretation of the density matrix, with a time development that is now dependent or whether we are aware of the initial state or not. In [123, 124], Diósi wrote a master equation linear in the state, which did not lead to negative probabilities. He added diffusion to the Aleksandrov-Gerasimenko brackets and obtained a time evolution corresponding to a constant Hamiltonian force. Alternative early examples of these kinds of hybrid dynamics include the works of Blanchard [125, 126], Diósi [127], Alicki [128], Poulin and Preskill [129]. The strength of CQ dynamics is that it holds for general Hamiltonians with dynamics continuous in configuration/phase space. Let us now understand that in greater detail.

2.3 CQ master equation formalism

In the CQ framework, one desires the dynamics of the state defined in Section 2.2 to retain its positivity, preserve the statistical interpretation of the density matrix, and give rise to positive probabilities when acting on half-entangled states. Therefore, the framework requires that the dynamics must be linear, completely positive (CP) and probability-preserving. If the dynamics are also time-local, then consistent CQ dynamics can be written in the form of a master equation [1], its unravelling in terms of stochastic differential equations [5, 116] or path integral [4, 130]. In this chapter, we will look at the first two, while the latter will be expanded upon in Chapter 3. One can, in theory, also go from the master equation to the corresponding path integral as described in [4]. We show this in Appendix B for the Newtonian limit of CQ gravity. With the additional assumption of time-locality, it has been shown in [1] that the dynamics can be written in the form of a CQ master equation

$$\frac{\partial \varrho(z,t)}{\partial t} = \int dz' \left(W^{\mu\nu}(z|z') L_\mu \varrho(z') L_\nu^\dagger \right) - \frac{1}{2} W^{\mu\nu}(z) \{L_\nu^\dagger L_\mu, \varrho\}_+, \tag{2.9}$$

where $\{\cdot,\cdot\}_+$ is the anti-commutator, L_μ are an arbitrary set of operators on the Hilbert space known as *Lindblad operators*. The matrix $W^{\mu\nu}(z|z')$ describes the conditional transition probability. The first term represents the likelihood of the classical degrees of freedom to transition from z' to z after the action of the Lindblad operators, while the second term is also known as the *no-event* term and it is related to the probability of nothing happening, and the equation has the interpretation of a balance equation for the rate of change of the CQ state. One can immediately see that the Classical-Quantum master equation is a natural generalisation of the differential Chapman-Kolmogorov equation evolving a probability distribution

$$\frac{\partial}{\partial t} p(z,t) = \int dz' [W(z|z',t)p(z',t) - W(z'|z,t)p(z,t)], \tag{2.10}$$

and the "GKSL" or "Lindblad" equation obtained for the evolution of the density matrix of an open quantum system [131, 132]

$$\frac{d}{dt}\rho(t) = -i[H,\rho(t)] + a^{ij}\left(L_i \rho L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_i, \rho\}_+ \right). \tag{2.11}$$

The required preservation of normalisation under the trace operation and the integral over the classical degrees of freedom defines

$$W^{\mu\nu}(z) = \int \mathrm{d}z' W^{\mu\nu}(z'|z). \tag{2.12}$$

Introducing a basis $L_\mu = (\mathbb{I}, L_\alpha)$ of Lindblad operators, the condition for Equation (2.9) to describe completely positive dynamics at all times is that the matrix

$$\begin{bmatrix} \delta(z,z') + W^{00}(z|z')\,\delta t & W^{0\beta}(z|z')\,\delta t \\ W^{\alpha 0}(z|z')\,\delta t & W^{\alpha\beta}(z|z')\,\delta t \end{bmatrix} \tag{2.13}$$

be a completely positive matrix kernel in (z,z') [3]. For notation simplicity, we shall take the dynamics to be autonomous, which we take to mean that $W^{\mu\nu}(z|z')$ are time-independent. More generally, one can allow for the time-dependent case. In the case of time-dependent couplings, one can also ask for the weaker condition that the dynamics describe a CP map only on initial states, whilst, at intermediate times, the dynamics need only describe CP evolution on the subset of states which are attainable via time evolution. In this case, the dynamics are non-Markovian; the master equation still takes the form of Equation (2.9) but the conditions for complete positivity need not hold at all times [1, 5, 133, 134, 135, 131].

The master equation (2.9) can be expanded in terms of the moments D_n of the transition amplitude $W^{\mu\nu}(z|z')$ [1, 3, 4],

$$D^{\mu\nu}_{n,i_1\dots i_n}(z') := \frac{1}{n!}\int dz\, W^{\mu\nu}(z|z')(z-z')_{i_1}\dots(z-z')_{i_n}. \tag{2.14}$$

It is important to note that the moments D_n are not independent since they must define complete positive dynamics and inherit the complete positivity conditions through the moments' expansion of the transition amplitudes. Inserting the moment expansion into the CQ master equation, we arrive at

$$\begin{aligned}\frac{\partial\varrho(z,t)}{\partial t} &= \sum_{n=1}^{\infty}(-1)^n\left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(D^{00}_{n,i_1\dots i_n}(z)\varrho(z,t)\right)\\ &\quad - i[\hat{H}(z),\varrho(z,t)] + D_0^{\alpha\beta}(z)L_\alpha\varrho(z,t)L_\beta^\dagger - \frac{1}{2}D_0^{\alpha\beta}\left\{L_\beta^\dagger L_\alpha,\varrho(z,t)\right\}_+\\ &\quad + \sum_{\mu\nu\neq 00}\sum_{n=1}^{\infty}(-1)^n\left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(D^{\mu\nu}_{n,i_1\dots i_n}(z)L_\mu\varrho(z,t)L_\nu^\dagger\right),\end{aligned} \tag{2.15}$$

It was shown in [2] that such master equations can be split into two classes, those which have continuous trajectories in the classical space, first discovered by Diósi in [127], and those with finite-sized jumps [126]. The formal difference between the two classes is encapsulated in the *CQ-Pawula theorem* formulated in [2], which states that for non-trivial CQ evolution, we must have infinitely many moments defined in Equation (2.15), or else the master equation takes the form

$$\begin{aligned}\frac{\partial\varrho(z,t)}{\partial t} &= \sum_{n=1}^{2}\frac{(-1)^n}{n!}\left(\frac{\partial^n}{\partial z_{i_1}\dots\partial z_{i_n}}\right)\left(D^{00}_{n,i_1\dots i_n}\,\varrho(z,t)\right)\\ &\quad - \frac{\partial}{\partial z_i}\left(D^{0\alpha}_{1,i}\,\varrho(z,t)L_\alpha^\dagger\right) - \frac{\partial}{\partial z_i}\left(D^{\alpha 0}_{1,i}\,L_\alpha\,\varrho(z,t)\right)\\ &\quad - i[H(z),\varrho(z,t)] + D_0^{\alpha\beta}\left(L_\alpha\varrho(z)L_\beta^\dagger - \frac{1}{2}\{L_\beta^\dagger L_\alpha,\varrho(z,t)\}_+\right),\end{aligned} \tag{2.16}$$

where $\{\cdot,\cdot\}_+$ is the anti-commutator, which is the most general form of the *continuous* master equation.

This expansion might appear cumbersome but can be readily interpreted as a sum of distinct effects. The first term represents the classical evolution of the system. It is composed of derivatives with respect to the classical degrees of freedom, where the first-order terms represent the Liouville evolution and the second-order terms, with coefficients $D_2^{00}(z)$, represent the diffusion in configuration/phase space. The second and third terms encode the back-reaction of the quantum degrees of freedom on the classical degrees of freedom, given by the coupling of derivatives with the Lindblad operators. These terms' coefficient is $D_1(z)$. The first term of the last line is the usual unitary quantum evolution, while the last two terms, with coefficient $D_0(z)$, are responsible for the decoherence of the quantum system, and can be traced back directly to the GKSL equation [131, 132]. If one compares Equation (2.16) to earlier attempts of hybrid dynamics evolution [117, 120, 121, 122, 123], one will immediately notice that this equation is linear in the density matrix and prevents the breakdown of the probabilistic interpretation. Revisiting the earlier example of the CQ qubit; we can expand the anticommutator of Eq. (2.8) as

$$\frac{\partial \varrho}{\partial t} = -i\lambda q[\hat{\sigma}_z, \varrho] - \frac{\partial}{\partial q}\left(\frac{p}{m}\varrho\right) + \frac{\partial}{\partial p}\left(\frac{\lambda}{2}\varrho\hat{\sigma}_z + \frac{\lambda}{2}\hat{\sigma}_z\varrho\right), \tag{2.17}$$

this allows us to identify $D_{1,p}^{0\sigma_z} = D_{1,p}^{\sigma_z 0} = -\frac{\lambda}{2}$. We can then implement the remaining terms and reorganise them to see how the CQ framework describes the evolution of the same physical system as

$$\begin{aligned}
\frac{\partial \varrho(q,p,t)}{\partial t} = &- \frac{\partial}{\partial q}\left(\frac{p}{m}\varrho(q,p,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial p^2}\left(D_{2,pp}^{00}\,\varrho(q,p,t)\right) \\
&- \frac{\partial}{\partial p}\left(D_{1,p}^{0,\sigma_z}\,\varrho(q,p,t)\,\hat{\sigma}_z\right) - \frac{\partial}{\partial p}\left(D_{1,p}^{\sigma_z,0}\,\hat{\sigma}_z\varrho(q,p,t)\right) \\
&- i[\hat{H}(p,q), \varrho(q,p,t)] + D_0^{\sigma_z,\sigma_z}\left(\hat{\sigma}_z\varrho(q,p,t)\,\hat{\sigma}_z - \frac{1}{2}\{\hat{\sigma}_z^2, \varrho(q,p,t)\}\right),
\end{aligned} \tag{2.18}$$

where the last term could be simplified further by recalling that $\hat{\sigma}_z^2 = \hat{\mathbb{I}}$, but we have chosen this form for ease of comparison with Eq. (2.16). Two realisations of the evolution described by Eq. (2.18) have been simulated in Figure 2.1, which can be found in [5]. The CQ evolution maintains the correlation between the classical and quantum state, providing a correct description for the hybrid dynamics, which a semiclassical theory akin to the semiclassical Einstein equations described in Chapter 1 would not be able to do.

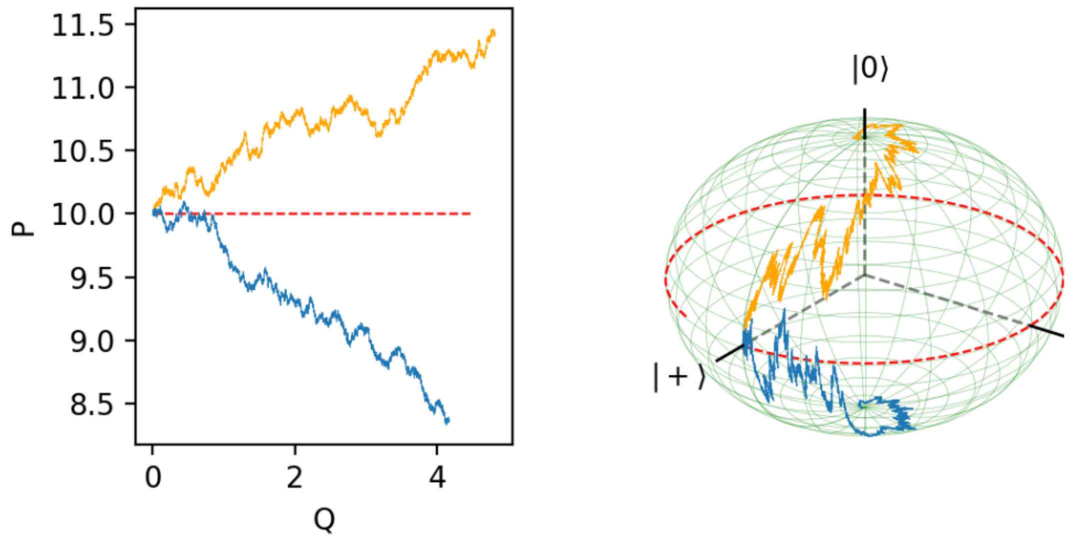


Figure 2.1: Classical-quantum trajectories, represented by a classical trajectory in phase space (left) and a quantum trajectory on the Bloch sphere (right). The orange and blue lines represent two possible realisations of the CQ trajectories described by Eq. (2.18). The red line would describe the predicted behaviour of a semiclassical theory akin to Eq. (1.2). One can see that the classical and quantum trajectories represent correlated random variables. This should be compared to the standard semi-classical result, for which correlations are lost. Figure courtesy of [5].

When analysing the time evolution described by this equation, we see that the decoherence terms and the additional diffusion in phase space contribute to suppressing the off-diagonal components of the CQ state and preserving the normalisation of phase space probabilities. The values of the decoherence and diffusion coefficients are not arbitrary. One can ask if there is any condition that needs to be imposed for the dynamics to preserve the state's positivity. Indeed, the question leads to a trade-off between the diffusion of the classical degrees of freedom and the decoherence of the density matrix.

2.4 The decoherence-diffusion trade off

While CQ systems, but also some of its precursors [124, 127], present decoherence in the quantum degrees of freedom and diffusion in the classical ones, the rates at which the two phenomena occur are not completely independent. In [3], it was shown that, in order for the dynamics to be completely positive, one must have

$$D_1^{br} D_0^{-1} D_1^{br\dagger} \preceq D_2, \tag{2.19}$$

for the matrix whose elements are the couplings

$$D_2 = D_{2,ij}^{\mu\nu}, \qquad D_1^{br} = D_{1,i}^{\alpha\mu}, \qquad D_0 = D_0^{\alpha\beta}, \tag{2.20}$$

where 'br' stands for *backreation*. Moreover, D_0 and D_2 are required to be positive semi-definite and obey

$$(\mathbb{I} - D_0 D_0^{-1}) D_1^{br} = 0, \tag{2.21}$$

which tells us that D_0 cannot vanish if there is non-zero back-reaction. Equation (2.19) implies that whenever there is back-reaction of a quantum system on a classical one, we *necessarily* have decoherence on the quantum system, as well as diffusion in the classical system, by an amount lower bounded by the coherence time. The trade-off in Equation (2.19) must hold for *all* Markovian classical-quantum dynamics. We refer to the trade-off in Equation (2.19) as the *decoherence-diffusion trade-off*, though strictly speaking, it is a trade-off between the diffusion D_2 and Lindbladian coupling D_0 entering into the master equation. A special case of this can be found in the condition for complete positivity of the constant force master equation of [127]. In Equation (2.16), the decoherence-diffusion trade-off reads

$$D_2^{00} \succeq D_1 D_0^{-1} D_1^{\dagger}, \qquad (\mathbb{I} - D_0 D_0^{-1}) D_1 = 0. \tag{2.22}$$

When the drift is generated by a CQ Hamiltonian, the back-reaction described by the D_1 term takes the form of the previously mentioned *Alexandrov-Gerasimenko bracket*, the decoherence-diffusion trade-off originating from the requirement of positivity simplifies to

$$4D_2 \succeq D_0^{-1}, \tag{2.23}$$

which, when saturated, has important consequences on the path integral formulation, as we will explore in detail in Chapter 3.

When the classical degrees of freedom are fields, for example when applying the framework to gravity, one needs to use the field-theoretic version of the moment expansion which was studied in [1, 3, 136]. In the field-theoretic case, the Lindblad operators can have spatial dependence, and the field-theoretic master equation follows by replacing discrete sums with integrals over space

$$\sum_{\nu\mu} D_n^{\mu\nu} L_\mu \varrho(z) L_\nu^\dagger \rightarrow \int dxdy\, D_n^{\mu\nu}(x,y) L_\mu(x) \varrho(z) L_\nu(y) \tag{2.24}$$

and replacing standard derivatives with functional derivatives. In other words, the spatial coordinate x acts like an index of the Lindblad operators and the matrices D_n.

In the field-theoretic case, one finds the same trade-off between coupling constants in Equation (2.19), but the moments are now matrix kernels representing diffusion and decoherence. In order for the dynamics to be completely positive $D_0(x,y), D_2(x,y)$ must also be positive kernels, where a positive kernel $f(x,y)$ is a kernel such that $\int dxdy\, a^*(x) f(x,y) a(y) \geq 0$ for any function $a(x)$. Written out explicitly, the trade-off in Equation (2.19) reads

$$\int dxdy [b^*(x), \alpha^*(x)] \begin{bmatrix} 2D_2(x,y) & D_1^{br}(x,y) \\ D_1^{br}(x,y) & D_0(x,y) \end{bmatrix} \begin{bmatrix} b(y) \\ \alpha(y) \end{bmatrix} \geq 0 \tag{2.25}$$

which should be positive for any position dependent vectors $b_\mu^i(x)$ and $a_\alpha(x)$. When viewing it as a matrix-kernel equation, this is equivalent to a trade-off between coupling constants in Equation (2.19).

It is clear that the stronger the interaction between the quantum and the classical systems, the greater the trade-off. Long coherence times are impossible without introducing enough diffusion in the classical degrees of freedom. One can visualise this phenomenon by thinking again of Figure 1.2 in Chapter 1. When the diffusion is high, the path of the test mass is marked by big random movements. The experimenter needs to wait a long time to understand if the planet's superposition is collapsed on the left or right, preserving the quantum state's coherence for longer. However, one should keep in mind that there is no need for an external observer; the classical system itself acts as the observer. As a result, the tradeoff between decoherence and diffusion is a crucial aspect of the CQ framework, which notably avoids the measurement problem.

The trade-off presented here is a key feature of the CQ framework, and it is precisely the reason why one can step around the no-go theorems which would prohibit sourcing a classical field with a superposition of quantum systems and would require the quantisation of gravity [44, 137, 138, 139], the trade-off makes it so that that measuring the state of the classical field does not automatically determine the state of the quantum system [1, 140]. Moreover, the trade-off can be exploited to look for an experimental signature not only of CQ but also of models of hybrid Newtonian mechanics such as [124] and any theory that treats gravity as fundamentally classical. This would be achieved by using the experimental signature to squeeze CQ theories through the measurement of decoherence times of mass superpositions and diffusion of the metric degrees of freedom, measured through Cavendish experiments [141, 142, 143] and measurements of Newton's constant [144, 145, 146]. As explained in [3], we think that lower bounds obtained from results on large molecules superposition decoherence times [147, 148, 149, 150] combined with small masses acceleration measures [151, 152, 153] already provide strong restrictions on theories where the spacetime metric is treated classically.

2.5 Unravelling of CQ master equation

When working with an open quantum system that is evolving according to the GKSL (Lindblad) equation [131, 132], it is often convenient to recast the dynamics in an unravelled form, where one follows the evolution of a *single pure quantum state* instead of following the entire density matrix. The state evolves stochastically in the Hilbert space, and the evolution of the quantum system can then be recovered by averaging over all paths. The advantages of this approach are multiple. Firstly, it is easily implementable in a computer simulation; the evolution of a single pure state, especially for larger systems, has lower computational complexity than the entire density matrix. Secondly, the unravelling approach is intuitively easier to grasp and offers a different perspective. The evolution of the system can be thought of as generated by continuous, deterministic dynamics accompanied by stochastic jumps of the wave function occurring stochastically whenever an interaction with the environment manifests.

Much like the GKSL equation, the CQ master equation can be unravelled to study the stochastic evolution of the quantum and classical degrees of freedom of a pure hybrid CQ state. Given a CQ state $\varrho(t, z) = \mathbb{E}[\delta(z - Z_t)\rho_t]$, the dynamics which generate the stochastic trajectories of the classical $\{Z_t\}_{t>0}$ and quantum $\{\rho_t\}_{t>0}$ degrees of freedom will induce the stochastic trajectory of the hybrid state $\varrho(z, t)$. The dynamics is positive and norm-preserving and can be written as a series of stochastic differential equations [5]

$$dZ_{t,i} = D_{1,i}(Z_t)dt + \langle D_{1,i}^{\alpha 0}(Z_t)L_\alpha + D_{1,i}^{\alpha 0}(Z_t)L_\alpha^\dagger\rangle dt + \sigma_{ij}(Z_t)dW_j, \tag{2.26}$$

$$d\rho_t = -i[H(Z_t), \rho_t]dt + D_0^{\alpha\beta}(Z_t)L_\alpha \rho L_\beta^\dagger dt - \frac{1}{2}D_0^{\alpha\beta}(Z_t)\{L_\beta^\dagger L_\alpha, \rho_t\}_+ dt$$
$$+ D_{1,j}^{\alpha 0}\sigma_{ij}^{-1}(Z_t)(L_\alpha - \langle L_\alpha\rangle)\rho_t dW_i + D_{1,j}^{\alpha 0}\sigma_{ij}^{-1}\rho_t(L_\alpha^\dagger - \langle L_\alpha^\dagger\rangle)(Z_t)dW_i, \tag{2.27}$$

where dW_i is the standard multivariate Wiener process and σ_{ij} is defined by $D_{2,ij}^{00} = \sigma_{ik}\sigma_{kj}^T$.

The first equation represents the path of the classical degrees of freedom Z_i through phase space. The first term is the usual classical evolution, and the second describes the back-reaction of the quantum degrees of freedom, appearing through the presence of the Lindblad operators. In contrast, the last term represents the random kicks that cause diffusion. The second equation allows one to simulate paths of the quantum state through Hilbert space. We can distinguish the standard unitary evolution, the decoherence terms (analogous to the GKSL equation but with a dependence on the classical phase space), and the noise in its trajectory manifested in the last two terms. One can notice how these coupled differential equations are much easier to simulate on a computer, and their averaged-out paths will recover the master equation formulation.

Moreover, in [5, 116, 126], it has been argued that the unravelling picture of the CQ state has an added ontological value with respect to the GKSL unravelling. The unravelling of the GKSL is highly non-unique due to the possibility of decomposing the same dynamics using different Lindblad operators. On the other hand, when unravelling the CQ master equation, if the assumption is made that each Lindblad operator has a well-defined effect on the classical degrees of freedom, the resulting unravelling of the dynamics will be unique when conditioned on the classical degrees of freedom. As a consequence, stochastic trajectories of the state can be associated with real physical trajectories, and the resulting collapse into a particular state is actually happening due to the physical interaction between the classical and quantum degrees of freedom. In other words, the unravelling allows us to determine the evolution of the quantum state conditioned on the classical trajectory, which remains pure if the decoherence diffusion trade-off is saturated [5].

In this chapter, we have introduced and discussed two of the three formulations of the CQ framework. In the next chapter, we will explore the path integral approach to Classical Quantum dynamics and understand its strengths and weaknesses.

Chapter 3

Classical Quantum Path Integrals

“Path integrals: because why take one path to your destination
when you can stumble over infinitely many and still claim
you’ve found the right way?”
$\sim$ ChatGPT

In this Chapter, we present the key aspects and ideas that lead to and justify the path integral formulation of the CQ framework. A complete and detailed derivation can be found in the original works [4, 130]. In this Chapter, we will first briefly explain why a CQ path integral formulation is well-suited to discuss field theories and gravity; then, we will look at path integrals in classical, quantum theories and CQ frameworks. Lastly, we will comment on the saturation of the decoherence diffusion trade-off and its effect on the path integral formulation. The path integral formulation will be key in the gravity applications of all the next Chapters.

The master equation formulation of the CQ framework is perfectly suited to highlight the consistency of the coupling between classic and quantum degrees of freedom. The evolution of the CQ state through the master equation in its continuous and jumping forms [1, 2] allows one to immediately appreciate how the different terms in the dynamical equation are tied to the time development of the individual classical and quantum degrees of freedom and to their backreaction on each other which lead to the effects of decoherence and diffusion discussed in Chapter 2. In addition, positivity conditions are easily accounted for, leading to the key property of decoherence-diffusion trade-off [3]. However, since the beginning, the CQ framework was formulated with the idea of being able to incorporate classical and quantum descriptions of field theories, from Klein-Gordon fields to gravity [1].

While, in Chapter 2, we spent a few words on adapting the master equation to describing fields through the replacement of discrete sums with integrals, as shown in Equation 2.24, this formalism is not the preferred choice for the description of quantum fields. Path integrals are, instead, the preferred tools for high-energy and gravitational physicists. Path integrals allow one to impose gauge and spacetime symmetries, impose constraints, and exploit modern notions of effective field theories [154]. It is challenging, albeit not necessarily impossible, to check and impose these ideas directly at the master equation level [136]. Moreover, the master equation formulation lacks many of the numerical methods that can be readily applied to path integrals when simulating the dynamics of a system.

However, the path integral formulation of the framework excels at all of these tasks. First of all, constraints can be directly imposed on trajectories through the use of delta functionals, allowing one to post-select the paths of the systems living on the constraint surface. Secondly, there exist many efficient numerical methods capable of simulating path integrals. For example, those involving Monte Carlo methods find great applicability everywhere from finance [155, 156, 157] to lattice gauge theories and quantum mechanics [158, 159]. Lastly, the path integral formulation allows one to impose space-time symmetries and gauge symmetries with ease, as it can exploit the presence of an action, benefiting from all the techniques and insights associated with it. This formalism makes it easier to formulate CQ dynamics in a covariant manner [130] and to enforce the necessary principles when studying effective field theories [154].

3.1 Path integrals in classical and quantum theories

Path integrals are very powerful tools that compute the evolution of a system by summing over all the possible allowed paths between an initial and final configuration of the system, where each path is weighted according to the probability of it being realized. The more probable a path, the greater the contribution to the final summation. Developed by Feynman in the 1940s [160], the essence of quantum path integrals lies in their ability to describe the probabilistic nature of quantum mechanics more intuitively. Unlike classical trajectories, where a system follows a definite path, quantum path integrals associate a probability amplitude to each path, given by the exponential of the classical action in units of Planck's constant.

$$\rho(\phi_f, t_f) = \mathcal{N} \int \mathcal{D}\phi(x)\, e^{\frac{i}{\hbar}\mathcal{S}[\phi]} \rho(\phi_i, t_i), \tag{3.1}$$

where $\rho(\phi, t)$ is the density operator, $\mathcal{N}$ the normalisation factor, $\phi(x)$ is a quantum field and $\mathcal{S}[\phi]$ is the associated action, for example, a scalar field with a potential

$$\mathcal{S}[\phi] = \int_{t_i}^{t_f} d^3\vec{x}dt\, \mathcal{L}[\phi] = \int_{t_i}^{t_f} d^3\vec{x}dt\, \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right). \tag{3.2}$$

Path integral methods were revolutionary for quantum field theory and allowed for the implementation of renormalization [161] and perturbative renormalisation [162], among a plethora of other useful computational techniques [163, 164, 165, 166].

Quantum path integrals can also be used to describe open quantum systems that present dissipative behaviour [167]

$$\rho(\phi_f^+, \phi_f^-, t_f) = \mathcal{N} \int \mathcal{D}\phi^+\mathcal{D}\phi^-\, e^{\frac{i}{\hbar}\left(\mathcal{S}[\phi^+]-\mathcal{S}[\phi^-]+\mathcal{S}_{FV}[\phi^+,\phi^-]\right)} \rho(\phi_i^+, \phi_i^-, t_i), \tag{3.3}$$

with action

$$\begin{aligned} \mathcal{S}[\phi^\pm] &= \int_{t_i}^{t_f} d^3\vec{x}dt\, \left(\frac{1}{2}\partial_\mu\phi^\pm\, \partial^\mu\phi^\pm - V(\phi^\pm)\right), \\ iS_{FV} &= \int_{t_i}^{t_f} d^3\vec{x}dt\, \left(D_0^{\alpha\beta} L_\alpha^+ L_\beta^{*-} - \frac{1}{2} D_0^{\alpha\beta}\left(L_\beta^{*-} L_\alpha^- + L_\beta^{*+} L_\alpha^+\right)\right), \end{aligned} \tag{3.4}$$

where $L_\alpha^+ = L_\alpha[\phi^+]$ and $L_\alpha^- = L_\alpha[\phi^-]$. The quantum path integral is doubled since it includes a path integral over both the bra and ket components of the density matrix, here represented using the $\pm$ notation. In the absence of the Feynman Vernon term S_{FV} [167] the path integral represents a quantum system evolving unitarily with an action $S[\phi]$. When the Feynman Vernon action S_{FV} is included, the path integral describes the path integral for dynamics undergoing Lindbladian evolution [132, 135] with Lindblad operators $L_\alpha(\phi)$. Because of the $\pm$ cross terms, the path integral no longer preserves the purity of the quantum state and there will generally be decoherence by an amount determined by D_0. Hence, we see that the decoherence coefficient appears in the path integral formulation to play a role analogous to that played in the master equation formulation of Chapter 2. The condition for complete positivity in this case requires

$D_0^{\alpha\beta}$ to be a positive semi-definite matrix, $D_0 \succeq 0$. As a simple example, we can take $D_0^{\alpha\beta} = D_0$ and take $L_\alpha^{\pm} = \phi^{\pm}(x)$ to be a local real scalar field. This will yield a Feynman-Vernon term

$$iS_{FV} = -D_0 \int_{t_i}^{t_f} dt dx \left(\phi^+(x) - \phi^-(x)\right)^2 . \tag{3.5}$$

As one can see, the effect of the influence functional is that off-diagonal terms in the density matrix where $\phi^+(x)$ is different to $\phi^-(x)$ are suppressed, the state decoheres in the $\phi(x)$ basis.

While path integrals for quantum mechanics are extensively known in the high energy and gravitational physics community, classical path integrals are perhaps less appreciated. However, stochastic systems that can be described by a master equation formalism can also be described by a so-called stochastic path integral, and a general equivalence exists between them [168, 169, 170, 171]. The idea behind these path integrals is that the probability distribution over the classical degrees of freedom $p(z,t)$ is evolved by summing over the intermediate steps that the stochastic systems can take, each weighted by the associated conditional transition probability $p(z,t|z',t')$. Much like the construction of quantum path integrals, the path is divided into infinitesimal steps, and the conditional transition probability is then expressed in terms of the Wiener process and summed over [172]. The result is a path integral where the integrand is the exponential of a functional originally formulated in [168] and known as the "Onsager-Machlup" action. Stochastic path integrals can be expressed both in configuration and phase space. As an example, one can consider a stochastic Brownian particle undergoing diffusion in phase space thanks to kicks in its momentum, such that $z = (q,p)$ and the path integral reads

$$p(q_f, p_f, t_f) = \mathcal{N} \int \mathcal{D}q\mathcal{D}p\, \delta\left(\dot{q} - \frac{\partial H}{\partial p}\right) e^{-\mathcal{S}_C[q,p]}\, p(q_i, p_i, t_i), \tag{3.6}$$

with classical action

$$\mathcal{S}_C[q,p] = \int_{t_i}^{t_f} \frac{dt}{2D_2} \left(\dot{p} + \frac{\partial H}{\partial q}\right)^2 , \tag{3.7}$$

where $H[q,p]$ is the Hamiltonian of the system and D_2 is the diffusion coefficient, which could depend on the degrees of freedom, subject to positivity requirements imposing $D_2^{-1} \geq 0$ (if we the diffusion coefficient is a matrix $D_{2,ij}$ the condition becomes $D_2^{-1} \succeq 0$). Looking at the path integral, one can see that it consists in a sum over all classical configurations (q,p) with a weighting given accordingly to the difference between the classical path and its expected force $-\frac{\partial H}{\partial q}$, by an amount characterized by the diffusion matrix D_2. In the case where the force is determined by a Lagrangian $\mathcal{L}_C$, the action $\mathcal{S}_C$ describes a suppression of paths away from the Euler-Lagrange equations, by an amount determined by the diffusion coefficient D_2. The role of the delta function $\delta\left(\dot{q} - \frac{\partial H}{\partial p}\right)$ is to impose Hamilton's equation of motion for the degree of freedom that does not feel the effects of the stochastic kicks directly. If there were no stochastic kicks at all, then the transition probability would simply be given by Hamilton's equations of motion and the integrand would just be a product of delta functions $\delta\left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta\left(\dot{p} + \frac{\partial H}{\partial q}\right)$.

3.2 Path integrals for CQ dynamics

Hybrid path integrals have appeared previously [173, 174, 175, 176, 177]. These may be valid when applied to some initial probability densities but generally lead to negative probabilities since the dynamics is not completely positive on all initial states. However, CQ path integrals produce dynamics which is CPTP on all states at all times. The path integral formulation can be derived starting from the master equation [4], but it can also be taken as the starting point [130]. As with the master equation approach, the dynamics of the hybrid system is linear in the density matrix, completely positive, and trace-preserving.

The classical-quantum state can be expanded in terms of its components, in this case, a continuous quantum degree of freedom ϕ:

$$\varrho(z,t) = \int d\phi^+ d\phi^- \, \varrho(z,\phi^+,\phi^-,t) \left|\phi^+\right\rangle \left\langle\phi^-\right| . \tag{3.8}$$

Where $\varrho(z,\phi^+,\phi^-,t) = \left\langle\phi^+\right| \varrho(z,t) \left|\phi^-\right\rangle$. Here, we double the degrees of freedom in order to define the density matrix – the ϕ^- field is the ket-field, while the ϕ^+ field is the bra-field. The general CQ configuration space path integral takes the form described in [4, 130]

$$\varrho(z_f,\phi_f^+,\phi_f^-,t_f) = \int_{z_i} \mathcal{N}\, \mathcal{D}z\mathcal{D}\phi^+\mathcal{D}\phi^- e^{\mathcal{I}[z,\phi^+,\phi^-,t_i,t_f]} \varrho(z_i,\phi_i^+,\phi_i^-,t_i), \tag{3.9}$$

where $z = (z_1, ..., z_n)$ are the classical degrees of freedom, $\mathcal{I}$ is the CQ action and it is implicitly understood that boundary conditions are to be imposed at t_f. The $\mathcal{N}$ normalisation factor is included in case the action does not preserve the norm of the state. The path integral gives each element of the density matrix at time t, given an initial density matrix.

According to the main result of [130], among all the possible completely positive actions, it is helpful to consider time-local CQ path integrals where the action takes the form of

$$\begin{aligned}\mathcal{I}(z,\phi^+,\phi^-,t_i,t_f) &= \mathcal{I}_{CQ}(z,\phi^+,t_i,t_f) + \mathcal{I}^*_{CQ}(z,\phi^-,t_i,t_f) \\ &\quad - \mathcal{I}_C(z,t_i,t_f) + \int_{t_i}^{t_f} dtd\vec{x} \sum_\gamma c^\gamma(z,x,t) L_\gamma(\phi^+) L^*_\gamma(\phi^-),\end{aligned} \tag{3.10}$$

which defines completely positive CQ dynamics.

In Equation (3.10), $\mathcal{I}_{CQ}[\phi^\pm]$ determines the CQ interaction the CQ interaction for the bra or ket ($\phi^\pm$) of the density matrix, and $\mathcal{I}_C(z,t_i,t_f)$ is a purely classical Fokker-Plank like action [171, 178], which should be positive definite, at least for large values of the classical variables z so that the path integral converges. If $\mathcal{I}_C$ is positive (semi) definite, the real part of $\mathcal{I}_{CQ}$ is negative (semi) definite, and $c^\gamma \geq 0$ is chosen such that the path integral will be convergent and define completely positive CQ dynamics [130]. From this point onward, we will also take $c^\gamma = 0$, the reason behind this choice will be explained in Section 3.3. However, one should keep in mind that all CQ path integrals with action given by Equation (3.10) are valid [130].

One can also go directly from the master equation picture in Equation (2.16) to a path integral picture whenever the master equation contains terms that are no more than quadratic in classical derivative operators [4]. In order to do so, one starts from the master equation, *Trotterizes* the dynamics in K steps from t_i to t_f and makes the identification with the short time expansion of the general CQ map where the conditional transition amplitudes have been Fourier transformed and written in terms of *response variables* conjugate to the classical degrees of freedom. Then, one proceeds to exponentiate the Lindblad operators by first transforming them in c-numbers. Afterwards, one inserts resolutions of the identity of the momenta associated with the quantum degrees of freedom and sends $K \to \infty$ to arrive at the path integral in phase space. Lastly, the response variables can be integrated out to obtain the phase space path integral formulation [4]. If the quantum momenta operators are no more than quadratic, they can be integrated out and the path integral can be written in configuration space. In this sense, Equation (3.10) allows for more general path integrals since one can include couplings that are higher than quadratic. In this case, the mapping between master equations and path integrals will not always be clear. The reduced phase space weak limit of gravitational CQ that we consider in Chapter 4 is of this type since, as we will see, imposing the constraint generates couplings that are higher than second order.

A natural class of theories introduced in [4, 130], which work especially well for classical and quantum fields are those derivable from a classical-quantum *proto-action*. The CQ proto-action is defined as

$$W_{CQ}[z,\phi] = \int dtd\vec{x}\,\Big(\mathcal{L}_C[z(x)] - \mathcal{V}_I[z(x),\phi(x)]\Big), \tag{3.11}$$

where $\mathcal{L}_C$ is the Lagrangian density of the classical field action $\mathcal{S}_c[z] = \int dtd\vec{x}\,\mathcal{L}_C[z]$, $\mathcal{V}_I$ is the interaction potential density $V_I[z,\phi] = \int dtd\vec{x}\,\mathcal{V}_I[z,\phi]$ and the classical and quantum degrees of freedom are now locally dependent on spacetime. The CQ action can then be written as

$$\begin{aligned}\mathcal{I}(z,\phi^+,\phi^-,t_i,t_f) &= i\mathcal{S}_Q[z,\phi^+] - i\mathcal{S}_Q[z,\phi^-] + i\mathcal{S}_{FV}[z,\phi^+,\phi^-] - \mathcal{S}_{diff}[z,\phi^+,\phi^-]\\ &= \int_{t_i}^{t_f} dtd\vec{x}\,\Bigg[i\mathcal{L}_Q[z,\phi^+] - i\mathcal{L}_Q[z,\phi^-] - \frac{1}{2}\frac{\delta\Delta W_{CQ}}{\delta z_\mu}D_{0,\mu\nu}[z(x)]\frac{\delta\Delta W_{CQ}}{\delta z_\nu}\\ &\qquad - \frac{1}{2}\frac{\delta\bar{W}_{CQ}}{\delta z_\mu}D^{-1}_{2,\mu\nu}[z(x)]\frac{\delta\bar{W}_{CQ}}{\delta z_\nu}\Bigg],\end{aligned} \tag{3.12}$$

where $\mathcal{S}_Q$ is the action for the quantum system, including the interaction potential. Here, the backreaction of the classical system on the quantum system defines a general form of the decoherence obtained via the *proto action difference*

$$\Delta W_{CQ}[z,\phi^+,\phi^-] \coloneqq W_{CQ}[z,\phi^+] - W_{CQ}[z,\phi^-], \tag{3.13}$$

while the backreaction of the quantum system on the classical system results in an average force obtained from the *proto action average*

$$\bar{W}_{CQ}[z,\phi^+,\phi^-] \coloneqq \frac{1}{2}\big(W_{CQ}[z,\phi^+] + W_{CQ}[z,\phi^-]\big). \tag{3.14}$$

To better understand the effect of the CQ action, we can again read Equation (3.12) in order. The first two terms represent the unitary evolution of the quantum degrees of freedom, which can be any quantum field theory Lagrangian. Given that we are evolving a density matrix we have that the left/right (bra/ket) branches are evolved separately by the $\pm$ terms. One can also add friction terms to Equation (3.12) though we shall not do this in the present work. Next, we find the decoherence term. It has a decoherence coefficient D_0 and is constructed from the variation of the difference between the left and right branches of the proto-action $\Delta W_{CQ}[z,\phi^+,\phi^-]$. This term is responsible for the decoherence of the system, penalising trajectories of the hybrid state that move further away from $\phi^+ = \phi^-$. This term does not affect the diagonal terms in the density matrix but does suppress the off-diagonal terms exponentially with time. Lastly, we have the diffusion term. Built from the variation of the $\pm$ averaged interaction $\bar{W}_{CQ}[z, \phi^+, \phi^-]$, this term has diffusion coefficient D_2 and penalises trajectories in which the classical degrees of freedom tend to deviate from their Euler-Lagrange equations of motion. In other words, the noise introduced in the classical degrees of freedom from the backreaction of the quantum degrees of freedom introduces a diffusion process that is visible from this term.

As a toy example, we could take a single classical degree of freedom describing a scalar field $z^\mu = \{q(x)\}$ coupled to a quantum scalar degree of freedom with spatial dependence (ex: a quantum field $\phi(x)$)

$$W_{CQ}[q, \phi] = \int dt d\vec{x} \left(\frac{1}{2} \dot{q}(x)^2 - q(x)\phi(x) \right). \tag{3.15}$$

Then, choosing for simplicity $D_0(q)^{\mu\nu} = D_0 \eta^{\mu\nu}$,

$$i\mathcal{S}_{FV}[q, \phi] := -\frac{1}{2} \int dt d\vec{x} \left(\frac{\delta \Delta W_{CQ}}{\delta z_\mu} D_0^{\mu\nu}(z) \frac{\delta \Delta W_{CQ}}{\delta z_\nu} \right) \tag{3.16}$$

$$= -\frac{1}{4} D_0 \int dt d\vec{x} \left(\phi^+(x) - \phi^-(x) \right)^2, \tag{3.17}$$

acts like a Feynman-Vernon term [167] which causes decoherence. As we mentioned above, the diagonal of the density matrix of Equation (3.8) occurs when $\phi^+ = \phi^-$ and, on these components of the density matrix, this term has no effect, while the greater the difference between the bra and ket fields, the more such paths are suppressed by the term in Equation (3.17).

In a similar manner D_2 tunes the averaged interaction term $\mathcal{S}_{diff}$, which is instead related to the diffusion of the classical degrees of freedom. Similarly to the decoherence term, paths that deviate from the Euler-Lagrange equations of motion, which are derived from the proto-action W_{CQ}, are suppressed due to stochastic diffusion. In the simple example of Equation (3.15), one obtains

$$\begin{aligned} \mathcal{S}_{diff}[q, \phi] &= \frac{1}{2} \int dt d\vec{x} \left(\frac{\delta \bar{W}_{CQ}}{\delta z_\mu} D^{-1}_{2,\mu\nu}(z) \frac{\delta \bar{W}_{CQ}}{\delta z_\nu} \right) \\ &= \frac{1}{2D_2} \int dt d\vec{x} \left(\ddot{q}(x) + \frac{\phi^+(x) + \phi^-(x)}{2} \right)^2 \end{aligned} \tag{3.18}$$

where the force on the classical field is produced by the average of the bra and ket quantum fields. This term allows for fluctuations around the force but acts to suppress large deviations from them.

Thanks to its form, it is possible to recognise how the CQ path integral is connected to the path integral formulation of open quantum systems. If the action $\mathcal{I}$ was only constructed out of the quantum degrees of freedom $\mathcal{I}[\phi^+, \phi^-] = i\mathcal{S}_Q[\phi^+] - i\mathcal{S}_Q[\phi^-] + i\mathcal{S}_{FV}[\phi^+, \phi^-]$, we would recover the standard decoherence and loss of purity behaviour related to open quantum systems. If the Feynman-Vernon term [167] was not present ($\mathcal{S}_{FV} = 0$), we would then recover standard unitary quantum mechanics. Much like open systems have a path integral formulation of their master equation version, the CQ path integral can, in certain circumstances, be directly thought of as coming from the master equation formulation [1, 4]. Nevertheless, according to [130], we can take the path integral as the starting point of the CQ framework. This allows for a simpler definition of the path integral, given that deriving a clean form from the master equation is not always possible [4].

The path integral constructs a CQ state at time t_f from a CQ state at time t_i. However, we are often interested in computing correlation functions for stationary states, and we would like to obtain information on correlation functions over arbitrary long times by taking the limit $t_i \to -\infty, t_f \to \infty$. In open systems, as well as when calculating scattering amplitudes, it is often assumed that the initial state in the infinite past does not affect the stationary state of the system so that there is a complete loss of memory of the initial state [178]. Under this assumption, it is possible to ignore the boundary term containing the initial CQ state $\varrho(z_i, \phi_i^{\pm}, t_i)$, arriving at the partition function

$$\mathcal{Z}[J^+, J^-, J_z] = \int \mathcal{D}z\mathcal{D}\phi^{\pm}\mathcal{N}\ e^{\mathcal{I}_{CQ}(z,\phi^+,\phi^-,-\infty,\infty) - i(J_+\phi^+ - J_-\phi^-) - J_z z} \tag{3.19}$$

such that we can use standard perturbation methods for computing correlation functions in CQ theories.

3.3 Decoherence-Diffusion and the purity of the state

The coefficients $D_0^{\mu\nu}, D_2^{\mu\nu}$ need to be positive definite matrices and kernels in the case of several classical degrees of freedom or fields . To ensure that the action takes the form of Equation

(3.10) and ensure complete positivity of the dynamics, one imposes the matrix inequality:

$$4D_2 \succeq D_0^{-1}. \tag{3.20}$$

This is again the *decoherence-diffusion trade-off* discussed in Chapter 2. The trade-off itself originates from positivity conditions on the master equation, on which we expand in Section 2.4, but was derived and explored in depth in [3]. Even in the path integral formulation, the physical meaning of this relationship between the decoherence and the diffusion coefficients tells us that, if we want to preserve the coherence of the quantum degrees of freedom for prolonged times, we will have a lot of noise introduced in our classical degrees of freedom. This can be seen from Equation (3.12) because the two coefficients regulate the amount of suppression for paths that maintain coherence of the quantum system and that diffuse away from the classical equations of motion. While the trade-off is in itself an equality, and hence valid for a wide range of parameters, we say that the trade-off is saturated when

$$4D_2 = D_0^{-1}. \tag{3.21}$$

The saturation of the trade-off is related to the last term in the CQ action of Equation (3.10), which reads

$$\int_{t_i}^{t_f} dt d\vec{x} \sum_{\gamma} c^{\gamma}(z, x, t) L_{\gamma}(\phi^{+}) L_{\gamma}^{*}(\phi^{-}). \tag{3.22}$$

This term contains cross terms between the bra and ket branches ϕ^+, ϕ^-, which sends pure states to mixed states and corresponds to including additional noise in the dynamics. It can be thought of as incorporating the loss of quantum information into the path integral through Lindbladian terms when the decoherence-diffusion trade-off is not saturated, much like a path integral version of a Kraus operation. Equation (3.12) is a special case of Equation (3.10) when the trade-off is satisfied, which can be seen by setting $c^{\gamma} = 0$ and expanding out the CQ action [130]. This is true for an arbitrary CQ proto-action W_{CQ}.

Since for $c^{\gamma} = 0$ Equation (3.10) contains no $\phi^{\pm}$ cross terms, the path integral preserves the purity of the quantum system. Pure quantum states are mapped to pure quantum states, and no information is lost. Moreover, conditioned on the classical trajectory, the quantum state evolution is deterministic, which provides a natural mechanism for wave-function collapse if the classical degrees of freedom are taken to be fundamental. Furthermore, because the classical degrees of freedom are themselves dynamical, unlike in spontaneous collapse models [104, 179, 180, 109], it is possible to make the dynamics covariant [130].

In the next chapters, we will apply the CQ framework to gravity, exploring the construction of diffeomorphism invariant CQ theories of gravity and the consequences of their weak field limit.

Part II

Classical-Quantum Gravitation

Chapter 4

Newtonian limit of quantum matter backreacting on classical spacetime

"Gravity is a contributing factor in nearly 73%
of all accidents involving falling objects."
$\sim$ Dave Barry

As we have seen in the previous chapters, consistent coupling of quantum and classical degrees of freedom exists so long as there is both diffusion of the classical degrees of freedom and decoherence of the quantum system. In this chapter, we derive the Newtonian limit of such classical-quantum theories of gravity. Consistent hybrid theories of Newtonian gravity have been studied via continuous measurement and feedback approaches [55, 110, 181, 182], and in [124] using a master equation for classical-quantum dynamics [125, 127, 129]. These approaches are all mathematically coherent and do not suffer from the problems of the standard semi-classical approach. As expected, the resulting master equation is linear in the combined hybrid state, preserves the classical-quantum split, and is completely positive on the quantum system. However, a derivation of the weak field limit from the complete general-relativistic theory is lacking for the CQ framework. This Chapter provides such derivation.

Our results are obtained both via the gauge fixing of the path integral theory of CQ general relativity based on Chapter 3 and via the CQ master equation approach described in Chapter 2. In each case, we find the same weak field dynamics. As described in Chapter 2, for any Markovian classical-quantum dynamics to be completely positive, which is required for the dynamics to be consistent when acting on half an entangled state and for mapping probability distributions to probability distributions, there must be a trade-off between the amount of decoherence on the quantum system and the amount of diffusion in the classical system. A relevant precursor to this result can be found in the constant force master-equation of Diósi [127]. More generally, since the trade-off can be shown to be a feature of all classical-quantum dynamics, this trade-off provides an experimental signature, not only of models of hybrid Newtonian dynamics such as [124], or of post-quantum theories of general relativity such as [1], but of *any* theory which treats gravity as being fundamentally classical. The metric necessarily diffuses away from what Einstein's General Relativity predicts. This signature *squeezes* classical-quantum theories of gravity from both sides: if one has shorter decoherence times for superpositions of different mass distributions, one necessarily has more diffusion of the gravitational metric. One can thus use Cavendish-type experiments to upper bound the amount of diffusion and coherence experiments to lower bound it, thus squeezing the parameter space of the theory from both sides. Therefore, the Newtonian potential diffuses by an amount bounded below by the decoherence rate into mass eigenstates. We also present our results as an unravelled system of Langevin stochastic differential equations (introduced in Chapter 2) for the trajectory of the hybrid classical-quantum state and provide a series of kernels that characterise correlations in stochastic dynamics. From these, theorists and experimentalists can choose to develop and test the parameters of CQ theories, possibly ruling out parts of the parameter space.

If such theories were in disagreement with experiments, this would provide an indirect test for the quantum nature of gravity. For example, in [3], the decoherence diffusion trade-off was used to rule out a large class of natural theories that we derive here, namely those which are ultra-local, non-relativistic and continuous in the classical phase space. Hence, a significant problem is studying and classifying consistent classical-quantum theories of gravity and their low energy limit further to reduce the parameter space of physically sensible CQ theories.

Moving away from fundamentally classical fields, we also mentioned in Chapter 1 that CQ theories of gravity could describe an effective quantum gravity regime whenever the gravitational field freedom behaves classically. In such a case, we expect that variants of the master equation and path integral found here will help describe this limit. However, an effective theory would probably be non-Markovian in some regimes, meaning that the decoherence-diffusion trade-off would not need to hold for all times [1]. We expand on this eventuality when discussing the weak-field results obtained here.

Let us briefly describe the structure of this chapter in detail.

In Section 4.1, we review a route to the Newtonian limit of classical gravity through a reduced action approach. We start from a reduced action by identifying the true degree of freedom of the non-relativistic Newtonian limit as the scalar perturbation of the metric, arriving at the Newtonian ADM Hamiltonian (4.11). This result will allow us to construct a hybrid gravitational system in the Newtonian limit in the master equation picture, where implementing the full GR constraint is challenging [136]. For reference and comparison, in Appendix A, we also summarise the standard derivation of the Newtonian limit, which uses the full Einstein's equations. Lastly, we present a stochastic classical analogue of the CQ theory for Newtonian gravity, where the field is sourced by a Markovian noise process. We see the role of the shift vector in the imposition of stochastic Newtonian constraints. In the end, it will turn out that this is the *actual classical limit* of the CQ theory when the matter degrees of freedom have completely decohered, and only the noise process remains in the classical degrees of freedom.

In Section 4.2, we derive the CQ Newtonian limit path integral as a gauge-fixed, non-relativistic limit of the diffeomorphism invariant CQ path integral for general relativity. The gauge fixing is informed by the reduced phase space approach to the Newtonian limit, and the non-relativistic limit is implemented by keeping leading terms in the speed of light c. This is one of the main results of this work. The dynamics is CP on the subset of states defined by the Newtonian limit. We find a generic prediction of CQ theories. The Newtonian potential diffuses away from its classical solution by an amount that depends on the decoherence rate into mass eigenstates.

In Section 4.3, we construct the weak field regime for the class of master equations with continuous back-reaction on the gravitational degrees of freedom (4.33). The effects of the dynamics are parameterized by a handful of functional parameters, which can be squeezed from experiments via the decoherence diffusion trade-off. In order to discuss experimental bounds, gravitational constraints on the CQ evolution must be imposed. The constrained master equation is then unravelled to derive one of the main results (4.44), determining the trajectory of any CQ state through its hybrid phase space.

We review how such theories are testable [3]. The most striking being the findings of [3] that classical-quantum theories of gravity, which are continuous in the gravitational degrees of freedom and produce only ultra-local, non-relativistic correlations, are already ruled out. Furthermore, when the coupling constants D_0, D_2 are constant kernels, we arrive at the Newtonian theory considered in [55, 110]. If one tried to minimise the amount of decoherence [55], one finds that theories with constant couplings are in tension with heating experiments if the Newtonian approximation is valid below scales of $10^{-15}m$ [55]. This calls for a study of relativistic corrections to hybrid theories.

Lastly, in Section 4.4, we compare and contrast with other models of semiclassical Newtonian gravity [55, 183, 184], and explain the bridge with the work of [110, 124] while highlighting the difference with previous measurement feedback and collapse models. We summarise how our main results have been cross-checked through the use of a variety of different methods. We conclude with a discussion of our results and comments on the theoretical and experimental challenges which remain open in constructing and testing theories with a classical gravitational field coupled with quantum matter.

4.1 Newtonian limit of classical GR

This section reviews the weak field and Newtonian limits of classical general relativity (GR), which motivates our study of the weak field and Newtonian limits of classical-quantum theories of gravity. By the weak field limit, we mean the linearised expansion of the metric around a flat Minkowski background, while by the Newtonian limit, we refer to a non-relativistic setting by taking the $c \to \infty$ limit and discarding terms with high powers of inverse c.

The Newtonian limit of GR is represented by a non-dynamical scalar perturbation of flat Minkowski spacetime expressed through the metric:

$$ds^2 = -c^2 \left(1 + \frac{2\Phi}{c^2}\right) dt^2 + \left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij} dx^i dx^j, \tag{4.1}$$

where Φ satisfies the gravitational Poisson equation. As a reminder, we present the usual derivation of this limit from a gauge fixing of the full Einstein theory in Appendix A. There, we start from a generalised scalar-vector tensor perturbation of the metric in the form of

$$ds^2 = -c^2\left(1+\frac{2\Phi}{c^2}\right)dt^2 + \frac{w_i}{c}(dtdx^i + dx^i dt) + \left[\left(1-\frac{2\psi}{c^2}\right)\delta_{ij} + \frac{2s_{ij}}{c^2}\right]dx^i dx^j, \tag{4.2}$$

where $\partial_i w^i = \partial_i s^{ij} = 0$ and we take the infinite c limit of Einstein's equations. When the stress-energy tensor is chosen to represent a pressureless dust distribution (a type of exact solution to the Einstein field equations where the gravitational field is generated solely by a perfect fluid with positive mass density and zero pressure for which $T^{\mu\nu} = m(x)U^\mu U^\nu$ with $m(x)$ being a mass distribution and U^μ the 4-velocity), only one non-dynamical scalar perturbation Φ remains at the end, and it is constrained to obey the gravitational Poisson's equation.

Based on the knowledge obtained from the full GR derivation, we present a derivation of the Hamiltonian formulation of the Newtonian limit, which starts directly from reducing the degrees of freedom to scalar perturbations. In the reduced degrees of freedom approach, we *first* assume that the relevant degrees of freedom are scalar perturbations. We shall also allow for vector perturbations at higher order in c, which are necessary to construct a consistent CQ theory.

As we will show in the rest of the main body, this provides us with a way of constructing the Newtonian limit of CQ theories via a reduction of the gravitational degrees of freedom, even in the absence of a complete CQ theory of GR which is positive on all possible states. This is explained in detail in Section 4.2, where we show that the problematic terms appearing in the CQ treatment of GR vanish in the Newtonian limit, validating the limit with a top-down approach. Regardless, we get the same results in both the path integral and the master equation approaches.

4.1.1 Newtonian limit via a scalar reduced action

To derive the Newtonian limit of GR via a reduced Hamiltonian, we take as a starting point the linearised Einstein Hilbert Lagrangian density, which is equivalent to the spin-2 field Fierz-Pauli action [185] for the metric perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$:

$$\mathcal{S}[h_{\mu\nu}] = \frac{c^4}{16\pi G}\int d^4x\, \mathcal{L}(h_{\mu\nu}), \tag{4.3}$$

$$\mathcal{L}(h_{\mu\nu}) = -\frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\mu h^{\rho\sigma}\partial_\rho h^\mu_\sigma - \frac{1}{4}\eta^{\mu\nu}\partial_\mu h^{\rho\sigma}\partial_\nu h_{\rho\sigma} + \frac{1}{4}\eta^{\mu\nu}\partial_\mu h\partial_\nu h. \tag{4.4}$$

We are interested in constructing CQ dynamics for a Newtonian theory, so we further make a Newtonian approximation of the metric. We take the ADM decomposition

$$\mathrm{d}s^2 = -(Nc\ \mathrm{d}t)^2 + g_{ij}\left(\ \mathrm{d}x^i + N^i c\ \mathrm{d}t\right)\left(\mathrm{d}x^j + N^j c\ \mathrm{d}t\right), \tag{4.5}$$

and make the weak field assumption

$$N = \left(1 + \frac{\Phi}{c^2}\right),\ N^i = \left(0 + \frac{n^i}{c^3}\right),\ g_{ij} = \left(1 - \frac{2\psi}{c^2}\right)\delta_{ij}. \tag{4.6}$$

The extra factor of c in the choice of shift-vector is related to the fact that, classically, the h_{0i} component occurs at a higher order than the h_{00}, h_{ij} components [186]. We assume that all fields vanish at infinity. In the purely classical case, we find that when the stress-energy tensor $T_{0i} = 0$, then $n^i = 0$, but we will show that in the combined CQ case $n^i \neq 0$ even in the absence of the stress-energy tensor. Instead, a non-zero n^i is required to preserve the theory's Hamiltonian constraint.

With the gauge fixing of Equation (4.6), the linearized action in Equation (4.3) is

$$\begin{aligned}\mathcal{S} = \frac{1}{8\pi G}\int d^4x\Big[&-\frac{3\dot{\psi}^2}{c^2} + \frac{\partial_i n^i}{2c^2}(\dot{\Phi} - \dot{\psi}) - \frac{\dot{n}^i}{2c^2}(\partial_i\Phi + 3\partial_i\psi) - \frac{1}{4c^2}\partial_i n^j\partial_j n^i \\ &+ \partial_i\psi\partial^i\psi - 2\partial_i\Phi\partial^i\psi + \frac{1}{4c^2}\partial_i n^j\partial_j n^i\Big].\end{aligned} \tag{4.7}$$

To go to the Hamiltonian picture, we first calculate the functional derivatives with respect to $\dot{\psi}$, $\dot{\phi}$ and $\dot{n}^i$ to find the conjugate momenta

$$\pi_\psi = -\frac{1}{16c^2G\pi}(12\dot{\psi} + \partial_i n^i),\ \pi_\Phi = \frac{\partial_i n^i}{16\pi c^2 G},\ \pi_i = -\frac{1}{16\pi c^2 G}(\partial_i\Phi + 3\partial_i\psi). \tag{4.8}$$

We see that Equation (4.8) defines two primary constraints. As a reminder, in Hamiltonian mechanics, a primary constraint is a relation between a coordinate and its conjugate momenta that holds regardless of the equation of motion. Here, the equations for π_Φ and π_i are constraints because $\dot{\pi}_\Phi$ and $\dot{\pi}_i$ will depend only on the initial state i.e. first order derivatives of the canonical coordinates, while only the evolution equation for π_ψ will depend on second order derivatives. We thus have two primary constraints:

$$\Pi_\Phi = \pi_\Phi - \frac{\partial_i n^i}{16\pi c^2 G} \approx 0, \tag{4.9}$$

$$\Pi_i = \pi_i + \frac{1}{16\pi c^2 G}(\partial_i\Phi + 3\partial_i\psi) \approx 0, \tag{4.10}$$

where ≈ 0 means *weakly zero* in the Dirac sense [187], that the quantity is set to 0 by constraining the initial data. In the $c \to \infty$ limit, Equations (4.9-4.10) become the constraints $\pi_\Phi, \pi_i \approx 0$, which enforce the usual Hamiltonian and momentum constraints.

One might worry about the fact that the constraints (4.9-4.10) derived from the (Fierz-Pauli) linearised gravity action appear to be different with respect to the constraints one obtains by first considering the constraints in the full ADM formalism and then linearising them. This has been studied in [188], where it was shown that the two forms are related by a canonical transformation. Alternatively, one could follow the approach of [187] and add a specific non-covariant term to the linearised action (4.3). This term vanishes on shell and simplifies the primary constraints to match them with those derived from the ADM formalism.

Since we are interested in the $c \to \infty$ limit, these distinctions do not matter, as we end up imposing the constraints $\pi_\Phi, \pi_i \approx 0$, which are equivalent to the primary constraints $\pi_N, \pi_{N_i} \approx 0$, where N, N^i are the lapse and shift vectors.

Using the definitions of conjugate momenta in Equation (4.8) and working to leading order in c, we arrive at the Newtonian Hamiltonian

$$H^{(gr)} = \int d^3x \left[-\frac{2\pi G c^2}{3}\pi_\psi^2 - \frac{1}{12}\pi_\psi \partial_i n^i + \frac{\partial_i \psi \partial_i \Phi}{4\pi G} - \frac{\partial_i \psi \partial_i \psi}{8\pi G} + \lambda_\Phi \pi_\Phi + \lambda^i \pi_i \right]. \tag{4.11}$$

We need to couple gravity with matter to find the Newtonian interaction Hamiltonian. When a matter action S_m is included, the coupling to the linear perturbation is found via $h_{\mu\nu}T^{\mu\nu}$, which is required to reproduce Einstein's equations. We shall consider the matter distribution to be that of a particle with mass density $m(x)$ and, because we are working in the non-relativistic limit, we shall assume that only T_{00} acts as a source for the gravitational field. The corresponding interaction Hamiltonian can be then easily written as

$$H^I = \int d^3x \, \mathcal{H}^I = \int d^3x \, \Phi(x) m(x), \tag{4.12}$$

such that the total Hamiltonian is given by $H_{tot} = H^{(gr)} + H^I$:

$$H_{tot} = \int d^3x \left[-\frac{2\pi G c^2}{3}\pi_\psi^2 - \frac{1}{12}\pi_\psi \partial_i n^i + \frac{\partial_i \psi \partial_i \Phi}{4\pi G} - \frac{\partial_i \psi \partial_i \psi}{8\pi G} + \lambda_\Phi \pi_\Phi + \lambda^i \pi_i + \Phi(x) m(x) \right]. \tag{4.13}$$

The dynamics associated with H_{tot} can be derived from Hamilton's equations and is given by:

$$\dot{\psi} = -\frac{4G\pi c^2 \pi_\psi}{3} - \frac{1}{12}\partial_i n^i, \ \dot{\pi}_\psi = \frac{\nabla^2(\Phi - \psi)}{4\pi G}, \ \dot{\Phi} = \lambda_\Phi, \ \dot{\pi}_\Phi = \frac{\nabla^2 \Phi}{4\pi G} - m, \ \dot{n}^i = \lambda^i, \ \dot{\pi}_i = -\frac{1}{12}\partial_i \pi_\psi \tag{4.14}$$

and we arrive at Newtonian gravity by solving Equation (4.14) since the constraint $\pi_\Phi \approx 0$ imposes

$$\frac{\nabla^2 \Phi}{4\pi G} - m \approx 0 \quad \Rightarrow \quad \Phi(t,x) = -G \int d^3x' \, \frac{m(x')}{|x - x'|} \tag{4.15}$$

on the potential Φ i.e, Φ must solve Poisson's equation. On the other hand, the constraint $\pi_i \approx 0$ imposes $\pi_\psi \approx 0$, where we have used the fact that π_ψ vanishes at infinity. Preservation of the $\pi_\psi \approx 0$ constraint imposes that $\Phi = \psi$. Moreover, the time derivative of the Newtonian potential directly dictates the Lagrange multiplier via $\lambda_\Phi = \dot{\Phi}$ and the divergence part of the shift vector via:

$$\dot{\Phi} = -\frac{1}{12}\partial_i n^i. \tag{4.16}$$

Since we assume a stationary source, where only T_{00} contributes, this imposes that $\partial_i n^i = 0$. Note this does not entirely fix the shift n^i, and solutions related by different choices of the shift vector will be gauge equivalent. In the classical theory, it is common to assume the gauge $n^i = 0$, in which case we arrive at the Newtonian metric of Equation (4.1), where Φ satisfies Poisson's equation.

We have arrived at the Newtonian limit of general relativity by making the Newtonian approximation on the metric in Equation (4.6) and then deriving the dynamics in the $c \to \infty$ limit. While deriving the Newtonian limit from a full GR approach requires a complete diffeomorphism invariant theory, we have seen that we can construct a consistent reduced theory by first identifying the correct degrees of freedom (in this case, scalar perturbations of the metric) and then writing down their dynamics according to a reduced Hamiltonian.

Before discussing how a quantum system's back-reaction on the classical Newtonian field is implemented through diffusion processes, we would like to mention our choice of gauge. The end goal of this work is to formulate the Newtonian limit of gravity for CQ-hybrid theories; we do not know if a complete CQ theory can be made fully diffeomorphism invariant. Regardless, our choice of gauge is motivated by the need to preserve the gravitational constraints. By choosing the gauge (i.e. coordinates) as in (4.6), we know that we have a way of consistently selecting trajectories that stay on the constraint surface, where the conjugate momenta vanish as described in this section.

4.1.2 The weak field classical limit

We will first consider a purely stochastic modification to classical general relativity. With the benefit of hindsight, it corresponds to taking the classical limit of the matter fields in the CQ theory. However, we present it here first, partly as a simple example and partly because, for some experiments, it is the regime of interest. It also provides an interesting analogy with quantum gravity. Note that although this limit is a stochastic theory of classical gravity, it is different to what is usually referred to as *stochastic gravity* [189], which is based on the semi-classical Einstein's equations and is interpreted as an effective theory. For example, the theory here is Markovian, while stochastic gravity needs to be non-Markovian in the case of statistical mixtures of states with a significant variance.

In the CQ case, we construct the Newtonian limit by taking the non-relativistic limit of the complete general relativistic theory, which leads to the relevant degrees of freedom being scalar perturbations of the metric of the form of Equation (4.6), and then considering a reduced CQ master equation governing the dynamics of the perturbations. Since we will be interested in describing the non-relativistic limit of a quantum mass interacting with classical gravity, the back-reaction on the gravitational field from the quantum matter is dominated by the T_{00} component, and any classical-quantum momentum constraint [136] will be unchanged since it does not involve matter. In particular, the back-reaction of the quantum system on the classical system enters through the π_Φ equation in (4.14). Loosely speaking, because quantum back-reaction must necessarily involve diffusion, the equation of motion for $\dot{\pi}_\Phi$ will be modified to include a stochastic term. To gain some intuition, we can consider the classical analogue by considering a Langevin equation for $\dot{\pi}_\Phi$

$$\dot{\pi}_\Phi = \frac{\nabla^2 \Phi}{4\pi G} - m - \sigma\xi, \tag{4.17}$$

where $\sigma(x)$ is a coefficient and $\xi(t, x)$ is a white noise process which we will relate to the D_2 coefficient appearing in the CQ master equation (2.9). We will later find that this stochasticity is all that is required to maintain complete positivity in the CQ case. With the modified dynamics for π_Φ, we find the constraint $\pi_\Phi \approx 0$ imposes $\frac{\nabla^2\Phi}{4\pi G} = m + \sigma\xi$ on the potential Φ. The momentum constraint π_i remains unchanged from the deterministic case, and its preservation again imposes the constraint $\Phi = \psi$. However, with the addition of the noise, the solution to the Newtonian constraint is no longer stationary but is instead given by a solution to the random Poisson equation

$$\Phi = -G \int dx' \, \frac{m(x') + \sigma(x')\xi(x', t)}{|x - x'|}. \tag{4.18}$$

Preservation of the Hamiltonian constraint in Equation (4.17) then determines λ_Φ and $\partial_i n^i$. In particular, with the gauge choice of Equation (4.6), we see that n^i is required in order for the theory to be consistent. The presence of diffusion, combined with the quantum back-reaction, will make the Newtonian potential $\psi = \Phi$ fluctuate, and its evolution will be determined by the shift vector $\partial_i n^i$ via Equation (4.14) or more immediately, via Equation (4.16) in particular. This is one of the key results of this chapter, and we will return to it when we discuss the master equation approach. Without allowing the shift to be a freely chosen gauge parameter, the momentum conjugate to the Newtonian potential Φ looks like it will diffuse off to infinity via the random walk process given by Equation (4.17).

We point out once again that this does not fix the shift n^i uniquely since we are free to add a divergenceless term and get the same solution to the equation of motion. Moreover, in a complete calculation, we expect that contributions from T_{0i} will also determine the components n^i without affecting the Newtonian contribution given by the h_{00} component. Regardless, performing higher-order calculations is beyond the scope of the current work.

In the stochastic case, we still find that $\Phi = \psi$ is set by the dynamics since the addition of noise in Equation (4.17) is the only modification to the theory. Hence, one can instead start with the metric perturbation:

$$N = \left(1 + \frac{\Phi}{c^2}\right), \; N^i = \left(0 + \frac{n^i}{c^3}\right), \; g_{ij} = \left(1 - \frac{2\Phi}{c^2}\right)\delta_{ij}. \tag{4.19}$$

and consider the dynamics obtained by setting $\Phi = \psi$ in Equation (4.11). One could even remove the kinetic term $-\frac{2\pi G c^2}{3}\pi_\psi^2$, which doesn't contribute to the equations of motion on the constraint surface, and the Lagrange multiplier term π_Φ, which is redundant as the Hamiltonian is also linear in π_ψ. In this case, the reduced Hamiltonian reads.

$$H^{(gr)} + H^I = \int d^3x \left[\frac{(\nabla\Phi)^2}{8\pi G} + m\Phi - \frac{1}{12}\pi_\Phi \partial_i n^i + \lambda^i \pi_i\right]. \tag{4.20}$$

Equations (4.19) and (4.20) are considerably simpler than Equations (4.6) and (4.11) but result in the same dynamics, we will use the former to describe the Newtonian limit of CQ theories.

Note that due to the white noise process, the metric perturbation will technically describe a probability measure, and we should use it to compute averaged quantities. This is true of both Φ and the shift vector n^i, which are now both stochastic quantities. In particular, averaging over a timescale ΔT and length scale ΔL, $\Delta L/\Delta T \ll c$, we have that $\frac{\Delta\Phi}{\Delta T} \sim \frac{n^i}{\Delta L}$, so that $\frac{\Delta L \Delta\Phi}{\Delta T} \sim n^i \ll c^3$ which verifies our initial assumption to include the perturbation h_{0i} as $\frac{n^i}{c^3}$ in Equation (4.6).

With this in mind, we now study the Newtonian limit of the complete CQ theory. In the $c \to \infty$ limit, we arrive at Poisson's equation on average. Still, because of the CQ interaction, the Newtonian limit also predicts diffusion around this solution according to Equation (4.18), with simultaneous decoherence on the quantum system. In Chapter 5, we study a diffeomorphism invariant theory of CQ scalar gravity and show that in the $c \to \infty$ limit, the results are quantitatively and qualitatively the same as in the reduced degrees of freedom approach. This gives us more confidence that our results are independent of our coordinate choice.

One can see the analogy between the quantum and the stochastic cases, where the order of operations of introducing the constraint and quantising/inserting noise matters. Unlike the quantum case, it is easier to see why the insertion of stochasticity has different effects depending on whether it is done before or after the phase space reduction. Imagine a system constrained to have spherical symmetry. If a non-spherically symmetric noise process is added, and then we project onto a spherically symmetric initial state, the noise process will continue and drive the system away from spherical symmetry unless the noise is also chosen to be spherically symmetric. On the other hand, if the system is constrained first so that all degrees of freedom can only depend on the radius, then the noise inserted can only be spherically symmetric. We comment more about this in Section 4.4. We are now in the position of deriving the Newtonian limit of the CQ path integral, utilising what we just discussed.

4.2 Newtonian limit from the gauge fixing of classical-quantum general relativity

In this section, we derive the Newtonian limit of the full general relativistic diffeomorphism invariant CQ path integral, which was first introduced in [130], here reported as Equation (4.22). To derive the Newtonian limit, we will start by gauge fixing the classical degrees of freedom using the gauge described in Equation (4.19), which is justified through the arguments of Section 4.1. We then take the non-relativistic limit by keeping only leading terms in the speed of light c.

This derivation leads to the same unravelled Newtonian limit that we will later obtain from the master equation in Equation (4.44) and acts as a sanity check for the theory introduced [130], showing that its constraints have a sensible non-relativistic limit. In the Newtonian limit, once the choice is made of keeping only the highest order terms in c, we find that the problematic off-diagonal terms appearing in the general relativistic CQ action of Equation (4.22) disappear. In other words, we show that the dynamics of Equation (4.22) defines completely positive dynamics on the subset of states defined by the Newtonian limit. We leave it as a question for further work whether the CQ constraints would be preserved in the more general case, and in particular if the dynamics of Equation (4.22) lead to stable dynamics which preserves the Newtonian limit once higher order terms are c are considered.

Consider the full diffeomorphism invariant theory of CQ general relativity, which, when coupled to a quantum mass density, has a path integral of the form:

$$\varrho(g_f, \phi_f^+, \phi_f^-, t_f) = \int \mathcal{D}g\mathcal{D}\phi^+\mathcal{D}\phi^- \ \mathcal{N}e^{\mathcal{I}_{CQ}[g,\phi^+,\phi^-_, t_i, t_f]}\varrho(g_i, \phi_i^+, \phi_i^-, t_i), \tag{4.21}$$

where $\mathcal{N}$ is a normalisation factor, and the action takes the form of:

$$\begin{aligned}\mathcal{I}_{CQ}[g,\phi^+,\phi^-,t_i,t_f] = \int_{t_i}^{t_f} dt d\vec{x} \Bigg[& i\big(\mathcal{L}_Q[g,\phi^+] - \mathcal{L}_Q[g,\phi^-]\big) \\ & - \frac{\mathrm{Det}[-g]}{8}\big(T^{\mu\nu}[\phi^+] - T^{\mu\nu}[\phi^-]\big)D_{0,\mu\nu\rho\sigma}[g]\big(T^{\rho\sigma}[\phi^+] - T^{\rho\sigma}[\phi^-]\big) \\ & - \frac{\mathrm{Det}[-g]c^8}{128\pi^2 G_N^2}\left(G^{\mu\nu} - \frac{8\pi G}{c^4}\bar{T}^{\mu\nu}[\phi^+,\phi^-]\right)D_{0,\mu\nu\rho\sigma}[g]\left(G^{\rho\sigma} - \frac{8\pi G}{c^4}\bar{T}^{\rho\sigma}[\phi^+,\phi^-]\right)\Bigg],\end{aligned} \tag{4.22}$$

where $\mathcal{L}_Q$ is the quantum Lagrangian density, including the appropriate metric factors, $\bar{T}[\phi^+,\phi^-]$ is the average of the left and right branches of the stress-energy tensor. We have taken D_0, D_2 to saturate the decoherence-diffusion trade-off (3.20) such that both the decoherence and diffusion coefficients are written in terms of D_0. Here, the bra and ket fields $\phi^{\pm}$ can be any quantum fields, but we shall consider pressureless dust $m^{\pm}(x)$ as a particular case.

We now take the couplings to be ultra-local, meaning that they will depend solely on the metric and not on its derivatives so that we can write them in terms of the generalised deWitt metric [59, 190]:

$$D_{0,\mu\nu\rho\sigma} = \frac{1}{2}\frac{D_0}{\sqrt{-g}}\big(g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} - 2\beta g_{\mu\nu}g_{\rho\sigma}\big). \tag{4.23}$$

To obtain the Newtonian limit, we write the path integral in an ADM formalism, described by summing over all paths of the lapse, shift, and spatial metric (N, N^i, γ_{ij}) and inserting the choice of matter field as a pressureless dust distribution. We then consider the action as a functional of the variables appearing in the ADM decomposition $\mathcal{I}_{CQ}[N, \vec{N}, \gamma_{ij}, m^+, m^-]$.

The Newtonian limit can then be understood as a gauge fixing of the complete theory, computing the transition amplitudes between the CQ states defined on hypersurfaces Σ_t:

$$\begin{aligned}\varrho(\gamma_f, m_f^+, m_f^-, t_f) = \int \mathcal{D}\gamma\mathcal{D}N\mathcal{D}\vec{N}\mathcal{D}m^+\mathcal{D}m^- \; \delta\left(\gamma_{ij} - \left(1 - \frac{2\Phi}{c^2}\right)\delta_{ij}\right)\delta\left(N - \left(1 + \frac{\Phi}{c^2}\right)\right) \\ \times\, \delta\left(N^i - \frac{n^i}{c^3}\right) e^{\mathcal{I}_{CQ}[N,\vec{N},\gamma_{ij},m^+,m^-,t_i,t_f]}\varrho(\gamma_i, m_i^+, m_i^-, t_i).\end{aligned} \tag{4.24}$$

Performing the delta functional integrals, we impose the Newtonian gauge. In particular, we have $g_{00} = -(cN)^2 \approx -(c^2 + 2\Phi)$, whilst $g_{ij} \approx \left(1 - \frac{2\Phi}{c^2}\right)$ and $g_{0i} = \frac{n^i}{c^3}$. The components of the Einstein tensor are calculated as

$$G_{00} = -2\nabla^2\Phi, \tag{4.25}$$

$$G_{0i} = -\frac{2}{c^5}\partial_0\partial_i\Phi + \frac{1}{2c^5}\nabla^2 n_i, \tag{4.26}$$

$$G_{ij} = -\frac{2}{c^4}\partial_t\partial_t\Phi. \tag{4.27}$$

Similarly, noting that $\det(-g) \approx c^2$, we see that due to the powers of c, the de-Witt metric is dominated by its 0000 component, which to leading order is given by

$$D_{0,0000} = D_0 c^3(1 - \beta). \tag{4.28}$$

Keeping only terms leading order in c, the path integral action in Equation (4.22) is dominated by terms only involving D_{0000} and leads to the Newtonian path integral:

$$\varrho(\Phi_f, m_f^+, m_f^-, t_f) = \mathcal{N}\int \mathcal{D}\Phi\mathcal{D}m^+\mathcal{D}m^-\ e^{\mathcal{I}_{CQ}[\Phi, m^+, m^-, t_i, t_f]}\varrho(\Phi_i, m_i^+, m_i^-, t_i), \tag{4.29}$$

with CQ action given by:

$$\begin{aligned}\mathcal{I}_{CQ}[\Phi, m^+, m^-, t_i, t_f] = \int_{t_i}^{t_f} dtd\vec{x}\ \Big[& i\big(\mathcal{L}_Q[m^+] - \mathcal{V}_I[\Phi, m^+] - \mathcal{L}_Q[m^-] + \mathcal{V}_I[\Phi, m^-]\big) \\ & - \frac{\tilde{D}_0}{2}\big(m^+(x) - m^-(x)\big)^2 - 2\tilde{D}_0\left(\frac{\nabla^2\Phi}{4\pi G} - \bar{m}(x)\right)^2\Big].\end{aligned} \tag{4.30}$$

were $\bar{m}(x) = \frac{1}{2}\big(m^+(x) + m^-(x)\big)$, we have redefined $\tilde{D}_0 = \frac{c^5 D_0}{4}(1-\beta)$, $\mathcal{L}_Q[m^\pm]$ is the matter Lagrangian in flat spacetime and $\mathcal{V}_I[\Phi, m^\pm] \propto m^\pm\Phi$ is the interaction potential coming from the expansion of $\sqrt{-g} \approx c - 2\Phi$ in the matter Lagrangian for curved spacetime. Sources coupled to the classical or quantum degrees of freedom could be added if needed.

We have arrived at the final form of the Newtonian CQ path integral. Since it gives the state of the gravitational field for any quantum state of matter, it can be thought of as the constraint equation, consistent with the proposal in [130]. This is one of the main results of this Chapter. Equation (4.30) describes an integral over paths of the classical Newtonian potentials and a doubled path integral over the quantum mass eigenstates $m^\pm$, which occur because the integral is a density matrix path integral over both bra and ket branches. Hence, provided $\beta \leq 1$, we find that the dynamics of the $c \to \infty$ limit of the full theory of Equation (4.22) gives rise to complete positive evolution, which describes a randomly sourced Poisson's equation with associated decoherence into mass eigenstates of the quantum state. This justifies from a top-down approach the Newtonian limit that will be derived in Equation (4.44) and gives rise to the hope that the theory [130] has constraints preserved in time. However, we highlight that, in this derivation, we have arrived at the Newtonian limit by gauge fixing the full theory and neglecting all the terms of higher order in c. Since we have neglected terms of higher order in c, we have implicitly eliminated the potentially positivity-violating terms of the full path integral (4.22). The terms involving D_{000i}, D_{0i0j} still arise but are higher order in c. Therefore, we have shown that the dynamics of Equation (4.22) defines completely positive dynamics on the *subset* of states defined by the Newtonian limit. Still, we have not shown that the dynamics are consistent away from this limit. For example, the evolution could be unstable for finite c, but we leave this as a question for future research. Nevertheless, a more general treatment is essential since we have not shown that the complete theory necessarily preserves the form of the Newtonian limit. A possible outcome might be that including higher-order terms in the calculation causes the dynamics to deviate from the correct limit. The clearest example of this hypothetical behaviour can be seen by considering that, in deriving the Newtonian limit, we have assumed that Poisson's equation holds on average at any scale. At this point, we can study how to obtain the Newtonian limit in the master equation formulation of the CQ framework.

4.3 Newtonian limit in the master equation formalism

In this section, we derive the Newtonian limit using the master equation formalism introduced in Chapter 2. The dynamics will match the behaviour derived in Equations (4.21) and (4.22). Before discussing the specifics of continuous and discrete master equations, we shall outline the general procedure and assumptions.

Assumption 1. *We assume that the CQ dynamics takes the form of Equation* (2.9)

Since Equation (2.9) is the most general form of Markovian, classical-quantum evolution, we expect this assumption to hold if we are treating this as a fundamental theory. If, on the other hand, it is an effective theory, the Markovian assumption may break down. We comment on the differences between a fundamental and effective theory in section 4.4 (see also the appendices of [5]).

Assumption 2. *In the weak field $c \to \infty$, the appropriate gravitational degrees of freedom are the perturbations of the metric in the form of Equation* (4.19)*.*

In particular, the leading order contribution which governs the geodesics of test particles is described by h_{00}. This is a bottom-up approach in the sense that we reduce the degrees of freedom in the action and then construct the CQ theory.

Assumption 3. *We assume that the purely classical part of the evolution is generated by the reduced Hamiltonian* (4.20)*, that the interaction between classical and quantum degrees of freedom is Hamiltonian and that it is governed by the reduced interaction Hamiltonian in Equation* (4.12)*, where the constraints $\pi_\Phi, \pi_i \approx 0$ should also be imposed.*

Specifically, we require that the first moment D_1 is chosen to reproduce the Newtonian back-reaction on average:

$$\begin{aligned} \mathrm{Tr}[\{H^I, \varrho\}] &= \int d^3x \ \mathrm{Tr}\left[\hat{m}(x)\frac{\delta\rho}{\delta\pi_\Phi(x)}\right] \\ &= -\sum_{\mu\nu\neq 00}\int d^3x \, \mathrm{Tr}\left[D^{\mu\nu}_{1,\pi_\Phi}(\Phi,\pi_\Phi,x)\, L_\mu(x)\frac{\delta\varrho}{\delta\pi_\Phi}L^\dagger_\nu(x)\right], \end{aligned} \tag{4.31}$$

which requires

$$\left\langle D_{1,\pi_\phi}(\Phi,\pi_\Phi,x)\right\rangle = -\langle\hat{m}(x)\rangle, \tag{4.32}$$

so that the dynamics are Hamiltonian on average. While this might appear to be a mild assumption, it does assume that the coupling strength and gravity itself is either unmodified at arbitrarily short distances, or at least that the short distance behaviour does not affect physics in the weak field regime. As a consequence of this coupling constant, a non-zero D_{1,π_ϕ} implies that there must be diffusion in the momenta conjugate to Φ.

Since the back-reaction of the quantum system on the classical system is associated with T_{00}, we expect the CQ momentum constraint to be unchanged from its classical counterpart, as it is not associated with any back-reaction. This was also found to be the case in a study of CQ gravitational constraints [136], and in the scalar gravity theory we consider in [7].

Assumption 4. *In this work, we will take the coefficients D_n entering the master equation to be minimally coupled, by which we mean they depend only on the Newtonian potential Φ, $D_n(\Phi)$ and not their conjugate momenta π_Φ.*

This assumption is motivated by the fact in Einstein's gravity, the mass density couples to the Newtonian potential and not its conjugate momenta, and we are imposing the constraint that $\pi_\Phi \approx 0$, which would make such terms vanish. Nonetheless, one could generalise the master equations to the non-minimally coupled case by considering couplings $D_n(\Phi) \to D_n(\Phi, \pi_\Phi)$ in all of the equations.

We now consider the dynamics consistent with assumptions 1-4. We only discuss the weak field limit for continuous master equations since, in this case, we can be more thorough and explicit. Then, we will use the unravelling of the weak field limit to impose the Newtonian constraints and obtain our main result as a set of coupled stochastic differential equations describing the Newtonian CQ dynamics.

4.3.1 Continuous gravitational back-reaction

In [2], it was shown that there are two classes of CQ master equations. One of the forms includes finite-sized jumps in the classical degrees of freedom due to the backreaction of the quantum part of the system, while in the other, the evolution remains continuous. The most general form of the CQ continuous master equation was then explicitly given as Equation (2.16). When the back-reaction is continuous, specifying that the first moment on average satisfies Equation (4.31) is enough to fix the terms of Equation (2.16), which correspond to the continuous back-reaction of the classical and quantum degrees of freedom onto each other. As discussed below, this is only the continuous part of the backreaction since the stochastic nature imposed by the Newtonian constraints on n^i will introduce further jumping backreaction terms. The continuous backreaction is composed of the decoherence and diffusion effects described by:

$$\begin{aligned}&\frac{1}{2}\int d^3x\big(\{\mathcal{H}^I(x),\varrho\}-\{\varrho,\mathcal{H}^I(x)\}\big)+\int d^3xd^3y\, D_2(\Phi;x,y)\frac{\delta^2\varrho}{\delta\pi_\Phi(x)\pi_\Phi(y)}\\&\quad+\frac{1}{2}\int d^3xd^3y\, D_0(\Phi;x,y)\left([\hat{m}(x),[\varrho,\hat{m}(y)]]\right),\end{aligned} \tag{4.33}$$

where π_Φ, and $D_0(\Phi;x,y), D_2(\Phi;x,y)$ are positive semi-definite kernels[1], and $\mathcal{H}^{\mathcal{I}}=\Phi(x)\hat{m}(x)$ is the interaction Hamiltonian density. Here, $\hat{m}(x)$ is the quantum mass density operator. For notational simplicity, we shall often suppress the dependence of the couplings D_0, D_2 on the Newtonian potential and write $D_0(x,y), D_2(x,y)$. The Lindbladian term, characterised by D_0, and the diffusion term D_2 are required for the back-reaction to be completely positive, which can be seen from the decoherence diffusion trade-off [3] apparent in Equation (2.16). Adding extra diffusion or decoherence terms is possible and still satisfies the conditions for the master equation to be completely positive. Still, here, we only consider the minimal amount of decoherence and diffusion required.

The entire master equation will also include terms associated with the pure Hamiltonian evolution of the quantum state, with the Hamiltonian given by Equation (4.20),

$$H^{(gr)}+H^I=\int d^3x\left[\frac{(\nabla\Phi)^2}{8\pi G}+\hat{m}\Phi-\frac{1}{12}\pi_\Phi\partial_i n^i+\lambda^i\pi_i\right], \tag{4.34}$$

[1] Recall a positive semi-definite kernel $f(x,y)$ is a kernel such that $\int dxdya^*(x)f(x,y)a(y)\geq 0$ for any function $a(x)$.

where the mass density $m(x)$ has been replaced by the operator $\hat{m}(x)$. This Hamiltonian contains Φ and n^i which will become stochastic degrees of freedom in order to impose the $\pi_\Phi \approx 0$ constraint. As a result, care must be taken, since this will result in the master equation containing additional Fokker-Plank and jump terms of the form

$$\alpha(\Phi, n^i)\frac{\delta^{k+l}\varrho}{\delta^k\Phi\delta^l\pi_\Phi} \quad k+l \geq 2 \tag{4.35}$$

associated with the correlations between Φ and n^i. Choosing n^i stochastically will back-react on Φ, giving a master equation with infinite terms that enforce the constraint by forcing the shift to have the necessary correlation with the stochastic gravitational field.

Since the only degree of freedom in Equation (4.14) associated with the matter back-reaction is π_Φ, up to these nuances, the choice of the possible master equation is therefore fully constrained up to the functional choices of the couplings $D_0(\Phi; x, y), D_2(\Phi; x, y)$, which are themselves constrained to satisfy the decoherence diffusion trade-off:

$$4D_2 \succeq D_0^{-1}. \tag{4.36}$$

Equation (4.36) is to be understood as a matrix kernel equation:

$$\int dxdy\ a(x)^*\big[4D_2(\Phi; x, y) - D_0^{-1}(\Phi; x, y)\big]a(y) \geq 0, \tag{4.37}$$

which must hold for an arbitrary function $a(x)$. In Equation (4.37), $D_0^{-1}(\Phi; x, y)$ is the generalised kernel inverse of the diffusion coupling $D_0(\Phi; x, y)$ [3] which is only required to be a positive semi-definite kernel. We give example kernels that satisfy the decoherence diffusion trade-off in Table 4.1. The decoherence diffusion condition in Equation (4.36) can be used to experimentally constrain fundamental theories with a classical gravitational field [3]. Before discussing the experimental bounds on the dynamics described by Equation (4.33), we must first impose the Newtonian constraint $\pi_\Phi \approx 0$.

4.3.2 Newtonian unravelling of the master equation

To arrive at the classical-quantum version of Poisson's equation, we must impose the constraint $\pi_\Phi \approx 0$ according to the Hamiltonian in Equation (4.20). In classical Hamiltonian dynamics, typically, one imposes constraints on initial data. One then enforces the preservation of the constraints by the dynamics via the Dirac procedure [187]. Because in Hamiltonian dynamics, there is an isomorphism between initial physical data and physical solutions to the dynamics, this procedure is equivalent to imposing the constraint on solutions to the dynamics since, by construction, the dynamics never leave the constraint surface.

In the non-deterministic case, where the dynamics contain randomness, there is no isomorphism between initial data and the solution space of the dynamics. Instead, one builds a probability distribution over the possible trajectories of the initial data according to the dynamics and imposes constraints on the trajectories.[2] As such, when we implement the constraint in the classical-quantum case, we have to ensure that the dynamics remain CP.

Given that we are imposing the constraint $\pi_\Phi \approx 0$ on the level of trajectories, it is more convenient to go to an unravelling picture, which enables us to discuss explicitly classical-quantum trajectories which satisfy the constraint. The unravelling picture of CQ dynamics clearly presents the Chapter's results. It also allows for an ontological interpretation of the trajectories and for the ease associated with simulating their time development with a computer.

The unravelling of the weak field master equation with continuous backreaction given by Equation (4.33) is derived by substituting the Hamiltonian drift terms in Equation (2.26) which is the general expression for the unravelling of the continuous master equation derived in [5]. Recalling that the mass density operator $\hat{m}$ is Hermitian, this results in the following coupled Itô stochastic differential equations:

[2]This is conceptually very similar to what is done in quantum theory when constraints are imposed via a path integral approach, where one associate to each path a measure given by the action, then selects only the paths which satisfy the constraint. Take for example, a Hamiltonian with $H(q,p) = H_0(q,p) + \lambda C(q,p)$. The phase space partition function for the theory is represented by $\mathcal{Z} = \int \mathcal{D}q\mathcal{D}p\mathcal{D}\lambda e^{\frac{i}{\hbar}\int dt[\dot{q}p - H(q,p) - \lambda C(q,p)]}$. Since the Hamiltonian is linear in λ, the path integral over the Lagrange multiplier in λ enforces a delta function over $\delta(C(q,p))$ so that the partition function reads $\mathcal{Z} = \int \mathcal{D}q\mathcal{D}p\delta(C(q,p))e^{\frac{i}{\hbar}\int dt[\dot{q}p - H(q,p)]}$ which can be interpreted as summing over all paths with weight $e^{\frac{i}{\hbar}\int dt[\dot{q}p - H(q,p)]}$ and then selecting only those that satisfy the constraint $C(q(t),p(t)) = 0$.

$$d\Phi_t = -\frac{1}{12}\partial_i n^i dt, \tag{4.38}$$

$$d\pi_{\Phi\, t} = \frac{\nabla^2\Phi_t}{4\pi G}dt - \langle \hat{m}(x)\rangle dt - \int d^3y\, \sigma(\Phi_t; x, y) dW_t(y), \tag{4.39}$$

$$\begin{aligned} d\rho_t = -i[\hat{H}_m + \hat{H}^I, \rho_t]dt + \frac{1}{2}\int d^3x\, d^3y\, D_0(\Phi_t; x, y)[\hat{m}(x), [\rho_t, \hat{m}(y)]]dt \\ + \frac{1}{2}\int d^3x\, d^3y\, \sigma^{-1}(\Phi_t; x, y)\big(\hat{m}(x)\rho_t + \rho_t\, \hat{m}(x) - 2\rho_t\langle \hat{m}(x)\rangle\big)dW_t(y) \end{aligned} \tag{4.40}$$

where $\hat{H}_m$ is the matter Hamiltonian, $\hat{H}^I = \int d^3x\, \hat{m}(x)\Phi(x)$ is the interaction Hamiltonian, $\langle\cdot\rangle$ is the usual expectation value, $\rho(t)$ is the normalized quantum state and $W_t(x)$ is a Wiener process in spacetime satisfying:

$$\mathbb{E}[dW_t(x)] = 0, \quad \mathbb{E}[dW_t(x)dW_t(y)] = \delta(x, y)dt. \tag{4.41}$$

In Equation (4.38), σ and its generalised inverse σ^{-1} are related to the diffusion coefficient D_2 appearing in Equation (4.33) via:

$$D_2(\Phi; x, y) = \int dw\ \sigma(\Phi; x, w)\sigma(\Phi; y, w). \tag{4.42}$$

One can verify that this unravelling is equivalent to the CQ master equation with continuous backreaction given by Equation (4.33) by using it to compute the evolution of the CQ state defined via:

$$\varrho(\Phi, \pi_\Phi, t) = \mathbb{E}[\delta(\Phi_t - \Phi)\delta(\pi_{\Phi\, t} - \pi_\Phi)\rho_t]. \tag{4.43}$$

We show this explicitly in Appendix C, which is enough to verify that this is the correct unravelling for the master equation since, differently from the purely quantum Lindblad equations, the unravelling is unique [5]. The classical Hamiltonian theory described in Equation (4.20), which we have made stochastic in Section 4.1, is equivalent to the dynamics for Φ in Equation (4.38) once we have imposed the $\pi_\Phi \approx 0$ constraint. The apparent difference is that, in the CQ case, the noise process is not added manually but emerges due to positivity requirements after directly coupling the quantum matter degrees of freedom with the classical Newtonian potential. Hence, the back-reaction in π_Φ turns it into a stochastic process. Moreover, the noise now is correlated with the quantum state, and the evolution of the quantum state itself involves decoherence due to the backreaction on the classical Newtonian potential. Indeed, the evolution of the quantum state is equivalent to that of a state undergoing continuous measurement of its mass. However, the quantum state is purely conditioned on the evolution of the Newtonian potential. It is only after tracing out the gravitational field that the decoherence is made manifest. We refer the reader interested in a detailed discussion to [5]. In particular, this defines a linear master equation.

The final step is imposing the constraint $\pi_\Phi \approx 0$ to arrive at the Newtonian limit. This can be done directly on the classical quantum evolution of Equation (4.38) or at the path integral level through delta functionals. We present here the former way, but the latter procedure is presented in Appendix B where, after constructing the path integral for the reduced gravitational degrees of freedom, we impose the Newtonian constraints at the level of trajectories.

To impose the constraint $\pi_\Phi \approx 0$ on Equation (4.38), one must choose n_i stochastically such that $\dot{\Phi} \approx \dot{\pi}_\Phi \approx 0$, where the equality is weak in the Dirac sense. Doing so turns Φ_t into a white noise variable with values given by the solution of Equation (4.44). However, naively replacing all occurrences of Φ_t with its solution in terms of an Itô white noise variable, particularly that which appears in $\hat{H}^I$, does not lead to completely positive and trace-preserving dynamics. Before the constraints are imposed, the dynamics of Φ_t are continuous. Thus, any back reaction from the quantum matter on Φ_t only returns to affect the quantum matter degrees of freedom at later times. To ensure that this time-ordering is maintained even in the limit that Φ_t no longer evolves continuously, one must be careful to ensure the action of $\hat{H}^I$ occurs after the other stochastic terms (for excellent further discussion on this issue of time-ordering, we refer the reader to [110], and [191, 192]). One possible way to ensure this is to write the unravelling of the density matrix in the Stratonovich formalism [193] and then impose the constraint that turns Φ_t, and hence $\hat{H}^I$, into white noise. The Stratonovich and Itô formalisms are two different approaches to interpreting stochastic differential equations. The Itô formalism is the most commonly used and is characterised by its non-anticipative nature, meaning the increments of the stochastic process are independent of the current state. This results in simpler mathematical properties, especially when using Itô's calculus. On the other hand, the Stratonovich formalism is more aligned with traditional calculus. It preserves the chain rule, making it useful in physical applications requiring an intuitive interpretation of stochastic integrals. The choice between the two formalisms depends on the specific problem and the desired properties of the solution.

Here, choosing the Stratonovich formalism allows us to correctly rewrite the unravelling such that when converting back into the Itô formalism, we pick up an extra decoherence term given by the backreaction of the stochastic gravitational field and allows us to get rid of the non-linear evolution terms arising from the solution of the noisy Poisson equation. This then gives the final form of unravelling in the Newtonian limit:

$$\frac{\nabla^2\Phi_t}{4\pi G} = \langle \hat{m}(x)\rangle + \int d^3y\,\sigma(\Phi_t;x,y)\xi_t(y), \tag{4.44}$$

$$\begin{aligned} d\rho_t = &-i\left[\hat{H}_m + \hat{V}_m, \rho_t\right]dt - i\int d^3x\,d^3y\,d^3y'\left[-G\,\frac{\hat{m}(x)\sigma(\Phi_t,y,y')}{|x-y|},\rho_t\right]dW_t(y') \\ &+\frac{1}{2}\int d^3x\,d^3y\,D_0(\Phi_t;x,y)\big([\hat{m}(x),[\rho_t,\hat{m}(y)]]\big)\,dt \\ &+\frac{1}{2}\int d^3x\,d^3y\,d^3y'\,[\hat{\sigma}(\Phi_t;x,y),[\rho_t,\hat{\sigma}(\Phi_t;x,y')]]\,dt \\ &+\frac{1}{2}\int d^3x\,d^3y\,\sigma^{-1}(\Phi_t;x,y)\big(\hat{m}(x)\rho_t + \rho_t\,\hat{m}(x) - 2\rho_t\langle\hat{m}(x)\rangle\big)\,dW_t(y), \end{aligned} \tag{4.45}$$

where $\xi_t(x) = \frac{dW_t(x)}{dt}$ is the formal definition of white noise, and

$$\begin{aligned} \hat{V}_m &= -\frac{G}{2}\int d^3x\,d^3y\frac{\hat{m}(x)\hat{m}(y)}{|x-y|}, \\ \hat{\sigma}(\Phi_t;x,y) &= -G\int d^3\,y''\frac{\hat{m}(x)\sigma(\Phi_t,y,y'')}{|x-y|}. \end{aligned} \tag{4.46}$$

These equations were first written down by Tilloy and Diósi in [110] and their derivation from a fundamental theory is a central result of our current work. In it, we notice how the Newtonian limit of CQ theories is described by a Newtonian potential diffusing around Poisson's equation by an amount defined by D_2, while simultaneously the density matrix decoheres into the mass eigenbasis by an amount determined by the Lindbladian coefficient D_0. In Equation (4.44), the Newtonian potential changes in time due to the random noise process $W(x)$ and its evolution fixes the divergence of the now stochastic n^i which, in general, will be correlated with the noise process appearing in the evolution of the quantum state. The fact that n^i is constrained not to vanish in order for the CQ theory to be consistent is another deviation from the standard Newtonian limit appearing in CQ gravity.

The details of the functional dependence of σ and D_0 on Φ_t have been left unspecified. However, three notable classes of functional dependence are worth highlighting. Firstly, they may not depend on Φ_t at all. In this case, the equations coincide with those of a continuous measurement of a quantum mass, where the measurement outcome is used to source the Newtonian potential, as given by equation (24) of [110]. Therefore, we have shown that such dynamics can be derived from classical-quantum theories of general relativity through a path integral approach or through the unravelling of a completely positive master equation that agrees with the Newtonian limit on expectation. Secondly, one may consider σ and D_0 to be dependent on a time integral of Φ_t, i.e. $\int dt f(t)\Phi_t$ for an arbitrary function $f(t)$. If allowed, such theories would be non-Markovian but still guaranteed to be CPTP. On the other hand, the final class of functional dependence is to allow a general Markovian functional of Φ_t. This will generically lead to additional terms, as was observed with H^I above, but these may not preserve the CPTP property of the dynamics. Exploring the details of these functional dependencies is an interesting question that we will leave open for future work.

In Equations (4.38) and (4.44), we have taken the drift to be local in x while we allow for the possibility that the decoherence and diffusion terms could have some range. In this case, the interaction law between the classical and quantum systems is still local, but non-local correlations can be created [194]. Significantly, if the Lindbladian coupling $D_0(\Phi; x, y)$ has some range, then despite the fact the CQ interaction is local, the master equation can, in principle, generate entanglement between two spatially separated quantum systems via the Lindbladian coefficient. However, this effect is likely to be small.

One can constrain other diffusion/decoherence kernels via Equation (4.44) and the decoherence diffusion trade-off in Equation (4.36). The Newtonian limit of CQ gravity predicts diffusion of the Newtonian potential by an amount depending on D_2. This can be upper bounded by precision mass experiments, which precisely measure the acceleration of particles. Conversely, coherence and heating experiments can be used to upper bound the inverse Lindbladian coefficient D_0^{-1}, which gives a lower bound on D_2 via the decoherence diffusion trade-off. Hence, when combined, it is possible to get an experimental squeeze on CQ theories. In [3] this was used to rule out ultra-local CQ theories, which are Equations (4.44) and (4.36) when the couplings are taken to be delta functions $D_0(x, y), D_2(x, y) \sim \delta(x, y)$.

Interestingly, when accounting for the stochasticity of the interaction Hamiltonian, the Newtonian limit we derive in Equation (4.36) contains a decoherence term proportional to σ^2, which gives bounds on decoherence due to constraints from anomalous heating. For the case where the coupling constants are independent of the Newtonian potential, the effects of the additional decoherence term were considered in [55]. In particular, it was shown that the choice of kernel giving rise to minimal decoherence is the Diósi-Penrose kernel $D_0(x, y) = \frac{G}{|x-y|}$. The precise amount of decoherence depends on the system's cut-off, and it was shown in [55] that theories with a cut-off below $10^{-15}m$ are inconsistent with experiments due to excess heating. This result calls for both an exploration of relativistic corrections to CQ theories, which we believe need to be considered at this scale, as well as experimental tests of gravity on smaller length scales.

4.4 Discussion

In this chapter, we have considered, on general grounds, the weak field limit of classical-quantum theories of gravity, which give rise to linear, completely positive dynamics. The master equation we derived is the weak field limit of the most straightforward realisation of the relativistic theory in [1]. In contrast, the path-integral we derive is the weak field limit of the manifestly covariant theory of [130]. Both approaches agree, as shown in Appendix B. The central new result is that we arrive at Equation (4.30) in the weak field limit.

We have here started from a fundamental, dynamical, and relativistic theory, and it is worthwhile to compare our limit to previous models proposed based on Newtonian gravity. An early model in which gravity is treated classically is the Schrödinger-Newton equation [121, 122, 183], which was also proposed as a model of gravitationally induced collapse of the wave-function [120, 123]. However, because the dynamics is non-linear in the wavefunction, it leads to instantaneous signalling [195, 196, 197] and a breakdown of the statistical mechanical interpretation of the density matrix. It is unrelated to the dynamics we have derived here, which is linear.

The master-equation approach used by us is more similar in spirit to that of Diósi's [124]. Indeed, the Newtonian back-reaction in Equation (4.33) has some similarities with the one considered in his work when the decoherence and diffusion kernels are chosen to be related to the Diósi-Penrose kernel (see Table 4.1). However, an essential difference between our master equation and Diósi's is that the latter contains diffusion in π_Φ, while here, we require $\pi_\Phi \approx 0$ as a constraint equation. This has to be the case here because in the weak field limit the kinetic energy term in Equation (4.13), $-\frac{2\pi G c^2}{3}\pi_\psi^2$ is negative and the theory would otherwise be unstable if not for the fact that we can choose $\pi_\psi = \pi_\Phi \approx 0$ in order to preserve the gauge fixing of the metric, Equations (4.6). In the Diósi model, the kinetic energy term is instead taken to be positive, and its inclusion results in dynamics in Φ, which is continuous. In contrast, the dynamics in Φ are discontinuous here since we take the $c \to \infty$ limit.

Another approach to deriving consistent classical-quantum theories is the measurement and feedback approaches of [181, 110, 99]. In these approaches, the classical degree of freedom is sourced by the outcomes of continuous measurements and by construction, such approaches are completely positive and lead to consistent coupling between classical and quantum degrees of freedom. As such, the dynamics for the density matrix of [181, 110, 99] undergoes a stochastic master equation evolution of the general form similar to the unravelling of the quantum state given in Equation (4.44). In the special case where $D_0(\Phi_t; x, y)$ and $\sigma(\Phi_t; x, y)$ do not depend on Φ_t, our Equation (4.45) can be put into the form of Equation (24) of [110]. When we impose the $\pi_\Phi \approx 0$ constraint and turn the Newtonian field into white noise, we pick up an extra decoherence term in the Itô formalism, as they do, which is necessary for the normalisation of the quantum state.

In [181, 110], the noise instead emerges because the Newtonian potential is modelled to be sourced by the outcomes of a continuous measurement of the mass operator. The behaviour in these models is qualitatively the same as those presented here when $D_0(\Phi_t; x, y)$ and $\sigma(\Phi_t; x, y)$ do not depend on Φ_t, meaning that the Newtonian potential diffuses by an amount that depends on the inverse of the strength of the measurement, whilst the quantum system decoheres into its mass eigenbasis because it's being continuously measured. Another difference is that these works generally utilised the mass density operator of a localised particle in the position basis (smeared by a Gaussian in Tilloy-Diósi), such that the decoherence of the quantum system emerges from mass measurements in the position basis of a point particle. The measurements are generally imagined to be carried out using an entangled measuring device to obtain the correlations required to get the Diósi-Penrose kernel. Here, matter is treated as a quantum field in the non-relativistic limit. However, one could consider the point-particle limit by writing the mass density operator as a sum of delta functions in the position basis and integrating over the $\mathcal{D}x^{\pm}$ position branches in the path integral.

Furthermore, [110, 181] imagine the results of a weak external measurement or collapse model as sourcing the gravitational field, and thus the Newtonian potential changes discontinuously, since the results of each measurement can be different. Here we emphasise that it is merely the coupling of quantum matter to the classical gravitational field which is responsible for the localisation of particles. The local time coordinate and the shift n_i are changing stochastically, as can be seen via Equations (4.38)-(4.40), in order to maintain the primary constraint $\pi_\Phi \approx 0$, while the quantum state and Newtonian potential stochastically change in lock-step together. No measurement postulate nor Born rule is needed, and there is no need to think about the ad-hoc field introduced in spontaneous collapse models [104, 109, 180, 179]. Instead, the fully classical treatment of the gravitational degrees of freedom acts to *classicalise* the quantum system. Although it appears as if the state of the matter fields undergoes decoherence, there is no decoherence if we condition on the gravitational field. The quantum state is pure conditioned on the classical trajectory when the decoherence vs diffusion trade-off is saturated [3, 5]. It is only when the gravitational field is integrated out that there appears to be loss of quantum information.

Although the theory considered here is not predicated on it explaining measurement or collapse, it may still suffer from anomalous heating of the quantum system [90, 106, 148, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208] which constrain collapse models. Since the decoherence couplings D_0 can be made arbitrarily small here, albeit at the expense of a large amount of diffusion, it is unclear the extent to which heating bounds constrain the theory, and more investigation is needed here. While the results of [55], suggest that the heating might be significant, there is evidence that relativistic effects need to be taken into account.

Although the primary motivation for studying the weak field limit of [1, 130] is to derive experimental bounds, several lessons can be learned for classical-quantum theories and attempts to quantise gravity. For example, we would like to point the reader to an analogy between quantising a theory and adding stochastic noise/diffusion to it when the theory has first-class constraints. When attempting to quantise a theory with constraints, it is well known that two possible approaches are Dirac quantisation and reduced phase space quantisation. The former consists of constructing a kinematical Hilbert space where the classical phase space functions that have been elevated to operators can act on the quantum states and then impose the constraints quantum mechanically as operator conditions to distinguish physical states. In other words, physical states are zero eigenvalues states of the constraint operator. On the other hand, reduced phase space quantisation first factorises the constraint surface with respect to the action of the gauge group generated by the constraints. This serves to identify the physical degrees of freedom directly at the classical level. Then, the resulting Hamiltonian system is quantised as a usual unconstrained system. The two procedures are not always equivalent, and the relationship between the approaches is discussed at length in the literature [209, 210, 211, 212, 213, 214].

In the same way, we could insert a noise process in a Hamiltonian system before or after reducing the phase space according to its constraints. In the main body, we start from the complete CQ theory of general relativity, where noise is present in the metric. We reduce it to the Newtonian limit by implementing the constraints. On the other hand, in Appendix B, we chose the latter approach, restricting the classical degrees of freedom before inserting them in the CQ framework, which implements a noise process. It is perhaps remarkable that the two procedures give the same theory here, while in the quantum case, they generally do not. In this work, we have arrived at this behaviour in complete generality from a reduction of the CQ degrees of freedom of the relativistic theory, with the diffusion of the Newtonian potential and decoherence on the quantum system described by the parameters $D_2(\Phi; x, y), D_0(\Phi; x, y)$ satisfying the decoherence/diffusion trade-off. The weak field CQ theories we studied gave a generic prediction: the Newtonian potential diffuses away from its average solution, and in order for the dynamics to be completely positive, the amount of the diffusion is lower bounded by the coherence time for masses in superposition. This is most elegantly described via the path integral formulation of Equation (4.30).

There are a number of proposals to test the quantum nature of gravity via gravitationally induced entanglement or coherence that are expected to be realizable within the next few decades with technological advancements [215, 216, 217, 218, 219, 220, 221, 222, 223, 224]. The idea is that if the underlying theory is local, then witnessing entanglement would imply that

Master Equation	Decoherence	Diffusion
Continuous non-relativistic (local)	$D_0(\Phi; x, y) = D_0(\Phi)\delta(x, y)$	$D_2(\Phi; x, y) = D_2(\Phi)\delta(x, y)$
Continous non-relativistic (Gaussian)	$D_0^{\alpha\beta}(x, y) = \frac{\lambda^{\alpha\beta}}{m_0^2} g_{\mathcal{N}(x,y)}$	$D_2(x, y) = \frac{1}{8} \frac{m_0^2}{r_0^3 \lambda} F(x, y) g_{\mathcal{N}(x,y)}$
Continuous non-relativistic (D.P)	$D_0^{\alpha\beta}(x, y) = \frac{D_0^{\alpha\beta}}{\lvert x-y \rvert}$	$D_2(x, y) = \frac{1}{8} \frac{(D_0^{-1})}{4\pi} \nabla_y^2(\delta(x, y))$

Table 4.1: Possible choices of kernels for the continuous master equations and the resulting diffusion/decoherence coefficients, which are assumed to saturate the trade-off in Equation (4.36). For a more detailed study of these kernels, including calculations of the diffusion and decoherence they produce, we refer the reader to [3].

gravity is not a classical field. Within the framework of consistent classical-quantum coupling, we are able to inquire from the other direction, asking about the general experimental signatures of treating the gravitational field as being classical.

If the Lindbladian coupling in Equation (4.33) $D_0(\Phi; x, y)$ is ultralocal, the dynamics do not generate entanglement between spatially separated regions, meaning that the models with local couplings parameterize the general form of the continuous master equation which would be ruled out by entanglement witnesses in GIE experiments. We give three examples of kernels D_0, D_2 for continuous master equations in Table 4.1. Models with ultralocal couplings form perhaps the most natural class of CQ dynamics. Non-relativistic versions of these models have already been ruled out by considerations of the decoherence diffusion trade-off [3]. In other words, (in line with assumptions 1-4) classical-quantum Newtonian theories of gravity, which have continuous gravitational degrees of freedom with local interactions and correlations, are already ruled out by experiment.

Here, we have considered the case where the gravitational field is fundamentally classical. As an effective theory of Newtonian gravity, we still expect the path integrals and the unravellings derived in this chapter to be valid dynamics with a time-local description [1]. However, in general, one expects an effective theory to be non-Markovian in some regimes, which means that the couplings D_0, D_2 need not be positive semi-definite for all times, [133, 134], nor satisfy the decoherence-diffusion trade-off for all times since this is a consequence of the Markovian assumption.

An open theoretical problem is the question of the existence of a complete diffeomorphism invariant theory of classical-quantum gravity. The theory of [1] requires a better understanding of the constraint structure [136], while the manifestly covariant candidate theory of [130], has not been proven to be completely positive norm-preserving. However, in the weak field limit it is, and we have thus been able to use it here. In principle, there does not appear to be any conceptual obstruction to such a theory. Indeed, in [7], which is presented in Chapter 5, we show that a diffeomorphism invariant theory can be completely positive by considering a scalar theory of CQ gravity based on the trace of trace realisation of the diffusion coefficient. We find that the Newtonian limit of the theory gives rise to qualitatively and quantitatively similar behaviours as the ones described here.

Chapter 5

Diffeomorphism invariant Classical-Quantum path integrals for Nordström gravity

"You may hate gravity, but gravity doesn't care."
∼ Clayton M. Christensen

Several models of classical-quantum Newtonian gravity have been proposed both via a master-equation approach [124] and measurement and feedback approach [55, 110, 184]. As found in Chapter 4, the weak field Newtonian limit of the general relativistic CQ theories of [1, 130] resembles the model of Diösi and Tilloy [110]. However, as seen in the last Chapter, the challenges presented by the presence of Hamiltonian constraints are not trivial. In the standard general relativistic treatment of the Newtonian limit, the dynamical components of the gravitational field are set to zero by the Hamiltonian and momentum constraints equations and the imposition of their preservation through time. Specifically, the conjugate momenta $\pi_\Phi, \pi_\Psi, \pi_i$ are constrained to vanish on the physical surface identified by such constraints [6].

On the other hand, in a CQ theory of gravity, consistent coupling of classical and quantum degrees of freedom result in a diffusion process being added to the Newtonian potential when the limit is taken. The consequence, detailed in Chapter 4, is that of generating a stochastic Newtonian constraint, which requires a stochastic shift vector for the dynamics to be consistent. Away from the Newtonian limit, the gravitational constraints in CQ theories are tightly related to notions of complete positivity and diffeomorphism invariance. Specifically, while the dynamics detailed in [1] are recognised as completely positive and norm-preserving, their diffeomorphism invariance has not been demonstrated. Conversely, the complete general relativistic covariant CQ path integral quoted in [130] is manifestly diffeomorphism invariant. However, its consistency has yet to be fully verified since the constraints' contributions to the path integral do not look completely positive. Additionally, the relationship between the dynamics of [1] and [130] remains unclear, partly due to the constraint algebra's ambiguity.

In this chapter, we construct the theory of quantum matter fields coupled with a classical self-contained scalar theory of gravity known as "Nordström gravity". The dynamics is constructed via the classical-quantum path integral of Chapter 3 and is completely positive, trace preserving (CPTP) and respects the classical-quantum split. Since Nordström gravity is a self-consistent and diffeomorphism invariant theory of gravity that does not require the Hamiltonian constraints, it can avoid their complications. The resulting theory is fully diffeomorphism invariant, although, like Nordström gravity, has a preferred background. In such a way, we sidestep the challenges of the constraints by presenting a toy model for hybrid classical-quantum scalar gravity that can be used to gain confidence in the weak field limit of [1, 130]. In this diffeomorphism invariant theory of CQ-gravity, the conformal factor now plays the role of the stochastic gravitational potential. We recover the same Newtonian limit behaviour of Chapter 4, suggesting that the resolution of the constraints utilised in it and the resulting dynamics were correct. The model also indicates the absence of tension between diffeomorphism invariance, stochastic theories, and any tension with the classical-quantum split. furthermore, the theory provides a model in which various questions of quantum and classical-quantum gravity can be explored in a simpler form. It is also a manifestly Lorentz-invariant theory of stochastic collapse [179, 104, 180, 109], as are the models of [225].

We now present the outline of the Chapter's sections.

In Section 5.1, we review Nordström's theory of gravity as a self-consistent theory of scalar gravity. The role of the dynamical field is played by a scalar conformal factor which evolves in a flat background. We discuss the theory's advantages and shortcomings in relation to general relativity and we reflect on why it is optimal as a toy model of classical quantum gravity.

In Section 5.2, we use the path integral approach to introduce and study a consistent diffeomorphism invariant theory of classical-quantum Nordström gravity. This provides a proof of principle that diffeomorphism invariant CQ dynamics can exist. In the Newtonian $c \to \infty$ limit we find that the theory gives rise to the Newtonian interaction on average. Due to the decoherence-diffusion trade-off, the dynamics necessarily involves diffusion away from the Newtonian solution, by an amount lower bounded by the decoherence rate into mass eigenstates. Though this example is to be understood as a toy model, it provides an instance where we have full control over the symmetries of the theory and gives support to the treatment of the more complete theory which we study in [6].

We conclude with a discussion in Section 5.3.

5.1 Nordström gravity

Nordström gravity [226, 227] was a first attempt at merging Newtonian gravity with relativity and ultimately led to the formulation of GR as it currently stands [228]. In its final formulation, Nordström Gravity can be thought of as a self-consistent scalar theory of gravity. It was the first metric theory of gravity, meaning it obeyed the equivalence principle. The classical theory is described through a conformally flat spacetime background which couples to matter via the equations:

$$\mathcal{R} = \frac{24\pi G}{c^4} T, \tag{5.1}$$

$$C_{\mu\nu\rho\sigma} = 0, \tag{5.2}$$

where $\mathcal{R}$ is the Ricci scalar, T denotes the trace of the stress-energy tensor for the matter degrees of freedom ϕ_m and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. Equation (5.1) is the dynamical equation of motion linking the Ricci scalar to the trace of the stress-energy tensor. The vanishing of the Weyl tensor in Equation (5.2) implies that the metric is conformally flat and always takes the form:

$$g_{\mu\nu} = e^{\frac{2\Phi}{c^2}} \eta_{\mu\nu}, \tag{5.3}$$

where Φ is the conformal degree of freedom and $\eta_{\mu\nu}$ is the Minkowsky metric.

Nordström gravity merges relativistic ideas of causality with Newtonian gravity but lacks many of the properties required for a full description of gravitational phenomena. One can immediately notice that the conformal metric couples only to the *trace* of the stress-energy tensor, and some forms of energy and momentum, like the stress-energy tensor for electromagnetic radiation, are traceless. As such, it does not predict the bending of light, in direct contrast with gravitational lensing effects observed by astronomers [229, 230]. Concerning other effects, Nordström gravity correctly predicts the result of the Pound-Rebka experiment for gravitational frequency shift but fails to predict the correct time delay factor and is missing subleading corrections to the acceleration of static test particles.

However, much like general relativity, Nordström's theory is *diffeomorphism invariant*, by which we mean that (g, Φ, ϕ_m) is a solution to the equations of motion if and only if (g^*, Φ^*, ϕ_m^*) is also a solution to the same equation's of motion, where * denotes the transformed variables after a diffeomorphism. In particular, conformal flatness is preserved under diffeomorphisms. The conformal flatness condition does not fix the conformal factor, which is the dynamical gravitational degree of freedom. For example, given the form of the metric in Equation (5.3), the Ricci scalar reads

$$\mathcal{R} = -\frac{6\tilde{\Box} e^{\frac{\Phi}{c^2}}}{c^2} e^{-\frac{3\Phi}{c^2}}, \tag{5.4}$$

where $\tilde{\Box} = \partial_\mu \partial_\nu \eta^{\mu\nu}$ is the *flat* space D'Alabertian. In the vacuum state ($T = 0$) the field equation is the wave equation for the scalar field $\tilde{\Box} e^{\frac{\Phi}{c^2}} = 0$. Therefore, the theory has a propagating conformal scalar degree of freedom. Still, this kind of gravitational wave differs from those predicted by general relativity as they are scalar waves and do not have a spin-2 mode.

Nonetheless, while being diffeomorphism invariant, Nordström's theory is intuitively background-dependent. It has a preferred frame given by the Minkowski metric due to the imposition that the metric be conformally flat. In particular, it admits a background-dependent formulation (which is still diffeomorphism invariant) where one stipulates that the metric takes the form of Equation (5.3), with the dynamics determined by Equation (5.1). For a more detailed discussion on the relationship between diffeomorphism invariance and background independence, we refer the reader to [231]. We will now show how Nordström gravity can be implemented in the CQ path integrals to study a diffeomorphism invariant self-consistent theory of gravity and its Newtonian limit.

5.2 CQ Path Integral for Nordström Gravity

In this section, we will first introduce CQ gravitational path integrals for general relativity, outlining the tension between the complete positivity of the dynamics and the gravitational constraints. Then, we shall construct the CQ path integral for Nordström gravity. Our choice of classical system will be the conformal part of the metric Φ. On the other hand, matter's degrees of freedom have a quantum nature, and we can study their backreaction with spacetime.

CQ general relativity

When writing the path integral for General relativity, we follow Chapter 3 and write a manifestly covariant path integral over 4-geometries g, of the form given by Equation (3.12)

$$\varrho(\Sigma_f, \phi_f^+, \phi_f^-, t_f) = \int \mathcal{D}g\mathcal{D}\phi^+\mathcal{D}\phi^- \mathcal{N}\ e^{\mathcal{I}_{CQ}[g,\phi^+,\phi^-,t_i,t_f]}\varrho(\Sigma_i, \phi_i^+, \phi_i^-, t_i). \tag{5.5}$$

with:

$$\begin{aligned}\mathcal{I}_{CQ}[g,\phi^+,\phi^-] = \int_{t_i}^{t_f} dtd\vec{x}\Bigg[&i\big(\mathcal{L}_{\mathcal{Q}}[g,\phi^+] - \mathcal{L}_{\mathcal{Q}}[g,\phi^-]\big)\\ &- \frac{\mathrm{Det}[-g]}{8}\big(T^{\mu\nu}[\phi^+] - T^{\mu\nu}[\phi^-]\big)D_{0,\mu\nu\rho\sigma}[g]\big(T^{\rho\sigma}[\phi^+] - T^{\rho\sigma}[\phi^-]\big)\\ &- \frac{\mathrm{Det}[-g]|c^8}{128\pi^2 G_N^2}\left(G^{\mu\nu} - \frac{8\pi G_N}{c^4}\bar{T}^{\mu\nu}[\phi^+,\phi^-]\right)D_{0,\mu\nu\rho\sigma}[g]\left(G^{\rho\sigma} - \frac{8\pi G_N}{c^4}\bar{T}^{\rho\sigma}[\phi^+,\phi^-]\right)\Bigg]\end{aligned} \tag{5.6}$$

where $\mathcal{L}_Q[g, \phi^\pm]$ is the Lagrangian for the quantum matter in curved spacetime and we have suppressed the metric dependence of all terms in Einstein equations for clarity. Σ_i and Σ_f are the initial and final spatial surfaces. The term $\bar{T}^{\mu\nu}[\phi^+, \phi^-]$ indicated the average of the bra and ket stress-energy tensors, as in Equation (3.14)

$$\bar{T}^{\mu\nu}[\phi^+,\phi^-] = \frac{1}{2}(T^{\mu\nu}[\phi^+] + T^{\mu\nu}[\phi^-]). \tag{5.7}$$

Here we have assumed that the decoherence-diffusion trade-off is saturated ($4D_0 = D_2^{-1}$). This is a manifestly diffeomorphism invariant hybrid path integral for general relativity, and it is fully characterised by the tensor density $D_{0,\mu\nu\,\rho\sigma}[g]$. As explained in [130], if we now choose D_0 such that it is positive semi-definite, we would have a completely positive treatment of semiclassical general relativity, where quantum fields $\phi(x)$ backreact of the classical metric

$g_{\mu\nu}$ inducing diffusion, and the classical spacetime continuously measures the quantum fields weakly. Unfortunately, choosing a positive semi-definite D_0 and capturing the transverse part of the Einstein equations is not possible in Lorentzian signature. However, one might choose

$$D_{0,\mu\nu\rho\sigma} = \frac{D_0(x)}{\sqrt{-g}} g_{\mu\nu} g_{\rho\sigma}. \tag{5.8}$$

This will lead to a positive semi-definite path integral describing suppressed trajectories as they diffuse away from the trace of Einstein's equations. We remind the reader that this differs from the choice made in Chapter 4

$$D_{0,\mu\nu\rho\sigma} = \frac{D_0(x)}{2\sqrt{-g}} (g_{\mu\rho} g_{\nu\sigma} + g_{\nu\rho} g_{\mu\sigma} - 2\beta g_{\mu\nu} g_{\rho\sigma}), \tag{5.9}$$

with D_0 a positive constant, which also captures the transverse part and which contains the constraints [130]. In Lorentzian signature, this is not positive semi-definite, but it does appear to be positive semi-definite, once normalised [232]. On the other hand, if one is happy with a consistent diffeomorphism invariant toy model of gravity in the CQ framework, one needs not to consider general relativity and could consider a scalar theory of gravity. In particular, Nordström gravity is an ideal candidate as a self-consistent theory of gravity that allows us to study the gravitational backreaction of spacetime and quantum matter without worrying about the constraints of general relativity or the positivity of the full CQ path integral for GR. We also take this chance to address the presence of higher-order derivatives in the path integral action. The reader may be concerned about such actions being associated with Hamiltonians unbounded from below, as shown by Ostrogradsky [233]. However, problems arise only when assuming that the action generates deterministic evolution. Onsager Machlup path integrals for stochastic processes often contain higher-order derivatives when presented in configuration space (see Chapter 12 of the work by Feynman-Hibbs [234]). The action is already composed of the equation of motion. Its variation should be interpreted as the most probable path between the set initial and final points of configuration space. For example, a stochastic harmonic oscillator will have a variation of its Onsager-Machlup action, which results in an apparent runaway of the solutions due to increasing the oscillation amplitudes. However, this increase is necessary to reach points normally outside the range of the equations of motion for fixed initial position and velocity. More about the relation between Ostrogradsky and CQ path integrals is discussed in Chapter 6.

CQ Nordstrom

To construct the CQ theory of Nordström gravity, we let ϕ_m denote the quantum matter degrees of freedom such that $\mathcal{L}_Q[g, \phi_m]$ is the matter Lagrangian (inclusive of the appropriate metric determinant factor). Nordström gravity can be derived classically from the action principle in the Jordan frame defined in [235], which we summarise in Appendix D. In that derivation, a Lagrange multiplier is used to impose the conformal flatness of the spacetime (5.3). Here, we are faced with two choices. We could insert the constraint in the proto-action directly or impose it through a delta functional. Given that in Nordström gravity, matter fields do not couple to the Weyl tensor; we do not expect the backreaction of the quantum degrees of freedom to break the conformal flatness of the metric. It is more sensible to choose the latter, imposing the constraint in a way akin to a gauge fixing of the classical degrees of freedom. Therefore, we construct the proto-action for Nordström gravity with matter as

$$W_{CQ}[g_{\mu\nu}, \phi_m] = -\frac{c^4}{48\pi G_N}\int d^4x\sqrt{-g}\,\mathcal{R} + \int d^4x\,\mathcal{L}_Q[g, \phi_m]. \tag{5.10}$$

While this action might look similar to the Einstein-Hilbert action, one should notice the different coefficients of the gravitational sector and the different relative signs between the gravitational and matter part, both are required to obtain the correct Nordström dynamical equation. We are now in the position to write down the CQ path integral. We choose the trace realisation of the decoherence coefficient in Equation (5.8) and impose the conformal flatness constraint as a delta functional through a Lagrange multiplier $\lambda_\mu^{\nu\rho\sigma}$

$$\varrho(\Sigma_f, \phi^+_{m,f}, \phi^-_{m,f}, t_f) = \int \mathcal{D}g\mathcal{D}\phi^+_m\mathcal{D}\phi^-_m\mathcal{D}\lambda_\mu^{\nu\rho\sigma}\,\mathcal{N}\,e^{\mathcal{I}_{CQ}(g_{\mu\nu},\phi^+_m,\phi^-_m,t_f,t_i)}\varrho(\Sigma_i, \phi^+_{m,i}, \phi^-_{m,i}, t_i), \tag{5.11}$$

where the CQ action is

$$\begin{aligned}\mathcal{I}_{CQ}(g_{\mu\nu}, \phi^+_m, \phi^-_m, t_f, t_i) = \int_{t_i}^{t_f} dt d\vec{x}\Bigg[\, & i(\mathcal{L}_Q[g, \phi^+_m] - \mathcal{L}_Q[g, \phi^-_m]) - \frac{D_0(x)}{2\sqrt{-g}}\frac{\delta\Delta W_{CQ}}{\delta g^{\mu\nu}}g_{\mu\nu}g_{\rho\sigma}\frac{\delta\Delta W_{CQ}}{\delta g^{\rho\sigma}} \\ & - \frac{2D_0(x)}{\sqrt{-g}}\frac{\delta\bar{W}_{CQ}}{\delta g^{\mu\nu}}g_{\mu\nu}g_{\rho\sigma}\frac{\delta\bar{W}_{CQ}}{\delta g^{\rho\sigma}} - i\lambda_\mu^{\nu\rho\sigma}C^\mu_{\nu\rho\sigma}\Bigg],\end{aligned} \tag{5.12}$$

and we have saturated the decoherence diffusion tradeoff:

$$4D_0[g] = D_2^{-1}[g]. \tag{5.13}$$

Once we integrate over the Lagrange multiplier, the delta function will ensure that we only sum over 4-geometries that are conformally flat $C^{\mu}_{\nu\rho\sigma} = 0$. Any conformally flat metric can be written as $g_{\mu\nu} = e^{2\frac{\Phi}{c^2}}\eta_{\mu\nu}$ for some Φ by definition. Therefore, we finally arrive at the Nordström hybrid path integral

$$\begin{aligned}\varrho(\Sigma_f, \phi^+_{m,f}, \phi^-_{m,f}, t_f) = &\int \mathcal{D}\Phi\mathcal{D}\phi^+_m\mathcal{D}\phi^-_m\,\mathcal{N}\,\varrho(\Sigma_i, \phi^+_{m,i}, \phi^-_{m,i}, t_i)\\ &\times \exp\Bigg[\int dtd\vec{x}\; i(\mathcal{L}_Q[\phi^+_m] - \mathcal{L}_Q[\phi^-_m])\\ &- \frac{\sqrt{-g}D_0(x)}{8}\big(T[\phi^+_m] - T[\phi^-_m]\big)^2 - \frac{\sqrt{-g}c^8 D_0(x)}{1152\,\pi^2 G_N^2}\left(\mathcal{R} - \frac{24\pi G_N}{c^4}\bar{T}[\phi^+_m, \phi^-_m]\right)^2\Bigg],\end{aligned} \tag{5.14}$$

where we have suppressed the Φ dependence in $\mathcal{R}$, T and $\mathcal{L}_Q$ to lighten the notation. When integrating over conformally flat metrics, we include any Jacobian factor in the measure $\mathcal{D}g^{C=0} \sim \frac{2}{c^2}e^{\frac{2\Phi}{c^2}}\mathcal{D}\Phi$, as it will not be relevant to the Newtonian limit of interest in this paper. In particular, to leading order we have that $Dg^{C=0} \sim \frac{2}{c^2}\mathcal{D}\Phi$. Since the action in Equation (5.14) contains quantum terms proportional to the square of the stress-energy tensor, a sufficient condition for the path integral to be normalised is that the purely quantum part of the action $\mathcal{L}_{\mathcal{Q}}[q, \phi^{\pm}]$ contains higher derivative kinetic terms $\sim \ddot{\phi}^2$ [236], which is suggestive that Equation (5.14) describes an effective theory [154]. The action has the effect of diffusing away from the bra/ket averaged Nordström equations whilst simultaneously decohering the quantum system according to the stress-energy tensor of the matter and the coupling $D_0[\Phi]$. Treated classically, the action is manifestly diffeomorphism invariant, which also includes the case where the diffeomorphism $\sigma : M \to M$ is dependent on the classical and quantum trajectories $\sigma[\Phi, \phi^{\pm}_m]$. However, just as for the classical theory, the CQ theory is not background-independent and has a preferred frame imposed by the requirement that the metric is conformally flat.

We now wish to compute the Newtonian limit of the theory in order to gain insight into the Newtonian limit of more general classical-quantum theories and compare and contrast it with [6]. We recall that the final goal is to search for low-energy experimental signatures of CQ by treating the gravitational field classically. To that end, much like in Chpater 4, we shall take the quantum degrees of freedom to be described by a pressureless dust distribution $\hat{T}^{\mu\nu} = \hat{m}(x)U^{\mu}U^{\nu}$ where $\hat{m}(x)^{\pm}$ is a (smeared) mass density. Using the conformally flat metric and our choice of matter, we can rewrite some of the quantities in the path integral as:

$$\sqrt{-g} = c\, e^{\frac{4\Phi}{c^2}},\ \mathcal{R}[\Phi] = -\frac{6\tilde{\Box}e^{\frac{\Phi}{c^2}}}{c^2}e^{-\frac{3\Phi}{c^2}},\ T^{\pm}[\Phi, m^{\pm}] = -e^{\frac{2\Phi}{c^2}}m^{\pm}(x), \tag{5.15}$$

and the path integral takes the form

$$\begin{aligned} \varrho(\Sigma_f, m_f^+, m_f^-, t_f) = \int & \mathcal{D}\Phi\mathcal{D}m^+\mathcal{D}m^-\mathcal{N}\varrho(\Sigma_i, m_i^+, m_i^-, t_i) \\ & \times \exp\Bigg[\int_{t_i}^{t_f} dtd\vec{x}\; i\big(\mathcal{L}_Q[\Phi, m^+] - \mathcal{L}_Q[\Phi, m^-]\big) \\ & \qquad - \frac{cD_0(x)e^{\frac{6\Phi}{c^2}}}{8}\big(m^+(x) - m^-(x)\big)^2 \\ & \qquad - \frac{c^7 D_0(x)}{192\,\pi^2 G_N^2}\left(-e^{\frac{\Phi}{c^2}}\tilde{\Box}e^{\frac{\Phi}{c^2}} + \frac{4\pi G_N e^{\frac{6\Phi}{c^2}}}{c^2}\bar{m}(x)\right)^2\Bigg], \end{aligned} \tag{5.16}$$

where $\bar{m}(x) = \frac{1}{2}\big(m^+(x) + m^-(x)\big)$.

We take the Newtonian $c \to \infty$ limit of the metric perturbations. Carrying out the transformations, we get

$$\sqrt{-g} = c\, e^{\frac{4\Phi}{c^2}} \approx c\left(1 + \frac{4\Phi}{c^2}\right) + \mathcal{O}\left(\frac{1}{c^3}\right), \tag{5.17}$$

$$e^{\frac{\Phi}{c^2}}\tilde{\Box}e^{\frac{\Phi}{c^2}} \approx \frac{\tilde{\Box}\Phi}{c^2} + \mathcal{O}\left(\frac{1}{c^4}\right), \tag{5.18}$$

$$\frac{4\pi G_N e^{\frac{6\Phi}{c^2}}}{c^2}\bar{m}(x) \approx \frac{4\pi G_N}{c^2}\bar{m}(x) + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{5.19}$$

To leading order in c, we then arrive at the Newtonian limit of the CQ scalar gravity theory

$$\varrho(\Sigma_f, m_f^+, m_f^-, t_f) = \int \mathcal{D}\Phi\mathcal{D}m^+\mathcal{D}m^-\,\mathcal{N}\, e^{\mathcal{I}_{CQ}[\Phi, m^+, m^-, t_i, t_f]}\varrho(\Sigma_i, m_i^+, m_i^-, t_i), \tag{5.20}$$

with CQ action given by:

$$\begin{aligned} \mathcal{I}_{CQ}[\Phi, m^+, m^-, t_i, t_f] = \int_{t_i}^{t_f} dtd\vec{x}\, \Bigg[& i\big(\mathcal{L}_Q[m^+] - \mathcal{V}_I[\Phi, m^+] - \mathcal{L}_Q[m^-] + \mathcal{V}_I[\Phi, m^-]\big) \\ & - \tilde{D}_0(x)\big(m(x)^+ - m(x)^-\big)^2 - \frac{c^2\tilde{D}_2^{-1}(x)}{6}\left(\frac{\tilde{\Box}\Phi}{4\pi G} - \bar{m}(x)\right)^2\Bigg]. \end{aligned} \tag{5.21}$$

where $\mathcal{V}_I[\Phi, m^{\pm}]$ is the interaction potential coming from the expansion of the metric determinant in the quantum Lagrangians. For example, in the spirit of Chapter 4, one could have

$\mathcal{V}_i = \Phi\, m^{\pm}$. We have defined $\tilde{D}_0(x) = \frac{cD_0(x)}{8}$, $\tilde{D}_2^{-1}(x) = \frac{cD_0(x)}{2}$ which are related to the decoherence and diffusion coefficients of the Newtonian potential. With these re-definitions, these coefficients relate to physically observable quantities: $\tilde{D}_0$ quantifies the suppression of quantum trajectories away from $m(x)^+ = m(x)^-$, which is the decohered trajectory. On the hand hand $\tilde{D}_2(x) = \frac{1}{2}\sigma_\Phi^2$ where σ_Φ quantifies the variance away from the semiclassical Newtonian solution. They saturate the decoherence-diffusion relation:

$$4\tilde{D}_0(x) = \tilde{D}_2^{-1}(x). \tag{5.22}$$

In Equation (5.21), we have explicitly kept the d'Alambert operator to highlight the fact that, differently from [6], ϕ is in principle still a dynamical variable as the ADM constraints do not constrain it. The d'Alambert operator is also required for normalisation [236]. However, in the slow-moving limit, we recover the randomly sourced Poisson equation and exactly match the Newtonian limit of [6].

Although the scalar theory is a toy model, it is worth highlighting some of its appealing features which we expect to apply to more general CQ theories. Firstly, as mentioned, we have both diffusion in the Newtonian potential and decoherence in the quantum system, by an amount quantified by Equation (5.22). More generally, we expect that the amount of diffusion in the Newtonian potential will be lower bounded by (5.22), which is an experimental signature of classical-quantum theories [3]. Indeed, we see the same decoherence-diffusion relation between the diffusion of the Newtonian potential away from its averaged solution and the decoherence rate into the mass eigenbasis in the Newtonian path integral of [6].

Secondly, expanding out Equation's (5.14) and (5.21), we see that all the $\pm$ cross terms cancel so that the path integral preserves the purity of the quantum state even though the state decoheres into the mass eigenbasis. Such classical quantum theories, therefore, provide a natural mechanism to describe wavefunction collapse via the interaction of a classical field with a quantum one. Moreover, the fact that the classical field is dynamical is enough to restore apparent diffeomorphism invariance in the theory, which can be seen via the diffeomorphism invariant action in (5.14). The fact that gravity interacts with matter through its stress-energy tensor provides an amplification mechanism by which small masses can maintain coherence whilst macroscopic objects will be decohered. Indeed, if one disregards the classical degrees of freedom, the resulting dynamics are very similar to the dynamics of collapse theories [179, 104, 180, 109], but we see that the complete theory is diffeomorphism invariant since we consider a *dynamical* classical field.

Let us finally comment on the continuity properties of the Nordström theory and its Newtonian limit. Typically, in path integral approaches, the path integral can be understood as an integral over paths that are (almost surely) continuous. The reason for this is that they typically involve kinetic terms $\frac{i}{\hbar}[\frac{1}{\delta t}(x_{t+\delta t} - x_t)]^2$ which give a highly oscillatory contribution to the path integral $\sim e^{\frac{i}{\hbar \delta t^2}}$ if the paths are discontinuous. For the Nordström theory, we expect similar behaviour for the gravitational field in the full path integral (Equation (5.14)) due to the $-\frac{\mathcal{R}^2}{4D_2}$ term which includes kinetic terms through $\tilde{\Box}\Phi$. However, in the $c \to \infty$ limit, we can neglect such terms, leading to a discontinuous path integral in Equation (5.21). This is a remnant of the approximation, and we expect that any physical, measurable quantity of interest should be smeared over a time scale to reflect this.

5.3 Discussion

In this chapter, we have constructed the covariant path integral of a full diffeomorphism invariant theory of CQ Nordstrom gravity and derived its Newtonian limit. The result matches with the one obtained in [6], where we started from general relativity and proceeded with gauge fixing the Newtonian metric. In both cases, the final path integral describes a classical Newtonian gravitational field diffusing around Poisson's equation of motion. At the same time, quantum matter degrees of freedom decohere into mass eigenstates due to the backreaction of the classical geometry. In order for the dynamics to be completely positive, the amount of diffusion is lower bounded by the coherence time for superpositions of mass distributions. Differently from [6], our choice of classical system was Nordström's scalar theory of gravity. While Nordström's theory does not accurately describe all gravitational phenomena, it is nonetheless a self-consistent theory of relativistic gravitation. The main appeal behind this choice is that it allows us to completely bypass any discussion or concern regarding gravitational constraints, serving as proof that there is no fundamental impediment to constructing a positive diffeomorphism invariant theory of CQ gravity and pointing to the fact that the path integral version of the framework [4, 130] is the most promising approach to constructing a complete theory.

Moreover, it provides us with a great toy model of a consistently covariant path integral of a diffeomorphism invariant theory of classical gravity coupled to quantum matter which obeys linear, completely positive dynamics.

It's worth noting that since the complete path integral of [130] is renormalisable in the gravitational degrees of freedom [9], one should expect this simpler theory to also be renormalisable. And since this theory is better understood in terms of normalisation, convergence properties and diffeomorphism invariance, we expect it to be an interesting toy model in which to explore the conceptual issues around classical-quantum theories of gravity.

Much like in Chapter 4, we intend these results to serve as a model for exploring questions about the presence of backreaction-induced gravitational noise and anomalous heating, which will clarify the accuracy of the CQ framework when it comes to the interaction of quantum fields with spacetime. Fortunately, we can look to proposals that aim to test the quantum nature of spacetime to test the theory. This includes proposals for the detection of gravitationally induced entanglement between masses in interferometric setups [215, 216, 217, 219, 220, 221, 222, 223, 237], c.f. [238] which may become feasible in the next decade or two. As explained in detail in [3], since the trade-off is in terms of the inverse Lindbladian coupling D_0^{-1} one can also constrain classical theories of gravity by bounds on anomalous heating of the quantum system [90, 106, 148, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208]. Other experimental ideas look for coherence or correlations in gravitational interactions [224, 239]. The variety of experimental proposals as well as new theoretical tools, suggest that probing the quantum vs classical nature of spacetime can be accomplished at low energy and is likely to shed light on attempts to reconcile quantum theory with general relativity.

We conclude by noting that measuring the existence of gravitational diffusion should not be interpreted as confirmation of a classical gravitational field by itself. Such effects could instead be caused by quantum theories of gravity, whose classical limit is effectively described by CQ dynamics. One such example is the regime usually considered when one is interested in vacuum fluctuations in the era of cosmological inflation, where quantum fluctuations interact with a classical space-time and where a probability distribution over space-time metrics with associated inhomogeneities describes the resulting state.

Chapter 6

Anomalous contribution to galactic rotation curves

"You know, dark matter matters."
$\sim$ Neil deGrasse Tyson

In this Chapter, we show that the stochastic behaviour of the gravitational sector of CQ theories modifies the expected general relativistic behaviour at low accelerations.

Systems undergoing stochastic motion can acquire extra terms in their evolution such that their expected (average) dynamics differ from their deterministic counterpart; one example of such contributions is extra *drift terms* appearing in the context of stochastic PDE. However, this deviation is not a necessary feature of all stochastic systems, and it might emerge only under certain boundary conditions or only in a subspace of the state space. In the case of the gravitational sector of CQ theories, this behaviour is relevant in the low acceleration regime when the variance in the acceleration produced by the gravitational field is high compared to that produced by the Newtonian potential. The result can be interpreted as an entropic force, causing a deviation from Einstein's theory of general relativity. This deviation will appear as a gravitational contribution without the presence of any visible matter. Part of this effect can be directly linked to a dark energy-like contribution, even if the gravitational theory under consideration has not been constructed with a cosmological term. The second contribution, anti-correlated to the first, has been used to fit galactic rotation curves without dark matter. These modifications are computed via the path integral formalism, where the most relevant terms known as *most probable paths* are analysed. Caution should be exercised as a greater understanding of this effect is needed before conclusions can be drawn, most likely through numerical simulations. A template for computing the deviation from general relativity is provided, which serves as an experimental signature of the Brownian motion of spacetime.

The Chapter outline is as follows:

Section 6.1, has two aims. The first is to show how stochastic path integrals give rise to *most probable paths* and discuss their meaning and interpretation. The second is to provide an example of an entropic force emerging due to boundary conditions by looking at a Brownian particle with a reflective wall. The entropic force is manifest in the average position of the particle, which is different from its deterministic counterpart.

Section 6.2 applies these concepts to the gravitational sector of CQ theories. First, their applications are shown in the weak field limit regime obtained in Chapter 4. Then, a Taylor expansion of the generalised Schwartzschild solution is performed, and it is shown how this gives rise to a dark energy contribution and how this contribution, together with a second term, can be used to fit the rotational curves of galaxies without the need for dark matter.

Lastly, Section 6.4 summarises the results and highlights the weaknesses of the calculations. Important assumptions were made to arrive at analytical results. This Chapter is expected to be the starting point of a fuller and more in-depth analysis of the connection between the CQ framework, dark energy and dark matter. The discussion is concluded by comparing the results of this paper with the theoretical results of the CQ framework in the context of tabletop experiments.

6.1 Most probable paths and entropic forces

Most Probable Paths

Consider a free particle undergoing Brownian motion with no drift. Given that there is no drift, the particle will accelerate only due to the random kicks in momentum caused by the presence of noise. An Onsager-Machlup stochastic path integral [168] can be written to obtain the probability of finding the particle at $q(t_f) = q_f$ given that at $t = 0$ it was at $q(0) = q_0$ as

$$
\begin{aligned}
\mathbb{P}(q_f|q_0) &= \frac{1}{\mathcal{N}} \int_{q_0}^{q_f} \mathcal{D}q \, e^{-\mathcal{S}_{OM}(q)} \\
\mathcal{S}_{OM}(q) &= \frac{1}{2D_2} \int_0^{t_f} (\ddot{q})^2 dt
\end{aligned}
\tag{6.1}
$$

Note that the path integral acts to suppress the probability of paths which do not satisfy $\ddot{q} = 0$, by an amount controlled by the diffusion constant D_2. The larger D_2 is, the more stochasticity we are likely to find in the realised paths. In other words, given that $\ddot{q} = 0$ is the equation of motion for the noise-free version of this system, the more a path deviates from the deterministic dynamics, the more it is suppressed. This is an equivalent description of the system as the one provided by the Langevin equation $\ddot{q} = F(q)/m + \xi(t)$, with F/m the drift produced by a deterministic force F (here set to 0), and $\xi(t)$ a stochastic white noise distribution process. The dynamics can also be described via the Fokker-Planck equation [240] or Ito calculus [241]. We refer the interested reader to [240] for a derivation of the Onsager-Machlup path integral, or [234] for a discussion of Brownian motion in the context of path integrals of similar form to Equation (6.1).

In the same fashion as in quantum theory, one can ask which of these paths contributes the most to the overall transition probability. These will be the paths that minimise the Onsager-Machlup Lagrangian. We can immediately see that the solutions to the deterministic equations of motion $\ddot{q} = 0$ will be extrema of the action, and they will be *global* minima as they lead to $\mathcal{S}_{OM} = 0$. However, the usual variational methods can find other minima of the action. In the specific, as shown also in [234], the extrema of this action are given by the fourth-order equation

$$
\frac{d^4}{dt^4} q(t) = 0, \tag{6.2}
$$

with general solution

$$
q(t) = \alpha_0 + \alpha_1 t + \frac{1}{2}\alpha_2 t^2 + \alpha_3 t^3. \tag{6.3}
$$

where α_2 has been rescaled to match the usual coefficient of acceleration terms.

One can see that the path space is much broader than the usual deterministic space of configurations. We call the solutions to this equation *most probable paths* (MPPs), adopting the language used in the study of diffusive dynamics [234, 242, 243]. The intuition behind the name is clear: given a set of boundary conditions, the equation determines the most probable path of the free particle connecting the initial to the final conditions. There are four constants of integration here, two more than the deterministic counterpart. Therefore, the two initial conditions supplied in the deterministic case are no longer sufficient. Random noise kicks mean that, in a fixed interval of time, multiple possible final conditions correspond to the same initial conditions, even if they are associated with different probabilities. In addition, the noise kicks allow the particle to reach positions and velocities that would not be permitted in the deterministic case, and the extra terms in the MPP allow precisely that. The two extra conditions can be imposed at the final time t_f or one at the final time and one extra at the initial time. Moreover, we point out that one could alternatively not fix one or more of these constants and then integrate them, but we will not do it here.

In order to have a transition probability, we fix the final position as previously mentioned to $q(t_f) = q_f$. Then, we fix the initial position and velocity q_0 and $\dot{q}(0) = v_0$, akin to the deterministic setting. The last condition could come from fixing the initial acceleration $\ddot{q}(0) = a_0$ or from setting some final condition, for example, the final velocity $q(t_f) = v_f$. These would give rise either to the transition probability $\mathbb{P}(q_f|q_0, v_0, a_0)$ or to $\mathbb{P}(q_f, v_f|q_0, v_0)$. Suppose we choose the first case and study the free particle with zero initial velocity and acceleration starting at a point q_0 and ending at a point q_f. It will then be possible to ask the question: given these conditions, what is the average final position of the particle? The question can be answered using the MPPs satisfying these initial conditions and then averaging over the final position q_f. Each path will contribute less the further away from the deterministic paths. If we substitute the MPP into the action of the Onsager-Machlup path integral and normalise over the final position, we will be summing over the most probable configurations leading to that final position, much like a saddle-point (Laplace's method) approximation.

With this method, the average final position for the free Brownian particle undergoing Brownian noise is

$$\langle q_f \rangle = \sqrt{\frac{6}{\pi D_2 t_f^3}} \int_{-\infty}^{\infty} dq_f \left(q_f \, e^{-\frac{6}{D_2 t_f^3}(q_f - q_0)^2} \right) = q_0, \tag{6.4}$$

which is the same as the deterministic position resulting from a free particle with no initial velocity in the absence of noise. We have recovered the well-known result stating that, given

Gaussian distributed random kicks with zero mean, the average final position will be the same as the initial one. While the outcome is not at all surprising, we used it as a simple benchmark to show how one can utilise the approach of the most probable path. We will now discuss the case where there is a divergence between the deterministic result and the average value computed through MPPs, which might initially be expected to align in the presence of noise with a zero mean.

Entropic Force

A canonical example of an entropic force is that due to a polymer which is initially curled up in a low entropy state but will unfurl or diffuse into a higher entropy state, with its ends exerting a force [244, 245]. Another is a gas in a box fitted with a piston on one side, which is slowly pushed out as the gas diffuses. Note that in Section 6.2 of this Chapter, we do not consider deriving gravity as an entropic force [246, 247, 248, 249, 250], but rather consider the entropic force that gravity exerts. This section aims to define entropic forces as applicable out of equilibrium and based only on the equations of motion. It will also give an example that can be solved in a similar manner to the gravitational case and has some identical features.

Consider Newton's law $F(q) = m\ddot{q}$. This is a deterministic equation, but we can consider the case where the system is in a probability distribution over q due to the presence of a mean-zero noise ξ,

$$\begin{aligned} &F(q) - m\ddot{q} = \xi, \\ &\langle \xi \rangle = 0, \end{aligned} \tag{6.5}$$

in which case, we still expect Newton's law to be satisfied on expectation

$$\langle F(q) - m\ddot{q} \rangle = 0. \tag{6.6}$$

The key concept here is that if the mean value of the force felt by the particle depends on the second and higher moment of its position, the particle will not generally follow its deterministic trajectory because the average of the position equation of motion is not the same as the equation of motion of the average position. A simple example is obtained by considering $F = -\alpha q^2$, which corresponds to the cubic potential $V(q) = \alpha q^3/3$. The time derivative of the particle's mean momentum obeys $\langle \dot{p} \rangle = \alpha \langle q^2 \rangle$, which can be significantly larger than $\langle q \rangle^2$.

Another example is the Brownian motion of a particle in a box with a piston. The presence of a wall on the other side suffices to ensure that the mean value of the particle's position q will change with time as the piston is pushed out. If there were no diffusion or wall, the particle's average position would not change. The wall placed at $q = 0$ makes it impossible for $\langle q_f \rangle = q_0$ when $\langle q^2 \rangle$ is non-zero. After all, given enough time, the reflecting boundary at the origin will skew the average final position in the direction opposite to the wall.

Indeed, as $\langle q^2 \rangle$ becomes greater and greater than $\langle q \rangle^2$ (possibly due to elapsed time or a temperature increase of the heath bath), the presence of the wall makes it so that the average final position will be further and further away from the mean. We will call this the *diffusion regime*, since the second moment of the observable is comparable to its variance and is influencing the observable equations of motion, in comparison to the case where the mean value of the observable is given by its deterministic value, which for a free Brownian particle corresponds to the final position being identical to its initial position $\langle q_f \rangle \approx q_0$. Therefore, we define the entropic force F_S to be

$$F_S(q) = F(\langle q \rangle) - \langle F(q) \rangle, \tag{6.7}$$

since it captures the extra force due to diffusion.

We will now explicitly show the example of a Brownian particle with a wall, and show that it has very similar features to the gravitational example later discussed in Section 6.2. Imagine the same particle presented in Section 6.1, but now suppose that there is a step function $V\Theta(-q)$ potential (we could take $V \to \infty$). This prevents the particle from going to negative values. We can express this by modifying the Onsager-Machlup Lagrangian to be

$$\mathcal{L}_{OM}(\ddot{q}) = \frac{1}{2D_2}\left(\frac{d^2}{dt^2}|q|\right)^2, \tag{6.8}$$

the variation of the Lagrangian provides the fourth-order Euler-Lagrange equation for the most probable paths:

$$\frac{d^4}{dt^4}|q| = 0, \tag{6.9}$$

with general solution

$$q_{MPP}(t) = \alpha_0 + \alpha_1 t + \frac{1}{2}\alpha_2 t^2 + \frac{1}{6}\alpha_3 t^3. \tag{6.10}$$

where α_2 has been rescaled again. Let us now go through the same process as before while pointing out a few more details.

When substituting back into the action, we see that the terms corresponding to the deterministic solution α_0 and α_1 (which is the global minimum) drop out due to the second-order time derivative. Therefore, we are always allowed to fix them through initial conditions on $q(0)$ and $\dot{q}(0)$. The action then takes the form of a bivariate Gaussian distribution, which when integrated from the initial time $t_0 = 0$ to the final time t_f becomes:

$$e^{-\mathcal{S}_{OM}} = \exp\left(-\frac{t_f}{3D_2}\left(3\alpha_2^2 + 3\alpha_3\alpha_2 t_f + \alpha_3^2 t_f^2\right)\right). \tag{6.11}$$

At this point, we can relate α_2 and α_3 to other known initial conditions or final conditions, and the action will act as the probability weight of the most probable path given the specified conditions. However, we will use it once again to find the average final position. Again, we assume that the particle starts with no acceleration and that, at the final time, it is at position q_f. In particular, we fix $q(0) = q_0 > 0$ and $\dot{q}(0) = \ddot{q}(0) = 0$, such that the particle begins on the right-hand side of the wall with zero initial velocity and acceleration. This fixes $\alpha_1 = \alpha_2 = 0$. The last condition is fixed by setting $q(t_f) = q_f$, the final position of the particle, arriving at

$$q(t) = q_0 + \frac{(q_f - q_0)t^3}{t_f^3}. \tag{6.12}$$

We can now substitute the solution into the Lagrangian to perform a saddle point approximation and integrate it up to the final time. We arrive at the action which determines the probability weighting of the most probable path given initial and final conditions:

$$\mathcal{S}_{OM}(q_f) = \frac{6(q_f - q_0)^2}{D_2 t_f^3}, \tag{6.13}$$

we can now integrate over all possible final positions to normalise the integral

$$\int_{-\infty}^{\infty} dq_f\, P(q_f|q_0, \dot{q}_0, \ddot{q}_0) = \frac{1}{\mathcal{N}}\int_{-\infty}^{\infty} dq_f\, e^{-\frac{6(q_f-q_0)^2}{D_2 t_f^3}} = 1, \tag{6.14}$$

to find

$$\mathcal{N} = \sqrt{\frac{D_2 \pi\, t_f^3}{6}}. \tag{6.15}$$

Up to now, it appears that everything is the same as the case without the wall. In particular, it seems as if the probability distribution of final positions can go into the negative values.

However, we are using that as a mathematical trick, and we can compute the average final position by keeping in mind that there is a wall at $q = 0$ such that

$$\begin{aligned} \langle q_f \rangle &= \sqrt{\frac{6}{D_2 \pi\, t_f^3}} \int_{-\infty}^{\infty} dq_f \, \left(|q_f| e^{-\frac{6}{D_2 t_f^3}(q_f - q_0)^2} \right) \\ &= \frac{1}{12} \left[6q_0 + \left(1 + \Gamma_{\mathcal{R}} \left(-\frac{1}{2}, 0, \frac{6q_0^2}{D_2 t_f^3} \right) \right) + \sqrt{\frac{6D_2}{\pi}} t_f^{3/2} e^{-\frac{6q_0^2}{D_2 t_f^3}} - 6q_0 \mathrm{Erf} \left(\sqrt{\frac{6}{D_2 t_f^3}} q_0 \right) \right], \end{aligned} \tag{6.16}$$

where Erf is the error function and $\Gamma_{\mathcal{R}}$ is the regularised Gamma function.

This solution is very insightful. As the diffusion vanishes $D_2 \to 0$ or the final time goes to zero $t_f \to 0$, the argument of the error function, the exponential and the regularised gamma function go to infinity. The error function and the exponential vanish while the gamma function becomes 1, leaving $\langle q_f \rangle = q_0$. As one would expect for a situation where there is either no diffusion or no time has elapsed, the final average position is the same as the initial one, which is also the deterministic behaviour. Even more interesting, the same happens when q_0 is very large; indeed, if the particle is very far from the wall, it will not feel its effect until enough time has passed, as it can be seen in Figure 6.1, such that there is an opposite effect between the growth of q_0 and that of t_f.

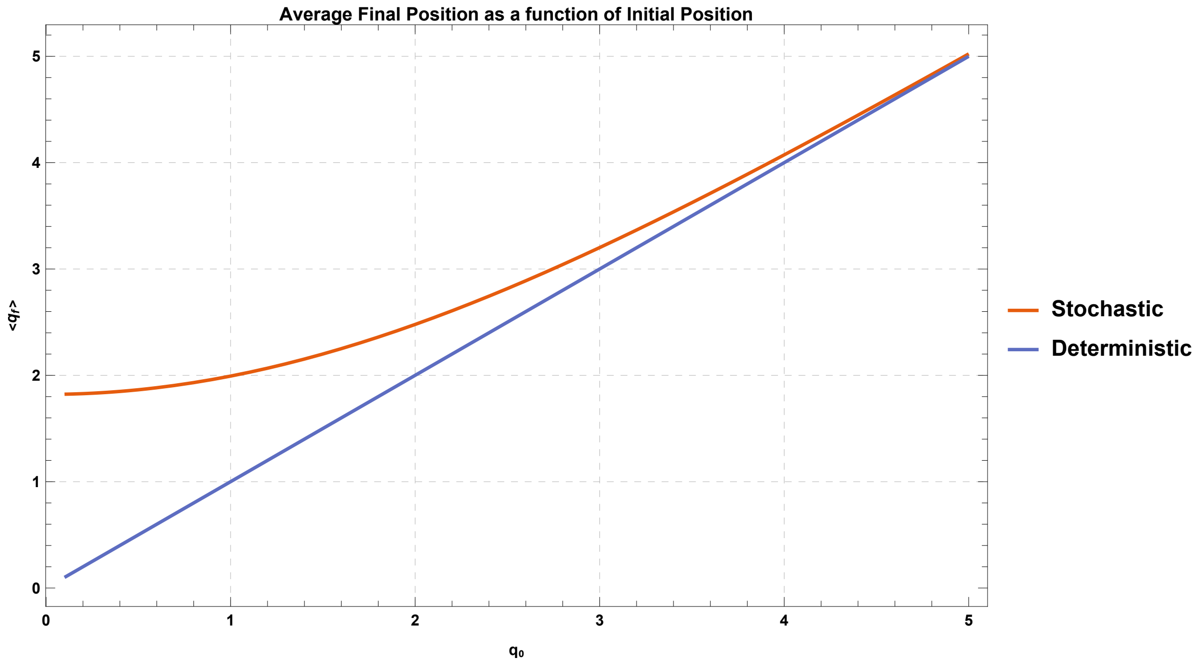


Figure 6.1: Average final position as a function of initial position according to Equation (6.16) for fixed final time $t_f = 5$ and diffusion coefficient $D_2 = \frac{1}{2}$. In the presence of a wall, the closer the Brownian particle starts to the reflective wall at $q = 0$, the more its average final position will diverge from its deterministic value. The particle is assumed to start with zero velocity and acceleration.

Lastly, one could assume the particle is not too far from the wall and perform a short time expansion to arrive at

$$\langle q_f \rangle = q_0 + \frac{1}{2}\sqrt{\frac{D_2}{6\pi} t_f^3}, \tag{6.17}$$

such that one sees that the average final position increases as $t^{3/2}$. In Figure 6.2, we show the probability density function of the final positions obtained from a Monte Carlo simulation.

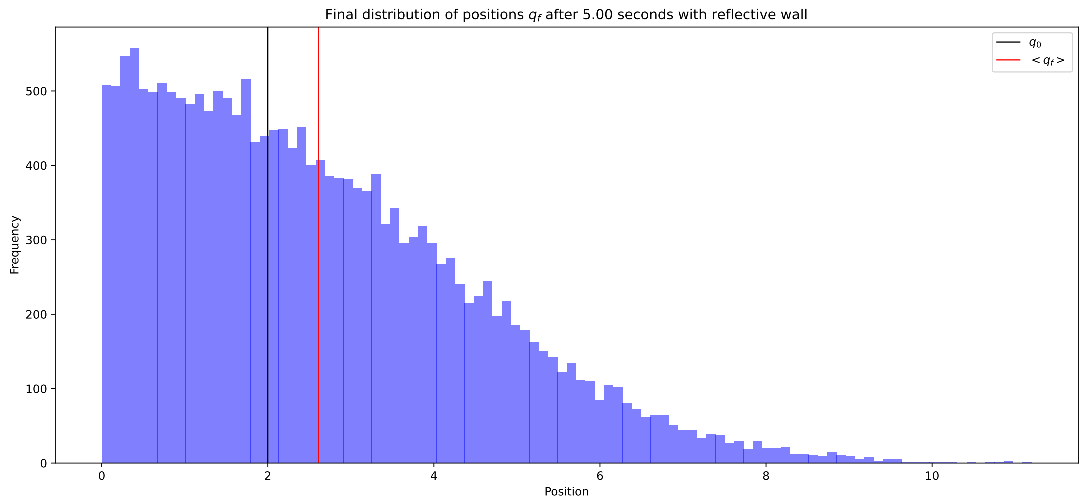


Figure 6.2: Monte Carlo simulation of the probability density function of the final position for the Brownian particle with a wall with $t_f = 5$. As one can see, the initial position is $q_0 = 2$. In the deterministic case, this should correspond to the final position in the absence of initial velocity and acceleration. However, the average final position is skewed to the right, as the presence of the wall produces an entropic force, which creates a deviation.

We will now apply these concepts to the gravitational sector of CQ theory and attempt to relate them to dark energy and dark matter.

6.2 Anomalous contribution to galaxy rotation curves

In 1964, the detection of the cosmic microwave background (CMB) by Penzias and Wilson substantiated a critical aspect of the Big Bang theory, underscoring the theory's postulate that the universe originated from a highly condensed and hot state and has been expanding ever since [251]. The dynamics of this expansion are influenced by the density and type of matter and energy present, with the total density's relation to the critical density being of particular interest. During the 1970s, scientific exploration was predominantly centred on models that hypothesised a universe composed entirely of baryonic matter. These models, however, faced significant challenges in explaining the formation of galaxies, primarily due to the minimal anisotropies in the CMB observed during this period. The breakthrough came in the early 1980s with the hypothesis that a universe dominated by cold dark matter (CDM) could resolve these discrepancies. This paradigm shift was further supported by the theory of cosmic inflation, which suggested a universe that approached critical density.

Research in the 1980s increasingly favoured models positing a universe with approximately 95% cold dark matter and 5% baryonic matter. These models showed promise in explaining the formation of galaxies and larger cosmic structures. Despite their successes, these models were not devoid of issues; they predicted a Hubble constant that was lower than observational data suggested and failed to account for the extent of galaxy clustering observed during the late 1980s and early 1990s [252]. The discovery of CMB anisotropy by the Cosmic Background Explorer (COBE) in 1992 led to significant revisions in the theoretical framework, with several modified CDM models coming under scrutiny. This period marked the advent of models such as ΛCDM, which incorporated dark energy, and hybrid models combining both cold and hot dark matter components [252]. The acceptance of the ΛCDM model was cemented following discoveries indicative of an accelerating universe in 1998, a conclusion supported by subsequent significant experiments such as the BOOMERanG microwave background experiment in 2000 and the 2dF Galaxy Redshift Survey in 2001. These studies indicated that the total (matter-energy) density of the universe was close to critical, while the baryonic and dark matter density was approximately 25%, suggesting the presence of a substantial component of dark energy [253, 254].

In addition, the ΛCDM model, while providing a robust framework for understanding the large-scale structure of the universe, also offers crucial insights into galaxy-scale phenomena, particularly through the study of the rotation curves of galaxies. Observations of galaxy rotation curves, which plot the rotational velocities of stars against their radial distance from the centre of the galaxy, reveal that the rotational velocities of stars do not decrease with increasing radius as would be expected if the visible matter alone were exerting gravitational influence. Instead, the velocities tend to flatten out, suggesting the presence of an additional, invisible mass component [255, 256]. This phenomenon cannot be explained by Newtonian mechanics and visible matter alone and is one of the primary observational evidence supporting the existence of dark matter. Within the framework of the ΛCDM model, dark matter is primarily cold (non-relativistic at the time of decoupling), which explains why it clumps and forms structures that do not disperse over time. These dark matter halos, as predicted by the model, extend well beyond the visible boundaries of galaxies and are believed to play a critical role in their formation and stability. The extended halos inferred from the rotation curves provide the necessary gravitational potential to account for the flat velocity profiles observed at large radii in spiral galaxies [255, 257]. Dark matter effects are observed in the CMB power-spectrum [258, 259], by gravitational lensing [260] such as that observed in the Bullet Cluster, through dispersion relations of elliptical galaxies [261], mass estimates of galaxy clusters [262], Further validation of the ΛCDM model through galaxy rotation curves comes from detailed simulations and observations of galaxy formation and evolution. These studies incorporate dark matter dynamics to simulate the observed properties of galaxies, including their rotation curves. Notably, the simulations under the ΛCDM paradigm successfully reproduce the thickness and distribution of galactic disks and the formation of galaxy clusters, aligning well with empirical data from observed galaxy rotations [263].

Despite its successes, ongoing research into the ΛCDM model seeks to resolve persistent discrepancies. At the cosmological scales, open questions such as the Hubble tension and the nearly uniform distribution of CMB perturbations across the celestial sphere believed to stem from minute thermal and acoustic irregularities at the epoch of recombination [264], still need to be addressed. At the galactic scales, the ΛCDM model faces challenges such as the "core-cusp problem" [265, 266, 267], where the predicted dense centres of dark matter halos (cusps) are not evident in the rotation curves of less massive galaxies, which instead show a more uniform distribution of dark matter (core), and the "missing satellite problem", referring to the discrepancy between the large number of small satellite galaxies predicted by simulations and the fewer number observed [268]. Lastly, the model lacks an explicit physical theory explaining the origins or nature of dark matter and energy. Despite large-scale efforts, neither dark energy nor dark matter have been directly detected [269, 270]. Their apparent existence is only felt through their gravitational field. Discoveries in physics are often indirect. Pauli conjectured the neutrino to exist in 1939 to account for energy conservation in β-decay, but the particle was first spotted through a signal in a particle detector 26 years later [271]. In the absence of any direct evidence for dark energy or dark matter, it is natural to wonder whether they could be unnecessary scientific constructs in the same manner as celestial spheres, aether, or the planet Vulcan, all of which were superseded by more straightforward explanations. Gravity has a long history of being a trickster.

Is it possible to explain dark matter-related effects without invoking a so-far invisible particle? In 1983, Mordehai Milgrom [272] realised that one could propose a theory modifying either the law of inertia or Newton's gravitational law at small accelerations such that

$$a = \begin{cases} a_N & \text{when } a \gg a_0, \\ \sqrt{a_0 a_N} & \text{when } a \ll a_0. \end{cases} \tag{6.18}$$

Here, a_N is the usual Newtonian acceleration, and a_0 a parameter of order $10^{-10}\ m/s^2$. With this modification, he could explain the flatness of galaxies rotation curves and the widely verified empirical relationship between the mass (or intrinsic luminosity) of a spiral galaxy and its asymptotic rotation velocity (or emission line width) known as Tully-Fischer relation [273]. He also observed that the acceleration parameter a_0 could be numerically related to fundamental constants of nature [272] such that, in a still unexplained coincidence:

$$a_0 \approx \frac{c^2}{2\pi}\sqrt{\frac{\Lambda}{3}}. \tag{6.19}$$

Milgrom named his theory Modified Newtonian Dynamics or MOND [274, 275, 276].

Several theories have been proposed to incorporate MOND-like phenomenology into a broader theoretical context. Notably, Bekenstein's TeVeS theory (Tensor–Vector–Scalar Gravity) and Milgrom's initial formulations have been foundational in this area [275, 276]. These approaches aim to modify the gravitational framework to allow for MOND's successes at galactic scales while attempting to remain consistent with General Relativity at larger scales. More recent theoretical explorations continue to expand on these ideas. Models that integrate aspects of quantum gravity and emergent gravity theories have been proposed too, suggesting that MOND phenomenology might arise from more fundamental physical principles [277, 278]. These theories often explore the low-energy limits of quantum theories of gravity, where they propose mechanisms by which MOND-like behaviour could naturally emerge. However, despite these theoretical efforts, a fully satisfying fundamental theory that reproduces MOND phenomenology across all scales has yet to emerge. The core challenge lies in the modification of gravitational theories at low energies. While it might be theoretically simpler to propose modifications at high energies—such as those near black holes or during the early universe—the constraints imposed by precise experimental measurements at low energies make it extremely difficult to alter theories without conflicting with observed phenomena [279]. Moreover, any successful low-energy modification must align with precise experimental data, including solar system dynamics and binary pulsar observations, which consistently support General Relativity with high precision. This requirement significantly restricts the space of viable theories. Since Milgrom's formulation of the original theory, various models that attempt to respect these experimental bounds while providing explanations for anomalous galactic dynamics have been proposed [280, 281]. However, it is also important to emphasise that MOND has yet to account for gravitational lensing results or the CMB power spectrum. Regardless of these shortcomings, it seems reasonable to wait before casting a final negative judgment, and it should be kept in mind that MOND should not be interpreted as a fundamental theory.

While MOND is the most famous alternative theory to dark matter, other interesting proposals can account for discrepancies between the observed galactic rotation curves and those predicted by general relativity. A known alternative to MOND initiated by Mannheim originates from fitting rotation curves to a spherically symmetric metric chosen to be a solution to a modification of general relativity known as *conformal gravity* [282, 283, 284], an approach we will discuss in more detail after having presented the results of the path integral.

We will now compute the effect on galactic rotation curves obtained as a consequence of treating general relativity in the CQ framework. To do so, we will use the path integral formulation of Chapter 3 and results from the weak field limit of CQ gravity obtained in Chapter 4. We would like to point out that the CQ framework was not developed with the idea of fitting rotation curves but rather to reconcile quantum theory and gravity. However, in [1], it was already hypothesised that diffusion in the metric degrees of freedom might lead to corrections akin to those explained by dark matter and dark energy. One of the results we obtain is that even when starting with no cosmological constant, stochastic fluctuations are such that one should typically expect a small one. We then find that the same stochastic fluctuations act as positive contributions to the mass that become relevant at a certain acceleration value, setting the scale for the divergence from deterministic general relativistic behaviour. Furthermore, we find that the acceleration scale γ_1 is obtained from cosmological parameters in a numerically similar relationship to the coincidence of Eq. (6.19). The fluctuation corresponding to the acceleration γ_1 is given by $\gamma_1 \approx \Lambda R_H$, with R_H the Hubble radius. This becomes $\gamma_1 \approx \sqrt{\Lambda}$ in a Λ-dominated universe, resulting in flat rotation curves in a region far from the galactic centre but with possible deviations at larger distances. One should keep in mind that these parameters correspond to boundary conditions of the EOM, which could be subject to change once a fuller understanding of the dynamics of the theory is achieved. Lastly, we point out that since quantum-classical theories of gravity are very restricted, a better understanding of the phenomena described in this Chapter will likely enable astrophysical tests to discern the quantum nature of spacetime.

Let us now proceed with the derivation. In this Chapter, we focus only on the classical limit of the theory. Quantum matter degrees of freedom are taken to be fully decohered. Regardless, classical degrees of freedom still undergo stochastic evolution. Moreover, we do not concern ourselves with the evolution of matter and thus only represent those degrees of freedom with a mass density distribution $m(x)$. In Chapter 4 (but the same action can also be obtained from the Nordström gravity route of Chapter 5), we have seen that such CQ path integrals look quite simple in the Newtonian limit, where the metric can be parametrised in terms of the Newtonian gravitational potential Φ. In this limit, the action of [130] was found to be [6]

$$\mathcal{I}_{CQ}[\Phi, m, t_i, t_f] = -\frac{D_0(1-\beta)}{G_N^2}\int_{t_i}^{t_f} dt d\vec{x} \left(\nabla^2\Phi - 4\pi G_N m(x)\right)^2. \tag{6.20}$$

While this action by itself allows for any $\beta < 1$, In [9], it has been observed that $\beta < \frac{1}{3}$ is required to ensure positivity of the full action. Regardless of bounds, it is naturally assumed for β to be of order $\mathcal{O}(1)$. In this Newtonian limit, $\beta < 1$ is a necessary and sufficient condition for the path integral to suppress paths away from Poisson's equation. We take this opportunity to point out that D_0/G_N^2 (here $c = 1$) is a dimensionless coupling constant which determines the scale of fluctuations and that $\beta = \frac{1}{3}$, together with the absence of matter, would correspond to a conformally invariant theory of gravity.

Drawing the connection with Section 6.1, we see that the action (6.20) is in the form of an equation of motion squared and has a global maximum when the equations of motion are satisfied

$$\langle \nabla^2\Phi - 4\pi G_N m \rangle = 0. \tag{6.21}$$

As shown in Chapter 4, this action derived as the weak field limit of [1, 130] is a path integral formulation of Diósi and Tilloy [110] model when a local noise kernel is chosen. Thanks to the linearity in Φ of Equation (6.21), if $m(x)$ is not given by a statistical mixture of different distributions but represents a single defined distribution, we obtain that the average equation of motion is identical to the equation of motion for the average and

$$\langle \nabla^2 \Phi - 4\pi G_N m \rangle = \nabla^2 \langle \Phi \rangle - 4\pi G_N m = 0. \tag{6.22}$$

Therefore, when $\nabla^2 \langle \Phi \rangle$ satisfies Poisson's equation, there is no difference between the expectation value of Φ and its value in the corresponding deterministic theory. Nonetheless, much like in the Brownian motion example of Section 6.1, the action of Equation (6.20) is extremised not only by the field configuration Φ which satisfies Poisson's equation, but also by more general configurations corresponding to a vanishing action variation (for fixed endpoints).

In the absence of matter, when $m(x) = 0$, the action of Equation (6.20) is extremised by the solution to the biharmonic differential equation

$$\nabla^4 \Phi = 0, \tag{6.23}$$

which general solution away from $x = 0$ is given by

$$\Phi_{MPP}(x) = -\frac{\kappa_m}{4\pi |x|} + \kappa_0 - 8\pi\, \kappa_1 |x| + \kappa_2 |x|^2. \tag{6.24}$$

The first two terms are the standard Newtonian potential plus an arbitrary constant term. In comparison, the last two additional terms do not satisfy the standard vacuum Poisson's equation and are, consequently, local rather than global extrema. However, they still make substantial contributions to the path integral. Note that the κ_m term and the κ_1 term are Green's functions for ∇^2 and ∇^4, respectively, and for this reason, we've explicitly put in the sign and factors of π. Solutions to the biharmonic equation with a source can be found, for example, in [285], the difference here being that the source is $4\pi\nabla^2\Phi$.

Once again, we would like to highlight the nature of this solution as that of a most probable field configuration given a set of (so far unspecified) boundary conditions. Much like in Section 6.1, these should not be thought of in the same way as the usual equations of motion. The presence of stochastic noise bends particle trajectories and field configurations out of shape with respect to their deterministic counterparts. Many new configurations are now reachable from the same initial conditions of the gravitational field. However, those parametrised by Equation (6.24) are the most probable. We further stress that these extra contributions do not necessarily need to relate to the matter distribution, even if a more significant discrepancy corresponds to a much lower probability of weighting in the path integral. The dominant contribution to the path integral comes from the usual solution to Poisson's equation. At the same time, the rest merely represent stochastic deviations from it, which are not too suppressed in the path integral given a set of boundary conditions. Other configurations also contribute, but their probability weighting will be even smaller.

Therefore, we will call the generalised field configurations such as those of Equation (6.24) *most probable paths*, using the same language of Section 6.1. Expanding on this terminology, the path integral clearly shows the presence of different contributions to the gravitational potential that can differ significantly from the usual solution to Poisson's equation. On the one hand, time fluctuations at short distance scales are required for the gravitational sector to be consistent with quantum theory and current known bounds on massive particle superpositions as discussed in Chapter 4 and Chapter 5. On the other hand, one expects deviations from Poisson's equation and the complete relativistic theory due to non-linearities in the full path integral. Lastly, one should also expect deviations arising due to the theory's dynamics. These could include temporal fluctuations generated during early times, stretched out during the universe's expansion akin to vacuum fluctuations of quantum fields, or fluctuations built up over cosmological or galactic time scales. This chapter will study the latter two types of fluctuations. In this case, for slow-enough dynamics, the final distribution of deviations away from Poisson's equation should be characterised by the action of Eq. (6.20) with the omitted integral over time. Therefore, unlike the Brownian motion path integral in Section 6.1, this path integral is first treated as non-dynamical, characterising the relative probabilities of various deviations away from the Newtonian potential, regardless of origin. Indeed, paths such as $\Phi_{MPP}(x)$ in Eq. (6.24) should be treated as static as they are inherited from the relativistic theory, and one would not expect them to fluctuate over time. Hence, the integral over dx^0 is dropped. To facilitate the tracking of units and dimensions, the dimensionless coupling constant D_0/G_N^2 is replaced by $D_{0,T}/G_N^2$, having units of distance[1].

In light of what we are trying to achieve, it is worth pointing out that all the κ should be considered constants. Moreover, the contribution of κ_1 to the most probable path has units of acceleration, and the κ_2 contribution would be a solution to general relativity if there were a constant matter density everywhere. Consequently, κ_2 has the same units as the cosmological constant. If one were to substitute the most probable path of Equation (6.24) into the CQ Newtonian action of (6.20), then the κ_m, $m(x)$ and κ_0 term would not contribute to the action if the Newtonian term is used as a Green's function for the matter distribution $m(x)$. They can, therefore, be set by the boundary conditions, as is done in solving Poisson's equation. For simplicity, let us consider only the κ_2 term for a moment. Differently from the Newtonian term, this would give an extra contribution to the action and would hence be suppressed by the path integral. Performing the substitution, we call the result the *MPP-action*

$$I_{MPP} = -\frac{D_{0,T}(1-\beta)}{4\pi G_N^2}\int d^3x\,(6\kappa_2)^2 \quad . \tag{6.25}$$

As mentioned in Section 6.1, substituting the most probable paths in the action is reminiscent of the on-shell actions used in quantum field theory, which capture the leading order terms to the path integral. If we included other terms in the action (such as the κ_1 term), the action would allow us to calculate the relative probabilities of the chosen field configurations regardless of whether they extremise it. Adding a source term $\mathcal{J}(x)O(\kappa)$ to the action with an arbitrary function $O(\kappa)$ of the parameters $\kappa = \{\kappa_m, \kappa_0, \kappa_1, \kappa_2\}$, we could construct a partition function

$$Z_{MPP}[J] = \mathcal{N}\int \mathcal{D}\kappa\, e^{\mathcal{I}_{CQ}[\Phi_{MPP}, m, \mathcal{J}]}, \tag{6.26}$$

[1] A note about the units used in this Chapter: the dimensionless coupling constant of the theory is $D_0 c^6/G_N^2$, which might appear different from what seen in Eq. (4.22), but the difference has to do with the chosen convention for the measure of the Einstein-Hilbert action (dx^4 or $cdtd\vec{x}$) which will then be used to build the CQ action. The two are fundamentally equivalent. D_0 is related to the decoherence rate of the theory, and there are different conventions as to how many powers of c are absorbed into it. In the master equation approach, such as [3], D_0 is quoted in units $\frac{m^3}{kg^2\,s}$ where it is the inverse of the gravitational diffusion coefficient $D_2 = 1/D_0$. Here and in other path integral papers, $D_0 c^3$ is quoted in $\frac{m^3}{kg^2\,s}$, while $D_{0,T}c^3$ has units $\frac{m^4}{kg^2\,s}$. Unless a comparison with experiments is made, the convention will always be that of $c = 1$.

with $\mathcal{N}$ the normalisation factor. This could be used to compute correlation functions of the κ parameters. However, this would not be necessary for this simple example as we can immediately notice that this can be considered as a normal distribution in κ_2, with a standard deviation scaling as $G_N/\sqrt{D_{0,T}V}$ with V being the spatial volume of the region under consideration.

We will show how κ_2 is equivalent to a small cosmological constant of arbitrary sign. We also emphasise that it appears as a necessary fluctuation even though the deterministic equations of motion would not allow it. Care should be taken with the MPP action. When tossing 1000 times a coin slightly biased towards heads, the most probable single configuration is composed only of heads. A more natural characterisation of the outcome would be in terms of the expected number of heads vs tails, which also characterises any local sample, provided it is sufficiently large. Note that unlike the Brownian motion path integral of Section 6.1, or the full CQ general relativistic one described, for example, in Equation (4.22) of Chapter 4, this is a non-dynamical path integral. However, if the time evolution is slow enough, we would expect it to characterise the final distribution.

Let us now move to the relativistic case. Here, the static solution is the appropriate metric for considering the effect of stochastic fluctuations over large distances. Therefore, we consider a spherically symmetric metric of the form

$$ds^2 = -e^{2\phi(r)}dt^2 + e^{-2\psi(r)}dr^2 + e^{-2\chi(r)}r^2\, d\Omega^2, \tag{6.27}$$

with Ω the 2-dimensional solid angle. In general relativity, one usually redefines r to reduce the metric to two free parameters before using Einstein's equation. We will do that here for simplicity, but it's important to note that further investigation of the sensibility of this sort of coordinate system is needed since the metric is undergoing stochastic changes, which would require one to redefine r to obtain a strictly static object constantly. Nonetheless, with the aim of focusing on radial contributions and deviations from general relativity, we redefine r to set $\chi(r) = 0$ and remove $e^{2\chi}$ on expectation. Then, we consider metrics of the generalised Schwarzschild form

$$\phi(r) = \psi(r) = \frac{1}{2}\log\left(1 - \frac{2F(r)}{r}\right). \tag{6.28}$$

If we were working purely with general relativity, Einstein's equation would require

$$F(r) = M, \tag{6.29}$$

recovering in this way the usual Schwarzschild solution. Under the assumption that the stochastic fluctuations cannot change the Lorentzian character of the metric, we point out that $1-\frac{2F(r)}{r}=0$ is a horizon which acts to bound $F(r)$. When restricting the full dynamical action of Equation (4.22) to its purely gravitational part, one can see that it can be written as

$$\mathcal{I}_{CQ} = -\frac{D_0}{G_N^2}\int d^4x\,\sqrt{-g}\big(\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} - \beta\,\mathcal{R}^2\big). \tag{6.30}$$

Given this choice of metric, the curvature terms appearing in the diffusion action take the following form

$$G^{\mu\nu}G_{\mu\nu} = \mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} = \frac{2}{r^2}F''(r)^2 + \frac{8}{r^4}F'(r)^2, \tag{6.31}$$

$$G^2 = \mathcal{R}^2 = \frac{4}{r^2}\left(\nabla^2 F(r)\right)^2. \tag{6.32}$$

When substituting the ansatz of Equation (6.28) into this action, we obtain the simple form

$$\mathcal{I}_F = -\frac{8\pi D_{0,T}}{G_N^2}\int dr\,\left((1-2\beta)F''(r)^2 + \frac{(4-8\beta)}{r^2}F'(r)^2 - \frac{8\beta}{r}F'(r)F''(r)\right), \tag{6.33}$$

where the angular part has already been integrated so that the path integral is a two-dimensional Gaussian distribution in the variables F' and F'. Here, F' has dimensions of acceleration, and F'' has dimensions of the cosmological constant. Motivated in part by the MPPs of Equation (6.24), let us consider the polynomial power expansion of $F(r)$

$$\begin{aligned}\frac{2F(r)}{r} &= \sum_{n=-\infty}^{\infty}\gamma_n r^n \\ &= \cdots + \frac{\gamma_m}{r} + \gamma_0 + \gamma_1 r + \gamma_2 r^2 + \cdots\,. \end{aligned} \tag{6.34}$$

In the second line, we have written the terms relevant to the length scales we are considering. The γ_m ends up dropping from the action, and we will subsequently set it to $2G_NM$ since at order r^{-1} it is the standard Schwarzschild term, which can here be determined from boundary conditions. Whether we include the γ_i corresponding to other higher or lower powers or transcendental functions like $\log(r)$ makes no difference for this discussion. Indeed, one can notice that the series is linear in the γ_i. When it is substituted back into the action, the coefficients follow a multivariate Gaussian distribution with zero mean and non-zero correlation. One could then perform the Gaussian integrals over all other γ_i that are not of interest, and the action for the remaining $\gamma_0, \gamma_1, \gamma_2$ would not change. We justify our interest only in these contributions by noting that negative powers of r would not contribute far away from the mass distribution of a galaxy, which is our zone of interest. On the other hand, the action would heavily suppress higher powers of the expansion once the radial integral is performed, as one can manually verify. These terms represent noise configurations that grow faster than the spacetime volume, and they can also be connected to fluctuations larger than our Hubble volume. For this reason, we only focus on the correlation between γ_0, γ_1 and γ_2 since including the other γ_i would not affect our conclusions.

We now substitute the power series of Equation (6.34) into the action and obtain

$$\mathcal{I}_\gamma = -\frac{8\pi D_{0,T}}{G_N^2}\int_0^{r_{max}} dr\,\big((5-18\beta)\gamma_1^2 + 18(1-4\beta)\gamma_2^2 r^2 + 18(1-4\beta)\gamma_1\gamma_2 r\big)\,, \tag{6.35}$$

for which we see that positivity imposes $\beta < \frac{1}{4}$. When integrated, the action becomes

$$\mathcal{I}_\gamma = -\frac{6\pi D_{0,T} V}{G_N^2}\left((5-18\beta)\frac{\gamma_1^2}{r_{max}^2} + 6(1-4\beta)\gamma_2^2 + 9(1-4\beta)\frac{\gamma_1\gamma_2}{r_{max}}\right). \tag{6.36}$$

We have dropped the constant term γ_0 for ease of presentation, as a constant added to the gravitational potential does not alter the conclusions nor contributes to the galactic rotation curves. Here, $V = \frac{4}{3}\pi r_{max}^3$ represents the spatial volume, and we could absorb it into the coupling constant $D_{0,T}/G_N^2$, which would renormalise it and give it units of Planck length to the 4-th power l_p^4, but we leave it in place to keep track of units. The analysis would not change much if we had integrated from an inner horizon r_{min} to r_{max}. Since we are considering large-scale fluctuations that exist throughout space and are naturally suppressed by a volume element, γ_2 is only suppressed by this amount because it corresponds to a constant noise configuration. We stress again that higher powers in the expansion of Equation (6.34) would be even more suppressed, representing configurations of the noise contribution that grow more than the spatial volume and fluctuations of a length scale which is not felt inside our Hubble volume, motivating us to integrate out these terms. This leaves us with the path integral over $\gamma = \{\gamma_1, \gamma_2\}$

$$Z_\gamma = \mathcal{N}\int \mathcal{D}\gamma\, e^{\mathcal{I}_\gamma[\gamma,m]}. \tag{6.37}$$

Much like in the Newtonian case, the γ_m term drops out of the action and can be fixed to match the general relativistic solution. Hence, integrating over 4-geometries is here limited to

4-geometries corresponding to the metric

$$ds^2 = -\left(1 - \frac{2MG_N}{r} - \gamma_1 r - \gamma_2 r^2\right) dt^2 + \left(1 - \frac{2MG_N}{r} - \gamma_1 r - \gamma_2 r^2\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (6.38)$$

Up to the extra constant term γ_0, this is equivalent to the *MK metric* of [285, 286]. This metric has been used to fit galactic rotation curves [282, 283, 284, 285, 287]. Unlike Mannheim, we will not fit γ_1; instead, we will determine it from the path integral. One should remember that while this metric is a solution to conformal gravity [286], it is not a solution to general relativity. Regardless, it contributes to the CQ path integral, as seen from Equation (6.36). While conformal gravity has issues deriving from negative norm ghosts, these same issues appear to be resolved for CQ in [9] (see also [288, 289, 290, 291, 292, 293, 294]). Moreover, criticisms of using the MK metric in fitting rotation curves akin to those presented in [295, 296, 297] are also not applicable to the classical-quantum theory of [1, 130] which is not conformally invariant, but rather scale-invariant without matter [9]. The matter action then breaks scale invariance. Here, the correct Newtonian potential plays the role of the dominant saddle contribution in our path integral. Before proceeding, we keep in mind that γ_2 corresponds to the cosmological constant term of Schwarzschild deSitter, while γ_1 contributes to the geodesic equation of stars far from the galactic centre.

Let us now explicitly normalise the path integral of Equation (6.37) and obtain the normalised probability distribution

$$\begin{aligned} f(\gamma) &= \frac{1}{\mathcal{N}} \exp\left(- \frac{6\pi D_{0,T} V}{G_N^2} \left((5 - 18\beta)\frac{\gamma_1^2}{r_{max}^2} + 6(1-4\beta)\gamma_2^2 + 9(1-4\beta)\frac{\gamma_1\gamma_2}{r_{max}} \right) \right), \\ \mathcal{N} &= \frac{r_{max}}{3\sqrt{3(1-4\beta)(13-36\beta)}} \frac{G_N^2}{D_{0,T} V}. \end{aligned} \quad (6.39)$$

which can be seen in the contour plot:

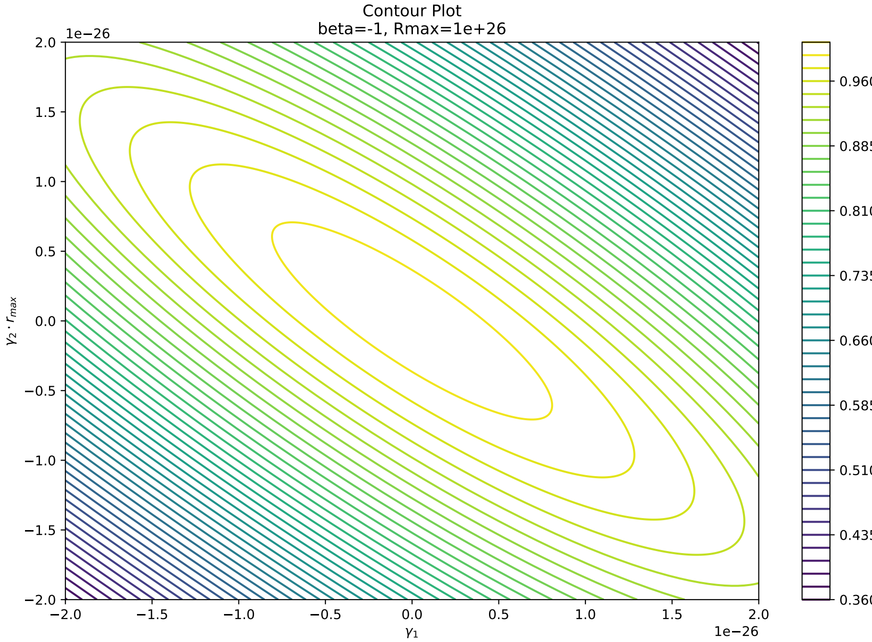


Figure 6.3: Contour plot of the probability distribution of γ_1 and γ_2 defined in Eq. (6.39). The negative correlation of the two variables is easily seen. To enhance the visibility of the plot, we have plotted γ_1 against $\gamma_2 * r_{max}$, and we chose a value of $r_{max} = R_H \approx 10^{26}\, m$, which represents the order of magnitude of the Hubble radius. We picked an indicative value of $\beta = -1$, but different beta will only tune the correlation as long as $\beta < \frac{1}{4}$.

The expectation values of γ_1 and γ_2 are zero, but the two random variables are normally distributed correlated. We can explicitly compute the second moments of γ and obtain the covariance matrix of the multivariate Gaussian distribution, which is composed of

$$\begin{aligned}
\Sigma_{11} &= \frac{2r_{max}^2}{3(13-36\beta)} \frac{G_N^2}{D_{0,T}V} \\
\Sigma_{22} &= \frac{(5-18\beta)}{9(13-36\beta)(1-4\beta)} \frac{G_N^2}{D_{0,T}V} \\
\Sigma_{12} &= -\frac{r_{max}}{2(13-36\beta)} \frac{G_N^2}{D_{0,T}V}
\end{aligned} \tag{6.40}$$

where Σ_{11} and Σ_{22} are the variances of γ_1 and γ_2 and the two variables have correlation coefficients given by

$$\rho_{12} = -\frac{3\sqrt{3}}{2\sqrt{2}} \sqrt{\frac{1-4\beta}{5-18\beta}}, \tag{6.41}$$

with the negative correlation easily seen from the plot in Figure 6.3 or by inspection of Equation (6.39). Let us pause and think about this distribution. The expected values of γ_1 and γ_2 are 0. The probability of both being exactly at their most probable value is zero. We expect them to be more likely in an interval around the distribution peak with a certain probability depending on $D_{0,T}$. Through observational cosmology, we know that our universe presents a positive cosmological constant Λ, which has to be manually inserted in Einstein's equations by hand and, in the weak field limit, contributes to the Newtonian potential as

$$\Phi = -\frac{GM}{r} - \frac{\Lambda}{3}r^2, \tag{6.42}$$

where the r^2 dependence is the sign of a global contribution. Emboldened by observations, we can use this to identify the γ_2 factor. In other words, given that we observe a specific value of $\gamma_2 = \frac{\Lambda}{3}$, we can ask what the expected value of γ_1 is. This can be computed easily by finding the conditional expectation

$$\begin{aligned} \mu_{\gamma_1|\gamma_2,r_{max}} &= \mu_{\gamma_1} + \rho_{12}\frac{\sigma_{\gamma_1}}{\sigma_{\gamma_2}}(\gamma_2 - \mu_{\gamma_2}) \\ &= -\frac{9}{2}\gamma_2 r_{max}\left(\frac{1-4\beta}{5-18\beta}\right), \end{aligned} \tag{6.43}$$

where we used the fact that $\mu_{\gamma_1} = \mu_{\gamma_2} = 0$. At this point, we make the substitution $\gamma_2 = \frac{\Lambda}{3}$ and choose $r_{max} = R_H = 1.37 * 10^{26}\, m$ to be the Hubble radius. This choice seems sensible because the Hubble radius gives a scale of the distance beyond which galaxies are receding from us faster than the speed of light due to the expansion of the Universe. Therefore, we arrive at:

$$\begin{aligned} \mu_{\gamma_1|\gamma_2,r_{max}} &= -\frac{3}{2}\Lambda R_H\left(\frac{1-4\beta}{5-18\beta}\right) \\ &= -2.28 * 10^{-26}\left(\frac{1-4\beta}{5-18\beta}\right). \end{aligned} \tag{6.44}$$

Restoring the units of c means multiplying the above expression by c^2, we obtain

$$\mu_{\gamma_1|\gamma_2,r_{max}} = -2.06 * 10^{-9}\left(\frac{1-4\beta}{5-18\beta}\right), \tag{6.45}$$

which, when $\beta \to 0$, tends to

$$\mu_{\gamma_1|\gamma_2,r_{max}} = -4.11 * 10^{-10}\; m/s^2, \tag{6.46}$$

as plotted in Figure 6.4.

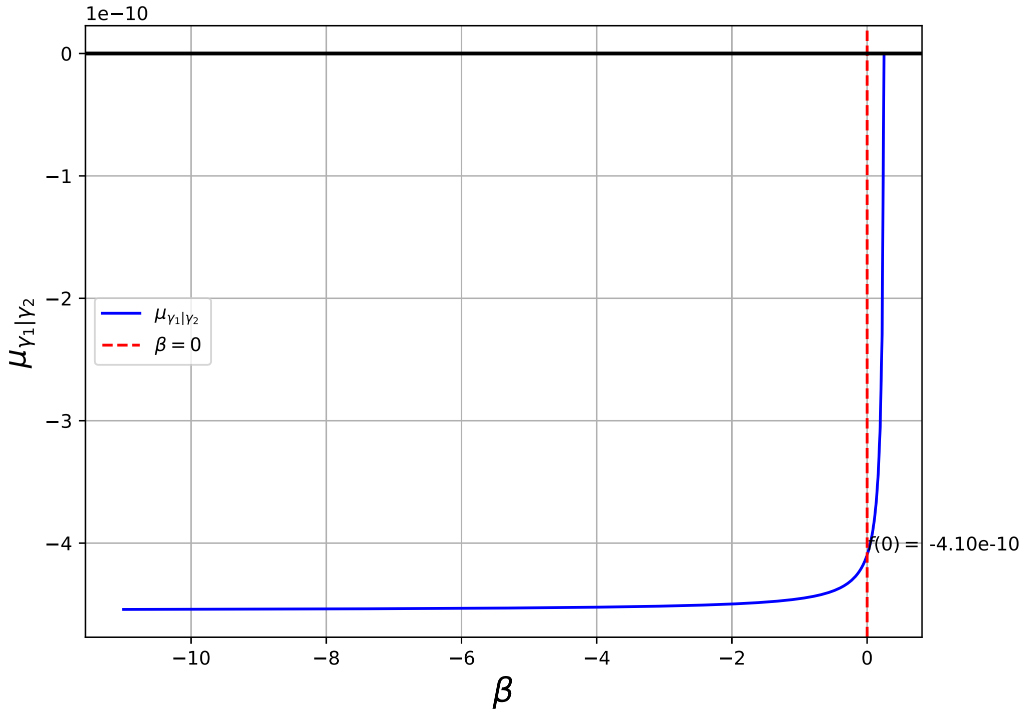


Figure 6.4: Conditional expectation of the value of γ_1 given the observed value of γ_2 as a function of β. Here $\gamma_2 = \frac{\Lambda}{3}$.

We can now more explicitly draw the connection between the cosmological constant and the acceleration regime of Equation (6.19). In a dark energy-dominated universe, the value of the Hubble radius can be expressed in terms of the cosmological constant as $R_H = \sqrt{\frac{3}{\Lambda}}$, meaning that the expected value of γ_1 is

$$\mu_{\gamma_1|\gamma_2,r_{max}} = -\frac{3\sqrt{3}}{2}\sqrt{\Lambda}\left(\frac{1-4\beta}{5-18\beta}\right) \approx \sqrt{\Lambda}. \tag{6.47}$$

Let us now conclude with a statistical analysis of the results. Given the observed value of Λ, we want to test two things. Firstly, we check how many standard deviations the observed value of γ_2 is from the predicted expected value of zero. Secondly, we derive how many standard deviations from its conditional expected value is the observed value of γ_1 needed to fit the data. We know the two values follow a bivariate Gaussian distribution so that we can utilise a Z-test. A Z-test is a statistical method that uses the standard normal distribution to calculate the probability of observing a test statistic as extreme as the one obtained, assuming the null hypothesis is true. Given that we have a free parameter $D_{0.T}$, this would allow us to understand the range of possible values of the decoherence constant required for the results to sit within 1 standard deviation of their expectation.

To perform the Z-test of γ_2, we recall that the observed value is $\gamma_2 = \frac{\Lambda}{3}$ and compute

$$Z_{\gamma_2} = \frac{\Lambda/3}{\sqrt{\Sigma_{22}}}, \tag{6.48}$$

where Σ_{22} is the variance of γ_2, and we use the value of the maximal radius as the Hubble radius. We get

$$Z_{\gamma_2} = \frac{\Lambda\sqrt{D_{0,T}V_H}}{G_N}\sqrt{\frac{(1-4\beta)(13-36\beta)}{5-18\beta}}. \tag{6.49}$$

We would like the Z score of γ_2 to be less than 1, such that the observed value of γ_2 lies within one standard deviation from the mean. Given that $V_H \approx 10^{79}$, plugging the values in the formula, we obtain

$$D_{0,T} \leq \frac{Z_{\gamma_2}^2 G_N^2}{V_H \Lambda^2} f(\beta), \tag{6.50}$$

with $f(\beta) = \sqrt{\frac{(1-4\beta)(13-36\beta)}{5-18\beta}}$. When substituting the back the units of c by dividing by c^3, the formula becomes

$$D_{0,T}c^3 \leq 1.34 \cdot 10^{-21} f(\beta)\ \frac{m^4}{s \cdot kg^2}. \tag{6.51}$$

Therefore, we see that we should be within the desired interval of one standard deviation for any value of $D_{0,T}$ with an order of magnitude less than $10^{-21} f(\beta)\ \frac{m^4}{s \cdot kg^2}$.

We can now perform the same computation for the observed value of γ_1. This could be the MOND value $\frac{a_0}{c^2} \approx 1.33 \cdot 10^{-27}$ or the value of γ_1 used by Mannheim [284]. The conditional variance of the result is given by

$$\begin{aligned} \sigma^2_{\gamma_1|\gamma_2, r_{max}} &= \Sigma_{11}(1-\rho_{12}^2) \\ &= \frac{G^2 r_{max}^2}{4D_{0,T}V(15-54\beta)}, \end{aligned} \tag{6.52}$$

which can be used to perform the Z-test together with the Hubble parameters

$$\begin{aligned} Z_{\gamma_1} &= \frac{\gamma_{obs} - \mu_{\gamma_1|\gamma_2, r_{max}}}{\sigma_{\gamma_1|\gamma_2, r_{max}}} \\ &= \frac{2\sqrt{D_{0,T}V_H(15-54\beta)}}{G_N R_H}\left(\gamma_{obs} + \frac{3(1-4\beta)\Lambda R_H}{2(5-18\beta)}\right). \end{aligned} \tag{6.53}$$

This can be rearranged to obtain

$$D_{0,T} \leq \frac{G_N^2 R_H^2 Z_{\gamma_1}^2}{V_H} \frac{(5-18\beta)}{3(2(5-18\beta)\gamma_{obs} + 3(1-4\beta)\Lambda R_H)^2}. \tag{6.54}$$

When substituting in the numbers, restoring units of c and setting $Z_{\gamma_1} = 1$, we obtain (for $\beta = -1$)

$$D_{0,T}c^3 \leq 1.09 \cdot 10^{-22} \ \frac{m^4}{s \cdot kg^2}, \tag{6.55}$$

which means that if $D_{0,T}c^3$ is such that the observed MOND acceleration is within one standard deviation of the conditional expected value, it will automatically be such that the observed value of Λ is within one standard deviation of the model.

So far, we have only considered correlations in larger-scale anomalous contributions rather than short-distance fluctuations. We would like to understand better how these arise from the local time-dependent fluctuations in (6.20). First, we can cross-check the obtained result with the theory's predictions for local time-dependent fluctuations. We present a suggestive calculation obtained by looking at the post-Newtonian expansion of the full theory. Here, we can see that local stochastic fluctuations lead to an acceleration scale, below which the laws of gravity are modified. This is most easily done in isotropic coordinates, and we refer the reader to Appendix E for more details. In these coordinates, the action is given by Eq. (E.9)

$$\begin{aligned} \mathcal{I}_{CQ} = -\frac{D_0 c^5}{64\pi^2 G_N^2} \int d^4x \, e^{\frac{2\Phi}{c^2}} \Bigg[& \left(\nabla^2\Phi - \frac{(\nabla\Phi)^2}{2c^2} - 4e^{-\frac{2\Phi}{c^2}}\pi G_N m\right)^2 + \frac{3}{c^4}(\nabla\Phi)^4 \\ & - 4\beta\left(\nabla^2\Phi - \frac{(\nabla\Phi)^2}{2c^2} - 4e^{-\frac{2\Phi}{c^2}}\pi G_N m\right)^2\Bigg], \end{aligned} \tag{6.56}$$

where powers of c have been re-inserted to highlight each term's contribution order. One can see that the terms containing the acceleration squared $(\nabla\Phi)^2/c^2$ play an important role. Taking $\beta = 0$ for simplicity since the argument doesn't change much if it's non-zero, let us also drop the $\frac{3}{c^4}(\nabla\Phi)^4$ as its inclusion would only enhance the argument we are about to present. Therefore, on expectation, the action implies that the scalar gravitational potential Φ must satisfy

$$\left\langle e^{\frac{\Phi}{c^2}} \left(\nabla^2\Phi - \frac{1}{2c^2}(\nabla\Phi)^2 - 4e^{-\frac{2\Phi}{c^2}}\pi G_N m(x)\right)\right\rangle = 0. \tag{6.57}$$

Immediately, one can notice that when $\langle(\nabla\Phi)^2\rangle >> \langle\nabla\Phi\rangle^2$, a deviation from the Newtonian limit of general relativity will be observed on average. Indeed, from Eq. (6.57), the extra variance acts like a positive mass term. We call the regime when $\langle(\nabla\Phi)^2\rangle >> \langle\nabla\Phi\rangle^2$, the *diffusion regime*. This regime implies that, when the acceleration $|\nabla\Phi|$ is small in comparison to its standard deviation, a deviation from the Newtonian law of gravity can be observed. In Section 6.1, we already defined an entropic force to be such a deviation from the deterministic equations. This is distinct from the entropic force used by Verlinde in the context of Holography, in which gravity itself is proposed as an entropic force acting as dark matter [298].

If the diffusion in the acceleration is relatively constant away from the galactic centre, then this naturally picks out a universal acceleration scale as in MOND phenomenology. Once the acceleration drops below the level set by the diffusion in $|\nabla\Phi|$, one observes a deviation from Newton's law and its post-Newtonian corrections. On the other hand, If $|\nabla\Phi|$ is above the diffusion regime, the expectation value $\langle\nabla\Phi\rangle$ obeys the post-Newtonian equations of motion, which explains why PPN tests of general relativity are unaffected by the stochastic fluctuations of [1, 130].

6.3 Gravitational fluctuations experiments

In [3], the constraints on temporal fluctuations of the gravitational field were computed from tabletop precision gravity measurements. These, in turn, constrain the value of D_0, which can then be compared with the constraints derived from galactic rotation curves. Given the two-point function of the theory, one can compute the variance of the local acceleration. This was done in [3] for several two-point functions. The *ultra-local, non-relativistic theory*, has a weak-field limit two-point function for $\Phi(x)$ given by

$$G_2(x,x') = \frac{G_N^2}{(4\pi)^2(1-\beta)D_0}\int d^3y\,\frac{\delta(t,t')}{|\vec{y}-\vec{x}||\vec{y}-\vec{x}'|} \quad . \tag{6.58}$$

This corresponds to the linearised weak field limit of GR with local white noise, a continuous model in phase space. This model was thought to be ruled out by experiments due to the divergences in the potential variance. Since $\langle\Phi(x)\Phi(x')\rangle$ diverges over finite distances with the volume of the ambient space unless D_2 depends strongly on the potential, this is an IR divergence. However, using dimensional regularisation, Eq. (6.58) can be shown to be equivalent to the two-point function

$$G(x,x') := -\frac{G_N^2}{8\pi(1-\beta)D_0}|\vec{x}-\vec{x}'|\delta(t,t'), \tag{6.59}$$

which is obtained by analytically continuing around the divergence. This was proven in [9], and we report it here in Appendix F. While Eq. (6.59) is not positive semi-definite at face value, a large enough constant can be added for any finite region of $|x-x'|$ to ensure this property is obeyed. It is important to remember that this constant does not affect the variance in acceleration, which is the physically meaningful quantity. Still, a fuller understanding of the physical implications is required. These two-point functions can be found from the non-relativistic limit of the two-point function for the trace of the scalar mode of the complete relativistic theory when the bare cosmological constant is taken to be zero [9].

The theory is consistent with the results when the regularised two-point function is used to place bounds on D_0 from tabletop experiments. This indicates that the IR divergence appears to be an artefact of the unregularised theory, leading to the conclusion that the ultra-local theory is not ruled out by experiment, even when the theory is linearised. One could compute the acceleration covariance matrix directly from Eq. (6.58)

$$\frac{\partial^2 G(x,x')}{\partial x_i \partial x'_j} = \frac{G_N^2}{8\pi(1-\beta)D_0}\frac{1}{(4\pi)^2}\int d^3y \left\{\frac{\delta_{ij}}{y|\vec{y}+\vec{z}|^3} - 3\frac{(y+z)_i(y+z)_j}{y|\vec{y}+\vec{z}|^5}\right\}\delta(t,t'), \tag{6.60}$$

where $\vec{z} = |\vec{x}-\vec{x}'|$ and which swaps the infrared divergence for an ultraviolet divergence for $\vec{y} \to -\vec{z}$. However, due to its equivalence with Eq. (6.59), we know this result to be finite in dimensional regularisation and be equivalent to the covariance matrix obtained from (6.59) as shown in [9] and reported in Appendix F

$$\frac{\partial^2 G(x,x')}{\partial x_i \partial x'_j} = \frac{G_N^2}{8\pi(1-\beta)D_0}\left(\delta_{ij} - \frac{|\vec{x}-\vec{x}'|_i|\vec{x}-\vec{x}'|_j}{|\vec{x}-\vec{x}'|^2}\right). \tag{6.61}$$

One can often obtain a more precise measurement of acceleration by monitoring the acceleration $a(t)$ over time and then applying a Fourier transform to obtain $\hat{a}(\omega)$. The Fourier transform of the acceleration variance is known as the *spectral density* $S_{aa}(\vec{x},\vec{x}';\omega)$. From [9], we know that the relativistic spectral density corresponding to the positive semi-definite propagator is derived to be

$$S_{aa}(\vec{x},\vec{x}';\omega) = \frac{D_G c^2}{4\pi(1-\beta)\epsilon|x-x'|(2i)}\left(k^2(\omega,+\epsilon)e^{ik(\omega,+\epsilon)|x-x'|} - k^2(\omega,-\epsilon)e^{-k(\omega,-\epsilon)|x-x'|}\right), \tag{6.62}$$

where $k(\omega, \pm\epsilon) \equiv \sqrt{\omega^2 \pm i\epsilon}$. Therefore, in the limit $\epsilon \to 0$

$$S_{aa}(\vec{x}, \vec{x}'; \omega) = \frac{D_G c^3}{4\pi(1-\beta)} \frac{\omega^2}{\epsilon|x - x'|} \sin\left(\frac{\omega}{c}|x - x'|\right). \tag{6.63}$$

For completeness, we note that one could also use the same methods to obtain the spectral densities starting from the retarded and advanced propagators, arriving at

$$S_{aa,R}(\vec{x}, \vec{x}', \omega) = \frac{D_G c^3}{4\pi(1-\beta)} \left(\frac{1}{|x - x'|} + \frac{i\omega}{2}\right) e^{i\omega|x-x'|}, \tag{6.64}$$

for the retarded spectral density and

$$S_{aa,A}(\vec{x}, \vec{x}', \omega) = \frac{D_G c^3}{4\pi(1-\beta)} \left(\frac{1}{|x - x'|} - \frac{i\omega}{2}\right) e^{-i\omega|x-x'|}, \tag{6.65}$$

for the advanced one. In the spectral densities above, prefactors have been fixed to correspond to the action of Eq. (6.20) in the weak field limit, where $D_G = \frac{G_N^2}{D_0 c^6}$ is the dimensionless prefactor.

At low frequencies, the spectral density of Eq. (6.63) can be Taylor expanded to obtain

$$S_{aa}(\vec{x}, \vec{x}'; \omega) = \frac{D_G c^2}{4\pi(1-\beta)} \frac{\omega^3}{\epsilon}, \tag{6.66}$$

where we take $\epsilon = \omega_0^2$ as an "induced Planck mass" mass [9]. In particular, recent experimental bounds obtained from the solar system [299] place the mass of the graviton to be $m_g < 10^{-23} eV$. Using the equivalence of $1\, eV = 2.42 \times 10^{14}\ Hz$, we get $\omega_0^2 = 4.84 \times 10^{-18}\ Hz$.

The low-frequency regime is relevant for most experiments, in which case Eq. (6.66) closely tracks the spectral density. Experimental upper bounds on the relevant spectral density have been collected in [300, 301]. Here, differential acceleration measurements such as those found in [302, 303] have been excluded since stochastic fluctuations at low frequency may affect both masses or paths equally, requiring some further understanding. The experiment of [304] uses a torsion pendulum with a moment of inertia $I \sim 10^{-7}\ kg\, m^2$ and finds a torque variance of order $10^{-27} \frac{(Nm)^2}{Hz}$ at the $\omega = 3\ mHz$ scale. The disk radius is of order cm, leading to $S_{aa}(\omega) \sim 10^{-17} \frac{(m/s^2)^2}{Hz}$ is of the same order of magnitude as for the sphere. Taking $1 - \beta$ to be of order unity leads to the dimensionless coupling constant being $G_N^2/D_0 c^6 \leq 10^{-43}$ ($D_0 c^3 \geq 10^{-3} \frac{m^3}{kg^2\, s}$). The experiment of [151] uses mm size masses, and reports $S_{aa}(\omega) \sim 10^{-18} \frac{(m/s^2)^2}{Hz}$, leading to $G_N^2/D_0 c^6 \leq 10^{-44}$ ($D_0 c^3 \geq 10^{-2}\ \frac{m^3}{kg^2\, s}$) also at the mHz scale.

6.4 Discussion

Let us reiterate the main accomplishment of this Chapter. We have found that, for our universe, one expects modifications to the metric parametrised by γ_1 and γ_2. The expected value of γ_1 is of $\mu_{\gamma_1|\gamma_2,r_{max}} \approx -10^{-26}\ m^{-1}$ (notice how from Fig 6.4 we see that for negative β we are insensitive to its value). Putting units of c back in, we have found that γ_1 is of the same order as the MOND acceleration $\gamma_1 \approx 10^{-10}\ m/s^2$. This was the value initially used to fit galactic rotation curves for larger spiral galaxies in the context of conformal gravity [283], giving roughly flat rotation curves in the region where the transition between the increasing $\gamma_1 r$ term is of the same order of the decreasing $\frac{G_N M}{r}$ term. If γ_1 were too large, it would run afoul of experimental bounds on solar system evolution. However, γ_1 needs to be adjusted as $\gamma_1(M) = \gamma_1(1 + M/10^{10}M_\odot)$ with $\gamma_1 \approx 10^{-28}m^{-1}$ to account for smaller dwarf galaxies where cold dark matter simulations also incur in the core-cusp problem [265, 266, 267]. Dark matter models also require novel properties, additional assumptions or further considerations [305]. Recently, new data required γ_2, to be taken as $\kappa \approx 10^{-50}m^{-2}$ [284] to "flatten the curve" while giving a slight rise at long distances, which is claimed to be observed [306] (see in contrast [307]). Without further considering the dynamics of the matter distribution, we have no reason, at this point, to select the constants in the most probable path because they correspond to boundary conditions. It is a reasonable choice to set γ_2 to be the value of the cosmological constant given its observation over the scale of the Hubble radius and then use this to predict γ_1. However, it would also be consistent to choose $\gamma_2 \approx \kappa$ over galactic distances of $r_{max} \approx 100\,kpc$ and obtain $\gamma_1 \approx 10^{-28}\,m^{-1}$, given that the observed rotation curves in each galaxy only extends to a radius of that order (the galactic disk of larger galaxies). This would require viewing γ_2 as a fluctuation which appears as if it were a dark energy contribution, akin to some form of quintessence [308, 309, 310]. While none of this would be inconsistent with observation, little can be said without a greater understanding of galaxy formation in a cosmological setting. Instead, the first conclusion we would like to draw is that the order of magnitude estimates suggest that the theory makes predictions broadly in line with current observations and advocates that simulations of the theory, combined with astrophysical observations, could be used to test its anomalous behaviour.

Efforts to understand the effect of these stochastic fluctuations in cosmology are initiated in [311], where some evidence is found that stochastic perturbations in the early universe can reproduce the phenomenology of dark matter cosmologically, without the need for any additional matter degrees of freedom. In particular, evidence is found that our current Λ cannot only result from large-scale fluctuations in the evolution of the scale factor. Cosmology studies using different models of stochastic fluctuations have been considered in [312, 313]. Other approaches have also tried to connect cosmology with the emergence of dark matter [314] and even attempt to explain dark energy as a fluctuation of the Newtonian gravitational constant [315]. To provide a template for further comparison between models and observation, we have also derived an estimate for the parameter $D_{0,T}/G_N^2$ at large distances in the case where we take the cosmological constant to be the result of stochastic fluctuations. While the complete analysis is detailed in Section 6.2, we repeat the crucial points here.

Since we expect to live in a typical universe, the variance in γ_2 should be of the order of Λ^2, such that the value witnessed value of γ_2 can be considered typical. From the above covariance matrix, one can see that this sets the value $D_{0,T}/G_N^2$ to be of the order of $D_{0,T}/G_N^2 \approx 1/\Lambda^2 V_H$, with V_H the Hubble volume. In units with c it is perhaps easiest to think in terms of a diffusion coefficient 4-density $\mathcal{D}_2 := G_N^2/D_{0,T}c^6 V_H \approx 10^{-104} m^{-4}$. It is fascinating to think that this result could explain two unanswered questions. Firstly, the small but non-zero value of the cosmological constant, at least in terms of $D_{0,T}$, especially since we found in [9] that a bare cosmological constant cannot be included if one wants to preserve complete positivity. Secondly, the numerical coincidence of the acceleration scale $a_0 \approx \sqrt{\Lambda}$ presented in [272] at which there is a deviation from the expected general relativistic behaviour. Moreover, we also did not need to fine-tune β as we found it is upper bounded by $\frac{1}{4}$, and its influence on the results flattens out quickly at negative values of order $\mathcal{O}(1)$.

Let us now highlight the weaknesses of these calculations, starting with the assumptions regarding the spacetime metric. To make calculations analytically tractable, we have restricted ourselves to spherically symmetric and static spacetimes, with metrics of the form of Equation (6.38). For spherically symmetric matter distributions, it is natural for the expectation value of the metric and its variance to be spherically symmetric. However, any realisation of the stochastic noise would be highly non-uniform and dynamical in time. At the same time, the metric ansatz used in this work is static and uniform over spheres of radius r. In a future refinement of this work, allowing ψ and ϕ to be different would be desirable. This would double the number of parameters in the action and may give further insight into galactic rotation curves. Static spacetimes were chosen as a consequence of not expecting large-scale fluctuations in the relativistic theory. The only large-scale anticipated fluctuations are assumed to be those already present, and we conjectured that given the scale, the γ_2 term could represent fluctuations baked in during the inflation era. The $\mathcal{R}^2$ term, which dominates the covariant path integral, could allow for Starobinski inflation [316, 317, 318], which is favoured by CMB data [319]. The dynamics of other contributions to the path integral have yet to be discovered.

On the topic of the path integral, let us be clear about some of its limitations. We have restricted ourselves to understanding the correlation in γ_1 and γ_2. They reflect different length scales of stochastic fluctuations. Still, there are correlations between them and higher powers in the expansion (lower powers, too, but they might not be relevant at large distances). Here, the full Gaussian distribution of Equation (6.33) may provide some insight into the distribution of what is currently considered dark matter. However, care is necessary since many different behaviours can be fit into a power series. Since we do not know what other terms may contribute in a relevant way, a fuller understanding of the probability distribution is required. It might also be appropriate to choose a different series expansion and perform a complete principal component analysis via the Kosambi–Karhunen–Loève theorem [320]. Another caveat of the action we wish to highlight concerns its positive definiteness. This is required to give finite probability distributions and suppress paths that deviate from Einstein's equation. While the action of the weak field limit derived in Chapter 4 has this property, the generalised deWitt metric defined of Equation (4.23) is not positive semidefinite. Nonetheless, the negative contributions to the path integral appear to correspond to non-dynamical degrees of freedom [9, 321]. One corresponds to the Gauss-Bonnet term, which in 4 spatial dimensions is a purely topological term and a total divergence. Since we don't sum over topologies, its bulk contribution is benign. The total divergence is usually discarded as a boundary term at spatial and temporal infinity, which does not affect local physics. However, whether this can be done here is less clear since the initial condition does not entirely determine the final condition. The other negative contribution corresponds to the magnetic part of the Weyl curvature, which is also non-dynamical in the sense of being made up of only first-time derivatives in the metric. This concern doesn't affect this calculation because the Weyl curvature term is positive definite on the metrics we have considered and enters the action with an overall minus sign if the Gauss-Bonnet identity is used. Nonetheless, care should be taken in extending this work to dynamical spacetimes [322] until this issue is better understood.

We must also renormalise $D_{0,T}$ by the volume element. While this does not affect the mean values that have been derived, nor does it modify the relative variances of γ_1 vs γ_2, it may make the $D_{0,T}$ we estimate here challenging to relate to that measured at shorter distance scales. A greater understanding of the renormalisation flow will be required to relate bounds on $D_{0,T}$ coming from astronomical data to the scale relevant for tabletop experiments such as those proposed in [3] based on the decoherence vs diffusion trade-off [3, 127]. Moreover, only larger scale fluctuations correlations have been considered. Understanding short-distance fluctuations, primarily how the former arises from the local time-dependent fluctuations present in (6.20), would be very important. Therefore, a necessary future step will be cross-checking the results given here with the CQ framework's predictions of local fluctuations.

Let us also address one objection presented in [323]. The criticism does not affect the rest of the work but raises an interesting discussion point in relation to Equation (6.24). In their note, the authors point out that in the Newtonian vacuum limit, a r term in the potential cannot satisfy matching boundary conditions for a localised source if it is to satisfy the MPP equation with matter $\nabla^4\Phi(x) = 4\pi G_N \nabla^2 m(x)$. This is true. However, the MPP is not an equation of motion but a parameterization of the most probable paths, and other paths, including the r term in vacuum, contribute. This can be seen by also inserting the κ_1 term into the action, which gives, in the spherically symmetric case,

$$\mathcal{I}'_{MPP} = -\frac{D_{0,T}(1-\beta)}{G_N^2}\int r^2 dr \left(6\kappa_2 + \frac{2\kappa_1}{|r|}\right)^2 . \tag{6.67}$$

One can notice that κ_1 is anti-correlated with the cosmological constant term κ_2. While suppressed, it still contributes enough to influence galactic rotation curves. This is demonstrated further by the path integral. If one observes an r^2 term, an extremal path corresponding to a fluctuation acting as a cosmological constant, then one should expect to see an r term. This remains unchanged when a localised matter distribution is present. In the actual calculation, we have considered a power law expansion. Still, as already mentioned in this discussion, a fuller principle component analysis would be helpful to determine the appropriate contributions in more detail. However, we would like to point out that the r dependent term is the one that contributes the most to the action among all the terms that significantly affect acceleration curves at large distances. Therefore, it does have a more significant role comparatively to all r^n terms with $n > 2$ or $n < -1$.

Lastly, let us compare the experiments discussed in Section 6.3 to our results on tabletop experiments performed on Earth. In this Chapter, we have found that stochastic fluctuations in the acceleration with a standard deviation of the MOND acceleration could explain both the small value of the cosmological constant and perhaps the flatness of galactic rotation curves. We have estimated the value of the diffusion 4-density $G_N^2/D_{0,T}V_H c^3$ to be of the order of Λ^2. This makes is impossible for terrestrial experiments being able to detect the long-range stochastic fluctuations discussed here, although it says little about shorter-range fluctuations.

Fortunately, it appears that the lower bounds on D_0 found in Section 6.3 are consistent with current upper bounds due to interference experiments and the decoherence vs diffusion trade-off. There, we found that the decoherence rate corresponding to the path integral of Equation (4.22) or its weak field limit [6] to be $\lambda = 2D_{0,T}c^3M^2/V_\lambda$, where M is the mass of the particle in the interference experiment, and V_λ is the volume of the wave-packet. Given $M \approx 10^{-24}\,kg$ for fullerene molecules, $V_\lambda \approx 10^{-25}\ m^3$ for the wave packet volume estimated in the experiment of [324], and a decoherence rate of $\lambda \geq 0.1\ s^{-1}$ [324], one obtains $D_0c^3 \leq 10^{24}\frac{m^3}{kg^2\,s}$. Therefore, one has a decoherence vs diffusion squeeze of $10^{-69} \leq D_G \leq 10^{-43}$. It should be noted that this decoherence rate is not the Diosi-Penrose rate [101, 102, 103], since the theory considered here is ultra-local and linear.

If we now consider secondary decoherence effects akin to those seen in [55] and Chapter 4. This is the additional decoherence caused by the stochastic contribution to the matter Hamiltonian, which results in additional decoherence on top of that given by the Lindblad operators. Since the mass term of the Hamiltonian (e.g. $\frac{1}{2}m^2\int\sqrt{g}\phi^2$ for a massive scalar field ϕ) depends on the fluctuating metric, it will source further decoherence. For the two-point function of Eq. (6.59), and separated mass distributions, this has been calculated to be

$$\lambda = 2D_0c^3\frac{M^2}{V_\lambda} + \frac{G_N^2}{D_0c^3}\frac{M^2}{8\pi\hbar^2}|x_L - x_R|, \tag{6.68}$$

where $|x_L - x_R|$ is the spatial separation of the superposition, which can be taken to be of the order of the diffraction grating spacing (typically $\sim 100nm$). We have also included the primary decoherence term in Eq (6.68). The lower bound on D_0c^3 of $10^{-3}\frac{m^3}{kg^2\,s}$ imply that the secondary decoherence is negligible at $\leq 10^{-8}\, s^{-1}$. This also suggests that measurements of gravity at short distances [325] are of interest in understanding this apparent behaviour, as well as conducting experiments to test the quantum vs classical nature of spacetime, with such experiments becoming more and more feasible [3, 51, 217, 224, 239, 326].

It is too early to make bold claims, and a greater understanding of the theoretical and experimental constraints is required. As already mentioned in Chapter 4 and Chapter 5 when discussing experimental results, it could be possible for the effects derived here to be the result of a fully quantum theory of gravity for which the CQ program describes as an effective theory. However, we are inclined to regard this possibility as unlikely. We do not expect stochastic spacetime fluctuations of this magnitude in a quantum theory of gravity. The parameter space of such an effective theory has been found by Isaac Layton in [327]. While this Chapter suggests that galactic rotation curves can undergo modification due to stochastic fluctuations, a phenomenon attributed to dark matter, it is vital to acknowledge the existence of separate, independent evidence supporting ΛCDM. In particular, the CMB power spectrum, gravitational lensing, the need for dark matter in the structure formation process, and various methods used to estimate the mass in galaxies. These now form an essential set of tools to test the CQ framework and its validity as a hybrid classical and quantum interaction theory.

Chapter 7

Closing Remarks

"It is only with the heart that one can see rightly;
what is essential is invisible to the eye."
∼ The Little Prince

Over the last 100 years, immense talent has been invested in the search for a unified quantum framework encompassing all fundamental interactions. Out of this search came a great deal of understanding of the relation between gravity, quantum theory and information. Should the framework proposed in this thesis be accurate in its most radical form, the long-anticipated concept of unification may prove unattainable. Instead, it is possible that we are moving towards an era in which fundamentally distinct theories coexist harmoniously. This would once again highlight nature's indifference to physicists' expectations, regardless of how certain we may be. If this paradigm shift occurs, it raises the question of how long it will take the scientific community to embrace the idea that the laws of nature may not conform to our human-scale ideal of "unified elegance". Some aspects of this situation resemble the early 20th century, when many believed the field of physics was nearing completion. Similarly, today, an implicit belief exists that the discovery of a quantum theory of gravity represents the final step toward a comprehensive understanding of the universe. However, this conviction echoes sentiments expressed at the close of the 19th century, just before revolutionary developments reshaped the field:

- **Albert A. Michelson (1894)**: "The more important fundamental laws and facts of physical science have all been discovered, and these are now so firmly established that the possibility of their ever being supplanted in consequence of new discoveries is exceedingly remote... Our future discoveries must be looked for in the sixth place of decimals" [328].

- **William Thomson, Lord Kelvin (1900)**: "There is nothing new to be discovered in physics now. All that remains is more and more precise measurement" [329].

- **Max Planck (1924)**: "I was advised [in 1874] to abandon physics, as almost everything is already discovered, and all that remains is to fill a few holes" [330].

In hindsight, these early 20th-century predictions appear shortsighted, especially as they preceded both the theory of relativity and quantum mechanics. One might rightfully think that being proven so wrong would serve as a cautionary tale for future generations. However, modern discourse often reveals a comparable confidence in approaching the limits of our understanding of the universe. For instance, looking at a few modern quotes, we find the following:

- **Carlo Rovelli (2001)**: "A quantum theory of gravity is essential if we want to understand the fundamental nature of space, time, and matter. Without it, our picture of the universe will remain incomplete." [331].

- **Roger Penrose (2004)**: "A quantum theory of gravity is necessary not just to unify our understanding of nature, but to fundamentally change our view of reality. Only through it can we grasp the true nature of spacetime and the cosmos." [332].

- **Lee Smolin (2006)**: "I have no doubt that we need a quantum theory of gravity in order to understand the very early universe and the ultimate fate of black holes" [75].

Although these statements may appear selectively chosen for emphasis, the underlying sentiment is widely shared within the scientific community, albeit to varying degrees. There appears to be a dichotomy between the conviction of inching ever closer to a theory of quantum gravity and the lack of experimental evidence about its quantum nature. It is acknowledged that this could change with new experimental results, and any definitive proof of gravity's quantum nature would be welcomed. However, while the contributions from the pursuit of quantum gravity are not to be underestimated, it raises the question of how long one should continue along this path before reassessing the foundational assumptions.

In conclusion, it seems appropriate to close this thesis with a cautionary note from the great physicist Richard Feynman:

"The first principle is that you must not fool yourself - and you are the easiest person to fool".

Bibliography

[1] Jonathan Oppenheim. A Postquantum Theory of Classical Gravity? *Phys. Rev. X*, 13 (4):041040, 2023. doi:10.1103/PhysRevX.13.041040.

[2] Jonathan Oppenheim, Carlo Sparaciari, Barbara Šoda, and Zachary Weller-Davies. The two classes of hybrid classical-quantum dynamics. *arXiv preprint*, 3 2022.

[3] Jonathan Oppenheim, Carlo Sparaciari, Barbara Šoda, and Zachary Weller-Davies. Gravitationally induced decoherence vs space-time diffusion: testing the quantum nature of gravity. *Nature Commun.*, 14(1):7910, 2023. doi:10.1038/s41467-023-43348-2.

[4] Jonathan Oppenheim and Zachary Weller-Davies. Path integrals for classical-quantum dynamics. *arXiv:2301.04677*, 2023. URL `https://arxiv.org/abs/2301.04677`.

[5] Isaac Layton, Jonathan Oppenheim, and Zachary Weller-Davies. A healthier semi-classical dynamics. *arXiv preprint*, 2023.

[6] Isaac Layton, Jonathan Oppenheim, Andrea Russo, and Zachary Weller-Davies. The weak field limit of quantum matter back-reacting on classical spacetime. *JHEP*, 08:163, 2023. doi:10.1007/JHEP08(2023)163.

[7] Jonathan Oppenheim, Andrea Russo, and Zachary Weller-Davies. Diffeomorphism invariant classical-quantum path integrals for Nordström gravity. *Phys. Rev. D*, 110(2):024007, 2024. doi:10.1103/PhysRevD.110.024007.

[8] Jonathan Oppenheim and Andrea Russo. Anomalous contribution to galactic rotation curves due to stochastic spacetime. *arXiv preprint*, 2024. URL `https://arxiv.org/abs/2402.19459`.

[9] Andrzej Grudka, Jonathan Oppenheim, Andrea Russo, and Muhammad Sajjad. Renormalisation of postquantum-classical gravity, 2024.

[10] Muhammad Sajjad, Andrea Russo, Maite Arcos, Andrzej Grudka, and Jonathan Oppenheim. A quantum oscillator interacting with a classical oscillator. *arXiv preprint*, 2024. URL https://arxiv.org/abs/2403.07479.

[11] Philipp A. Hoehn, Andrea Russo, and Alexander R. H. Smith. Matter relative to quantum hypersurfaces. *Phys. Rev. D*, 109(10):105011, 2024. doi:10.1103/PhysRevD.109.105011.

[12] Juan F. Pedraza, Andrea Russo, Andrew Svesko, and Zachary Weller-Davies. Lorentzian Threads as Gatelines and Holographic Complexity. *Phys. Rev. Lett.*, 127(27):271602, 2021. doi:10.1103/PhysRevLett.127.271602.

[13] Juan F. Pedraza, Andrea Russo, Andrew Svesko, and Zachary Weller-Davies. Sewing spacetime with Lorentzian threads: complexity and the emergence of time in quantum gravity. *JHEP*, 02:093, 2022. doi:10.1007/JHEP02(2022)093.

[14] Juan F. Pedraza, Andrea Russo, Andrew Svesko, and Zachary Weller-Davies. Computing spacetime. *Int. J. Mod. Phys. D*, 31(14):2242010, 2022. doi:10.1142/S021827182242010X.

[15] Hermann Minkowski. Raum und zeit. *Physikalische Zeitschrift*, 10:104–111, 1909.

[16] A. Einstein. Zur Elektrodynamik bewegter Körper. *Annalen Phys.*, 322(10):891–921, 1905. doi:10.1002/andp.19053221004.

[17] Albert Einstein. Zur Allgemeinen Relativitätstheorie. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, 1915:778–786, 1915. [Addendum: Sitzungsber.Preuss.Akad.Wiss.Berlin (Math.Phys.) 1915, 799–801 (1915)].

[18] Max Planck. Zur theorie des gesetzes der energieverteilung im normalspectrum. *Verhandlungen der Deutschen Physikalischen Gesellschaft*, 2:237–245, 1900.

[19] Albert Einstein. über einen die erzeugung und verwandlung des lichtes betreffenden heuristischen gesichtspunkt. *Annalen der Physik*, 322(6):132–148, 1905.

[20] Niels Bohr. On the constitution of atoms and molecules. *Philosophical Magazine*, 26(151): 1–25, 1913.

[21] Werner Heisenberg. über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik. *Zeitschrift für Physik*, 43(3-4):172–198, 1927.

[22] Louis de Broglie. *Recherches sur la théorie des quanta (Researches on the quantum theory)*. PhD thesis, University of Paris, 1924.

[23] Erwin Schrödinger. An undulatory theory of the mechanics of atoms and molecules. *Physical Review*, 28(6):1049, 1926.

[24] Vladimir Fock. Über die invariante form der wellen- und der bewegungsgleichungen für einen geladenen massenpunkt. *Zeitschrift für Physik*, 39(2-3):226–232, 1926.

[25] Paul AM Dirac. The quantum theory of the electron. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 117 (778):610–624, 1928.

[26] Wolfgang Pauli and Werner Heisenberg. Zur quantendynamik der wellenfelder. *Zeitschrift für Physik*, 56(1-2):1–61, 1929.

[27] Paul AM Dirac. A theory of electrons and protons. *Proceedings of the Royal Society of London. Series A, containing papers of a mathematical and physical character*, 126(801): 360–365, 1930.

[28] Sin-Itiro Tomonaga. On a relativistically invariant formulation of the quantum theory of wave fields. *Progress of Theoretical Physics*, 1(2):27–42, 1946.

[29] Julian Schwinger. Quantum electrodynamics. i. a covariant formulation. *Physical Review*, 74(10):1439, 1948.

[30] Richard P Feynman. Space-time approach to quantum electrodynamics. *Physical Review*, 76(6):769, 1949.

[31] Chen Ning Yang and Robert Mills. Conservation of isotopic spin and isotopic gauge invariance. *Physical Review*, 96(1):191, 1954.

[32] Sheldon L Glashow. Partial-symmetries of weak interactions. *Nuclear Physics*, 22(4): 579–588, 1961.

[33] Steven Weinberg. A model of leptons. *Physical Review Letters*, 19(21):1264, 1967.

[34] Abdus Salam. Weak and electromagnetic interactions. In *Elementary Particle Theory: Relativistic Groups and Analyticity*, volume Nobel Symposium No. 8, pages 367–377. Almqvist & Wiksell, 1968.

[35] Murray Gell-Mann. A schematic model of baryons and mesons. *Physics Letters*, 8(3): 214–215, 1964.

[36] George Zweig. An su(3) model for strong interaction symmetry and its breaking. *CERN Reports No.8182/TH.401 and No.8419/TH.412*, 1964.

[37] Peter Higgs. Broken symmetries and the masses of gauge bosons. *Physical Review Letters*, 13(16):508, 1964.

[38] Matvei Bronstein. Republication of: Quantum theory of weak gravitational fields. *General Relativity and Gravitation*, 44:267–283, 2012.

[39] Bryce S. DeWitt. New directions for research in the theory of gravitation, 1953. URL https://static1.squarespace.com/static/5852e579be659442a01f27b8/t/587800e715d5dbb77e25374c/1484259563896/dewitt.pdf. Accessed on [Access Date].

[40] Dean Rickles and Cécile M. DeWitt. *The Necessity of Gravitational Quantization*. Max-Planck-Gesellschaft zur Förderung der Wissenschaften, 2011. doi:10.34663/9783945561294-30. URL https://edition-open-sources.org/sources/5/toc.html.

[41] Yakir Aharonov and Daniel Rohrlich. *Quantum Paradoxes: Quantum theory for the perplexed*. Wiley-VCH, 2011.

[42] Laser interference. Institute for Quantum Computing - University of Waterloo, 2024. https://uwaterloo.ca/institute-for-quantum-computing/outreach/quantum-toolbox/laser-interference.

[43] Bryce S DeWitt. Definition of commutators via the uncertainty principle. *Journal of Mathematical Physics*, 3(4):619–624, 1962.

[44] Kenneth Eppley and Eric Hannah. The necessity of quantizing the gravitational field. *Foundations of Physics*, 7(1-2):51–68, 1977.

[45] J Caro and LL Salcedo. Impediments to mixing classical and quantum dynamics. *Physical Review A*, 60(2):842, 1999.

[46] LL Salcedo. Absence of classical and quantum mixing. *Physical Review A*, 54(4):3657, 1996.

[47] Debendranath Sahoo. Mixing quantum and classical mechanics and uniqueness of planck's constant. *Journal of Physics A: Mathematical and General*, 37(3):997, 2004.

[48] Daniel R Terno. Inconsistency of quantum—classical dynamics, and what it implies. *Foundations of Physics*, 36(1):102–111, 2006.

[49] LL Salcedo. Statistical consistency of quantum-classical hybrids. *Physical Review A*, 85 (2):022127, 2012.

[50] Carlos Barceló, Raúl Carballo-Rubio, Luis J Garay, and Ricardo Gómez-Escalante. Hybrid classical-quantum formulations ask for hybrid notions. *Physical Review A*, 86(4): 042120, 2012.

[51] Chiara Marletto and Vlatko Vedral. Gravitationally induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity. *Physical review letters*, 119(24):240402, 2017.

[52] Thomas D. Galley, Flaminia Giacomini, and John H. Selby. A no-go theorem on the nature of the gravitational field beyond quantum theory, 2021.

[53] Mark Albers, Claus Kiefer, and Marcel Reginatto. Measurement analysis and quantum gravity. *Phys. Rev. D*, 78:064051, Sep 2008. doi:10.1103/PhysRevD.78.064051. URL https://link.aps.org/doi/10.1103/PhysRevD.78.064051.

[54] Adrian Kent. Simple refutation of the eppley–hannah argument. *Classical and Quantum Gravity*, 35(24):245008, 2018. ISSN 1361-6382. doi:10.1088/1361-6382/aaea20. URL http://dx.doi.org/10.1088/1361-6382/aaea20.

[55] Antoine Tilloy and Lajos Diósi. Principle of least decoherence for newtonian semiclassical gravity. *Physical Review D*, 96(10):104045, 2017.

[56] C. J. Isham. Quantum gravity. In M. A. H. MacCallum, editor, *General Relativity and Gravitation*, pages 99–129. Cambridge University Press, Cambridge, 1987.

[57] Claus Kiefer. *Quantum gravity*, volume 124. Clarendon, Oxford, 2004. ISBN 978-0-19-958520-5.

[58] P. A. M. Dirac. The theory of gravitation in hamiltonian form. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 246(1246):333–343, 1958.

[59] Bryce S. DeWitt. Quantum theory of gravity. i. the canonical theory. *Physical Review*, 160(5):1113–1148, 1967. doi:10.1103/physrev.160.1113.

[60] Edward Anderson. The problem of time in quantum gravity, 2012.

[61] Abhay Ashtekar. New variables for classical and quantum gravity. *Physical Review Letters*, 57:2244–2247, 1986.

[62] Carlo Rovelli and Lee Smolin. Loop space representation of quantum general relativity. *Nuclear Physics B*, 331:80–152, 1990.

[63] Thomas Thiemann. *Modern Canonical Quantum General Relativity*. Cambridge University Press, Cambridge, 2007.

[64] John C Baez. Spin foam models. *Classical and Quantum Gravity*, 15(7):1827–1858, 1998.

[65] John W Barrett and Louis Crane. Relativistic spin networks and quantum gravity. *Journal of Mathematical Physics*, 39(6):3296–3302, 1998.

[66] Alejandro Perez. The spin foam approach to quantum gravity. *Living Reviews in Relativity*, 16(1):3, 2013.

[67] Carlo Rovelli and Lee Smolin. Discreteness of area and volume in quantum gravity. *Nuclear Physics B*, 442:593–619, 1995. Erratum-ibid. B 456: 753, 1995.

[68] Abhay Ashtekar, John C. Baez, Alejandro Corichi, and Kirill Krasnov. Quantum geometry and black hole entropy. *Physical Review Letters*, 80(5):904–907, 1998.

[69] Joel Scherk and John H. Schwarz. Dual Models for Nonhadrons. *Nucl. Phys. B*, 81: 118–144, 1974. doi:10.1016/0550-3213(74)90010-8.

[70] T. Yoneya. Connection of Dual Models to Electrodynamics and Gravidynamics. *Prog. Theor. Phys.*, 51:1907–1920, 1974. doi:10.1143/PTP.51.1907.

[71] Michael B. Green, J. H. Schwarz, and Edward Witten. *SUPERSTRING THEORY. VOL. 1: INTRODUCTION.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 7 1988. ISBN 978-0-521-35752-4.

[72] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12 2007. ISBN 978-0-511-25227-3, 978-0-521-67227-6, 978-0-521-63303-1. doi:10.1017/CBO9780511816079.

[73] J. Polchinski. *String theory. Vol. 2: Superstring theory and beyond.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12 2007. ISBN 978-0-511-25228-0, 978-0-521-63304-8, 978-0-521-67228-3. doi:10.1017/CBO9780511618123.

[74] Peter Woit. *Not even wrong: The Failure of String Theory & the Continuing Challenge to Unify the Laws of Physics.* Vintage, 2006.

[75] Lee Smolin. *The Trouble with Physics: The Rise of String Theory, The Fall of a Science, and What Comes Next.* Houghton Mifflin, 2006.

[76] T. Banks. A Critique of pure string theory: Heterodox opinions of diverse dimensions, 6 2003.

[77] Leonard Susskind. The Anthropic landscape of string theory, 2 2003.

[78] Michael R. Douglas. The Statistics of string / M theory vacua. *JHEP*, 05:046, 2003. doi:10.1088/1126-6708/2003/05/046.

[79] P. Adamson, I. Anghel, A. Aurisano, G. Barr, M. Bishai, A. Blake, G. J. Bock, D. Bogert, S. V. Cao, T. J. Carroll, C. M. Castromonte, R. Chen, S. Childress, J. A. B. Coelho, L. Corwin, D. Cronin-Hennessy, J. K. de Jong, S. De Rijck, A. V. Devan, N. E. Devenish, M. V. Diwan, C. O. Escobar, J. J. Evans, E. Falk, G. J. Feldman, W. Flanagan, M. V. Frohne, M. Gabrielyan, H. R. Gallagher, S. Germani, R. A. Gomes, M. C. Goodman, P. Gouffon, N. Graf, R. Gran, K. Grzelak, A. Habig, S. R. Hahn, J. Hartnell, R. Hatcher, A. Holin, J. Huang, J. Hylen, G. M. Irwin, Z. Isvan, C. James, D. Jensen, T. Kafka, S. M. S. Kasahara, G. Koizumi, M. Kordosky, A. Kreymer, K. Lang, J. Ling, P. J. Litchfield, P. Lucas, W. A. Mann, M. L. Marshak, N. Mayer, C. McGivern, M. M. Medeiros, R. Mehdiyev, J. R. Meier, M. D. Messier, W. H. Miller, S. R. Mishra, S. Moed Sher, C. D. Moore, L. Mualem, J. Musser, D. Naples, J. K. Nelson, H. B. Newman, R. J. Nichol, J. A. Nowak, J. O'Connor, M. Orchanian, R. B. Pahlka, J. Paley, R. B. Patterson, G. Pawloski, A. Perch, M. M. Pfützner, D. D. Phan, S. Phan-Budd, R. K. Plunkett, N. Poonthottathil, X. Qiu, A. Radovic, B. Rebel, C. Rosenfeld, H. A. Rubin, P. Sail, M. C. Sanchez, J. Schneps, A. Schreckenberger, P. Schreiner, R. Sharma, A. Sousa, N. Tagg, R. L. Talaga, J. Thomas, M. A. Thomson, X. Tian, A. Timmons, J. Todd, S. C. Tognini, R. Toner, D. Torretta, G. Tzanakos, J. Urheim, P. Vahle, B. Viren, A. Weber, R. C. Webb, C. White, L. Whitehead, L. H. Whitehead, S. G. Wojcicki, and R. Zwaska. Constraints on large extra dimensions from the minos experiment. *Phys. Rev. D*, 94:111101, Dec 2016. doi:10.1103/PhysRevD.94.111101. URL `https://link.aps.org/doi/10.1103/PhysRevD.94.111101`.

[80] Katherine J. Mack and Robert McNees. Bounds on extra dimensions from micro black holes in the context of the metastable Higgs vacuum. *Phys. Rev. D*, 99(6):063001, 2019. doi:10.1103/PhysRevD.99.063001.

[81] Don N Page and CD Geilker. Indirect evidence for quantum gravity. *Physical Review Letters*, 47(14):979, 1981.

[82] S. W. Hawking. Black hole explosions. *Nature*, 248:30–31, 1974. doi:10.1038/248030a0.

[83] S. W. Hawking. Particle Creation by Black Holes. *Commun. Math. Phys.*, 43:199–220, 1975. doi:10.1007/BF02345020. [Erratum: Commun.Math.Phys. 46, 206 (1976)].

[84] Alan H. Guth. The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems. *Phys. Rev. D*, 23:347–356, 1981. doi:10.1103/PhysRevD.23.347.

[85] Misao Sasaki. Large Scale Quantum Fluctuations in the Inflationary Universe. *Progress of Theoretical Physics*, 76(5):1036–1046, 11 1986. ISSN 0033-068X. doi:10.1143/PTP.76.1036. URL https://doi.org/10.1143/PTP.76.1036.

[86] Alan H. Guth and So-Young Pi. Fluctuations in the new inflationary universe. *Phys. Rev. Lett.*, 49:1110–1113, Oct 1982. doi:10.1103/PhysRevLett.49.1110. URL https://link.aps.org/doi/10.1103/PhysRevLett.49.1110.

[87] Alan H. Guth and So-Young Pi. Quantum mechanics of the scalar field in the new inflationary universe. *Phys. Rev. D*, 32:1899–1920, Oct 1985. doi:10.1103/PhysRevD.32.1899. URL https://link.aps.org/doi/10.1103/PhysRevD.32.1899.

[88] Sidney Coleman. Black holes as red herrings: topological fluctuations and the loss of quantum coherence. *Nuclear Physics B*, 307(4):867–882, 1988.

[89] David J Gross. Is quantum gravity unpredictable? *Nuclear Physics B*, 236(2):349–367, 1984.

[90] T. Banks, M. E. Peskin, and L. Susskind. Difficulties for the evolution of pure states into mixed states. *Nuclear Physics B*, 244:125–134, September 1984. doi:10.1016/0550-3213(84)90184-6.

[91] S. W. Hawking. Breakdown of predictability in gravitational collapse. *Phys. Rev. D*, 14:2460–2473, Nov 1976. doi:10.1103/PhysRevD.14.2460. URL http://link.aps.org/doi/10.1103/PhysRevD.14.2460.

[92] S. W. Hawking. The Unpredictability of Quantum Gravity. *Commun. Math. Phys.*, 87:395–415, 1982. doi:10.1007/BF01206031.

[93] Ahmed Almheiri, Donald Marolf, Joseph Polchinski, and James Sully. Black Holes: Complementarity or Firewalls? *JHEP*, 02:062, 2013. doi:10.1007/JHEP02(2013)062.

[94] Pisin Chen, Yen Chin Ong, and Dong-han Yeom. Black Hole Remnants and the Information Loss Paradox. *Phys. Rept.*, 603:1–45, 2015. doi:10.1016/j.physrep.2015.10.007.

[95] Ahmed Almheiri, Thomas Hartman, Juan Maldacena, Edgar Shaghoulian, and Amirhossein Tajdini. The entropy of Hawking radiation. *Rev. Mod. Phys.*, 93(3):035002, 2021. doi:10.1103/RevModPhys.93.035002.

[96] Geoffrey Penington. Entanglement Wedge Reconstruction and the Information Paradox. *JHEP*, 09:002, 2020. doi:10.1007/JHEP09(2020)002.

[97] Eugene P. Wigner. The Problem of Measurement. *Am. J. Phys.*, 31(1):6–15, 1963. doi:10.1119/1.1969254.

[98] Lajos Diósi, Nicolas Gisin, and Walter T Strunz. Quantum approach to coupling classical and quantum dynamics. *Physical Review A*, 61(2):022108, 2000.

[99] Lajos Diósi and Jonathan J. Halliwell. Coupling classical and quantum variables using continuous quantum measurement theory. *Physical Review Letters*, 81(14):2846–2849, Oct 1998. ISSN 1079-7114. doi:10.1103/physrevlett.81.2846. URL http://dx.doi.org/10.1103/PhysRevLett.81.2846.

[100] Josef M. Jauch. *The Problem of Measurement in Quantum Mechanics*. Springer Netherlands, Dordrecht, 1973. ISBN 978-94-010-2602-4. doi:10.1007/978-94-010-2602-4_33. URL https://doi.org/10.1007/978-94-010-2602-4_33.

[101] Frederick Karolyhazy. Gravitation and quantum mechanics of macroscopic objects. *Il Nuovo Cimento A (1965-1970)*, 42(2):390–402, 1966.

[102] Lajos Diósi. Models for universal reduction of macroscopic quantum fluctuations. *Physical Review A*, 40(3):1165, 1989.

[103] Roger Penrose. On gravity's role in quantum state reduction. *General relativity and gravitation*, 28(5):581–600, 1996.

[104] Philip M. Pearle. Combining Stochastic Dynamical State Vector Reduction With Spontaneous Localization. *Phys. Rev. A*, 39:2277–2289, 1989. doi:10.1103/PhysRevA.39.2277.

[105] G. C. Ghirardi, A. Rimini, and T. Weber. Unified dynamics for microscopic and macroscopic systems. *Phys. Rev. D*, 34:470–491, Jul 1986. doi:10.1103/PhysRevD.34.470. URL https://link.aps.org/doi/10.1103/PhysRevD.34.470.

[106] Gian Carlo Ghirardi, Alberto Rimini, and Tullio Weber. Unified dynamics for microscopic and macroscopic systems. *Physical review D*, 34(2):470, 1986.

[107] N. Gisin and I. C. Percival. The quantum-state diffusion model applied to open systems. *Proc. R. Soc. London Ser. A*, 447:189N, 1992.

[108] Stephen L Adler and Todd A Brun. Generalized stochastic schrödinger equations for state vector collapse. *Journal of Physics A: Mathematical and General*, 34(23):4797, 2001.

[109] Angelo Bassi and GianCarlo Ghirardi. Dynamical reduction models. *Physics Reports*, 379(5-6):257–426, Jun 2003. ISSN 0370-1573. doi:10.1016/s0370-1573(03)00103-0. URL http://dx.doi.org/10.1016/S0370-1573(03)00103-0.

[110] Antoine Tilloy and Lajos Diósi. Sourcing semiclassical gravity from spontaneously localized quantum matter. *Physical Review D*, 93(2):024026, 2016.

[111] Charles Kivunja. Distinguishing between theory, theoretical framework, and conceptual framework: A systematic review of lessons from the field. *International Journal of Higher Education*, 7(6), 2018. doi:10.5430/ijhe.v7n6p44.

[112] R. F. Werner. The classical limit of quantum theory, 4 1995.

[113] Maximilian Schlosshauer. Decoherence, the Measurement Problem, and Interpretations of Quantum Mechanics. *Rev. Mod. Phys.*, 76:1267–1305, 2004. doi:10.1103/RevModPhys.76.1267.

[114] Wojciech Hubert Zurek. Decoherence and the transition from quantum to classical—revisited. In *Quantum Decoherence*, pages 1–31. Springer, 2006.

[115] Alessandro Coppo, Nicola Pranzini, and Paola Verrucchi. Threshold size for the emergence of classical-like behavior. *Phys. Rev. A*, 106(4):042208, 2022. doi:10.1103/PhysRevA.106.042208.

[116] Jonathan Oppenheim, Carlo Sparaciari, Barbara Šoda, and Zachary Weller-Davies. Objective trajectories in hybrid classical-quantum dynamics. *Quantum*, 7:891, January 2023. ISSN 2521-327X. doi:10.22331/q-2023-01-03-891. URL https://doi.org/10.22331/q-2023-01-03-891.

[117] Wayne Boucher and Jennie Traschen. Semiclassical physics and quantum fluctuations. *Physical Review D*, 37(12):3522, 1988.

[118] IV Aleksandrov and Z Naturf. 36a, 902 (1981); a. anderson. *Phys. Rev. Lett*, 74:621, 1995.

[119] V. I. Gerasimenko. Dynamical equations of quantum-classical systems. *Theoretical and Mathematical Physics*, 50(1):49–55, 1982. doi:10.1007/BF01027604. URL https://doi.org/10.1007/BF01027604.

[120] Roger Penrose. Quantum computation, entanglement and state reduction. *Philosophical transactions-royal society of London series a mathematical physical and engineering sciences*, pages 1927–1937, 1998.

[121] Lev P Pitaevskii. Vortex lines in an imperfect bose gas. *Sov. Phys. JETP*, 13(2):451–454, 1961.

[122] Eugene P Gross. Structure of a quantized vortex in boson systems. *Il Nuovo Cimento (1955-1965)*, 20(3):454–477, 1961.

[123] Lajos Diosi. A universal master equation for the gravitational violation of quantum mechanics. *Physics letters A*, 120(8):377–381, 1987.

[124] Lajos Diósi. The gravity-related decoherence master equation from hybrid dynamics. In *Journal of Physics-Conference Series*, volume 306, page 012006, 2011.

[125] Philippe Blanchard and Arkadiusz Jadczyk. On the interaction between classical and quantum systems. *Physics Letters A*, 175(3-4):157–164, 1993.

[126] Philippe Blanchard and Arkadiusz Jadczyk. Event-enhanced quantum theory and piecewise deterministic dynamics. *Annalen der Physik*, 507(6):583–599, 1995. https://arxiv.org/abs/hep-th/9409189.

[127] Lajos Diosi. Quantum dynamics with two planck constants and the semiclassical limit. *arXiv preprint quant-ph/9503023*, 1995.

[128] Robert Alicki and Stanisław Kryszewski. Completely positive bloch-boltzmann equations. *Physical Review A*, 68(1):013809, 2003.

[129] David Poulin and John Preskill. Information loss in quantum field theories, 2017. URL http://online.kitp.ucsb.edu/online/qinfo-c17/poulin/. Frontiers of Quantum Information Physics, KITP.

[130] Jonathan Oppenheim and Zachary Weller-Davies. Covariant path integrals for quantum fields back-reacting on classical space-time. *arXiv:2302.07283*, 2023. URL https://arxiv.org/abs/2302.07283.

[131] Vittorio Gorini, Alberto Frigerio, Maurizio Verri, Andrzej Kossakowski, and E. C. G. Sudarshan. Properties of Quantum Markovian Master Equations. *Rept. Math. Phys.*, 13:149, 1978. doi:10.1016/0034-4877(78)90050-2.

[132] G. Lindblad. On the generators of quantum dynamical semigroups. *Commun. Math. Phys.*, 48:119–130, 1976.

[133] Michael J. W. Hall, James D. Cresser, Li Li, and Erika Andersson. Canonical form of master equations and characterization of non-markovianity. *Physical Review A*, 89(4), Apr 2014. ISSN 1094-1622. doi:10.1103/physreva.89.042120. URL http://dx.doi.org/10.1103/PhysRevA.89.042120.

[134] Heinz-Peter Breuer, Elsi-Mari Laine, Jyrki Piilo, and Bassano Vacchini. Colloquium: Non-markovian dynamics in open quantum systems. *Reviews of Modern Physics*, 88(2), Apr 2016. ISSN 1539-0756. doi:10.1103/revmodphys.88.021002. URL http://dx.doi.org/10.1103/RevModPhys.88.021002.

[135] Vittorio Gorini, Andrzej Kossakowski, and E. C. G. Sudarshan. Completely Positive Dynamical Semigroups of N Level Systems. *J. Math. Phys.*, 17:821, 1976. doi:10.1063/1.522979.

[136] Jonathan Oppenheim and Zachary Weller-Davies. The constraints of post-quantum classical gravity. *JHEP*, 02:080, 2022. doi:10.1007/JHEP02(2022)080.

[137] Cécile M. DeWitt and Dean Rickles. *The role of gravitation in physics: Report from the 1957 Chapel Hill Conference*, volume 5. epubli, 2011.

[138] Richard P Feynman, Fernando B Morinigo, and William G Wagner. Feynman lectures on gravitation. *European Journal of Physics*, 24(3):330, may 2003. doi:10.1088/0143-0807/24/3/702. URL https://dx.doi.org/10.1088/0143-0807/24/3/702.

[139] Yakir Aharonov and Daniel Rohrlich. *Quantum Paradoxes: Quantum Theory for the Perplexed*. Wiley-VCH, 09 2003. doi:10.1002/9783527619115. pp212-213.

[140] David Poulin and John Preskill. Information loss in quantum field theories. Frontiers of Quantum Information Physics, KITP, 2017. URL http://online.kitp.ucsb.edu/online/qinfo-c17/poulin/.

[141] Henry Cavendish. Xxi. experiments to determine the density of the earth. *Philosophical Transactions of the Royal Society of London*, 88:469–526, 1798.

[142] Gabriel G Luther and William R Towler. Redetermination of the newtonian gravitational constant g. *Physical Review Letters*, 48(3):121, 1982.

[143] Jens H. Gundlach and Stephen M. Merkowitz. Measurement of newton's constant using a torsion balance with angular acceleration feedback. *Phys. Rev. Lett.*, 85:2869–2872, Oct 2000. doi:10.1103/PhysRevLett.85.2869. URL https://link.aps.org/doi/10.1103/PhysRevLett.85.2869.

[144] Terry Quinn. Measuring big g. *Nature*, 408(6815):919–920, 2000.

[145] GT Gillies and CS Unnikrishnan. The attracting masses in measurements of g: an overview of physical characteristics and performance. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 372(2026):20140022, 2014.

[146] Christian Rothleitner and Stephan Schlamminger. Invited review article: Measurements of the newtonian constant of gravitation, g. *Review of Scientific Instruments*, 88(11):111101, 2017.

[147] Markus Arndt, Olaf Nairz, Julian Vos-Andreae, Claudia Keller, Gerbrand Van der Zouw, and Anton Zeilinger. Wave–particle duality of c60 molecules. *nature*, 401(6754):680–682, 1999.

[148] Stefan Nimmrichter, Klaus Hornberger, and Klemens Hammerer. Optomechanical sensing of spontaneous wave-function collapse. *Physical review letters*, 113(2):020405, 2014.

[149] Stefan Gerlich, Yaakov Y Fein, and Markus Arndt. Interferometric tests of wave-function collapse. In *Do Wave Functions Jump?*, pages 385–399. Springer, 2021.

[150] Angelo Bassi, Kinjalk Lochan, Seema Satin, Tejinder P. Singh, and Hendrik Ulbricht. Models of wave-function collapse, underlying theories, and experimental tests. *Rev. Mod. Phys.*, 85:471–527, Apr 2013. doi:10.1103/RevModPhys.85.471. URL `https://link.aps.org/doi/10.1103/RevModPhys.85.471`.

[151] Tobias Westphal, Hans Hepach, Jeremias Pfaff, and Markus Aspelmeyer. Measurement of gravitational coupling between millimetre-sized masses. *Nature*, 591(7849):225–228, 2021. doi:10.1038/s41586-021-03250-7.

[152] Jonas Schmöle, Mathias Dragosits, Hans Hepach, and Markus Aspelmeyer. A micromechanical proof-of-principle experiment for measuring the gravitational force of milligram masses. *Classical and Quantum Gravity*, 33(12):125031, May 2016. doi:10.1088/0264-9381/33/12/125031.

[153] J. Lee and Robert M. Wald. Local symmetries and constraints. *J. Math. Phys.*, 31: 725–743, 1990. doi:10.1063/1.528801.

[154] H Georgi. Effective field theory. *Annual Review of Nuclear and Particle Science*, 43(1): 209–252, 1993. doi:10.1146/annurev.ns.43.120193.001233. URL `https://doi.org/10.1146/annurev.ns.43.120193.001233`.

[155] Pietro Capuozzo, Emanuele Panella, Tancredi Schettini Gherardini, and Dimitri D. Vvedensky. Path integral monte carlo method for option pricing. *Physica A: Statistical Mechanics and its Applications*, 581, 2021. ISSN 0378-4371.

doi:https://doi.org/10.1016/j.physa.2021.126231. URL https://www.sciencedirect.com/science/article/pii/S0378437121005045.

[156] Guido Montagna, Oreste Nicrosini, and Nicola Moreni. A path integral way to option pricing. *Physica A: Statistical Mechanics and its Applications*, 310(3):450–466, 2002. ISSN 0378-4371. doi:https://doi.org/10.1016/S0378-4371(02)00796-3. URL https://www.sciencedirect.com/science/article/pii/S0378437102007963.

[157] Phelim P. Boyle. Options: A monte carlo approach. *Journal of Financial Economics*, 4 (3):323–338, 1977. ISSN 0304-405X. doi:https://doi.org/10.1016/0304-405X(77)90005-8. URL https://www.sciencedirect.com/science/article/pii/0304405X77900058.

[158] M Creutz and B Freedman. A statistical approach to quantum mechanics. *Annals of Physics*, 132(2):427–462, 1981. ISSN 0003-4916. doi:https://doi.org/10.1016/0003-4916(81)90074-9. URL https://www.sciencedirect.com/science/article/pii/0003491681900749.

[159] Michael Creutz, Laurence Jacobs, and Claudio Rebbi. Monte carlo computations in lattice gauge theories. *Physics Reports*, 95(4):201–282, 1983. ISSN 0370-1573. doi:https://doi.org/10.1016/0370-1573(83)90016-9. URL https://www.sciencedirect.com/science/article/pii/0370157383900169.

[160] R. P. Feynman. Space-time approach to non-relativistic quantum mechanics. *Rev. Mod. Phys.*, 20:367–387, Apr 1948. doi:10.1103/RevModPhys.20.367. URL https://link.aps.org/doi/10.1103/RevModPhys.20.367.

[161] Kenneth G. Wilson. The renormalization group: Critical phenomena and the kondo problem. *Rev. Mod. Phys.*, 47:773–840, Oct 1975. doi:10.1103/RevModPhys.47.773. URL https://link.aps.org/doi/10.1103/RevModPhys.47.773.

[162] Joseph Polchinski. Renormalization and effective lagrangians. *Nuclear Physics B*, 231 (2):269–295, 1984. ISSN 0550-3213. doi:https://doi.org/10.1016/0550-3213(84)90287-6. URL https://www.sciencedirect.com/science/article/pii/0550321384902876.

[163] Steven Weinberg. *The Quantum theory of fields. Vol. 1: Foundations.* Cambridge University Press, 6 2005. ISBN 978-0-521-67053-1, 978-0-511-25204-4. doi:10.1017/CBO9781139644167.

[164] Steven Weinberg. *The quantum theory of fields. Vol. 2: Modern applications.* Cambridge University Press, 8 2013. ISBN 978-1-139-63247-8, 978-0-521-67054-8, 978-0-521-55002-4. doi:10.1017/CBO9781139644174.

[165] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory.* Addison-Wesley, Reading, USA, 1995. ISBN 978-0-201-50397-5.

[166] M. Srednicki. *Quantum field theory.* Cambridge University Press, 1 2007. ISBN 978-0-521-86449-7, 978-0-511-26720-8.

[167] R.P. Feynman and F.L. Vernon. The theory of a general quantum system interacting with a linear dissipative system. *Annals of Physics*, 281(1):547–607, 2000. ISSN 0003-4916. doi:https://doi.org/10.1006/aphy.2000.6017. URL https://www.sciencedirect.com/science/article/pii/S0003491600960172.

[168] L. Onsager and S. Machlup. Fluctuations and irreversible processes. *Phys. Rev.*, 91:1505–1512, Sep 1953. doi:10.1103/PhysRev.91.1505. URL https://link.aps.org/doi/10.1103/PhysRev.91.1505.

[169] Mark Iosifovich Freidlin and Alexander D Wentzell. Random perturbations. In *Random perturbations of dynamical systems*, pages 15–43. Springer, 1998.

[170] Markus F Weber and Erwin Frey. Master equations and the theory of stochastic path integrals. *Reports on Progress in Physics*, 80(4):046601, Mar 2017. ISSN 1361-6633. doi:10.1088/1361-6633/aa5ae2. URL http://dx.doi.org/10.1088/1361-6633/aa5ae2.

[171] Hagen Kleinert. *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets.* WORLD SCIENTIFIC, 5th edition, 2009. doi:10.1142/7305. URL https://www.worldscientific.com/doi/abs/10.1142/7305.

[172] H.S. Wio. *Path Integrals for Stochastic Processes: An Introduction.* World Scientific, 2013. ISBN 9789814449045. URL https://books.google.it/books?id=jpG6CgAAQBAJ.

[173] Fei Wang and Nancy Makri. Quantum-classical path integral with a harmonic treatment of the back-reaction. *The Journal of Chemical Physics*, 150(18):184102, 2019. doi:10.1063/1.5091725. URL https://doi.org/10.1063/1.5091725.

[174] Roberto Lambert and Nancy Makri. Quantum-classical path integral. i. classical memory and weak quantum nonlocality. *The Journal of Chemical Physics*, 137(22):22A552, 2012. doi:10.1063/1.4767931. URL https://doi.org/10.1063/1.4767931.

[175] Roberto Lambert and N. Makri. Quantum-classical path integral. ii. numerical methodology. *The Journal of chemical physics*, 137 22:22A553, 2012.

[176] Nancy Makri. Quantum-classical path integral: A rigorous approach to condensed phase dynamics. *International Journal of Quantum Chemistry*, 115(18):1209–1214, 2015. doi:https://doi.org/10.1002/qua.24975. URL https://onlinelibrary.wiley.com/doi/abs/10.1002/qua.24975.

[177] Gerard McCaul and Denys I. Bondar. How to win friends and influence functionals: Deducing stochasticity from deterministic dynamics, 2020.

[178] L M Sieberer, M Buchhold, and S Diehl. Keldysh field theory for driven open quantum systems. *Reports on Progress in Physics*, 79(9):096001, Aug 2016. ISSN 1361-6633. doi:10.1088/0034-4885/79/9/096001. URL http://dx.doi.org/10.1088/0034-4885/79/9/096001.

[179] G.C. Ghirardi, A. Rimini, and T Weber. A model for a unified quantum description of macroscopic and microscopic systems. In *Quantum Probability and Applications, L. Accardi et al. (eds)*. Springer, Berlin., 1985.

[180] Gian Carlo Ghirardi, Philip Pearle, and Alberto Rimini. Markov processes in hilbert space and continuous spontaneous localization of systems of identical particles. *Phys. Rev. A*, 42:78–89, Jul 1990. doi:10.1103/PhysRevA.42.78. URL https://link.aps.org/doi/10.1103/PhysRevA.42.78.

[181] D. Kafri, J. M. Taylor, and G. J. Milburn. A classical channel model for gravitational decoherence. *New J. Phys.*, 16:065020, 2014. doi:10.1088/1367-2630/16/6/065020.

[182] D Kafri, G J Milburn, and J M Taylor. Bounds on quantum communication via newtonian gravity. *New Journal of Physics*, 17(1):015006, jan 2015. doi:10.1088/1367-2630/17/1/015006. URL https://doi.org/10.1088/1367-2630/17/1/015006.

[183] Remo Ruffini and Silvano Bonazzola. Systems of self-gravitating particles in general relativity and the concept of an equation of state. *Physical Review*, 187(5):1767, 1969.

[184] D Kafri, JM Taylor, and GJ Milburn. A classical channel model for gravitational decoherence. *New Journal of Physics*, 16(6):065020, 2014.

[185] M. Fierz and W. Pauli. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. *Proc. Roy. Soc. Lond.*, A173:211–232, 1939. doi:10.1098/rspa.1939.0140.

[186] Sean M. Carroll. Lecture notes on general relativity, 1997. URL https://arxiv.org/abs/gr-qc/9712019.

[187] P.A.M. Dirac. *Lectures on Quantum Mechanics*. Belfer Graduate School of Science, monograph series. Dover Publications, 2001. ISBN 9780486417134. URL https://books.google.co.uk/books?id=GVwzb1rZW9kC.

[188] K. R. Green, N. Kiriushcheva, and S. V. Kuzmin. Analysis of Hamiltonian formulations of linearized General Relativity. *Eur. Phys. J. C*, 71:1678, 2011. doi:10.1140/epjc/s10052-011-1678-2.

[189] Bei Lok Hu and Enric Verdaguer. Stochastic gravity: Theory and applications. *Living Reviews in Relativity*, 11(1):3, 2008.

[190] Domenico Giulini and Claus Kiefer. Wheeler-dewitt metric and the attractivity of gravity. *Physics Letters A*, 193(1):21–24, 1994. doi:10.1016/0375-9601(94)00651-2.

[191] H. M. Wiseman and G. J. Milburn. Quantum theory of optical feedback via homodyne detection. *Phys. Rev. Lett.*, 70:548–551, Feb 1993. doi:10.1103/PhysRevLett.70.548. URL https://link.aps.org/doi/10.1103/PhysRevLett.70.548.

[192] L. Diosi and N. Gisin. Comment on “quantum theory of optical feedback via homodyne detection”. *Phys. Rev. Lett.*, 72:4053–4053, Jun 1994. doi:10.1103/PhysRevLett.72.4053. URL `https://link.aps.org/doi/10.1103/PhysRevLett.72.4053`.

[193] B. K. Øksendal. *Stochastic differential equations: An introduction with applications*. Springer, 2013.

[194] Jonathan Oppenheim and Benni Reznik. Fundamental destruction of information and conservation laws, 2009. URL `https://arxiv.org/abs/0902.2361`.

[195] Nicolas Gisin. Stochastic quantum dynamics and relativity. *Helv. Phys. Acta*, 62(4): 363–371, 1989.

[196] Nicolas Gisin. Weinberg's non-linear quantum mechanics and supraluminal communications. *Physics Letters A*, 143(1-2):1–2, 1990.

[197] Joseph Polchinski. Weinberg's nonlinear quantum mechanics and the einstein-podolsky-rosen paradox. *Physical Review Letters*, 66(4):397, 1991.

[198] LE Ballentine. Failure of some theories of state reduction. *Physical Review A*, 43(1):9, 1991.

[199] Philip Pearle, James Ring, Juan I Collar, and Frank T Avignone. The csl collapse model and spontaneous radiation: an update. *Foundations of physics*, 29(3):465–480, 1999.

[200] Angelo Bassi, Emiliano Ippoliti, and Bassano Vacchini. On the energy increase in space-collapse models. *Journal of Physics A: Mathematical and General*, 38(37):8017, 2005.

[201] Stephen L Adler. Lower and upper bounds on csl parameters from latent image formation and igm heating. *Journal of Physics A: Mathematical and Theoretical*, 40(12):2935, 2007.

[202] Kinjalk Lochan, Suratna Das, and Angelo Bassi. Constraining continuous spontaneous localization strength parameter λ from standard cosmology and spectral distortions of cosmic microwave background radiation. *Physical Review D*, 86(6):065016, 2012.

[203] Mohammad Bahrami, Angelo Bassi, and Hendrik Ulbricht. Testing the quantum superposition principle in the frequency domain. *Physical Review A*, 89(3):032127, 2014.

[204] Franck Laloë, William J Mullin, and Philip Pearle. Heating of trapped ultracold atoms by collapse dynamics. *Physical Review A*, 90(5):052119, 2014.

[205] Mohammad Bahrami, M Paternostro, Angelo Bassi, and H Ulbricht. Proposal for a noninterferometric test of collapse models in optomechanical systems. *Physical Review Letters*, 112(21):210404, 2014.

[206] Daniel Goldwater, Mauro Paternostro, and PF Barker. Testing wave-function-collapse models using parametric heating of a trapped nanosphere. *Physical Review A*, 94(1): 010104, 2016.

[207] Antoine Tilloy and Thomas M Stace. Neutron star heating constraints on wave-function collapse models. *Physical Review Letters*, 123(8):080402, 2019.

[208] Sandro Donadi, Kristian Piscicchia, Catalina Curceanu, Lajos Diósi, Matthias Laubenstein, and Angelo Bassi. Underground test of gravity-related wave function collapse. *Nature Physics*, pages 1–5, 2020.

[209] K. V. Kuchar. Hamiltonian Dynamics of Gauge Systems. *Phys. Rev. D*, 34:3031–3043, 1986. doi:10.1103/PhysRevD.34.3031.

[210] K. V. Kuchar. Covariant Factor Ordering of Gauge Systems. *Phys. Rev. D*, 34:3044–3057, 1986. doi:10.1103/PhysRevD.34.3044.

[211] Joseph D. Romano and Ranjeet S. Tate. Dirac Versus Reduced Space Quantization of Simple Constrained Systems. *Class. Quant. Grav.*, 6:1487, 1989. doi:10.1088/0264-9381/6/10/017.

[212] K. Schleich. Is reduced phase space quantization equivalent to Dirac quantization? *Class. Quant. Grav.*, 7:1529–1538, 1990. doi:10.1088/0264-9381/7/8/028.

[213] R. Loll. Noncommutativity of constraining and quantizing: A U(1) gauge model. *Phys. Rev. D*, 41:3785–3791, 1990. doi:10.1103/PhysRevD.41.3785.

[214] A. Ashtekar. *Lectures on nonperturbative canonical gravity*, volume 6. WorldScientific, 1991. doi:10.1142/1321.

[215] Dvir Kafri and J. M. Taylor. A noise inequality for classical forces. *arXiv preprint*, 2013.

[216] D. Kafri, G. J. Milburn, and J. M. Taylor. Bounds on quantum communication via Newtonian gravity. *New J. Phys.*, 17(1):015006, 2015. doi:10.1088/1367-2630/17/1/015006.

[217] Sougato Bose, Anupam Mazumdar, Gavin W Morley, Hendrik Ulbricht, Marko Toroš, Mauro Paternostro, Andrew A Geraci, Peter F Barker, MS Kim, and Gerard Milburn. Spin entanglement witness for quantum gravity. *Physical review letters*, 119(24):240401, 2017.

[218] Chiara Marletto and Vlatko Vedral. Why we need to quantise everything, including gravity. *npj Quantum Information*, 3(1):29, 2017.

[219] Ryan J. Marshman, Anupam Mazumdar, and Sougato Bose. Locality and entanglement in table-top testing of the quantum nature of linearized gravity. *Physical Review A*, 101 (5), May 2020. doi:10.1103/physreva.101.052110.

[220] Julen S. Pedernales, Kirill Streltsov, and Martin B. Plenio. Enhancing gravitational interaction between quantum systems by a massive mediator. *arXiv preprint*, 2021.

[221] Daniel Carney, Holger Müller, and Jacob M. Taylor. Testing quantum gravity with interactive information sensing. *arXiv preprint*, 2021.

[222] Marios Christodoulou, Andrea Di Biagio, Markus Aspelmeyer, Časlav Brukner, Carlo Rovelli, and Richard Howl. Locally mediated entanglement through gravity from first principles, 2022. URL `https://arxiv.org/abs/2202.03368`.

[223] Daine L. Danielson, Gautam Satishchandran, and Robert M. Wald. Gravitationally Mediated Entanglement: Newtonian Field vs. Gravitons. *arXiv preprint*, 12 2021.

[224] Ludovico Lami, Julen S. Pedernales, and Martin B. Plenio. Testing the quantum nature of gravity without entanglement. *arXiv preprint*, 2023.

[225] Jonathan Oppenheim and Zachary Weller-Davies. Lorentz invariant information destruction is not lorentz covariant. unpublished draft, 2020.

[226] G.Nordström. Zur theorie der gravitation vom standpunkt des relativitätsprinzips. *Annalen-der-Physik*, 42:533, 1913.

[227] Finn Ravndal. Scalar gravitation and extra dimensions, 2004.

[228] John D. Norton. Einstein, nordström and the early demise of scalar, lorentz-covariant theories of gravitation. *Archive for History of Exact Sciences*, 45(1):17–94, 1992. doi:10.1007/BF00375886. URL https://doi.org/10.1007/BF00375886.

[229] F. W. Dyson, A. S. Eddington, and C. Davidson. A Determination of the Deflection of Light by the Sun's Gravitational Field, from Observations Made at the Total Eclipse of May 29, 1919. *Phil. Trans. Roy. Soc. Lond. A*, 220:291–333, 1920. doi:10.1098/rsta.1920.0009.

[230] Matthias Bartelmann. Gravitational lensing. *Classical and Quantum Gravity*, 27(23): 233001, nov 2010. doi:10.1088/0264-9381/27/23/233001. URL https://dx.doi.org/10.1088/0264-9381/27/23/233001.

[231] Oliver Pooley. Background independence, diffeomorphism invariance, and the meaning of coordinates, 2015. URL https://arxiv.org/abs/1506.03512.

[232] Isaac Layton, Jonathan Oppenheim, and Emanuelle Panella. Normalisation of classical-quantum path integrals. Manuscript in preparation, 2023.

[233] Michael Ostrogradsky. *Mémoire sur les équations différentielles relatives au problème des isopérimètres.* Mémoires de l'Académie Impériale des Sciences de Saint-Pétersbourg, 6e série, 1850.

[234] Richard P Feynman and Albert R Hibbs. *Quantum mechanics and path integrals.* McGraw-Hill, New York., 1965. Section 12.6 on Brownian motion.

[235] Nathalie Deruelle. Nordström's scalar theory of gravity and the equivalence principle. *General Relativity and Gravitation*, 43(12):3337–3354, 2011. doi:10.1007/s10714-011-1247-x.

[236] Zachary Weller-Davies. A model of quantum gravity with dynamical collapse. manuscript in preparation, 2024.

[237] C. Marletto and V. Vedral. Gravitationally induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity. *Phys. Rev. Lett.*, 119:240402, Dec 2017. doi:10.1103/PhysRevLett.119.240402. URL `https://link.aps.org/doi/10.1103/PhysRevLett.119.240402`.

[238] Kirill Streltsov, Julen Simon Pedernales, and Martin Bodo Plenio. On the significance of interferometric revivals for the fundamental description of gravity. *Universe*, 8(2):58, 2022.

[239] Serhii Kryhin and Vivishek Sudhir. Distinguishable consequence of classical gravity on quantum matter. *arXiv preprint arXiv:2309.09105*, 2023.

[240] Hannes Risken and Hannes Risken. *Fokker-planck equation*. Springer, 1996.

[241] C. W. Gardiner. *Handbook of stochastic methods for physics, chemistry and the natural sciences*, volume 13 of *Springer Series in Synergetics*. Springer-Verlag, Berlin, third edition, 2004. ISBN 3-540-20882-8.

[242] Detlef Dürr and Alexander Bach. The onsager-machlup function as lagrangian for the most probable path of a diffusion process. *Communications in Mathematical Physics*, 60: 153–170, 1978.

[243] Ying Chao and Jinqiao Duan. The onsager–machlup function as lagrangian for the most probable path of a jump-diffusion process. *Nonlinearity*, 32(10):3715, 2019.

[244] Richard M Neumann. The entropy of a single gaussian macromolecule in a noninteracting solvent. *The Journal of Chemical Physics*, 66(2):870–871, 1977.

[245] Nico Roos. Entropic forces in brownian motion. *American Journal of Physics*, 82(12): 1161–1166, 2014.

[246] Kurt Jacobs. Twenty open problems in quantum control, 2014.

[247] Thanu Padmanabhan. Thermodynamical aspects of gravity: new insights. *Reports on Progress in Physics*, 73(4):046901, 2010.

[248] Erik Verlinde. On the origin of gravity and the laws of newton. *Journal of High Energy Physics*, 2011(4):1–27, 2011.

[249] Matt Visser. Conservative entropic forces. *Journal of High Energy Physics*, 2011(10): 1–22, 2011.

[250] Mark Van Raamsdonk. Building up space-time with quantum entanglement. *International Journal of Modern Physics D*, 19(14):2429–2435, 2010.

[251] Arno A. Penzias and Robert Woodrow Wilson. A Measurement of excess antenna temperature at 4080-Mc/s. *Astrophys. J.*, 142:419–421, 1965. doi:10.1086/148307.

[252] George F. Smoot et al. Structure in the COBE differential microwave radiometer first year maps. *Astrophys. J. Lett.*, 396:L1–L5, 1992. doi:10.1086/186504.

[253] P. de Bernardis et al. A Flat universe from high resolution maps of the cosmic microwave background radiation. *Nature*, 404:955–959, 2000. doi:10.1038/35010035.

[254] Matthew Colless et al. The 2dF Galaxy Redshift Survey: Spectra and redshifts. *Mon. Not. Roy. Astron. Soc.*, 328:1039, 2001. doi:10.1046/j.1365-8711.2001.04902.x.

[255] Vera C Rubin, W Kent Ford Jr, and Norbert Thonnard. Extended rotation curves of high-luminosity spiral galaxies. iv-systematic dynamical properties, sa through sc. *Astrophysical Journal, Part 2-Letters to the Editor, vol. 225, Nov. 1, 1978, p. L107-L111.*, 225:L107–L111, 1978.

[256] V. C. Rubin, N. Thonnard, and W. K. Ford, Jr. Rotational properties of 21 SC galaxies with a large range of luminosities and radii, from NGC 4605 /R = 4kpc/ to UGC 2885 /R = 122 kpc/. *Astrophys. J.*, 238:471, 1980. doi:10.1086/158003.

[257] Yoshiaki Sofue and Vera Rubin. Rotation curves of spiral galaxies. *Ann. Rev. Astron. Astrophys.*, 39:137–174, 2001. doi:10.1146/annurev.astro.39.1.137.

[258] Peter AR Ade, Nabila Aghanim, M Arnaud, Mark Ashdown, J Aumont, Carlo Baccigalupi, AJ Banday, RB Barreiro, JG Bartlett, N Bartolo, et al. Planck 2015 results-xiii. cosmological parameters. *Astronomy & Astrophysics*, 594:A13, 2016.

[259] Gary Hinshaw, JL Weiland, RS Hill, N Odegard, D Larson, CL Bennett, J Dunkley, B Gold, MR Greason, N Jarosik, et al. Five-year wilkinson microwave anisotropy probe* observations: data processing, sky maps, and basic results. *The Astrophysical Journal Supplement Series*, 180(2):225, 2009.

[260] AN Taylor, S Dye, Thomas J Broadhurst, N Benitez, and E Van Kampen. Gravitational lens magnification and the mass of abell 1689. *The Astrophysical Journal*, 501(2):539, 1998.

[261] SM Faber and Robert E Jackson. Velocity dispersions and mass-to-light ratios for elliptical galaxies. *Astrophysical Journal, vol. 204, Mar. 15, 1976, pt. 1, p. 668-683.*, 204:668–683, 1976.

[262] Steven W Allen, August E Evrard, and Adam B Mantz. Cosmological parameters from observations of galaxy clusters. *Annual Review of Astronomy and Astrophysics*, 49:409–470, 2011.

[263] Volker Springel, SDM White, Adrian Jenkins, Carlos Frenk, Naoki Yoshida, Liang Gao, Julio Navarro, R. Thacker, Darren Croton, John Helly, Joe Peacock, Shaun Cole, Peter Thomas, H. Couchman, August Evrard, Joerg Colberg, and Frazer Pearce. Simulations of the formation, evolution and clustering of galaxies and quasars. *Nature*, 435, 06 2005. doi:10.1038/nature03597.

[264] P. A. R. Ade et al. Planck 2015 results. XIII. Cosmological parameters. *Astron. Astrophys.*, 594:A13, 2016. doi:10.1051/0004-6361/201525830.

[265] Ben Moore. Evidence against dissipation-less dark matter from observations of galaxy haloes. *Nature*, 370(6491):629–631, 1994.

[266] W. J. G. de Blok. The Core-Cusp Problem. *Adv. Astron.*, 2010:789293, 2010. doi:10.1155/2010/789293.

[267] Se-Heon Oh, Deidre A Hunter, Elias Brinks, Bruce G Elmegreen, Andreas Schruba, Fabian Walter, Michael P Rupen, Lisa M Young, Caroline E Simpson, Megan C Johnson,

et al. High-resolution mass models of dwarf galaxies from little things. *The Astronomical Journal*, 149(6):180, 2015.

[268] B. Moore, S. Ghigna, F. Governato, G. Lake, Thomas R. Quinn, J. Stadel, and P. Tozzi. Dark matter substructure within galactic halos. *Astrophys. J. Lett.*, 524:L19–L22, 1999. doi:10.1086/312287.

[269] J. Silk et al. *Particle Dark Matter: Observations, Models and Searches*. Cambridge Univ. Press, Cambridge, 2010. ISBN 978-1-107-65392-4. doi:10.1017/CBO9780511770739.

[270] E. Aprile et al. Dark Matter Results from 225 Live Days of XENON100 Data. *Phys. Rev. Lett.*, 109:181301, 2012. doi:10.1103/PhysRevLett.109.181301.

[271] C. L. Cowan, F. Reines, F. B. Harrison, H. W. Kruse, and A. D. McGuire. Detection of the free neutrino: A Confirmation. *Science*, 124:103–104, 1956. doi:10.1126/science.124.3212.103.

[272] Mordehai Milgrom. A modification of the newtonian dynamics as a possible alternative to the hidden mass hypothesis. *Astrophysical Journal, Part 1 (ISSN 0004-637X), vol. 270, July 15, 1983, p. 365-370. Research supported by the US-Israel Binational Science Foundation.*, 270:365–370, 1983.

[273] R. B. Tully and J. R. Fisher. A New method of determining distances to galaxies. *Astron. Astrophys.*, 54:661–673, 1977.

[274] Robert H Sanders and Stacy S McGaugh. Modified newtonian dynamics as an alternative to dark matter. *Annual Review of Astronomy and Astrophysics*, 40(1):263–317, 2002.

[275] Jacob D Bekenstein. The modified newtonian dynamics—mond and its implications for new physics. *Contemporary Physics*, 47(6):387–403, 2006.

[276] Mordehai Milgrom. The mond paradigm. *arXiv preprint arXiv:0801.3133*, 2008.

[277] Kimet Jusufi and Ahmad Sheykhi. Apparent dark matter inspired by einstein equation of state, 2024.

[278] Ricardo Gallego Torromé. On the emergence of gravity in a framework of emergent quantum mechanics, 2024.

[279] Felix Finster, José M. Isidro, Claudio F. Paganini, and Tejinder P. Singh. Theoretically motivated dark electromagnetism as the origin of relativistic modified newtonian dynamics. *Universe*, 10(3):123, March 2024. ISSN 2218-1997. doi:10.3390/universe10030123. URL http://dx.doi.org/10.3390/universe10030123.

[280] Mordehai Milgrom. The modified dynamics as a vacuum effect. *Physics Letters A*, 253 (5–6):273–279, March 1999. ISSN 0375-9601. doi:10.1016/s0375-9601(99)00077-8. URL http://dx.doi.org/10.1016/S0375-9601(99)00077-8.

[281] Mordehai Milgrom. The mond paradigm of modified dynamics, 2013. URL http://www.scholarpedia.org/article/The_MOND_paradigm_of_modified_dynamics#:~:text=MOND%20is%20an%20alternative%20paradigm,one%20adheres%20to%20standard%20dynamics. and references therein. Accessed: 2024-04-27.

[282] Philip D Mannheim. Are galactic rotation curves really flat? *The Astrophysical Journal*, 479(2):659, 1997.

[283] Philip D Mannheim and Demosthenes Kazanas. Exact vacuum solution to conformal weyl gravity and galactic rotation curves. *Astrophysical Journal, Part 1 (ISSN 0004-637X), vol. 342, July 15, 1989, p. 635-638. Research supported by the National Academy of Sciences.*, 342:635–638, 1989.

[284] Philip D Mannheim and James G O'Brien. Fitting galactic rotation curves with conformal gravity and a global quadratic potential. *Physical Review D*, 85(12):124020, 2012.

[285] Philip D Mannheim and Demosthenes Kazanas. Newtonian limit of conformal gravity and the lack of necessity of the second order poisson equation. *General Relativity and Gravitation*, 26:337–361, 1994.

[286] Ronald J Riegert. Birkhoff's theorem in conformal gravity. *Physical review letters*, 53(4): 315, 1984.

[287] James G. O'Brien, Thomas L. Chiarelli, and Philip D. Mannheim. Universal properties of galactic rotation curves and a first principles derivation of the tully–fisher relation. *Physics Letters B*, 782:433–439, July 2018. ISSN 0370-2693. doi:10.1016/j.physletb.2018.05.060. URL `http://dx.doi.org/10.1016/j.physletb.2018.05.060`.

[288] Diego Buccio, John F. Donoghue, Gabriel Menezes, and Roberto Percacci. Physical running of couplings in quadratic gravity, 2024.

[289] Philip D Mannheim. Ghost problems from pauli–villars to fourth-order quantum gravity and their resolution. *International Journal of Modern Physics D*, 29(14):2043009, 2020.

[290] Bob Holdom and Jing Ren. Quadratic gravity: from weak to strong. *International Journal of Modern Physics D*, 25(12):1643004, 2016.

[291] John F Donoghue and Gabriel Menezes. On quadratic gravity. *arXiv preprint arXiv:2112.01974*, 2021.

[292] Damiano Anselmi. On the quantum field theory of the gravitational interactions. *Journal of High Energy Physics*, 2017(6):1–20, 2017.

[293] Alberto Salvio, Alessandro Strumia, and Hardi Veermäe. New infra-red enhancements in 4-derivative gravity. *The European Physical Journal C*, 78:1–12, 2018.

[294] Alberto Salvio. Bicep/keck data and quadratic gravity. *Journal of Cosmology and Astroparticle Physics*, 2022(09):027, 2022.

[295] Keith Horne. Conformal gravity rotation curves with a conformal higgs halo. *Monthly Notices of the Royal Astronomical Society*, 458(4):4122–4128, 2016.

[296] MP Hobson and AN Lasenby. Conformal gravity does not predict flat galaxy rotation curves. *Physical Review D*, 104(6):064014, 2021.

[297] Michael Hobson and Anthony Lasenby. Conformally-rescaled schwarzschild metrics do not predict flat galaxy rotation curves. *The European Physical Journal C*, 82(7):585, 2022.

[298] Erik P. Verlinde. Emergent gravity and the dark universe. *SciPost Physics*, 2(3), May 2017. ISSN 2542-4653. doi:10.21468/scipostphys.2.3.016. URL `http://dx.doi.org/10.21468/SciPostPhys.2.3.016`.

[299] Claudia de Rham, J. Tate Deskins, Andrew J. Tolley, and Shuang-Yong Zhou. Graviton mass bounds. *Reviews of Modern Physics*, 89(2), May 2017. ISSN 1539-0756. doi:10.1103/revmodphys.89.025004. URL `http://dx.doi.org/10.1103/RevModPhys.89.025004`.

[300] D Carney, G Krnjaic, D C Moore, C A Regal, G Afek, S Bhave, B Brubaker, T Corbitt, J Cripe, N Crisosto, A Geraci, S Ghosh, J G E Harris, A Hook, E W Kolb, J Kunjummen, R F Lang, T Li, T Lin, Z Liu, J Lykken, L Magrini, J Manley, N Matsumoto, A Monte, F Monteiro, T Purdy, C J Riedel, R Singh, S Singh, K Sinha, J M Taylor, J Qin, D J Wilson, and Y Zhao. Mechanical quantum sensing in the search for dark matter. *Quantum Science and Technology*, 6(2):024002, 2021. ISSN 2058-9565. doi:10.1088/2058-9565/abcfcd. URL `http://dx.doi.org/10.1088/2058-9565/abcfcd`.

[301] Martijn Janse, Dennis G Uitenbroek, Loek van Everdingen, Jaimy Plugge, Bas Hensen, and Tjerk H Oosterkamp. Current experimental upper bounds on spacetime diffusion. *arXiv preprint arXiv:2403.08912*, 2024.

[302] Peter Asenbaum, Chris Overstreet, Tim Kovachy, Daniel D Brown, Jason M Hogan, and Mark A Kasevich. Phase shift in an atom interferometer due to spacetime curvature across its wave function. *Physical review letters*, 118(18):183602, 2017.

[303] Michele Armano, H Audley, J Baird, P Binetruy, Michael Born, D Bortoluzzi, Eleonora Castelli, A Cavalleri, A Cesarini, AM Cruise, et al. Beyond the required lisa free-fall performance: new lisa pathfinder results down to 20 μ hz. *Physical review letters*, 120(6):061101, 2018.

[304] Charles D Hoyle, DJ Kapner, Blayne R Heckel, EG Adelberger, JH Gundlach, U Schmidt, and HE Swanson. Submillimeter tests of the gravitational inverse-square law. *Physical Review D*, 70(4):042004, 2004.

[305] Andrew Pontzen and Fabio Governato. Cold dark matter heats up. *Nature*, 506(7487): 171–178, 2014.

[306] James G. O'Brien and Philip D. Mannheim. Fitting dwarf galaxy rotation curves with conformal gravity. *Mon. Not. Roy. Astron. Soc.*, 421:1273, 2012. doi:10.1111/j.1365-2966.2011.20386.x.

[307] Margot M. Brouwer et al. The weak lensing radial acceleration relation: Constraining modified gravity and cold dark matter theories with KiDS-1000. *Astron. Astrophys.*, 650: A113, 2021. doi:10.1051/0004-6361/202040108.

[308] R. R. Caldwell, Rahul Dave, and Paul J. Steinhardt. Cosmological imprint of an energy component with general equation of state. *Phys. Rev. Lett.*, 80:1582–1585, 1998. doi:10.1103/PhysRevLett.80.1582.

[309] Ivaylo Zlatev, Li-Min Wang, and Paul J. Steinhardt. Quintessence, cosmic coincidence, and the cosmological constant. *Phys. Rev. Lett.*, 82:896–899, 1999. doi:10.1103/PhysRevLett.82.896.

[310] Bharat Ratra and P. J. E. Peebles. Cosmological Consequences of a Rolling Homogeneous Scalar Field. *Phys. Rev. D*, 37:3406, 1988. doi:10.1103/PhysRevD.37.3406.

[311] Jonathan Oppenheim, Emanuele Panella, and Andrew Pontzen. A stochastic cosmology, 2024.

[312] Yoann L. Launay, Gerasimos I. Rigopoulos, and E. P. S. Shellard. Stochastic inflation in general relativity, 2024.

[313] Thomas Colas. *Open Effective Field Theories for primordial cosmology : dissipation, decoherence and late-time resummation of cosmological inhomogeneities*. PhD thesis, Institut d'astrophysique spatiale, France, AstroParticule et Cosmologie, France, APC, Paris, 2023.

[314] David E. Kaplan, Tom Melia, and Surjeet Rajendran. The classical equations of motion of quantized gauge theories, part i: General relativity, 2023.

[315] Marco de Cesare, Fedele Lizzi, and Mairi Sakellariadou. Effective cosmological constant induced by stochastic fluctuations of Newton's constant. *Phys. Lett. B*, 760:498–501, 2016. doi:10.1016/j.physletb.2016.07.015.

[316] Alexei A Starobinsky. A new type of isotropic cosmological models without singularity. *Physics Letters B*, 91(1):99–102, 1980.

[317] Alexander Vilenkin. Classical and quantum cosmology of the starobinsky inflationary model. *Physical Review D*, 32(10):2511, 1985.

[318] Lei-Hua Liu, Tomislav Prokopec, and Alexei A Starobinsky. Inflation in an effective gravitational model and asymptotic safety. *Physical Review D*, 98(4):043505, 2018.

[319] Yashar Akrami, Frederico Arroja, M Ashdown, J Aumont, Carlo Baccigalupi, M Ballardini, Anthony J Banday, RB Barreiro, Nicola Bartolo, S Basak, et al. Planck 2018 results-x. constraints on inflation. *Astronomy & Astrophysics*, 641:A10, 2020.

[320] M. Loève. *Probability Theory Vol. II*, volume 46 of *Graduate Texts in Mathematics*. Springer-Verlag, 4 edition, 1978. ISBN 978-0-387-90262-3. URL `https://link.springer.com/book/9780387902623`.

[321] Isaac Layton, Jonathan Oppenheim, and Emanuele Panella. Normalisation and interpretations of generalised stochastic path integrals. Manuscript in preparation, 2023.

[322] John D Barrow and Sigbjørn Hervik. The weyl tensor in spatially homogeneous cosmological models. *Classical and Quantum Gravity*, 19(20):5173, 2002.

[323] Mark P. Hertzberg and Abraham Loeb. Comment on "anomalous contribution to galactic rotation curves due to stochastic spacetime", 2024.

[324] Stefan Gerlich, Sandra Eibenberger, Mathias Tomandl, Stefan Nimmrichter, Klaus Hornberger, Paul Fagan, Jens Tüxen, Marcel Mayor, and Markus Arndt. Quantum interference of large organic molecules. *Nature communications*, 2:263, 04 2011. doi:10.1038/ncomms1263.

[325] Tobias Westphal, Hans Hepach, Jeremias Pfaff, and Markus Aspelmeyer. Measurement of gravitational coupling between millimetre-sized masses. *Nature*, 591(7849):225–228, 2021.

[326] Daniel Carney, Philip CE Stamp, and Jacob M Taylor. Tabletop experiments for quantum gravity: a user's manual. *Classical and Quantum Gravity*, 36(3):034001, 2019.

[327] Isaac Layton and Jonathan Oppenheim. The classical-quantum limit. *arXiv preprint arXiv:2310.18271*, 2023.

[328] Albert Abraham Michelson. Dedication of the ryerson physical laboratory, university of chicago. Speech at the University of Chicago, Ryerson Physical Laboratory, 1894. The quote is often cited in historical discussions about the state of physics at the time.

[329] William (Lord Kelvin) Thomson. Address to the british association for the advancement of science, 1900. It should be noted that this phrase is attributed to a few people, and it might not have been Kelvin to pronounce it, but one of his contemporaries.

[330] Damien Broderick. *Year Million: Science at the Far Edge of Knowledge*. Atlas & Co., New York, 2008. This book contains a collection of essays including one that recounts Max Planck's 1924 lecture discussing the state of physics as portrayed to him by Philipp von Jolly.

[331] Carlo Rovelli. *Quantum Gravity*. Cambridge University Press, Cambridge, UK, 2001.

[332] Roger Penrose. *The Road to Reality: A Complete Guide to the Laws of the Universe*. Jonathan Cape, London, UK, 2004.

[333] Clifford M. Will. *Theory and Experiment in Gravitational Physics*. Cambridge University Press, 2 edition, 2018. doi:10.1017/9781316338612.

[334] Clifford M. Will. On the unreasonable effectiveness of the post-Newtonian approximation in gravitational physics. *Proc. Nat. Acad. Sci.*, 108:5938, 2011. doi:10.1073/pnas.1103127108.

Appendix A

Weak field and Newtonian limit of Einstein's equation

"These appendices serve the same purpose as a human appendix:
you can live without it, but it's nice to have it just in case."
$\sim$ A. Russo

In this Appendix, we recall the Newtonian limit of Einstein's equations [186]. To begin, we perform a scalar-vector-tensor decomposition of the metric

$$ds^2 = -c^2\left(1+\frac{2\Phi}{c^2}\right)dt^2 + \frac{w_i}{c}\left(dtdx^i + dx^i dt\right) + \left[\left(1-\frac{2\psi}{c^2}\right)\delta_{ij} + \frac{2s_{ij}}{c^2}\right]dx^i dx^j, \tag{A.1}$$

where s_{ij} is traceless and the factors of c ensure that the fields Φ, ψ, w_i, s_{ij} all have dimensions c^2. To arrive at the Newtonian limit, we choose the transverse gauge, which amounts to taking a gauge such that $\partial_i w^i = 0$, $\partial_i s^{ij} = 0$. We shall also assume we take the rest frame of a particle with mass density $m(x)$, so that $T_{00} = c^4 m(x)$. Before the gauge choice, the Einstein equation's $G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$ look like:

$$G_{00} = 2\nabla^2\psi = 8\pi G m(x) \tag{A.2}$$

$$G_{0i} = \frac{2}{c^3}\partial_0\partial_i\psi - \frac{1}{2c^2}\nabla^2 w_i = 0 \tag{A.3}$$

$$G_{ij} = \frac{1}{c^2}(\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi-\psi) - \frac{1}{c^3}\partial_0\partial_{(i}w_{j)} + \frac{2}{c^2}\delta_{ij}\partial_0\partial^0\psi - \frac{1}{c^2}\Box s_{ij} = 0. \tag{A.4}$$

While after, they reduce to:

$$G_{00} = \nabla^2\psi = 4\pi G m(x) \tag{A.5}$$

$$G_{0i} = \frac{2}{c^3}\partial_0\partial_i\psi - \frac{1}{2c^2}\nabla^2 w_i = 0 \tag{A.6}$$

$$G_{ij} = \frac{1}{c^2}(\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \psi) - \frac{2}{c^4}\partial_0\partial_0\psi + \frac{1}{c^4}\partial_0\partial_0 s_{ij} - \frac{1}{c^2}\nabla^2 s_{ij} = 0. \tag{A.7}$$

We remind the reader that the G_{00} and G_{0i} components are first order in time derivatives and hence are the constraints on the initial data of the theory, whilst G_{ij} describes the dynamics.

With this in mind, let us arrive at the Newtonian limit. We first solve the G_{00} component of the Einstein equation, which is the Poisson equation for ψ. We see from G_{0i} and the solution for G_{00} that $\partial_0\partial_i\psi = 0$, which imposes that there can be no vector perturbations $w_i = 0$. Conversely, we see from the momentum constraint G_{0i} that if there are no vector perturbations, $w_i = 0$ then the constraint $\partial_i\partial_0\psi = 0$ must be imposed.

To obtain the final form of the Newtonian limit. We take the trace of G_{ij} to see which imposes that $\psi = \Phi = const$, which in combination with the fact that $\partial_0\partial_i\psi = 0$ imposes $s_{ij} = 0$.

Altogether, we are then left with the Newtonian metric

$$ds^2 = -c^2\left(1 + \frac{2\Phi}{c^2}\right)dt^2 + \left(1 - \frac{2\Phi}{c^2}\right)\delta_{ij}dx^i dx^j, \tag{A.8}$$

where Φ solves the Poisson equation due to the G_{00} component of Einstein's equation

$$\nabla^2\Phi = 4\pi G m(x). \tag{A.9}$$

Appendix B

Equivalence of the weak field path integral and master equation

In this Appendix, we arrive at the path integral (4.29) from the master equation (4.33). We will use the result of [4] to arrive at the path integral derived from the correspondence between CQ master equations and path integrals. This derivation shows that the two approaches given in Chapter 4, i.e. formulating the CQ weak field limit by either gauge fixing the full path integral or constructing the master equation from the reduced Hamiltonian, are equivalent. We refer the reader directly to [4] for more details on deriving CQ path integrals.

We start by recalling that because the equation of motion of the weak field total Hamiltonian (4.13):

$$\dot{\psi} = -\frac{4G\pi c^2 \pi_\psi}{3} - \frac{1}{12}\partial_i n^i,\ \dot{\pi}_\psi = \frac{\nabla^2(\Phi - \psi)}{4\pi G},\ \dot{\Phi} = \lambda_\Phi,\ \dot{\pi}_\Phi = \frac{\nabla^2 \Phi}{4\pi G} - m,\ \dot{n}^i = \lambda^i,\ \dot{\pi}_i = -\frac{1}{12}\partial_i \pi_\psi \tag{B.1}$$

only associate back-reaction to π_Φ, the path integral takes the form:

$$\begin{aligned}\varrho(z_f, m_f^+, m_f^-, t_f) = \mathcal{N}\int \mathcal{D}z\mathcal{D}m^+\mathcal{D}m^- \delta\left(\dot{\psi} + \frac{4G\pi c^2\pi_\psi}{3} + \frac{1}{12}\partial_i n^i\right)\delta\left(\dot{\pi}_\psi - \frac{\nabla^2(\Phi-\psi)}{4\pi G}\right)\delta(\dot{\Phi} - \lambda_\Phi) \\ \times\, \delta(\dot{n}^i - \lambda^i)\delta\left(\dot{\pi}_i + \frac{1}{12}\partial_i\psi\right)\delta(\pi_\Phi) e^{\mathcal{I}_{CQ}[z, m^+, m^-, t_i, t_f]}\varrho(z_i, m_i^+, m_i^-, t_i)\end{aligned} \tag{B.2}$$

where the last delta function imposes the Newtonian limit constraint $\pi_\Phi \approx 0$. In Equation (B.2), for the sake of clarity, we have summarised all the classical degrees of freedom with z such that the functional measure over the classical functions $\mathcal{D}z$ represents:

$$\mathcal{D}z = \mathcal{D}\psi\, \mathcal{D}\pi_\psi\, \mathcal{D}\vec{n}\, \mathcal{D}\vec{\pi}\, \mathcal{D}\lambda_\Phi\, \mathcal{D}\vec{\lambda}\, \mathcal{D}\Phi\, \mathcal{D}\pi_\Phi \tag{B.3}$$

and the hybrid action $\mathcal{I}_{CQ}$ is:

$$\mathcal{I}_{CQ}[z, m^+, m^-, t_i, t_f] = \int_{t_i}^{t_f} dt d\vec{x} \Big[i\big(\mathcal{L}_Q[m^+] - \mathcal{V}_I[\Phi, m^+] - \mathcal{L}_Q[m^-] + \mathcal{V}_I[\Phi, m^-]\big) - \frac{D_0[z]}{2}\big(m^+(x) - m^-(x)\big)^2 - \frac{1}{2D_2[z]}\left(\dot{\pi}_\Phi - \frac{\nabla^2\Phi}{4\pi G} + \frac{1}{2}\big(m^+(x) + m^-(x)\big)\right)^2 \Big]. \tag{B.4}$$

where $\mathcal{L}_Q[m^\pm]$ is the matter Lagrangian and $\mathcal{V}_I[\Phi, m^\pm] = \Phi(x)m^\pm(x)$. The CQ interaction term in Equation (B.4) is the path integral version of Equation (4.33). This correspondence was derived explicitly in [4] and takes the same form as the Hamiltonian CQ path integrals in [4, 130].

Just as in the deterministic case, one can then reduce the system to describe it in terms of the Newtonian potential alone. Performing all of the delta integrals in Equation (B.2), we arrive at the hybrid Newtonian CQ path integral in terms of Φ alone,

$$\varrho(\Phi_f, m_f^+, m_f^-, t_f) = \mathcal{N} \int \mathcal{D}\Phi \mathcal{D}m^+ \mathcal{D}m^- \; e^{\mathcal{I}_{CQ}[\Phi, m^\pm, t_i, t_f]} \varrho(\Phi_i, m_i^+, m_i^-, t_i), \tag{B.5}$$

where the CQ action is given by:

$$\mathcal{I}_{CQ}[\Phi, m^+, m^-, t_i, t_f] = \int_{t_i}^{t_f} dt d\vec{x} \Big[i\big(\mathcal{L}_Q[m^+] - \mathcal{V}_I[\Phi, m^+] - \mathcal{L}_Q[m^-] + \mathcal{V}_I[\Phi, m^-]\big) - \frac{D_0[\Phi]}{2}\big(m^+(x) - m^-(x)\big)^2 - \frac{1}{2D_2[\Phi]}\left(\frac{\nabla^2\Phi}{4\pi G} - \frac{1}{2}\big(m^+(x) + m^-(x)\big)\right)^2 \Big] \tag{B.6}$$

which takes the same form as the one derived from general relativity in Equation (4.29). The apparent difference in form of this path integral with the unravelling in Equations (4.44) and (4.45) arises due to the fact that the path integral in (B.5) is unnormalised; one may normalise this path-integral by computing a Gaussian integral over $\nabla^2\Phi$ and upon doing so one finds the appearance of secondary decoherence and $\hat{V}_m$ terms.

Appendix C

Unravelling of CQ theories

In this Appendix, we show how to recover the continuous backreaction term of Equation (4.33) from the unravelled equation of Equation (4.38). Given the stochastic nature of $\partial_i n^i$, it is not possible to recover a closed form, but we can see how the correlation terms emerge due to the divergence of the shift being a white noise process.

We start by defining the CQ state as:

$$\varrho(\Phi, \pi_\Phi.t) = \mathbb{E}[\delta(\Phi_t - \Phi)\delta(\pi_{\Phi\, t} - \pi_\Phi)\rho_t]. \tag{C.1}$$

When we now take the total differential of the CQ state, we have to apply Itô's rule:

$$\begin{aligned} d\varrho = \mathbb{E}[&d\delta(\Phi_t - \Phi)\delta(\pi_{\Phi\, t} - \pi_\Phi)\rho_t + \delta(\Phi_t - \Phi)d\delta(\pi_{\Phi\, t} - \pi_\Phi)\rho_t \\ &+ \delta(\Phi_t - \Phi)\delta(\pi_{\Phi\, t} - \pi_\Phi)d\rho_t + d\delta(\Phi_t - \Phi)d\delta(\pi_{\Phi\, t} - \pi_\Phi)\rho_t \\ &+ d\delta(\Phi_t - \Phi)\delta(\pi_{\Phi\, t} - \pi_\Phi)d\rho_t + \delta(\Phi_t - \Phi)d\delta(\pi_{\Phi\, t} - \pi_\Phi)d\rho_t + \cdots], \end{aligned} \tag{C.2}$$

where higher terms of order $\mathcal{O}(dt^2)$ or higher are immediately discarded.

We will now start to unpack the terms one at a time. Recalling that Φ and π_Φ are functionals, we have to pay attention to how their total derivatives are expanded. We keep only terms of order less than $\mathcal{O}(dt^2)$. usually, this is enough to guarantee a closed form for a continuous master equation. Unfortunately, the fact that $\partial_i n^i \approx \frac{dW_t}{dt}$ means that any power of these terms will never be greater than $\mathcal{O}(dt^2)$. Therefore, we will represent all these terms and their product with terms of order $\mathcal{O}(dt)$ as dots "$\cdots$". We write explicitly only the terms that lead to the continuous part of the master equation and the continuous backreaction.

$$\begin{aligned} d\delta(\Phi_t - \Phi) &= \int d^3z \, \frac{\delta}{\delta \Phi_t(z)} \delta(\Phi_t(x) - \Phi(x)) d\Phi_t(z) \\ &+ \int d^3z d^3w \, \frac{\delta^2}{\delta\Phi_t(z)\delta\Phi_t(w)} \delta(\Phi_t(x) - \Phi(x)) d\Phi_t(z) d\Phi_t(w) + \cdots \\ &= \int d^3z \, \frac{\delta}{\delta\Phi_t(z)} \delta(\Phi_t - \Phi) \left(-\frac{1}{12} \partial_i n^i \right) dt + \cdots \\ &= \frac{1}{12} \int d^3z \, \frac{\delta}{\delta\Phi(z)} \delta(\Phi_t - \Phi) \partial_i n^i dt + \cdots , \end{aligned} \tag{C.3}$$

where in the last line we have used the property of delta functions stating that $\delta_{\Phi_t}\delta(\Phi_t - \Phi) = -\delta_\Phi \delta(\Phi_t - \Phi)$ to change the functional derivative variable. Therefore, the first term reduces to:

$$\mathbb{E}[d\delta(\Phi_t - \Phi)\delta(\pi_{\Phi\, t} - \pi_\Phi)\rho_t] = \frac{1}{12} \int d^3x \, \frac{\delta \varrho}{\delta \Phi(x)} \partial_i n^i dt + \cdots , \tag{C.4}$$

where we have used the definition of the CQ state.

Proceeding, we have:

$$\begin{aligned} d\delta(\pi_{\Phi\, t} - \pi_\Phi) &= \int d^3z \, \frac{\delta}{\delta \pi_{\Phi\, t}(z)} \delta(\pi_{\Phi\, t}(x) - \pi_\Phi(x)) d\pi_{\Phi\, t}(z) \\ &+ \int d^3z d^3w \, \frac{\delta^2}{\delta\pi_{\Phi\, t}(z)\delta\pi_{\Phi\, t}(w)} \delta(\pi_{\Phi\, t}(x) - \pi_\Phi(x)) d\pi_{\Phi\, t}(z) d\pi_{\Phi\, t}(w) \\ &= -\int d^3z \, \frac{\delta}{\delta\pi_\Phi(z)} \delta(\pi_{\Phi\, t} - \pi_\Phi) \left(\frac{\nabla^2\Phi}{4\pi G} - \langle \hat{m} \rangle \right) dt \\ &- \int d^3z d^3y \, \frac{\delta}{\delta\pi_\Phi(z)} \delta(\pi_{\Phi\, t} - \pi_\Phi) \sigma(\Phi, x, y) dW_t(y) \\ &+ \int d^3z d^3w d^3y d^3y' \, \frac{\delta^2 \delta(\pi_{\Phi\, t} - \pi_\Phi)}{\delta\pi_\Phi(z)\delta\pi_\Phi(w)} \sigma(\Phi, z, y)\sigma(\Phi, w, y') dW(y) dW(y'). \end{aligned} \tag{C.5}$$

When we now average over the noise, we use the properties of the Wiener process (4.41) and the definition of the diffusion coefficient (4.42) to arrive at:

$$\begin{aligned} \mathbb{E}[\delta(\Phi_t - \Phi)\, d\delta(\pi_{\Phi\, t} - \pi_\Phi)\rho_t] = &-\int d^3x \, \frac{\delta\varrho}{\delta\pi_\Phi(x)} \left(\frac{\nabla^2\Phi}{4\pi G} - \langle \hat{m} \rangle \right) dt \\ &+ \int d^3x d^3y \, \frac{\delta^2}{\delta\pi_\Phi(x)\delta\pi_\Phi(y)} (D_2(\Phi, x, y)\varrho) dt. \end{aligned} \tag{C.6}$$

Moving on to the next term, we find

$$\begin{aligned} d\rho_t = &\frac{\partial \rho_t}{\partial t} dt + \frac{\partial \rho_t}{\partial W_t} dW_t + \frac{1}{2} \frac{\partial^2 \rho}{\partial W_t^2} dW_t^2 \\ &- i[H_m, \rho_t] dt + \frac{1}{2} \int d^3x\, d^3y\, D_0(\Phi_t; x, y)[\hat{m}(x), [\rho_t, \hat{m}(y)]] dt \\ &+ \frac{1}{2} \int d^3x\, d^3y\, \sigma^{-1}(\Phi_t; x, y) \big(\hat{m}(x)\rho_t + \rho_t\, \hat{m}(x) - 2\rho_t \langle \hat{m}(x) \rangle \big) dW(y), \end{aligned} \tag{C.7}$$

which gives:

$$\mathbb{E}[\delta(\Phi_t-\Phi)\delta(\pi_{\Phi\,t}-\pi_\Phi)\,d\rho_t] = -i[H_m,\varrho]dt + \frac{1}{2}\int d^3x\,d^3y\,D_0(\Phi;x,y)[\hat{m}(x),[\varrho,\hat{m}(y)]]dt. \quad \text{(C.8)}$$

At this point, we need to consider the expectations of terms with mixed derivatives. After a closer inspection, we notice that, keeping in mind that we will be averaging over the noise, only one term is relevant for the continuous part of the master equation, specifically:

$$\begin{aligned} d\delta(\pi_{\Phi\,t}-\pi_\Phi)\,d\rho_t = -\frac{1}{2}\int d^3x\,d^3z d^3y d^3w \frac{\delta}{\delta\pi_{\Phi\,t}(z)}\delta(\pi_{\Phi\,t}-\pi_\Phi)\sigma(\Phi,x,y)\sigma^{-1}(\Phi,x,w) \\ \times\big(\hat{m}(x)\rho+\rho\,\hat{m}(x)-2\rho\langle\hat{m}(x)\rangle\big)dW(y)dW(w), \end{aligned} \quad \text{(C.9)}$$

while the other surviving mixed terms will include the correlations between the stochastic shift and the classical gravitational field. When we average over the noise, we can integrate dw over the delta function $\delta(y-w)$ coming from the Wiener processes to use $\int d^3y\,\sigma(\Phi,x,y)\sigma^{-1}(\Phi,x,y) = \mathbb{1}$. Therefore, we arrive at:

$$\mathbb{E}[\delta(\Phi_t-\Phi)\,d\delta(\pi_{\Phi\,t}-\pi_\Phi)\,d\rho_t] = \int d^3x\left(\frac{1}{2}\hat{m}(x)\frac{\delta\varrho}{\delta\pi_\Phi(x)} + \frac{1}{2}\frac{\delta\varrho}{\delta\pi_\Phi(x)}\,\hat{m}(x) - \frac{\delta\varrho}{\delta\pi_\Phi(x)}\langle\hat{m}(x)\rangle\right)dt. \quad \text{(C.10)}$$

Finally, we can sum all the terms and divide by dt to arrive at the Master Equation:

$$\begin{aligned} \frac{\partial\varrho}{\partial t} &= \frac{1}{12}\int d^3x\,\frac{\delta\varrho}{\delta\Phi(x)}\partial_i n^i - \int d^3x\,\frac{\delta\varrho}{\delta\pi_\Phi(x)}\left(\frac{\nabla^2\Phi}{4\pi G}-\langle\hat{m}\rangle\right) \\ &\quad - i[H_m,\varrho] + \frac{1}{2}\int d^3y\,D_0(\Phi;x,y)[\hat{m}(x),[\varrho,\hat{m}(y)]] \\ &\quad + \int d^3x\left(\frac{1}{2}\hat{m}(x)\frac{\delta\varrho}{\delta\pi_\Phi(x)} + \frac{1}{2}\frac{\delta\varrho}{\delta\pi_\Phi(x)}\,\hat{m}(x) - \frac{\delta\varrho}{\delta\pi_\Phi(x)}\langle\hat{m}(x)\rangle\right) \\ &\quad + \int d^3x d^3y\,\frac{\delta^2}{\delta\pi_\Phi(x)\delta\pi_\Phi(y)}(D_2(\Phi,x,y)\varrho) + \cdots. \end{aligned} \quad \text{(C.11)}$$

We notice how the terms proportional to the expectation value $\langle\hat{m}\rangle$ simplify, and we arrive at:

$$\begin{aligned} \frac{\partial\varrho}{\partial t} &= -i[H_m,\varrho] + \frac{1}{12}\int d^3x\,\frac{\delta\varrho}{\delta\Phi(x)}\partial_i n^i - \frac{1}{4\pi G}\int d^3x\,\frac{\delta\varrho}{\delta\pi_\Phi(x)}\nabla^2\Phi \\ &\quad + \frac{1}{2}\int d^3y\,D_0(\Phi;x,y)[\hat{m}(x),[\varrho,\hat{m}(y)]] + \int d^3x d^3y\,\frac{\delta^2}{\delta\pi_\Phi(x)\delta\pi_\Phi(y)}(D_2(\Phi,x,y)\varrho) \\ &\quad + \frac{1}{2}\int d^3x\left(\hat{m}(x)\frac{\delta\varrho}{\delta\pi_\Phi(x)} + \frac{\delta\varrho}{\delta\pi_\Phi(x)}\,\hat{m}(x)\right) + \cdots, \end{aligned} \quad \text{(C.12)}$$

which we can rewrite as:

$$\frac{\partial \varrho}{\partial t} = -i[H_m, \varrho] + \frac{1}{12}\int d^3x\, \frac{\delta \varrho}{\delta \Phi(x)}\partial_i n^i - \frac{1}{4\pi G}\int d^3x\, \frac{\delta \varrho}{\delta \pi_\Phi(x)}\nabla^2\Phi$$
$$+ \frac{1}{2}\int d^3x\big(\{\mathcal{H}^I(x), \varrho\} - \{\varrho, \mathcal{H}^I(x)\}\big) + \int d^3x d^3y\, D_2(\Phi; x, y)\{\mathcal{H}_C(x), \{\varrho, \mathcal{H}_C(y)\}\} \quad \text{(C.13)}$$
$$+ \frac{1}{2}\int d^3x d^3y\, D_0(\Phi; x, y)\,([\hat{m}(x), [\varrho, \hat{m}(y)]]) + \cdots ,$$

where $\mathcal{H}_I$ and $\mathcal{H}_C$ are the interaction Hamiltonian density and the Hamiltonian density for the classical degrees of freedom as specified in Section 4.1 and "$\cdots$" includes the jumping terms and the correlation terms present due to the stochastic nature of $\partial_i n^i$.

Appendix D

Nordström gravity from action variation

In this Appendix, we summarise the derivation of the Nordström equation of motion from an action principle as discussed in detail in [235]. To begin, one considers an action similar to that of Equation (5.10)

$$\mathcal{S}[g_{\mu\nu}, \phi_m, \lambda_\mu^{\nu\rho\sigma}] = -\frac{c^3}{48\pi G_N} \int d^4x \sqrt{-g} \left(\mathcal{R} + \lambda_\mu^{\nu\rho\sigma} C^\mu_{\nu\rho\sigma}\right) + \int d^4x \mathcal{L}_Q[g, \phi_m]. \tag{D.1}$$

where $\lambda_\mu^{\nu\rho\sigma} C^\mu_{\nu\rho\sigma}$ is the Weyl tensor which is constrained to vanish through the Lagrange multiplier $\lambda_\mu^{\nu\rho\sigma}$. As one may notice, here the Lagrange multiplier is directly inserted in the action. Moreover, the factor of c^4 of Equation (5.10) is here c^3. This is because the authors of [235] define their stress-energy tensor with an extra factor of c, but the ultimate result is the same. When this action is varied with respect to the matter degrees of freedom, the Lagrange multiplier and the metric, one obtains for its extrema:

$$\frac{\delta \mathcal{S}}{\delta \phi_m} = \nabla_\nu T_m^{\mu\nu} = 0, \quad \frac{\delta \mathcal{S}}{\delta \lambda_\mu^{\nu\rho\sigma}} = C^\mu_{\nu\rho\sigma} = 0 \tag{D.2}$$

which impose energy conservation and conformal flatness of the metric on the constraint surface $g_{\mu\nu} - e^{\frac{2\Phi}{c^2}} \eta_{\mu\nu} \approx 0$. Lastly,

$$\frac{\delta \mathcal{S}}{\delta g^{\mu\nu}} = \begin{cases} \mathcal{R} - \frac{24\pi G_N}{c^4} T_m = 0 & \text{Trace part} \\ \frac{\partial^\alpha \partial_\beta \lambda^\beta_{\mu\alpha\nu}}{\phi^2} = -\frac{24\pi G_N}{c^4} \left(T_{m,\mu\nu} - \mathcal{R}_{\mu\nu} - \frac{1}{4} g_{\mu\nu}(T_m + \mathcal{R})\right) & \text{Traceless part} \end{cases} \tag{D.3}$$

This leaves the equation of Jordan's frame version of Nordström's final theory to be

$$\mathcal{R} - \frac{24\pi G_N}{c^4} T_m = 0, \quad \nabla_\nu T_m^{\mu\nu} = 0, \quad C^\mu_{\nu\rho\sigma} = 0. \tag{D.4}$$

The traceless part can be written in terms of flat covariant derivatives using the conformal flatness of the metric

$$\mathcal{R}_{\mu\nu} = -e^{-\frac{\Phi}{c^2}} \partial_\mu \partial_\nu e^{\frac{\Phi}{c^2}} - \eta_{\mu\nu}\, e^{-\frac{\Phi}{c^2}} \tilde{\Box} e^{\frac{\Phi}{c^2}} + 4 e^{-\frac{2\Phi}{c^2}} \partial_\mu e^{\frac{\Phi}{c^2}} \partial_\nu e^{\frac{\Phi}{c^2}} - \eta_{\mu\nu}\, e^{-\frac{2\Phi}{c^2}} \partial_\rho \partial^\rho e^{\frac{\Phi}{c^2}} \tag{D.5}$$

$$\mathcal{R} = -\frac{6 \tilde{\Box} e^{\frac{\Phi}{c^2}}}{c^2} e^{-\frac{3\Phi}{c^2}}. \tag{D.6}$$

which is often found in the literature written using the notation $e^{\frac{\Phi}{c^2}} = \phi$, which results in

$$\mathcal{R}_{\mu\nu} = -\frac{\partial_\mu \partial_\nu \phi}{\phi} - \eta_{\mu\nu} \frac{\tilde{\Box}\phi}{\phi} + 4 \frac{\partial_\mu \phi \partial_\nu \phi}{\phi^2} - \eta_{\mu\nu} \frac{\partial_\rho \partial^\rho \phi}{\phi^2}, \tag{D.7}$$

$$\mathcal{R} = -\frac{6\tilde{\Box}\phi}{c^2 \phi^3}. \tag{D.8}$$

Once the solution for the conformal factor ϕ is known, the traceless equation is an equation of motion for the Lagrange multiplier $\lambda_{\mu\nu\rho\sigma}$. However, the system is undefined as λ has 10 components (same symmetries as Weyl tensor), but the system of equation composed of (D.2) and (D.5), being traceless, has only 9. Classically this is not a problem as the Lagrange multiplier does not enter the equation of motion for the Nordström field ϕ. In the CQ we can exploit this by choosing the decoherence/diffusion coefficient to be positive semi-definite in the form of (5.8), allowing us only to consider diffusion away from the trace of the variation, which is exactly the Nordström field equation.

Appendix E

The stochastic action for the isotropic metric, and the Newtonian limit

For the purpose of this Appendix, we only consider a static matter distribution with negligible contributions from matter pressure, frame velocity and specific energy density. In other words, we are only interested in higher-order corrections coming from the gravitational potential Φ itself. We implicitly choose a homogeneous isotropic universe in which resides an isolated Post-Newtonian system with coordinates such that the outer region far from the isolated system is in freefall with respect to the surrounding cosmological model but at rest with respect to a frame in which the universe appears isotropic. It is then possible to show that one can construct a local quasi-Cartesian system in which metric and matter degrees of freedom can all be evaluated consistently with the Post-Newtonian approximation. Lastly, one might need to take into account the extent of preferred frame effects including frame dragging and the coordinate velocity of the frame relative to the mean rest frame of the universe. All the aforementioned effects can be summarised through what is known as the Parametrised Post-Newtonian formalism (PPN), whose first formulation dates back to Eddington in 1922. When formulated in a coordinate frame moving along with the physical system of interest, post-Newtonian effects can be summarised through the metric (with units of c):

$$\begin{aligned} g_{00} &\approx -c^2\left(1+\frac{2\Phi}{c^2}+\frac{2\beta\Phi^2}{c^4}+f(\alpha_i,\beta,\gamma,\zeta_i,V_i,W_i)\right)+\mathcal{O}(c^6),\\ g_{ij} &\approx \left(1-\frac{2\gamma\Phi}{c^2}\right)\delta_{ij}+\mathcal{O}(c^4),\\ g_{0i} &\approx h(\alpha_i,\gamma,\zeta_i,V_i,W_i)+\mathcal{O}(c^5), \end{aligned} \tag{E.1}$$

where α_i, ζ_i with $i=\{1,2,3\}$ represents respectively the extent of preferred frame effects and the extent of failure in the conservation of energy, β measures the amount of nonlinearity in the superposition law for gravity, γ the amount of curvature produced by a unit rest mass and V_i, W_i effects related to the frame velocity [333, 334]. The strength of the Parametrised Post Newtonian formalism is that it can be applied to theories of gravity outside of general relativity. However, to describe the post-Newtonian limit of general relativity one takes $\alpha_i=\zeta_i=0$ and $\beta=\gamma=1$, which is what we will do in this paper.

Given these premises, we write the isotropic metric as

$$ds^2=-c^2e^{\frac{2\Phi}{c^2}}dt^2+e^{-\frac{2\Phi}{c^2}}\delta_{ij}dx^idx^j. \tag{E.2}$$

One may worry that the exponential form of this metric may not be consistent at higher orders in the expansion, for example, not all terms in the expansion may be physically relevant. However, for all effects and purposes, in this paper, we will never exceed order $\mathcal{O}(c^4)$, such that the metric matches perfectly with the PPN formalism. For the matter distribution, we will take the Stress-Energy tensor to be that of pressureless dust, being given by

$$T_{00}=m\,e^{2\Phi},\quad T^{ij}=0,\quad T^{0i}=0. \tag{E.3}$$

Using the isotropic metric (with $c=1$), the components of the CQ action become

$$G^{\mu\nu}G_{\mu\nu}=e^{4\Phi}\left(3\left(\nabla\Phi\right)^4+\left(\left(\nabla\Phi\right)^2-2\left(\nabla^2\Phi\right)\right)^2\right), \tag{E.4}$$

$$G^2=4e^{4\Phi}\left(\left(\nabla\Phi\right)^2-\nabla^2\Phi\right)^2, \tag{E.5}$$

The coupling to matter can also be deduced from the diffusion term of the CQ action (as in Eq. (5.6)), since in the classical limit the system decoheres and for a decohered system, there is no distinction between $\bar{T}_{\mu\nu}$ and $T_{\mu\nu}$. The components are given by

$$G^{\mu\nu}T_{\mu\nu} = -m\, e^{2\Phi}\left((\nabla\Phi)^2 - 2\left(\nabla^2\Phi\right)\right), \tag{E.6}$$

$$T^{\mu\nu}T_{\mu\nu} = m^2, \tag{E.7}$$

$$T^{\mu}_{\mu} = -m. \tag{E.8}$$

where T is the trace of the stress energy tensor.

The full action for the isotropic metric is thus

$$\begin{aligned}\mathcal{I} = -\frac{D_0 c^5}{64\pi^2 G_N^2}\int d^4x\; e^{\frac{2\Phi}{c^2}}\Bigg[&\left(\nabla^2\Phi - \frac{(\nabla\Phi)^2}{2c^2} - 4e^{-\frac{2\Phi}{c^2}}\pi G m\right)^2 + \frac{3}{c^4}(\nabla\Phi)^4 \\ &- 4\beta\left(\nabla^2\Phi - \frac{(\nabla\Phi)^2}{2c^2} - 4e^{-\frac{2\Phi}{c^2}}\pi G m\right)^2\Bigg]\end{aligned} \tag{E.9}$$

where we have put in powers of c as one can use it to perform an expansion in powers of $1/c^2$. One immediately sees that at $0'th$ order in $1/c^2$, we recover the Newtonian action of (6.20).

Appendix F

Dimensional regularisation of the IR divergence

Here, we show that the two-point function Eq. (6.58)

$$G_2(\vec{x}-\vec{x}') = \frac{1}{(4\pi)^2}\int d^3y\,\frac{1}{|\vec{y}-\vec{x}||\vec{y}-\vec{x}'|}, \tag{F.1}$$

discussed in Chapter 6, after dimensional regularisation, *i.e.* on employing analytic continuation in the dimension $d = 3-2\epsilon$, gives the two-point function of Eq. (6.59),

$$G(\vec{x}-\vec{x}') = -\frac{1}{8\pi}|\vec{x}-\vec{x}'|. \tag{F.2}$$

Thus, we start from

$$\int d^{3-2\epsilon}y'\,\frac{1}{|\vec{y}\,'-\vec{x}||\vec{y}\,'-\vec{x}'|} = \int d^{3-2\epsilon}y\,\frac{1}{y|\vec{y}+\vec{z}|} \tag{F.3}$$

where we shifted $\vec{y}\,' = \vec{y}+\vec{x}$ and set $\vec{z} = \vec{x}-\vec{x}'$. Combining denominators using Feynman parameters and using standard loop-integral formulae for the resulting integral:

$$\begin{aligned}
\int d^{3-2\epsilon}y\,\frac{1}{y|\vec{y}+\vec{z}|} &= \frac{\Gamma(1)}{\Gamma^2(\frac{1}{2})}\int d^{3-2\epsilon}y\int_0^1 d\alpha_1 d\alpha_2\,\frac{\delta(1-\alpha_1-\alpha_2)\alpha_1^{-\frac{1}{2}}\alpha_2^{-\frac{1}{2}}}{\alpha_1 y^2+\alpha_2(\vec{y}+\vec{z})^2}\\
&= \frac{1}{\pi}\int d^{3-2\epsilon}y\int_0^1 d\alpha\,\frac{(1-\alpha)^{-\frac{1}{2}}\alpha^{-\frac{1}{2}}}{(\vec{y}+\alpha\vec{z})^2+\alpha(1-\alpha)z^2}\\
&= \frac{1}{\pi}\frac{(2\pi)^{3-2\epsilon}}{(4\pi)^{\frac{3}{2}-\epsilon}}\Gamma\left(-\tfrac{1}{2}+\epsilon\right)\int_0^1 d\alpha\left[\alpha(1-\alpha)z^2\right]^{\frac{1}{2}-\epsilon}\alpha^{-\frac{1}{2}}(1-\alpha)^{-\frac{1}{2}}\\
&= (\pi z^2)^{1/2-\epsilon}\,\Gamma(-\tfrac{1}{2}+\epsilon)\int_0^1 d\alpha\,\alpha^{-\epsilon}(1-\alpha)^{-\epsilon} \quad\rightarrow\quad -2\pi z\,,
\end{aligned} \tag{F.4}$$

where in the last step we recognize that the expression has a finite limit as $\epsilon \to 0$. This finite answer is the result of analytically continuing around the pole at $d = 2$ ($\epsilon = \frac{1}{2}$). Substituting the result into (F.1) we get (F.2) as claimed.

Had we computed the variance of the acceleration directly we would avoid the infrared divergence:

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x'_j}G_2(\vec{x}-\vec{x}') = -\frac{\partial^2}{\partial z_i \partial z_j}G_2(z) = \frac{1}{(4\pi)^2}\int d^3y \left\{ \frac{\delta_{ij}}{y|\vec{y}+\vec{z}|^3} - 3\frac{(y+z)_i(y+z)_j}{y|\vec{y}+\vec{z}|^5} \right\} \tag{F.5}$$

but replace it with an ultraviolet divergence (seen here in the limit $\vec{y} \to -\vec{z}$). Despite this, from above we expect the result to be finite in dimensional regularisation and be what we would obtain by using (F.2):

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x'_j}G_2(\vec{x}-\vec{x}') = -\frac{\partial^2}{\partial z_i \partial z_j}G(z) = \frac{1}{8\pi z}\left(\delta_{ij} - \frac{z_i z_j}{z^2}\right) . \tag{F.6}$$

Using Feynman parametrisation, (F.5) can be written as

$$\frac{\delta_{ij}}{8\pi^3}\int d^{3-2\epsilon}y \int_0^1 d\alpha \, \frac{(1-\alpha)^{-\frac{1}{2}}\alpha^{\frac{1}{2}}}{\left[(\vec{y}+\alpha\vec{z})^2+\alpha(1-\alpha)z^2\right]^2} - \frac{1}{2\pi^3}\int d^{3-2\epsilon}y \int_0^1 d\alpha \, \frac{(1-\alpha)^{-\frac{1}{2}}\alpha^{\frac{3}{2}}(y+z)_i(y+z)_j}{\left[(\vec{y}+\alpha\vec{z})^2+\alpha(1-\alpha)z^2\right]^3} , \tag{F.7}$$

This can again be tackled by standard loop-integral formulae, and indeed the result verifies (F.6).