

Incorporating Income Disparity and Utility Heterogeneity in Resource Allocation

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Problem definition: Efficient allocation of a divisible resource and its associated costs among consumers is a critical issue in many societal decision-making scenarios. This decision is especially difficult when consumers have heterogeneous incomes and private levels of resource utility. Common approaches in practice often overlook either utility heterogeneity or income disparity, leaving a significant gap between potential and actual outcomes. We develop and analyze a resource allocation method to bridge this gap. **Methodology/results:** Our model incorporates consumer heterogeneity in both income levels and private utility from the resource. We formulate the problem of allocating the resource and its associated costs or savings as a mechanism design problem, aiming to maximize aggregate consumer welfare. We propose a mechanism that offers consumers income-dependent menus (IDMs) of quantity and cost (or savings) options, and uncover structural properties of these menus. Our IDM approach significantly outperforms the considered alternatives in a numerical study calibrated using real-world data. **Managerial implications:** In the realm of resource allocation where both income and resource utility levels are heterogeneous, significant welfare gains can be realized by judiciously leveraging the dual dimensions of heterogeneity. Implementing our IDM approach ensures that consumers receive and contribute to resources in a manner that reflects their financial capacities and utility levels, maximizing overall welfare.

Key words: Resource allocation, resource utility heterogeneity, income disparity, efficiency, incentive compatibility, water scarcity

1. Introduction

How can a societal decision-maker effectively allocate a limited resource and its associated cost among consumers with disparate levels of income and utility from the resource? This question is pertinent in various contexts, such as the allocation of essential resources and their costs in regions facing scarcity. For instance, in water-stressed regions, decision-makers must prioritize basic survival needs for water at minimal or no cost to all consumers. After meeting the basic needs, the allocation of the remaining water and costs should account for consumer heterogeneity. The utility derived from water use varies significantly among consumers and is typically unobservable to decision-makers. Moreover, wealthier households may have a higher willingness to pay for additional water, whereas lower-income households may struggle to afford more water beyond their basic needs, even if their utility from more water is high. These considerations add layers of complexity to the allocation problem, as exemplified in the Chennai metropolitan area, where a tanker booking system ([Chennai Metropolitan Water Supply & Sewerage Board 2022](#)) and many water projects ([Shuler 2022](#)) have been implemented to supply water to residents. These complexities are reflective of similar issues that arise in any context where limited resources and costs must be distributed among diverse populations.

In this paper, we address both income disparity and heterogeneous utility in resource allocation. The unique feature of this problem is the coupled allocation problems—one for the resource itself and one for its associated cost—that must be solved under the dual dimensions of consumer characteristics: unobservable resource utility and observable income levels. Our proposed allocation scheme is an income-dependent menu (IDM) approach, in which consumers at each income level are offered a menu of pairs of resource quantity and cost to choose from. These multiple menus (one for each income level) are jointly optimized to maximize the aggregate welfare.

The extensive literature on resource allocation (detailed in [Section 2](#)) often employs mechanisms such as auctions or contracts to elicit heterogeneous preferences. However, traditional approaches based on quasi-linear preferences (risk-neutral with respect to money) do not differentiate consumers with differing financial capacities. There is a body of literature on mechanism design with non-quasi-linear preferences that capture income effects. A stream within this literature considers the allocation of multiple units of goods to consumers with multi-unit demand or the allocation of an infinitely divisible good to consumers, which is closely related to our study. The broad conclusion of this stream of literature is that, without quasi-linear preferences, there does not exist

an allocation mechanism that achieves efficiency, individual rationality, and incentive compatibility (see [Serizawa 2002](#) and other references in Section 2). Our IDM approach ensures individual rationality and incentive compatibility; moreover, our numerical results in the context of water allocation demonstrate that the IDM approach can significantly reduce the efficiency gap between simple approaches and the first-best solution.

We also explore the structural properties of the optimal IDM. We show that under the optimal IDM, consumers with higher resource utility receive more of the resource and shoulder more cost, and we further identify situations under which the optimal cost allocations bring financial utilities among consumer groups as close as possible and situations where they are as far apart as possible. Additionally, we are able to transform the original multi-menu mechanism design problem into a convex optimization problem under a special case, which facilitates numerical optimization. The optimized menus generally feature lower costs allocated to lower-income consumers. To our knowledge, our work is the first to extend mechanism design approaches to jointly optimize multiple menus and characterize their structures.

Because the IDM approach alleviates cost burdens on low-income consumers, it aligns with the objectives of various ratepayer-funded subsidy programs designed to assist low-income consumers. These programs typically finance subsidies through bill charges that are *not* linked to any specific resource or project (e.g., the Tiered Assistance Program (TAP) rider surcharges by [Philadelphia Water Department 2024](#); see also [LIHEAP Clearinghouse 2025](#)).

In parallel, utilities frequently undertake a variety of projects to manage costs. For instance, a municipal water utility not currently facing water scarcity might still develop rainwater harvesting or stormwater capture projects to reduce reliance on traditional water sources, while also minimizing stormwater discharge that could otherwise cause sewer overflows and pollution ([Environmental Protection Agency 2023](#)). Such projects can either impose a net cost on the utility or generate net savings. Under prevailing practices, utilities often distribute any net cost or savings associated with these projects across all customers by adjusting rates uniformly (e.g., rate changes by [Philadelphia Water Department 2024](#)). This approach provides no opportunity for consumers who are willing to pay a premium for such projects to contribute directly. Consequently, some consumers' higher willingness to pay goes untapped.

In contrast, the IDM approach translates consumers' willingness to pay for a specific resource into funds that support both the resource and low-income consumers. This method is more effective

than simply allocating net costs (or savings) through uniform rate changes or funding subsidies through bill charges that are unrelated to any specific resource. The IDM approach achieves better allocation efficiency by accounting for heterogeneity in both resource utilities and incomes.

The IDM approach also bears some resemblance to price discrimination. It blends second-degree price discrimination (as the menu offers pairs of prices and quantities to differentiate consumers with unobservable valuations) and third-degree price discrimination (as consumers are segmented based on their income level and offered different menus). Unlike price discrimination, which typically aims to maximize the seller’s revenue, our goal is to maximize total consumer welfare.

In this paper, we also consider simpler allocation schemes that do not account for income disparity, resource utility heterogeneity, or both. We demonstrate the advantage of our IDM approach over these alternatives using a realistic example of allocating water and associated cost in Chennai. In this case, the IDM approach is able to significantly reduce the efficiency gap between the basic resource allocation scheme and the first-best allocation.

2. Related Literature

This paper builds on the theory of mechanism design and proposes income-dependent menus for allocating a divisible resource among consumers with heterogeneous utility and incomes. In this section, we review the relevant literature in resource allocation, mechanism design, and their applications in operations management.

It is well-established that if the true preferences of all individuals are known, Pareto-efficient resource allocation can be achieved through the Walrasian equilibrium (see [Just et al. 2004](#) and [Jehle and Reny 2011](#)). In this framework, market-clearing prices are determined, and each individual buys or sells at these prices to maximize their own utility. However, when individuals’ preferences are self-reported, calculating equilibrium prices and allocating resources based on the reported preferences introduces the potential for misreporting and market manipulation. Consequently, a central focus of the extensive literature on resource allocation is developing allocation mechanisms that achieve efficiency, individual rationality, and strategy-proofness (also known as dominant strategy incentive compatibility), meaning that truthfully reporting preferences is a dominant strategy. These are referred to as the desired properties in the rest of our review.

With quasi-linear preferences, where consumers are risk-neutral with respect to money, it is well known that the Vickrey-Clarke-Groves (VCG) mechanism ([Vickrey 1961](#), [Clarke 1971](#), [Groves 1973](#)) achieves the desired properties. For a comprehensive treatment of mechanism design under

quasi-linear preferences, refer to [Myerson \(1989\)](#), [Klemperer \(2004\)](#), [Maskin \(2008\)](#), and [Börger \(2015\)](#).

Optimizing resource allocation becomes more challenging when preferences are non-quasi-linear. To advance this line of research, additional restrictions are often imposed. A significant body of work has focused on the setting where goods are indivisible and each individual demands at most one unit of good. This problem has been studied in the context of matching markets by [Crawford and Knoer \(1981\)](#), [Quinzii \(1984\)](#), [Demange and Gale \(1985\)](#), [Alkan and Gale \(1990\)](#) and in the context of auctions by [Maskin and Riley \(1984\)](#), [Demange et al. \(1986\)](#), [Saitoh and Serizawa \(2008\)](#). The overarching finding is that, with single-unit demands, there exists a resource allocation mechanism, known as the minimum price Walrasian rule, that achieves the desired properties. More recent work in this area extends the settings to study agent entry and exit ([Alaei et al. 2016](#)) and multiple dimensions of private information ([Baisa 2017](#), [Morimoto and Serizawa 2015](#), [Kazumura et al. 2020b](#), [Sakai and Serizawa 2023](#)).

This paper studies the allocation of a *divisible* good and its associated cost or benefit under additive utilities that exhibit risk aversion with respect to money. Unlike the quasi-linear preference domain or the unit-demand setting, resource allocation in our framework is known to be particularly challenging. In his seminal work, [Hurwicz \(1972\)](#) establishes that no strategy-proof, Pareto-efficient, and individually rational mechanism exists with two individuals and two goods when the preference domain encompasses a sufficiently broad class of classical preferences. [Serizawa \(2002\)](#) extends this impossibility result to any finite number of consumers and goods.

There is a small but developing stream of literature that studies resource allocation when demand can involve multiple units. [Kazumura and Serizawa \(2016\)](#) show an impossibility result similar to [Serizawa \(2002\)](#). Positive results regarding the existence of a mechanism with the desired properties can be obtained under various restrictions. For example, [Dobzinski et al. \(2012\)](#) identify such a mechanism under a special utility function that is linear in both the number of goods and money, but with a budget constraint that is public information and only two bidders. They also prove the impossibility result if the budget is private information. Other work in the area of budget-constrained auctions includes [Malakhov and Vohra \(2008\)](#), [Pai and Vohra \(2014\)](#), and [Velez \(2023\)](#). In a special setting with two bidders and a single-dimensional private type, [Baisa \(2020\)](#) shows the existence of a mechanism that achieves the desired properties. For general settings with multi-dimensional types, he shows that no mechanism satisfies the desired properties. [Shinozaki](#)

(2023) considers demands of at most two units and obtains the existence result. [Ma et al. \(2018\)](#) characterize the largest non-quasi-linear utility domain within which mechanisms with the desired properties exist. For general settings where mechanisms with the desired properties do not exist, [Kazumura et al. \(2020a\)](#) prove several properties of strategy-proof mechanisms. In this paper, we aim to explore the structural properties of optimal mechanisms. We assume a mild departure from quasi-linear utility: the consumer preferences in our model are represented by the sum of resource utility and financial utility that are concave in resource quantity and money, respectively. We assume that individuals have two-dimensional types: a private resource utility parameter and an observable income level. In contrast to the prior work, we propose income-dependent menus (IDMs), where we jointly design multiple menus (one for each income level) of quantity and money options. We theoretically explore the structures of the optimal IDM and numerically show that the efficiency loss is small in the context of allocating water among residents in Chennai.

This paper also contributes to the growing literature on applying mechanism design to address various problems in operations management. For example, there are a range of applications in specific operations contexts, including supply chain management ([Belloni et al. 2017](#)), humanitarian operations ([Zhang et al. 2020](#)), revenue management and service design (in the context of priority queues, see [Gurvich et al. 2019](#)), healthcare ([Su and Zenios 2006](#)), and electric vehicle charging ([Wu et al. 2022](#)).

We illustrate our resource allocation method using an example of water allocation in a water-scarce region, contributing to the literature on managing water resources. [Drake and Spinler \(2013\)](#) identify that market mechanisms typically fail for resources like water that are not priced in a free market, with equity goals being particularly challenging to achieve. Our proposed approach offers a framework for a societal decision-maker to manage the allocation of limited water, incorporating equity considerations arising from income disparity. Water allocation can occur at various levels, such as between municipalities (see [Murali et al. 2015](#)) or between farms (see [Dawande et al. 2013](#)). [Liu et al. \(2023\)](#) study the planning of stormwater retention infrastructure designed to safeguard against extreme rainfalls. In this paper, we apply our resource allocation approach to guide how a societal decision-maker, such as a municipal water authority, can allocate water among households, taking into account their heterogeneous utility and income levels.

3. Designing Resource Allocation Schemes: Model and Analysis

In this section, we consider the problem faced by a societal decision-maker in allocating a given amount of a divisible resource and its associated net *monetary* benefit or cost to a community of individuals or households (consumers hereafter). Different consumers derive different *non-monetary* benefits from using the resource, reflected in their heterogeneous utilities from the resource. The net monetary benefit or cost associated with the resource is shared by all consumers, who also differ in their income levels. The decision-maker aims to allocate both the resource and its associated net cost or savings to maximize the utilitarian social welfare (i.e., the sum of consumer utilities).

Let $S > 0$ (for size) denote the total amount of the resource to be allocated and M (for money) denote the net monetary benefit brought by the resource. If $M < 0$, it represents a net cost. In the long run, the societal decision-maker may adjust the size S , with the knowledge of the relationship between S and M as well as their optimal allocations. Our analysis focuses on the optimal allocation in the short run, where we assume that S and M are known with certainty at the start of the planning horizon, before the allocation is made.

3.1 Modeling Consumer Heterogeneity and Welfare

Corresponding to the resource S and money M , consumers are heterogeneous along two dimensions:

(i) Resource utility functions, parameterized by $g > 0$. Consumers with a higher value of g derive greater utility from consuming the resource. The value of g is private to each consumer and non-contractible, but we assume that the distribution of g across consumers and the functional form of the utility are known to the decision-maker.

(ii) Income $f > 0$. We assume that consumers with a lower income f have a higher marginal utility on money. To differentiate from the resource utility in (i), we refer to this utility on money as *financial utility*. Unlike parameter g , the income level is verifiable information and known to the decision-maker. This aligns with existing assistance programs for low and moderate-income households, including those offered by utility firms. For instance, the Philadelphia Water Department and the District of Columbia Water and Sewer Authority run income-qualified assistance programs that require income verification ([Philadelphia Water Department 2024](#), [District of Columbia Water and Sewer Authority 2024](#)).

When a consumer characterized by parameters (f, g) receives a resource quantity $s \in [0, S]$ and a monetary benefit m (if $m < 0$, it represents a cost), the consumer derives a resource utility denoted

by $G(s, g)$ and a financial utility denoted by $F(f + m)$. We assume that consumers experience diminishing marginal utilities in both financial benefits and resource amounts. In addition, we assume that the marginal resource utility $\partial G(s, g)/\partial s$ is increasing in g . This assumption may not hold in all situations in reality, but when it holds, it allows us to derive some structural properties of the optimal resource allocations. These assumptions are formally stated below.

Assumption 1 *The utility functions satisfy:*

- (i) $F(\cdot)$ is a strictly concave and increasing function;
- (ii) For any $g > 0$, $G(s, g)$ is strictly concave and increasing in s , with $G(0, g) = 0$; $G(s, g)$ is strictly supermodular in (s, g) for $s \geq 0$ and $g > 0$.
- (iii) $F(\cdot)$ and $G(\cdot, \cdot)$ are twice continuously differentiable.

Furthermore, for a consumer characterized by (f, g) , the total utility (welfare) from an allocated resource-money pair (s, m) is

$$u(s, m; f, g) = G(s, g) + F(f + m). \quad (1)$$

Equation (1) and the strict concavity of $F(\cdot)$ in Assumption 1(i) generalize the widely used quasi-linear utilities that lack income effects. Our utility model captures income effects while retaining the additive form to maintain tractability: consumers with a higher income f are willing to pay more for the same amount of resource (income effect); consumers with the same parameter g derive the same resource utility $G(s, g)$, regardless of their financial status or utility (allowing for tractable analysis). Note that F and G are not measured in monetary units; they measure consumer utility from the resource and money. Using Assumption 1(ii), it is straightforward to show that $G(s, g)$ strictly increases in g for any $s > 0$. The strictness of the properties in Assumption 1(ii) and the differentiability in Assumption 1(iii) serve for analytical convenience.

To account for a minimum requirement for essential resources such as water, we can replace $G(0, g) = 0$ in Assumption 1(ii) by $G(s_{\min}, g) = 0$ and modify the feasible set (i.e., modify constraint (6), which will be shortly introduced, to $s_{ij} \geq s_{\min}$), where $s_{\min} > 0$ is the minimum requirement for the resource. Equivalently, we let S be the remaining resource after allocating $s_{\min}$ to every consumer and let the decision be the allocation beyond $s_{\min}$. With this definition of S and decision variables, the analysis in this section can be directly applied to situations with a minimum requirement, and the assumption $G(0, g) = 0$ means that the utility from the minimum resource amount is normalized to zero.

In practice, allocation schemes typically deal with a finite number of groups. Accordingly, we consider I income levels and J resource utility parameter values. Specifically, we assume $f \in \{f_i : i \in \mathcal{I}\}$ and $g \in \{g_j : j \in \mathcal{J}\}$, where $\mathcal{I} = \{1, \dots, I\}$ and $\mathcal{J} = \{1, \dots, J\}$. Without loss of generality, these parameters are sorted in ascending order: $f_1 < f_2 < \dots < f_I$ and $g_1 < g_2 < \dots < g_J$. We refer to i as the *income index*, and j as the *resource utility index*. We refer to the consumers with parameter pair (f_i, g_j) as group (f_i, g_j) or simply group (i, j) .

Let N be the total number of consumers in the decision-maker's jurisdiction. Let $\theta_{ij} \in (0, 1)$ denote the fraction of consumers that belong to group (i, j) , with $\sum_{i \in \mathcal{I}, j \in \mathcal{J}} \theta_{ij} = 1$. As discussed at the beginning of this subsection, we assume that the decision-maker knows all the θ_{ij} fractions.

Consumers in each group (i, j) have identical concave utility functions and thus should receive the same allocations. Let s_{ij} and m_{ij} respectively denote the amount of resource and monetary allocations for each consumer in group (i, j) . Then, the aggregate welfare is

$$N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} u(s_{ij}, m_{ij}; f_i, g_j) \theta_{ij}, \quad (2)$$

which is the objective that the decision-maker seeks to maximize.

3.2 First-Best Benchmark

To obtain an upper bound on (2), we present and analyze the first-best solution that maximizes the aggregate welfare subject to individual rationality constraints (every consumer is willing to accept the allocation). We denote the first-best aggregate welfare by $W_{\mathcal{B}}$, obtained by solving the following problem:

$$W_{\mathcal{B}} = \max_{\{s_{ij}, m_{ij}\}} N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} (G(s_{ij}, g_j) + F(f_i + m_{ij})) \theta_{ij}, \quad (3)$$

$$\text{subject to: } N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} s_{ij} \theta_{ij} = S, \quad (4)$$

$$N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} m_{ij} \theta_{ij} = M, \quad (5)$$

$$s_{ij} \geq 0, \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \quad (6)$$

$$G(s_{ij}, g_j) + F(f_i + m_{ij}) \geq F(f_i), \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J}. \quad (7)$$

The first-best solution ensures Pareto efficiency (no consumer can be strictly better off without making at least one of the consumers worse off) and individual rationality, but does not achieve incentive compatibility under our setting of allocating a divisible resource with non-quasi-linear

utilities (Hurwicz 1972, Serizawa 2002). We will add incentive compatibility constraints in Section 3.3.

Note that if $M \geq 0$, there always exists a feasible solution to (3)-(7), but if M is too negative, there may not exist a feasible allocation. Throughout this paper, we assume that M is not too negative so that our allocation problems are feasible (in particular, it is still feasible with the additional incentive compatibility constraints in Section 3.3).

One may expect that the first-best solution will always allocate more resource to consumers with a higher resource utility. However, this is not true in general, as shown in the following example.

Example 1 Suppose $N = 2000$, $G(s, g) = g \ln(1 + s)$, and $F(x) = \ln(x)$; all consumers have the same income level $f_1 = 100$ but they have two different resource utility parameters: $g_1 = 1.5 < g_2 = 2$, with equal proportions ($\theta_{12} = \theta_{22} = 1/2$). Suppose $S = 10,000$ units of resource and an associated cost of $M = -190,000$ are to be allocated. It can be shown that the (unique) first-best allocation is $s_{11}^{\text{FB}} = 5.46$, $s_{12}^{\text{FB}} = 4.54$, $m_{11}^{\text{FB}} = -93.91$, and $m_{12}^{\text{FB}} = -96.09$. Note that $s_{11}^{\text{FB}} > s_{12}^{\text{FB}}$ although $g_1 < g_2$, i.e., more resource is allocated to the consumer group with the lower resource utility parameter g_1 . This is because when allocating a cost ($M < 0$), to meet the individual rationality constraints, consumers with g_1 would require more resource to compensate for the same loss in financial utility than consumers with g_2 . That is, if the cost were equally allocated, every consumer with g_1 would require $s_{11} = 6.37$ units of resource to meet the individual rationality constraint, leaving only $s_{12} = 3.63$ units to every consumer with g_2 . It is thus optimal to shift some cost and resource from group g_1 to group g_2 while maintaining group g_1 's individual rationality constraint, but the optimal shift is not large enough to reverse $s_{11} > s_{12}$, as a further shift would hurt the total financial utility more than the gain in the total resource utility. In this example, increasing S or M sufficiently will lead to a more intuitive solution involving $s_{11}^{\text{FB}} < s_{12}^{\text{FB}}$.

Despite this complication, the first-best *monetary* allocation has the monotonicity property with respect to income levels. (Throughout the paper, monotonicity is used in its weak sense unless otherwise stated.) Lemma 1 confirms the intuition that consumers with higher income shoulder higher costs (or receive lower benefits). The proof leverages the concavity of utility functions, although it is non-trivial (see all proofs in the online appendix). We also show in Lemma 1 that the first-best monetary allocation decreases in resource utility index, resulting from the concavity of utility functions and the supermodularity of $G(s, g)$. Thus, although consumers with a higher resource utility may or may not receive more resource (see Example 1), they will shoulder a higher

cost or receive a lower benefit.

Lemma 1 *The first-best monetary allocation m_{ij}^{FB} decreases in the income index $i \in \mathcal{I}$ and decreases in the resource utility index $j \in \mathcal{J}$.*

The individual rationality constraint in (7) may be binding only for some groups. When it is non-binding, the first-best quantity allocation has a more explicit expression and has the intuitive monotonicity property, as stated in Lemma 2.

Lemma 2 *The first-best solution $\{(s_{ij}^{FB}, m_{ij}^{FB}) : i \in \mathcal{I}, j \in \mathcal{J}\}$ satisfies:*

- (i) *For any $i \in \mathcal{I}$, if the individual rationality constraint in (7) is not binding for group (i, j) , then it is not binding for all groups (i, k) with $k > j$.*
- (ii) *There exists $\lambda > 0$ such that for all (i, j) with non-binding individual rationality constraints:*

$$s_{ij}^{FB} = \begin{cases} G_j'^{-1}(\lambda), & \text{if } \lambda < G_j'(0), \\ 0, & \text{if } \lambda \geq G_j'(0), \end{cases} \quad (8)$$

where $G_j(s) \equiv G(s, g_j)$. In (8), the optimal quantity s_{ij}^{FB} depends only on the resource utility index j and increases in j .

Lemma 2(i) implies that for any income level $i \in \mathcal{I}$, there exists a threshold $\hat{j}(i) \in \{1, \dots, J+1\}$, such that the individual rationality constraint is binding for $j < \hat{j}(i)$ and non-binding otherwise. When it is non-binding, Lemma 2(ii) further states that the first-best allocated quantities increase in the resource utility index. Note that the first-best allocated quantities can decrease in j when $j \leq \hat{j}(i)$ (see Example 1). These results rely on all parts of Assumption 1.

Lemma 2(ii) also states that, when the individual rationality constraint is non-binding, consumers allocated with positive quantities (first branch in (8)) have the same marginal resource utility λ , whereas consumers without any allocation have a marginal resource utility no more than λ . This result is driven by the concavity of resource utility functions (supermodularity is not needed).

The first-best solution serves as an upper bound on the total welfare, and we will show later that our proposed allocation mechanism is able to close a large portion of the gap between welfare under simple approaches and this upper bound.

3.3 Our Proposal: Income-Dependent Menu (IDM)

The first-best allocation in the previous section requires the decision-maker to know the resource

utility parameters of all consumers, which is impractical. In this section, we propose an allocation scheme that offers consumers at each income level a menu of resource quantity and monetary benefit/cost options (i.e., pairs of (s, m)) to choose from. Every consumer is included, receiving a menu choice. Within each income group, consumers with resource utility parameter g_j (unobservable to the decision-maker ex-ante) self-select into a menu option designed for them. This is a mechanism design problem that optimizes multiple menus jointly. Specifically, the decision-maker determines a menu $\{(s_{ij}, m_{ij}) : j \in \mathcal{J}\}$ for every income group $i \in \mathcal{I}$, such that each consumer in group (i, j) willingly chooses the menu item (s_{ij}, m_{ij}) . The welfare maximization problem can be formulated as follows:

$$W_{\mathcal{M}} = \max_{\{s_{ij}, m_{ij}\}} N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} (G(s_{ij}, g_j) + F(f_i + m_{ij})) \theta_{ij}, \quad (9)$$

subject to: (4), (5), (6),

$$G(s_{ij}, g_j) + F(f_i + m_{ij}) \geq F(f_i), \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \quad (10)$$

$$G(s_{ij}, g_j) + F(f_i + m_{ij}) \geq G(s_{ik}, g_j) + F(f_i + m_{ik}), \quad \forall i \in \mathcal{I}, \forall j, k \in \mathcal{J}. \quad (11)$$

The incentive compatibility constraints (11) ensure that no consumer has an incentive to pretend to have a different resource utility parameter. Unlike the classic mechanism design problem with one menu, our problem involves I menus (one for each income level), and these menus are linked by the resource and benefit/cost constraints (4)-(5). The mechanism design problem (9)-(11) endogenously decides whether each consumer group receives an allocation or not, because the optimization also considers $(s_{ij}, m_{ij}) = (0, 0)$, which corresponds to no-allocation for group (i, j) . Note that if consumer group (i, j) does not receive an allocation, the incentive compatibility constraints (11) must still hold so that they will prefer to choose $(s_{ij}, m_{ij}) = (0, 0)$.

As discussed in Section 2, allocating a divisible resource with non-quasi-linear utilities does not, in general, permit us to achieve efficiency, incentive compatibility, and individual rationality at the same time. We remark that, if the utility functions were quasi-linear and M is not too negative, then the problem in (9)-(11) indeed achieves the first-best consumer welfare as in (3)-(7). To see this, note that quasi-linear utilities (i.e., $F(f + m) = f + m$) reduce the objective in (9) to $N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} G(s_{ij}, g_j) \theta_{ij}$, because the total financial utility is a constant. We can then solve the problem of allocating the resource S first to maximize this objective, and then construct a monetary allocation that ensures the individual rationality and the incentive compatibility constraints.¹

¹This construction is straightforward based on the simplified constraints in (13)-(14).

The problem (9)-(11) is difficult to solve because the incentive compatibility constraints (11), in general, form a non-convex feasible region (unless an additional assumption is imposed; see discussion at the end of this section). In order to aid solution and uncover insights on the optimal allocation, we prove some structural properties of the optimal solutions in Propositions 1 and 2.

Proposition 1 *An optimal income-dependent menu, $\{(s_{ij}^*, m_{ij}^*) : i \in \mathcal{I}, j \in \mathcal{J}\}$ that solves (9)-(11), has the following properties:*

- (i) Resource quantity s_{ij}^* increases in the resource utility index $j \in \mathcal{J}$, $\forall i \in \mathcal{I}$;
- (ii) Monetary benefit m_{ij}^* decreases in the resource utility index $j \in \mathcal{J}$, $\forall i \in \mathcal{I}$;
- (iii) $s_{ij}^* < s_{i,j+1}^*$ if and only if $m_{ij}^* > m_{i,j+1}^*$, $\forall j \in \mathcal{J} \setminus \{J\}$, $\forall i \in \mathcal{I}$.

The proof of Proposition 1 relies on the supermodularity of $G(s, g)$ and the additive form of the utility function in Assumption 1. The implication of Proposition 1 is that, for any given income level, it is optimal to incentivize consumers with higher resource utility parameters to choose a lower benefit (or higher cost) in return for a larger quantity of the resource. In effect, this mechanism provides consumers with high resource utility parameters an opportunity to financially support the resource, thereby reflecting their willingness to pay for it. For example, the optimal menu provides a “basic option” (s_{i1}^*, m_{i1}^*) intended for consumers with the lowest resource utility parameter g_1 ; a consumer with a higher g_j may choose a higher resource quantity $s_{ij}^* > s_{i1}^*$ and pay a premium of $m_{i1}^* - m_{ij}^* > 0$ for the additional units of the resource.

To further explore the structural properties of the optimal resource allocation scheme, we provide a compact, equivalent formulation of the problem in (9)-(11). This simplified formulation is based on the approach in the literature that, under certain conditions, it suffices to enforce the individual rationality constraint for consumers with the lowest type and incentive compatibility constraints for adjacent types (refer to Lovejoy 2006 and Wu et al. 2022). For completeness, these results are formally stated and proved in Lemmas A.1 and A.2 in Online Appendix A; the required conditions are monotonicity and supermodularity of $G(s, g)$. This leads to the following simplified problem:

$$W_{\mathcal{M}} = \max_{\{s_{ij}, m_{ij}\}} N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} (G(s_{ij}, g_j) + F(f_i + m_{ij})) \theta_{ij}, \quad (12)$$

subject to: (4), (5), (6),

$$G(s_{i1}, g_1) + F(f_i + m_{i1}) \geq F(f_i), \quad \forall i \in \mathcal{I}, \quad (13)$$

$$\begin{aligned}
G(s_{i,j+1}, g_j) - G(s_{ij}, g_j) &\leq F(f_i + m_{ij}) - F(f_i + m_{i,j+1}) \\
&\leq G(s_{i,j+1}, g_{j+1}) - G(s_{ij}, g_{j+1}), \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J} \setminus \{J\}.
\end{aligned} \tag{14}$$

Obtained by arranging the incentive compatibility constraints for adjacent groups (i.e., consumers in group (i, j) have no incentive to pretend to be in group $(i, j + 1)$ and vice versa), (14) states that the financial utility difference between any two adjacent groups is bounded above and below by resource utility differences. Specifically, the lower bound in (14) implies that if a consumer with utility parameter g_j pretends to have parameter g_{j+1} , they receive more resource and less money (according to Proposition 1), but the gain in resource utility does not exceed the financial utility loss; the upper bound in (14) implies that if a consumer with g_{j+1} pretends to have g_j , their financial utility gain does not exceed the loss in resource utility.

Leveraging the properties of the objective function in (12), we are able to characterize the structure of the optimal resource allocation by identifying when the upper bound or the lower bound of (14) is binding, as formalized in the next proposition. The proof of Proposition 2 uses all parts of Assumption 1.

Proposition 2 *An optimal income-dependent menu, $\{(s_{ij}^*, m_{ij}^*) : i \in \mathcal{I}, j \in \mathcal{J}\}$ that solves (9)-(11) or (12)-(14), has the following structure: For any given $i \in \mathcal{I}$, there exists $\hat{j} \in \mathcal{J} \cup \{0\}$, such that*

- (i) $\forall j > \hat{j}, F(f_i + m_{ij}^*) - F(f_i + m_{i,j+1}^*) = G(s_{i,j+1}^*, g_j) - G(s_{ij}^*, g_j);$
- (ii) $\forall j < \hat{j}, F(f_i + m_{ij}^*) - F(f_i + m_{i,j+1}^*) = G(s_{i,j+1}^*, g_{j+1}) - G(s_{ij}^*, g_{j+1});$
- (iii) *If $j = \hat{j}$, (14) may hold with strict inequalities.*

Furthermore, if constraint (13) is not binding for a given $i \in \mathcal{I}$, then $\hat{j} = 0$, implying that part (i) holds for the given i and $\forall j \in \mathcal{J}$.

Propositions 1 and 2 reveal the following structure of the optimal mechanism: Within each income group, as the resource utility parameter decreases from J to 1, the allocated resource amount decreases while the monetary allocation increases (Proposition 1). The monetary allocation should increase as slowly as possible, following the lower bound in (14), and then as fast as possible, following the upper bound in (14). Only at the transition point, can the increase fall between the lower and upper bounds. If such a transition point exists, the individual rationality constraint (13) for consumers with the lowest resource utility must be binding. In other words, the need for the fastest increase in monetary allocation arises from the requirement to satisfy the individual rationality constraint for the consumer with the lowest resource utility parameter.

In traditional mechanism design problems, the principal is motivated by profit-seeking objectives to drive prices apart from each other—the principal sets high prices for high-valuation consumers, leaving only the information rent so that high-valuation consumers will not choose the option designed for low-valuation consumers. In contrast, Proposition 2(i) and the last statement in Proposition 2 reveal that, within each income group, the decision-maker aims to keep the monetary allocation as close as possible (i.e., following the lower bound), except when necessary to satisfy the individual rationality constraint. This is driven by the concave financial utility function (part of the consumer welfare) in the objective of the decision-maker.

Proposition 2(ii) shows that it may still be optimal to require the maximum possible differences in monetary allocation between adjacent groups, but the motivation is completely different: A profit-seeking principal aims to increase the price for high-valuation consumers as much as possible, whereas the societal decision-maker aims to decrease the price for low-valuation consumers as much as needed to satisfy the individual rationality constraint.

Propositions 1 and 2 provide structural properties for the optimal income-dependent menus that solve the problem in (9)-(11). Although this problem in general presents numerical challenges because the incentive compatibility constraints (11) form a non-convex feasible region, it can be converted to a convex optimization problem under a special case.

Assumption 2 *The resource utility function has the form: $G(s, g_j) = g_j \bar{G}(s)$, for all $j \in \mathcal{J}$.*

Under Assumptions 1 and 2, for every $(i, j) \in \mathcal{I} \times \mathcal{J}$, let $v_{ij} \equiv g_j \bar{G}(s_{ij})$ and $w_{ij} \equiv F(f_i + m_{ij})$ respectively denote the resource utility and financial utility of an individual in group (i, j) when choosing the intended menu item (s_{ij}, m_{ij}) . Using the change of variables, the problem in (9)-(11) can be written as:

$$W_{\mathcal{M}}(S, M) = \max_{\{v_{ij}, w_{ij}\}} N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} (v_{ij} + w_{ij}) \theta_{ij}, \quad (15)$$

$$\text{subject to: } N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} \bar{G}^{-1}(v_{ij}/g_j) \theta_{ij} \leq S, \quad (16)$$

$$N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} (F^{-1}(w_{ij}) - f_i) \theta_{ij} \leq M, \quad (17)$$

$$v_{ij} \geq 0, \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \quad (18)$$

$$v_{ij} + w_{ij} \geq F(f_i), \quad \forall i \in \mathcal{I}, \forall j \in \mathcal{J}, \quad (19)$$

$$v_{ij} + w_{ij} \geq \frac{g_j}{g_k} v_{ik} + w_{ik}, \quad \forall i \in \mathcal{I}, \forall j, k \in \mathcal{J}. \quad (20)$$

The constraints (16) and (17) are written as inequality constraints, so that they define convex sets of the decision variables, rendering (15)-(20) a convex optimization problem, which greatly simplifies our numerical optimization in Section 4. At optimality, (16) and (17) hold with equality. Furthermore, we can show that the optimal value is concave in the resource amount and monetary benefit, as stated in the following lemma.

Lemma 3 *The welfare under the optimal income-dependent menu, $W_{\mathcal{M}}(S, M)$ in (15), is concave in (S, M) .*

Finally, the problem of designing multiple menus jointly in (9)-(11) can be viewed as a bi-level problem of optimizing the quantity and money allocations across income groups (high level) and optimizing the menu of quantity and money within each income group (low level). Let S_i and M_i be the resource and money allocated to income group i , and let $W_{\mathcal{M}_i}(S_i, M_i)$ denote the optimal aggregate welfare of income group i . Then, the high-level problem is

$$\max_{\{S_i, M_i\}} \left\{ \sum_{i \in \mathcal{I}} W_{\mathcal{M}_i}(S_i, M_i) : \sum_{i \in \mathcal{I}} S_i = S, \sum_{i \in \mathcal{I}} M_i = M, S_i \geq 0, i \in \mathcal{I} \right\}. \quad (21)$$

Lemma 4 *The objective function of the high-level problem in (21) is concave in $(S_i, M_i : i \in \mathcal{I})$.*

3.4 Additional Approaches for Comparison

In addition to the first-best upper bound in Section 3.2 and the optimal mechanism in Section 3.3, we present two other practical resource-monetary allocation approaches. The first approach ignores both dimensions of consumer heterogeneity, and the second approach offers an income-dependent allocation while ignoring heterogeneous resource utilities. Although these approaches are sub-optimal, their simplicity is attractive and it is useful to compare them with the optimal mechanism.

3.4.1 Income-Agnostic Cost (or Benefit) and Quantity

In this approach, consumers choose to receive either no allocation or an equal share of the resource S/N_1 and its cost or benefit M/N_1 , where $N_1 \leq N$. Under this simple allocation scheme, the resource is allocated on a first-come-first-served basis. If all consumers prefer the allocation over nothing and that consumers from all groups completely randomly decide when to sign up and the first N_1 consumers successfully sign up, the resulting welfare is:

$$W_{\mathcal{C}} = N_1 \sum_{i \in \mathcal{I}, j \in \mathcal{J}} u(S/N_1, M/N_1; f_i, g_j) \theta_{ij} + (N - N_1) \sum_{i \in \mathcal{I}, j \in \mathcal{J}} u(0, 0; f_i, g_j) \theta_{ij}. \quad (22)$$

It is not difficult to see that the concavity of the utility functions in Assumption 1 implies

that the aggregate utility in (22) increases in N_1 , the number of consumers who receive an allocation. Thus, it would be sensible to lower the allocation to S/N , so that all consumers have equal allocations. Thus, the best aggregate utility from the income-agnostic approach is

$$W_{\mathcal{A}} = N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} (G(S/N, g_j) + F(f_i + M/N)) \theta_{ij}. \quad (23)$$

We remark that equally allocating the resource and cost or benefit among all consumers can be far from being efficient due to the diverse resource utility and income levels.

The approach in (23) will always satisfy the individual rationality constraints when $M > 0$, as every consumer receives $M/N > 0$ and $S/N > 0$. However, if $M < 0$ (allocating a net cost), some consumers may opt not to receive any allocation. Thus, in the case of $M < 0$, we need to search for the best subset of consumers for whom an even allocation of resource S and cost M is feasible (i.e., satisfying their individual rationality constraints). To this end, we iterate over all possible subsets of consumer groups. For each subset, if even allocation is feasible, we calculate the resulting welfare. Finally, we choose the subset with the highest welfare. Note that if M is negative enough, there may be no feasible income-agnostic benefit and quantity offering, although other allocation schemes may still be feasible.

3.4.2 Income-Dependent Cost (or Benefit) and Quantity

The second alternative scheme is an income-dependent offering, in which consumers are qualified based on their incomes for different amounts of resource and benefits or costs. That is, consumers at income level i receive a benefit/cost of m_i and s_i units of the resource.

Observe that this scheme subsumes two more restrictive schemes: (a) uniform allocation quantities and income-dependent benefits/costs; and (b) uniform benefits/costs and income-dependent allocation quantities.

We first consider the case of $M \geq 0$ and we impose $m_i \geq 0$ as well. Then, the problem can be written as

$$\begin{aligned} W_{\mathcal{D}} = \max_{\{s_i, m_i\}} & N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} (G(s_i, g_j) + F(f_i + m_i)) \theta_{ij}, \\ \text{subject to: } & N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} s_i \theta_{ij} = S, \quad N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} m_i \theta_{ij} = M, \quad s_i \geq 0, \quad m_i \geq 0, \quad \forall i \in \mathcal{I}, \end{aligned} \quad (24)$$

which can be decomposed into

$$\max_{\{s_i \geq 0\}} \sum_{i \in \mathcal{I}, j \in \mathcal{J}} G(s_i, g_j) \theta_{ij}, \quad \text{subject to: } N \sum_{i \in \mathcal{I}} s_i \theta_i = S, \quad (25)$$

$$\max_{\{m_i \geq 0\}} \sum_{i \in \mathcal{I}} F(f_i + m_i) \theta_i, \quad \text{subject to: } N \sum_{i \in \mathcal{I}} m_i \theta_i = M, \quad (26)$$

where $\theta_i = \sum_{j \in \mathcal{J}} \theta_{ij}$.

Proposition 3 *The income-dependent benefit and quantity allocation has the following properties:*

- (i) *If the income distribution and the resource utility parameter distribution are independent, i.e., θ_{ij}/θ_i depends only on j , then it is optimal to set the allocation s_i to be equal for all income levels: $s_i = S/N$, $\forall i \in \mathcal{I}$.*
- (ii) *The optimal benefit m_i decreases in the income index i .*

Proposition 3(i) intuitively means that when income information has no predictive power for the resource utility parameter, there is no advantage of allocating resource differently across income levels. In the case of green energy allocation, [Gustafson et al. \(2019\)](#) find that individuals with higher incomes are not more likely to pay more for renewable energy, with little to no correlation observed between income and willingness to pay for clean electricity. Part (ii) does not require the independence condition in part (i), but if the independence condition holds, then $m_i/s_i = m_i N/S$ also decreases in i , i.e., a higher net benefit *rate* is offered to lower-income consumers.

Similar to the discussion at the end of Section 3.4.1, the case of $M < 0$ requires us to search for the best subset of consumers to participate in the allocation of S and M . We iterate over all possible subsets of consumer groups. For each subset, if the income-dependent cost/benefit and quantity allocation solved from (25)-(26) is feasible, we calculate the resulting welfare. Finally, we choose the subset with the highest welfare. We employ such a numerical approach in Section 4, which deals with allocating water and associated cost $M < 0$.

4. Numerical Analysis

In this section, we present a calibrated numerical study demonstrating the application of our resource allocation mechanism. The study focuses on allocating water and its associated cost ($M < 0$) in a low-income neighborhood in Chennai, India. In this study, the specific allocations we obtain are naturally driven by our choice of utility functions. When data are limited, we make our best effort to estimate the parameters.

Chennai is a water-scarce region in southern India. In this section, we address the challenge

of alleviating water shortages in the economically disadvantaged Ward 155 in Chennai, calibrating parameters based on actual data. According to a survey conducted by Venkatachalam (2015), the average household water availability in Ward 155 is approximately 7114 L (liters) per month, or 237 L per day. This falls significantly short of the Ministry of Urban Development's per capita recommendation of 135 L per day (Venkatachalam 2015). When multiplied by the average household size of 4.02 (Knoema 2011), the daily recommendation amounts to about 543 L per household, indicating a shortfall of approximately 306 L per day per household.

4.1 Parameter Calibration

Ward 155 is home to 75,748 individuals (Population Census Data 2011), approximately 18,843 households. A willingness-to-pay study by Venkatachalam (2015) surveyed a representative sample of these households to determine how much they would be willing to pay for an additional 375 L of water per day. Building on this information, we consider a project that provides, on average, an additional 375 L of water per day per household. To ensure all households' water access improves, we first allocate a fixed quantity of $s_{\min} = 213$ L per day to each household at no cost to them, bringing their daily guaranteed availability to 450 L per day. The subsequent allocation of the remaining water amounts to a total volume of $S = (375 - 213) \times 75,748/4.02 = 3,052,531$ L per day (as explained in Section 3.1, S is the resource quantity after allocating $s_{\min}$ to all consumers). We consider the subsidized cost of supplying water to this ward being the lowest rate of 2.50 Rupees/m³ or 0.0025 Rupees/L (The Energy and Resources Institute 2010). Thus, the cost of the additional water for all households (375 L per day per household) is $M = -17,665$ Rupees per day.

We consider two income groups, the low-income group being the first two rows of Table 5 in Venkatachalam (2015) and the next two rows corresponding to high-income consumers. This categorization leads to $\theta_1 = 0.7442$ and $\theta_2 = 0.2558$. The first two rows of Table 5 in Venkatachalam (2015) correspond to monthly income brackets of up to Rupees 5000/month and Rupees 5001-10000/month, and the next two rows correspond to monthly income brackets of Rupees 10001-15000/month and over 15000/month. Mapping the four income brackets to their midpoints with increments of Rupees 5000 (Rupees 2500, 7500, 12500, and 17500), and weighing the rows by the number of households in them, we get a low income level of Rupees 6953/month and a high income level of Rupees 14091/month. Converting these quantities into per day incomes assuming a 30 day month, we get $f_1 = \text{Rupees } 232/\text{day}$ and $f_2 = \text{Rupees } 470/\text{day}$. Consistent with Assumption 1, we set the financial utility function $F(x) = \ln(x)$, and the resource utility function $G(s, g) = g \ln(1 +$

$\frac{s}{237+213}$), where s measures the water allocation above the minimum allocation of $s_{\min} = 213$ L. Accordingly, when $s = 0$, the resource utility normalizes to zero (as discussed in Section 3.1, the utility from having the minimum resource allocation is normalized to zero). Without this water project, the resource utility would be negative, corresponding to $s = -213$. Having very little water will drive utility to $-\infty$, corresponding to $s \rightarrow -(237 + 213)$, where 237 is the current average water consumption per household.

We calibrate the distribution of the resource utility parameter g based on Venkatachalam (2015), who found that 216 of 302 surveyed households were not willing to pay any additional money for 375 L of additional water, while the remaining 86 households were willing to pay varying amounts, with a minimum of 50 Rupees per month, a maximum of 300 Rupees per month, and an average of 137.21 Rupees per month. We represent this distribution by considering a population with four different resource utility levels, corresponding to daily willingness to pay (WTP) values as follows: $\text{WTP} = 1/30$ (a proxy for none), $50/30$, $137.21/30$, and $300/30$. To find the values of the resource utility parameter g corresponding to these WTP values, we solve the following equation that reflects the WTP for 375 L of water as asked in the survey: $F(292.9) + G(-213, g) = F(292.9 - \text{WTP}) + G(375 - 213, g)$. The left side of this equation represents the utility of an average household at the time of the survey (i.e., without the focal water project, $s = -213$); $f = 292.9$ is the weighted average of f_1 and f_2 . The right side is the total utility of an average household with 375 L of additional water per day. Solving the equation for g under each considered value of WTP, we have $g_1 = 0.0001$, $g_2 = 0.0060$, $g_3 = 0.0166$, and $g_4 = 0.0366$. We then find θ values that make the group means and standard deviations consistent with Table 5 in Venkatachalam (2015); the resulting consumer groups are shown in Table 1.

Table 1: Chennai Ward 155: Consumer group characteristics (f_i, g_j) and proportions (θ_{ij})

	$g_1 = 0.0001$	$g_2 = 0.0060$	$g_3 = 0.0166$	$g_4 = 0.0366$	Row sum
$f_1 = 232$ Rupees/day	0.532	0.039	0.158	0.015	0.744
$f_2 = 470$ Rupees/day	0.183	0.0016	0.065	0.0066	0.256

The number in row f_i (income) and column g_j (resource utility parameter) represents the fraction of consumers θ_{ij}

4.2 Results

To compare various water allocation schemes ranging from the most basic income-agnostic approach (Section 3.4.1) to the most sophisticated income-dependent menu (Section 3.3), we normalize the

aggregate welfare under the income-agnostic approach to zero and show the welfare improvement achieved by alternative schemes in Table 2. The table also shows that the welfare gap between $W_{\mathcal{A}}$ (income-agnostic cost and quantity) and $W_{\mathcal{B}}$ (first-best) can be narrowed to different extents. For completeness, we also include an “income-agnostic menu” approach that offers a single menu of costs and quantities for all consumers, regardless of their income levels. The theoretical expression of the welfare under this approach is complex and hence not included in Section 3.

Table 2: Welfare comparison across various water allocation program designs

	Income-Agnostic Cost & Quantity $W_{\mathcal{A}}$	Income-Agnostic Menu	Income-Dep. Cost & Quantity $W_{\mathcal{D}}$	Income-Dep. Menu $W_{\mathcal{M}}$	First-Best $W_{\mathcal{B}}$
Welfare improvement relative to $W_{\mathcal{A}}$	0	1.52	13.22	17.54	20.31
Gap closed	0%	7.46%	65.10%	86.37%	100%

Under all approaches, consumers receive $s_{\min} = 213$ L of water at no cost and can opt not to receive additional water $s \geq 0$ at a cost. Note that the $s < 0$ part of the utility function is used only for parameter estimation; all allocation schemes require $s \geq 0$.

Under the baseline income-agnostic cost and quantity approach, all consumers are given the choice of 2092.35 L of additional water daily at a cost of 10.88 Rupees per day; only low-income consumers in group g_4 and high income consumers in groups g_3 and g_4 take this option. The income-agnostic menu approach designs menu items such that consumers in groups g_3 and g_4 choose varying amounts of additional water. This approach marginally improves welfare compared with the vanilla income-agnostic approach, closing the welfare gap from the first-best by 7.46%.

A significant improvement can be achieved when income disparity is considered: the income-dependent cost and quantity approach reduces the gap between the baseline and the first-best welfare levels by 65.1%. Under this scheme, it is optimal to offer low-income consumers the option of an additional 430.9 L of water per day at a daily cost of 1.24 Rupees, and high-income consumers 1227.1 L of water at a cost of 10.15 Rupees. This offer is accepted by groups g_3 and g_4 in both income groups. The superior performance of this approach over the income-agnostic menu suggests that leveraging income heterogeneity is more critical than addressing resource utility heterogeneity in this context.

Given that the gaps closed by the income-agnostic menu (ignoring income disparity) and income-dependent cost and quantity approach (ignoring utility heterogeneity) are 7.46% and 65.10%, respectively, one might expect that our IDM approach (respecting both dimensions of heterogeneity) would not be able to move the needle much towards the first-best welfare. However, the IDM approach succeeds in closing about 86% of the gap between the baseline and the first-best welfare level.

Table 3: The optimal income-dependent menu (IDM) for water allocation program

	Menu for low-income households			Menu for high-income households		
Allocation $s_{\min} + s^*$ (L/day)	213.00 [‡]	561.87	1515.19	213.00 [‡]	1264.64	3172.20
Cost $-m^*$ (Rupees/day)	0.00 [‡]	0.82	3.81	0.00 [‡]	9.30	22.93

[‡] Intended for consumer groups g_1 and g_2

The optimal IDM solution, as shown in Table 3, addresses both income disparity and resource utility heterogeneity. This approach allows households to choose the amount of water (in addition to the free 213 L) they wish to purchase, with low-income households eligible for a discounted rate. Although there are four utility levels, the optimal allocations for consumer groups g_1 and g_2 are the same: solely the free allocation of 213 L. Households are presented with all three options applicable to their income level and choose the option that best meets their needs. Households in g_3 and g_4 groups (constituting about 25% of the population in view of Table 1) derive greater utility from additional water and opt for the 2nd and 3rd options on the menu, respectively. Consequently, these households bear the costs, while those in g_1 and g_2 groups choose to receive only 213 L at no cost.

4.3 Discussion

The water allocation example demonstrates the application of the methods discussed in our paper to ensure essential resources are allocated at the minimum quantity without burdening households, while offering consumers with the options to pay for quantities beyond the minimum. In this example, only households in groups g_3 and g_4 opt to purchase additional water. This holds true across the income-agnostic cost and quantity scheme, the income-dependent cost and quantity scheme, and the IDM approach. The IDM approach stands out by offering differentiated options to g_3 and g_4 households and by providing discounts to low-income households.

We also note from Table 3 that no consumers receive a net benefit under these schemes. This is because the water project is costly for Ward 155 community and every household receives 213 L for free. Thus, there is insufficient water to generate additional willingness-to-pay.

Should the IDM approach be difficult to implement, the income-dependent cost and quantity approach remains a feasible alternative, closing a reasonably large gap between the income-agnostic and first-best approaches.

The welfare estimates in this example inherently depend on the selected utility functions. We opted for logarithmic utility functions as they strike a balance between linearity and concavity, positioning them in the middle of the constant relative risk aversion (CRRA) utility class. This choice moderately reflects the intention for high-income individuals to bear a greater proportion of the costs or receive a reduced share of the benefits.

5. Concluding Remarks

In this paper, we address the challenge of optimally allocating a limited resource and its associated costs or savings among consumers with (private) heterogeneous utilities from the resource and disparate incomes. We study various alternative allocation schemes to the current practice and demonstrate using a realistic case study that the income-dependent menu (IDM) approach succeeds in closing most of the gap between the naive approach and the theoretical first-best. We also uncover various structural properties of the optimal IDM.

The strong performance of our IDM approach, exemplified by our application to water allocation, is attributed to two features of this approach. First, even if there is no income disparity, the menu approach allows the consumers who value the resource more to receive more of the resource and also pay more for it. Second, considering income disparity, the IDM approach jointly designs multiple menus (one for each income level) so that the low-income consumers shoulder less of the cost or receive more of the savings.

To conclude this paper, we discuss strategies to overcome IDM implementation challenges and potential directions for future research.

Overcoming IDM Implementation Challenges

Although the IDM approach performs well theoretically, there can be a number of practical challenges in implementing it. First, the observability of income in our model requires the decision-maker in resource allocation to collaborate with government agencies to verify household income.

This is possible if the decision-maker either is a government entity or has a partnership with the government. Such a partnership will allow the decision-maker to convey the income thresholds used to divide income groups for the IDM purpose to the government (e.g., thresholds may be set at 200% of the federal poverty line or tied to the median income in the jurisdiction), and the government will then confirm the income group of each household. Thus, the decision-maker does not need to know the actual household incomes.

Second, if different decision-makers (e.g., water, electricity, and gas utility companies) offer similar IDM programs, it will unnecessarily increase the administrative burden as well as adding to households' confusion. Thus, ensuring coordination and avoiding overlap across programs is important. For example, if electricity and gas are provided by two different companies, they should consider a joint IDM program to allow consumers to choose menu items based on their overall energy use instead of separating into gas and electric.

Third, there is a risk that some individuals may misreport their income to qualify for a menu designed for a low-income group. Addressing these risks requires implementing robust income reporting and tax processes on the government side.

Fourth, individuals who are just above the income threshold may feel unfairly excluded from the IDM for the low-income group, which could generate dissatisfaction and administrative burdens. In practice, multiple thresholds can be used, e.g., households earning below 150% of the federal poverty line receive a larger subsidy than those earning between 150% and 200%. In addition, as income can vary from year to year, companies can consider a buffer zone around the threshold (e.g., 5-10%) to allow households just above the cutoff to remain eligible for subsidies for a short period.

Avenues for Future Research

Our work opens various avenues for further exploration. First, although this paper focuses on a short-term allocation problem with a fixed resource quantity and cost, our IDM scheme's higher allocation efficiency compared with simpler approaches suggests that integrating it into a long-term planning framework could incentivize increasing the resource size relative to simpler approaches. In a one-shot planning setting, if the decision-maker knows the cost as a function of the resource size, then an optimization can be performed to determine the optimal resource size. In multi-period setting, a dynamic resource allocation scheme that incorporates both income and utility heterogeneity would further advance the theory and expand its applications to situations where the size of the resource and its associated cost/benefit evolve over time. Second, it is worth considering a

setting where the decision-maker is not a socially oriented entity but a regulated for-profit firm. How the firm would allocate and charge for the resource and how it should be regulated are important research questions. Finally, in light of the documented mixed consumer behavioral changes—some show increased usage while others adopt more conservative behavior (Awaysheh et al. 2024)—it is crucial to examine how the firm could allocate the resource while incentivizing sustainable consumption patterns. These areas provide fertile ground for future research, contributing to a deeper understanding of resource allocation in both theoretical and practical contexts.

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Incorporating Income Disparity and Utility Heterogeneity in Resource Allocation: Online Appendices

A. Optimizing the Income-Dependent Menu: Compact Formulation

In this appendix, we drive the simplified constraints in (13) and (14).

Lemma A.1 *In the problem (9)-(11), the set of individual rationality constraints in (10) can be reduced to:*

$$G(s_{i1}, g_1) + F(f_i + m_{i1}) \geq F(f_i), \quad \forall i \in \mathcal{I}. \quad (27)$$

Lemma A.1 states that we only need to ensure the individual rationality constraint for the consumers with $j = 1$, the lowest resource utility parameter. The proof shows that the individual rationality constraints for $j > 1$ are implied by the first individual rationality constraint and the incentive compatibility constraints.

Proof of Lemma A.1: First, consider the incentive compatibility constraint (11) for any i and for $j > k = 1$.

$$G(s_{ij}, g_j) + F(f_i + m_{ij}) \geq G(s_{i1}, g_j) + F(f_i + m_{i1}).$$

Because $G(s, g)$ is increasing in g , we have $G(s_{i1}, g_j) + F(f_i + m_{i1}) \geq G(s_{i1}, g_1) + F(f_i + m_{i1})$.

Therefore, if the individual rationality constraint for group $j = 1$ in (27) is satisfied, the above two inequalities lead to $G(s_{ij}, g_j) + F(f_i + m_{ij}) \geq F(f_i)$ for $j > 1$. In other words, the individual rationality constraint for $j > 1$ is implied by the incentive compatibility constraints and the individual rationality constraint for $j = 1$. $\square$

Lemma A.2 *In the problem (9)-(11), the set of incentive compatibility constraints in (11) can be reduced to:*

$$G(s_{i,j+1}, g_j) - G(s_{ij}, g_j) \leq F(f_i + m_{ij}) - F(f_i + m_{i,j+1}) \leq G(s_{i,j+1}, g_{j+1}) - G(s_{ij}, g_{j+1}), \forall i, \forall j < J.$$

Lemma A.2 states that we only need to ensure the incentive compatibility constraints that prevent adjacent deviations, i.e., a consumer in group j has no incentive to mimic a consumer in group $j + 1$ and vice versa.

Proof of Lemma A.2: Consider indices j , x , and k , such that $1 \leq j < x < k \leq J$. The incentive-compatibility constraints ensure that consumers within the same income group i , but with different resource utility indices, j , x , and k , have no incentive to mimic each other. In what follows, we

prove that the incentive-compatibility constraints between groups (i, j) and (i, x) , together with those between groups (i, x) and (i, k) , imply the incentive-compatibility constraints between groups (i, j) and (i, k) .

First, if consumer (i, j) will not mimic being in group (i, x) and consumer (i, x) will not mimic being in group (i, k) , we must have

$$G(s_{ij}, g_j) + F(f_i + m_{ij}) \geq G(s_{ix}, g_j) + F(f_i + m_{ix}), \quad (28)$$

$$G(s_{ix}, g_x) + F(f_i + m_{ix}) \geq G(s_{ik}, g_x) + F(f_i + m_{ik}). \quad (29)$$

Second, using the same logic as in the proof of Proposition 1, we can show that the incentive-compatibility constraints between groups (i, x) and (i, k) along with the supermodularity of $G(s, g)$ imply that $s_{ix} \leq s_{ik}$. Then, the supermodularity of $G(s, g)$ and $g_j < g_x$ imply that

$$G(s_{ik}, g_x) - G(s_{ix}, g_x) \geq G(s_{ik}, g_j) - G(s_{ix}, g_j). \quad (30)$$

Summing up the three inequalities (28)-(30), we have

$$G(s_{ij}, g_j) + F(f_i + m_{ij}) \geq G(s_{ik}, g_j) + F(f_i + m_{ik}).$$

That is, consumer (i, j) will not mimic being in group (i, k) .

Similarly, we can show that if consumer (i, k) will not mimic being in group (i, x) and consumer (i, x) will not mimic being in group (i, j) , then consumer (i, k) will not mimic being in group (i, j) .

Therefore, the incentive-compatibility constraints between adjacent groups (i, j) and $(i, j + 1)$ for all $j < J$ imply the full set of incentive-compatibility constraints in (11). $\square$

B. Proofs

Proof of Lemma 1: We first prove that the first-best monetary allocation m_{ij}^{FB} (weakly) decreases in the income index i for any given utility index j . For an arbitrary feasible solution to (3)-(7) that does not satisfy this property, i.e., one where $m_{ij} < m_{kj}$ with $i < k$, for some $i, k \in \mathcal{I}$ and some $j \in \mathcal{J}$, we will prove that this solution can be strictly improved.

Consider a consumer in group (i, j) and another consumer in group (k, j) . Because $f_i < f_k$ and $m_{ij} < m_{kj}$, we have $f_i + m_{ij} < f_k + m_{kj}$. We propose a new allocation for these two consumers. (Similarly, we can propose new allocations for all consumers in these two groups, and the remainder of the proof will only require slight modifications to account for θ_{ij} and θ_{kj} .) We consider two cases.

Case 1: Non-binding individual rationality constraint for (k, j) : $G(s_{kj}, g_j) + F(f_k + m_{kj}) > F(f_k)$.

We propose a new monetary allocation, $\hat{m}_{ij} = m_{ij} + \epsilon_1$ and $\hat{m}_{kj} = m_{kj} - \epsilon_1$, where $\epsilon_1 > 0$ is small enough so that consumer (k, j) 's individual rationality constraint remains non-binding and that $f_i + m_{ij} < f_i + \hat{m}_{ij} < f_k + \hat{m}_{kj} < f_k + m_{kj}$. Then, the strict concavity of $F(\cdot)$ implies that

$$F(f_i + \hat{m}_{ij}) + F(f_k + \hat{m}_{kj}) > F(f_i + m_{ij}) + F(f_k + m_{kj}),$$

which strictly improves the sum of the utilities of these two consumers.

Case 2: Binding individual rationality constraint for (k, j) : $G(s_{kj}, g_j) + F(f_k + m_{kj}) = F(f_k)$.

This binding constraint implies $m_{ij} < m_{kj} \leq 0$. Furthermore, we have $s_{ij} > s_{kj}$ because

$$G(s_{ij}, g_j) \geq F(f_i) - F(f_i + m_{ij}) > F(f_k) - F(f_k + m_{ij}) > F(f_k) - F(f_k + m_{kj}) = G(s_{kj}, g_j),$$

where the first inequality is the individual rationality constraint for (i, j) , the second inequality follows from the strict concavity of $F(\cdot)$, and the third inequality follows from $m_{ij} < m_{kj}$.

Consider a new monetary allocation, $\hat{m}_{ij} = m_{ij} + \epsilon_2$ and $\hat{m}_{kj} = m_{kj} - \epsilon_2$, and a new resource allocation, $\hat{s}_{ij} = s_{ij} - \epsilon_3$ and $\hat{s}_{kj} = s_{kj} + \epsilon_3$, satisfying the following properties: consumer (k, j) 's individual rationality constraint remains binding, $G(\hat{s}_{kj}, g_j) + F(f_k + \hat{m}_{kj}) = F(f_k)$, and $\epsilon_2 > 0$ and $\epsilon_3 > 0$ are small enough such that $f_i + m_{ij} < f_i + \hat{m}_{ij} < f_k + \hat{m}_{kj} < f_k + m_{kj}$ and $s_{kj} < \hat{s}_{kj} < \hat{s}_{ij} < s_{ij}$. Then, the strict concavity of $G(\cdot, g_j)$ and $F(\cdot)$ implies that the new allocation changes consumer (i, j) 's resource and financial utilities respectively by:

$$\begin{aligned} G(\hat{s}_{ij}, g_j) - G(s_{ij}, g_j) &> G(s_{kj}, g_j) - G(\hat{s}_{kj}, g_j), \\ F(f_i + \hat{m}_{ij}) - F(f_i + m_{ij}) &> F(f_k + m_{kj}) - F(f_k + \hat{m}_{kj}). \end{aligned}$$

Summing the above two inequalities and noting that the sum on the right-side vanishes to zero because the individual rationality constraint for (k, j) remains binding under the new allocation, we have $G(\hat{s}_{ij}, g_j) + F(f_i + \hat{m}_{ij}) - G(s_{ij}, g_j) - F(f_i + m_{ij}) > 0$, i.e., the utility of consumer (i, j) is strictly higher under the new allocation, while the utility of consumer (k, j) does not change.

In both cases, we have found feasible allocations that strictly improve the total utility. Therefore, the first-best monetary allocation m_{ij}^{FB} must (weakly) decrease in the income index i .

Next, we prove that m_{ij}^{FB} (weakly) decreases in the resource utility index j for any income index i . For an arbitrary feasible solution to (3)-(7) that does not satisfy this property, i.e., one where $m_{ij} < m_{ik}$ with $j < k$, for some $i \in \mathcal{I}$ and $j, k \in \mathcal{J}$, we will prove that this solution can be strictly improved.

Consider a consumer in group (i, j) and another consumer in group (i, k) . We propose a new

allocation for these two consumers.

Case 1: Non-binding individual rationality constraint for (i, k) : $G(s_{ik}, g_k) + F(f_i + m_{ik}) > F(f_i)$.

We propose a new monetary allocation, $\hat{m}_{ij} = m_{ij} + \epsilon_1$ and $\hat{m}_{ik} = m_{ik} - \epsilon_1$, where $\epsilon_1 > 0$ is small enough so that consumer (i, k) 's individual rationality constraint remains non-binding and that $m_{ij} < \hat{m}_{ij} < \hat{m}_{ik} < m_{ik}$. Then, the strict concavity of $F(\cdot)$ implies that

$$F(f_i + \hat{m}_{ij}) + F(f_i + \hat{m}_{ik}) > F(f_i + m_{ij}) + F(f_i + m_{ik}),$$

which strictly improves the original allocation.

Case 2: Binding individual rationality constraint for (i, k) : $G(s_{ik}, g_k) + F(f_i + m_{ik}) = F(f_i)$.

This binding constraint implies $m_{ij} < m_{ik} \leq 0$. Furthermore, we have $s_{ij} > s_{ik}$ because

$$G(s_{ij}, g_j) \geq F(f_i) - F(f_i + m_{ij}) > F(f_i) - F(f_i + m_{ik}) = G(s_{ik}, g_k) > G(s_{ik}, g_j),$$

where the first inequality is the individual rationality constraint for (i, j) , the second inequality follows from $m_{ij} < m_{ik}$, and the last inequality is because $G(s, g)$ strictly increases in g and $g_j < g_k$.

Consider a new monetary allocation, $\hat{m}_{ij} = m_{ij} + \epsilon_2$ and $\hat{m}_{ik} = m_{ik} - \epsilon_2$, and a new resource allocation, $\hat{s}_{ij} = s_{ij} - \epsilon_3$ and $\hat{s}_{ik} = s_{ik} + \epsilon_3$, satisfying the following properties: consumer (i, k) 's individual rationality constraint remains binding, $G(\hat{s}_{ik}, g_k) + F(f_i + \hat{m}_{ik}) = F(f_i)$, and $\epsilon_2 > 0$ and $\epsilon_3 > 0$ are small enough such that $m_{ij} < \hat{m}_{ij} < \hat{m}_{ik} < m_{ik}$ and $s_{ik} < \hat{s}_{ik} < \hat{s}_{ij} < s_{ij}$. Then, the new allocation changes consumer (i, j) 's resource and financial utilities respectively by:

$$G(\hat{s}_{ij}, g_j) - G(s_{ij}, g_j) > G(s_{ik}, g_j) - G(\hat{s}_{ik}, g_j) > G(s_{ik}, g_k) - G(\hat{s}_{ik}, g_k),$$

$$F(f_i + \hat{m}_{ij}) - F(f_i + m_{ij}) > F(f_i + m_{ik}) - F(f_i + \hat{m}_{ik}),$$

where the first and last inequalities are due to the strict concavity of $G(\cdot, g_j)$ and $F(\cdot)$, and the second inequality follows from the supermodularity of $G(s, g)$.

Summing the above inequalities and noting that the sum on the right-side vanishes to zero because the individual rationality constraint for (i, k) remains binding under the new allocation, we have $G(\hat{s}_{ij}, g_j) + F(f_i + \hat{m}_{ij}) - G(s_{ij}, g_j) - F(f_i + m_{ij}) > 0$, i.e., the utility of consumer (i, j) is strictly higher under the new allocation, while the utility of consumer (i, k) does not change.

In both cases, we have found feasible allocations that strictly improve the total utility. Therefore, the first-best monetary allocation m_{ij}^{FB} must (weakly) decrease in j . $\square$

Proof of Lemma 2: (i) We prove the first statement of the lemma by contradiction. Suppose the first-best solution to (3)-(7) satisfies the following: for some $i \in \mathcal{I}$ and $j < k$,

$$G(s_{ij}^{\text{FB}}, g_j) + F(f_i + m_{ij}^{\text{FB}}) > F(f_i) \quad \text{and} \quad G(s_{ik}^{\text{FB}}, g_k) + F(f_i + m_{ik}^{\text{FB}}) = F(f_i). \quad (31)$$

We will prove that such a solution can be strictly improved, contradicting to the fact that it is the first-best.

Consider a consumer in group (i, j) and another consumer in group (i, k) . We propose a new allocation for these two consumers. Since $j < k$, Lemma 1 leads to $m_{ij}^{\text{FB}} \geq m_{ik}^{\text{FB}}$. Thus, we consider the following two cases.

Case 1: $m_{ij}^{\text{FB}} = m_{ik}^{\text{FB}}$. In this case, we must have $G(s_{ij}^{\text{FB}}, g_j) > G(s_{ik}^{\text{FB}}, g_k)$ in view of (31). This implies $s_{ij}^{\text{FB}} > s_{ik}^{\text{FB}}$ because $g_j < g_k$. Therefore,

$$G'_j(s_{ij}^{\text{FB}}) < G'_j(s_{ik}^{\text{FB}}) < G'_k(s_{ik}^{\text{FB}}) \quad (32)$$

where the first inequality is due to the strict concavity of $G_j(s) \equiv G(s, g_j)$ and the second inequality follows from the supermodularity of $G(s, g)$.

We propose a new resource allocation, $\hat{s}_{ij} = s_{ij}^{\text{FB}} - \epsilon$ and $\hat{s}_{ik} = s_{ik}^{\text{FB}} + \epsilon$, where ϵ is small so that consumer (i, j) 's individual rationality constraint remains non-binding and the order of marginal resource utilities in (32) remains unchanged. This new allocation shifts a small amount of resource from consumer (i, j) with a lower marginal utility to consumer (i, k) with a higher marginal utility, thereby strictly improving the original allocation.

Case 2: $m_{ij}^{\text{FB}} > m_{ik}^{\text{FB}}$, which leads to

$$F'(f_i + m_{ij}^{\text{FB}}) < F'(f_i + m_{ik}^{\text{FB}}). \quad (33)$$

In this case, the order of marginal utilities $G'_j(s_{ij}^{\text{FB}})$ and $G'_k(s_{ik}^{\text{FB}})$ gives three subcases:

Case 2a: $G'_j(s_{ij}^{\text{FB}}) < G'_k(s_{ik}^{\text{FB}})$. In this case, the same improvement strategy as in Case 1 applies.

Case 2b: $G'_j(s_{ij}^{\text{FB}}) > G'_k(s_{ik}^{\text{FB}})$. In this case, consumer (i, j) has a higher marginal resource utility but a lower marginal financial utility than consumer (i, k) . Thus, we propose a new allocation, $\hat{s}_{ij} = s_{ij}^{\text{FB}} + \epsilon_1$, $\hat{s}_{ik} = s_{ik}^{\text{FB}} - \epsilon_1$, $\hat{m}_{ij} = m_{ij}^{\text{FB}} - \epsilon_2$, and $\hat{m}_{ik} = m_{ik}^{\text{FB}} + \epsilon_2$, satisfying the following properties: (31) continues to hold, and $\epsilon_1 > 0$ and $\epsilon_2 > 0$ are small enough such that $m_{ij}^{\text{FB}} > \hat{m}_{ij} > \hat{m}_{ik} > m_{ik}^{\text{FB}}$ and $G'_j(s_{ij}^{\text{FB}}) > G'_j(\hat{s}_{ij}) > G'_k(\hat{s}_{ik}) > G'_k(s_{ik}^{\text{FB}})$. Then, the new allocation strictly improves both the sum of financial utilities and the sum of resource utilities.

Case 2c: $G'_j(s_{ij}^{\text{FB}}) = G'_k(s_{ik}^{\text{FB}})$. In this case, we propose the same improvement strategy as in

Case 2b, which leads to a resource utility loss of order $o(\epsilon_1)$ due to $G'_j(s_{ij}^{\text{FB}}) = G'_k(s_{ik}^{\text{FB}})$. Recall that ϵ_1 and ϵ_2 are chosen such that consumer (i, k) 's individual rationality constraint remains binding, implying that ϵ_1 and ϵ_2 are of the same order of magnitude. On the other hand, the improvement strategy improves the financial utilities by an amount of order $O(\epsilon_2)$ in view of (33), which dominates the loss of order $o(\epsilon_1)$ for small enough ϵ_1 and ϵ_2 . Hence, there exist ϵ_1 and ϵ_2 such that the new allocation strictly improves the original allocation.

(ii) The Lagrangian for the first-best allocation problem in (3)-(7) is:

$$\begin{aligned} \mathcal{L} = & N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} (G_j(s_{ij}) + F(f_i + m_{ij})) \theta_{ij} + \lambda \left(S - N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} s_{ij} \theta_{ij} \right) + \sum_{i \in \mathcal{I}, j \in \mathcal{J}} \mu_{ij} s_{ij} \\ & + \sum_{i \in \mathcal{I}, j \in \mathcal{J}} \xi_{ij} \left(G_j(s_{ij}) + F(f_i + m_{ij}) - F(f_i) \right) + \eta \left(M - N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} m_{ij} \theta_{ij} \right), \end{aligned}$$

where $G_j(s) \equiv G(s, g_j)$.

Because the constraints in (4)-(7) define a non-empty (when M is not too negative, as discussed after (7)), convex, and bounded feasible region, and the objective in (3) is strictly concave, the first-best solution exists and is unique. The first-best solution must satisfy the Karush–Kuhn–Tucker (KKT) conditions. We leverage a subset of the KKT conditions for the purpose of this proof.

If at the first-best solution, the individual rationality constraint is non-binding for consumer group (i, j) , then $\xi_{ij} = 0$ and the first-best resource allocation s_{ij}^{FB} must satisfy the following subset of KKT conditions:

$$N\theta_{ij}(G'_j(s_{ij}^{\text{FB}}) - \lambda) + \mu_{ij} = 0, \quad (34)$$

$$\mu_{ij} s_{ij}^{\text{FB}} = 0, \quad (35)$$

$$\mu_{ij} \geq 0, \quad (36)$$

$$s_{ij}^{\text{FB}} \geq 0. \quad (37)$$

If $\lambda < G'_j(0)$, then s_{ij}^{FB} cannot be 0, as zero allocation would violate either (34) or (36). Therefore, in this case, we must have $s_{ij}^{\text{FB}} > 0$, which implies $\mu_{ij} = 0$ due to (35). Hence, s_{ij}^{FB} must satisfy $\lambda = G'_j(s_{ij}^{\text{FB}})$ due to (34).

If $\lambda \geq G'_j(0)$, then $\lambda > G'_j(s)$ for all $s > 0$, because $G'_j(s)$ strictly decreasing in s . In this case, s_{ij}^{FB} cannot be positive, because $s_{ij}^{\text{FB}} > 0$ implies $\lambda > G'_j(s_{ij}^{\text{FB}})$, which requires $\mu_{ij} > 0$ to satisfy (34), which then violates (35). Hence, in this case, $s_{ij}^{\text{FB}} = 0$ and $\mu_{ij} = N\theta_{ij}(\lambda - G'_j(0))$.

To summarize, the solution for s_{ij}^{FB} is:

$$s_{ij}^{\text{FB}} = \begin{cases} G_j'^{-1}(\lambda), & \text{if } \lambda < G_j'(0), \\ 0, & \text{if } \lambda \geq G_j'(0), \end{cases}$$

which is exactly (8).

Clearly, s_{ij}^{FB} depends only on j . To see that s_{ij}^{FB} increases in j , note the following properties. First, if j is small such that $\lambda \geq G_j'(0)$, then $s_{ij}^{\text{FB}} = 0$. Second, $G_j'(s)$ decreases in s (convexity) and increases in j (supermodularity of $G(s, g)$). Thus, its inverse $G_j'^{-1}(\lambda)$ decreases in λ and increases in j . $\square$

Proof of Proposition 1: We prove a stronger result that any feasible solution to (9)-(11) satisfies the conditions in the proposition.

(i) The incentive compatibility constraints in (11) ensure that consumers within the same income group i , but with different resource utility parameters g_j and g_k , have no incentive to mimic each other. That is, $\forall i \in \mathcal{I}, \forall j, k \in \mathcal{J}$, a feasible solution must satisfy:

$$G(s_{ik}, g_j) - G(s_{ij}, g_j) \leq F(f_i + m_{ij}) - F(f_i + m_{ik}) \leq G(s_{ik}, g_k) - G(s_{ij}, g_k). \quad (38)$$

Consider $j < k$ and thus $g_j < g_k$. We aim to prove $s_{ij} \leq s_{ik}$ by contradiction. Suppose $s_{ij} > s_{ik}$ holds. Then, the strict supermodularity of $G(s, g)$ (Assumption 1) implies that

$$G(s_{ik}, g_j) - G(s_{ij}, g_j) > G(s_{ik}, g_k) - G(s_{ij}, g_k),$$

which contradicts (38). Therefore, $s_{ij} \leq s_{ik}$ for $j < k$.

(ii) From part (i), for $j < k$, we have $s_{ij} \leq s_{ik}$ and thus $G(s_{ik}, g_j) - G(s_{ij}, g_j) \geq 0$. Then, (38) implies that $F(f_i + m_{ij}) - F(f_i + m_{ik}) \geq 0$, which holds only if $m_{ij} \geq m_{ik}$. Thus, we conclude that m_{ij} decreases in j for any feasible solution.

(iii) If $s_{ij} < s_{ik}$, we have $G(s_{ik}, g_j) - G(s_{ij}, g_j) > 0$, because $G(s, g)$ strictly increases in s . Then, (38) implies that $F(f_i + m_{ij}) - F(f_i + m_{ik}) > 0$, which holds only if $m_{ij} > m_{ik}$.

Conversely, if $m_{ij} > m_{ik}$, we have $F(f_i + m_{ij}) - F(f_i + m_{ik}) > 0$, since $F(\cdot)$ is a strictly increasing function. Then, the second inequality in (38) implies that $G(s_{ik}, g_k) - G(s_{ij}, g_k) > 0$, which holds only if $s_{ij} < s_{ik}$. $\square$

Proof of Proposition 2: For a given income group $i \in \mathcal{I}$ and a feasible resource allocation $\{s_{ij}, j \in \mathcal{J}\}$, we define

$$L_j = G(s_{i,j+1}, g_j) - G(s_{ij}, g_j), \quad U_j = G(s_{i,j+1}, g_{j+1}) - G(s_{ij}, g_{j+1}), \quad j \in \mathcal{J} \setminus \{J\}.$$

The proof of Proposition 1 shows that the feasible s_{ij} must be (weakly) increasing in j . This, along with the supermodularity of $G(s, g)$, implies $0 \leq L_j \leq U_j$.

Given $\{s_{ij}, j \in \mathcal{J}\}$ and the corresponding L_j 's and U_j 's, we optimize the monetary allocation $\{m_{ij}, j \in \mathcal{J}\}$. Equivalently, we change decision variables and optimize $w_j = F(f_i + m_{ij})$.

The incentive compatibility constraints (14) for the given i can be written as

$$L_j \leq w_j - w_{j+1} \leq U_j, \quad \forall j \in \mathcal{J} \setminus \{J\}. \quad (39)$$

The structural property we aim to prove can be restated as: There exists $\hat{j} \in \mathcal{J} \cup \{0\}$, such that

- (i) $\forall j > \hat{j}, w_j^* - w_{j+1}^* = L_j$;
- (ii) $\forall j < \hat{j}, w_j^* - w_{j+1}^* = U_j$.

Note that part (iii) of the property in Proposition 2 is implied by (i) and (ii).

If a feasible monetary allocation does not have the above structure, it must have the following property: There exist k and l with $1 \leq k < l < J$, such that

$$w_k - w_{k+1} < U_k \quad \text{and} \quad w_l - w_{l+1} > L_l. \quad (40)$$

In what follows, we show that such a solution can be strictly improved.

We construct another feasible solution as follows:

$$\tilde{w}_j = w_j, \quad \text{for } j = 1, \dots, k, \quad (41)$$

$$\tilde{w}_j = w_j + \delta_n, \quad \text{for } j = k+1, \dots, l, \quad (42)$$

$$\tilde{w}_j = w_j + \delta_p, \quad \text{for } j = l+1, \dots, J, \quad (43)$$

where $\delta_n < 0$ and $\delta_p > 0$ are chosen so that the solution remains feasible, as detailed below. First, the individual rationality constraint (13) is satisfied, because w_1 remains unchanged by (41). Second, to verify that $\tilde{w}_j$'s satisfy the incentive compatibility constraints (14) or (39), note that $\tilde{w}_j - \tilde{w}_{j+1} = w_j - w_{j+1}$ for $j \neq k, l$, so we only need to check (39) for $j = k, l$. Because of (40), we can choose $\delta_n < 0$ and $\delta_p > 0$ small enough so that (39) remains satisfied:

$$\tilde{w}_k - \tilde{w}_{k+1} = w_k - w_{k+1} - \delta_n < U_k,$$

$$\tilde{w}_l - \tilde{w}_{l+1} = w_l - w_{l+1} + \delta_n - \delta_p > L_l.$$

Finally, satisfying the budget constraint (5) by changing $\{m_{ij}, j \in \mathcal{J}\}$ for the given i requires keeping $\sum_{j \in \mathcal{J}} m_{ij} \theta_{ij}$ constant. Since $w_j = F(f_i + m_{ij})$, we equivalently keep $\sum_{j \in \mathcal{J}} (H(w_j) - f_i) \theta_{ij}$ constant, where $H(w) = F^{-1}(w)$, or equivalently, we must ensure

$$\sum_{j \in \mathcal{J}} H(\tilde{w}_j) \theta_{ij} = \sum_{j \in \mathcal{J}} H(w_j) \theta_{ij}. \quad (44)$$

Assumption 1 ensures sufficient differentiability for applying Taylor's expansion: $H(\tilde{w}_j) = H(w_j) + \delta_n H'(w_j) + O(\delta_n^2), \forall j = k+1, \dots, l$, and $H(\tilde{w}_j) = H(w_j) + \delta_p H'(w_j) + O(\delta_p^2), \forall j = l+1, \dots, J$. Substituting the expanded expressions into (44), canceling terms on both sides, we have

$$\delta_n \sum_{j=k+1}^l H'(w_j) \theta_{ij} + \delta_p \sum_{j=l+1}^J H'(w_j) \theta_{ij} = O(\delta_p^2).$$

Therefore, we can choose a small value of $\delta_p > 0$ and set

$$\delta_n = -\delta_p \frac{\sum_{j=l+1}^J H'(w_j) \theta_{ij}}{\sum_{j=k+1}^l H'(w_j) \theta_{ij}} + O(\delta_p^2),$$

so that the values $\{\tilde{w}_j : j \in \mathcal{J}\}$ in (41)-(43) correspond to a feasible monetary allocation.

It remains to be shown that $\{\tilde{w}_j : j \in \mathcal{J}\}$ strictly improves the objective (12) under $\{w_j : j \in \mathcal{J}\}$. Note that this perturbation affects the objective only through $\sum_{j \in \mathcal{J}} w_j \theta_{ij}$ for the given i . The perturbed value of this term is:

$$\sum_{j=1}^J \tilde{w}_j \theta_{ij} = \sum_{j=1}^k w_j \theta_{ij} + \sum_{j=k+1}^l (w_j + \delta_n) \theta_{ij} + \sum_{j=l+1}^J (w_j + \delta_p) \theta_{ij} = \sum_{j=1}^J w_j \theta_{ij} + \delta_n \sum_{j=k+1}^l \theta_{ij} + \delta_p \sum_{j=l+1}^J \theta_{ij}.$$

Thus, the change in the objective value is

$$\delta_n \sum_{j=k+1}^l \theta_{ij} + \delta_p \sum_{j=l+1}^J \theta_{ij} = -\delta_p \frac{\sum_{j=l+1}^J H'(w_j) \theta_{ij}}{\sum_{j=k+1}^l H'(w_j) \theta_{ij}} \sum_{j=k+1}^l \theta_{ij} + \delta_p \sum_{j=l+1}^J \theta_{ij} + O(\delta_p^2). \quad (45)$$

Because $H(w) = F^{-1}(w)$ is strictly convex and increasing in w , and w_j is (weakly) decreasing in j ($w_j - w_{j+1} \geq L_j \geq 0$ in (39)), we have $H'(w_j) \geq H'(w_{j+1})$ for all $j < J$. Furthermore, the strict inequality holds at $j = l$, i.e., $H'(w_l) > H'(w_{l+1})$, because $w_l - w_{l+1} > L_l$ is the assumed property of the feasible allocation and $L_l \geq 0$. Therefore,

$$\frac{\sum_{j=l+1}^J H'(w_j) \theta_{ij}}{\sum_{j=k+1}^l H'(w_j) \theta_{ij}} \leq \frac{H'(w_{l+1}) \sum_{j=l+1}^J \theta_{ij}}{H'(w_l) \sum_{j=k+1}^l \theta_{ij}} < \frac{\sum_{j=l+1}^J \theta_{ij}}{\sum_{j=k+1}^l \theta_{ij}}. \quad (46)$$

Inequality (46) implies that the value in (45) is positive when δ_p is small enough. Therefore, $\{w_j : j \in \mathcal{J}\}$ satisfying $w_k - w_{k+1} < U_k$ and $w_l - w_{l+1} > L_l$ with $k < l$ cannot be optimal. This completes the proof of the properties (i)-(iii) stated in the proposition.

The final statement in the proposition is that if the individual rationality constraint is not binding for a given $i \in \mathcal{I}$, then property (i) holds for the given i and $\forall j \in \mathcal{J}$. We prove this statement by contradiction. Suppose the optimal solution is such that $F(f_i + m_{ij}^*) - F(f_i + m_{i,j+1}^*) > G(s_{i,j+1}^*, g_j) - G(s_{ij}^*, g_j)$ for some $j \in \mathcal{J} \setminus \{J\}$. Then, we can construct a solution such that

$$\tilde{m}_{ij'} = m_{ij'}^* - \epsilon_1, \text{ for } j' \leq j, \quad \text{and} \quad \tilde{m}_{ij'} = m_{ij'}^* + \epsilon_2, \text{ for } j' > j,$$

where $\epsilon_1 > 0$ and $\epsilon_2 > 0$ are small enough so that this new solution satisfies the individual rationality constraint in (13) and the incentive compatibility constraints in (14). In addition, ϵ_1 and ϵ_2 are such that the budget constraint in (5) is satisfied: $\sum_{j' \in \mathcal{J}} \tilde{m}_{ij'} \theta_{ij'} = \sum_{j' \in \mathcal{J}} m_{ij'}^* \theta_{ij'}$. Then, by the Karamata's inequality and the strict concavity of the financial utility function, we can conclude that

$$\sum_{j' \in \mathcal{J}} F(f_i + \tilde{m}_{ij'}) \theta_{ij} > \sum_{j' \in \mathcal{J}} F(f_i + m_{ij'}^*) \theta_{ij}.$$

Note that the original Karamata's inequality applies to cases with equal weights, but its generalization to unequal weights is intuitive. This can be seen by refining the unequal weights into many small, equal weights. A formal proof can follow the structure of the original proof of Karamata's inequality. $\square$

Proof of Lemma 3: Consider any two different pairs (S^a, M^a) and (S^b, M^b) for which the problem (15)-(20) is feasible. Because the problem is a convex optimization on a compact convex set, the optimal solution exists, which is denoted as $\{v_{ij}^a, w_{ij}^a\}$ and $\{v_{ij}^b, w_{ij}^b\}$, respectively.

Next, we show that the problem (15)-(20) with $(S, M) = (\frac{S^a + S^b}{2}, \frac{M^a + M^b}{2})$ is feasible and a feasible solution is $\{v_{ij}^m, w_{ij}^m\} = \{\frac{v_{ij}^a + v_{ij}^b}{2}, \frac{w_{ij}^a + w_{ij}^b}{2}\}$. It is immediate to see that $\{v_{ij}^m, w_{ij}^m\}$ satisfy the linear constraints (18)-(20). It remains to verify the nonlinear constraints (16) and (17):

$$\begin{aligned} N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} \bar{G}^{-1}(\frac{1}{2}(v_{ij}^a + v_{ij}^b)/g_j) \theta_{ij} &\leq N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} \frac{1}{2}(\bar{G}^{-1}(v_{ij}^a/g_j) + \bar{G}^{-1}(v_{ij}^b/g_j)) \theta_{ij} = S, \\ N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} (F^{-1}(\frac{1}{2}(w_{ij}^a + w_{ij}^b)) - f_i) \theta_{ij} &\leq N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} \frac{1}{2}(F^{-1}(w_{ij}^a) - f_i + F^{-1}(w_{ij}^b) - f_i) \theta_{ij} = M, \end{aligned}$$

where the inequalities follow from the convexity of $\bar{G}^{-1}(\cdot)$ and $F^{-1}(\cdot)$, and the equalities are due to the optimality of $\{v_{ij}^a, w_{ij}^a\}$ and $\{v_{ij}^b, w_{ij}^b\}$. Therefore, we have

$$W_{\mathcal{M}}(\frac{S^a + S^b}{2}, \frac{M^a + M^b}{2}) \geq N \sum_{i \in \mathcal{I}, j \in \mathcal{J}} \frac{1}{2}(v_{ij}^a + v_{ij}^b + w_{ij}^a + w_{ij}^b) \theta_{ij} = \frac{1}{2}(W_{\mathcal{M}}(S^a, M^a) + W_{\mathcal{M}}(S^b, M^b)),$$

where the inequality follows from the feasibility of $\{v_{ij}^m, w_{ij}^m\}$ and the equality is due to the optimality

of $\{v_{ij}^a, w_{ij}^a\}$ and $\{v_{ij}^b, w_{ij}^b\}$. $\square$

Proof of Lemma 4: In the lower-level problem, we allocate S_i and M_i within income group i that has a population of $N\theta_i$, which is a special case of (15)-(20) with one income group. Therefore, the optimal aggregate welfare of income group i , $W_{\mathcal{M}_i}(S_i, M_i)$, is concave in (S_i, M_i) according to Lemma 3. Hence, the objective function in (21) is concave in $(S_i, M_i : i \in \mathcal{I})$. $\square$

Proof of Proposition 3: (i) The resource allocation problem is:

$$\max_{\{s_i \geq 0\}} \sum_{i \in \mathcal{I}, j \in \mathcal{J}} G(s_i, g_j) \theta_{ij}, \quad \text{subject to: } N \sum_{i \in \mathcal{I}} s_i \theta_i = S.$$

Consider a feasible solution in which allocation are not equal, i.e., $s_i < s_k$ for some $i, k \in \mathcal{I}$. Define $\bar{s} = \frac{s_i \theta_i + s_k \theta_k}{\theta_i + \theta_k}$. Replacing s_i and s_k by $\bar{s}$ clearly respects the resource constraint $\sum_{i \in \mathcal{I}} s_i \theta_i = S$. Because $\bar{s}$ is a convex combination of s_i and s_k , the strict concavity of $G(\cdot, g_j)$ implies

$$G(\bar{s}, g_j) > G(s_i, g_j) \frac{\theta_i}{\theta_i + \theta_k} + G(s_k, g_j) \frac{\theta_k}{\theta_i + \theta_k}, \quad \forall j \in \mathcal{J}. \quad (47)$$

By the proposition's assumption, we have that $\theta_{ij}/\theta_{kj} = \theta_i/\theta_k$. Thus, (47) implies

$$G(\bar{s}, g_j) (\theta_{ij} + \theta_{kj}) > G(s_i, g_j) \theta_{ij} + G(s_k, g_j) \theta_{kj}, \quad \forall j \in \mathcal{J}.$$

Summing over j , we conclude that the solution with $s_i < s_k$ can be strictly improved because

$$\sum_{j \in \mathcal{J}} G(\bar{s}, g_j) \theta_{ij} + \sum_{j \in \mathcal{J}} G(\bar{s}, g_j) \theta_{kj} > \sum_{j \in \mathcal{J}} G(s_i, g_j) \theta_{ij} + \sum_{j \in \mathcal{J}} G(s_k, g_j) \theta_{kj}.$$

Therefore, the optimal allocation are equal.

(ii) The cost allocation problem is:

$$\max_{\{m_i \geq 0\}} \sum_{i \in \mathcal{I}} F(f_i + m_i) \theta_i, \quad \text{subject to: } N \sum_{i \in \mathcal{I}} m_i \theta_i = M.$$

The proof is parallel to part (i) and abbreviated. If a feasible solution has $m_i < m_k$ for $i < k$, then increasing m_i and decreasing m_k while maintaining the budget constraint will strictly improve the objective value due to the strict concavity of the financial utility function $F(\cdot)$. Thus, at optimality, m_i must decrease in i . $\square$