UNIVERSAL SEQUENCES OF LINES IN $\mathbb{R}^d$

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Abstract. What is a universal sequence of oriented and unoriented lines in d-space? Here we give partial answers to this question, and to the analogous one for k-flats.

1. Introduction

Assume $a_1, \ldots, a_N$ is a sequence of points in $\mathbb{R}^d$ in general position meaning that no $d+1$ of these $N \geq d+1$ points lie on a hyperplane. In other words, for every subsequence $1 \leq i_1 < \ldots < i_{d+1} \leq N$ the determinant of the $(d+1) \times (d+1)$ matrix

$$\begin{pmatrix} a_{i_1} & a_{i_2} & \cdots & a_{i_{d+1}} \\ 1 & 1 & 1 \cdots & 1 \end{pmatrix} \tag{1.1}$$

is different from zero. Write $\det(a, i_1, \ldots, i_{d+1})$ for this determinant. Observe that the general position condition simply means that the $d+1$ tuples of the sequence are outside the zero set of $m = \binom{N}{d+1}$ polynomials $p_1, \ldots, p_m$. These polynomials split the set of sequences of N points into finitely many cells, where a cell is the set of point sequences of length N such that, for every $i \in [m]$, the sign of p_i is constant, $+1$ or -1, on the sequence.

The m signs of these polynomials are referred to as the *order-type* of the sequence $a_1, a_2, \ldots, a_N$. Moreover it is known that these polynomials split the sets of sequences of N points into finitely many connected components as well; their number is finite according to a famous theorem of Oleinik–Petrovskii [9], Milnor [7], and Thom [14]. As a matter of fact, the upper bounds given by these authors imply upper bounds on the number of order-types, much below the immediate bound 2^m. For more details see Goodman and Pollack [5] and Alon [1]. On the other hand, it is known that there is no upper bound on the number of connected components described by a single order type, and, in fact, a single order type can be as topologically complicated as essentially any algebraic variety, see Mnëv [8] and Richter-Gebert [13].

A useful and famous (and probably folklore) result says that for every d and $n \geq d+1$ there is an $N = N(d, n)$ such that the following holds. Every sequence $a_1, \ldots, a_N$ of points in $\mathbb{R}^d$ in general position contains a

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homogeneous subsequence $b_1, \dots, b_n$ of length n; homogeneous meaning that the determinants $\det(b, i_1, \dots, i_{d+1})$ have the same sign for all sequences $1 \le i_1 < \dots < i_{d+1} \le n$. Subsequence means, as usual, that $b_j = a_{k_j}$ for all $j \in [n]$ where $1 \le k_1 < \dots < k_n \le N$. Here $[n]$ denotes the set $\{1, 2, \dots, n\}$. This result actually follows from Ramsey theorem [12]: the $d+1$ tuples of our sequence are coloured by $+1$ or by -1, so there is a large subsequence all of whose $d+1$ tuples are of the same colour. More precisely, for every n there is an N such that every $d+1$ tuple of an n element subsequence of $a_1, \dots, a_N$ carries the same colour. When $d = 2$ this method gives a proof of the famous Erdős–Szekeres theorem, see [3] and [4], but with weaker bounds on N. We remark that in this paper an r tuple of points (or objects) always means an ordered r tuple, that is, a sequence of points (or objects) of length r.

Assume $0 < t_1 < \dots < t_n$ and consider the sequence of points $\gamma(t_1), \dots, \gamma(t_n)$ from the moment curve $\gamma(t) = (t, t^2, \dots, t^d), t \in \mathbb{R}$. Observe that the determinant $\det(\gamma(t), i_1, \dots, i_{d+1})$ is positive for every sequence $1 \le i_1 < \dots < i_{d+1} \le n$. Set $\gamma^*(t) = (t, t^2, \dots, t^{d-1}, -t^d)$. Analogously for the sequence $\gamma^*(t_1), \dots, \gamma^*(t_n)$ the corresponding determinants are all negative.

The property that all determinants have the same sign is *universal*: for every n, every long enough sequence of points in $\mathbb{R}^d$ has a subsequence of length n with this property. The two examples with γ and γ^* show that there are exactly two kinds of *universal sequences*. In the language of polynomials and cells, universality says that our subsequence lies in a cell C, that is, there is a subsequence $b_1, \dots, b_n$ such that C is on the positive side of each polynomial $\det(b, i_1, \dots, i_{d+1})$, or on the negative side of each such polynomial.

Here comes another example. Again, let $a_1, \dots, a_N$ be a sequence of points in $\mathbb{R}^d$ in general position. Consider a subsequence $b_1, \dots, b_{d+2}$ and set $B = \{b_1, \dots, b_{d+2}\}$. Note B has a unique Radon partition [11], that is $[d+2] = X \cup Y$ with $X, Y \neq \emptyset$ and X, Y are disjoint and $\operatorname{conv}\{b_i : i \in X\} \cap \operatorname{conv}\{b_j : j \in Y\} \neq \emptyset$, for concreteness we assume $1 \in X$. So each subsequence $b_1, \dots, b_{d+2}$ defines a subset $X \subset [d+2]$ (with $1 \in X \neq [d+2]$). Ramsey theorem applies again and implies that, for N large enough, there is a subsequence $b_1, \dots, b_n$ of the a_i such that X is the same subset of $[d+2]$ for every subsequence $c_1, \dots, c_{d+2}$ of the b_i sequence. What is the universal type or what are the universal types for this *Radon property*? It turns out that the answer is simple: X and Y are interlacing subsets of $[d+2]$, that is, $X = \{1, 3, 5, \dots\}$ and $Y = \{2, 4, \dots\}$. The proof is left to the interested reader. One can check that in the above examples with $\gamma(t)$ and $\gamma^*(t)$ the universal sequences indeed have interlacing Radon partitions.

2. Definition of universality

What is a universal sequence of lines, and of oriented lines in $\mathbb{R}^d$? How many types are there of them? This is the main topic in this article. We have some partial results for the case of lines and also for the same question with k-flats in $\mathbb{R}^d$. The definition of universality (or rather its metadefinition) requires four conditions or steps.

(i) First we define when an ordered r tuple of k-flats is in *general position*. This usually means that they do not lie in the zero set of a finite number of well-defined polynomials that determine cells in the space of r tuples of k-flats. Here we also require that there are at least two such cells.

(ii) Next we define a *property* of an ordered s tuple $A_1, \dots, A_s$ of k-flats whose r tuples are in general position, of course $s \geq r$. This property is a function F on s tuples of k-flats that take values in a finite set M. Elements of M will be called *types*. In our first example this finite set M is $\{1, -1\}$ and in the second M is the family of all $X \subset [d+2]$ with $1 \in X \neq [d+2]$.

(iii) Assume $B_1, \dots, B_n$ is a sequence of k-flats (oriented or unoriented) in $\mathbb{R}^d$ all of whose r tuples are in general position. F maps subsequences of length $s \geq r$ of this sequence to elements of M. The sequence $B_1, \dots, B_n$ is called *homogeneous* relative to *property* F if F maps all of its s tuples to the same element (or type) $m \in M$. A type $m \in M$ is *universal* if, for every $n \geq s$ there is a homogeneous sequence $B_1, \dots, B_n$ whose s tuples are all mapped to m.

(iv) Next comes Ramsey: For every $n \geq s$ there is N such that the following holds. Assume $A_1, \dots, A_N$ is a sequence of k-flats in $\mathbb{R}^d$ all of whose r tuples are in general position. As M is finite, the Ramsey theorem implies the existence of a homogeneous subsequence $B_1, \dots, B_n$ of the sequence $A_1, \dots, A_N$. Of course N depends on n, r, s and the map F as well. The finiteness of M implies further that there is a universal element m in M. The corresponding sequences $B_1, \dots, B_n$ are called *universal* of type m (relative to *property* F).

The question is what types $m \in M$ are universal, how many of them are there, and how the universal sequences look like.

3. Oriented lines

We are going to use some notation from exterior algebra, for instance $u_1 \wedge \ldots \wedge u_d$ is the determinant of the matrix whose columns are the vectors $u_i \in \mathbb{R}^d, i \in [d]$, and $u_1 \wedge \ldots \wedge u_{d-1}$ is a vector in $\mathbb{R}^d$, the wedge product of the u_is.

An oriented line L in $\mathbb{R}^d$ $d \geq 2$ is given by a pair (a, v) with $a, v \in \mathbb{R}^d$ and $v \neq 0$ and $L = \{a + tv : t \in \mathbb{R}\}$. In fact it is the equivalence class of such pairs where (a, v) and $(\bar{a}, \bar{v})$ are equivalent (or represent the same line) if $\bar{a} = a + \alpha v$ for some $\alpha \in \mathbb{R}$ and $\bar{v} = \beta v$ with $\beta > 0$. An ordered

$d-1$ tuple of lines $L_1, \dots, L_{d-1}$, with L_i represented by (a_i, v_i) is in *general position* if the numbers $h_i := a_i \wedge v_1 \wedge \dots \wedge v_{d-1}$, $i \in [d-1]$ are all distinct. General position then means that the system $(a_i, v_i), i \in [d-1]$ avoids the zero set of the polynomials $(a_i - a_j) \wedge v_1 \wedge \dots \wedge v_{d-1}$, for distinct $i, j \in [d-1]$. In particular, $u = v_1 \wedge \dots \wedge v_{d-1}$ is a non-zero vector in $\mathbb{R}^d$. We mention that $h_i = u \cdot v_i$, scalar product.

The $d-1$ real numbers h_i come in increasing order as $h_{j_1} < h_{j_2} < \dots < h_{j_{d-1}}$. They define a permutation σ of $[d-1]$ via $\sigma(i) = j_i$. As is easy to check this permutation σ does not depend on the choice of the pair (a_i, v_i) representing L_i: σ depends only on the $d-1$ tuple $L_1, \dots, L_{d-1}$. Observe that every permutation of $[d-1]$ can occur. Here we need $d \geq 3$ as for $d = 2$ there is only one h.

Another way to see the permutation σ is to consider the hyperplane $H = \frac{1}{d-1}(L_1 + \dots + L_{n-1})$ which is the Minkowski average of the lines. Because of the general position assumption H is indeed a hyperplane, its outer normal is $u = v_1 \wedge \dots \wedge v_{d-1} \neq 0$. A suitably translated copy, say H_i of H contains L_i. The hyperplanes $H_1, \dots, H_{d-1}$ intersect the line whose direction is u in order σ in distinct points.

This time F maps $L_1, \dots, L_{d-1}$ to σ and the set of permutations is finite, of size $(d-1)!$. Ramsey theorem applies and gives the following.

Theorem 3.1. *For integers $n \geq d \geq 3$ there is a number N such that every sequence $L_1, \dots, L_N$ of oriented lines in $\mathbb{R}^d$ whose $d-1$ tuples are in general position contains a homogeneous subsequence $K_1, \dots, K_n$ of type σ for some permutation σ of $[d-1]$. In other words F maps every $d-1$ tuple of $K_1, \dots, K_n$ to the same type σ.* □

Corollary 3.1. *For oriented lines in $\mathbb{R}^d$ there are universal types, that is permutations σ of $[d-1]$. There are at most $(d-1)!$ of them.* □

This implies in particular, that in $\mathbb{R}^3$ there are at most two types. In fact there are exactly two types in $\mathbb{R}^3$. This can be seen from the example of the one sheeted hyperboloid whose equation is $x^2 + y^2 = z^2 + 1$. This hyperboloid contains two sets of lines, see Figure 1. Given a line L on the hyperboloid, it is associated with the pair (a, v). Orient L by requiring that the z component of v is positive. The two sets of lines are shown in Figure 1.

The question is how many of the possible $(d-1)!$ types are universal. A partial answer is the content of the next result.

Theorem 3.2. *There are at least $2^{d-1} - 2$ different universal permutations of $[d-1]$.*

The proof is in Section 7. It uses rapidly increasing sequences and RI matrices that are related to the stretched grid and to the stretched diagonal, that come from a paper by Bukh, Nivasch, Matoušek [2], and are also connected to a construction of Pór [10]. The necessary background is given in Section 6.

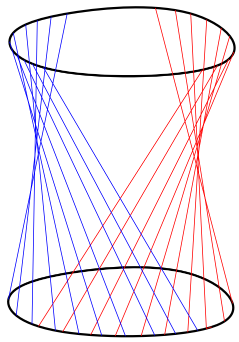

FIGURE 1. Two sets of universal oriented lines in $\mathbb{R}^3$.

We remark here that the red lines in Figure 1 form a *continuous and universal family* of lines. More precisely, define $a(t) = (\cos t, \sin t, 0)$ and $v(t) = (-\sin t, \cos t, 1)$. Let $L(t)$ be the oriented line given by the pair $(a(t), v(t)), t \in [0, \pi)$. When $0 < t_1 < t_2 < \ldots < t_n < \pi$ the sequence of lines $L(t_1), \ldots, L(t_n)$ is homogeneous of type identity, as one can check directly. That's why we call the family of lines $L(t), t \in [0, \pi)$ a continuous and universal family of lines. Section 9 gives the proper definition, and examples of such families in every dimension. In Section 10 we prove that the type of such a family is either the identity or its reverse.

4. Unoriented lines

The setting with unoriented lines (or simply lines) in $\mathbb{R}^d$ is similar to the oriented ones. But for instance in $\mathbb{R}^3$ there are pairs (L_1, L_2) and (K_1, K_2) of oriented lines in general position that belong to distinct cells (defined in this case by a single polynomial), see Figure 2, they are of distinct types. This does not hold for lines: these pairs (when unoriented) belong to the same cell. In fact, they can be carried to each other by a homotopy through general position pairs. This is not the case for triples of lines. That is, there are triples (L_1, L_2, L_3) and (K_1, K_2, K_3) of lines in $\mathbb{R}^3$ (in general position) such that belong to distinct cells, as we shall see soon.

A line L in $\mathbb{R}^d$ is given by a pair (a, v) with $a, v \in \mathbb{R}^d$ and $v \neq 0$ and $L = \{a + tv : t \in \mathbb{R}\}$. In fact it is the equivalence class of such pairs where (a, v) and $(\bar{a}, \bar{v})$ are equivalent (or represent the same line) if $\bar{a} = a + \alpha v$ for some $\alpha \in \mathbb{R}$ and $\bar{v} = \beta v$ with $\beta \neq 0$. Given an ordered d tuple of lines $L_1, \ldots, L_d$, with L_i represented by (a_i, v_i), define $u_j = v_1 \wedge \ldots \wedge v_d$ where v_j is missing from the wedge product. The d tuple $L_1, \ldots, L_d$ is in *general position* if, for every $j \in [d]$, the numbers $h_{j,i} := a_i \wedge v_j$, $i \in [d] \setminus \{j\}$ are all distinct. General position again means that the system $(a_i, v_i), i \in [d]$ avoids the zero set of certain polynomials. Note that we assume here $d > 2$.

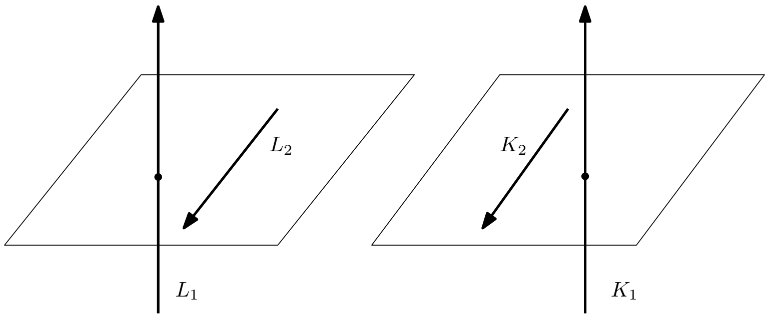

FIGURE 2. Non-homotopic pairs of oriented lines in $\mathbb{R}^3$.

For every $j \in [d]$ the numbers $h_{j,i}, i \in [d] \setminus \{j\}$ define a permutation σ_j of $[d] \setminus \{j\}$, namely, $h_{j,\sigma_j(1)} < h_{j,\sigma_j(2)} < \ldots < h_{j,\sigma_j(d)}$ where again $h_{\sigma_j(j)}$ is not defined and is missing from the list. Note that L_i is represented by both (a_i, v_i) and $(a_i, -v_i)$. Consequently $\sigma_j = (\sigma_j(1), \ldots, \sigma_j(d))$ and $\sigma_j^* = (\sigma_j(d), \ldots, \sigma_j(1))$ represent the same ordering, of course $\sigma_j(j)$ is missing again.

So the map F for universality associates with $L_1, \ldots, L_d$ d pairs of permutations $\{\sigma_1, \sigma_1^*\}, \ldots, \{\sigma_d, \sigma_d^*\}$. The values of F are from a finite set M, of size $((d-1)!/2)^d$, so Ramsey theorem works again:

Theorem 4.1. *For integers n, d with $n \geq d \geq 3$ there is a number N such that every sequence $L_1, \ldots, L_N$ of lines in $\mathbb{R}^d$ whose d tuples are in general position contains a homogeneous subsequence $K_1, \ldots, K_n$. In other words, F maps every d tuple $K_{i_1}, \ldots, K_{i_d}$ to the same type $\{\sigma_1, \sigma_1^*\}, \ldots, \{\sigma_d, \sigma_d^*\}$.* □

A direct corollary is that there are universal types for unoriented lines in $\mathbb{R}^d$ and that there are universal sequences of unoriented lines of length n for every $n \geq d$. Their number is at most $((d-1)!/2)^d$. We are going to reduce this number to $(d-1)!/2$. Observe first that σ_j is a permutation of $[d] \setminus \{j\}$ which is a linearly ordered set of $d-1$ elements. So we can consider σ_j a permutation of $[d-1]$, and σ_j^* is its reverse permutation.

Lemma 4.2. *Assume $L_1, \ldots, L_n$ is a homogeneous sequence of lines in $\mathbb{R}^d$, $n > d \geq 3$ with permutation pairs $\{\sigma_j, \sigma_j^*\}$ of $[d-1]$ for every $j \in [d]$. Then $\{\sigma_1, \sigma_1^*\} = \{\sigma_2, \sigma_2^*\} = \ldots = \{\sigma_d, \sigma_d^*\}$.*

Proof. It suffices to consider $n = d+1$. We show that $\sigma_1 = \sigma_d$. For the ordered d tuple $L_1, \ldots, L_d$ $u_1 = v_2 \wedge \ldots \wedge v_d$, and the permutation σ_1 (of $[d-1]$) is determined by the increasing rearrangement of the numbers $a_2 \wedge u_1, a_3 \wedge u_1, \ldots, a_d \wedge u_1$. We check the permutation σ_d for the ordered d tuple $L_2, \ldots, L_{d+1}$. The corresponding u vector is exactly the previous $u_1 = v_2 \wedge \ldots \wedge v_d$ because the last vector v_{d+1} has to be deleted. So σ_d is given by the increasing rearrangement of the same numbers $a_2 \wedge u_1, a_3 \wedge u_1, \dots, a_d \wedge u_1$. Then $\sigma_1 = \sigma_d$ indeed. The proof of the other cases $\sigma_1 = \sigma_j$, $j < d$ is identical. □

Theorem 4.3. *The number of universal types for unoriented lines in $\mathbb{R}^d$ is at least $2^{d-2} - 1$.*

Remark. Assume that $L_1, \dots, L_n$ $(n \geq d)$ is a universal sequence of oriented lines of type σ, a permutation of $[d-1]$. Forgetting their orientation, the same sequence of (unoriented) lines becomes a universal sequence of unoriented lines. Indeed, every $d-1$ tuple of the lines is ordered by σ or σ^* as one can check directly.

5. Universal k-flats when $d \equiv 1 \bmod k$

Next we consider oriented k-flats in $\mathbb{R}^d$ under the condition that $d \equiv 1 \bmod k$, or, in different form $d = rk + 1$ with $r \geq 2$ and integer. In this case the method we used for lines works so we only give a sketch. An oriented k-flat A is given by a pair (a, B) where $a \in \mathbb{R}^d$ and B is an ordered set of k linearly independent vectors: $B = (v_1, \dots, v_k)$. The linear span of B, lin B, is defined as

$$\text{lin } B = \left\{ \sum_1^k \alpha_i v_i : \alpha_i \in \mathbb{R} \text{ for all } i \in [k] \right\}.$$

Then $A = a + \text{lin } B$. In fact A is given by an equivalence class of such pairs where (a, B) and $(\bar{a}, \bar{B})$ are equivalent if $\bar{a} = a + \sum_1^k \alpha_i v_i$ with some real numbers α_i, and if there is a linear transformation $T : \text{lin } B \to \text{lin } B$ with positive determinant that carries the basis of B to the basis of $\bar{B}$, that is $Tv_i = \bar{v_i}$ for $i \in [k]$, where $\bar{B} = (\bar{v_1}, \dots, \bar{v_k})$. Define $u(B) = v_1 \wedge \ldots \wedge v_k$.

Given an r tuple $A_1, \dots, A_r$ of k flats with A_i represented by (a_i, B_i) set

$$u = u(B_1) \wedge \ldots \wedge u(B_r).$$

The condition $d \equiv 1 \bmod k$ implies that u is a vector in $\mathbb{R}^d$. The r tuple $A_1, \dots, A_r$ is in *general position* if the numbers $h_i := a_i \wedge u$, $i \in [r]$ are all distinct. General position then means that the system $(a_i, B_i), i \in [r]$ avoids the zero set of certain polynomials: $(a_i - a_j) \wedge u$, for distinct $i, j \in [r]$, in particular $u \neq 0$. The increasing rearrangement of the numbers $h_1, \dots, h_r$ defines a permutation π of $[r]$. We observe that π depends only on the r tuple $A_1, \dots, A_r$ and not on the representation by (a_i, B_i) of the k-flats A_i.

Same way as before F maps the r tuple $A_1, \dots, A_r$ to the permutation π, and the universality scheme gives the following result.

Theorem 5.1. *Assume $d = kr+1$ where d, k, r are integers, $r \geq 2, k \geq 1$. There is a universal permutation π of $[r]$. This means that for every integer $n > r$ there is a number N such that every sequence $A_1, \dots, A_N$ of oriented k-flats in $\mathbb{R}^d$ whose r tuples are in general position contains* *a subsequence* $D_1, \dots, D_n$ *such that* F *maps every* r *tuple* $D_{i_1}, \dots, D_{i_r}$ *to the permutation* π. □

So under the above conditions there are universal sequences of k-flats. Evidently, there are at most $r!$ different types of them. A straightforward modification of the proof of Theorem 3.2 shows that their number is at least $2^r - 2$.

6. Rapidly increasing entries

Let M be a $D \times m$ matrix with entry $a(i,j)$ in row i and column j, we assume D is fixed and m is large, much larger than D. The entries in M are *rapidly increasing* if every $a(i,j) \geq 1$ is an integer and, for fixed $i \in [D]$, $a(i,j+1)$ is much larger than $a(i,j)$ for all $j \in [m-1]$, and further, $a(i+1,1)$ is larger than $a(i,m)$. Such a matrix is called an RI matrix. Similar matrices with various purposes were constructed by Bukh, Nivasch, and Matoušek [2] and by Pór [10].

The main feature of an RI matrix is that if M^* is a $k \times k$ submatrix of M (here $k \in [D]$ and $k \geq 2$), then $\det M^*$ is essentially equal to the product of the entries on the main diagonal of M^*. The meaning of "essentially equal" is made precise the following way. Given a small $\varepsilon > 0$, the "much larger" in the definition of an RI matrix can be chosen so large that, with P denoting the product of the entries on the main diagonal of M^*, we have $|\det M^* - P| < \varepsilon P$. With a slight but very convenient abuse of notation we will write this as

$$\det M^* = (1 \pm \varepsilon)P. \tag{6.1}$$

The "much larger" condition in the definition of rapidly increasing is in fact $Dm - 1$ conditions. It follows from the results of [2] and [10] that they can be chosen so that equation (6.1) holds. We will need one further requirement, namely

$$a(i,j-1)a(i,j) < \varepsilon a(i,j+1) \quad \text{for all } i \in [D] \text{ and } j \in \{2, \dots, m-1\}. \tag{6.2}$$

Theorem 6.1. *For integers* D, m *with* $m > D \geq 1$ *and for every* $\varepsilon > 0$ *there is an* $D \times m$ *RI matrix* M *satisfying (6.1) and (6.2).*

The proof is postponed to Section 8. The extra condition (6.2) implies that $a(i,j) < \varepsilon a(i,j-1)a(i,j+1)$ for all $i \in [D]$ and $j \in \{2, \dots, m-1\}$. This follows from the inequality

$$a(i,j) < \frac{\varepsilon a(i,j+1)}{a(i,j-1)} \leq \varepsilon a(i,j+1)a(i,j-1)$$

as $a(i, j-1) \geq 1$. We will use the following consequences of the extra condition. For all $i \in [D]$

$$\begin{aligned} a(i,j) &< \varepsilon a(i,j-1)a(i,J) \text{ when } J > j, \text{ and} \\ a(i,J)a(i,j) &< \varepsilon a(i,j+1) \text{ when } J < j. \end{aligned} \tag{6.3}$$

7. Proof of Theorem 3.2

We are going to give examples of universal permutations other than the identity and the reverse identity. These examples are number sequences generated by writing the numbers $1, \ldots, d-1$ one by one, starting by 1 and appending the next number either on the left or the right of the current sequence, for instance

$$9, 8, 5, 4, 1, 2, 3, 6, 7, 10$$

or the same but starting from $d-1$ and going downwards. We call these permutations *two-sided stacked*. It is easy to see that there are at most $2^{d-1} - 2$ of them. Indeed, starting with 1 we have 2^{d-2} choices to go left or right, and the same number when starting with $d-1$. But the sequence $1, 2, \ldots, d-1$ and its reverse are counted twice.

If v is a d dimensional vector let $[v]_i$ denote its ith component. Let M be $2d \times m$ RI matrix satisfying conditions (6.2). The parameter $\varepsilon > 0$ will be specified later. A typical column C of M is a $2d$ dimensional vector, to be denoted by $C = (v, B)$ where v, B are d-dimensional vectors and the ordering within C is as follows. The components of v and B alternate: $[v]_1 < [B]_1 < [v]_2 < \ldots < [B]_d$ where $<$ means in fact"much larger" because of the RI condition.

Define the vector $b \in \mathbb{R}^d$ by

$$[b]_i = \prod_{j=d+1-i}^{d} [B]_j \text{ so } \frac{[b]_i}{[b]_{i-1}} = [B]_{d+1-i}.$$

We will choose a sequence $\gamma_1, \ldots, \gamma_m$ of positive reals, to be specified later, that grow faster than anything else so far. When v is the vector in the column $C = (v, B)$ of M we write $(v)_{-k}$ for the corresponding vector k position before the column of v. Let $\delta = (\delta_1, \ldots, \delta_d)$ be a ± 1 vector, that is each δ_i is either 1 or -1. For a fixed column $C = (v, B)$ of M define $a = \gamma(\sum_{j=1}^{d} \delta_j [b]_j (v)_{-j})$. This defines the oriented line corresponding to the pair (a, v). We will only consider a subset of these lines.

Assume $m = n(d+1)$, $n \geq d$ are integers. The sequence in our example consists of n oriented lines, each corresponding to a column of the form $C_{x(d+1)}$, $x \in [n]$. Call such a column *special*. Select $d-1$ special columns and let $v_1, \ldots, v_{d-1}$ be the v vectors (in increasing order) of the selected columns. We will only consider the following $(d-1)(d+1)$ vectors that come in this order (but not necessary consecutively) in M:

$$(v_1)_{-d}, \ldots, (v_1)_{-1}, v_1, (v_2)_{-d}, \ldots, (v_2)_{-1}, v_2, \ldots, v_{d-1}.$$

The line L_i corresponds to the pair (a_i, v_i), $i \in [d-1]$.

We have to estimate how large $\det(a_i, v_1, \dots, v_{d-1})$ is. Define

$$P = [v_1]_1 \cdot \ldots \cdot [v_{i-1}]_{i-1} \cdot [v_i]_{i+1} \cdot \ldots \cdot [v_{d-1}]_d.$$

We are going to show first that, with the convenient notation introduced in (6.1),

$$\begin{aligned}(7.1)\qquad &\det(a_i, v_1, \dots, v_{d-1}) = \\ &(1 \pm (d+1)\varepsilon)\gamma_i P(-1)^{i-1}\delta_{d+1-i}[b_i]_{d+1-i}[(v_i)_{-(d+1-i)}]_i.\end{aligned}$$

Using the definition of the vector a_i we see that

$$\begin{aligned}&\det(a_i, v_1, \dots, v_{d-1}) = \gamma_i \textstyle\sum_{j=1}^d \delta_j [b_i]_j \det((v_i)_{-j}, v_1, \dots, v_{d-1}) \\ &= \gamma_i \textstyle\sum_{j=1}^d \delta_j [b_i]_j (-1)^{i-1} \det(v_1, \dots, v_{i-1}, (v_i)_{-j}, v_i, \dots, v_{d-1}).\end{aligned}$$

The properties of the RI matrix imply that last determinant is essentially equal to the product of the entries on the main diagonal of the corresponding matrix. This product equals

$$[v_1]_1 \cdot \ldots \cdot [v_{i-1}]_{i-1} \cdot [(v_i)_{-j}]_i \cdot [v_i]_{i+1} \cdot \ldots \cdot [v_{d-1}]_d = P \cdot [(v_i)_{-j}]_i$$

so the product P is a common factor here, implying that

$$\det(a_i, v_1, \dots, v_{d-1}) = \gamma_i P(-1)^{i-1} \sum_{j=1}^d \delta_j [b_i]_j [(v_i)_{-j}]_i.$$

Consider i fixed and set $T_j = [b_i]_j[(v_i)_{-j}]_i$, so the last sum is $\sum_{j=1}^d \delta_j T_j$. Which is the dominant term here? We claim that it is T_{d+1-i}. We are going to show this by proving that for $j \le d+1-i$

$$(7.2)\qquad T_{j-1} < \varepsilon T_j,$$

and for $j > d+1-i$

$$(7.3)\qquad T_j < \varepsilon T_{j-1}$$

Indeed for $j \le d+1-i$ we have

$$\frac{[b_i]_j}{[b_i]_{j-1}} = [B_i]_{d+1-j} \ge [v_i]_i.$$

Because of (6.3) we have $[(v_i)_{-(j-1)}]_i < \varepsilon[v_i]_i[(v_i)_{-j}]_i$ implying that $[b_i]_{j-1}[(v_i)_{-(j-1)}]_i < \varepsilon[b_i]_j[(v_i)_{-j}]_i$, which is exactly (7.2). Similarly, for $j > d+1-i$ we have

$$\frac{[b_i]_j}{[b_i]_{j-1}} = [B_i]_{d+1-j} \le [B_i]_{i-1} \le [v_{i-1}]_i.$$

In view of (6.3) $[(v_i)_{-(j-1)}]_i[v_{i-1}]_i \le \varepsilon[(v_i)_{-j}]_i$ implying that $[b_i]_j[(v_i)_{-j}]_i \le \varepsilon[b_i]_{j-1}[(v_i)_{-j-1}]_i$. This is again the same as (7.3).

So the dominant term in the sum $\sum_{j=1}^d \delta_j [b_i]_j [(v_i)_{-j}]_i$ is the one $j = d+1-i$, and we have $T_j \le \varepsilon T_{d+1-i}$ for all $j \ne d+1-i$, each $T_j \ge 1$ of course. It follows that

$$\sum_{j=1}^d \delta_j [b_i]_j [(v_i)_{-j}]_i = (1 \pm (d-1)\varepsilon)\delta_{d+1-i} T_{d+1-i}$$

extending the notation of (6.1). We choose now $\varepsilon < \frac{1}{K(d+1)}$ with K large, $K = 100$ or 1000, say. It is easy to check that equation (7.1) holds true. More importantly, with this choice of ε the factor $(1 \pm (d+1)\varepsilon)$ is between $(1 - \frac{1}{K})$ and $(1 + \frac{1}{K})$, so it is very close to one.

In order to determine the permutation σ of $[d-1]$ for the sequence of lines $L_1, \dots, L_{d-1}$ we have to check, for all pairs $h < i$, the sign of

$$\det(a_i - a_h, v_1, \dots, v_{d-1}) = \det(a_i, v_1, \dots, v_{d-1}) - \det(a_h, v_1, \dots, v_{d-1}).$$

Observe that we only use γ_i for special columns so it suffices to choose $\gamma_{d+1}, \gamma_{2(d+1)}, \dots, \gamma_{n(d+1)}$. We introduce the notation $\gamma_x^* = \gamma_{x(d+1)}$

Claim 7.1. *The sequence $\gamma_1^*, \dots, \gamma_n^*$ can be chosen so that for all $h < i$*

$$|\det(a_h, v_1, \dots, v_{d-1})| < \varepsilon |\det(a_i, v_1, \dots, v_{d-1})|. \tag{7.4}$$

Proof. The vector v_i resp. v_h comes from a special column $C_{x(d+1)}$ and $C_{y(d+1)}$ with $y < x$. Then v_i is in fact $v_{x(d+1)}$ and the index i can be any number in $[d-1]$ except 1 because $h < i$. Similarly v_h coincides with $v_{y(d+1)}$ and h can be any number in $[d-1]$ except $d-1$.

We define γ_x^* recursively, starting with $\gamma_1^* = 1$. Assume γ_z^* has been defined for all $z < x$, is a positive integer, and satisfies (7.4). The possible values of $|\det(a_i, v_1, \dots, v_{d-1})|$ disregarding the factors γ_x^* and $(1 \pm (d+1)\varepsilon)$ (the latter is between $(1 - \frac{1}{K})$ and $(1 + \frac{1}{K})$) are of the form

$$P[b_i]_{d+1-i}[(v_i)_{d+1-i}]_i$$

for all available choices of i and $v_1, \dots, v_{d-1}$ with $v_i = v_{x(d+1)}$, P also varies. This is a finite set Z_x (say) of positive integers. The possible values of $|\det(a_h, v_1, \dots, v_{d-1})|$ disregarding the factor $(1 \pm (d+1)\varepsilon)$ is of the form

$$\gamma_y^* P[b_h]_{d+1-h}[(v_h)_{d+1-h}]_h$$

for all choices of $y < x$, $h < d$ and $v_1, \dots, v_{d-1}$ with $v_h = v_{y(d+1)}$, P varies again. This is another finite set, V_x (say), of positive integers.

It is clear that there is an integer γ_x^* so that $\max V_x < \frac{\varepsilon}{10} \min Z_x$. Bringing back the factors $1 \pm (d+1)\varepsilon$ finishes the proof. □

The claim shows that the sign of $\det(a_i - a_h, v_1, \dots, v_{d-1})$ coincides with that of $\det(a_i, v_1, \dots, v_{d-1})$ when $h < i$. The sign of the last expression the same as the sign of $(-1)^{i-1}\delta_{d+1-i}$ because of (7.1). We can decide this sign by choosing δ_{d+1-i} any way we like. This means that when $L_{x(d+1)}$ is in position i of the sequence $L_1, \dots, L_{d-1}$, in the corresponding permutation σ of $[d-1]$, either every $h < i$ will come before i, or every $h < i$ will come after i, depending on the choice of δ_{d+1-i}. This is exactly what two-sided stacked means, finishing the proof of Theorem 3.2. □

There are exactly two resp. six universal permutations for $d = 3$ and $d = 4$ because in these cases $(d-1)!$ and $2^{d-1} - 2$ coincide. For $d = 5$ the number of universal permutations is at least 14 and at most 24. The unresolved cases are the permutations 2143, 2413, 1324, 4132, 1423 and their reverses 3412, 3142, 4231, 1423, 4132, of course none of them two-sided stacked. We don't know if anyone of them is universal. but we believe that none of them is.

8. Proof of Theorem 6.1

We define $a(i,j)$ by induction, and for that we require a new condition, slightly stronger than (6.1) which is described next.

Consider a $k \times k$ submatrix ($k \geq 2$) M^* of M whose bottom right entry is $a(i,j)$. Let P be the product of the elements on its main diagonal. A *diagonal*, Δ, of M^* is a set of entries containing exactly one entry from every row and column of M^*. Write $\prod \Delta$ for the product of the elements in Δ. We are going to require that for every diagonal Δ except the main one

$$\prod \Delta < \eta P. \tag{8.1}$$

Here we choose $\eta > 0$ to be small, namely $\eta = \varepsilon/d!$. As $\det M^*$ is the sum of all $\prod \Delta$, each taken with a well-defined sign ± 1, $\det M^* = P + \sum(\pm 1)\prod \Delta$ where the sum is taken over all diagonals except the main one. It follows that $|\det M^* - P| \leq (k! - 1)\eta P$. The choice $\eta = \varepsilon/d! < \varepsilon/(k!-1)$ ensures that $\det M^* = (1 \pm \varepsilon)P$, exactly condition (6.1).

The point in the following induction argument is that conditions (8.1) and (6.2) plus the RI condition only give lower bounds on the next entry $a(i,j)$, and there are finitely many such lower bounds. This is always easy to satisfy by choosing $a(i,j)$ large enough.

Observe that (8.1) only matters when $i \geq 2$. For $i = 1$ we set $a(1,1) = 1$ and $a(1,2) = 2$. When $i = 1$ condition (6.2) says that

$$a(1, j-2)a(1, j-1) < \varepsilon a(1,j)$$

and only needed for $j = 3, 4, \ldots, m$ and is easy to satisfy by induction on j.

So $i \geq 2$ and we assume $a(I,J)$ have been determined and satisfy both (8.1) and (6.2) for all pairs (I,J) when $I < i$ and $J \in [m]$, and when $I = i$ and $J < j$. The cases $j = 1, 2$ are simple but need special treatment.

Defining $a(i,1)$ is easy: the only condition is $a(i,1) > a(i-1,m)$ so $a(i,1) = a(i-1,m) + 1$ will do. Defining $a(i,2)$ is similarly easy:

it has to be larger than $a(i,1)$ and condition (8.1) is meaningful only for 2×2 submatrices whose bottom right entry is $a(i,2)$. In this case it requires that $a(i,1)a(I,2) < \eta a(I,1)a(i,2)$ for all $I < i$. These are only lower bounds on $a(i,2)$, and there are finitely many of them. So choosing $a(i,2)$ large enough we are done with $a(i,1)$ and $a(i,2)$.

We assume from now on that $j \geq 3$ (and $i \geq 2$). Consider a $k \times k$ ($k \geq 2$) submatrix M^* of M whose bottom right entry is $a(i,j)$. Write Q for product of the elements on the main diagonal of M^* except $a(i,j)$. The requirement is that ηQ be larger than $\prod \Delta$ for every (but the main) diagonal Δ of M^*. We consider two cases separately.

Case 1 when $a(i,j) \notin \Delta$. For such diagonals $\prod \Delta$ is a positive integer, independent of $a(i,j)$. There are finitely many such integers, corresponding to every possible choice of $k \in \{2,3,\dots,i\}$ and M^* and Δ, their maximum is an integer $H(i,j)$. Moreover, let $Q(i,j)$ be the minimum of the product of the entries on the main diagonal of M^* except $a(i,j)$ for every possible choice of k and M^*. Condition (8.1) requires that $H(i,j) < \eta a(i,j)Q(i,j)$. We can clearly choose $a(i,j)$ so large that this inequality is satisfied.

Case 2 when $a(i,j) \in \Delta$. For these diagonals $\prod \Delta = a(i,j) \prod \Delta^*$ where Δ^* is the corresponding diagonal of the $(k-1)\times(k-1)$ submatrix M° that you get after deleting row i and column j from M^*.

Assume first that $k \geq 3$. Let $a(i^\circ, j^\circ)$ be the bottom right entry on the main diagonal Δ° of M°. Condition (8.1) of the induction hypothesis for the pair $a(i^\circ, j^\circ)$ implies that $\prod \Delta^* < \eta \prod \Delta^\circ$ for all diagonals of M° except the main one. Multiplying by $a(i,j)$ we conclude that for all diagonals Δ of M^* with $a(i,j) \in \Delta$ (except the main one) $\prod \Delta < \eta a(i,j) \prod \Delta^\circ = \eta Q$. This shows that condition (8.1) is automatically satisfied for the pair (i,j) in Case 2 when $k \geq 3$.

Assume now that $k = 2$. Condition (8.1) says now that for all $I < i$ and $J < j$,

$$a(i,J)a(I,j) < \eta a(I,J)a(i,j).$$

This is again a finite set of lower bounds in $a(i,j)$. Let $H^*(i,j)$ be the maximum of these lower bonds.

It is easy to deal with (6.2) which, in the present case $j \geq 3$, requires that

$$a(i,j-1)a(i,j-2) < \varepsilon a(i,j).$$

Finally we choose an integer $a(i,j)$ larger than the maximum of the three numbers

$$\frac{H(i,j)}{\eta Q(i,j)},\ H^*(i,j),\ \frac{a(i,j-1)a(i,j-2)}{\varepsilon}.$$

9. Continuous and universal families of lines

In this section and the next we work with oriented lines in $\mathbb{R}^d$. Let $I \subset \mathbb{R}$ be an open interval, for instance $I = (0,1)$ will do. Suppose that, for each $t \in I$, $L(t) = (v(t), a(t))$ is an (oriented) line in $\mathbb{R}^d$. We say that $L(t)$ is *continuous* if

- $v(t)$ is continuous,
- as $t \to t_0$ we have $\text{dist}(L(t), L(t_0)) \to 0$, where dist is the standard Euclidean distance, the infimum of the distances between any two points from the lines.

The family $L(t)$ is *universal* if the type of the lines $L(t_1), \dots, l(t_{d-1})$ is the same permutation σ of $[d-1]$ for every $t_1 < \dots < t_{d-1}$, and σ is the *type* of such a family.

A simple example is the family of lines $L(t) = (a(t), v(t))$ with $t \in (0,1)$, say, given by $a(t) = (t, 0, \dots, 0)$ and $v(t) = (1, t, \dots, t^{d-1})$. When $0 < t_1 < \dots < t_{d-1}$, suitably translated copies of the hyperplane $H = \frac{1}{d-1}(L(t_1) + \dots + L(t_{d-1}))$ contain the lines $L(t_i)$ in the order $L(t_1), \dots, L(t_{d-1})$. This is easy to see: the translated copy of H containing the line $L(t_i)$ passes through the point $a(t_i)$ and these points come in order $a(t_1), \dots, a(t_{d-1})$ on the line $L = \{a(s) : s \in \mathbb{R}\}$. So this is a continuous and universal family of lines and their type is the identity. But in this example all the lines intersect the line L. A more generic example comes next. Some preparations are needed.

Let $0 < t_1 < \dots < t_{d-1}$ be an increasing sequence of real numbers. The Vandermonde determinant

$$V_0 = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{d-2} & t_2^{d-2} & \cdots & t_{d-1}^{d-2} \end{vmatrix} = \prod_{i=2, i>j}^{d-1} (t_i - t_j)$$

in positive. Following the terminology of [6] we say that V_0 is the *principal* Vandermondian. The generalized Vandermonde determinant is, assuming $0 < b_1 < \dots < b_{d-1}$,

$$\begin{vmatrix} t_1^{b_1} & t_2^{b_1} & \cdots & t_{d-1}^{b_1} \\ t_1^{b_2} & t_2^{b_2} & \cdots & t_{d-1}^{b_2} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{b_{d-1}} & t_2^{b_{d-1}} & \cdots & t_{d-1}^{b_{d-1}} \end{vmatrix}.$$

This is the principal Vandermondian when $\{b_1, \dots, b_{d-1}\}$ coincides with $\{0, \dots, d-2\}$. When $1 \le j \le d-1$ and $\{b_1, \dots, b_{d-1}\} = \{0, 1, \dots, d-1\} \setminus \{d-1-j\}$ we say that the corresponding generalized Vandermonde determinant is the *secondary* Vandermondian V_j.

Observe that if $j = 0$ would be allowed, the definition would give back the principal Vandermondian V_0.

Let $E_j = \sum t_1 t_2 \cdots t_j$ be the elementary symmetric function where we add up all the products of any j different variables. Observe that $E_0 = 1$ and that E_1 is the sum of all variables, and $E_j > 0$ since all t_i are positive.

Theorem 1 from [6] states that $\frac{V_j}{V_0} = E_j$ and therefore we have

Fact 9.1. *The secondary Vandermondians $V_1, \dots, V_{d-1}$ are all positive.*

Assume now that $0 < t_1 < \dots < t_{d-1}$ and $a_1, a_2, \dots, a_d > 0$ and define the matrix

$$A = \begin{pmatrix} a_1 & 1 & 1 & \cdots & 1 \\ -a_2 & t_1 & t_2 & \cdots & t_{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{d-1}a_d & t_1^{d-1} & t_2^{d-1} & \cdots & t_{d-1}^{d-1} \end{pmatrix}.$$

The determinant of A when expanding by the first column is

$$\det A = a_1 V_{d-1} + a_2 V_{d-2} + \dots + a_d V_0 > 0. \tag{9.1}$$

We can give now the more generic example of a continuous and universal family of lines. Assume $a_i(t) > 0$ is a continuous and increasing function on $t \in (0, \infty)$ for $i \in [d]$. For $t > 0$ set

- $v(t) = (1, t, t^2, \dots, t^{d-1})$,
- $a(t) = (a_1(t), -a_2(t), \dots, (-1)^{d-1}a_d(t))$.

Theorem 9.1. *The family of lines $L(t) = (v(t), a(t))$, $t > 0$ is continuous and universal and its type is the identity.*

The proof follows directly from (9.1).

10. Uniqueness of continuous type

Assume $L(t) = (v(t), a(t))$, $(t \in I)$ is a continuous family of lines in $\mathbb{R}^d$. Here $I \subset \mathbb{R}$ is an open interval (possibly infinite). We assume that $v(t)$ is a unit vector for all $t \in I$. We call the identity permutation and its reverse *trivial*. The previous section contains examples of continuous and universal families of lines whose types are trivial. The target in this section is to prove the converse, namely, that the type of such a family of lines is always trivial, at least when $d \geq 5$.

Theorem 10.1. *Let $d \geq 5$. If $L(t)$ is a continuous and universal family of lines, then its type is trivial.*

The same holds for $d = 3$: the type of a continuous and universal family of lines in $\mathbb{R}^3$ is trivial because this is the only type available in $\mathbb{R}^3$. We believe that the same holds in $\mathbb{R}^4$ but our method does not work in that case.

Before the proof we need a few auxiliary lemmas. By definition, if $L(t)$ is a continuous and universal family of lines, then for any $t_1 < \ldots < t_{d-1}$ the vectors $v(t_1), \ldots, v(t_{d-1})$ are linearly independent. The following claim extends this to any d such vectors.

Lemma 10.2. *The vectors $v(t_1), \ldots, v(t_d)$ are linearly independent when $t_1 < \ldots < t_d$.*

Corollary 10.1. *Let $L(t)$ be a continuous family of lines, U be proper linear subspace of $\mathbb{R}^d$ and J some open interval $J \subset I$. Then either $L(t)$ has trivial type, or there exists a sub-interval $J' \subset J$ such that $v(t) \notin U$ for all $t \in J'$.*

For the next lemma we need something like the derivative of $v(t)$. Unfortunately $v(t)$ may not be a differentiable function.

For any $t, s \in I$ let $\delta v(t,s) = v(s) - v(t)$ and define the normalized change as $\Delta v(t,s) = \frac{\delta v(t,s)}{||\delta v(t,s)||}$; note that $\delta v(t,s) \neq 0$ because $v(t)$ and $v(s)$ are linearly independent. Since $\Delta v(t,s)$ lies in a compact set (namely the unit sphere), it has limit, $v'(t)$, from the right for every t. That is for every t there exists a sequence $s_i > t$ with limit t such that the limit of $\Delta v(t, s_i)$ is $v'(t)$, and $v'(t) \neq 0$. (There could be several different values of $v'(t)$ that could work, we just choose one of them.)

Observe that $v(t)$ and $v'(t)$ are orthogonal and that for every $t, s \in I$ with $t \neq s$ $\text{lin}\,\{v(t), v(s)\} = \text{lin}\,\{v(t), \Delta v(t,s)\}$.

Lemma 10.3. *Let $L(t)$ be a continuous and universal family of lines in $\mathbb{R}^d$, $d \geq 5$. Then either $L(t)$ has trivial type, or there exist $t, s \in I$ such that the four vectors $v(t), v'(t), v(s), a(s) - a(t)$ are linearly independent.*

Proof of Theorem 10.1 using the previous lemmas. Let σ be the type of $L(t)$ and assume on the contrary that σ is not the identity or its reverse. Then there exist integers j, k such that $\sigma(j) < \sigma(k) < \sigma(j+1)$ or $\sigma(j+1) < \sigma(k) < \sigma(j)$. We can assume without loss of generality that the first case occurs.

By Lemma 10.3 we can choose t_j, t_k such that the four vectors $v(t_j), v'(t_j), v(t_k)$ and $a(t_k) - a(t_j)$ are linearly independent. Next we choose t_i one by one for $i \in [d-1] \setminus \{j, j+1, k\}$ so that $t_1 < \ldots < t_j < t_{j+2} < \ldots < t_{d-1}$ and that the d vectors $v(t_1), \ldots, v(t_j), v'(t_j), v(t_{j+2}), \ldots, v(t_{d-1}), a(t_k) - a(t_j)$ are linearly independent. For this we use Corollary 10.1 by defining U to be the linear span of the previous (at most $d-1$) vectors and choose J to reflect the relative position of the next t_i.

Let $s_1 > s_2 > \ldots$ be an infinite decreasing sequence with limit t_j and $\lim_{i \to \infty} \Delta v(t_j, s_i) = v'(t_j)$. We can assume that $t_j < s_i < t_{j+2}$ for every i.

Set $H_i = \text{lin}\,\{v(t_1), \ldots, v(t_j), v(s_i), v(t_{j+2}), \ldots, v(t_{d-1})\}$ which is a $(d-1)$-dimensional subspace because these vectors are linearly independent. It is clear that H_i remains unchanged if in its definition $v(s_i)$

is replaced by $\Delta v(t_j, s_i)$. The orthogonal vector to H_i is

$$w_i = v(t_1) \wedge \ldots \wedge v(t_j) \wedge \Delta v(t_j, s_i) \wedge v(t_{j+2}) \wedge \ldots \wedge v(t_{d-1}).$$

Let $u_i = \frac{w_i}{||w_i||}$ be the unit vector orthogonal to H_i.

The limit of H_i is $H = \text{lin}\,\{v(t_1), \ldots, v(t_j), v'(t_j), v(t_{j+2}), \ldots, v(t_{d-1})\}$ and the limit of u_i is u, a unit vector orthogonal to H.

The order of the lines $L(t_1), \ldots, L(t_j), L(s_i), L(t_{j+2}), \ldots, L(t_{d-1})$ is the same as the order of the real numbers

$$u_i \cdot a(t_1), \ldots, u_i \cdot a(t_j), u_i \cdot a(s_i), u_i \cdot a(t_{j+2}), \ldots, u_i \cdot a(t_{d-1}).$$

Since $L(s_i)$ plays the role of the $(j+1)$st line

$$u_i \cdot a(t_j) < u_i \cdot a(t_k) < u_i \cdot a(s_i)$$

The distance of the two lines $L(t_j)$ and $L(s_i)$ is at least the distance of any two hyperplanes through the two lines, particularly the ones parallel to H_i, which is $u_i \cdot (a(s_i) - a(t_j))$. Consequently the limit of $u_i \cdot a(t_j)$ and $u_i \cdot a(s_i)$ is the same, namely $u \cdot a(t_j)$. Therefore the limit of the previous inequality is

$$u \cdot a(t_j) \leq u \cdot a(t_k) \leq u \cdot a(t_j)$$

and we have $u \cdot (a(t_k) - a(t_j)) = 0$. This contradicts the choice of $t_1, \ldots, t_j, t_{j+2}, \ldots, t_{d-1}$ as u is orthogonal to every one of the d vectors $v(t_1), \ldots, v(t_j), v'(t_j), v(t_{j+2}), \ldots, v(t_{d-1}), a(t_k) - a(t_j)$ that are linearly independent. □

Proof of Lemma 10.2. Assume that the vectors $v(t_1), \ldots, v(t_d)$ are not linearly independent. Their linear span is then a $(d-1)$-dimensional subspace. Let z be its normal vector. The numbers $h_i = z \cdot a(t_i)$, $(i \in [d])$ come in the order $h_{i_1}, ..., h_{i_d}$, and $\pi = (i_1, ..., i_d)$ is a permutation of $[d]$. We can assume that $i_1 < i_d$ (by replacing z by $-z$ if necessary). The permutation σ_j of $[d-1]$ comes from π by deleting the entry $j \in [d]$. Universality implies that, for every j, σ_j is the *same ordering* of $d-1$ linearly ordered elements. Assume $d = i_h$, then in σ_j the largest element is in position $h-1$ if $j \in \{i_1, ..., i_{h-1}\}$ and in position h if $j \in \{i_{h+1}, ..., i_d\}$ implying that $\{i_{h+1}, ..., i_d\} = \emptyset$. It follows that $h = d$. The same argument works for the kth largest element of $[d]$ (by backward induction on k): it has to be in position $d-k+1$ in π. □

Proof of Corollary 10.1. Let $S \subset I$ be the set of all t, such that $v(t) \in U$. Observe that $|S| \leq d-1$ as otherwise U coincides with $\mathbb{R}^d$ because any d direction vectors are linearly independent by Lemma 10.2. We can choose $J \subset I$ to be any open interval avoiding S. □

Proof of Lemma 10.3. Assume on the contrary that the four vectors $v(t), v'(t), v(s), a(s) - a(t)$ are linearly dependent for every pair $t < s$. Set

$$W(t, s) = \text{lin}\,\{v(t), v'(t), v(s), a(s) - a(t)\}.$$

The vectors $v(t), v(s), a(s) - a(t)$ are linearly independent because any two lines of the family are skew. Consequently $\dim W(t,s) = 3$ and $W(t,s) = \text{lin}\,\{v(t), v(s), a(s) - a(t)\}$.

Given $t_0 < t_1 < t_2$ set

$$W = \text{lin}\,\{v(t_0), v'(t_0), v(t_1), a(t_1) - a(t_0), v(t_2), a(t_2) - a(t_0)\}.$$

Observe that W is the span of the two 3-dimensional subspaces $W(t_0, t_1)$ and $W(t_0, t_2)$ that intersect in an at least 2-dimensional subspace because both contain $\text{lin}\,\{v(t_0), v'(t_0)\}$. Thus the dimension of W is at most 4.

Moreover W contains $W(t_1, t_2) = \text{lin}\,\{v(t_1), v(t_2), a(t_2) - a(t_1)\}$ as these three vectors all belong to W. Then $v'(t_1) \in W$ since $v'(t_1) \in W(t_1, t_2)$.

Assume next that there exist $t_0 < t_1$ such that the four vectors $v(t_0), v'(t_0), v(t_1), v'(t_1)$ are linearly independent. Then there exists $t_2 > t_1$ such that $v(t_0), v'(t_0), v(t_1), v'(t_1), v(t_2)$ are linearly independent because otherwise $v(t_2)$ lies in $\text{lin}\,\{v(t_0), v'(t_0), v(t_1), v'(t_1), v(t_2)\}$ for every $t_2 > t_1$ contradicting Lemma 10.2. It follows that the linear span of $v(t_0), v'(t_0), v(t_1), v'(t_1), v(t_2)$ is 5-dimensional. But it is a subspace of W and $\dim W \le 4$, a contradiction showing that the four vectors $v(t_0), v'(t_0), v(t_1), v'(t_1)$ are indeed linearly dependent for every $t_0 < t_1$. Note that this is the point where the condition $d > 4$ is used.

So any two of the 2-dimensional subspaces $\text{lin}\,\{v(t), v'(t)\}$ intersect in an at least 1-dimensional subspace. There are two possibilities that can happen.

Case 1. All subspaces $\text{lin}\,\{v(t), v'(t)\}$ are contained in some 3 dimensional space. This is impossible as there exist $d \ge 4$ direction vectors that are independent.

Case 2. There exists some vector u contained in $\text{lin}\,\{v(t), v'(t)\}$ for every t. In this case choose an increasing sequence $t_0 < \ldots < t_n$ such that u together with any $d-1$ different direction vectors $v(t_i)$ are linearly independent. This goes by induction on n and for $n \le d-1$ a simple application of Corollary 10.1 works. When $n \ge d$ and we have defined $t_0, \ldots, t_{n-1}$, then t_n is found by repeated, actually $\binom{n-1}{d-2}$-fold, applications of Corollary 10.1.

For every t_i, $i \in [n]$, define $\gamma_i \in \mathbb{R}$ by

$$a(t_i) - a(t_0) = \gamma_i u + \alpha_i v(t_i) + \beta_i v(t_0)$$

where α_i, β_i are some real numbers. The existence of $\gamma_i, \alpha_i, \beta_i$ follows from the fact that $W(t_0, t_i) = \text{lin}\,\{v(t_0), v(t_i), a(t_i) - a(t_0)\}$ and $u \in \text{lin}\,\{v(t_0), v'(t_0)\} \subset W(t_0, t_i)$. Note that γ_i remains the same even if $a(t_i) \in L(t_i)$ is replaced by another point $a(t_i) + \delta v(t_i) \in L$.

Observe that

$$\begin{aligned} a(t_i) - a(t_j) &= a(t_i) - a(t_0) - (a(t_j) - a(t_0)) \\ &= (\gamma_i - \gamma_j)u + \alpha_i v(t_i) - \alpha_j v(t_j) + (\beta_i - \beta_j)v(t_0). \end{aligned}$$

Assume $n > d^2$. A classic result of Erdős and Szekeres [3] shows then that there exists a subsequence of the t_i of length $(d-2)$ such that the corresponding γ_i form an increasing (or decreasing) sequence. For simplicity assume that this subsequence is $t_1, \ldots, t_{d-2}$ and the γ_i are increasing. Let $w = v(t_0) \wedge v(t_1) \wedge \ldots \wedge v(t_{d-2})$ be the orthogonal vector to all lines $L(t_i)$. Then the order of the lines $L(t_i)$ depends on the order of the numbers $w \cdot a(t_i)$ and therefore on the signs of $w \cdot (a(t_i) - a(t_j)) = (\gamma_i - \gamma_j) w \cdot u$.

So the order of all the lines but $L(t_0)$ is increasing, meaning that in σ the last $d-2$ element come in increasing order. But now by symmetry, choose t_n as the anchor of the γ_is and we get that the first $d-2$ elements in σ are come in increasing order. Therefore σ is identity permutation. □

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