*Sbornik*: *Mathematics* **204**:12 1728–1744 *Matematicheskiĭ Sbornik* **204**:12 31–48
DOI 10.1070/SM2013v204n12ABEH004358

# Nonmaximality of known extremal metrics on torus and Klein bottle

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**Abstract.** The El Soufi-Ilias theorem establishes a connection between minimal submanifolds of spheres and extremal metrics for eigenvalues of the Laplace-Beltrami operator. Recently, this connection was used to provide several explicit examples of extremal metrics. We investigate the properties of these metrics and prove that none of them is maximal.

Bibliography: 24 titles.

**Keywords:** extremal metrics, bipolar surface, Otsuki tori, Lawson tau-surfaces.

## § 1. Introduction

Let $M$ be a closed surface and $g$ be a Riemannian metric on $M$. Then the Laplace-Beltrami operator $\Delta$ acts on the space of smooth functions on $M$ through the formula

$$\Delta f = -\frac{1}{\sqrt{|g|}}\,\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial x^j}\right).$$

It is known that the spectrum of $\Delta$ is discrete and consists only of eigenvalues. Moreover, the multiplicity of any eigenvalue is finite and the sequence of eigenvalues tends to infinity. We denote this sequence by

$$0 = \lambda_0(M,g) < \lambda_1(M,g) \leqslant \lambda_2(M,g) \leqslant \lambda_3(M,g) \leqslant \cdots,$$

where the eigenvalues are listed according to their multiplicities.

For a fixed $M$ the following quantities can be considered as functionals on the space of all Riemannian metrics on $M$,

$$\Lambda_i(M,g) = \lambda_i(M,g)\operatorname{Area}(M,g).$$

Some recent papers [1]–[10] deal with finding the supremum of these functionals in the space of all Riemannian metrics on $M$.

An upper bound for $\Lambda_1(M,g)$ in terms of the genus of $M$ was provided in the paper [10] and the existence of such a bound for $\Lambda_i(M,g)$ was proved in [6].

The research of the author was partially supported by a Dobrushin Fellowship and by Simonsium Fellowship.

*AMS* 2010 *Mathematics Subject Classification.* Primary 58E11.

Exact upper bounds, $\sup \Lambda_i(M, g)$, are known for a limited number of functionals: $\Lambda_1(\mathbb{S}^2, g)$ (see [4]), $\Lambda_1(\mathbb{RP}^2, g)$ (see [7]), $\Lambda_1(\mathbb{T}^2, g)$ (see [8]), $\Lambda_1(\mathbb{Kl}, g)$ (see [1], [5]), $\Lambda_2(\mathbb{S}^2, g)$ (see [9]). We refer the reader to the introduction to [11] for more details.

The functional $\Lambda_i(M, g)$ depends continuously on $g$ but this functional is not differentiable. However, it is known that for an analytic deformation $g_t$ of the initial metric $g$ the left and right derivatives of $\Lambda_i(M, g_t)$ with respect to $t$ exist; see, for example, [3], [12] and [13]. This motivates the following definition.

**Definition 1** (see [2], [8]). A Riemannian metric $g$ on a closed surface $M$ is called an *extremal* metric for a functional $\Lambda_i(M, g)$ if for any analytic deformation $g_t$ such that $g_0 = g$ the following inequality holds:

$$\frac{d}{dt}\Lambda_i(M, g_t)\Big|_{t=0+} \leqslant 0 \leqslant \frac{d}{dt}\Lambda_i(M, g_t)\Big|_{t=0-}.$$

**Definition 2.** A metric $g$ is called a *maximal* metric for a functional $\Lambda_i(M, g)$ if for any metric $h$ on $M$

$$\Lambda_i(M, g) \geqslant \Lambda_i(M, h).$$

The question of whether there exists a smooth maximal metric is itself not trivial. For example, there is no smooth maximal metric for $\Lambda_2(\mathbb{S}^2, g)$ (see [9]).

The list of known extremal metrics is longer than the list of known exact upper bounds for $\Lambda_i(M, g)$, but until now their maximality has not been studied. Here we fill this gap, and investigate the maximality of all the known extremal metrics. The list of currently known extremal metrics is as follows.

(A) Metrics on the Otsuki tori $O_{p/q}$, which were studied in [11].

(B) Metrics on the Lawson tori and Klein bottles $\tau_{m,k}$, studied in [14].

(C) Metrics on the surfaces $\widetilde{\tau}_{m,k}$ bipolar to Lawson surfaces, studied in [15].

(D) Metrics on the bipolar surfaces $\widetilde{O}_{p/q}$ to Otsuki tori, studied in [16].

In what follows the Klein bottle is denoted by $\mathbb{K}$.

The definitions of these surfaces are given in the following sections. The main result in our paper is the following theorem.

**Theorem 1.** *There are no maximal metrics among the metrics* (A)–(D) *apart from* $\widetilde{\tau}_{3,1}$.

*Remark* 1. The metric on the Lawson bipolar Klein bottle, $\widetilde{\tau}_{3,1}$, is maximal for the functional $\Lambda_1(\mathbb{K}, g)$, see [1], [5].

We also prove the following proposition.

**Proposition 1.** *The metric on the Clifford torus is extremal for an infinite number of functionals* $\Lambda_i(M, g)$, *but it is not maximal for any of them.*

That the metric on the Clifford torus is extremal for an infinite number of functionals $\Lambda_i(M, g)$ is known but, to the best of the author's knowledge, has not yet been published. In the present paper we fill this gap.

In what follows we use the notation $K(k)$, $E(k)$ and $\Pi(n,k)$ for elliptic integrals of the first, second and third kind respectively, see [17],

$$K(k)=\int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\,dx, \qquad E(k)=\int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx,$$
$$\Pi(n,k)=\int_0^1 \frac{1}{(1-nx^2)\sqrt{1-x^2}\sqrt{1-k^2x^2}}\,dx.$$

The paper is organized in the following way. In § 2 we prove lower bounds for $\sup\Lambda_n(\mathbb{T}^2,g)$ and $\sup\Lambda_n(\mathbb{K},g)$. These bounds are used throughout the paper to prove that the metrics (A)–(D) are nonmaximal. In § 3.2 we recall a connection between extremal metrics and minimal submanifolds of the unit sphere. Subsection 3.3 contains a description of Otsuki tori as SO(2)-invariant minimal submanifolds of $\mathbb{S}^3$ of cohomogeneity 1. Estimates for the extremal metrics (A)–(D) are established in § 3.4 and §§ 4–6, respectively, and this completes the proof of Theorem 1. Finally, § 7 contains the proof of Proposition 1.

## § 2. Lower bounds for $\sup\Lambda_n$

The aim of this section is to prove the following proposition (cf. Corollary 4 in [18]).

**Proposition 2.** *The following inequalities hold*:

$$\sup\Lambda_n(\mathbb{T}^2,g)\geqslant 8\pi\left(n-1+\frac{\pi}{\sqrt{3}}\right),$$
$$\sup\Lambda_n(\mathbb{K},g)\geqslant 8\pi(n-1)+12\pi E\left(\frac{2\sqrt{2}}{3}\right),$$

*where $E(k)$ stands for the elliptic integral of the second kind.*

**2.1. Attaching handles using the method due to Chavel-Feldman.** Let $M$ be a compact smooth Riemannian manifold of dimension $n\geqslant 2$. We pick two distinct points $p_1,p_2\in M$. For $\varepsilon>0$ we define

$B_\varepsilon$ := the union of open geodesic balls of radius $\varepsilon$ about $p_1$ and $p_2$,
$\Omega_\varepsilon$ := $M\setminus B_\varepsilon$,
$\Gamma_\varepsilon$ := $\partial B_\varepsilon=\partial\Omega_\varepsilon$.

Here the number $\varepsilon$ is chosen to be less than a quarter of the injectivity radius of $M$ and less than a quarter of the distance between $p_1$ and $p_2$ if $p_1$ and $p_2$ lie in the same connected component of $M$. We say that the manifold $M_\varepsilon$ is obtained from $M$ by adding a handle across $\Gamma_\varepsilon$ if

1) $\Omega_\varepsilon$ is isometrically embedded in $M_\varepsilon$,
2) there exists a diffeomorphism $\Psi_\varepsilon\colon M_\varepsilon\setminus\Omega_{2\varepsilon}\to[-1,1]\times\mathbb{S}^{n-1}$ such that

$$M_\varepsilon\setminus\Omega_\varepsilon=\Psi_\varepsilon^{-1}\left(\left[-\frac{1}{2},\frac{1}{2}\right]\times\mathbb{S}^{n-1}\right).$$

We denote the spectrum of the Laplace-Beltrami operator on $M$ and $M_\varepsilon$ by $\lambda_j$ and $\lambda_j(\varepsilon)$, respectively. In [19], Chavel and Feldman obtained a sufficient condition for

the convergence $\lambda_j(\varepsilon) \to \lambda_j$ as $\varepsilon$ tends to 0. In order to formulate this condition we need the following definition.

**Definition 3.** For any compact connected Riemannian manifold $X$ of dimension $n \geqslant 2$, the *isoperimetric constant* $c_1(X)$ is defined by

$$c_1(X) = \inf_Y \frac{(\mathrm{vol}_{n-1}(Y))^n}{(\min(\mathrm{vol}_n(X_1), \mathrm{vol}_n(X_2)))^{n-1}},$$

where $\mathrm{vol}_k$ stands for $k$-dimensional Riemannian measure, and $Y$ ranges over all compact $(n-1)$-dimensional submanifolds of $X$ that divide $X$ into two open submanifolds $X_1$, $X_2$ with common boundary $Y$.

**Theorem 2** (Chavel and Feldman [19]). *Assume that $M_\varepsilon$ is connected for any $\varepsilon$ and there exists a constant $c > 0$ such that $c_1(M_\varepsilon) \geqslant c$ for all $\varepsilon > 0$. Then $\lim_{\varepsilon\to 0} \lambda_j(\varepsilon) = \lambda_j$ for all $j = 1, 2, \dots$ .*

*Remark* 2. Taking $Y = \Gamma_\varepsilon$, the assumption in the above theorem implies that

$$\lim_{\varepsilon\to 0} \mathrm{vol}_n(M_\varepsilon) = \mathrm{vol}_n(M).$$

In the same paper the existence of such $M_\varepsilon$ is established for any surface $M$ and almost any pair of points $p_1, p_2 \in M$.

**Theorem 3.** *Let $M$ be a compact 2-dimensional Riemannian manifold with Gaussian curvature $K$ and let*

$$\widetilde{M} = (M \setminus K^{-1}(0)) \cup \operatorname{int} K^{-1}(0)$$

*be an open, dense subset of $M$. Suppose that $p_1, p_2 \in \widetilde{M}$ and one of the following possibilities occurs*:

- *$M$ is connected*;
- *$M$ has two connected components and the $p_i$ lie in different connected components.*

*Then $M_\varepsilon$ can be constructed so that the assumption of Theorem 2 holds. In particular*, $\mathrm{Area}(M_\varepsilon) \to \mathrm{Area}(M)$ *as* $\varepsilon \to 0$.

*Remark* 3. Note that Chavel and Feldman [19] only considered the case of a connected manifold $M$. However, their arguments can be extended almost unchanged to the non-connected case stated above.

**2.2. Proof of Proposition 2.** Consider the flat equilateral torus $\tau_{\mathrm{eq}}$ with a lattice of equilateral triangles. After a suitable rescaling of the metric we have $\mathrm{Area}(\tau_{\mathrm{eq}}) = 4\pi^2/\sqrt{3}$ and $\lambda_1(\tau_{\mathrm{eq}}) = 2$. The Euclidean sphere $\mathbb{S}^2$ of volume $4\pi$ also has $\lambda_1(\mathbb{S}^2) = 2$. Take $n-1$ copies of $\mathbb{S}^2$ denoted by $S_i$, $i = 1, 2, \dots, n-1$. Thus for $T_n = \tau_{\mathrm{eq}} \coprod_{i=1}^{n-1} S_i$ we have $\lambda_n(T_n) = 2$ and therefore $\Lambda_n(T_n) = 8\pi\left(n - 1 + \pi/\sqrt{3}\right)$. Successive applications of Theorem 3 yield the existence of a sequence $M_\varepsilon$, diffeomorphic to torus, such that $\Lambda_n(M_\varepsilon) \to \Lambda_n(T_n)$ as $\varepsilon$ tends to 0. This observation completes the proof of the first inequality.

The second inequality can be proved in the same way. The only difference is that instead of $\tau_{\mathrm{eq}}$ one has to use the Lawson bipolar Klein bottle $\widetilde{\tau}_{3,1}$ (see § 5 for a definition). It was proved in [5] that $\Lambda_1(\widetilde{\tau}_{3,1}) = 12\pi E\left(2\sqrt{2}/3\right)$. By a suitable rescaling of the metric on $\widetilde{\tau}_{3,1}$, one can assume that $\lambda_1(\widetilde{\tau}_{3,1}) = 2$ and then apply the construction in the previous paragraph.

## § 3. Otsuki tori

**3.1. The connection with minimal submanifolds of the sphere.** Let $\psi\colon M \looparrowright \mathbb{S}^n$ be a minimal immersion in the unit sphere with canonical metric $g_{\mathrm{can}}$. We denote the Laplace-Beltrami operator on $M$ associated with the metric $\psi^* g_{\mathrm{can}}$ by $\Delta$. We introduce Weyl's eigenvalue counting function

$$N(\lambda) = \#\{i \mid \lambda_i(M, g) < \lambda\}.$$

The following theorem provides an approach for constructing explicit examples of (smooth) extremal metrics.

**Theorem 4** (El Soufi and Ilias [3]). *Let $\psi\colon M \looparrowright \mathbb{S}^n$ be a minimal immersion in the unit sphere $\mathbb{S}^n$ endowed with the canonical metric $g_{\mathrm{can}}$.*

*Then the metric $\psi^* g_{\mathrm{can}}$ on $M$ is extremal for the functional $\Lambda_{N(2)}(M, g)$.*

Therefore, if we start with a minimal submanifold $N$ of the unit sphere and compute $N(2)$, then the metric induced on $N$ by this immersion is extremal for the functional $\Lambda_{N(2)}(N, g)$. However, for a given minimal submanifold there is no algorithm for computing the exact value of $N(2)$. Nevertheless, this approach was successfully realized by Penskoi in the papers [11] and [14] for the metrics (A) and (B), and also by the author in [16] for the metrics (D). Some of the ideas in this approach were used in [15] for the metrics (C).

**3.2. A reduction theorem for minimal submanifolds.** Let $M$ be a Riemannian manifold equipped with a metric $g'$ and let $G$ be a compact group acting on $M$ by isometries. For every point $x \in M$ we shall denote the stability subgroup of $x$ by $G_x$.

**Definition 4.** For two points $x, y \in M$ we say that $x \preccurlyeq y$ if $G_x \subset gG_yg^{-1}$ for some $g \in G$. The orbit $Gx$ is an *orbit of principal type* if for any point $y \in M$ one has $x \preccurlyeq y$.

Let $M^*$ be the union of all orbits of principal type; then $M^*$ is an open dense submanifold of $M$ (see [20]). Moreover, $M^*/G$ carries a natural Riemannian metric $g$ defined by the formula $g(X, Y) = g'(X', Y')$, where $X$ and $Y$ are tangent vectors at $x \in M^*/G$, and $X'$ and $Y'$ are tangent vectors at $x' \in \pi^{-1}(x) \subset M^*$ (where $\pi$ is the projection) such that $X'$ and $Y'$ are orthogonal to the orbit $\pi^{-1}(x)$ and $d\pi(X') = X$, $d\pi(Y') = Y$.

Let $f\colon N \looparrowright M$ be a $G$-invariant immersed submanifold, that is, a manifold equipped with an action of $G$ by isometries such that $g \cdot f(x) = f(g \cdot x)$ for any $x \in N$.

**Definition 5.** The *cohomogeneity* of a $G$-invariant immersed submanifold $N$ is the number $\dim N - \nu$, where $\nu$ is the dimension of the orbits of principal type.

For $x \in M^*/G$ we define a volume function $V(x)$ by the formula $V(x) = \mathrm{Vol}(\pi^{-1}(x))$. Also for each integer $k \geqslant 1$ we define a metric $g_k = V^{2/k} g$.

**Proposition 3** (Hsiang and Lawson [21]). *Let $f\colon N \looparrowright M^*$ be a $G$-invariant immersed submanifold of cohomogeneity $k$, and let $M^*/G$ be equipped with the metric $g_k$. Then $f\colon N \looparrowright M^*$ is minimal if and only if $\overline{f}\colon N/G \looparrowright M^*/G$ is minimal.*

**3.3. Otsuki tori.** Otsuki tori were introduced by Otsuki in [22]. We recall the concise description given by Penskoi in [11]. For more details see § 1.2 of [11]. Consider the action of SO(2) on the three-dimensional unit sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ given by the formula

$$\alpha \cdot (x, y, z, t) = (\cos \alpha x + \sin \alpha y, -\sin \alpha x + \cos \alpha y, z, t),$$

where $\alpha \in [0, 2\pi)$ is a coordinate on SO(2). The space of orbits $\mathbb{S}^3/\operatorname{SO}(2)$ is the closed half-sphere $\mathbb{S}^2_+$,

$$q^2 + z^2 + t^2 = 1, \qquad q \geqslant 0,$$

where the point $(q, z, t)$ corresponds to the orbit $(q \cos \alpha, q \sin \alpha, z, t) \in \mathbb{S}^3$. The space of principal orbits $(\mathbb{S}^3)^*/\operatorname{SO}(2)$ is the open half-sphere

$$\mathbb{S}^2_{>0} = \{(q, z, t) \in \mathbb{S}^2 | q > 0\}.$$

We introduce the spherical coordinates in the space of orbits,

$$\begin{cases} t = \cos \varphi \sin \theta, \\ z = \cos \varphi \cos \theta, \\ q = \sin \varphi. \end{cases}$$

Since we are looking for minimal submanifolds of cohomogeneity 1, Hsiang-Lawson's metric is given by the formula

$$V^2(d\varphi^2 + \cos^2 \varphi d\theta^2) = 4\pi^2 \sin^2 \varphi (d\varphi^2 + \cos^2 \varphi d\theta^2). \tag{3.1}$$

**Definition 6.** An immersed minimal SO(2)-invariant two-dimensional torus in $\mathbb{S}^3$, such that its image by the projection $\pi\colon \mathbb{S}^3 \to \mathbb{S}^3/\operatorname{SO}(2)$ is a closed geodesic in $(\mathbb{S}^3)^*/\operatorname{SO}(2)$ endowed with the metric (3.1), is called an *Otsuki torus*.

The following proposition was proved in [11].

**Proposition 4.** *Apart from the particular case given by the equation $\psi = \pi/4$, Otsuki tori are in one-to-one correspondence with rational numbers $p/q$ such that*

$$\frac{1}{2} < \frac{p}{q} < \frac{\sqrt{2}}{2}, \qquad p, q > 0, \quad (p, q) = 1.$$

**Definition 7.** We denote the Otsuki torus corresponding to $p/q$ by $O_{p/q}$. Following [11], we reserve the term 'Otsuki tori' for the tori $O_{p/q}$.

In order to fix our notation we give a sketch of the proof of Proposition 4.

*Proof.* We shall use the standard notation for the coefficients of the metric (3.1),

$$E = 4\pi^2 \sin^2 \varphi, \qquad G = 4\pi^2 \sin^2 \varphi \cos^2 \varphi.$$

As we know, the velocity vector of a geodesic has constant length. Suppose this length equals 1. This assumption, together with the equation of geodesics for $\ddot{\theta}$,

yields the following two equations:

$$\dot\theta = \frac{\sin a\cos a}{2\pi\cos^2\varphi\sin^2\varphi}, \tag{3.2}$$

$$\dot\varphi^2 = \frac{\sin^2\varphi\cos^2\varphi - \sin^2 a\cos^2 a}{4\pi^2\sin^4\varphi\cos^2\varphi}, \tag{3.3}$$

where $a$ is the minimum value of $\varphi$ on the geodesic. Then the geodesic is situated in the annulus $a \leqslant \varphi \leqslant \pi/2 - a$. We choose a natural parameter $t$ such that $\varphi(0) = a$.

We denote the difference between the value of $\theta$ corresponding to $\varphi = a$ and the value of $\theta$ closest to it corresponding to $\varphi = \pi/2 - a$ by $\Omega(a)$. It is clear that

$$\Omega(a) = \sin a\cos a\int_a^{\pi/2-a}\frac{d\varphi}{\cos\varphi\sqrt{\sin^2\varphi\cos^2\varphi - \sin^2 a\cos^2 a}}.$$

The geodesic is closed if and only if $\Omega(a) = p\pi/q$. The rest of the proof follows from the following properties of the function $\Omega(a)$ (see [22]):

1) $\Omega(a)$ is continuous and monotonic on $(0, \pi/4]$,
2) $\lim_{a\to 0+}\Omega(a) = \pi/2$ and $\Omega(\pi/4) = \pi/\sqrt{2}$.

**3.4. Estimates for $\Lambda_{2p-1}(O_{p/q})$.** According to [11], the metric on an Otsuki torus $O_{p/q}$ is extremal for the functional $\Lambda_{2p-1}(\mathbb{T}^2, g)$. The goal of this section is to prove the following proposition.

**Proposition 5.** *For all $p$, $q$ such that $(p, q) = 1$ and $1/2 < p/q < \sqrt{2}/2$, the following inequality holds*:

$$8\pi\left(2p - 2 + \frac{\pi}{\sqrt{3}}\right) > \Lambda_{2p-1}(O_{p/q}).$$

In order to prove Proposition 5 we have to prove several auxiliary propositions.

**Proposition 6.** *If $a \in (0, \pi/4)$ is such that $\Omega(a) = p\pi/q$, then*

$$\Lambda_{2p-1}(O_{p/q}) = 8\pi q\cos a E\left(\sqrt{1 - \tan^2 a}\right).$$

*Proof.* We shall use the notation in Proposition 4. As we know,

$$\dot\varphi = \pm\frac{\sqrt{G - c^2}}{\sqrt{EG}},$$

where $c = 2\pi\sin a\cos a$. Therefore, the length of the segment on the geodesic $\pi(O_{p/q})$ between the closest points with $\varphi = a$ and $\varphi = \pi/2 - a$ is equal to $2\pi I$, where

$$I = \int_a^{\pi/2-a}\frac{\sin\varphi}{\sqrt{1 - \sin^2 a\cos^2 a/(\sin^2\varphi\cos^2\varphi)}}\,d\varphi.$$

We express $I$ in terms of elliptic integrals,

$$I = \int_{\sin a}^{\cos a} \frac{x\sqrt{1-x^2}}{\sqrt{x^2(1-x^2)-\cos^2 a\sin^2 a}}\,dx = \frac{1}{2}\int_{\sin^2 a}^{\cos^2 a} \frac{\sqrt{1-u}}{\sqrt{u(1-u)-\cos^2 a\sin^2 a}}\,du$$
$$= \frac{1}{2}\int_0^1 \frac{\sqrt{(1-\sin^2 a)-(\cos^2 a-\sin^2 a)t}}{\sqrt{t(1-t)}}\,dt = \frac{1}{2}\cos a\int_0^1 \frac{\sqrt{1-(1-\tan^2 a)t}}{\sqrt{t(1-t)}}\,dt$$
$$= \cos a\int_0^1 \frac{\sqrt{1-(1-\tan^2 a)y^2}}{\sqrt{1-y^2}}\,dy = \cos aE\Big(\sqrt{1-\tan^2 a}\Big).$$

Here we have used the following changes of variables:

$$\cos\varphi = x, \qquad x^2 = u, \qquad u = (\cos^2 a-\sin^2 a)t+\sin^2 a, \qquad t = y^2.$$

Since the maps $\theta\mapsto\theta+\theta_0$ and $\theta\mapsto\theta_0-\theta$ are isometries, the length of the geodesic $\pi(O_{p/q})$ is equal to $4\pi q\cos aE(\sqrt{1-\tan^2 a})$. By Proposition 13 in [11], $\Lambda_{2p-1}(O_{p/q})$ is equal to twice the length of the geodesic $\pi(O_{p/q})$.

**Proposition 7.** *If $k\in[0,1]$ the following inequality holds*:

$$K(k)-\frac{2}{2-k^2}E(k)\geqslant 0.$$

*Proof.* We expand the left-hand side using the definitions of $E$ and $K$,

$$K(k)-\frac{2}{2-k^2}E(k) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}-\frac{2}{2-k^2}\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\,d\theta$$
$$= \frac{k^2}{2-k^2}\int_0^{\pi/2}\frac{2\sin^2\theta-1}{\sqrt{1-k^2\sin^2\theta}}\,d\theta.$$

Since the integrand is negative on $(0,\pi/4)$ and positive on $(\pi/4,\pi/2)$, one has

$$\int_0^{\pi/2}\frac{2\sin^2\theta-1}{\sqrt{1-k^2\sin^2\theta}}\,d\theta = \int_0^{\pi/4}\frac{2\sin^2\theta-1}{\sqrt{1-k^2\sin^2\theta}}\,d\theta+\int_{\pi/4}^{\pi/2}\frac{2\sin^2\theta-1}{\sqrt{1-k^2\sin^2\theta}}\,d\theta$$
$$\geqslant \int_0^{\pi/4}\frac{2\sin^2\theta-1}{\sqrt{1-k^2/2}}\,d\theta+\int_{\pi/4}^{\pi/2}\frac{2\sin^2\theta-1}{\sqrt{1-k^2/2}}\,d\theta$$
$$= -\frac{1}{\sqrt{1-k^2/2}}\int_0^{\pi/2}\cos 2\theta\,d\theta = 0.$$

We introduce the notation

$$\Phi(a) = \cos aE\Big(\sqrt{1-\tan^2 a}\Big).$$

**Proposition 8.** *The function $\Phi(a)$ is nondecreasing and $\Phi'(a) < 1/2$ for any $a \in (0, \pi/4)$. In particular, $1 = \Phi(0) \leqslant \Phi(a) \leqslant \Phi(\pi/4) = \pi/(2\sqrt{2})$.*

**Corollary 1.** *The following inequalities hold:*

$$4\sqrt{2}\pi^2 q \geqslant \Lambda_{2p-1}(O_{p/q}) \geqslant 8\pi q. \tag{3.4}$$

*Remark* 4. We should point out that while the manuscript was in preparation inequality (3.4) appeared in the paper [23].

*Proof of Proposition* 8. Recall the following formulae for the derivatives of elliptic integrals:

$$\frac{dE(k)}{dk} = \frac{E(k) - K(k)}{k}, \qquad \frac{dK(k)}{dk} = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}, \tag{3.5}$$

$$\frac{\partial \Pi(n,k)}{\partial n} = \frac{1}{2(k^2-n)(n-1)}\left(E(k) + \frac{(k^2-n)}{n}K(k) + \frac{(n^2-k^2)}{n}\Pi(n,k)\right),$$
$$\frac{\partial \Pi(n,k)}{\partial k} = \frac{k}{n-k^2}\left(\frac{E(k)}{k^2-1} + \Pi(n,k)\right). \tag{3.6}$$

We introduce the notation $\beta = \sqrt{1 - \tan^2 a}$. Then

$$\begin{aligned}\Phi'(a) &= \cos a\left(-2\tan a \frac{E(\beta) - K(\beta)}{2\cos^2 a(1-\tan^2 a)}\right) - \sin a E(\beta)\\ &= -\sin a\left(E(\beta) + \frac{E(\beta) - K(\beta)}{\cos^2 a - \sin^2 a}\right)\\ &= \frac{\sqrt{(1-\beta^2)(2-\beta^2)}}{\beta^2}\left(K(\beta) - \frac{2}{2-\beta^2}E(\beta)\right).\end{aligned} \tag{3.7}$$

Now the fact that the function $\Phi(a)$ is monotonic follows from Proposition 7.

To prove the second part, we return to formula (3.7). We have

$$\begin{aligned}\Phi'(a) &= -\sin a\left(\frac{2\cos^2 a E(\beta) - K(\beta)}{\cos^2 a - \sin^2 a}\right)\\ &= -\frac{\sin a}{\cos^2 a - \sin^2 a}\int_0^{\pi/2} \frac{2\cos^2 a(1-\beta^2\sin^2\theta) - 1}{\sqrt{1-\beta^2\sin^2\theta}}\,d\theta\\ &= \sin a\int_0^{\pi/2} \frac{2\sin^2\theta - 1}{\sqrt{1-\beta^2\sin^2\theta}}\,d\theta \leqslant \sin a\int_{\pi/4}^{\pi/2} \frac{2\sin^2\theta - 1}{\sqrt{1-\beta^2}}\,d\theta\\ &= -\cos a\int_{\pi/4}^{\pi/2}\cos 2\theta\,d\theta = \cos a\frac{\sin 2\theta}{2}\Big|_{\pi/2}^{\pi/4} \leqslant \frac{1}{2}.\end{aligned}$$

This completes the proof of Proposition 8.

**Proposition 9.** *The function* $(2/\pi)\Omega(a) - \Phi(a)$ *is increasing on the interval* $(0, \pi/4)$.

*Proof.* In [23] the following formula was proved:

$$\Omega(a) = \frac{1}{\sin a}\,\Pi\left(-\frac{\cos 2a}{\sin^2 a}, \sqrt{1-\tan^2 a}\right).$$

Using (3.6), we obtain the following formula:

$$\frac{d\Omega(a)}{da} = \frac{1}{\cos a \cos 2a} K\left(\sqrt{1-\tan^2 a}\right) - \frac{2\cos a}{\cos 2a} E\left(\sqrt{1-\tan^2 a}\right).$$

Recall the notation $\beta(a) = \sqrt{1-\tan^2 a}$. Then

$$\Omega'(a) = \frac{(2-\beta^2)^{3/2}}{\beta^2}\left(K(\beta) - \frac{2}{2-\beta^2}E(\beta)\right), \tag{3.8}$$

$$\Omega(a) = \sqrt{\frac{2-\beta^2}{1-\beta^2}}\,\Pi\left(-\frac{\beta^2}{1-\beta^2}, \beta\right).$$

Moreover, by formula (3.7) we find that

$$\Phi'(a) = \frac{\sqrt{(1-\beta^2)(2-\beta^2)}}{\beta^2}\left(K(\beta) - \frac{2}{2-\beta^2}E(\beta)\right).$$

The inequality $2/\pi(2-\beta^2) - \sqrt{1-\beta^2} > 0$ and Proposition 7 imply the inequality

$$\frac{2}{\pi}\Omega'(a) - \Phi'(a) = \frac{\sqrt{2-\beta^2}}{k^2}\left(K(\beta) - \frac{2}{2-\beta^2}E(\beta)\right)\left(\frac{2}{\pi}(2-\beta^2) - \sqrt{1-\beta^2}\right) > 0.$$

**Corollary 2.** *If* $a \in [1/5, \pi/4]$, *then*

$$\frac{2}{\pi}\Omega(a) - \Phi(a) > \frac{2\sqrt{3}-\pi}{3\sqrt{3}}.$$

*Proof.* Using the tables of elliptic integrals given in the book [17], for instance, we obtain the inequality

$$\frac{2}{\pi}\Omega\left(\frac{1}{5}\right) - \Phi\left(\frac{1}{5}\right) > \frac{2\sqrt{3}-\pi}{3\sqrt{3}}.$$

The rest of the proof follows as the function on the left-hand side is monotonic.

**Proposition 10.** *If* $\xi \in [0, 1/5]$, *then*

$$\Omega'(\xi) > \frac{\pi}{4}\left(\frac{\pi}{\sqrt{3}} - 1\right)^{-1}.$$

*Proof.* By formula (3.8) for $\xi \in [0, 1/5]$ we see that

$$\Omega'(\xi) = \frac{(2-\beta(\xi)^2)^{3/2}}{\beta(\xi)^2}\left(K(\beta(\xi)) - \frac{2}{2-\beta(\xi)^2}E(\beta(\xi))\right)$$
$$\geqslant K\left(\beta\left(\frac{1}{5}\right)\right) - 2\frac{2-\beta^2(1/5)}{\beta^2(1/5)}E\left(\beta\left(\frac{1}{5}\right)\right).$$

In the last inequality we used the facts that $K(k)$ is an increasing function, whilst $E(k)$ and $\beta(a)$ are decreasing functions. The table of the elliptic integrals in [17] provides the inequality

$$K\left(\beta\left(\frac{1}{5}\right)\right) - 2\frac{2-\beta^2(1/5)}{\beta^2(1/5)}E\left(\beta\left(\frac{1}{5}\right)\right) > \frac{\pi}{4}\left(\frac{\pi}{\sqrt{3}} - 1\right)^{-1},$$

which completes the proof.

*Proof of Proposition 5.* We want to prove that

$$8\pi\left(2p - 2 + \frac{\pi}{\sqrt{3}}\right) > 8\pi q\Phi(a),$$

where $\Omega(a) = p\pi/q$. This inequality is equivalent to the following:

$$2\frac{p}{q} - \frac{2\sqrt{3}-\pi}{q\sqrt{3}} > \Phi(a).$$

Since $\Omega(a) = p\pi/q$, it is sufficient to prove that

$$\frac{2}{\pi}\Omega(a) - \Phi(a) > \frac{2\sqrt{3}-\pi}{q\sqrt{3}}. \tag{3.9}$$

Since $q \geqslant 3$, Corollary 2 shows that inequality (3.9) holds for $a \in [1/5, \pi/4]$. In order to prove this inequality for $a \in [0, 1/5]$ we note that by Proposition 8

$$\frac{2}{\pi}\Omega(a) - \Phi(a) = \frac{2}{\pi}(\Omega(a) - \Omega(0)) - (\Phi(a) - \Phi(0))$$
$$= a\left(\frac{2}{\pi}\Omega'(\xi) - \Phi'(\eta)\right) \geqslant a\left(\frac{2}{\pi}\Omega'(\xi) - \frac{1}{2}\right)$$

for some $\xi, \eta \in (0, a)$. Moreover,

$$\frac{1}{2q}\pi \leqslant \frac{2p-q}{2q}\pi = \frac{p}{q}\pi - \frac{1}{2}\pi = \Omega(a) - \Omega(0) = a\Omega'(\xi),$$

or

$$\frac{1}{q} < \frac{2a}{\pi}\Omega'(\xi).$$

Therefore, (3.9) follows from the inequality

$$\frac{2}{\pi}\Omega'(\xi) - \frac{1}{2} > \frac{2}{\pi}\left(2 - \frac{\pi}{\sqrt{3}}\right)\Omega'(\xi),$$

or the inequality

$$\Omega'(\xi) > \frac{\pi}{4}\left(\frac{\pi}{\sqrt{3}} - 1\right)^{-1}.$$

The last inequality follows easily from Proposition 10.

## § 4. Lawson surfaces

A Lawson tau-surface $\tau_{m,k}$ is an immersed surface in the sphere $\mathbb{S}^3$ defined by the doubly-periodic immersion of $\mathbb{R}^2$ in $\mathbb{R}^4$ given by the formula

$$(\cos mx \cos y,\ \sin mx \cos y,\ \cos kx \sin y,\ \sin kx \sin y).$$

It was introduced by Lawson in [24]. He also proved that for each pair $\{m, k\}$ such that $m \geqslant k \geqslant 1$ and $(m, k) = 1$ the surfaces $\tau_{m,k}$ are distinct compact immersive surfaces in $\mathbb{S}^3$. Assume that $(m, k) = 1$; if both $m$ and $k$ are odd then $\tau_{m,k}$ is a torus, which we call a Lawson torus. Otherwise $\tau_{m,k}$ is a Klein bottle, called a Lawson Klein bottle.

**Proposition 11** (Penskoi [14]). *Let $\tau_{m,k}$ be a Lawson surface. Then the induced metric on $\tau_{m,k}$ is an extremal metric for the functional $\Lambda_j(M, g)$, where*

$$j = 2\left[\frac{\sqrt{m^2 + k^2}}{2}\right] + m + k - 1, \tag{4.1}$$

*$M = \mathbb{T}^2$ if both $m$, $k$ are odd and $M = \mathbb{K}$ otherwise.*

*The corresponding value of the functional is*

$$\Lambda_j(\tau_{m,k}) = 8\pi m E\left(\frac{\sqrt{m^2 - k^2}}{m}\right).$$

**Proposition 12.** *Let $j$ be defined by formula (4.1). If $\tau_{m,k}$ is a Lawson torus, then*

$$\Lambda_j(\tau_{m,k}) < 8\pi\left(j - 1 + \frac{\pi}{\sqrt{3}}\right).$$

*If $\tau_{m,k}$ is a Klein bottle, then*

$$\Lambda_j(\tau_{m,k}) < 8\pi(j - 1) + 12\pi E\left(\frac{2\sqrt{2}}{3}\right).$$

*Proof.* It is sufficient to show that

$$j \geqslant m E\left(\frac{\sqrt{m^2 - k^2}}{m}\right). \tag{4.2}$$

Note that the function

$$\varphi(x) = 1 + x - E\left(\sqrt{1 - x^2}\right)$$

is positive on the interval $[0, 1]$. Indeed,

$$\begin{aligned} E(x) &= \int_0^{\pi/2} \sqrt{1 - x^2 \sin^2\psi}\, d\psi \leqslant \int_0^{\pi/2} \left(\sqrt{1 - \sin^2\psi} + \sqrt{(1 - x^2)\sin^2\psi}\right) d\psi \\ &= 1 + \sqrt{1 - x^2}. \end{aligned}$$

Now divide both sides of (4.2) by $m$ and denote the ratio $k/m \in [0, 1]$ by $x$. Since

$$\left[\sqrt{\frac{m^2 + k^2}{2}}\right] \geqslant \left[\frac{m + k}{2}\right] \geqslant \left[\frac{m + 1}{2}\right] > \frac{m}{2},$$

inequality (4.2) follows from the positivity of $\varphi(x)$.

## § 5. Bipolar surfaces to Lawson surfaces

Let $I\colon N \looparrowright \mathbb{S}^3$ be a minimal immersion. The Gauss map $I^*\colon N \to \mathbb{S}^3$ is defined pointwise as the image of the unit normal in $\mathbb{S}^3$ translated to the origin in $\mathbb{R}^4$. Then the exterior product $\widetilde{I} = I \wedge I^*$ is an immersion of $N$ in $\mathbb{S}^5 \subset \mathbb{R}^6$. In [24] Lawson proved that this immersion is minimal. The image $\widetilde{I}(N)$ is called the bipolar surface to $N$.

We denote the bipolar surface to the surface $\tau_{m,k}$ by $\widetilde{\tau}_{m,k}$. Lapointe proved in [15] that

- if $mk \equiv 0 \pmod 2$, then $\widetilde{\tau}_{m,k}$ is a torus carrying an extremal metric for the functional $\Lambda_{4m-2}(\mathbb{T}^2, g)$ and
$$\Lambda_{4m-2}(\widetilde{\tau}_{m,k}) = 16\pi m E\left(\frac{\sqrt{m^2-k^2}}{m}\right);$$
- if $mk \equiv 1 \pmod 4$, then $\widetilde{\tau}_{m,k}$ is a torus carrying an extremal metric for the functional $\Lambda_{2m-2}(\mathbb{T}^2, g)$ and
$$\Lambda_{2m-2}(\widetilde{\tau}_{m,k}) = 8\pi m E\left(\frac{\sqrt{m^2-k^2}}{m}\right);$$
- if $mk \equiv 3 \pmod 4$, then $\widetilde{\tau}_{m,k}$ is a Klein bottle carrying an extremal metric for $\Lambda_{m-2}(\mathbb{K}\mathrm{l}, g)$ and
$$\Lambda_{m-2}(\widetilde{\tau}_{m,k}) = 4\pi m E\left(\frac{\sqrt{m^2-k^2}}{m}\right).$$

**Proposition 13.** *If $mk \equiv 1 \pmod 4$, then the following inequality holds*:
$$\Lambda_{2m-2}(\widetilde{\tau}_{m,k}) < 8\pi\left(2m - 3 + \frac{\pi}{\sqrt{3}}\right).$$
*If $mk \equiv 0 \pmod 2$, then the following inequality holds*:
$$\Lambda_{4m-2}(\widetilde{\tau}_{m,k}) < 8\pi\left(4m - 3 + \frac{\pi}{\sqrt{3}}\right).$$
*If $mk \equiv 3 \pmod 4$, then the following inequality holds*:
$$8\pi(m-3) + 12\pi E\left(\frac{2\sqrt{2}}{3}\right) > \Lambda_{m-2}(\widetilde{\tau}_{m,k}).$$

*Proof.* In order to prove the first inequality it is sufficient to prove that
$$mE\left(\frac{\sqrt{m^2-k^2}}{m}\right) \leqslant (2m-2). \tag{5.1}$$
It is well-known that $E(\widetilde{k}) \leqslant \pi/2$ for $\widetilde{k} \in [0,1]$. This implies that it is sufficient to prove that
$$\pi m \leqslant 4m - 4.$$

This inequality holds for $m \geqslant 5$. The statement for $\widetilde{\tau}_{1,1}$ follows from the fact that $\widetilde{\tau}_{1,1}$ is a Clifford torus and $\Lambda_1(\tau_{1,1}) = 4\pi^2$.

In the same way, in order to prove the second inequality in Proposition 13 it is sufficient to prove that

$$\pi m \leqslant 4m - 3 + \frac{\pi}{\sqrt{3}}.$$

This inequality holds for $m \geqslant 2$.

The third inequality is equivalent to the following:

$$2(m-3) + 3E\left(\frac{2\sqrt{2}}{3}\right) > mE\left(\frac{\sqrt{m^2-k^2}}{m}\right).$$

Since $E(\widetilde{k}) < \pi/2$, it is sufficient to prove that

$$\left(2 - \frac{\pi}{2}\right)m > 6 - 3E\left(\frac{2\sqrt{2}}{3}\right).$$

This inequality holds for $m \geqslant 7$. For the exceptional case $\{m,k\} = \{5,3\}$ one has to verify the third inequality explicitly using the tables of elliptic integrals in [17].

## § 6. Bipolar surfaces to Otsuki tori

The following proposition was proved in [16].

**Proposition 14.** *The bipolar surface $\widetilde{O}_{p/q}$ to an Otsuki torus $O_{p/q}$ is a torus.*

*If $q$ is odd, then the metric on the bipolar Otsuki torus $\widetilde{O}_{p/q}$ is extremal for the functional $\Lambda_{2q+4p-2}(\mathbb{T}^2, g)$ and $\Lambda_{2q+4p-2}(\widetilde{O}_{p/q}) < 4\sqrt{2}q\pi^2$.*

*If $q$ is even, then the metric on the bipolar Otsuki torus $\widetilde{O}_{p/q}$ is extremal for the functional $\Lambda_{q+2p-2}(\mathbb{T}^2, g)$ and $\Lambda_{q+2p-2}(\widetilde{O}_{p/q}) < 2\sqrt{2}q\pi^2$.*

**Proposition 15.** *If $q$ is even, then the following inequality holds*:

$$\Lambda_{q+2p-2}(\widetilde{O}_{p/q}) < 8\pi\left(q + 2p - 3 + \frac{\pi}{\sqrt{3}}\right).$$

*If $q$ is odd, then*

$$\Lambda_{2q+4p-2}(\widetilde{O}_{p/q}) < 8\pi\left(2q + 4p - 3 + \frac{\pi}{\sqrt{3}}\right).$$

*Proof.* If $q$ is even, then we have

$$8\pi\left(q + 2p - 3 + \frac{\pi}{\sqrt{3}}\right) > 8\pi(q + 2p - 2) > 12\pi q > 2\sqrt{2}\pi^2 q.$$

We have used the inequalities $2p > q$ and $p > 1$ in order to prove the last inequality. In the same way, if $q$ is odd, then we have

$$8\pi\left(2q + 4p - 3 + \frac{\pi}{\sqrt{3}}\right) > 8\pi(2q + 4p - 2) > 24\pi q > 4\sqrt{2}\pi^2 q.$$

Now it is easy to see that Propositions 5, 12, 13 and 15 together with Proposition 2 imply Theorem 1.

## § 7. The Clifford torus

We will represent the Clifford torus as a flat torus with a square lattice with edges equal to $2\pi$. In this case the Laplace-Beltrami operator coincides up to a sign with the classical two-dimensional Laplace operator. Therefore, using separation of variables we obtain a basis of the following form for the eigenfunctions:

$$\sin nx \sin my, \qquad \sin nx \cos ly, \qquad \cos kx \sin my, \qquad \cos kx \cos ly,$$

where $n, m \in \mathbb{N}$ and $k, l \in \mathbb{Z}_{\geqslant 0}$. The corresponding eigenvalues are equal to $n^2+m^2$, $n^2+l^2$, $k^2+m^2$, $k^2+l^2$.

**Proposition 16.** *Weyl's counting function $N(\lambda)$ for the Clifford torus is equal to the number of integer points in the open disk of radius $\sqrt{\lambda}$ with centre at the origin of $\mathbb{R}^2$.*

*Proof.* We introduce a one-to-one correspondence $\nu$ between eigenfunctions and integer points in $\mathbb{R}^2$. We set

$$\begin{cases} \nu(\sin nx \sin my) = (n, m), \\ \nu(\sin nx \cos ly) = (n, -l), \\ \nu(\cos kx \sin my) = (-k, m), \\ \nu(\cos kx \cos ly) = (-k, -l). \end{cases}$$

We also note that the eigenvalue of the function $f$ is equal to the square of the distance between $(0, 0)$ and $\nu(f)$. This observation completes the proof.

**7.1. Proof of Proposition 1.** It is easy to check that the set of functions

$$(\sin kx,\ \cos kx,\ \sin ky,\ \cos ky)$$

forms an isometric immersion of the Clifford torus in the unit sphere. The same is true for the set

$$(\sin kx \sin ky,\ \sin kx \cos ky,\ \cos kx \sin ky,\ \cos kx \cos ky)$$

and the set

$$\begin{aligned} (\sin kx \sin ly,\ \sin kx \cos ly,\ \cos kx \sin ly,\ \cos kx \cos ly,\\ \sin lx \sin ky,\ \sin lx \cos ky,\ \cos lx \sin ky,\ \cos lx \cos ky), \end{aligned}$$

where $k \neq l$. Therefore, according to Theorem 4, the metric on the Clifford torus is extremal for the functionals $\Lambda_{N(r^2)}(\mathbb{T}^2, g)$, where $r^2 = n^2 + m^2$ with $n, m \in \mathbb{Z}$, and $\Lambda_{N(r^2)}(\mathbb{T}_{Cl}) = 4\pi^2 r^2$.

Let $B_r$ be a disc of radius $r$. Then the simple estimate

$$N(r^2) \geqslant \operatorname{Area}(B_{r-\sqrt{2}/2}) = \pi\left(r - \frac{\sqrt{2}}{2}\right)^2$$

holds.

So it is sufficient to prove that

$$2\left(r - \frac{\sqrt{2}}{2}\right)^2 > r^2,$$

and this inequality holds for $r^2 \geqslant 6$. For $r^2 < 6$ we have the inequality

$$8\pi N(r^2) > 4\pi r^2.$$

This inequality can be obtained by a direct enumeration of all possible values of $r^2$. This completes the proof of Proposition 1.

The author thanks A. V. Penskoi for posing this problem, for fruitful discussions and for his invaluable help in the preparation of the manuscript.

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Received 27/FEB/13 and 10/JUN/13
Translated by M. KARPUKHIN