SPECTRAL MOMENT FORMULAE FOR GL(3) × GL(2) L-FUNCTIONS I: THE CUSPIDAL CASE

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Abstract. Spectral moment formulae of various shapes have proven very successful in studying the statistics of central L-values. In this article, we establish, in a completely explicit fashion, such formulae for the family of $\mathrm{GL}(3) \times \mathrm{GL}(2)$ Rankin–Selberg L-functions using the period integral method. Our argument does not rely on either the Kuznetsov or Voronoi formulae. We also prove the essential analytic properties and derive explicit formulae for the integral transform of our moment formulae. We hope that our method will provide deeper insights into moments of L-functions for higher-rank groups.

1. Introduction

1.1. **Background.** The study of L-values at the central point $s = 1/2$ has taken center stage in many branches of number theory over the past decades due to their profound arithmetic significance. A variety of perspectives have enriched our understanding of the nature of central L-values. In particular, a statistical perspective can offer valuable insights. Fundamental questions in this direction include the determination of (non-)vanishing and sizes of these L-values. An effective way to approach problems of this sort is via **Moments of L-functions**. Techniques from analytic number theory have proven fruitful in estimating the sizes of moments of all kinds. Moreover, spectacular results can be obtained when moment estimates join forces with arithmetic geometry and automorphic representations.

This line of investigation is nicely exemplified by the landmark result of Conrey–Iwaniec [CI00]. Let χ be a real primitive Dirichlet character $(\bmod\, q)$ with q odd and square-free. The main object of [CI00] is the cubic moment of GL(2) automorphic L-functions of the congruence subgroup $\Gamma_0(q)$ twisted by χ. An *upper bound* of Lindelöf strength in the q-aspect was established therein. When combining this upper bound with the celebrated Waldspurger formula [Wa81], the famous Burgess 3/16-bound for Dirichlet L-functions was improved for the first time since the 1960's. In fact, [CI00] proved the bound

$$L\left(\frac{1}{2}, \chi\right) \ll_\epsilon q^{\frac{1}{6}+\epsilon}. \tag{1.1}$$

Understanding the effects of a sequence of intricate arithmetic and analytic transformations constitutes a significant part of moment calculations as illustrated by [CI00]. Surprisingly, such a sequence of [CI00] ends up in a single elegant *identity* showcasing a duality between the cubic average over a basis of GL(2) automorphic forms (Maass or holomorphic) and the fourth moment of GL(1) L-functions. This remarkable phenomenon was uncovered relatively recently by Petrow [Pe15]. His work consists of new elaborate analysis (see also Young [Y17]) building upon the foundation of [CI00]. Further contributions to this topic include Frolenkov [Fr20] and earlier works of Ivić [Iv01, Iv02], which studied other aspects of the problem. In its basic form, the identity roughly takes the shape

$$\sum_{\substack{f:\ GL(2)\text{-Maass/} \\ \text{Holomorphic}}} L\left(\frac{1}{2}, f\right)^3 = \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt + (***), \tag{1.2}$$

where the weight functions for the moments are suppressed and $(***)$ represents certain polar contributions.

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Besides its structural elegance, the identity (1.2) comes with immediate applications. It leads to sharp moment estimates as a consequence of exact evaluation. As an extra benefit, it streamlines the analysis in the traditional, approximate approaches. In [Pe15], this identity was referred to as a *'Motohashi-type identity'*. Previously, Motohashi [Mo93, Mo97] discovered a similar identity, but with the test function chosen on the fourth moment side, effectively reversing the direction of analysis performed by [CI00, Pe15, Y17, Iv01, Iv02]. It greatly enhances our understanding of the fourth moment of the ζ-function. Recent works by Young [Y11], Blomer et al. [BHKM20], Topacogullari [To21] and Kaneko [Ka21+] extending Motohashi's work to Dirichlet L-functions.

In [CI00, Introduction], Conrey–Iwaniec further envisioned the possibilities and challenges of extending their method to a setting involving a GL(3) automorphic form. This is natural because the cubic moment of GL(2) L-functions can be regarded as the first moment of GL(3) × GL(2) Rankin–Selberg L-functions, averaged over a basis of GL(2) automorphic forms, where the GL(3) automorphic form is a minimal-parabolic Eisenstein series. It is anticipated that advances in harmonic analysis of GL(3) could provide new perspectives towards the Conrey–Iwaniec method. Furthermore, the GL(3) set-up introduces an important new example: the first moment for the GL(3) × GL(2) family involving a GL(3) *cusp form*, which necessitates the use of genuine GL(3) techniques.

In the decade following [CI00], two key breakthroughs made this extension possible for GL(3). Firstly, Miller–Schmid [MS06] (see also [GoLi06, IT13]) developed the *GL*(3) *Voronoi formula*, making it usable for various analytic applications. Notably, the Hecke combinatorics of GL(3) associated to twisting and ramifications are considerably more involved than the classical GL(2) counterpart. Secondly, Li [Li11] successfully applied the GL(3) Voronoi formula together with new techniques of her own to obtain strong *upper bounds* for the first moment of GL(3) × GL(2) Rankin–Selberg L-functions in the GL(2) spectral aspect. As a corollary, she obtained the first instance of subconvexity for GL(3) automorphic L-functions.

1.2. **Main Results.** The purpose of this article is to further the investigation of GL(3) × GL(2) moments of L-functions. However, we will depart from the standard approaches taken in the existing literature. We are interested in understanding the *intrinsic mechanisms* and examining the *essential ingredients* that lead directly to the complete structure of these moments, including both main terms and off-diagonals. Addressing these aspects carefully is crucial for enabling generalizations to higher-rank groups. We find that the formalism of *period integrals* for GL(3) is particularly effective in achieving these objectives.

We are ready to state the main result of this article, which is the moment identity of Motohashi type behind the work of [Li11].

Theorem 1.1. Let

- Φ be a fixed, Hecke-normalized Maass cusp form of $SL_3(\mathbb{Z})$ with the Langlands parameters $(\alpha_1, \alpha_2, \alpha_3) \in (i\mathbb{R})^3$, and $\widetilde{\Phi}$ be the dual form of Φ;
- $(\phi_j)_{j=1}^{\infty}$ be an orthogonal basis of **even**, Hecke-normalized Maass cusp forms of $SL_2(\mathbb{Z})$ which satisfy $\Delta\phi_j = (1/4 - \mu_j^2)\phi_j$;
- $L(s, \phi_j \otimes \Phi)$ and $L(s, \Phi)$ be the Rankin–Selberg L-function of the pair (ϕ_j, Φ) and the standard L-function of Φ respectively, where Λ denotes the corresponding complete L-functions;
- $\mathcal{C}_\eta$ $(\eta > 40)$ be the class of holomorphic functions H defined on the vertical strip $|\operatorname{Re}\mu| < 2\eta$ such that $H(\mu) = H(-\mu)$ and has rapid decay:

$$H(\mu) \ \ll \ e^{-2\pi|\mu|} \qquad\qquad (|\operatorname{Re}\mu| \ < \ 2\eta).$$

- For $H \in \mathcal{C}_\eta$, $(\mathcal{F}_\Phi H)(s_0, s)$ is the integral transform defined in equation (7.6) and it only depends on the Langlands parameters of Φ.

Then on the domain $\frac{1}{4}+\frac{1}{200}<\sigma<\frac{3}{4}$, we have the following moment identity:

$$\sum_{j=1}^{\infty} H(\mu_j)\frac{\Lambda(s,\phi_j\otimes\widetilde{\Phi})}{\langle\phi_j,\phi_j\rangle}+\int_{(0)} H(\mu)\,\frac{\Lambda(s+\mu,\widetilde{\Phi})\Lambda(1-s+\mu,\Phi)}{|\Lambda(1+2\mu)|^2}\,\frac{d\mu}{4\pi i}$$
$$=\frac{\pi^{-3s}}{2}\,L(2s,\Phi)\int_{(0)}\frac{H(\mu)}{|\Gamma(\mu)|^2}\prod_{i=1}^{3}\Gamma\left(\frac{s+\mu-\alpha_i}{2}\right)\Gamma\left(\frac{s-\mu-\alpha_i}{2}\right)\frac{d\mu}{2\pi i}$$
$$+\frac{1}{2}\,L(2s-1,\Phi)\left(\mathcal{F}_\Phi H\right)(2s-1,s)$$
$$+\frac{1}{2}\int_{(1/2)}\zeta\left(2s-s_0\right)L\left(s_0,\Phi\right)\left(\mathcal{F}_\Phi H\right)\left(s_0,s\right)\frac{ds_0}{2\pi i}. \tag{1.3}$$

The function $s\mapsto\left(\mathcal{F}_\Phi H\right)(2s-1,s)$ can be computed explicitly, see Theorem 1.2 below.

The temperedness assumption $(\alpha_1,\alpha_2,\alpha_3)\in(i\mathbb{R})^3$ for our fixed Maass cusp form Φ is very mild — it merely serves as a simplification of our exposition (when applying Stirling's formula in Section 8) and can be removed with a little more effort. In fact, all Maass cusp forms of $\mathrm{SL}_3(\mathbb{Z})$ are conjectured to be tempered and it was proved in [Mil01] that the non-tempered forms constitute a density zero set.

We have made no attempt to enlarge the class of test functions for Theorem 1.1 since this is not the focus of this article (but is doable by more refined analysis). The regularity assumptions of $\mathcal{C}_\eta$ essentially follow from those of the Kontorovich–Lebedev inversion (see Section 5.2). As in [GK13, GSW21, GSW23+, Bu20], the class $\mathcal{C}_\eta$ already includes good test functions that are useful in a number of applications and allows us to deduce a version of Theorem 1.1 for incomplete L-functions (see Remark 5.27).

Also, we have obtained the analytic properties and several explicit expressions for the integral transform $\left(\mathcal{F}_\Phi H\right)(s_0,s)$. They are written in terms of Mellin-Barnes integrals or hypergeometric functions as in [Mo93, Mo97]. We do not record the full formulae here but refer the readers to Section 10 for the detailed discussions. However, we record an interesting identity of special functions as follow:

Theorem 1.2 (Theorem 10.2)**.** For $1/2+1/100<\sigma<1$, we have

$$\left(\mathcal{F}_\Phi H\right)(2s-1,s)\;=\;\pi^{\frac{1}{2}-s}\prod_{i=1}^{3}\frac{\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}\right)}{\Gamma\left(1-s-\frac{\alpha_i}{2}\right)}\cdot\int_{(0)}\frac{H(\mu)}{|\Gamma(\mu)|^2}\prod_{i=1}^{3}\prod_{\pm}\Gamma\left(\frac{1-s+\alpha_i\pm\mu}{2}\right)\frac{d\mu}{2\pi i}. \tag{1.4}$$

There are actually two additional identities of Barnes type that account for the origins and the combinatorics of six (out of eight) of the off-diagonal main terms for the cubic moment of GL(2) L-functions. The results align nicely with the predictions of the *'Moment Conjecture'* (or *'Recipe'*) of [CFKRS05]. We refer the interested readers to our papers [Kw23a+, Kw23b+].

1.3. **Follow-up Works.** The current work aims to illustrate the key ideas and address the main analytic issues of our period integral approach. It is the simplest to illustrate all these using the cuspidal case for Φ. However, this is by no means the end of the scope of our method. In our upcoming works [Kw23a+, Kw23b+], we demonstrate the versatility of our method by:

(1) Providing a new proof of the cubic moment identity (1.2) (actually for the more general *'shifted moment'*) with a number of technical advantages, as well as a new unified way of extracting the full set of main terms. There are considerable recent interests in understanding the deep works of [Mo93, Mo97] and [CI00] from different perspectives, e.g., Nelson [Ne19+], Wu [Wu21+], Balkanova–Frolenkov–Wu [BFW21+].
(2) Establishing a Motohashi's formula of GL(3) in the non-archimedean aspect which dualizes GL(2) twists of Hecke eigenvalues into GL(1) twists by Dirichlet characters. This offers insights into the celebrated works of Young [Y11] and Blomer et al. [BHKM20] on the fourth moment of Dirichlet L-functions. In their works, this change of structures was the result of a long sequence of spectral/harmonic transformations and it was surprising and useful to observe such a phenomenon.

2. Outline

In Section 3, we discuss the technical features of the method used in this article and draw comparisons with the current literature. In Section 4, we include a sketch of our arguments to demonstrate the essential ideas of our method and sidestep the technical points. In Section 5, we collect the essential notions and results for later parts of the article.

The proof of Theorem 1.1 is divided into four sections. In Section 6, we prove the key identity of this article (see Corollary 6.2). In Section 7, we develop such an identity into moments of L-functions on the region of absolute convergence. In particular, the intrinsic structure of the problem allows one to easily see the shape of the dual moment (see Proposition 7.2). In Section 8, we obtain the region of holomorphy and growth of the archimedean transform. In Section 9, a step-by-step analytic continuation argument is performed based on the analytic information obtained in Section 8.

In Section 10, we prove Theorem 1.2. and provide several explicit formulae of the integral transforms.

3. Technical Features of Our Method

3.1. **Period Reciprocity.** Our work adds a new instance to the recent banner *'Period Reciprocity'* which seeks to uncover the underlying structures of moments of L-functions through the lenses of period integrals. The general philosophy of this method is to evaluate a period integral in two distinct manners. Under favorable circumstances, the intrinsic structures of period integrals would lead to interesting, non-trivial moment identities, say connecting two different-looking families of L-functions.

In our case, the generalized Motohashi-type phenomenon of Theorem 1.1 at $s = 1/2$ will be shown to be an intrinsic property of a given Maass cusp form Φ of $\mathrm{SL}_3(\mathbb{Z})$ via the following trivial identity

$$\int_0^1 \left[\int_0^\infty \Phi\begin{pmatrix} \begin{pmatrix} y_0 & \\ & y_0\end{pmatrix}\begin{pmatrix} 1 & u \\ & 1\end{pmatrix} & \\ & 1\end{pmatrix} d^\times y_0\right] e(-u)\,du = \int_0^\infty \left[\int_0^1 \Phi\begin{pmatrix} \begin{pmatrix} 1 & u \\ & 1\end{pmatrix}\begin{pmatrix} y_0 & \\ & y_0\end{pmatrix} & \\ & 1\end{pmatrix} e(-u)\,du\right] d^\times y_0. \tag{3.1}$$

Roughly speaking, Theorem 1.1 follows from (1). spectrally-expanding the innermost integral on the left in terms of a basis of GL(2) automorphic forms, and (2). computing the innermost integral on the right in terms of the GL(3) Fourier-Whittaker period. A sketch of this will be provided in Section 4. In practice, it turns out to be convenient to work with a more general set-up

$$\int_{SL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})} P(g;h)\Phi\begin{pmatrix} g & \\ & 1\end{pmatrix} |\det g|^{s-\frac{1}{2}}\, dg \tag{3.2}$$

so as to bypass certain technical difficulties, where $P(*;h)$ is a Poincaré series of $\mathrm{SL}_2(\mathbb{Z})$.

The current examples for Period Reciprocity occur rather sporadically, and there is currently no systematic method for constructing new examples. Also, techniques differ greatly in each known instance (see [MV06, MV10, Ne19+], [Bl12a], [Nu20+], [JN21+], [Za21, Za20+]). This stands in stark contrast to the more traditional 'Kuznetsov–Voronoi' framework (see Section 3.2). However, Period Reciprocity seems to address some of the technical complications more softly than the Kuznetsov–Voronoi approach. We shall elaborate more in the upcoming subsections.

Regarding the 'classical' Motohashi phenomenon (1.2), Michel and Venkatesh [MV06, MV10] proposed a strategy that was very recently developed into a fully rigorous method by Nelson [Ne19+] through the use of regularized period integrals, incorporating new insights from automorphic representations. This article provides an alternative approach, which not only includes (1.2) but also generalizes several related instances of this phenomenon. We address the structural and analytic aspects of the formulae rather differently using unipotent integration for GL(3) and method of analytic continuation. (We begin by considering (3.2) for $\mathrm{Re}\, s \gg 1$.) For further discussions, see Section 4.

We would also like to mention the works of Wu [Wu21+] and Balkanova–Frolenkov–Wu [BFW21+] in which an interesting framework in terms of tempered distributions and relative trace formula of Godement–Jacquet type was developed to address the phenomenon (1.2).

3.2. **Comparisons with the Conrey–Iwaniec–Li Method.** The celebrated works of Conrey–Iwaniec [CI00] and Li [Li09, Li11] are known for their successful analysis based on the Kuznetsov trace formulae and summation formulae of Poisson/Voronoi type. Their accomplishments include a delicate treatment of the arithmetic of exponential sums as well as the stationary phase analysis.

The Kuznetsov trace formula (or more generally the relative trace formula) has been a cornerstone in the analytic theory of L-functions over the past few decades. In the context of Theorem 1.1, which involves summing over a basis of even Maass forms for $\mathrm{SL}_2(\mathbb{Z})$ over a basis of even Maass forms for $\mathrm{PGL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R})$), it is an equality of the shape

$$\sum_j H(\mu_j)\,\frac{\lambda_j(n)\overline{\lambda_j(m)}}{L(1,\mathrm{Ad}^2\phi_j)} + (\mathrm{cts}) \;=\; \delta_{m=n}\int_{\mathbb{R}} H(\mu)\,d_{\mathrm{spec}}\mu \;+\; \sum_{\pm}\sum_c \frac{S(\pm m,n;c)}{c}\mathcal{J}^{\pm}\left(\frac{4\pi\sqrt{mn}}{c}\right) \tag{3.3}$$

between the spectral bilinear form of Hecke eigenvalues and the geometric expansion, which consists of Kloosterman sums $S(m,n;c)$ and oscillatory integrals $\mathcal{J}^+$ and $\mathcal{J}^-$ involving the J-Bessel and K-Bessel function in their kernels respectively. These two pieces have to be treated separately.

As noticed by [CI00, Li09, Li11, Bl12b] and a number of subsequent works, the J-Bessel piece is particularly interesting due to its striking technical features. These features are crucial for achieving significant cancel- lations in geometric sums and integrals, a property that appears to be distinctive to higher-rank settings. (In view of this, readers may wish to compare with Liu–Ye's analysis in the GL(2) settings, see [LY02].) More concretely, Li [Li11] was able to apply the GL(3) Voronoi formula *twice*, which were surprisingly non-involutary, because of a subtle cancellation taking place between the *arithmetic phase* coming from Voronoi and the *analytic phase* coming from the J-Bessel transform.

The treatment of the J-Bessel piece in the Kuznetsov–Voronoi approach is crucial for analyzing more general moments of L-functions, including those involving non-selfdual L-functions or non-central L-values, as demonstrated in Theorem 1.1.

In our period integral approach, the Kuznetsov formula, the Voronoi formula, and the approximate functional equation, which belong to the standard toolbox in analytic number theory, are completely avoided. This is motivated by several conceptual reasons, which we will now explain:

- Firstly, since the GL(3)×GL(2) L-functions on the spectral side are interpreted as period integrals, we never need to open up those L-functions into Dirichlet series. As a result, averaging over the Hecke eigenvalues of our basis of GL(2) Maass forms using the Kuznetsov formula is unnecessary.
- Secondly, the standard L-function of GL(3) takes part in the arithmetic of the dual side of our moment identity (1.3). The standard L-function is constructed solely from the GL(3) Hecke eigenvalues, whereas the GL(3) Voronoi formula involves *general* GL(3) Fourier coefficients due to arithmetic twisting. It is thus reasonable to expect a proof of (1.3) that does not rely on the $GL(3)$ Voronoi formula of [MS06] nor the full Fourier expansions of [JPSS]. The set-up (3.1) already suggests that our method meets such an expectation, but see Proposition 6.1 for full details.
- Thirdly, we do not encounter any intermediate exponential sums (e.g., Kloosterman/Ramanujan sums), slow-decaying/very oscillatory special functions, nor shifted convolution sums, which are necessary components in [Iv01, Iv02, Fr20] for (1.2). Also, we handle the archimedean component of (1.3) in a unified manner, rather than handling the J- and K-Bessel pieces separately as done in [CI00, Li09, Li11]. We directly work with the GL(3) Whittaker function associated with the automorphic form Φ.
- Fourthly, we take advantage of the equivariance of the Whittaker functions under unipotent translations which helps to simplify many formulae.

Our period integral approach offers several technical advantages and is fundamentally distinct from the Kuznetsov–Voronoi approach. Indeed, our method is *local* and the key result Proposition 6.1 can be easily phrased in terms of adeles (see (4.7)), whereas the Kuznetsov–Voronoi approach is *global* and *non-adelic*. In this article, we focus on the level 1 case and the spectral aspect as a proof of concept and thus we use the classical language of real groups. In our upcoming work, we wish to extend our approach in various non-archimedean aspects.

3.3. **Prospects for Higher-Rank.** Once we reach GL(3), the geometric expansion for the Kuznetsov formula becomes significantly more intricate and presents a number of obstacles in generalizing the Kuznetsov-based approaches to moments of L-functions of higher-rank:

Remark 3.1 (**Oscillatory Integrals**)**.** In GL(2), a couple of coincidences allow us to identify the oscillatory integrals with some well-studied special functions, see [Mo97], [I02]. However, such such phenomena do not occur in GL(3), where unexpected analytic difficulties arise; see Buttcane [Bu13, Bu16]. The complicated formulae for the oscillatory integrals make the Kuznetsov trace formula for GL(3) challenging to apply, see Blomer–Buttcane [BlBu20].

Remark 3.2 (**Kloosterman Sums**)**.** The GL(3) Kloosterman sums, e.g.,

$$S(m_1, m_2, n_1, n_2; D_1, D_2) := \sum_{\substack{B_1\,(D_1),\, B_2\,(D_2)\\ C_1\,(D_1),\, C_2\,(D_2)}}^{\dagger} e\left(\frac{m_1B_1 + n_1(Y_1D_2 - Z_1B_2)}{D_1}\right) e\left(\frac{m_2B_2 + n_2(Y_2D_1 - Z_2B_1)}{D_2}\right), \tag{3.4}$$

are clearly much harder to work with than the usual one, where the definitions of Y_i, Z_i's along with a couple of congruence and coprimality conditions are suppressed. There are two other Kloosterman sums for GL(3). See [Bu13] for details.

As discussed in Section 3.2, further transformations of the exponential sums from the Kuznetsov formulae encode important arithmetic information of the moment of L-functions. Blomer–Buttcane [BlBu20] has demonstrated this approach for (3.4) after applying a four-fold Poisson summation. However, beyond this specific instance, the general applicability of such transformations to (3.4) remains unclear. On the other hand, applications of Voronoi formulae for GL(3) (see [CI00, Li09, Li11, Bl12b, BK19a, BK19b]) and for GL(4) (see [BLM19, CL20]) are currently limited to the usual Kloosterman sums of GL(2), with complications arising quickly beyond this familiar context.

Conceptually speaking, the challenges associated with Remark 3.1–3.2 stem from the *Bruhat decomposition*, which is fundamental to the framework of relative trace formulae in general. However, ideas from Period Reciprocity offers a way to bypass the Bruhat decomposition and the related geometric sums and integrals, which is a welcoming feature.

Regarding Remark 3.1, the advantages of our method are visible even in the context of Theorem 1.1. Even though we work with the group GL(3) on the dual side, the oscillatory factor in our approach (see (6.8)) is actually simpler than the ones encountered in the 'Kuznetsov–Voronoi' approaches (see [Li11]). It is more structured in two key ways: (1). it arises naturally from the definition of the archimedean Whittaker function, and (2). it serves as an important constituent of the exact Motohashi structure, the exact structures of the main terms predicted by [CFKRS05], as well as for the analytic continuation past $\operatorname{Re} s = 1/2$. Furthermore, our approach is devoid of integrals over non-compact subsets of the unipotent subgroups (or the complements) which are known to result in intricate dual calculations and exponential phases in case of GL(3) Voronoi formula (cf. [IT13, Section 4]) and Kuznetsov formulae (cf. [Gold, Chapter 11]).

It is worth pointing out the crucial archimedean ingredient in our proof generalizes to GL(n) through *Stade's formula* (see [St01]), which allows us to rewrite the archimedean part completely in terms of integrals Γ-functions. This representation is sufficient for our purposes and possesses remarkable recursive structures beneficial for further analytic manipulations, as detailed in Section 10. Another notable recent application of Stade's formula can be found in [GSW21, GSW23+]. We anticipate that our method will provide insights into the structures of archimedean transforms, pave the way for generalizing to moments of higher-rank L-functions and overcome the technical challenges posed by the 'Kuznetsov–Voronoi' method. We shall return to this subject in our upcoming works, together with treatment of the non-archimedean places.

4. Informal Sketch and Discussion

To assist the readers, we first outline the main ideas of this article, before diving into any of the analytic subtleties of our actual argument. In fact, this represents the most intrinsic picture of our method and facilitates comparisons with the strategy of Michel–Venkatesh [MV06]. The style of this section will be largely informal — we shall suppress the constant multiples (say those 2's and π's), assume convergence, and set aside the treatment of the main terms.

According to [MV06], the classical Motohashi formula can be understood as an intrinsic property of the GL(2) Eisenstein series (denoted by E^* below) via the 'regularized' geodesic period

$$\int_0^\infty |E^*(iy)|^2 \ d^\times y,$$

which can be evaluated in two ways according to $|E^*|^2$ and $E^* \cdot \overline{E^*}$ respectively:

(1) (GL(2) spectral expansion)

$$\sum_{\phi:\ GL(2)} \left\langle |E^*|^2, \phi \right\rangle \int_0^\infty \phi(iy)\ d^\times y \ = \ \sum_{\phi:\ GL(2)} \Lambda\left(1/2, \phi\right)^2 \cdot \Lambda\left(1/2, \phi\right) \ + \ (\cdots) \tag{4.1}$$

(2) (GL(1) × GL(1) expansion, or the Mellin–Plancherel formula)

$$\int_{(1/2)} |\widetilde{E^*}(s)|^2 \frac{ds}{2\pi i} \ = \ \int_{\mathbb{R}} \left|\Lambda\left(1/2+it\right)^2\right|^2 \frac{dt}{2\pi}. \tag{4.2}$$

This seemingly simple sketch turns out to require rather sophisticated regularizations but was skillfully executed by Nelson [Ne19+] very recently.

We now turn to our sketch of the (generalized) Motohashi phenomenon as described in Theorem 1.1. Let Φ be a Maass cusp form of $\mathrm{SL}_3(\mathbb{Z})$. As mentioned in the introduction, our starting point is the trivial identity

$$\int_0^1 \left[\int_0^\infty \Phi\begin{pmatrix} y_0\left(\begin{smallmatrix} 1 & u \\ & 1\end{smallmatrix}\right) & \\ & 1\end{pmatrix} d^\times y_0\right] e(-u)\ du \ = \ \int_0^\infty \left[\int_0^1 \Phi\begin{pmatrix} \left(\begin{smallmatrix} 1 & u \\ & 1\end{smallmatrix}\right)y_0 & \\ & 1\end{pmatrix} e(-u)\ du\right] d^\times y_0. \tag{4.3}$$

For symmetry, observe that the right side of (4.3) can be written as

$$\int_0^\infty \left[\int_0^1 \widetilde{\Phi}\left[\begin{pmatrix}1 & & \\ & 1 & u \\ & & 1\end{pmatrix}\begin{pmatrix}y_0 & & \\ & 1 & \\ & & 1\end{pmatrix}\right] e(-u)\, du\right] d^\times y_0 \tag{4.4}$$

with $\widetilde{\Phi}(g) := \Phi({}^t g^{-1})$ being the dual form of Φ.

Remark 4.1. Indeed, the center-invariance of Φ implies that

$$(4.3) \ = \ \int_0^\infty \int_0^1 \Phi\left[\begin{pmatrix}1 & u & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ & & y_0\end{pmatrix}\right] e(-u)\ du\ d^\times y_0.$$

Let $w_\ell := \left(\begin{smallmatrix} & & -1 \\ & 1 & \\ 1 & & \end{smallmatrix}\right)$. The observation

$$\begin{pmatrix}1 & & \\ & 1 & \\ & & y_0\end{pmatrix} \ = \ w_\ell^{-1}\begin{pmatrix}y_0 & & \\ & 1 & \\ & & 1\end{pmatrix}w_\ell \quad \text{and} \quad \begin{pmatrix}1 & & \\ & 1 & \\ & -u & 1\end{pmatrix} \ = \ w_\ell\begin{pmatrix}1 & u & \\ & 1 & \\ & & 1\end{pmatrix}w_\ell^{-1}$$

together with the left and right invariance of Φ by w_ℓ further rewrite (4.3) as

$$\int_0^\infty \int_0^1 \Phi\left[\begin{pmatrix}1 & & \\ & 1 & \\ & -u & 1\end{pmatrix}\begin{pmatrix}y_0 & & \\ & 1 & \\ & & 1\end{pmatrix}\right] e(-u)\ du\ d^\times y_0$$

$$= \int_0^\infty \int_0^1 \widetilde{\Phi}\left[\begin{pmatrix}1 & & \\ & 1 & u \\ & & 1\end{pmatrix}\begin{pmatrix}y_0 & & \\ & 1 & \\ & & 1\end{pmatrix}\right] e(-u)\ du\ d^\times y_0.$$

As an overview of our strategy,

(1) *Similar to* Michel–Venkatesh's strategy, the integral over $(0,\infty)$ (or the center $Z_{GL_2}^+(\mathbb{R})$) yields the Rankin–Selberg L-functions on the spectral side and a t-integral on the dual side;

(2) *Different from* Michel–Venkatesh's strategy, our approach introduces an extra integral over $[0,1]$ (or the quotient $U_2(\mathbb{Z})\backslash U_2(\mathbb{R})$ of the unipotent subgroup U_2 of GL(2)). This integral results in Whittaker functions as weight functions on the spectral side, and leads to a product of two distinct L-functions on the dual side;

(3) The Mellin-Plancherel of (4.2) is *replaced by* two Fourier expansions over $\mathbb{Z}\backslash\mathbb{R}$ below.

In fact, the *unipotent* nature of our period method is crucial in realizing the spectral duality for the fourth moment of Dirichlet L-functions (see [Kw23b+]), as well as in ensuring the abundance of admissible test functions on the spectral side, but these features will not be displayed in this section.

4.1. **The GL(2) (spectral) side.** This side is relatively straight-forward and gives the desired GL(3) × GL(2) moment. Regard Φ as a function of $L^2(\Gamma_2\backslash\mathfrak{h}^2)$ via

$$(\mathrm{Proj}_2^3\,\Phi)(g) \;:=\; \int_0^\infty \Phi\begin{pmatrix} y_0 g & \\ & 1\end{pmatrix} d^\times y_0 \qquad (g \in \mathfrak{h}^2),$$

which in turn can be expanded spectrally as

$$(\mathrm{Proj}_2^3\,\Phi)(g) \;=\; \sum_j \frac{\langle \mathrm{Proj}_2^3\,\Phi, \phi_j\rangle}{||\phi_j||^2}\,\phi_j(g) \;+\; \frac{\langle \mathrm{Proj}_2^3\,\Phi, 1\rangle}{||1||^2}\cdot 1 \;+\; (\mathrm{cont}).$$

The spectral coefficients $\langle \mathrm{Proj}_2^3\,\Phi, \phi_j\rangle$ are precisely the GL(3)×GL(2) Rankin–Selberg L-functions. Hence,

$$\text{LHS of (4.3)} \;=\; \int_0^1 \left(\mathrm{Proj}_2^3\,\Phi\right)\begin{pmatrix}1 & u\\ & 1\end{pmatrix} e(-u)\,du \;=\; \sum_j W_{\mu_j}(1)\,\frac{\Lambda\left(1/2, \phi_j\otimes\Phi\right)}{||\phi_j||^2} \;+\; (\mathrm{cont}), \tag{4.5}$$

where $\mu \mapsto W_\mu(1)$ is a weight function.

4.2. **The GL(1) (dual) side.** In view of Point (3) above, we evaluate the innermost integral of (4.4) in terms of the Fourier–Whittaker periods for $\widetilde{\Phi}$, denoted by $(\widehat{\widetilde{\Phi}})_{(\cdot,\cdot)}$ (see Definition 5.12). From Proposition 6.1, (4.4) is given by

$$\int_0^\infty\int_0^1\int_0^1 \widetilde{\Phi}\left[\begin{pmatrix}1 & & u_{1,3}\\ & 1 & u_{2,3}\\ & & 1\end{pmatrix}\begin{pmatrix}y_0 & & \\ & 1 & \\ & & 1\end{pmatrix}\right] e(-u_{2,3})\,du_{1,3}\,du_{2,3}\,d^\times y_0$$
$$+ \sum_{a_0\in\mathbb{Z}-\{0\}}\;\sum_{a_1\in\mathbb{Z}-\{0\}} \int_0^\infty \left(\widehat{\widetilde{\Phi}}\right)_{(1,a_1)}\left[\begin{pmatrix}1 & & \\ a_0 & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}y_0 & & \\ & 1 & \\ & & 1\end{pmatrix}\right] d^\times y_0. \tag{4.6}$$

The first line of (4.6) corresponds to the diagonal term and is precisely the integral representation of the standard L-function of $\widetilde{\Phi}$. It is equal to $L(1,\widetilde{\Phi})Z_\infty(1,\widetilde{\Phi})$, where $Z_\infty(\cdot,\widetilde{\Phi})$ is the GL(3) local zeta integral at ∞. The second line of (4.6) is the off-diagonal contribution, denoted by OD_Φ below, and is expressed in terms of the Fourier coefficients of $\widetilde{\Phi}$:

$$\mathrm{OD}_\Phi \;=\; \sum_{a_0\in\mathbb{Z}-\{0\}}\;\sum_{a_1\in\mathbb{Z}-\{0\}} \frac{\mathcal{B}_{\widetilde{\Phi}}(1,a_1)}{|a_1|}\int_0^\infty \left(\widehat{\widetilde{\Phi}}\right)_{(1,1)}\left[\begin{pmatrix}a_1/a_0 & & \\ 1 & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}y_0 & & \\ & 1 & \\ & & 1\end{pmatrix}\right] d^\times y_0. \tag{4.7}$$

It can be further explicated as

$$\mathrm{OD}_\Phi \;=\; \sum_{a_0\in\mathbb{Z}-\{0\}}\;\sum_{a_1\in\mathbb{Z}-\{0\}} \frac{\mathcal{B}_{\widetilde{\Phi}}(1,a_1)}{|a_1|}\int_0^\infty W_{\alpha(\Phi)}\left(\left|\frac{a_1}{a_0}\right|\frac{y_0}{1+y_0^2},\;1\right) e\left(\frac{a_1}{a_0}\frac{y_0^2}{1+y_0^2}\right) d^\times y_0 \tag{4.8}$$

using the GL(3) Whittaker function $W_{\alpha(\Phi)}$, where the oscillatory factor $e(\cdots)$ originates from the unipotent translation of Whittaker functions.

Roughly speaking, (4.8) suggests some forms of (multiplicative) convolutions between the GL(3) and GL(1) data at both the archimedean and the non-archimedean places:

(1) (Archimedean) We apply the Mellin inversion formula for $W_{\alpha(\Phi)}$, a standard result in GL(3) theory, together with the local functional equation for GL(1) in the form:

$$e(x) + e(-x) \;=\; \int_{-i\infty}^{i\infty} \frac{\Gamma_{\mathbb{R}}(u)}{\Gamma_{\mathbb{R}}(1-u)} \, |x|^{-u} \, \frac{du}{2\pi i} \qquad (x \;\neq\; 0); \tag{4.9}$$

(2) (Non-archimedean) Observe the following identity of the double Dirichlet series:

$$\sum_{a_0 \neq 0} \sum_{a_1 \neq 0} \frac{\mathcal{B}_{\Phi}(a_1, 1)}{|a_1|} \left| \frac{a_1}{a_0} \right|^{1-s_0-u} \;=\; L(s_0+u, \widetilde{\Phi}) \zeta(1-s_0-u). \tag{4.10}$$

We thus arrive at

$$\mathrm{OD}_{\Phi} \;=\; \int_{(1/2)} \zeta\,(1-s_0)\, L(s_0, \widetilde{\Phi}) \cdot (\cdots)\, \frac{ds_0}{2\pi i}, \tag{4.11}$$

where '$(\cdots)$' stands for a certain integral transform that can be described purely in terms of Γ-functions.

Remark 4.2.

(1) In (3.2), the test function h of the Poincaré series $P(*; h)$ will be transformed into the Kontorovich-Lebedev transform $h^{\#}$ on the GL(2) side (see (5.25)) and into the Mellin transform $\widetilde{h}$ on the GL(1) side (see (7.6)). This is consistent with the sketch above.

(2) Readers may wish to compare the integral transforms obtained in the sketch with the one described in [BFW21+, Section 1.3].

Remark 4.3. The choices of unipotent subgroups have been important in the constructions of various L-series for the group GL(3):

- $\left\{ \begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{pmatrix} \right\}$ for the standard L-function;
- $\left\{ \begin{pmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{pmatrix} \right\}$ for Bump's double Dirichlet series ([Bump84]);
- $\left\{ \begin{pmatrix} 1 & & \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$ or $\left\{ \begin{pmatrix} 1 & * & \\ & 1 & \\ & & 1 \end{pmatrix} \right\}$ for the Motohashi phenomenon of this article.

5. Preliminary

The analytic theory of automorphic forms for the group GL(3) has undergone considerable development in the past decade. Readers should beware that the recent articles in the field (e.g., [Bu13, Bu16, Bu20, GSW21]) have adopted a different set of conventions and normalizations from those in the standard text [Gold]. Nevertheless, [Gold] remains a useful reference as it thoroughly documents many standard results and their proofs.

In this article, we follow the more recent conventions (closest to [Bu20]), which is better aligned with the theory of automorphic representation. We will summarize the essential notions and results below with extra attention on the archimedean calculations involving Whittaker functions, as they play a key role in our analysis.

5.1. Notations and Conventions.

Throughout this article, we use the following notations: $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ $(s \in \mathbb{C})$; $e(x) := e^{2\pi i x}$ $(x \in \mathbb{R})$; $\Gamma_n := \mathrm{SL}_n(\mathbb{Z})$ $(n \geqslant 2)$. Without otherwise specified, our test function H lies in the class $\mathcal{C}_\eta$ and $H = h^{\#}$. We will often use the same symbol to denote a function (in s) and its analytic continuation.

We will frequently encounter contour integrals of the shape

$$\int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} (\cdots)\, \frac{ds_1}{2\pi i} \cdots \frac{ds_k}{2\pi i}$$

where the contours involved should follow Barnes' convention: they pass to the right of all of the poles of the gamma functions in the form $\Gamma(s_i + a)$ and to the left of all of the poles of the gamma functions in the form $\Gamma(a - s_i)$.

We also adopt the following set of conventions:

(1) All Maass cusp forms will be simultaneous eigenfunctions of the Hecke operators and will be either even or odd. Also, their first Fourier coefficients are equal to 1. In this case, the forms are said to be **Hecke-normalized**. Note that there are no odd form for $\mathrm{SL}_3(\mathbb{Z})$, see [Gold, Proposition 9.2.5].
(2) Our fixed Maass cusp form Φ of $\mathrm{SL}_3(\mathbb{Z})$ is assumed to be **tempered at** ∞, i.e., its Langlands parameters are purely imaginary.
(3) Denote by θ the best progress towards the Ramanujan conjecture for the Maass cusp forms of $\mathrm{SL}_3(\mathbb{Z})$. We have $\theta \leqslant \frac{1}{2} - \frac{1}{10}$, see [Gold, Theorem 12.5.1].

5.2. **(Spherical) Whittaker Functions & Transforms.** In the rest of this article, all Whittaker functions will refer to the spherical ones. The Whittaker function of $\mathrm{GL}_2(\mathbb{R})$ is more familiar and is given by

$$W_\mu(y) \ := \ 2\sqrt{y}K_\mu(2\pi y) \tag{5.1}$$

for $\mu \in \mathbb{C}$ and $y > 0$. Under this normalization, the following holds:

Proposition 5.1. For $\mathrm{Re}(w + \frac{1}{2} \pm \mu) > 0$, we have

$$\int_0^\infty W_\mu(y) y^w \, d^\times y \ = \ \frac{\pi^{-w-\frac{1}{2}}}{2} \Gamma\left(\frac{w+\frac{1}{2}+\mu}{2}\right)\Gamma\left(\frac{w+\frac{1}{2}-\mu}{2}\right). \tag{5.2}$$

Proof. Standard, see [Mo97, Equation (2.5.2)] for instance. □

For the group $\mathrm{GL}_3(\mathbb{R})$, we first introduce the function

$$I_\alpha(y_0, y_1) \ = \ I_\alpha\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix} \ := \ y_0^{1-\alpha_3} y_1^{1+\alpha_1}$$

for $y_0, y_1 > 0$ and $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)} := \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3 : \alpha_1 + \alpha_2 + \alpha_3 = 0\}$. Then the Whittaker function for $GL_3(\mathbb{R})$, denoted by $W_\alpha(y_0, y_1) = W_\alpha\left(\begin{smallmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{smallmatrix}\right)$, is defined in terms of *Jacquet's integral*:

$$\prod_{1\leqslant j<k\leqslant 3} \Gamma_{\mathbb{R}}(1+\alpha_j-\alpha_k) \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} I_\alpha\left[\left(\begin{smallmatrix} & & 1\\ & -1 & \\ 1 & & \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & u_{1,2} & u_{1,3}\\ & 1 & u_{2,3}\\ & & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{smallmatrix}\right)\right] e(-u_{1,2}-u_{2,3}) \, du_{1,2}\, du_{1,3}\, du_{2,3} \tag{5.3}$$

for $y_0, y_1 > 0$ and $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)}$. See [Gold, Chapter 5.5] for details.

Remark 5.2. Notice the difference in the normalization of I_α here compared to that in [Gold, Equation 5.1.1]. Also, the Whittaker functions discussed here are the *complete* Whittaker functions as defined in [Gold].

Moreover, the Whittaker function of $\mathrm{GL}_3(\mathbb{R})$ admits the following useful Mellin-Barnes representation commonly known as the *Vinogradov-Takhtadzhyan formula*:

Proposition 5.3. Assume $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)}$ is tempered, i.e., $\mathrm{Re}\, \alpha_i = 0$ $(i = 1, 2, 3)$. Then for any $\sigma_0, \sigma_1 > 0$,

$$W_{-\alpha}(y_0, y_1) \ = \ \frac{1}{4} \int_{(\sigma_0)}\int_{(\sigma_1)} G_\alpha(s_0, s_1) y_0^{1-s_0} y_1^{1-s_1} \frac{ds_0}{2\pi i}\frac{ds_1}{2\pi i}, \qquad y_0, y_1 \ > \ 0, \tag{5.4}$$

where

$$G_\alpha(s_0, s_1) \ := \ \frac{\prod_{i=1}^{3} \Gamma_{\mathbb{R}}(s_0+\alpha_i)\Gamma_{\mathbb{R}}(s_1-\alpha_i)}{\Gamma_{\mathbb{R}}(s_0+s_1)}. \tag{5.5}$$

Proof. This can be verified (up to the constant 1/4) by a brute force yet elementary calculation, i.e., checking the right side of (5.4) satisfies the differential equations of GL(3) (see [Bump84, pp. 38-39]). For a cleaner proof starting from (5.3), see [Bump84, Chapter X]. □

Remark 5.4. Notice the sign convention of the α_i's in formula (5.4) — it is consistent with [Bu20] but is opposite to that of [Gold, Equation (6.1.4)–(6.1.5)].

Corollary 5.5. For any $-\infty < A_0, A_1 < 1$, we have

$$|W_{-\alpha}(y_0,y_1)| \ \ll \ y_0^{A_0}y_1^{A_1}, \qquad y_0,y_1 \ > \ 0, \tag{5.6}$$

where the implicit constant depends only on α, A_0, A_1.

Proof. Follows directly from Proposition 5.3 by contour shifting. ☐

We will need the explicit evaluation of the $\mathrm{GL}_3(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$ Rankin–Selberg integral. It is a consequence of the *Second Barnes Lemma* stated as follows.

Lemma 5.6. For $a,b,c,d,e,f \in \mathbb{C}$ with $f = a+b+c+d+e$, we have

$$\int_{-i\infty}^{i\infty} \frac{\Gamma(w+a)\Gamma(w+b)\Gamma(w+c)\Gamma(d-w)\Gamma(e-w)}{\Gamma(w+f)}\ \frac{dw}{2\pi i} = \frac{\Gamma(d+a)\Gamma(d+b)\Gamma(d+c)\Gamma(e+a)\Gamma(e+b)\Gamma(e+c)}{\Gamma(f-a)\Gamma(f-b)\Gamma(f-c)}. \tag{5.7}$$

The contours of integration must adhere to Barnes' convention; see Section 5.1.

Proof. See Bailey [Ba64]. ☐

Proposition 5.7. Let W_μ and $W_{-\alpha}$ be the Whittaker functions of $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{GL}_3(\mathbb{R})$ respectively. For $\operatorname{Re} s \gg 0$, we have

$$\mathcal{Z}_\infty\left(s;\ W_\mu,\ W_{-\alpha}\right) \ := \ \int_0^\infty\int_0^\infty W_\mu(y_1)W_{-\alpha}(y_0,y_1)(y_0^2y_1)^{s-\frac{1}{2}}\ \frac{dy_0dy_1}{y_0y_1^2} \ = \ \frac{1}{4}\prod_{\pm}\prod_{k=1}^{3}\Gamma_{\mathbb{R}}\left(s\pm\mu-\alpha_k\right). \tag{5.8}$$

Proof. See [Bump88]. ☐

The following pair of integral transforms plays an important role in the archimedean aspect of this article.

Definition 5.8. Let $h : (0,\infty) \to \mathbb{C}$ and $H : i\mathbb{R} \to \mathbb{C}$ be measurable functions with $H(\mu) = H(-\mu)$. Let $W_\mu(y) := 2\sqrt{y}K_\mu(2\pi y)$. Then the Kontorovich-Lebedev transform of h is defined by

$$h^{\#}(\mu) \ := \ \int_0^\infty h(y)W_\mu(y)\ \frac{dy}{y^2}, \tag{5.9}$$

whereas its inverse transform is defined by

$$H^{\flat}(y) \ = \ \frac{1}{4\pi i}\int_{(0)} H(\mu)W_\mu(y)\ \frac{d\mu}{|\Gamma(\mu)|^2}, \tag{5.10}$$

provided the integrals converge absolutely. Note: the normalization constant $1/4\pi i$ in (5.10) is consistent with that in [Mo97] and [I02].

Definition 5.9. Let $\mathcal{C}_\eta$ be the class of holomorphic functions H on the vertical strip $|\operatorname{Re}\mu| < 2\eta$ such that

(1) $H(\mu) = H(-\mu)$,
(2) H has rapid decay in the sense that

$$H(\mu) \ \ll \ e^{-2\pi|\mu|} \qquad (|\operatorname{Re}\mu| \ < \ 2\eta). \tag{5.11}$$

In this article, we take $\eta > 40$ without otherwise specifying.

By contour-shifting and Stirling's formula, we have

Proposition 5.10. For any $H \in \mathcal{C}_\eta$, the integral (5.10) defining $H^\flat$ converges absolutely. Moreover, we have

$$H^\flat(y) \ \ll \ \min\{y, y^{-1}\}^\eta \qquad (y \ > \ 0). \tag{5.12}$$

Proof. See [Mo97, Lemma 2.10]. □

Proposition 5.11. Under the same assumptions of Proposition 5.10, we have

$$(h^{\#})^{\flat}(g) \;=\; h(g) \qquad \text{and} \qquad (H^{\flat})^{\#}(\mu) \;=\; H(\mu). \tag{5.13}$$

Proof. See [Mo97, Lemma 2.10]. It is a consequence of the Rankin–Selberg calculation for $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$. □

5.3. **Automorphic Forms of GL(2) and GL(3).** Let

$$\mathfrak{h}^2 := \left\{ \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix} : u \in \mathbb{R},\ y > 0 \right\}$$

with its invariant measure given by $y^{-2}dudy$. Let $\Delta := -y^2\left(\partial_x^2 + \partial_y^2\right)$. An automorphic form $\phi : \mathfrak{h}^2 \to \mathbb{C}$ of $\Gamma_2 = \mathrm{SL}_2(\mathbb{Z})$ satisfies $\Delta\phi = \left(\frac{1}{4} - \mu^2\right)\phi$ for some $\mu = \mu(\phi) \in \mathbb{C}$. It is often handy to identify μ with the pair $(\mu, -\mu) \in \mathfrak{a}_{\mathbb{C}}^{(2)}$.

For $a \in \mathbb{Z} - \{0\}$, the a-th Fourier coefficient of ϕ, denoted by $\mathcal{B}_\phi(a)$, is defined by

$$(\widehat{\phi})_a(y) \;:=\; \int_0^1 \phi\left[\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix}\right] e(-au)\, du \;=\; \frac{\mathcal{B}_\phi(a)}{\sqrt{|a|}} \cdot W_{\mu(\phi)}(|a|y). \tag{5.14}$$

In the case of the Eisenstein series of Γ_2, i.e.,

$$\phi \;=\; E(z;\mu) \;:=\; \frac{1}{2} \sum_{\gamma \in U_2(\mathbb{Z})\backslash\Gamma_2} I_\mu(\mathrm{Im}\,\gamma z) \qquad (z \in \mathfrak{h}^2), \tag{5.15}$$

where $I_\mu(y) := y^{\mu + \frac{1}{2}}$, it is well-known that $\Delta E(*;\mu) = \left(1/4 - \mu^2\right) E(*;\mu)$ and the Fourier coefficients $\mathcal{B}(a;\mu)$ of $E(*;\mu)$ is given by

$$\mathcal{B}(a;\mu) \;=\; \frac{|a|^\mu \sigma_{-2\mu}(|a|)}{\Lambda(1+2\mu)}, \tag{5.16}$$

where

$$\Lambda(s) \;:=\; \pi^{-s/2}\Gamma(s/2)\zeta(s) \qquad \text{and} \qquad \sigma_{-2\mu}(|a|) \;:=\; \sum_{d|a} d^{-2\mu}.$$

The series (5.15) converges absolutely for $\mathrm{Re}\,\mu > 1/2$ and it admits a meromorphic continuation to $\mathbb{C}$.

Next, let

$$\mathfrak{h}^3 \;:=\; \left\{ \begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{pmatrix} \begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} : u_{i,j} \in \mathbb{R},\ y_k > 0 \right\}.$$

Let $\Phi : \mathfrak{h}^3 \to \mathbb{C}$ be a Maass cusp form of Γ_3 as defined in [Gold, Definition 5.1.3]. In particular, there exists $\alpha = \alpha(\Phi) \in \mathfrak{a}_{\mathbb{C}}^{(3)}$ such that for any $D \in Z(U\mathfrak{gl}_3(\mathbb{C}))$ (the center of the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_3(\mathbb{C})$), we have

$$D\Phi \;=\; \lambda_D \Phi \qquad \text{and} \qquad DI_\alpha \;=\; \lambda_D I_\alpha$$

for some $\lambda_D \in \mathbb{C}$. The triple $\alpha(\Phi)$ is said to be the *Langlands parameters* of Φ.

Definition 5.12. Let $m = (m_1, m_2) \in (\mathbb{Z} - \{0\})^2$ and $\Phi : \mathfrak{h}^3 \to \mathbb{C}$ be a Maass cusp form of $\mathrm{SL}_3(\mathbb{Z})$. For any $y_0, y_1 > 0$, the integral defined by

$$(\widehat{\Phi})_{(m_1,m_2)}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix} \;:=\; \int_0^1\int_0^1\int_0^1 \Phi\left[\begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{pmatrix}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1 \end{pmatrix}\right] e\left(-m_1u_{2,3} - m_2u_{1,2}\right)\, du_{1,2}\, du_{1,3}\, du_{2,3}. \tag{5.17}$$

is said to be the (m_1, m_2)-th **Fourier-Whittaker period** of Φ. Moreover, the (m_1, m_2)-th **Fourier coefficient** of Φ is the complex number $\mathcal{B}_\Phi(m_1, m_2)$ for which

$$(\widehat{\Phi})_{(m_1,m_2)}\begin{pmatrix} y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix} = \frac{\mathcal{B}_\Phi(m_1,m_2)}{|m_1m_2|}\, W_{\alpha(\Phi)}^{\operatorname{sgn}(m_2)}\begin{pmatrix} (|m_1|y_0)(|m_2|y_1) & & \\ & |m_1|y_0 & \\ & & 1\end{pmatrix} \tag{5.18}$$

holds for any $y_0, y_1 > 0$.

Remark 5.13.

(1) The multiplicity-one theorem of Shalika (see [Gold, Theorem 6.1.6]) guarantees the well-definedness of the Fourier coefficients for Φ.
(2) If Φ is Hecke-normalized (cf. Section 5.1.(1)), then $\mathcal{B}_\Phi(1, n)$ can be shown to be a Hecke eigenvalue of Φ (see [Gold, Section 6.4]).

5.4. **Automorphic L-functions.** The Maass cusp forms Φ and ϕ below are Hecke-normalized and their Langlands parameters are denoted by $\alpha \in \mathfrak{a}_{\mathbb{C}}^{(3)}$ and $\mu \in \mathfrak{a}_{\mathbb{C}}^{(2)}$ respectively. Let $\widetilde{\Phi}(g) := \Phi\left({}^t g^{-1}\right)$ be the dual form of Φ. It is not hard to show that the Langlands parameters of $\widetilde{\Phi}$ are given by $-\alpha$.

Definition 5.14. Suppose Φ and ϕ are Maass cusp forms of Γ_3 and Γ_2 respectively. For $\operatorname{Re} s \gg 1$, the Rankin–Selberg L-function of Φ and ϕ is defined by

$$L\left(s, \phi\otimes\Phi\right) := \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{\mathcal{B}_\phi(m_2)\mathcal{B}_\Phi(m_1,m_2)}{\left(m_1^2m_2\right)^s}. \tag{5.19}$$

Although we do not make use of the Dirichlet series for $L(s, \phi\otimes\Phi)$ in this article, it is frequently used in the literature, especially in the 'Kuznetsov–Voronoi' method. We take this opportunity to indicate our normalization in terms of Dirichlet series so as to faciliate conversion and comparison, and to correct some minor inaccuracies in Section 12.2 of [Gold].

Proposition 5.15. Suppose Φ and ϕ are Maass cusp forms of Γ_3 and Γ_2 respectively. In addition, assume that ϕ is even. Then for any $\operatorname{Re} s \gg 1$, we have

$$\int_{\Gamma_2\backslash GL_2(\mathbb{R})} \phi(g)\widetilde{\Phi}\begin{pmatrix} g & \\ & 1\end{pmatrix}|\det g|^{s-\frac{1}{2}}\, dg = \frac{1}{2}\Lambda(s,\phi\otimes\widetilde{\Phi}), \tag{5.20}$$

where

$$\Lambda(s,\phi\otimes\widetilde{\Phi}) := L_\infty(s,\phi\otimes\widetilde{\Phi})L(s,\phi\otimes\widetilde{\Phi}) \tag{5.21}$$

and

$$L_\infty(s,\phi\otimes\widetilde{\Phi}) := \prod_{k=1}^{3}\Gamma_{\mathbb{R}}\left(s\pm\mu-\alpha_k\right). \tag{5.22}$$

Proof. The assumption on the parity of ϕ is missing in [Gold]. Also, the pairing should be taken over the quotient $\Gamma_2\backslash \mathrm{GL}_2(\mathbb{R})$ instead of $\Gamma_2\backslash\mathfrak{h}^2$ in [Gold].

As a brief sketch, we replace $\widetilde{\Phi}\left(\begin{smallmatrix} g & \\ & 1\end{smallmatrix}\right)$ by its Fourier-Whittaker expansion (see [Gold, Theorem 5.3.2]) on the left side of (5.20) and unfold. Then one may extract the Dirichlet series in (5.19) by using (5.14) and (5.17). The integral of Whittaker functions can be computed by Proposition 5.7. □

In the rest of this article, we will often make use of the shorthands $(\mathbb{P}_2^3\Phi)(g) := \Phi\left(\begin{smallmatrix} g & \\ & 1\end{smallmatrix}\right)$ and the pairing

$$\left(\phi, (\mathbb{P}_2^3\Phi)\cdot|\det *|^{\overline{s}-\frac{1}{2}}\right)_{\Gamma_2\backslash GL_2(\mathbb{R})}$$

for the integral on the left side of (5.20). By the rapid decay of Φ at ∞, this integral converges absolutely for any $s \in \mathbb{C}$ and uniformly on any compact subset of $\mathbb{C}$. Thus, the L-function $L(s,\phi\otimes\widetilde{\Phi})$ admits an entire continuation.

Remark 5.16.

(1) When ϕ is even, the involution $g \mapsto {}^t g^{-1}$ gives the functional equation

$$\Lambda(s, \phi \otimes \widetilde{\Phi}) = \Lambda(1-s, \phi \otimes \Phi).$$

(2) When ϕ is odd, the right side of (5.20) is identical to 0 and hence *does not* provide an integral representation for $\Lambda(s, \phi \otimes \widetilde{\Phi})$. One must alter Proposition 5.15 accordingly in this case, say using the raising/lowering operators, or proceed adelically with an appropriate choice of test vector at ∞. However, we shall not go into these as our spectral average is taken over even Maass forms of Γ_2 only.

(3) As discussed in Section 3.2, the roles of parities and root numbers are rather intricate in the study of moments of L-functions, especially regarding the archimedean integral transforms.

Definition 5.17. Let $\Phi : \mathfrak{h}^3 \to \mathbb{C}$ be a Maass cusp form of Γ_3. For $\operatorname{Re} s \gg 1$, the standard L-function of Φ is defined by

$$L(s, \Phi) := \sum_{n=1}^{\infty} \frac{\mathcal{B}_{\Phi}(1,n)}{n^s}. \tag{5.23}$$

In the rest of this article, we will not make use of the integral representation of $L(s, \Phi)$, i.e., the first line of (4.6) with $\widetilde{\Phi}$ replaced by Φ. It suffices to note that $L(s, \Phi)$ admits an entire continuation and satisfies the following functional equation:

Proposition 5.18. Let $\Phi : \mathfrak{h}^3 \to \mathbb{C}$ be a Maass cusp form of Γ_3. For any $s \in \mathbb{C}$, we have

$$\Lambda(s, \Phi) = \Lambda(1-s, \widetilde{\Phi}), \tag{5.24}$$

where

$$\Lambda(s, \Phi) := L_{\infty}(s, \Phi) L(s, \Phi) \tag{5.25}$$

and

$$L_{\infty}(s, \Phi) := \prod_{k=1}^{3} \Gamma_{\mathbb{R}}(s + \alpha_k). \tag{5.26}$$

Proof. See [Gold, Chapter 6.5] or [JPSS]. □

Furthermore, since ϕ and Φ are assumed to be Hecke-normalized, the standard L-functions $L(s, \phi)$ and $L(s, \Phi)$ admit Euler products of the form:

$$L(s, \phi) = \prod_p \prod_{j=1}^{2} \left(1 - \beta_{\phi,j}(p) p^{-s}\right)^{-1}, \qquad L(s, \Phi) = \prod_p \prod_{k=1}^{3} \left(1 - \alpha_{\Phi,k}(p) p^{-s}\right)^{-1} \tag{5.27}$$

for $\operatorname{Re} s \gg 1$. Then one can show that

$$L(s, \phi \otimes \Phi) = \prod_p \prod_{j=1}^{2} \prod_{k=1}^{3} \left(1 - \beta_{\phi,j}(p) \alpha_{\Phi,k}(p) p^{-s}\right)^{-1} \tag{5.28}$$

by Cauchy's identity, see the argument of [Gold, Proposition 7.4.12].

Proposition 5.19. For $\operatorname{Re}(s \pm \mu) \gg 1$, we have

$$\left(E(*; \mu), \left(\mathbb{P}_2^3 \Phi\right) \cdot |\det *|^{\bar{s} - \frac{1}{2}}\right)_{\Gamma_2 \backslash GL_2(\mathbb{R})} = \frac{1}{2} \frac{\Lambda(s+\mu, \widetilde{\Phi}) \Lambda(s-\mu, \widetilde{\Phi})}{\Lambda(1+2\mu)}. \tag{5.29}$$

Proof. Parallel to Proposition 5.15. Meanwhile, we make use of (5.16). □

Remark 5.20. By analytic continuation, (5.20) and (5.29) hold for $s \in \mathbb{C}$ and away from the poles of $E(*; \mu)$. In fact, the rapid decay of Φ at ∞ guarantees the pairings converge absolutely.

5.5. **Calculation on the Spectral Side.** As noted before, our approach diverges from the "Kuznetsov-Voronoi" method from the outset. We express the moment of $GL(3) \times GL(2)$ L-functions via the period integral in Proposition 5.15 using a Poincaré series.

Definition 5.21. Let $a \geqslant 1$ be an integer and $h \in C^\infty(0,\infty)$. The Poincaré series of Γ_2 is defined as

$$P^a(z;h) \ := \sum_{\gamma \in U_2(\mathbb{Z})\backslash\Gamma_2} h(a \operatorname{Im}\gamma z) e\left(a \operatorname{Re}\gamma z\right) \qquad (z \ \in \ \mathfrak{h}^2) \tag{5.30}$$

provided the series converges absolutely.

It is not hard to see that if the bounds

$$h(y) \ \ll \ y^{1+\epsilon} \quad (\text{as} \ \ y \to 0) \quad \text{ and } \quad h(y) \ \ll \ y^{\frac{1}{2}-\epsilon} \quad (\text{as} \ \ y \to \infty) \tag{5.31}$$

are satisfied, then the Poincaré series $P^a(z;h)$ converges absolutely and represents an L^2-function. In this article, we take $h := H^\flat$ with $H \in \mathcal{C}_\eta$ and $\eta > 40$. By Proposition 5.10, the conditions in (5.31) are clearly met. We will often use the shorthand $P^a := P^a(*;h)$. Also, we denote the Petersson inner product on $\Gamma_2\backslash\mathfrak{h}^2$ by $\langle \cdot\,,\,\cdot \rangle$, defined as

$$\langle \phi_1,\, \phi_2 \rangle \ := \ \int_{\Gamma_2\backslash\mathfrak{h}^2} \phi_1(g)\overline{\phi_2(g)}\, dg$$

with dg being the invariant measure on $\mathfrak{h}^2$.

Lemma 5.22. Let ϕ be a Maass cusp form of Γ_2, $\Delta\phi = \left(1/4 - \mu^2\right)\phi$, and $\mathcal{B}_\phi(a)$ be the a-th Fourier coefficient of ϕ. Then

$$\left\langle P^a,\, \phi\right\rangle \ = \ a^{1/2}\cdot\overline{\mathcal{B}_\phi(a)}\cdot h^{\#}\left(\overline{\mu}\right).$$

Proof. Replace P^a in $\left\langle P^a,\, \phi\right\rangle$ by its definition and unfold, we easily find that

$$\left\langle P^a,\, \phi\right\rangle \ = \ \int_0^\infty h(ay)\,\overline{\widehat{(\phi)}_a(y)}\,\frac{dy}{y^2}.$$

The result follows at once upon plugging-in (5.14) and making the change of variable $y \to a^{-1}y$. ☐

Similarly, the following holds away from the poles of $E(*;\mu)$:

Lemma 5.23.

$$\left\langle P^a, E(*;\mu)\right\rangle \ = \ a^{1/2}\cdot\frac{a^{\overline{\mu}}\sigma_{-2\overline{\mu}}(a)}{\Lambda(1+2\overline{\mu})}\cdot h^{\#}\left(\overline{\mu}\right). \tag{5.32}$$

Proposition 5.24 (Spectral Expansion)**.** Suppose $f \in L^2(\Gamma_2\backslash\mathfrak{h}^2)$ and $\langle f,1\rangle = 0$. Then

$$f(z) \ = \ \sum_{j=1}^\infty \frac{\left\langle f,\,\phi_j\right\rangle}{\left\langle \phi_j,\phi_j\right\rangle}\cdot\phi_j(z) \ + \ \int_{(0)} \left\langle f,\, E(*;\mu)\right\rangle\cdot E(z;\mu)\,\frac{d\mu}{4\pi i} \qquad (z \ \in \ \mathfrak{h}^2) \tag{5.33}$$

where $(\phi_j)_{j\geqslant 1}$ is any orthogonal basis of Maass cusp forms for Γ_2.

Proof. See [Gold, Theorem 3.16.1]. ☐

Proposition 5.25. Let Φ be a Maass cusp form of Γ_3 and P^a be a Poincaré series of Γ_2. Then

$$\begin{aligned} 2a^{-1/2}\left(P^a, (\mathbb{P}_2^3\Phi)\cdot|\det *|^{\overline{s}-\frac{1}{2}}\right)_{\Gamma_2\backslash GL_2(\mathbb{R})} \ = \ & \sum_{j=1}^{\infty}{}' \ h^{\#}\left(\overline{\mu_j}\right)\,\frac{\overline{\mathcal{B}_j(a)}\Lambda(s,\phi_j\otimes\widetilde{\Phi})}{\langle\phi_j,\phi_j\rangle} \\ & + \int_{(0)} h^{\#}\left(\mu\right)\frac{\sigma_{-2\mu}(a)a^{-\mu}\Lambda(s+\mu,\widetilde{\Phi})\Lambda(1-s+\mu,\Phi)}{|\Lambda(1+2\mu)|^2}\,\frac{d\mu}{4\pi i} \end{aligned} \tag{5.34}$$

for any $s \in \mathbb{C}$, where the sum is restricted to an orthogonal basis (ϕ_j) of even Hecke-normalized Maass cusp forms for Γ_2 with $\Delta\phi_j = (1/4-\mu_j^2)\phi_j$ and $\mathcal{B}_j(a) := \mathcal{B}_{\phi_j}(a)$.

Proof. Substitute the spectral expansion of P^a as in (5.33) into the pairing $(P^a,\ (\mathbb{P}_2^3\Phi)\cdot|\det *|^{\overline{s}-\frac{1}{2}})_{\Gamma_2\backslash GL_2(\mathbb{R})}$. The inner products involved have been computed in Lemmas 5.22–5.23 and Definitions 5.15–5.19. ☐

Remark 5.26. Good control over spectral aspects and integral transforms, along with flexibility in choosing test functions on the spectral side, are crucial in applications. Also, this helps eliminate extraneous polar contributions (e.g., those not predicted by [CFKRS05]) for Eisenstein cases. These explain the preference of Kuznetsov-based methods over period-based methods (see the discussions in [Bl12a, Nu20+, Za21, Za20+]).

While our method is period-based, it accommodates a broad class of test functions similar to the Kuznetsov approaches, thanks to the transforms in Definition 5.8. These transforms, generalized to $GL(n)$ as in [GK12], have significantly contributed to the development of higher-rank Kuznetsov formulae (see [GK13, GSW21, GSW23+, Bu20]).

Our method effectively combines the strengths of both Kuznetsov and period approaches, balancing precision in the archimedean aspect with structural insights into the non-archimedean aspect.

Remark 5.27. Within our class $\mathcal{C}_\eta$ of test functions, a good choice of test function is given by

$$H(\mu) \ := \ \left(e^{((\mu+iM)/R)^2}+e^{((\mu-iM)/R)^2}\right)\frac{\Gamma(2\eta+\mu)\Gamma(2\eta-\mu)}{\prod_{i=1}^3\Gamma\big(\frac{1/2+\mu-\alpha_i}{2}\big)\Gamma\big(\frac{1/2-\mu-\alpha_i}{2}\big)}, \tag{5.35}$$

where $\eta>40$, $M\gg 1$, and $R=M^\gamma\ \ (0<\gamma\leqslant 1)$. In (5.35),

- the factor $e^{((\mu+iM)/R)^2}+e^{((\mu-iM)/R)^2}$ serves as a smooth cut-off for $|\mu_j|\in[M-R,M+R]$ and gives the needed decay in Proposition 5.10;
- the factors $\prod_{i=1}^3\Gamma\left(\frac{1/2+\mu-\alpha_i}{2}\right)\Gamma\left(\frac{1/2-\mu-\alpha_i}{2}\right)$ cancel out the archimedean factors of $\Lambda\big(1/2,\phi_j\otimes\widetilde{\Phi}\big)$ on the spectral expansion (5.34) and in the diagonal contribution (6.9);
- the factors $\Gamma(2\eta+\mu)\,\Gamma(2\eta-\mu)$ balance off the exponential growth from $d\mu/|\Gamma(\mu)|^2$, $||\phi_j||^{-2}$ and $|\Lambda(1+2i\mu)|^{-2}$. Also, a large enough region of holomorphy of (5.35) is maintained so that $h(y):=H^\flat(y)$ has sufficient decay at 0 and ∞.

Remark 5.28. One might consider using an automorphic kernel instead of a Poincaré series for Theorem 1.1. While this offers more structural flexibility, the analysis of the spherical transforms becomes quite complicated (see [Z79], [Bu13]). The Poincaré series approach appears better suited to the analytic number theory of higher-rank groups.

6. Basic Identity for Dual Moment

6.1. **Unipotent Integration.** We are ready to work on the dual side of our moment formula. To simplify our argument, we will consider $P=P^a(*;h)$ with $a=1$ in the following. Suppose $\operatorname{Re}s>1+\theta/2$, where θ was defined in Section 5.1. We start by substituting the definition of P into the pairing in (5.34). We find upon unfolding that

$$\begin{aligned}&\big(P,\,\mathbb{P}_2^3\Phi\cdot|\det *|^{\overline{s}-\frac{1}{2}}\big)_{\Gamma_2\backslash GL_2(\mathbb{R})}\\ &\quad = \int_0^\infty\int_0^\infty h(y_1)\cdot(y_0^2y_1)^{s-\frac{1}{2}}\cdot\int_0^1\widetilde{\Phi}\left[\begin{pmatrix}1 & u_{1,2} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right]e(u_{1,2})\,du_{1,2}\,\frac{dy_0dy_1}{y_0y_1^2}.\end{aligned} \tag{6.1}$$

The main task of this section is to compute the inner, 'incomplete' unipotent integral in (6.1) in terms of the Fourier–Whittaker periods of Φ (see Definition 5.12), which are relevant in constructing various L-functions associated with Φ, as discussed in Section 5.4.

While this can be achieved using the *full* Fourier expansion of [JPSS] (see [Gold, Theorem 5.3.2]) and simplifying, we opt for a self-contained and conceptual treatment, which follows from two one-dimensional Fourier expansions and the automorphy of Φ. Essentially, this is where 'summation formulae' come into play in our method, presented in an elementary, clean, and global manner.

Proposition 6.1. For any automorphic function Φ of Γ_3, we have, for any $y_0, y_1 > 0$,

$$\int_0^1 \Phi\left[\begin{pmatrix}1 & u_{1,2} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] e(-u_{1,2})\, du_{1,2} = \sum_{a_0,a_1=-\infty}^{\infty} (\widehat{\Phi})_{(a_1,1)}\left[\begin{pmatrix}1 & & \\ & 1 & \\ & -a_0 & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right]. \tag{6.2}$$

Proof. Firstly, we Fourier-expand along the abelian subgroup $\left\{\begin{pmatrix}1 & & * \\ & 1 & \\ & & 1\end{pmatrix}\right\}$:

$$\int_0^1 \Phi\left[\begin{pmatrix}1 & u_{1,2} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] e(-u_{1,2})\, du_{1,2} = \sum_{a_0=-\infty}^{\infty} \int_{\mathbb{Z}^2\backslash\mathbb{R}^2} \Phi\left[\begin{pmatrix}1 & u_{1,2} & u_{1,3} \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] e(-u_{1,2} - a_0 \cdot u_{1,3})\, du_{1,2}\, du_{1,3}. \tag{6.3}$$

Secondly, for each $a_0 \in \mathbb{Z}$, consider a unimodular change of variables of the form $(u_{1,2}, u_{1,3}) = (u_{1,2}', u_{1,3}') \cdot \begin{pmatrix}1 & \\ -a_0 & 1\end{pmatrix}$. One can readily observe that

$$\begin{pmatrix}1 & u_{1,2} & u_{1,3} \\ & 1 & \\ & & 1\end{pmatrix} = \begin{pmatrix}1 & & \\ & 1 & \\ & a_0 & 1\end{pmatrix}\begin{pmatrix}1 & u_{1,2}' & u_{1,3}' \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ & -a_0 & 1\end{pmatrix}.$$

Together with the automorphy of Φ with respect to Γ_3, we have

$$\int_0^1 \Phi\left[\begin{pmatrix}1 & u_{1,2} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] e(-a_2 \cdot u_{1,2})\, du_{1,2} = \sum_{a_0=-\infty}^{\infty} \int_{\mathbb{Z}^2\backslash\mathbb{R}^2} \Phi\left[\begin{pmatrix}1 & u_{1,2}' & u_{1,3}' \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ & -a_0 & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] e(-u_{1,2}')\, du_{1,2}'\, du_{1,3}'. \tag{6.4}$$

The result follows from a third and final Fourier expansion along the abelian subgroup $\left\{\begin{pmatrix}1 & & \\ & 1 & * \\ & & 1\end{pmatrix}\right\}$:

$$\int_0^1 \Phi\left[\begin{pmatrix}1 & u_{1,2} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] e(-u_{1,2})\, du_{1,2} = \sum_{a_0,a_1=-\infty}^{\infty} \int_0^1\int_0^1\int_0^1 \Phi\left[\begin{pmatrix}1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ & 1 & \\ & -a_0 & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] e(-u_{1,2} - a_1 \cdot u_{2,3})\, du_{1,2}\, du_{1,3}\, du_{2,3}.$$

□

We then explicate Proposition 6.1 when Φ is a Maass cusp form of Γ_3. This constitutes the *basic identity* of the present article. Theorem 1.1 is a natural consequence of this identity and the diagonal/off-diagonal structures on the dual side become apparent (see Proposition 7.2).

Corollary 6.2. Suppose Φ is a Maass cusp form of Γ_3. Then

$$\int_0^1 \Phi\left[\begin{pmatrix}1 & u_{1,2} & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix}\right] e(-u_{1,2})\, du_{1,2}$$
$$= \sum_{a_1\neq 0} \frac{\mathcal{B}_\Phi(a_1,1)}{|a_1|}\cdot W_{\alpha(\Phi)}\left(|a_1|y_0, y_1\right)$$
$$+ \sum_{a_0\neq 0}\sum_{a_1\neq 0} \frac{\mathcal{B}_\Phi(a_1,1)}{|a_1|}\cdot W_{\alpha(\Phi)}\left(\frac{|a_1|y_0}{1+(a_0y_0)^2},\ y_1\sqrt{1+(a_0y_0)^2}\right)$$
$$\cdot\, e\left(-\frac{a_0a_1y_0^2}{1+(a_0y_0)^2}\right). \tag{6.5}$$

Proof. By cuspidality, $(\widehat{\Phi})_{(0,1)} \equiv 0$. The result follows from a straight-forward linear algebra calculation:

$$\begin{pmatrix}1 & & \\ & 1 & \\ & -a_0 & 1\end{pmatrix}\begin{pmatrix}y_0y_1 & & \\ & y_0 & \\ & & 1\end{pmatrix} \equiv \begin{pmatrix}1 & & \\ & 1 & -\frac{a_0y_0^2}{1+(a_0y_0)^2} \\ & & 1\end{pmatrix}\begin{pmatrix}\frac{y_0}{1+(a_0y_0)^2}\cdot y_1\sqrt{1+(a_0y_0)^2} & & \\ & \frac{y_0}{1+(a_0y_0)^2} & \\ & & 1\end{pmatrix} \tag{6.6}$$

under the right quotient by $O_3(\mathbb{R})\cdot\mathbb{R}^\times$. This can be verified by the formula stated in [Bu18, Section 2.4] or the mathematica command *IwasawaForm[]* in the *GL(n)pack* (*gln.m*). The user manual and the package can be downloaded from Kevin A. Broughan's website: https://www.math.waikato.ac.nz/~kab/glnpack.html. ☐

6.2. **Initial Simplification and Absolute Convergence.** We temporarily restrict ourselves to the vertical strip $1+\frac{\theta}{2} < \sigma := \operatorname{Re} s < 4$. As we will see, this guarantees absolute convergence of sums and integrals.

Suppose $H \in \mathcal{C}_\eta$ with $\eta > 40$ (see Proposition 5.10). Then the bound (5.12) for $h := H^\flat$ implies its Mellin transform $\widetilde{h}(w) := \int_0^\infty h(y)y^w\, d^\times y$ is holomorphic on the strip $|\operatorname{Re} w| < \eta$. Substituting (6.5) into (6.1), applying the changes of variables $y_0 \to |a_1|^{-1}y_0$ and $y_0 \to |a_0|^{-1}y_0$ to the first and second piece of the resultant, we have

$$\left(P,\ \mathbb{P}_2^3\Phi\cdot|\det *|^{\overline{s}-\frac{1}{2}}\right)_{\Gamma_2\backslash GL_2(\mathbb{R})} = 2\cdot L(2s,\Phi)\cdot\int_0^\infty\int_0^\infty h(y_1)\cdot(y_0^2y_1)^{s-\frac{1}{2}}\cdot W_{-\alpha(\Phi)}(y_0,y_1)\,\frac{dy_0dy_1}{y_0y_1^2} \tag{6.7}$$
$$+\ \mathrm{OD}_\Phi(s),$$

where

Definition 6.3.

$$\mathrm{OD}_\Phi(s) := \sum_{a_0\neq 0}\sum_{a_1\neq 0}\frac{\mathcal{B}_\Phi(1,a_1)}{|a_0|^{2s-1}|a_1|}\cdot\int_0^\infty\int_0^\infty h(y_1)\cdot(y_0^2y_1)^{s-\frac{1}{2}}\cdot e\left(\frac{a_1}{a_0}\cdot\frac{y_0^2}{1+y_0^2}\right)$$
$$\cdot\, W_{-\alpha(\Phi)}\left(\left|\frac{a_1}{a_0}\right|\cdot\frac{y_0}{1+y_0^2},\ y_1\sqrt{1+y_0^2}\right)\frac{dy_0dy_1}{y_0y_1^2}. \tag{6.8}$$

Proposition 6.4. When $H \in \mathcal{C}_\eta$ and $4 > \sigma > \frac{1+\theta}{2}$, we have

$$\int_0^\infty\int_0^\infty h(y_1)\cdot(y_0^2y_1)^{s-\frac{1}{2}}\cdot W_{-\alpha(\Phi)}(y_0,y_1)\,\frac{dy_0dy_1}{y_0y_1^2}$$
$$= \frac{\pi^{-3s}}{8}\cdot\int_{(0)}\frac{H(\mu)}{|\Gamma(\mu)|^2}\cdot\prod_{i=1}^3\Gamma\left(\frac{s+\mu-\alpha_i}{2}\right)\Gamma\left(\frac{s-\mu-\alpha_i}{2}\right)\frac{d\mu}{2\pi i}. \tag{6.9}$$

Proof. From Proposition 5.11, we have

$$\int_0^\infty \int_0^\infty h(y_1) \cdot (y_0^2 y_1)^{s-\frac{1}{2}} \cdot W_{-\alpha(\Phi)}(y_0, y_1) \, \frac{dy_0 dy_1}{y_0 y_1^2}$$
$$= \frac{1}{2} \cdot \int_{(0)} \frac{H(\mu)}{|\Gamma(\mu)|^2} \cdot \int_0^\infty \int_0^\infty W_\mu(y_1) W_{-\alpha(\Phi)}(y_0, y_1)(y_0^2 y_1)^{s-\frac{1}{2}} \, \frac{dy_0 dy_1}{y_0 y_1^2} \, \frac{d\mu}{2\pi i}.$$

The y_0, y_1-integrals can be evaluated by Proposition 5.7 and (6.9) follows. Moreover, the right side of (6.9) is holomorphic on $\sigma > 0$. □

Proposition 6.5. The off-diagonal $\mathrm{OD}_\Phi(s)$ converges absolutely when $4 > \sigma > 1 + \frac{\theta}{2}$ and $H \in \mathcal{C}_\eta$ ($\eta > 40$).

Proof. Upon inserting absolute values, breaking up the y_0-integral into $\int_0^1 + \int_1^\infty$, and applying the bounds (5.6) and $|\mathcal{B}_\Phi(1, a_1)| \ll |a_1|^\theta$, observe that

$$\mathrm{OD}_\Phi(s) \ll \sum_{a_0=1}^\infty \sum_{a_1=1}^\infty \frac{1}{a_0^{2\sigma-1} a_1^{1-\theta}} \left(\int_{y_0=1}^\infty + \int_{y_0=0}^1 \right) \int_{y_1=0}^\infty |h(y_1)| (y_0^2 y_1)^{\sigma-\frac{1}{2}} \big(\frac{a_1 a_0^{-1} y_0}{1+y_0^2} \big)^{A_0} \big(y_1 \sqrt{1+y_0^2} \big)^{A_1} \, \frac{dy_0 dy_1}{y_0 y_1^2},$$

where the implicit constant depends only on Φ, A_0, A_1 with $-\infty < A_0, A_1 < 1$. We are allowed to choose different A_0, A_1 in different ranges of the y_0, y_1-integrals.

The convergence of both of the series is guaranteed if

$$A_0 \; < \; -\theta \quad \text{and} \quad \sigma \; > \; 1 - A_0/2. \tag{6.10}$$

We now show that if (6.10) and

$$A_1 \; < \; A_0 - 2\sigma + 1 \tag{6.11}$$

both hold, then the y_0-integrals converge. Indeed, observe that $2\sigma + A_0 - 2 > -1$ (by (6.10)), and

$$\int_{y_0=0}^1 y_0^{2\sigma+A_0-2} \left(1+y_0^2\right)^{\frac{A_1}{2}-A_0} \, dy_0 \; \asymp_{A_0,A_1} \; \int_{y_0=0}^1 y_0^{2\sigma+A_0-2} \, dy_0.$$

So, the last integral converges. Also, (6.10) and (6.11) imply $A_1 < \min\{1, 2A_0\}$ and thus,

$$\int_{y_0=1}^\infty y_0^{2\sigma+A_0-2} \left(1+y_0^2\right)^{\frac{A_1}{2}-A_0} \, dy_0 \; \leqslant \; \int_{y_0=1}^\infty y_0^{2\sigma+A_1-A_0-2} \, dy_0.$$

The last integral converges because of (6.11).

For the y_1-intgeral, the integrals

$$\int_{y_1=1}^\infty |h(y_1)| y_1^{\sigma+A_1-\frac{5}{2}} \, dy_1 \quad \text{and} \quad \int_{y_1=0}^1 |h(y_1)| y_1^{\sigma+A_1-\frac{5}{2}} \, dy_1$$

converge whenever $H \in \mathcal{C}_\eta$ (we then have (5.12)) and

$$\eta \; > \; \big|\sigma + A_1 - 3/2\big|. \tag{6.12}$$

Let $\delta := \sigma - 1 - (\theta/2)$ (> 0). In view of (6.10) and (6.11), we may take $A_0 := -\theta - \delta$ and $A_1 := -2\theta - 1 - 4\delta$. Also, (6.12) trivially holds as $\eta > 40$ and $\sigma < 4$. The result follows. □

Remark 6.6. Readers may notice the similarity between (3.2) and the inner product construction of the Kuznetsov formula. Indeed, $\mathbb{P}_2^3\Phi$ is an infinite sum of Poincaré series for $\mathrm{SL}_2(\mathbb{Z})$ due to its Fourier expansion, though we never adopt this perspective in this article. This serves as a GL(3) × GL(2) analog to the Kuznetsov formula. However, there are key differences. Our moment identity equates two unfoldings, rather than comparing spectral and geometric expansions.

The second difference is technical. In the Kuznetsov formula, the oscillatory factors can be eliminated to obtain a 'primitive' trace formula, see [GK13], [Zh14], [GSW21]. However, this does not work here — we have yet to analytically continue into the critical strip in Proposition 6.5. Here, the oscillatory factor in $\mathrm{OD}_\Phi(s)$ is crucial, arising naturally from the abstract characterization of Whittaker functions.

7. Structure of the Off-diagonal

Fix $\epsilon := 1/100$ (say), $0 < \phi < \pi/2$, and consider the domain $1 + \theta/2 + \epsilon < \sigma < 4$ in this section to maintain absolute convergence. We will stick with this choice of ϵ for the rest of this article and the number ϕ here should not pose any confusion with the basis of cusp forms (ϕ_j) of Γ_2. We define a perturbed version of $\mathrm{OD}_\Phi(s)$ as follows:

$$\mathrm{OD}_\Phi(s;\phi) \;:=\; \sum_{a_0\neq 0}\sum_{a_1\neq 0} \frac{\mathcal{B}_\Phi(1,a_1)}{|a_0|^{2s-1}|a_1|} \int_0^\infty \int_0^\infty h(y_1)(y_0^2 y_1)^{s-\frac{1}{2}} W_{-\alpha(\Phi)}\left(\left|\frac{a_1}{a_0}\right| \frac{y_0}{1+y_0^2},\; y_1\sqrt{1+y_0^2}\right) \cdot e\left(\frac{a_1}{a_0}\frac{y_0^2}{1+y_0^2};\phi\right) \frac{dy_0 dy_1}{y_0 y_1^2}, \tag{7.1}$$

where

$$e\,(x;\phi) \;:=\; \int_{(\epsilon)} |2\pi x|^{-u} e^{iu\phi\,\mathrm{sgn}(x)}\Gamma(u)\, \frac{du}{2\pi i} \qquad (x\in\mathbb{R}-\{0\}). \tag{7.2}$$

In Proposition 7.3, we show that

$$\lim_{\phi\to\pi/2} \mathrm{OD}_\Phi(s;\phi) \;=\; \mathrm{OD}_\Phi(s) \tag{7.3}$$

on a smaller region of absolute convergence.

Remark 7.1. The goals of this section is to obtain an expression of $\mathrm{OD}_\Phi(s;\phi)$ that

- reveals the structure of the dual moment;
- can be analytically continued into the critical strip;
- and will allow us to pass to the limit $\phi\to\pi/2$ (in the critical strip).

Given these considerations, it is natural to work on the dual side of the Mellin transforms, which also allows for the separation of variables. The main result of this section is as follows:

Proposition 7.2 (Dual Moment)**.** Let $H\in\mathcal{C}_\eta$ $(\eta>40)$ and $\phi\in(0,\pi/2)$. On the vertical strip

$$1+\frac{\theta}{2}+\epsilon \;<\; \sigma \;<\; 4, \tag{7.4}$$

we have

$$\mathrm{OD}_\Phi(s;\phi) \;=\; \frac{1}{4}\int_{(1+\theta+2\epsilon)} \zeta\,(2s-s_0)\, L\,(s_0,\Phi)\cdot \sum_{\delta=\pm} (\mathcal{F}_\Phi^{(\delta)}H)\,(s_0,s;\phi)\, \frac{ds_0}{2\pi i}, \tag{7.5}$$

where the transform of H is given by

$$(\mathcal{F}_\Phi^{(\delta)}H)\,(s_0,s;\phi) \;:=\; \int_{(15)}\int_{(\epsilon)} \widetilde{h}\,(s-s_1-1/2)\,\mathcal{G}_\Phi^{(\delta)}\,(s_1,u;\; s_0,s;\;\phi)\, \frac{du}{2\pi i}\frac{ds_1}{2\pi i}, \tag{7.6}$$

with $h := H^\flat$, $G_\Phi := G_{\alpha(\Phi)}$ as defined in (5.5), and

$$\mathcal{G}_\Phi^{(\delta)}\,(s_1,u;\; s_0,s;\;\phi) \;:=\; G_\Phi\,(s_0-u,s_1)\,(2\pi)^{-u} e^{i\delta\phi u}\Gamma(u) \frac{\Gamma\left(\frac{u+1-2s+s_1-s_0}{2}\right)\Gamma\left(\frac{2s-s_0-u}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}-s_0\right)}. \tag{7.7}$$

Proof. Plug-in the expression for $W_{-\alpha(\Phi)}$ from Proposition 5.3 into $\mathrm{OD}_\Phi(s;\phi)$ with

$$\sigma_1 \;:=\; 15 \quad\text{and}\quad 1+\theta \;<\; \sigma_0 \;<\; 2\sigma-1-\epsilon. \tag{7.8}$$

Insert absolute values to the resulting expression, the sums and integrals are bounded by

$$\sum_{\delta:=\mathrm{sgn}(a_0a_1)=\pm} \left(\sum_{a_0\neq 0}\frac{1}{|a_0|^{2\sigma-\sigma_0-\epsilon}}\right)\left(\sum_{a_1\neq 0}\frac{|\mathcal{B}_\Phi(1,a_1)|}{|a_1|^{\sigma_0+\epsilon}}\right)\left(\int_{(\sigma_0)}\int_{(\sigma_1)} |G_\Phi(s_0,s_1)|\,|ds_0||ds_1|\right) \cdot\left(\int_{(\epsilon)} |e^{i\delta\phi u}\Gamma(u)|\,|du|\right)\left(\int_0^\infty y_0^{-\sigma_0-2\epsilon+2\sigma}(1+y_0^2)^{\sigma_0+\epsilon-\frac{1+\sigma_1}{2}}\, d^\times y_0\right)\left(\int_0^\infty |h(y_1)| y_1^{\sigma-\sigma_1-\frac{1}{2}}\, d^\times y_1\right). \tag{7.9}$$

Observe that:

- by Stirling's formula, the s_0, s_1, u-integrals converge as long as

$$\sigma_0,\ \sigma_1,\ \epsilon\ >\ 0, \qquad \phi\ \in\ (0,\pi/2); \tag{7.10}$$

- the y_0-integral converges as long as

$$\sigma_0+2\epsilon\ <\ 2\sigma\ <\ \sigma_1-\sigma_0+1; \tag{7.11}$$

- by the bound $|\mathcal{B}_\Phi(1,a_1)|\ \ll\ |a_1|^\theta$, the a_0-sum and the a_1-sum converge as long as

$$2\sigma-1\ >\ \sigma_0+\epsilon\ >\ 1+\theta. \tag{7.12}$$

Under (7.8), items (7.10), (7.11), (7.12) hold. Moreover, the y_1-integral converges by (5.12) and $H\in\mathcal{C}_\eta$ ($\eta>40$). Now, upon rearranging sums and integrals, and noticing that $\mathcal{B}_\Phi(1,a_1)=\mathcal{B}_\Phi(1,-a_1)$, we have

$$\begin{aligned}\mathrm{OD}_\Phi(s;\phi)\ =\ 2\sum_{\delta=\pm}\int_{(\sigma_0)}\int_{(\sigma_1)}\int_{(\epsilon)} \frac{G_\Phi(s_0,s_1)}{4}(2\pi)^{-u}e^{i\delta\phi u}\Gamma(u)\left(\int_0^\infty h(y_1)y_1^{s-s_1-\frac{1}{2}}d^\times y_1\right)\\ \cdot\left(\int_0^\infty y_0^{-s_0-2u+2s}(1+y_0^2)^{s_0+u-\frac{1+s_1}{2}}d^\times y_0\right)\left(\sum_{a_0=1}^\infty\sum_{a_1=1}^\infty\frac{\mathcal{B}_\Phi(1,a_1)}{a_0^{2s-1}a_1}\left(\frac{a_1}{a_0}\right)^{1-s_0-u}\right)\frac{ds_0}{2\pi i}\frac{ds_1}{2\pi i}\frac{du}{2\pi i}.\end{aligned} \tag{7.13}$$

Recall the integral identity

$$\int_0^\infty y_0^v(1+y_0^2)^A\ d^\times y_0\ =\ \frac{1}{2}\ \frac{\Gamma\left(-A-\frac{v}{2}\right)\Gamma\left(\frac{v}{2}\right)}{\Gamma(-A)} \tag{7.14}$$

for $0<\operatorname{Re}v<-2\operatorname{Re}A$. It follows that

$$\begin{aligned}\mathrm{OD}_\Phi(s;\phi)\ =\ 2\sum_{\delta=\pm}\int_{(\sigma_0)}\int_{(\sigma_1)}\int_{(\epsilon)} \zeta\left(2s-s_0-u\right)L\left(s_0+u;\Phi\right)\cdot\widetilde{h}\left(s-s_1-1/2\right)\\ \cdot\frac{G_\Phi(s_0,s_1)}{4}(2\pi)^{-u}e^{i\delta\phi u}\Gamma(u)\cdot\frac{1}{2}\ \frac{\Gamma\left(s-\frac{s_0}{2}-u\right)\Gamma\left(\frac{1+s_1-s_0}{2}-s\right)}{\Gamma\left(\frac{1+s_1}{2}-s_0-u\right)}\ \frac{ds_0}{2\pi i}\frac{ds_1}{2\pi i}\frac{du}{2\pi i}.\end{aligned} \tag{7.15}$$

We pick the contour $(\sigma_0):=(1+\theta+\epsilon)$, thus imposing (7.4). To isolate the non-archimedean part of $\mathrm{OD}_\Phi(s;\phi)$, we change variables to $s_0'=s_0+u$. Substituting the expression for $G_\Phi\left(s_0'-u,s_1\right)$ (see (5.5)), we obtain (7.5)–(7.7). The absolute convergence proven earlier also ensures the holomorphy of the integral transform $(\mathcal{F}_\Phi^{(\delta)}h)\left(s_0',\ s;\ \phi\right)$ on the domain:

$$\sigma\ <\ 4 \qquad \text{and} \qquad 1+\theta+\epsilon\ <\ \sigma_0'\ <\ 2\sigma-1. \tag{7.16}$$

This completes the proof. ☐

Proposition 7.3. For $4>\sigma>(3+\theta)/2$ and $H\in\mathcal{C}_\eta$, we have

$$\lim_{\phi\to\pi/2}\ \mathrm{OD}_\Phi(s;\phi)\ =\ \mathrm{OD}_\Phi(s). \tag{7.17}$$

Proof. Let $\epsilon:=1/100$, $\sigma_1:=15$, and pick any σ_0 satisfying

$$\frac{3}{2}+\theta+\epsilon\ <\ \sigma_0\ <\ 2\sigma-1-\epsilon. \tag{7.18}$$

Denote by $\mathcal{C}_\epsilon$ the indented path consisting of the line segments:

$$-\frac{1}{2}-\epsilon-i\infty\ \to\ -\frac{1}{2}-\epsilon-i\ \to\ \epsilon-i\ \to\ \epsilon+i\ \to\ -\frac{1}{2}-\epsilon+i\ \to\ -\frac{1}{2}-\epsilon+i\infty.$$

Replace $e(x;\phi)$ in (7.13) by the expression:

$$e(x;\phi)\ =\ \int_{\mathcal{C}_\epsilon}|2\pi x|^{-u}e^{iu\phi\,\mathrm{sgn}(x)}\Gamma(u)\ \frac{du}{2\pi i}. \tag{7.19}$$

Note that $\left|e^{iu\phi\,\mathrm{sgn}(x)}\Gamma(u)\right|\ll_\epsilon(1+|\operatorname{Im}u|)^{-1-\epsilon}$ for $u\in\mathcal{C}_\epsilon$ and $\phi\in(0,\pi/2]$. Insert absolute values in (7.13). The resulting sums and integrals converge absolutely when $\phi\in(0,\pi/2]$ and (7.18) holds, which can be seen by the same argument following (7.9). Apply dominated convergence and shift the contour of the u-integral to $-\infty$, the residual series obtained is exactly $e\left(\frac{a_1}{a_0}\frac{y_0^2}{1+y_0^2}\right)$. This completes the proof. $\square$

Now, $\mathrm{OD}_\Phi(s;\phi)$ is expressed as Mellin–Barnes integrals. The Γ-factors from Proposition 5.3 and (7.2) alone are not sufficient for our goals (see Remark 7.1 and (7.10)–(7.12)). The three extra Γ-factors brought by the y_0-integral, which mix all variables of integrations, will play an important role in Section 8-9.

8. Analytic Properties of the Archimedean Transform

In (7.5), the factors $\zeta(2s-s_0)$ and $L(s_0,\Phi)$ are known to admit holomorphic continuation and have polynomial growth in vertical strips, except on the line $2s-s_0=1$. We also examine the archimedean part of (7.5), i.e., the integral transform

$$(\mathcal{F}_\Phi^{(\delta)}H)(s_0,s;\phi) := \int_{(15)}\int_{(\epsilon)} \widetilde{h}(s-s_1-1/2)\mathcal{G}_\Phi^{(\delta)}(s_1,u;s_0,s;\phi)\,\frac{du}{2\pi i}\frac{ds_1}{2\pi i}, \tag{8.1}$$

where $h := H^\flat$ and $\mathcal{G}_\Phi^{(\delta)}(\cdots)$ as defined in (7.7). In Section 7, we have shown that when $\phi\in(0,\pi/2)$, the function $(s_0,s)\mapsto(\mathcal{F}_\Phi^{(\delta)}H)(s_0,s;\phi)$ is holomorphic on the domain (7.16), i.e.,

$$\sigma < 4 \qquad \text{and} \qquad 1+\theta+\epsilon < \sigma_0 < 2\sigma-1.$$

In this section, we establish a larger region of holomorphy for $(s_0,s)\mapsto(\mathcal{F}_\Phi^{(\delta)}H)(s_0,s;\phi)$ that holds for $\phi\in(0,\pi/2]$. We write

$$s = \sigma+it, \quad s_0=\sigma_0+it_0, \quad s_1=\sigma_1+it_1, \quad \text{and} \quad u=\epsilon+iv,$$

with $\epsilon := 1/100$. It is sufficient to consider s inside the rectangular box $\epsilon<\sigma<4$ and $|t|\leqslant T$, for any given $T\geqslant 1000$. Moreover, $\alpha_k := i\gamma_k\in i\mathbb{R}$ ($k=1,2,3$) by our assumptions on Φ. The main result of this section can be stated as follows:

Proposition 8.1. Suppose $H\in\mathcal{C}_\eta$.

(1) For any $\phi\in(0,\pi/2]$, the transform $(\mathcal{F}_\Phi^{(\delta)}H)(s_0,s;\phi)$ is holomorphic on the domain

$$\sigma_0 > \epsilon, \quad \sigma < 4, \quad \text{and} \quad 2\sigma-\sigma_0-\epsilon > 0. \tag{8.2}$$

(2) Whenever $(\sigma_0,\sigma)\in$ (8.2), $|t|<T$, and $\phi\in(0,\pi/2)$, the transform $(\mathcal{F}_\Phi^{(\delta)}H)(s_0,s;\phi)$ has exponential decay as $|t_0|\to\infty$. Note: The explicit estimate is stated in the proof below and the implicit constant depends only on T and Φ.

Remark 8.2. The domain (8.2) is chosen in a way that the function $(s_0,s)\mapsto\mathcal{G}_\Phi^{(\delta)}(s_1,u;\ s_0,s;\ \phi)$ is holomorphic on (8.2) when $\operatorname{Re}s_1=\sigma_1\geqslant 15$ and $\operatorname{Re}u=\epsilon$. Moreover, if we have $15\leqslant\sigma_1\leqslant\eta-1/2$ and (8.2), then $s-s_1-1/2$ lies inside the region of holomorphy of $\widetilde{h}$.

Remark 8.3. As we shall see in Proposition 9.2, the region of holomorphy (8.2) is essentially optimal in terms of σ_0.

Proof. The proof is based on a careful application of the Stirling estimate

$$|\Gamma(a+ib)| \asymp_a (1+|b|)^{a-\frac{1}{2}}\,e^{-\frac{\pi}{2}|b|} \qquad (a\neq 0,\,-1,\,-2,\,\ldots,\quad b\in\mathbb{R}) \tag{8.3}$$

to the kernel function $\mathcal{G}_\Phi^{(\delta)}(s_1,u;\ s_0,s;\ \phi)$. The following set of conditions will be repeated throughout the proof:

$$\begin{cases} \qquad 0 < \phi \leqslant \pi/2, \\ \sigma_0 > \epsilon, \quad \sigma < 4, \quad 2\sigma-\sigma_0-\epsilon > 0, \\ \operatorname{Re}s_1 = \sigma_1 \geqslant 15, \quad \operatorname{Re}u = \epsilon. \end{cases} \tag{8.4}$$

Assuming (8.4), we apply (8.3) to the kernel function (7.7). It follows that

$$\begin{aligned}|\mathcal{G}^{(\delta)}_{\Phi}(s_1,u;s_0,s;\phi)| \asymp{}& (1+|v|)^{\epsilon-\frac12}\, e^{-\left(\frac{\pi}{2}-\phi\right)|v|}\cdot\prod_{k=1}^{3}(1+|t_1-\gamma_k|)^{\frac{\sigma_1-1}{2}}\, e^{-\frac{\pi}{4}|t_1-\gamma_k|}\\ &\cdot\prod_{k=1}^{3}(1+|t_0-v+\gamma_k|)^{\frac{\sigma_0-\epsilon-1}{2}}\, e^{-\frac{\pi}{4}|t_0-v+\gamma_k|}\cdot(1+|2t-t_0-v|)^{\frac{2\sigma-1-\sigma_0-\epsilon}{2}}\, e^{-\frac{\pi}{4}|2t-t_0-v|}\\ &\cdot(1+|v-2t+t_1-t_0|)^{\frac{\epsilon-2\sigma+\sigma_1-\sigma_0}{2}}\, e^{-\frac{\pi}{4}|v-2t+t_1-t_0|}\\ &\cdot(1+|t_1-2t_0|)^{-\left(\frac{\sigma_1}{2}-\sigma_0\right)}\, e^{\frac{\pi}{4}|t_1-2t_0|}\cdot(1+|t_0+t_1-v|)^{-\frac{\sigma_0+\sigma_1-\epsilon-1}{2}}\, e^{\frac{\pi}{4}|t_0+t_1-v|},\end{aligned}\tag{8.5}$$

where the implicit constant depends at most on σ_1. Note that the domain (8.2) for (σ,σ_0) is bounded and thus the estimate is uniform in σ,σ_0,ϵ. This will be assumed for all estimates in the rest of this section.

Let $\mathcal{P}^{\Phi}_{s}(t_0,t_1,v)$ be the 'polynomial part' of (8.5) and the 'exponential phase' of (8.5) be

$$\mathcal{E}^{\Phi}_{s}(t_0,t_1,v) := \sum_{k=1}^{3}\{|t_1-\gamma_k|+|t_0-v+\gamma_k|\}+|2t-t_0-v|+|v-2t+t_1-t_0|-|t_1-2t_0|-|t_0+t_1-v|.$$

We first examine $\mathcal{E}^{\Phi}_{s}(t_0,t_1,v)$ of (8.5), which determines the effective support of $\left(\mathcal{F}^{(\delta)}_{\Phi}H\right)(s_0,s;\phi)$. By the triangle inequality and the fact $\gamma_1+\gamma_2+\gamma_3=0$, we have

$$|\mathcal{G}^{(\delta)}_{\Phi}(s_1,u;\,s_0,s;\,\phi)| \ll_{\sigma_1} e^{\pi T}\cdot\mathcal{P}^{\Phi}_{s}(t_0,t_1,v)\cdot\exp\left(-\frac{\pi}{4}\mathcal{E}(t_0,t_1,v)\right)\cdot e^{-\left(\frac{\pi}{2}-\phi\right)|v|}\tag{8.6}$$

with

$$\mathcal{E}(t_0,t_1,v) := 3|t_1|+3|t_0-v|-|t_1-2t_0|+|v+t_1-t_0|+|t_0+v|-|t_0+t_1-v|,\tag{8.7}$$

whenever we have (8.4) and $|t|\leqslant T$,

Claim 8.4. For any $t_0,t_1,v\in\mathbb{R}$, we have $\mathcal{E}(t_0,t_1,v)\geqslant 0$. Equality holds if and only if

$$t_1 = 0 \quad\text{and}\quad t_0-v = 0.\tag{8.8}$$

Proof. Adding up the inequalities $|t_1|+|t_0-v|\geqslant|t_0+t_1-v|$ and $|v+t_1-t_0|+|t_0+v|\geqslant|t_1-2t_0|$, we have

$$\mathcal{E}(t_0,t_1,v) \geqslant 2\,(|t_1|+|t_0-v|) \geqslant 0.\tag{8.9}$$

The equality case is apparent. ☐

Claim 8.5. When (8.4) and $|t|\leqslant T$ hold, the integral

$$\iint\limits_{\substack{(\operatorname{Re}s_1,\operatorname{Re}u)=(\sigma_1,\epsilon),\\ (t_1,v):(8.11)\text{ holds}}} \widetilde{h}(s-s_1-1/2)\mathcal{G}^{(\delta)}_{\Phi}(s_1,u;\,s_0,s;\,\phi)\,\frac{du}{2\pi i}\,\frac{ds_1}{2\pi i}\tag{8.10}$$

has exponential decay as $|t_0|\to\infty$, where

$$|t_1| > \log^2(3+|t_0|) \quad\text{or}\quad |v-t_0| > \log^2(3+|t_0|).\tag{8.11}$$

Proof. In the case of (8.11), we have

$$\mathcal{E}(t_0,t_1,v) > \log^2(3+|t_0|)+|t_1|+|t_0-v|\tag{8.12}$$

from (8.9). The polynomial part $\mathcal{P}^{\Phi}_{s}(t_0,t_1,v)$ can be crudely bounded by

$$\mathcal{P}^{\Phi}_{s}(t_0,t_1,v) \ll_{\Phi,\sigma_1,T} [(1+|t_1|)\,(1+|v-t_0|)\,(1+|t_0|)]^{A(\sigma_1)},\tag{8.13}$$

where $A(\sigma_1)>0$ is some constant.

Putting (8.12), (8.13), and the bound $e^{-\left(\frac{\pi}{2}-\phi\right)|v|} \leqslant 1$ ($\phi \in (0, \pi/2]$) into (8.6), we obtain

$$|\mathcal{G}_{\Phi}^{(\delta)}(s_1, u; s_0, s; \phi)| \ll_{\Phi,\sigma_1,T} (1+|t_0|)^{A(\sigma_1)} e^{-\frac{\pi}{4}\log^2(3+|t_0|)} \cdot [(1+|t_1|)(1+|v-t_0|)]^{A(\sigma_1)} e^{-\frac{\pi}{4}[|t_1|+|t_0-v|]} \tag{8.14}$$

whenever (8.11), (8.4), and $|t| \leqslant T$ hold. The boundedness of $\widetilde{h}$ on vertical strips implies that (8.10) is

$$\ll_{\sigma_1,\Phi,T} (1+|t_0|)^{A(\sigma_1)} e^{-\frac{\pi}{4}\log^2(3+|t_0|)}. \tag{8.15}$$

This proves Claim 8.5. □

Now, let $\phi \in (0, \pi/2]$ and consider $(\mathcal{F}_{\Phi}^{(\delta)} H)(s_0, s; \phi)$ as a function on the bounded domain

$$(\sigma_0, \sigma) \in (8.2), \qquad |t|, |t_0| \leqslant T. \tag{8.16}$$

When $|t_1| > \log^2(3+T)$ or $|v| > T + \log^2(3+T)$, observe that (8.11) is satisfied and from (8.14),

$$|\mathcal{G}_{\Phi}^{(\delta)}(s_1, u; s_0, s; \phi)| \ll_{\Phi,T} [(1+|t_1|)(1+|v|)]^{A(15)} \cdot e^{-\frac{\pi}{4}[|t_1|+|v|]}. \tag{8.17}$$

The last function is clearly jointly integrable with respect to t_1, v, and by Remark 8.2, $\left(\mathcal{F}_{\Phi}^{(\delta)} H\right)(s_0, s; \phi)$ is a holomorphic function on (8.16). Since the choice of T is arbitrary, we arrive at the first conclusion of Proposition 8.1.

In the remaining part of this section, we prove the second assertion of Proposition 8.1. We estimate the contribution from

$$|t_1| \leqslant \log^2(3+|t_0|) \quad \text{and} \quad |v-t_0| \leqslant \log^2(3+|t_0|), \tag{8.18}$$

where the complementary part has been treated in Claim 8.5.

It suffices to restrict to the effective support (8.8). The polynomial part can be essentially computed by substituting $t_1 := 0$ and $v := t_0$. More precisely, when (8.18) and $|t_0| \gg_T 1$ hold, there are only two possible scenarios for the factors $1+|(\cdots)|$ in (8.5): either $1+|(\cdots)| \asymp |t_0|$, or $\log^{-C}(3+|t_0|) \ll 1+|(\cdots)| \ll \log^{C}(3+|t_0|)$ for some absolute constant $C>0$.

In the case of (8.18), we apply the bounds $e^{-\frac{\pi}{4}\mathcal{E}(t_0,t_1,v)} \leqslant 1$ and $e^{-\left(\frac{\pi}{2}-\phi\right)|v|} \leqslant e^{-\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)|t_0|}$ for $|t_0| \gg 1$ to (8.6). As a result, if we also have (8.4), $|t|<T$, and $|t_0|>8T$, then

$$|\mathcal{G}_{\Phi}^{(\delta)}(s_1, u; s_0, s; \phi)| \ll_{\sigma_1,\Phi,T} |t_0|^{7-\frac{\sigma_1}{2}} e^{-\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)|t_0|} \log^{B(\sigma_1)}|t_0| \tag{8.19}$$

and

$$\iint\limits_{\substack{(\operatorname{Re} s_1, \operatorname{Re} u)=(\sigma_1,\epsilon),\\ (t_1,v):\ (8.18)\text{ holds}}} \widetilde{h}(s-s_1-1/2)\mathcal{G}_{\Phi}^{(\delta)}(s_1, u;\ s_0, s;\ \phi)\ \frac{du}{2\pi i}\,\frac{ds_1}{2\pi i}$$
$$\ll_{\sigma_1,\Phi,T} |t_0|^{7-\frac{\sigma_1}{2}} e^{-\frac{1}{2}\left(\frac{\pi}{2}-\phi\right)|t_0|} \log^{4+B(\sigma_1)}|t_0|, \tag{8.20}$$

where $B(\sigma_1)>0$ is some constant. If $\phi<\pi/2$, then there is exponential decay in (8.20) as $|t_0| \to \infty$. Therefore, the second conclusion of the proposition follows from (8.20) and (8.15) (putting $\sigma_1 = 15$). □

9. Analytic Continuation of the Off-diagonal (Proof of Theorem 1.1)

Recall that

$$\mathrm{OD}_{\Phi}(s;\phi) = \frac{1}{4}\int_{(1+\theta+2\epsilon)} \zeta(2s-s_0)\, L(s_0,\Phi) \cdot \sum_{\delta=\pm} (\mathcal{F}_{\Phi}^{(\delta)} H)(s_0, s; \phi)\, \frac{ds_0}{2\pi i} \tag{9.1}$$

for $1+\frac{\theta}{2}+\epsilon<\sigma<4$ and $\phi \in (0, \pi/2)$, see Proposition 7.2.

9.1. **Step 1:** We first obtain a holomorphic continuation of $\mathrm{OD}_\Phi(s;\phi)$ up to $\mathrm{Re}\, s > \frac{1}{2}+\epsilon$ by shifting the s_0-integral to the left.

Fix any $\phi \in (0, \pi/2)$ and $T \geqslant 1000$. We first restrict ourselves to

$$1+\frac{\theta}{2}+2\epsilon \;<\; \sigma \;<\; 4, \qquad |t| \;<\; T. \tag{9.2}$$

Clearly, the pole $s_0 = 2s-1$ of $\zeta(2s-s_0)$ is on the right of the contour $\mathrm{Re}\, s_0 = 1+\theta+2\epsilon$ of the integral (7.5).

Let $T_0 \gg 1$. The rectangle with vertices $2\epsilon \pm iT_0$ and $(1+\theta+2\epsilon) \pm iT_0$ in the s_0-plane lies inside the region of holomorphy (8.2) of $(\mathcal{F}_\Phi^{(\delta)}H)(s_0,s;\phi)$. The contribution from the horizontal segments $[2\epsilon \pm iT_0, (1+\theta+2\epsilon)\pm iT_0]$ tends to 0 as $T_0 \to \infty$ by the exponential decay of $(\mathcal{F}_\Phi^{(\delta)}H)(s_0,\ s;\ \phi)$ (see Proposition 8.1), which surely counteracts the polynomial growth from $L(s_0,\Phi)$ and $\zeta(2s-s_0)$. As a result, we may shift the line of integration to $\mathrm{Re}\, s_0 = 2\epsilon$ and no pole is crossed. Hence,

$$\mathrm{OD}_\Phi(s;\phi) \;=\; \frac{1}{4}\int_{(2\epsilon)} \zeta\left(2s-s_0\right) L\left(s_0,\Phi\right) \cdot \sum_{\delta=\pm} \left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(s_0,s;\phi\right) \frac{ds_0}{2\pi i} \tag{9.3}$$

on (9.2). The right side of (9.3) is holomorphic on

$$\frac{1}{2}+\epsilon \;<\; \sigma \;<\; 4, \qquad |t| \;<\; T \tag{9.4}$$

and serves as an analytic continuation of $\mathrm{OD}_\Phi(s;\phi)$ to (9.4) by using Proposition 8.1. Note that $\sigma > \frac{1}{2}+\epsilon$ implies the holomorphy of $\zeta(2s-s_0)$.

9.2. **Step 2: Crossing the Polar Line (Shifting the s_0-integral again).** Consider a subdomain of (9.4):

$$\frac{1}{2}+\epsilon \;<\; \sigma \;<\; \frac{3}{4}, \qquad |t| \;<\; T. \tag{9.5}$$

Different from Step 1, the pole $s_0 = 2s-1$ is now inside the rectangle with vertices $2\epsilon \pm iT_0$ and $\frac{1}{2} \pm iT_0$ provided $T_0 > 4T$. Such a rectangle lies in the region of holomorphy (8.2) of $(\mathcal{F}_\Phi^{(\delta)}H)(s_0,s;\phi)$. When $\phi < \pi/2$, the exponential decay of $(\mathcal{F}_\Phi^{(\delta)}H)(s_0,s;\phi)$ once again allows us to shift the line of integration from $\mathrm{Re}\, s_0 = 2\epsilon$ to $\mathrm{Re}\, s_0 = 1/2$, crossing the pole of $\zeta(2s-s_0)$ which has residue -1. In other words,

$$\begin{aligned}\mathrm{OD}_\Phi(s;\phi) \;=\; & \frac{1}{4}\, L(2s-1,\Phi) \sum_{\delta=\pm} \left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(2s-1,s;\phi\right) \\ & + \frac{1}{4}\int_{(1/2)} \zeta\left(2s-s_0\right) L\left(s_0,\Phi\right) \cdot \sum_{\delta=\pm} \left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(s_0,s;\phi\right) \frac{ds_0}{2\pi i}.\end{aligned} \tag{9.6}$$

On the line $\mathrm{Re}\, s_0 = 1/2$, observe that $s \mapsto \left(\mathcal{F}_\Phi^{(\delta)}H\right)(s_0,s;\phi)$ is holomorphic on $\sigma > \frac{1}{4}+\frac{\epsilon}{2}$ by (8.2); whereas $s \mapsto \zeta(2s-s_0)$ is holomorphic on $\sigma < 3/4$ as $2\sigma - \sigma_0 < 1$. As a result, the function $s \mapsto \int_{(1/2)}(\cdots)\,\frac{ds_0}{2\pi i}$ in (9.6) is holomorphic on the vertical strip

$$\frac{1}{4}+\frac{\epsilon}{2} \;<\; \sigma \;<\; \frac{3}{4}, \tag{9.7}$$

which is sufficient for our purpose.

Proposition 8.1 asserts that the function $s \mapsto \left(\mathcal{F}_\Phi^{(\delta)}H\right)(2s-1,s;\phi)$ is holomorphic on $1/2+\epsilon < \sigma < 4$. However, it actually admits a continuation to the domain $\epsilon < \sigma < 4$ as we will see in Proposition 9.2.

9.3. **Step 3: Putting Back $\phi \to \pi/2$ — Shifting the s_1-integral and Refining Step 1-2.** By using estimate (8.14) and dominated convergence, we have

$$\lim_{\phi \to \pi/2} \left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(2s-1,s;\phi\right) \;=\; \left(\mathcal{F}_\Phi^{(\delta)}H\right)\left(2s-1,s;\pi/2\right) \tag{9.8}$$

for $1/2+\epsilon < \sigma < 4$ and $|t| < T$. However, for the continuous part of (9.6), we need a follow-up of Proposition 8.1 in order to pass to the limit $\phi \to \pi/2$. Essentially, thanks to the structure of the Γ's in Proposition 5.3 and the analytic properties of $\widetilde{h}$, it is possible to shift the line of integration of the s_1-integral to gain sufficient polynomial decay.

Proposition 9.1. Let $H \in \mathcal{C}_\eta$. There exists a constant $B = B_\eta$ such that whenever $(\sigma_0, \sigma) \in$ (8.2), $|t| < T$, and $|t_0| \gg_T 1$, we have the estimate

$$|(\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \pi/2)| \ll |t_0|^{8-\frac{\eta}{2}} \log^B |t_0|, \tag{9.9}$$

where the implicit constant depends only on η, T, Φ.

Proof. On domain (8.2), observe that the vertical strip $\operatorname{Re} s_1 \in [15, \eta - \frac{1}{2}]$ contains no pole of the function $s_1 \mapsto \mathcal{G}_\Phi^{(\delta)}(s_1, u;\ s_0, s;\ \phi)$, and it lies within the region of holomorphy of $\widetilde{h}$ (see Remark 8.2). The estimate (8.14) allows us to shift the line of integration from $\operatorname{Re} s_1 = 15$ to $\operatorname{Re} s_1 = \eta - \frac{1}{2}$ in (7.6). Notice that the estimates done in Proposition 8.1 works for $\phi = \pi/2$ too. In particular, from (8.20) and (8.15), the bound (9.9) follows by taking $\sigma_1 := \eta - \frac{1}{2}$ therein (after the contour shift). This completes the proof. ☐

Suppose $(3+\theta)/2 < \sigma < 4$. By Proposition 7.3, equation (7.5) and equation (9.3), we have

$$\begin{aligned} \mathrm{OD}_\Phi(s) &= \lim_{\phi \to \pi/2} \mathrm{OD}_\Phi(s; \phi) \\ &= \lim_{\phi \to \pi/2} \frac{1}{4} \int_{(1+\theta+2\epsilon)} \zeta(2s - s_0) L(s_0, \Phi) \cdot \sum_{\delta = \pm} (\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi) \frac{ds_0}{2\pi i} \\ &= \lim_{\phi \to \pi/2} \frac{1}{4} \int_{(2\epsilon)} \zeta(2s - s_0) L(s_0, \Phi) \cdot \sum_{\delta = \pm} (\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi) \frac{ds_0}{2\pi i}. \end{aligned} \tag{9.10}$$

Proposition 9.1 ensures enough polynomial decay and hence the absolute convergence of (9.11) at $\phi = \pi/2$:

$$\mathrm{OD}_\Phi(s) = \frac{1}{4} \int_{(2\epsilon)} \zeta(2s - s_0) L(s_0, \Phi) \cdot \sum_{\delta = \pm} (\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \phi) \frac{ds_0}{2\pi i}. \tag{9.11}$$

Now, (9.11) serves as an analytic continuation of $\mathrm{OD}_\Phi(s)$ to the domain $1/2 + \epsilon < \sigma < 4$.

On the smaller domain $1/2 + \epsilon < \sigma < 3/4$, the expressions (9.10) and (9.6) are equal. Then

$$\begin{aligned} \mathrm{OD}_\Phi(s) = (9.10) &= \frac{1}{4} L(2s-1, \Phi) \sum_{\delta = \pm} (\mathcal{F}_\Phi^{(\delta)} H)(2s-1, s; \pi/2) \\ &\quad + \frac{1}{4} \int_{(1/2)} \zeta(2s - s_0) L(s_0, \Phi) \cdot \sum_{\delta = \pm} (\mathcal{F}_\Phi^{(\delta)} H)(s_0, s; \pi/2) \frac{ds_0}{2\pi i} \end{aligned} \tag{9.12}$$

by dominated convergence and Proposition 8.1. The last integral is holomorphic on $1/4 + \epsilon/2 < \sigma < 3/4$.

In the following, we write $(\mathcal{F}_\Phi H)(s_0, s) := (\mathcal{F}_\Phi^+ H)(s_0, s;\ \pi/2) + (\mathcal{F}_\Phi^- H)(s_0, s;\ \pi/2)$. Duplication and reflection formulae of Γ-functions in the form:

$$2^{-u}\Gamma(u) = \frac{1}{2\sqrt{\pi}} \cdot \Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{u+1}{2}\right) \quad \text{and} \quad \Gamma\left(\frac{1+u}{2}\right)\Gamma\left(\frac{1-u}{2}\right) = \pi \sec \frac{\pi u}{2},$$

lead to

$$\begin{aligned} (\mathcal{F}_\Phi H)(s_0, s) = \sqrt{\pi} \int_{(\eta - 1/2)} \widetilde{h}(s - s_1 - 1/2) \pi^{-s_1} \frac{\prod_{i=1}^3 \Gamma\left(\frac{s_1 - \alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} - s_0\right)} \\ \cdot \int_{(\epsilon)} \frac{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{s_1 - s_0 + u}{2} + \frac{1}{2} - s\right) \prod_{i=1}^3 \Gamma\left(\frac{s_0 - u + \alpha_i}{2}\right)\Gamma\left(s - \frac{s_0 + u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right)\Gamma\left(\frac{s_0 - u + s_1}{2}\right)} \frac{du}{2\pi i} \frac{ds_1}{2\pi i}. \end{aligned} \tag{9.13}$$

In Section 10, we will work with this expression further.

9.4. Step 4: Continuation of the Residual Term — Shifting the u-integral.

Proposition 9.2. Let $H \in \mathcal{C}_\eta$. The function $s \mapsto (\mathcal{F}_\Phi H)(2s-1, s)$ can be holomorphically continued to the vertical strip $\epsilon < \sigma < 4$ except at the three simple poles: $s = (1-\alpha_i)/2 \quad (i = 1, 2, 3)$, where $(\alpha_1, \alpha_2, \alpha_3)$ are the Langlands parameters of the Maass cusp form Φ.

Proof. We will prove a stronger result in Proposition 10.2. However, a simpler argument suffices for the time being. Suppose $1/2 + \epsilon < \sigma < 4$ and $s_0 = 2s - 1$. In (9.13), we shift the line of integration from $\operatorname{Re} u = \epsilon$ to $\operatorname{Re} u = -1.9$:

$$(\mathcal{F}_\Phi H)(2s-1, s) = 2\sqrt{\pi} \prod_{i=1}^{3} \Gamma\Big(s - \frac{1}{2} + \frac{\alpha_i}{2}\Big) \int_{(\eta - \frac{1}{2})} \widetilde{h}(s - s_1 - 1/2) \frac{\pi^{-s_1} \prod_{i=1}^{3} \Gamma\left(\frac{s_1 - \alpha_i}{2}\right) \Gamma\left(\frac{s_1}{2} + 1 - 2s\right)}{\Gamma\left(\frac{1+s_1}{2} + 1 - 2s\right) \Gamma\left(s - \frac{1}{2} + \frac{s_1}{2}\right)} \frac{ds_1}{2\pi i}$$
$$+ \sqrt{\pi} \int_{(\eta - \frac{1}{2})} \int_{(-1.9)} (\text{Same as the integrand of (9.13)}) \frac{du}{2\pi i} \frac{ds_1}{2\pi i}.$$

By Stirling's formula and the same argument following (8.17), the integrals above represent holomorphic functions on $\epsilon < \sigma < 4$. □

9.5. Step 5: Conclusion.

Apply Proposition 9.2 to (9.12) and observe that the poles of $s \mapsto (\mathcal{F}_\Phi H)(2s-1, s)$ are exactly the trivial zeros of the the arithmetic factor $L(2s-1, \Phi)$ in (9.6). We conclude that the product of functions $s \mapsto L(2s-1, \Phi)(\mathcal{F}_\Phi H)(2s-1, s)$ is holomorphic on $\epsilon < \sigma < 4$ and thus (9.12) provides a holomorphic continuation of $\mathrm{OD}_\Phi(s)$ to the vertical strip $1/4 + \epsilon/2 < \sigma < 3/4$. By the rapid decay of Φ at ∞, the pairing $s \mapsto (P, \mathbb{P}_2^3 \Phi \cdot |\det *|^{\overline{s} - \frac{1}{2}})_{\Gamma_2 \backslash GL_2(\mathbb{R})}$ represents an entire function. Putting (5.25), (6.9) and (9.12) together, we arrive at Theorem 1.1.

Remark 9.3. Readers might have noticed that the analytic continuation procedure in our case (for a moment of automorphic L-functions of degree 6) is much more involved than the ones for degree 4 (cf. the second moment formula of GL(2) of Iwaniec-Sarnak/Motohashi). To some extent, this is hinted by the presence of off-diagonal main terms in our case when Φ is specialized to be an Eisenstein series, whereas this does not happen in the degree 4 cases. See [CFKRS05, pp. 35] for further discussions.

However, the subtle arithmetic differences of the off-diagonals are the deeper causes. More specifically, the arithmetic in Iwaniec-Sarnak/Motohashi is given by a shifted Dirichlet series of two divisor functions and the holomorphy of the dual side in the critical strip simply rests on the absolute convergence of such a Dirichlet series. However, the absolute convergence provided by Proposition 7.2 is very much insufficient in our case—we must move the contour judiciously so that the L-functions present in the off-diagonal take value on $\operatorname{Re} s_0 = 1/2$ (when $s = 1/2$).

10. Explication of the off-diagonal — Main Terms and Integral Transform

The power of spectral summation formulae (including Theorem 1.1) is encoded in the archimedean transformations. It is important to obtain very explicit expressions for the transformations, usually in terms of *special functions*. While the special functions for GL(2) exhibit numerous symmetries and identities, this is less true for higher-rank groups, leaving much to explore.

Nevertheless, there has been some success in higher-rank cases. For example, Stade [St01, St02] computed the Mellin transforms and certain Rankin–Selberg integrals of Whittaker functions for $\mathrm{GL}_n(\mathbb{R})$; Goldfeld et al. [GK13, GSW21, GSW23+] obtained (harmonic-weighted) spherical Weyl laws of $\mathrm{GL}_3(\mathbb{R})$, $\mathrm{GL}_4(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{R})$ with strong power-saving error terms; and Buttcane developed the Kuznetsov formulae for $\mathrm{GL}_3(\mathbb{R})$. These works heavily rely on *Mellin–Barnes integrals*, suggesting this approach effectively handles the archimedean aspects of higher-rank problems.

In this final section, we continue such investigation and record several formulae for the archimedean transform $(\mathcal{F}_\Phi H)(s_0, s)$.

Lemma 10.1. Suppose $H \in \mathcal{C}_\eta$ and $h := H^\flat$. On the vertical strip $-\frac{1}{2} < \operatorname{Re} w < \eta$, we have

$$\widetilde{h}(w) := \int_0^\infty h(y) y^w \, d^\times y = \frac{\pi^{-w-\frac{1}{2}}}{4} \int_{(0)} H(\mu) \frac{\Gamma\left(\frac{w+\frac{1}{2}+\mu}{2}\right)\Gamma\left(\frac{w+\frac{1}{2}-\mu}{2}\right)}{|\Gamma(\mu)|^2} \frac{d\mu}{2\pi i}, \tag{10.1}$$

Proof. Since $H \in \mathcal{C}_\eta$, both sides of (10.1) converge absolutely on the strip $-1/2 < \operatorname{Re} w < \eta$ by Stirling's formula and Proposition 5.11. Substituting the definition of h as in (5.10) into $\widetilde{h}(w)$, the result follows from equation (5.2).

□

10.1. **The Off-diagonal Main Term in Theorem 1.1.** In this subsection, we will show that the off-diagonal main term of Theorem 1.1 (i.e., $L(2s-1,\Phi)\left(\mathcal{F}_\Phi H\right)(2s-1,s)/2$) matches up with the prediction of [CFKRS05]. This follows immediately from proving a Mellin–Barnes integral identity, after which the matching follows from the functional equation (5.24).

The proof is more involved than that of Proposition 5.7, as the u-integral (see Section 7) adds intricacies. However, the introduction of new Γ-factors reveals symmetries in the u-integral, leading to several cancellations and reductions.

Theorem 10.2. Suppose $\frac{1}{2}+\epsilon < \sigma < 1$. Then

$$\left(\mathcal{F}_\Phi H\right)(2s-1,s) \;=\; \pi^{\frac{1}{2}-s}\cdot\prod_{i=1}^{3}\frac{\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}\right)}{\Gamma\left(1-s-\frac{\alpha_i}{2}\right)}\int_{(0)}\frac{H(\mu)}{|\Gamma(\mu)|^2}\prod_{i=1}^{3}\prod_{\pm}\Gamma\left(\frac{1-s+\alpha_i\pm\mu}{2}\right)\frac{d\mu}{2\pi i}. \tag{10.2}$$

Proof. Suppose $\frac{1}{2}+\epsilon < \sigma < 4$. When $s_0 = 2s-1$, observe that the factor $\Gamma(\frac{1-u}{2})$ in the denominator of (9.13) cancels with the factor $\Gamma(s-\frac{s_0+u}{2})$ in the numerator of (9.13). This gives

$$\begin{aligned}\left(\mathcal{F}_\Phi H\right)(2s-1,s) \;&=\; \sqrt{\pi}\int_{(\eta-1/2)}\widetilde{h}(s-s_1-1/2)\frac{\pi^{-s_1}\prod_{i=1}^{3}\Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma(\frac{1+s_1}{2}+1-2s)}\\ &\cdot\int_{(\epsilon)}\frac{\Gamma(\frac{u}{2})\Gamma(\frac{u+s_1}{2}+1-2s)\prod_{i=1}^{3}\Gamma(s-\frac{1}{2}+\frac{\alpha_i-u}{2})}{\Gamma(s-\frac{1}{2}+\frac{s_1-u}{2})}\frac{du}{2\pi i}\frac{ds_1}{2\pi i}.\end{aligned} \tag{10.3}$$

We make the change of variable $u \to -2u$ and take

$$(a,b,c;d,e) \;=\; (s-1/2+\alpha_1/2,\; s-1/2+\alpha_2/2,\; s-1/2+\alpha_3/2;\; 0,\; s_1/2+1-2s)$$

in (5.7). Notice that

$$(a+b+c)+d+e = 3\,(s-1/2)+s_1/2+1-2s = s-1/2+s_1/2 \;(:= f)$$

because of $\alpha_1+\alpha_2+\alpha_3 = 0$. We find the u-integral is equal to

$$2\cdot\prod_{i=1}^{3}\frac{\Gamma(s-\frac{1}{2}+\frac{\alpha_i}{2})\Gamma(\frac{1}{2}-s+\frac{s_1+\alpha_i}{2})}{\Gamma(\frac{s_1-\alpha_i}{2})}. \tag{10.4}$$

Notice that the three Γ-factors in denominator of the last expression cancel with the three in the numerator of the first line of (10.3). Hence, we have

$$\left(\mathcal{F}_\Phi H\right)(2s-1,s) = 2\sqrt{\pi}\cdot\prod_{i=1}^{3}\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}\right)\int_{(\eta-1/2)}\widetilde{h}(s-s_1-1/2)\frac{\pi^{-s_1}\prod_{i=1}^{3}\Gamma(\frac{1}{2}-s+\frac{s_1+\alpha_i}{2})}{\Gamma(\frac{1+s_1}{2}+1-2s)}\frac{ds_1}{2\pi i}. \tag{10.5}$$

We must now further restrict to $1/2+\epsilon < \sigma < 1$. We shift the line of integration to the left from $\operatorname{Re} s_1 = \eta-1/2$ to $\operatorname{Re} s_1 = \sigma_1$ satisfying $2\sigma-1 < \sigma_1 < \sigma$, It is easy to see no pole is crossed and we may now apply Lemma 10.1:

$$\begin{aligned}\left(\mathcal{F}_\Phi H\right)(2s-1,s) \;&=\; \frac{\pi^{\frac{1}{2}-s}}{2}\cdot\prod_{i=1}^{3}\Gamma\left(s-\frac{1}{2}+\frac{\alpha_i}{2}\right)\\ &\cdot\int_{(0)}\frac{H(\mu)}{|\Gamma(\mu)|^2}\int_{(\sigma_1)}\frac{\prod_{i=1}^{3}\Gamma(\frac{1}{2}-s+\frac{s_1+\alpha_i}{2})\cdot\Gamma(\frac{s-s_1+\mu}{2})\Gamma(\frac{s-s_1-\mu}{2})}{\Gamma\left(\frac{1+s_1}{2}+1-2s\right)}\frac{ds_1}{2\pi i}\frac{d\mu}{2\pi i}.\end{aligned} \tag{10.6}$$

For the s_1-integral, apply the change of variable $s_1 \to 2s_1$ and (5.7) the second time but with

$$(a,b,c;d,e) \;=\; \big(1/2-s+\alpha_1/2,\; 1/2-s+\alpha_2/2,\; 1/2-s+\alpha_3/2;\; (s+\mu)/2, (s-\mu)/2\big). \tag{10.7}$$

Oberserve that

$$(a+b+c)+(d+e) \;=\; 3\,(1/2-s)+s \;=\; 3/2-2s\ (:=f).$$

The s_1-integral is thus equal to

$$\prod_{i=1}^{3}\frac{\prod_{\pm}\Gamma(\frac{1-s+\alpha_i\pm\mu}{2})}{\Gamma(1-s-\frac{\alpha_i}{2})}$$

and the result follows.

□

10.2. **Integral Transform.** Based on the experience of Stade [St01, St02], we *do not* expect the Mellin-Barnes integrals of $(\mathcal{F}_\Phi H)(s_0,s)$ (see (10.12) below) to be completely reducible as in Proposition 10.2 if (s_0,s) is in a *general position*. However, reductions can occur if the integrals take certain special forms, most clearly seen when expressed as *hypergeometric functions*.

We define

$$\begin{aligned}{}_4\widehat{F}_3\left(\begin{matrix}A_1 & & A_2 & & A_3 & & A_4\\ & B_1 & & B_2 & & B_3 & \end{matrix}\,\middle|\, z\right) &:= \frac{\Gamma(A_1)\Gamma(A_2)\Gamma(A_3)\Gamma(A_4)}{\Gamma(B_1)\Gamma(B_2)\Gamma(B_3)}\cdot {}_4F_3\left(\begin{matrix}A_1 & & A_2 & & A_3 & & A_4\\ & B_1 & & B_2 & & B_3 & \end{matrix}\,\middle|\, z\right)\\ &:= \sum_{n=0}^{\infty}\frac{\Gamma(A_1+n)\Gamma(A_2+n)\Gamma(A_3+n)\Gamma(A_4+n)}{\Gamma(B_1+n)\Gamma(B_2+n)\Gamma(B_3+n)}\frac{z^n}{n!}.\end{aligned} \tag{10.8}$$

The series converges absolutely when $|z|<1$ and $A_1,A_2,A_3,A_4\notin\mathbb{Z}_{\leqslant 0}$; and on $|z|=1$ if

$$\operatorname{Re}\left(B_1+B_2+B_3-A_1-A_2-A_3-A_4\right) \;>\; 0.$$

In fact, our hypergeometric functions are of *Saalschütz* type, i.e., $B_1+B_2+B_3-A_1-A_2-A_3-A_4=1$. Only such special type of hypergeometric functions at $z=1$ possess many functional relations and integral representations, see [M12].

Proposition 10.3. Suppose $H\in\mathcal{C}_\eta$ and $h:=H^\flat$. On the region $\sigma_0\;>\;\epsilon$, $\sigma\;<\;4$, and $2\sigma-\sigma_0-\epsilon\;>\;0$, we have $\left(\mathcal{F}_\Phi H\right)(s_0,s)$ equal to $2\pi^{3/2}$ times

$$\begin{aligned}&\int_{(\eta-1/2)}\widetilde{h}(s-s_1-1/2)\,\frac{\prod_{i=1}^{3}\Gamma(\frac{s_1-\alpha_i}{2})}{\Gamma(\frac{1+s_1}{2}-s_0)}\,\pi^{-s_1}\sec\frac{\pi}{2}\,(2s+s_0-s_1)\\ &\qquad\cdot\,{}_4\widehat{F}_3\left(\begin{matrix}s-\frac{s_0}{2} & & \frac{s_0+\alpha_1}{2} & & \frac{s_0+\alpha_2}{2} & & \frac{s_0+\alpha_3}{2}\\ & 1/2 & & \frac{s_0+s_1}{2} & & s+\frac{1}{2}+\frac{s_0-s_1}{2} & \end{matrix}\,\middle|\,1\right)\frac{ds_1}{2\pi i}\\ &\quad-\int_{(\eta-1/2)}\widetilde{h}(s-s_1-1/2)\,\frac{\prod_{i=1}^{3}\Gamma(\frac{s_1-\alpha_i}{2})}{\Gamma(\frac{1+s_1}{2}-s_0)}\,\pi^{-s_1}\sec\frac{\pi}{2}\,(2s+s_0-s_1)\\ &\qquad\cdot\,{}_4\widehat{F}_3\left(\begin{matrix}\frac{1}{2}-s_0+\frac{s_1}{2} & & \frac{1}{2}-s+\frac{s_1+\alpha_1}{2} & & \frac{1}{2}-s+\frac{s_1+\alpha_2}{2} & & \frac{1}{2}-s+\frac{s_1+\alpha_3}{2}\\ & \frac{1}{2}-s+s_1 & & 1-s-\frac{s_0-s_1}{2} & & \frac{3}{2}-s-\frac{s_0-s_1}{2} & \end{matrix}\,\middle|\,1\right)\frac{ds_1}{2\pi i}.\end{aligned} \tag{10.9}$$

Proof. By Stirling's formula, we can shift the line of integration of the u-integral in (9.13) to $-\infty$. The residual series obtained can then be identified in terms of hypergeometric series as asserted in the present proposition. This can also be verified by *InverseMellinTransform[]* command in mathematica. More systematically, one rewrites the u-integral in the form of a Meijer's G-function. The conversion between Meijer's G-functions and generalized hypergeometric functions is known as *Slater's theorem*, see [PBM90, Chapter 8]. □

Recently, the articles [BBFR20, BFW21+] have brought in powerful asymptotic analysis of hypergeometric functions to study moments, yielding sharp spectral estimates. Our class of admissible test functions in Theorem 1.1 is broad enough for such prospects, see Remark 5.27.

Next, we establish the existence of a kernel function for the integral transform $\left(\mathcal{F}_\Phi H\right)(s_0,s)$ when integrating against a chosen test function $H(\mu)$ on the spectral side. This formula serves as a step toward a more practical result for $\left(\mathcal{F}_\Phi H\right)(s_0,s)$. While the proof requires care, it is relatively manageable for our case. However, this is not always true; for example, in the spectral Kuznetsov formulae for GL(2) and GL(3), kernel existence can be more challenging, as noted by [Bu16, Mo97].

Proposition 10.4. Suppose $H \in \mathcal{C}_\eta$. On the domain

$$\sigma_0 \; > \; \epsilon \; := \; 1/100, \quad \sigma \; < \; 4, \quad 2\sigma - \sigma_0 - \epsilon \; > \; 0, \quad \sigma_0 + 2\sigma - 1 - \epsilon \; > \; 0, \quad 1 + \epsilon - \sigma_0 - \sigma \; > \; 0, \tag{10.10}$$

we have

$$(\mathcal{F}_\Phi H)(s_0, s) \;=\; \frac{\pi^{\frac{1}{2}-s}}{4} \int_{(0)} \frac{H(\mu)}{|\Gamma(\mu)|^2} \cdot \mathcal{K}(s_0, s; \alpha, \mu) \, \frac{d\mu}{2\pi i}, \tag{10.11}$$

where the kernel function $\mathcal{K}(s_0, s; \alpha, \mu)$ is given explicitly by the double Barnes integrals

$$\mathcal{K}(s_0, s; \alpha, \mu) \;:=\; \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{\Gamma\left(\frac{s-s_1+\mu}{2}\right) \Gamma\left(\frac{s-s_1-\mu}{2}\right) \prod_{i=1}^3 \Gamma\left(\frac{s_1-\alpha_i}{2}\right)}{\Gamma\left(\frac{1+s_1}{2} - s_0\right)} \cdot \frac{\Gamma\left(\frac{u}{2}\right) \Gamma\left(\frac{s_1-s_0+u}{2} + \frac{1}{2} - s\right) \prod_{i=1}^3 \Gamma\left(\frac{s_0-u+\alpha_i}{2}\right) \Gamma\left(s - \frac{s_0+u}{2}\right)}{\Gamma\left(\frac{1-u}{2}\right) \Gamma\left(\frac{s_0-u+s_1}{2}\right)} \, \frac{du}{2\pi i} \frac{ds_1}{2\pi i}, \tag{10.12}$$

and the contours follow the Barnes convention.

Remark 10.5.

(1) The domain (10.10) is certainly non-empty as it includes our point of interest $(\sigma_0, \sigma) = (1/2, 1/2)$.

(2) The contours of (10.12) may be taken explicitly as the vertical lines $\operatorname{Re} u = \epsilon$ and $\operatorname{Re} s_1 = \sigma_1$ with

$$\sigma_0 + 2\sigma - 1 - \epsilon \; < \; \sigma_1 \; < \; \sigma. \tag{10.13}$$

Proof. Suppose

$$\sigma_0 \; > \; \epsilon, \qquad \sigma \; < \; 4, \qquad \text{and} \qquad 2\sigma - \sigma_0 - \epsilon \; > \; 0 \tag{10.14}$$

as in Proposition 8.1. Recall the expression (9.13) for $(\mathcal{F}_\Phi H)(s_0, s)$. This time, we shift the line of integration of the s_1-integral to $\operatorname{Re} s_1 = \sigma_1$ satisfying

$$\sigma_1 \; < \; \sigma \tag{10.15}$$

and no pole is crossed during this shift as long as

$$\sigma_1 \; > \; 0 \qquad \text{and} \qquad \sigma_1 \; > \; \sigma_0 + 2\sigma - 1 - \epsilon. \tag{10.16}$$

Now, assume (10.10). The restrictions (10.14), (10.15), (10.16) hold and such a line of integration for the s_1-integral exists. Upon shifting the line of integration to such a position, substituting (10.1) into (9.13) and the result follows. ☐

The second step is to apply a very useful rearrangement of the Γ-factors in the $(n-1)$-fold Mellin transform of the GL(n) spherical Whittaker function as discovered in Ishii-Stade [IS07]. We shall only need the case of $n = 3$ which we describe as follows. Recall

$$G_\alpha(s_1, s_2) \;:=\; \pi^{-s_1-s_2} \cdot \frac{\prod_{i=1}^3 \Gamma\left(\frac{s_1+\alpha_i}{2}\right) \Gamma\left(\frac{s_2-\alpha_i}{2}\right)}{\Gamma\left(\frac{s_1+s_2}{2}\right)} \tag{10.17}$$

from Proposition 5.3. The First Barnes Lemma, i.e.,

$$\int_{-i\infty}^{i\infty} \Gamma(w+\alpha) \Gamma(w+\mu) \Gamma(\gamma-w) \Gamma(\delta-w) \, \frac{dw}{2\pi i} \;=\; \frac{\Gamma(\alpha+\gamma) \Gamma(\alpha+\delta) \Gamma(\mu+\gamma) \Gamma(\gamma+\delta)}{\Gamma(\alpha+\mu+\gamma+\delta)}, \tag{10.18}$$

can be applied *in reverse* such that (10.17) can be rewritten as

$$G_\alpha(s_1, s_2) \;=\; \pi^{-s_1-s_2} \cdot \Gamma\left(\frac{s_1+\alpha_1}{2}\right) \Gamma\left(\frac{s_2-\alpha_1}{2}\right) \cdot \int_{-i\infty}^{i\infty} \Gamma\left(z + \frac{s_1}{2} - \frac{\alpha_1}{4}\right) \Gamma\left(z + \frac{s_2}{2} + \frac{\alpha_1}{4}\right) \Gamma\left(\frac{\alpha_2}{2} + \frac{\alpha_1}{4} - z\right) \Gamma\left(\frac{\alpha_3}{2} + \frac{\alpha_1}{4} - z\right) \frac{dz}{2\pi i}, \tag{10.19}$$

see [IS07, Section 2]. Although (10.19) is less symmetric than (10.17), it more clearly displays the recursive structure of the GL(3) Whittaker function in terms of the K-Bessel function.

Theorem 10.6. Suppose $\operatorname{Re} s_0 = \operatorname{Re} s = 1/2$ and $\operatorname{Re} \alpha_i = \operatorname{Re} \mu = 0$. Then $\mathcal{K}(s_0, s; \alpha, \mu)$ is equal to

$$4 \cdot \gamma\left(-\frac{s_0+\alpha_1}{2}\right) \prod_{\pm} \Gamma\left(\frac{s \pm \mu - \alpha_1}{2}\right) \cdot \int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \Gamma(s+t)\Gamma\left(\frac{1-\alpha_1}{2}+t\right)\Gamma\left(\frac{\alpha_2}{2}+\frac{\alpha_1}{4}-z\right)\Gamma\left(\frac{\alpha_3}{2}+\frac{\alpha_1}{4}-z\right)\prod_{\pm}\Gamma\left(\frac{-s\pm\mu}{2}+\frac{\alpha_1}{4}+z-t\right) \cdot \frac{\gamma\left(t+\frac{s_0}{2}\right)\gamma\left(\frac{\alpha_1}{4}-z-\frac{s_0}{2}\right)}{\gamma\left(\frac{\alpha_1}{4}+t-z\right)} \frac{dz}{2\pi i}\frac{dt}{2\pi i}, \tag{10.20}$$

where the contours may be explicitly taken as the vertical lines $\operatorname{Re} t = a$ and $\operatorname{Re} z = b$ satisfying

$$-1/2 < a < -1/4, \quad -1/4 < b < 0, \quad \text{and} \quad b - a > 1/4 \tag{10.21}$$

and

$$\gamma(x) := \frac{\Gamma(-x)}{\Gamma\left(\frac{1}{2}+x\right)}. \tag{10.22}$$

Remark 10.7.

(1) The assumptions in Theorem 10.6 cover the most interesting cases of Theorem 1.1, particularly on the critical line and for tempered forms, though they are not strictly necessary. These were chosen for a clean description of the contours (10.21).

(2) Furthermore, if either of the following holds:
  (a) the cusp form Φ is fixed, allowing implicit constants to depend on $\alpha(\Phi)$;
  (b) or $\Phi = E_{\min}^{(3)}(*;\alpha)$, where the 'shifts' α_i's are small as in [CFKRS05] (i.e., $\ll 1/\log R$, per Remark 5.27),

then by continuity, it suffices to assume $\alpha_1 = \alpha_2 = \alpha_3 = 0$. With $s = 1/2$, this leads to a simpler formula for (10.20):

$$4 \cdot \gamma\left(-\frac{s_0}{2}\right)\prod_{\pm}\Gamma\left(\frac{1/2\pm\mu}{2}\right)\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\Gamma\left(\frac{1}{2}+t\right)^2\Gamma(-z)^2\prod_{\pm}\Gamma\left(\frac{-1/2\pm\mu}{2}+z-t\right) \cdot \frac{\gamma\left(t+\frac{s_0}{2}\right)\gamma\left(-z-\frac{s_0}{2}\right)}{\gamma(t-z)}\frac{dz}{2\pi i}\frac{dt}{2\pi i}.$$

(3) For analytic applications involving Whittaker functions for GL(n), the formula from [IS07] has proven more effective than the ones obtained previously. For example:
  (a) Buttcane [Bu20] used the formula (10.19) to significantly simplify the archimedean Rankin-Selberg calculation for GL(3), earlier done by Stade [St93].
  (b) In [GSW23+], it was crucial for deriving strong bounds for Whittaker functions and their inverse transforms, and the Weyl law.

(This was pointed out to the author by Prof. Eric Stade and Prof. Dorian Goldfeld. The author would like to thank their comments here.)

(4) Finally, Stirling's formula shows that the integrand in the Mellin-Barnes representation (10.20) decays exponentially as $|\operatorname{Im} z|, |\operatorname{Im} t| \to \infty$, independent of $|\operatorname{Im} s_0|$. This advantage is not shared by the integrand in (8.1).

Proof of Theorem 10.6. Substitute (10.19) into (10.12) rearrange the integrals, we find that

$$\begin{aligned}\mathcal{K}(s_0, s; \alpha, \mu) &:= \int_{-i\infty}^{i\infty} \frac{\Gamma\left(\frac{s-s_1+\mu}{2}\right)\Gamma\left(\frac{s-s_1-\mu}{2}\right)\Gamma\left(\frac{s_1-\alpha_1}{2}\right)}{\Gamma\left(\frac{1+s_1}{2}-s_0\right)} \\ &\cdot \int_{-i\infty}^{i\infty} \Gamma\left(\frac{\alpha_2}{2}+\frac{\alpha_1}{4}-z\right)\Gamma\left(\frac{\alpha_3}{2}+\frac{\alpha_1}{4}-z\right)\Gamma\left(z+\frac{s_1}{2}+\frac{\alpha_1}{4}\right) \\ &\cdot \int_{-i\infty}^{i\infty} \frac{\Gamma(\frac{u}{2})\Gamma(\frac{s_1-s_0+u}{2}+\frac{1}{2}-s)\Gamma(s-\frac{s_0+u}{2})\Gamma(\frac{s_0-u+\alpha_1}{2})\Gamma(z+\frac{s_0-u}{2}-\frac{\alpha_1}{4})}{\Gamma(\frac{1-u}{2})} \\ &\qquad \frac{du}{2\pi i}\frac{dz}{2\pi i}\frac{ds_1}{2\pi i}. \end{aligned} \tag{10.23}$$

The innermost u-integral, originally of ${}_4F_3(1)$-type (Saalschütz), reduces to a ${}_3F_2(1)$-type (non-Saalschütz), allowing further transformations. We apply the following Barnes integral identity for ${}_3F_2(1)$-type (see Bailey [Ba64]):

$$\begin{aligned}&\int_{-i\infty}^{i\infty} \frac{\Gamma(a+u)\Gamma(b+u)\Gamma(c+u)\Gamma(f-u)\Gamma(-u)}{\Gamma(e+u)}\frac{du}{2\pi i} \\ &= \frac{\Gamma(b)\Gamma(c)\Gamma(f+a)}{\Gamma(f+a+b+c-e)\Gamma(e-b)\Gamma(e-c)} \\ &\cdot \int_{-i\infty}^{i\infty} \frac{\Gamma(a+t)\Gamma(e-c+t)\Gamma(e-b+t)\Gamma(f+b+c-e-t)\Gamma(-t)}{\Gamma(e+t)}\frac{dt}{2\pi i}. \end{aligned} \tag{10.24}$$

Make a change of variable $u \to -2u$ and take

$$\begin{aligned} a &= s-\frac{s_0}{2}, \quad b = \frac{s_0+\alpha_1}{2}, \quad c = z+\frac{s_0}{2}-\frac{\alpha_1}{4}, \\ f &= \frac{s_1-s_0}{2}+\frac{1}{2}-s, \quad e = 1/2 \end{aligned} \tag{10.25}$$

in (10.24), the u-integral of (10.23) can be written as

$$\begin{aligned}&2\cdot\frac{\Gamma\left(\frac{s_0+\alpha_1}{2}\right)\Gamma\left(z+\frac{s_0}{2}-\frac{\alpha_1}{4}\right)\Gamma\left(\frac{1+s_1}{2}-s_0\right)}{\Gamma\left(\frac{s_1}{2}+z+\frac{\alpha_1}{4}\right)\Gamma\left(\frac{1-s_0-\alpha_1}{2}\right)\Gamma\left(\frac{1}{2}-z-\frac{s_0}{2}+\frac{\alpha_1}{4}\right)} \\ &\cdot\int_{-i\infty}^{i\infty} \frac{\Gamma\left(t+s-\frac{s_0}{2}\right)\Gamma\left(t+\frac{1}{2}-z-\frac{s_0}{2}+\frac{\alpha_1}{4}\right)\Gamma\left(t+\frac{1}{2}-\frac{s_0+\alpha_1}{2}\right)\Gamma\left(\frac{s_0+s_1}{2}+z-s+\frac{\alpha_1}{4}-t\right)\Gamma(-t)}{\Gamma\left(\frac{1}{2}+t\right)}\frac{dt}{2\pi i}. \end{aligned} \tag{10.26}$$

Putting this back into (10.23). Observe that two pairs of Γ-factors involving s_1 will be cancelled and we can then execute the s_1-integral. More precisely,

$$\begin{aligned}\frac{1}{2}\cdot\mathcal{K}(s_0, s; \alpha, \mu) &= \frac{\Gamma\left(\frac{s_0+\alpha_1}{2}\right)}{\Gamma\left(\frac{1-s_0-\alpha_1}{2}\right)} \cdot \int_{-i\infty}^{i\infty} \frac{dt}{2\pi i}\frac{\Gamma\left(t+s-\frac{s_0}{2}\right)\Gamma\left(t+\frac{1}{2}-\frac{s_0+\alpha_1}{2}\right)\Gamma(-t)}{\Gamma\left(\frac{1}{2}+t\right)} \\ &\cdot\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\frac{\Gamma\left(\frac{\alpha_2}{2}+\frac{\alpha_1}{4}-z\right)\Gamma\left(\frac{\alpha_3}{2}+\frac{\alpha_1}{4}-z\right)\Gamma\left(z+\frac{s_0}{2}-\frac{\alpha_1}{4}\right)}{\Gamma\left(\frac{1}{2}-z-\frac{s_0}{2}+\frac{\alpha_1}{4}\right)}\Gamma\left(t+\frac{1}{2}-z-\frac{s_0}{2}+\frac{\alpha_1}{4}\right) \\ &\cdot\int_{-i\infty}^{i\infty}\frac{ds_1}{2\pi i}\Gamma\left(\frac{s_0+s_1}{2}+z-s+\frac{\alpha_1}{4}-t\right)\Gamma\left(\frac{s-s_1+\mu}{2}\right)\Gamma\left(\frac{s-s_1-\mu}{2}\right)\Gamma\left(\frac{s_1-\alpha_1}{2}\right). \end{aligned} \tag{10.27}$$

Applying (10.18) once again, we obtain

$$\begin{aligned}\frac{1}{4}\cdot\mathcal{K}(s_0,s;\alpha,\mu) \ = \ & \frac{\Gamma\left(\frac{s_0+\alpha_1}{2}\right)}{\Gamma\left(\frac{1-s_0-\alpha_1}{2}\right)}\,\Gamma\left(\frac{s+\mu-\alpha_1}{2}\right)\Gamma\left(\frac{s-\mu-\alpha_1}{2}\right) \\ & \cdot\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\,\Gamma\left(s+t-\frac{s_0}{2}\right)\Gamma\left(\frac{1-\alpha_1}{2}+t-\frac{s_0}{2}\right)\frac{\Gamma(-t)}{\Gamma\left(\frac{1}{2}+t\right)} \\ & \quad\cdot\int_{-i\infty}^{i\infty}\frac{dz}{2\pi i}\,\Gamma\left(\frac{\alpha_2}{2}+\frac{\alpha_1}{4}-z\right)\Gamma\left(\frac{\alpha_3}{2}+\frac{\alpha_1}{4}-z\right)\frac{\Gamma\left(z+\frac{s_0}{2}-\frac{\alpha_1}{4}\right)}{\Gamma\left(\frac{1}{2}-z-\frac{s_0}{2}+\frac{\alpha_1}{4}\right)} \\ & \qquad\cdot\Gamma\left(\frac{-s+\mu}{2}+\frac{\alpha_1}{4}+\frac{s_0}{2}+z-t\right)\Gamma\left(\frac{-s-\mu}{2}+\frac{\alpha_1}{4}+\frac{s_0}{2}+z-t\right) \\ & \qquad\quad\cdot\frac{\Gamma\left(\frac{1}{2}+\frac{\alpha_1}{4}-\frac{s_0}{2}-z+t\right)}{\Gamma\left(-\frac{\alpha_1}{4}+\frac{s_0}{2}+z-t\right)}. \end{aligned} \tag{10.28}$$

A final cleaning can be performed via the change of variables $t\to t+\frac{s_0}{2}$. This leads to (10.20) and completes the proof.

□

11. Notes

Remark 11.1 (Note added in Dec. 2021)**.** The first version of our preprint appeared on Arxiv in December 2021. Peter Humphries has kindly informed the author that the moment of Theorem 1.1 arises naturally from the context of the L^4-norm problem of GL(2) Maass forms and can also be investigated under another set of 'Kuznetsov-Voronoi' method (see [BK19a, BK19b, BLM19]) that is distinct from [Li09, Li11]. This is his on-going work with Rizwanur Khan.

Remark 11.2 (Note added in Oct. 2022/Apr. 2023)**.** The preprint of Humphries-Khan has now appeared, see [HK22+]. The spectral moments considered in [HK22+] and the present paper are distinct in a number of ways. In one case, our spectral moments coincide when both $\Phi=\widetilde{\Phi}$ and $s=1/2$ hold true, but otherwise extra twistings by root numbers are present in the one considered by [HK22+]. This would then lead to different conclusions in view of the Moment Conjecture of [CFKRS05] (see the discussions in Section 3.2). In the other case, our spectral moments differ by a full holomorphic spectrum and thus give rise to distinct conclusions in applications toward non-vanishing (say). All these result in different ways of making choices of test functions, as well as different shapes of the dual sides. The self-duality assumption was used in [HK22+] to annihilate two of the terms in their proof, but no such treatment is necessary for our method.

There is also the recent preprint of Biró [Bi22+] which studies another instance of reciprocity closely related to ours, but with the decomposition '$4=2\times 2$' on the dual side instead. His integral construction consists of a product of an automorphic kernel with a copy of θ-function and Maass cusp form of $SL_2(\mathbb{Z})$ attached to each variable. The integration is taken over both variables and over the quotient $\Gamma_0(4)\backslash\mathfrak{h}^2$. See equation (3.15) therein.

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