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ORIGINAL PAPER

A cancellation theorem for metacyclic group rings

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Abstract

A ring Λ has stably free cancellation when every stably free Λ-module is free. Let $G = C_p \rtimes C_q$ be a finite metacyclic group where p is an odd prime and q is a positive integral divisor of $p-1$. We show that the group ring $\mathcal{R}[G]$ has stably free cancellation when $\mathcal{R} = \mathbb{Z}[t_1, t_1^{-1}, \dots t_m, t_m^{-1}, x_1, \dots x_n]$ is a ring of mixed polynomials and Laurent polynomials over the integers. As a consequence, when $C_\infty^{(m)}$ is the free abelian group of rank m then the integral group ring $\mathbb{Z}[G(p,q) \times C_\infty^{(m)}]$ has stably free cancellation.

Keywords Stably free module · Locally free module · Milnor square

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A module S over a ring Λ is *stably free* when $S \oplus \Lambda^m \cong \Lambda^n$ for integers $m \le n$. By saying a stably free module is *nontrivial* we mean it is not free. The ring Λ has *stably free cancellation* when any stably free module is free; that is:

SFC: $S \oplus \Lambda^m \cong \Lambda^n \implies S \cong \Lambda^{n-m}$.

In this paper we study the question of stably free cancellation for group algebras $\mathcal{R}[G]$ where $\mathcal{R}$ is a ring of generalized integral polynomials

$$\mathcal{R} = \mathbb{Z}[t_1, t_1^{-1}, \dots t_m, t_m^{-1}, x_1, \dots x_n] \qquad (*)$$

and $G = G(p,q)$ is a finite metacyclic group of the form

$$G(p,q) = C_p \rtimes C_q \qquad (**)$$

where p is an odd prime, q is a positive integral divisor of $p-1$ and C_q acts via the canonical imbedding $C_q \hookrightarrow \mathrm{Aut}(C_p)$. It is known that the integral group ring

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$\mathbb{Z}[G(p,q)]$ has stably free cancellation; this is proved explicitly in (8.4) below. We extend this to group rings with more general coefficients by proving:

Theorem I *$\mathcal{R}[G(p,q)]$ has stably free cancellation when $\mathcal{R}$ is a ring of generalized integral polynomials.*

When $n = 0$ we may identify $\mathcal{R}[G(p,q)]$ with the group ring $\mathbb{Z}[G(p,q) \times C_\infty^{(m)}]$ where $C_\infty^{(m)} = \underbrace{C_\infty \times \cdots \times C_\infty}_{m}$ is the free abelian group of rank m. Hence we have:

Corollary II *$\mathbb{Z}[G(p,q) \times C_\infty^{(m)}]$ has stably free cancellation for all $m \geq 1$.*

In Evans (2017) claimed a proof of Corollary II for all $m \geq 1$ when q is prime. Unfortunately Evans' original argument was fallacious. In a retraction and partial correction (Evans 2018) the cancellation property, for q prime, was established for stably free modules of rank ≥ 3. However the case of stably free modules of rank 2 remained problematic. In this paper we give, *ab initio*, a complete proof, when q is an arbitrary integral divisor of $p-1$, of the more general Theorem I.

The problem of stably free cancellation arises naturally within the context of non-simply connected topology. If G is a finitely presented group the existence of stably free modules which are not free greatly complicates the homotopy theory of spaces with fundamental group G. This becomes apparent when, for example, one attempts to solve the $D(2)$ problem of C.T.C. Wall (Johnson 2003a, b, 2021; Wall 1965) for the fundamental group G. When G is finite the question of whether the integral group ring $\mathbb{Z}[G]$ admits nontrivial stably free modules is determined almost entirely by the structure of the real group ring $\mathbb{R}[G]$. With the convention that $M_n(R) = 0$ when $n = 0$, the Wedderburn-Maschke structure theorem shows that $\mathbb{R}[G]$ decomposes as a product of matrix algebras

$$\mathbb{R}[G] \cong \prod_{i=1}^{a} M_{d_i}(\mathbb{R}) \times \prod_{j=1}^{b} M_{e_j}(\mathbb{C}) \times \prod_{k=1}^{c} M_{f_k}(\mathbb{H})$$

where $\mathbb{H}$ is the division ring of Hamiltonian quaternions. By adapting the cancellation theorem of Jacobinski (1968), Swan showed in Swan (1983) that $\mathbb{Z}[G]$ has stably free cancellation provided that, in the quaternionic matrix factors, no f_k takes the value 1. This is the celebrated 'Eichler condition' (Eichler 1938; Swan 1970). The extent to which the converse holds is studied in the paper of Nicholson (2021).

For infinite groups the situation is much less well understood. In Bass (1964) Bass showed that $\mathbb{Z}[F]$ has stably free cancellation when F is a nonabelian free group. As an addendum to the Quillen-Suslin solution (Lam 2006) of the Serre Conjecture, Swan showed in Swan (1978) that $\mathbb{Z}[C_\infty^{(m)}]$ has stably free cancellation where $C_\infty^{(m)}$ is the free abelian group of rank m. However, these examples apart, it would seem that the existence of nontrivial stably free modules is a relatively common occurence (Artamonov 1981; Berridge and Dunwoody 1979). The simplest examples known to the author are those of O'Shea (O' Shea 2012), see also (Johnson 2011, p. 180) who showed that if F is a nonabelian free group and Φ is a group of order p^n, where p is prime and $n \geq 2$, then $\mathbb{Z}[F \times \Phi]$ has infinitely many isomorphically distinct stably free modules.

By contrast we show that the integral group rings $\mathbb{Z}[C^{(m)} \times G(p,q)]$ do have stably free cancellation when $G(p,q)$ is the semidirect product $C_p \rtimes C_q$ where p is an odd prime, q is a positive integal divisor of $p-1$ and where C_q acts on C_p via the imbedding $C_q \hookrightarrow C_{p-1} \cong \mathrm{Aut}(C_p)$. This is equivalent to studying the stably free cancellation problem for the group rings $\mathcal{R}[G(p,q)]$ where $\mathcal{R} = \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}]$ is the ring of integral Laurent polynomials in m variables $t_1, \ldots, t_m$.

The fact that $\mathbb{Z}[G(p,q)]$ has stably free cancellation is a direct consequence of the theorem of Swan-Jacobinski (Swan 1970). However this powerful theorem does not extend to more general coefficient rings. As both $\mathcal{R}[G(p,q)]$ and $\mathbb{Z}[G(p,q)]$ can be described by means of Milnor squares (Milnor 1971), we proceed instead by comparing their defining squares using the induction techniques introduced by Quillen and Suslin (2006) and enhanced by Swan (1978). An essential feature is the remarkable theorem of Suslin (1977) which allows us to lift sufficiently many invertible elements through the ring homomorphisms of the Milnor squares.

We point out that such comparisons fail in general. If $Q(8)$ is the quaternion group of order 8 then, as an exception to the Swan-Jacobinski criterion, $\mathbb{Z}[Q(8)]$ has stably free cancellation. However $\mathcal{R}[Q(8)]$ has infinitely many isomorphically distinct stably free modules (Johnson 2011, Chapter 12, p. 208).

Finally we point out that when q is even the injectivity of the operator homomorphism $C_q \to \mathrm{Aut}(C_p)$ would seem to be an essential requirement. The group

$$\Gamma_{4p} = \langle x, y \mid x^p = 1, y^4 = 1, yxy^{-1} = x^{p-1} \rangle$$

which is known both as the binary dihedral group and the quaternion group of order $4p$ is a metacyclic group of the form $C_p \rtimes C_4$ in which the operator homomorphism is not injective. When $p \geq 5$ is prime $\mathbb{Z}[\Gamma_{4p}]$ fails to have stably free cancellation (Swan 1983). In this case again, $\mathcal{R}[\Gamma_{4p}]$ has infinitely many nontrivial stably free modules (Kamali 2010).

We wish to thank the referee for some suggestions which have helped to clarify the exposition.

1 Locally free modules

All modules in this paper should be understood to be right modules. By a *fibre square* we shall mean a commutative diagram of ring homomorphisms

$$\left\{ \begin{array}{ccc} \Lambda & \xrightarrow{\pi_-} & \Lambda_- \\ \downarrow^{\pi_+} & & \downarrow \varphi_- \\ \Lambda_+ & \xrightarrow{\varphi_+} & \Lambda_0 . \end{array} \right. \qquad (\mathfrak{F})$$

in which $\Lambda \cong \varprojlim(\varphi_+, \varphi_-)$. Such a fibre square satisfies Milnor's patching condition when at least one of φ_+, φ_- is surjective. Under this condition Milnor, in Milnor (1971), classified the projective modules over Λ in terms of projective modules over Λ_+, Λ_- as follows: let P_+, P_- be projective modules over Λ_+, Λ_- respectively and suppose

$$\alpha : P_+ \otimes_{\Lambda_+} \Lambda_0 \xrightarrow{\simeq} P_- \otimes_{\Lambda_-} \Lambda_0$$

is an isomorphism over Λ_0. Then there is a well defined projective Λ-module denoted by $P = (P_+, P_-; \alpha)$, with the property that $P \otimes_\Lambda \Lambda_\sigma \cong P_\sigma$ for $\sigma \in \{+, -\}$; moreover, up to isomorphism, every projective module over Λ is obtained in this way. As a special case we may take $P_\sigma = \Lambda_\sigma^{(k)}$ $P_\sigma \otimes_{\Lambda_\sigma} \Lambda_0 = \Lambda_0^{(k)}$ so that $P_\sigma \otimes_{\Lambda_\sigma} \Lambda_0 = \Lambda_0^{(k)}$ and $\alpha \in GL_k(\Lambda_0)$. In this case we write

$$\mathcal{L}(\alpha) = (\Lambda_+^{(k)}, \Lambda_-^{(k)}; \alpha).$$

$\mathcal{L}(\alpha)$ is then said to be *locally free of rank* k with respect to $\mathfrak{F}$ or simply $\mathfrak{F}$-*locally free of rank* k. At the referee's suggestion we stress that *local freeness* in the sense used here should not be confused with the notion of *local freeness at a prime* p which occurs frequently elsewhere in the literature; for example, in (Swan (1983)). We define

$$\overline{GL_k(\mathfrak{F})} = GL_k(\Lambda_+)\backslash GL_k(\Lambda_0)/GL_k(\Lambda_-).$$

When $\mathfrak{F}$ is a Milnor square, Milnor's classification theorem (Milnor 1971, pp. 20–24; see also Lemma A4 of (Swan (1983)), Appendix A) gives a bijection:

$$\{\mathfrak{F} - \text{locally free modules of rank } k\} \overset{\simeq}{\longleftrightarrow} \overline{GL_k(\mathfrak{F})}. \tag{1.1}$$

If S is a stably free module of rank k over Λ then $S_\sigma = S \otimes_\Lambda \Lambda_\sigma$ is stably free over Λ_σ. Hence if Λ_σ has property SFC then $S_\sigma \cong \Lambda_\sigma^{(k)}$. Thus we have:

Let S be a stably free module of rank k over Λ; if Λ_+ and Λ_- both have property SFC then S is locally free with respect to $\mathfrak{F}$. (1.2)

2 The rings Ω_0 and Ω

Throughout this paper m and n will denote fixed non-negative integers such that $m + n > 0$. For any ring R, we adopt the notation that $R[\mathbf{t}, \mathbf{t}^{-1}]$ denotes the ring of Laurent polynomials;

$$R[\mathbf{t}, \mathbf{t}^{-1}] = R[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}] \tag{2.1}$$

$R[\mathbf{x}]$ will denote the ring of ordinary polynomials;

$$R[\mathbf{x}] = R[x_1, \ldots, x_n] \tag{2.2}$$

and $R[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$ will denote the ring of mixed polynomials

$$R[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}] = R[\mathbf{t}, \mathbf{t}^{-1}] \otimes_R R[\mathbf{x}] = R[t_1, t_1^{-1}, \ldots t_m, t_m^{-1}, x_1, \ldots x_n]. \tag{2.3}$$

There is an augmentation homomorphism $\epsilon : R[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}] \to R$ determined by

$$\epsilon(t_i) = 1 \quad \text{for} \quad 1 \le i \le m \, ; \quad \epsilon(x_j) = 0 \quad \text{for} \quad 1 \le j \le n$$

ϵ is left inverse to the inclusion $i : R \hookrightarrow R[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$. In addition we fix the following:

p	:	an odd prime ;
$\mathbb{F}_p$	:	the field with p elements;
ζ_p	=	$\exp(\frac{2\pi i}{p})$;
q	:	a positive integer which divides $p-1$;
d	=	$(p-1)/q$.

Then C_q imbeds as a subgroup of the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and we denote by

$$A = \mathbb{Z}(\zeta_p)^{C_q}$$

the fixed ring under the Galois action of C_q. Then

$$A \otimes_{\mathbb{Z}} \mathbb{R} \cong \begin{cases} \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{d} & \text{if } q \text{ is even;} \\ \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{d/2} & \text{if } q \text{ is odd.} \end{cases} \tag{2.4}$$

It is known (cf (Birch 1967) p. 87; (Hasse (1962)), p. 220) that p ramifies completely in A. We denote by $\mathfrak{p}$ the unique prime in A over p so that

$$p = u\,\mathfrak{p}^d \text{ for some unit } u \in A^* \tag{2.5}$$

Let $\natural : \mathbb{Z} \to \mathbb{F}_p$ and $\nu : A \to A/\mathfrak{p} = \mathbb{F}_p$ be the canonical homomorphisms and denote by Ω_0 the pullback ring in the following fibre square

$$\begin{array}{ccc} \Omega_0 & \xrightarrow{\pi_-} & A \\ \downarrow^{\pi_+} & & \downarrow \nu \\ \mathbb{Z} & \xrightarrow{\natural} & \mathbb{F}_p\,. \end{array} \tag{$\mathfrak{L}_0$}$$

Noting that $\Omega_0 \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R} \times A \otimes_{\mathbb{Z}} \mathbb{R}$ then:

$$\Omega_0 \otimes_{\mathbb{Z}} \mathbb{R} \cong \begin{cases} \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{d+1} & \text{if } q \text{ is even;} \\ \mathbb{R} \times \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{d/2} & \text{if } q \text{ is odd.} \end{cases} \tag{2.6}$$

Thus Ω_0 satisfies the Eichler condition so by the Swan-Jacobinski theorem (Swan 1970)

$$\Omega_0 \text{ has property } SFC. \tag{2.7}$$

If R is a commutative ring we denote its Krull dimension (Eisenbud 2004, p. 227) by $\mathrm{Kdim}(R)$.

Proposition 2.8 $\mathrm{Kdim}(\Omega_0) = 1$.

Proof For a direct product we have $\mathrm{Kdim}(R_1 \times R_2) = \max\{\mathrm{Kdim}(R_1), \mathrm{Kdim}(R_2)\}$. As A is a Dedekind domain then $\mathrm{Kdim}(A) = 1$. Consequently $\mathrm{Kdim}(\mathbb{Z} \times A) = 1$. As Ω_0 is a subring of $\mathbb{Z} \times A$ then $\mathrm{Kdim}(\Omega_0) \leq 1$. However, Ω_0 has a subring isomorphic to $\mathbb{Z}$ so that $1 \leq \mathrm{Kdim}(\Omega_0)$, whence the conclusion. □

We define $\Omega = \Omega_0[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$. Tensoring $\mathfrak{L}_0$ with $\mathbb{Z}[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$ gives the fibre square

$$\begin{cases} \begin{array}{ccc} \Omega & \xrightarrow{\pi_-} & A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}] \\ \downarrow^{\pi_+} & & \downarrow \nu \\ \mathbb{Z}[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}] & \xrightarrow{\natural} & \mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]\,. \end{array} \end{cases} \tag{$\mathfrak{L}$}$$

As $\natural$ and ν are both surjective then $\mathfrak{L}$ is a Milnor square.

3 Almost surjectivity for $k \geq 3$

In general, for any commutative ring $\mathbb{A}$, $GL_k(\mathbb{A})$ is a semidirect product

$$GL_k(\mathbb{A}) = SL_k(\mathbb{A}) \rtimes \mathbb{A}^* \tag{3.1}$$

where $\mathbb{A}^*$ is imbedded in $GL_k(\mathbb{A})$ via the diagonal matrices

$$u \mapsto \begin{pmatrix} u & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

and $SL_k(\mathbb{A}) = \{X \in GL_k(\mathbb{A}) \mid \det(X) = 1\}$. A ring homomorphism $\psi : \mathbb{B} \to \mathbb{A}$ induces homomorphisms $\psi_* : GL_k(\mathbb{B}) \to GL_k(\mathbb{A})$ for each $k \geq 2$. We shall say that ψ is *almost surjective for* k when $SL_k(\mathbb{A}) \subset \mathrm{Im}(\psi_* : GL_k(\mathbb{B}) \to GL_k(\mathbb{A}))$.

Let $\epsilon(i, j) \in M_k(\mathbb{A})$ denote the basic matrix $\epsilon(i, j)_{r,s} = \delta_{i,r}\delta_{j,s}$. We denote by $E_k(\mathbb{A})$ (cf. (Suslin 1977)) the subgroup of $GL_k(\mathbb{A})$ generated by the elementary transvections $E(i, j; \lambda) = I_k + \lambda\epsilon(i, j)$ where $i \neq j$ and $\lambda \in \mathbb{A}$. Evidently we have

$$E_k(\mathbb{A}) \subset SL_k(\mathbb{A}). \tag{3.2}$$

A theorem Suslin (1977) shows that:

$$\text{For any field } \mathbb{F},\ E_k(\mathbb{F}[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) = SL_k(\mathbb{F}[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) \text{ when } k \geq 3. \tag{3.3}$$

If $\psi : \mathbb{B} \to \mathbb{A}$ is a surjective ring homomorphism then the induced homomorphism $\psi : E_k(\mathbb{B}) \to E_k(\mathbb{A})$ is surjective for all $k \geq 2$. As $\nu : A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}] \to \mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$ is surjective and $\mathbb{F}_p$ is a field then, by (3.3):

$$\nu : E_k(A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) \to SL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) \text{ is surjective for } k \geq 3. \tag{3.4}$$

It now follows from (3.1) that:

$$\nu : A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}] \to \mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}] \text{is almost surjective for each } k \geq 3. \tag{3.5}$$

Let $\alpha \in SL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}])$ and consider the $\mathfrak{L}$-locally free module $\mathcal{L}(\alpha)$ of rank k obtained by glueing by α:

$$\begin{array}{ccc} \mathcal{L}(\alpha) & \xrightarrow{\pi_-} & A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(k)} \\ \downarrow^{\pi_+} & & \downarrow \nu \\ \mathbb{Z}[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(k)} & \xrightarrow{\natural} & \mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(k)}. \end{array} \tag{3.6}$$

By **(3.5)** $\alpha \in \mathrm{Im}(\nu_* : GL_k(A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) \to GL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]))$ when $k \geq 3$. It now follows from Milnor's classification (Milnor 1971) that:

$$\text{If } \alpha \in SL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) \text{ then } \mathcal{L}(\alpha) \cong \Omega^{(k)} \text{ for } k \geq 3. \tag{3.7}$$

4 Almost surjectivity for $k = 2$

Now consider the case $k = 2$; if $\alpha \in SL_2(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}])$ and $\mathrm{Id} \in GL_1(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}])$ then $\alpha \oplus \mathrm{Id} \in SL_3(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}])$ and $\mathcal{L}(\alpha) \oplus \Omega \cong \mathcal{L}(\alpha \oplus \mathrm{Id})$; hence by (3.7):

$$\mathcal{L}(\alpha) \oplus \Omega \cong \Omega^{(3)} \quad \text{if } \alpha \in SL_2(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) \tag{4.1}$$

We improve on (4.1) as follows:

Theorem 4.2 *If* $\alpha \in SL_2(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}])$ *then* $\mathcal{L}(\alpha) \cong \Omega^{(2)}$.

Proof $\mathcal{L}(\alpha)$ is a projective module of rank 2 over $\Omega = \Omega_0[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$. By (2.8) $\mathrm{rk}(\mathcal{L}(\alpha)) > \mathrm{Kdim}(\Omega_0)$. Moreover, by (4.1), $[\mathcal{L}(\alpha)] = 0 \in \widetilde{K}_0(\Omega)$. It now follows from a theorem of Swan ((Swan 1978), Theorem 1.1) that $\mathcal{L}(\alpha)$ is induced from Ω_0; that is, there exists a projective module Q over Ω_0 such that $\mathcal{L}(\alpha) \cong i_*(Q)$ where $i : \Omega_0 \hookrightarrow \Omega$ is the canonical inclusion. Let $\epsilon : \Omega \to \Omega_0$ be the ring homomorphism uniquely specified by the assignments $\epsilon(t_i) = 1$ and $\epsilon(x_j) = 0$. Then $\epsilon \circ i = \mathrm{Id}_{\Omega_0}$. In particular, $\epsilon_*(\Omega) = \Omega_0$ and $\epsilon_*(\mathcal{L}(\alpha)) \cong Q$. Thus applying ϵ_* to (4.1) we see that

$$Q \oplus \Omega_0 \cong \Omega_0^{(3)}.$$

It follows from (2.7) that $Q \cong \Omega_0^{(2)}$ and hence $\mathcal{L}(\alpha) \cong i_*(\Omega_0^{(2)}) = \Omega^{(2)}$. □

We arrive at the following:

Theorem 4.3 $\nu : A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}] \to \mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$ *is almost surjective for* $k = 2$.

Proof Let $\alpha \in SL_2(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}])$. We claim that $\alpha \in \mathrm{Im}(\nu_*)$. Thus let $\mathcal{L}(\alpha)$ be the $\mathfrak{L}$-locally free Ω-module obtained by glueing via α

$$\begin{array}{ccc} \mathcal{L}(\alpha) & \longrightarrow & A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])^{(2)} \\ \downarrow & & \downarrow \nu \\ \mathbb{Z}[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]^{(2)} & \stackrel{\natural}{\longrightarrow} & \mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]^{(2)}. \end{array}$$

By (4.2), $\mathcal{L}(\alpha) \cong \Omega^{(2)}$. However, $\Omega^{(2)}$ is the $\mathfrak{L}$-locally free module of rank 2 obtained by glueing via $I_2 \in GL_2(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])$ thus:

$$\mathcal{L}(I_2) = \left\{ \begin{array}{ccc} \Omega^{(2)} & \longrightarrow & A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]^{(2)} \\ \downarrow & & \downarrow \nu \\ \mathbb{Z}[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]^{(2)} & \stackrel{\natural}{\longrightarrow} & \mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]^{(2)}. \end{array} \right.$$

By Milnor's classification (Milnor 1971) there exist $\beta \in GL_2(\mathbb{Z}[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])$ and $\gamma \in GL_2(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])$ such that $\alpha = \natural_*(\beta)\cdot I_2 \cdot \nu_*(\gamma) = \natural_*(\beta)\cdot \nu_*(\gamma)$. However, if $j : \mathbb{Z} \hookrightarrow A$ is the canonical inclusion then the following diagram commutes

$$\begin{array}{ccc} \mathbb{Z} & \stackrel{j}{\longrightarrow} & A \\ & {\scriptstyle\natural}\searrow \quad \swarrow{\scriptstyle\nu} & \\ & \mathbb{F}_p & \end{array}$$

and induces a commutative diagram of group homomorphisms

$$\begin{array}{ccc} GL_2(\mathbb{Z}[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]) & \stackrel{j_*}{\longrightarrow} & GL_2(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]) \\ & {\scriptstyle\natural_*}\searrow \quad \swarrow{\scriptstyle\nu_*} & \\ & GL_2(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]) & \end{array}$$

In particular, $\natural_*(\beta) = \nu_*(j_*(\beta))$, so $\alpha = \nu_*(j_*(\beta)\cdot\gamma) \in \mathrm{Im}(\nu_*)$ as claimed. □

Taking (3.5) and (4.3) together we see that:

$$\nu : A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}] \to \mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}] \text{ is almost surjective for all } \quad k \geq 2. \tag{4.4}$$

5 The quasi-triangular ring $\mathcal{T}_q(A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}], \mathfrak{p})$

We denote by $\mathcal{T}_q(A, \mathfrak{p})$ the subring of $M_q(A)$ consisting of *quasi-triangular matrices*

$$\mathcal{T}_q(A, \mathfrak{p}) \;=\; \{X = (x_{rs})_{1 \le r,s \le q} \in M_q(A) \mid x_{rs} \in (\mathfrak{p}) \text{ if } r > s\}.$$

We identify $A/\mathfrak{p}$ with $\mathbb{F}_p$. There is then a Milnor fibre square

$$(\mathfrak{T}_0) \qquad \begin{array}{ccc} \mathcal{T}_q(A, \mathfrak{p}) & \xrightarrow{\subset} & M_q(A) \\ \downarrow \natural & & \downarrow \nu \\ \mathcal{T}_q(\mathbb{F}_p) & \xrightarrow{\subset} & M_q(\mathbb{F}_p). \end{array}$$

where

$$\mathcal{T}_q(A/\mathfrak{p}) \;=\; \{X = (x_{rs})_{1 \le r,s \le q} \in M_q(A/\mathfrak{p}) \mid x_{rs} = 0 \text{ if } r > s\}.$$

Tensoring $(\mathfrak{T}_0)$ with $\mathbb{Z}[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$ gives another fibre square

$$(\mathfrak{T}) \qquad \begin{array}{ccc} \mathcal{T}_q(A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}], \mathfrak{p}) & \xrightarrow{\subset} & M_q(A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) \\ \downarrow \natural & & \downarrow \nu \\ \mathcal{T}_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]) & \xrightarrow{\subset} & M_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]). \end{array}$$

From $(\mathfrak{T}_0)$ we then have a commutative diagram of surjective ring homomorphisms

$$\textbf{(5.1)} \qquad \begin{array}{ccc} \mathcal{T}_q(A, \mathfrak{p}) & & \\ \downarrow \varphi_+ & \searrow \natural & \\ & & \mathcal{T}_q(A/\mathfrak{p}) \;=\; \mathcal{T}_q(\mathbb{F}_p) \\ \downarrow & \swarrow \tau & \\ \mathbb{F}_p^{(q)} & & \end{array}$$

where τ is the canonical ring homomorphism to $\mathbb{F}_p^{(q)} = \underbrace{\mathbb{F}_p \times \cdots \times \mathbb{F}_p}_{q}$ with kernel the two-sided ideal of strictly upper triangular matrices. On taking mixed polynomials we have a corresponding commutative diagram of surjections

$$\begin{array}{ccc} & \mathcal{T}_q(A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}], \mathfrak{p}) & \\ & \Big\downarrow \quad \searrow^{\natural} & \\ \varphi_+ = \natural & \qquad\quad \mathcal{T}_q(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) & \\ & \Big\downarrow \quad \swarrow_{\tau} & \\ & \mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)} & \end{array} \tag{5.2}$$

$\mathcal{T}_q(A[\mathbf{t}, \mathbf{t}^{-1}], \mathfrak{p})$ contains the diagonal subring $A[\mathbf{t}, \mathbf{t}^{-1}]^{(q)}$ which maps surjectively onto $\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}]^{(q)}$ under φ_+. We proceed to describe $GL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)})$ in detail. When $k = 1$, $GL_1(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) = \mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^*$. We identify $\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}]$ with the group ring $\mathbb{F}_p[C_\infty^{(m)}]$ so that $\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}] \cong \mathcal{R}[C_\infty^{(m)}]$ where $\mathcal{R}$ is the polynomial ring $\mathbb{F}_p[\mathbf{x}]$. Hence $\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^* = \mathcal{R}[C_\infty^{(m)}]^*$. For $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ we denote by $\mathbf{t}^{\mathbf{a}}$ the group element $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \ldots t_m^{a_m}$. The group $C_\infty^{(m)}$ satisfies Higman's 'two unique products' condition (Higman 1940). As $\mathcal{R} = \mathbb{F}_p[\mathbf{x}]$ is an integral domain then, by Higman's theorem, $\mathbb{F}_p[\mathbf{x}][C_\infty^{(m)}]$, and hence $\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$, has only trivial units; that is:

$$\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^* = \{u \cdot \mathbf{t}^{\mathbf{a}} \mid u \in \mathbb{F}_p^*, \quad \mathbf{a} \in \mathbb{Z}^m\} \tag{5.3}$$

We now suppose that $k \geq 2$. If $\alpha \in GL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}])$ then we may write $\alpha = \mathbf{w} \cdot E$ where $\mathbf{w} \in \mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^*$ and $E \in SL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}])$. Taking the above description of the unit group $\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^*$ and writing $\mathbf{w} = u \cdot \mathbf{t}^{\mathbf{a}}$ when $k \geq 2$ we have:

$$GL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) = \{u \cdot \mathbf{t}^{\mathbf{a}} \cdot E \mid u \in \mathbb{F}_p^*, \quad \mathbf{a} \in \mathbb{Z}^m, \quad E \in SL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}])\} \tag{5.4}$$

Now consider $GL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)}) = \underbrace{GL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) \times \ldots \cdots \times GL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}])}_{q}$. When $k = 1$ we have

$(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)})^* = \{ (u_1 \cdot \mathbf{t}^{\mathbf{a_1}}, \ldots, u_q \cdot \mathbf{t}^{\mathbf{a_q}}) \mid u_i \in \mathbb{F}_p^*, \;\; \mathbf{a}_i \in \mathbb{Z}^{(m)}\}$. We write this expression in the form

$$(u_1 \cdot \mathbf{t}^{\mathbf{a_1}}, \ldots, u_q \cdot \mathbf{t}^{\mathbf{a_q}}) = \mathbf{u} \cdot \mathbf{T}(\mathbf{a_1}, \ldots \mathbf{a_q}) \tag{5.5}$$

where $\mathbf{u} = (u_1, \ldots, u_q) \in (\mathbb{F}_p^{(q)})^*$ and $\mathbf{T}(\mathbf{a_1}, \ldots \mathbf{a_q}) = (\mathbf{t}^{\mathbf{a_1}}, \ldots, \cdot \mathbf{t}^{\mathbf{a_q}}) \in (\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}]^{(q)})^*$.

Likewise, when $k \geq 2$ an element $\alpha \in GL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}]^{(q)})$ takes the form

$$\alpha = (u_1 \cdot \mathbf{t}^{\mathbf{a_1}} \cdot E_1, \ldots, u_q \cdot \mathbf{t}^{\mathbf{a_q}} \cdot E_q)$$

where $u_i \in \mathbb{F}_p^*, \;\; \mathbf{a}_i \in \mathbb{Z}^m, \;\; E_i \in SL_k(\mathbb{F}_p([\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}])\}$. We write this expression as

$$\alpha = (u_1 \cdot \mathbf{t}^{\mathbf{a_1}} \cdot E_1, \ldots, u_q \cdot \mathbf{t}^{\mathbf{a_q}} \cdot E_q) = \mathbf{u} \cdot \mathbf{T}(\mathbf{a}_1, \ldots, \mathbf{a}_q) \cdot \mathbf{E} \tag{5.6}$$

where $\mathbf{E} = (E_1, \ldots, E_q) \in SL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)})$. It follows from (4.4) that we have inclusions for each $k \geq 2$:

$$\mathrm{SL}_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)}) \subset \mathrm{Im}(\natural : GL_k(A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)}) \to \mathrm{GL}_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)})) \tag{5.7}$$

Moreover $GL_k(A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)}) \subset GL_k(\mathcal{T}_q(A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}], \mathfrak{p}))$. We obtain the following which is essential in the unit lifting arguments of §7:

$$\natural : \mathcal{T}_q(A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}], \mathfrak{p}) \to \mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)} \text{ is almost surjective for all } k \geq 2 \tag{5.8}$$

In the rest of this section we establish the SFC property for the relevant corner rings in the Milnor square decomposition of $\mathcal{T}_q(A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}], \mathfrak{p})$. As A is a Dedekind domain, it follows from Swan's addendum to the solution of the Serre Conjecture (Swan 1978, Theorem 1.1; see also Lam 2006, p. 189) that:

$$A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}] \text{ has property SFC.} \tag{5.9}$$

and likewise

$$\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}] \text{ has property SFC.} \tag{5.10}$$

If the rings $R_1, \ldots R_q$ all have property SFC it follows easily that the product ring $R_1 \times \cdots \times R_q$ also has property SFC. It now follows from (5.10) that:

$$\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)} \text{ has property SFC.} \tag{5.11}$$

If I is a nilpotent two sided ideal in R and R/I has property SFC it is a straightforward consequence of Nakayama's Lemma (Magurn 2002) that R also has property

SFC. As the kernel of the canonical homomorphism $\tau : \mathcal{T}_q(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) \to \mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)}$ is nilpotent it follows from (5.11) and Nakayama's Lemma that:

$$\mathcal{T}_q(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) \ \textit{text has property SFC}. \tag{5.12}$$

If R has property SFC then by Morita's Theorem(Magurn 2002) the matrix ring $M_q(R)$ also has property SFC. It now follows from (5.9) that:

$$M_q(A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]) \text{ has property SFC.} \tag{5.13}$$

6 Stably free modules over $\mathcal{T}_q(A[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}], \mathfrak{p})$

For any ring R and any integers $k \geq 1$, $q \geq 2$ there is a ring isomorphism, 'block decomposition', m

$$\widehat{\ } : M_{kq}(R) \to M_k(M_q(R))$$

defined as follows; if $X = (x_{rs})_{1 \leq r,s \leq dq} \in M_{kq}(R)$ and $1 \leq i, j \leq k$ then

$$\widehat{X} = (X(i, j))_{1 \leq i, j \leq k}$$

where $X(i, j) \in M_q(R)$ is given by $X(i, j)_{rs} = x_{q(i-1)+r,\, q(j-1)+s}$. We denote the inverse isomorphism by $\mu : M_k(M_q(R)) \to M_{kq}(R)$. Then μ and $\widehat{\ }$ induce mutually inverse isomorphisms

$$GL_k(M_q(R)) \xrightarrow{\mu} GL_{kq}(R) \xrightarrow{\widehat{\ }} GL_k(M_q(R)).$$

We now specialize to the case where $R = \mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$. If $\alpha \in GL_k(M_q(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]))$ then we may write

$$\mu(\alpha) = \Delta \cdot E \tag{6.1}$$

where $E \in SL_{kq}(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}])$ and $\Delta \in GL_{kq}(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}])$ is given by

$$\Delta_{rs} = \begin{cases} \det(\mu(\alpha)) & r = s = 1 \\ 1 & r = s \neq 1 \\ 0 & r \neq s \end{cases}.$$

Noting that $GL_k(\mathcal{T}_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])) \subset GL_k(M_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]))$ we see that

$$\widehat{\Delta} \in GL_k(\mathcal{T}_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])) \tag{6.2}$$

Moreover, as $E \in SL_{kq}(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])$ then by (4.4) we may write $E = \nu(\mathcal{E})$ where $\mathcal{E} \in GL_{kq}(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])$ and so

$$\widehat{E} = \nu(\widehat{\mathcal{E}}) \tag{6.3}$$

where $\widehat{\mathcal{E}} \in GL_k(M_q(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]))$. Applying $\widehat{\ }$ to (6.1) we have shown:

$$\begin{aligned} &\text{If } \alpha \in GL_k(M_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])) \text{ then } \alpha = \widehat{\Delta}\cdot\widehat{E} \\ &\qquad \text{where } \widehat{\Delta} \in GL_k(\mathcal{T}_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])) \\ &\qquad \text{and } \widehat{E} \in \mathrm{Im}[\nu : GL_k(M_q(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])) \to GL_k(M_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]))] \end{aligned} \tag{6.4}$$

Theorem 6.5 $\mathcal{T}_q(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}],\mathfrak{p})$ *has property* SFC.

Proof Let S be a stably free module of rank k over $\mathcal{T}_q(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}],\mathfrak{p})$. We claim that $S \cong \mathcal{T}_q(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}],\mathfrak{p})^{(k)}$. By (5.12) and (5.13), both $\mathcal{T}_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])$ and $M_q(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])$ have property SFC. It follows that S is locally free with respect to

$$\begin{array}{ccc} \mathcal{T}_q(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}],\mathfrak{p}) & \hookrightarrow & M_q(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]) \\ \downarrow \natural & & \downarrow \nu \\ \mathcal{T}_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]) & \hookrightarrow & M_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]). \end{array} \tag{6.6}$$

Hence $S \cong \mathcal{L}(\alpha)$ for some $\alpha \in GL_k(M_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1}]))$. By (6.4) there exist

(i) $\widehat{\Delta} \in GL_k(\mathcal{T}_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]))$,
(ii) $\widehat{E} \in \mathrm{Im}(\nu : GL_k(M_q(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])) \to GL_k(M_q(\mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])))$

such that $\alpha = \widehat{\Delta}\cdot\widehat{E}$. As ν is surjective then (6.6) satisfies Milnor's patching condition. By Milnor's isomorphism criterion ((Milnor 1971) we see that $\mathcal{L}(\alpha) \cong \mathcal{L}(\mathrm{Id}_k)$. Hence, as claimed, $S \cong \mathcal{L}(\mathrm{Id}_k) = \mathcal{T}_q(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}],\mathfrak{p})^{(k)}$. □

7 An induction theorem for locally free modules

We now consider the metacyclic group $G(p,q) = C_p \rtimes C_q$ where p is an odd prime, q is a positive integral divisor of $p-1$ and C_q acts via the canonical imbedding

$C_q \hookrightarrow \mathrm{Aut}(C_p)$. We observe that $\mathbb{Z}[G(p,q)]$ occurs in a Milnor fibre square of the following form (see, for example. p.187 of (Johnson (2021))):

$$\text{(7.1)} \qquad \begin{array}{ccc} \mathbb{Z}[G(p,q)] & \longrightarrow & \mathcal{T}_q(A,\mathfrak{p}) \\ \downarrow \natural & & \downarrow \natural \\ \mathbb{Z}[C_q] & \longrightarrow & \mathbb{F}_p[C_q]. \end{array}$$

Put $\Lambda = \mathbb{Z}[G(p,q)]$. As $C_q \subset \mathbb{F}_p^*$, the polynomial $y^q - 1$ factorizes into linear factors over $\mathbb{F}_p$ so that $\mathbb{F}_p[C_q] \cong \mathbb{F}_p^{(q)}$ and we may rewrite (7.1) as

$$(\mathfrak{S}_0) \qquad \begin{array}{ccc} \Lambda & \longrightarrow & \mathcal{T}_q(A,\mathfrak{p}) \\ \downarrow \natural & & \downarrow \natural \\ \mathbb{Z}[C_q] & \longrightarrow & \mathbb{F}_p^{(q)}. \end{array}$$

As in (1.1), $\mathfrak{S}_0$-locally free modules of rank k are classified by

$$\overline{GL_k(\mathfrak{S}_0)} = GL_k(\mathbb{Z}[C_q]) \backslash GL_k(\mathbb{F}_p^{(q)}) / GL_k(\mathcal{T}_q(A,\mathfrak{p})).$$

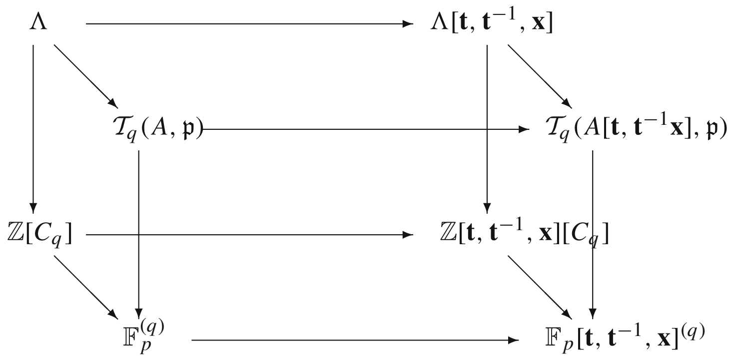
Tensoring $(\mathfrak{S}_0)$ with $\mathbb{Z}[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]$ gives the following fibre square which again satisfies Milnor's patching condition.

$$(\mathfrak{S}) \qquad \begin{array}{ccc} \Lambda[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}] & \longrightarrow & \mathcal{T}_q(A[\mathbf{t},\mathbf{t}^{-1}\mathbf{x}],\mathfrak{p}) \\ \downarrow \natural & & \downarrow \natural \\ \mathbb{Z}[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}][C_q] & \longrightarrow & \mathbb{F}_p[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]^{(q)} \end{array}$$

Likewise, $\mathfrak{S}$-locally free modules of rank k are classified by

$$\overline{GL_k(\mathfrak{S}[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}])} = GL_k(\mathbb{Z}[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}][C_q]) \backslash GL_k(\mathbb{F}_p^{(q)}[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]) / GL_k(\mathcal{T}_q(A[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}],\mathfrak{p}))$$

Let $\iota : \Lambda \to \Lambda[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}]$ denote the canonical inclusion; then ι has a canonical left inverse $\epsilon : \Lambda[\mathbf{t},\mathbf{t}^{-1},\mathbf{x}] \to \Lambda$ defined by $\epsilon(t_i) = 1$ and $\epsilon(x_j) = 0$. Moreover ι induces a mapping of squares $\iota : \mathfrak{S}_0 \to \mathfrak{S}$:



and hence a mapping $\iota_* : \overline{GL_k(\mathfrak{S}_0)} \to \overline{GL_k(\mathfrak{S})}$. Recall that, in the notation of §5,

$$\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^* = \{u \cdot \mathbf{t}^{\mathbf{a}} \mid u \in \mathbb{F}_p^*, \quad \mathbf{a} \in \mathbb{Z}^m\} \tag{7.2}$$

$$(u_1 \cdot \mathbf{t}^{\mathbf{a_1}}, \ldots, u_q \cdot \mathbf{t}^{\mathbf{a_q}}) = \mathbf{u} \cdot \mathbf{T}(\mathbf{a_1}, \ldots \mathbf{a_q}) \tag{7.3}$$

where $\mathbf{u} = (u_1, \ldots, u_q) \in (\mathbb{F}_p^{(q)})^*$ and $\mathbf{T}(\mathbf{a_1}, \ldots \mathbf{a_q}) = (\mathbf{t}^{\mathbf{a_1}}, \ldots, \cdot\mathbf{t}^{\mathbf{a_q}}) \in (\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}]^{(q)})^*$. The expression $\mathbf{T}(\mathbf{a_1}, \ldots \mathbf{a_q})$ can be equally be regarded as an element of $\mathcal{T}_q(A[\mathbf{t}, \mathbf{t}^{-1}], \mathfrak{p})^*$. Consequently the canonical map $\iota : (\mathbb{F}_p^{(q)})^* \to GL_1(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)})/GL_1(\mathcal{T}_q(A[\mathbf{t}, \mathbf{t}^{-1}], \mathfrak{p}))$ is surjective. It follows immediately that:

$$\iota : \overline{GL_1(\mathfrak{S}_0)} \longrightarrow \overline{GL_1(\mathfrak{S})} \quad \text{is surjective.} \tag{7.4}$$

Likewise, when $k \geq 2$ an element $\alpha \in GL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)})$ takes the form

$$\alpha = (u_1 \cdot \mathbf{t}^{\mathbf{a_1}} \cdot E_1, \ldots, u_q \cdot \mathbf{t}^{\mathbf{a_q}} \cdot E_q) = \mathbf{u} \cdot \mathbf{T}(\mathbf{a}_1, \ldots, \mathbf{a}_q) \cdot \mathbf{E} \tag{7.5}$$

where $\mathbf{E} = (E_1, \ldots, E_q) \in SL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}]^{(q)})$ so when $k \geq 2$ it follows from (5.8) that any element $\mathbf{T}(\mathbf{a}_1, \ldots, \mathbf{a}_q) \cdot \mathbf{E} \in GL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)})$ lifts to $GL_k(\mathcal{T}_q(A[\mathbf{t}, \mathbf{t}^{-1}], \mathfrak{p}))$. Hence the canonical map $\psi : (\mathbb{F}_p^{(q)})^* \to GL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(q)}) / GL_k(\mathcal{T}_q(A[\mathbf{t}, \mathbf{t}^{-1}], \mathfrak{p}))$ is surjective for all $k \geq 2$, from which, in conjunction with (7.4), it follows that:

$$\iota : \overline{GL_k(\mathfrak{S}_0)} \longrightarrow \overline{GL_k(\mathfrak{S})} \quad \text{is surjective for all } k \geq 1 \tag{7.6}$$

Hence we obtain:

Theorem 7.7 *Let P be a locally free module with respect to $\mathfrak{S}$; then there is a module P_0, locally free with respect to $\mathfrak{S}_0$, such that $P = \iota_*(P_0)$.*

8 Stably free cancellation for $G(p, q) \times C_\infty^{(m)}$

An easy calculation from (7.1) shows that $\mathbb{R}[G(p,q)] \cong \mathbb{R}[C_q] \times M_q(A \otimes_{\mathbb{Z}} \mathbb{R})$. Moreover,

$$\mathbb{R}[C_q] \cong \begin{cases} \mathbb{R} \times \mathbb{R} \times \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{(q-2)/2} & q \text{ even;} \\ \\ \mathbb{R} \times \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{(q-1)/2} & q \text{ odd.} \end{cases} \tag{8.1}$$

Recalling that $d = (p-1)/q$ then

$$M_q(A \otimes_{\mathbb{Z}} \mathbb{R}) \cong \begin{cases} \underbrace{M_q(\mathbb{R}) \times \cdots \times M_q(\mathbb{R})}_{d} & \text{if } q \text{ is even;} \\ \\ \underbrace{M_q(\mathbb{C}) \times \cdots \times M_q(\mathbb{C})}_{d/2} & \text{if } q \text{ is odd.} \end{cases} \tag{8.2}$$

In any case, $\mathbb{R}[G(p,q)]$ has no quaternionic factor so that:

$$\mathbb{Z}[G(p,q)] \text{ satisfies the Eichler condition.} \tag{8.3}$$

In consequence of (8.3) and the theorem of Swan-Jacobinski (Swan 1970) we see that:

$$\mathbb{Z}[G(p,q)] \text{ has the } SFC \text{ property.} \tag{8.4}$$

We note also that:

$$\mathbb{Z}[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}][C_q] \text{ has property } SFC. \tag{8.5}$$

In the case $n = 0$, (8.5) follows from the main theorem of (Johnson 2014). However the argument given there continues to hold, with the same justification, on replacing $\mathbb{Z}[\mathbf{t}, \mathbf{t}^{-1}]$ by $\mathbb{Z}[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$ as the theorem of Swan in (Swan 1978) applies equally in either case.

Theorem 8.6 $\Lambda[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$ *has property* SFC.

Proof Let S be a stably free module of rank k over $\Lambda[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$. We claim that $S \cong \Lambda[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(k)}$. It follows from (1.2), (6.5) and (8.5) that S is locally free with respect to $\mathfrak{S}$. By (7.7), there exists a module S_0, locally free with respect to $\mathfrak{S}_0$, such that $S \cong \iota_*(S_0)$. Let $\epsilon : \Lambda[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}] \to \Lambda$ be the canonical left inverse to ι. As S is stably free, then $S_0 = \epsilon_*(S)$ is also stably free. By (8.4) $S_0 \cong \Lambda^{(k)}$ and hence $S \cong \iota_*(\Lambda^{(k)}) = \Lambda[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]^{(k)}$. □

Noting that $\Lambda[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}] = \mathcal{R}[G(p, q)]$ where $\mathcal{R} = \mathbb{Z}[\mathbf{t}, \mathbf{t}^{-1}, \mathbf{x}]$ we see that (8.6) proves Theorem I of the Introduction. In the special case where the purely polynomial variables x_j are absent we have:

$$\mathbb{Z}[G(p, q) \times C_\infty^{(m)}] \cong \Lambda[\mathbf{t}, \mathbf{t}^{-1}] \tag{8.7}$$

As a consequence we obtain the following which is Corollary II of the Introduction:

Corollary 8.8 $\mathbb{Z}[G(p, q) \times C_\infty^{(m)}]$ *has property* SFC.

In conclusion we point out that when $q = 2$ the proof of (8.8) is somewhat simpler than the general case. Then $G(p, 2)$ is simply the dihedral group

$$D_{2p} = \langle x,\ y \mid x^p = y^2 = 1, yxy^{-1} = x^{-1} \rangle$$

and $A = \mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ is the real subring of the cyclotomic integers $\mathbb{Z}[\zeta_p]$. In this case we are aided by the well known fact (Birch 1967; Hasse 1962) that the induced map on unit groups $\nu : \mathbb{Z}[\zeta_p + \zeta_p^{-1}]^* \to \mathbb{F}_p^*$ is surjective. In consequence, when $q = 2$, the homomorphism $\nu : GL_k(A[\mathbf{t}, \mathbf{t}^{-1}]) \to GL_k(\mathbb{F}_p[\mathbf{t}, \mathbf{t}^{-1}])$ is surjective for $k \geq 1$ which is a stronger statement than (4.4).

Data Availability Not applicable.

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