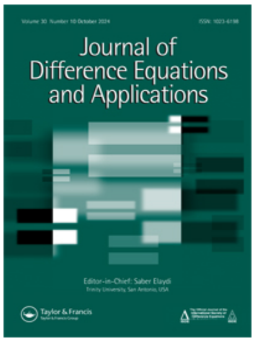

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Convexity of non-compact carrying simplices in logarithmic coordinates

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ABSTRACT
We study a non-compact version of the carrying simplex for the planar Leslie–Gower and planar Ricker maps when they are written in logarithmic variables. We show that for both of these models there is a convex (unbounded) invariant set X_∞, and all orbits are attracted to X_∞. For the Leslie–Gower map, which is injective, the boundary of X_∞ globally attracts all orbits and we identify it with a non-compact carrying simplex. As the Ricker map is not invertible, the boundary of X_∞ may not be invariant. We establish conditions on the parameters of the Ricker map which guarantee that there is a convex non-compact carrying simplex when $r, s < 1$ which maps into a compact carrying simplex in the standard untransformed coordinates.

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1. Introduction

Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_{++} = (0, \infty)$ and $F : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be a continuous function. Consider the following planar difference equation:

$$\begin{aligned} \mathbf{x}_0 &\in \mathbb{R}^2_+, \\ \mathbf{x}_{n+1} &:= F(\mathbf{x}_n) = F^{n+1}(\mathbf{x}_0), \quad n \in \mathbb{N} := \{0, 1, 2, \ldots\}. \end{aligned} \tag{1}$$

In this article we are interested in a special invariant curve of (1) known as the *carrying simplex*. Hirsch's definition [9] of a carrying simplex, when applied to the above system, is as follows.

Definition 1.1: We call $\Sigma \subset \mathbb{R}^2_+ - \{\mathbf{0}\}$ a carrying simplex if

(CS1) Σ is compact and invariant.
(CS2) For any $\mathbf{x} \in \mathbb{R}^2_+ - \{\mathbf{0}\}$ there exists $\mathbf{y} \in \Sigma$ such that $\lim_{n\to\infty} |F^n(\mathbf{x}) - F^n(\mathbf{y})| = 0$. (asymptotic completeness)
(CS3) Σ is unordered. (i.e. for $(x_1, y_1), (x_2, y_2) \in \Sigma$, if $x_1 < x_2$, then $y_2 < y_1$, and if $y_1 < y_2$, then $x_2 < x_1$)

When it exists, the carrying simplex Σ is thus a compact and invariant manifold for (1) that attracts $\mathbb{R}^2_+ - \{\mathbf{0}\}$ and that has the special property that Σ is the graph of a decreasing

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Dedicated to Saber Elaydi on the occasion of his 80th birthday.

and continuous function. To date, and to the best of the authors' knowledge, planar carrying simplices for discrete dynamics have been studied exclusively in the context of retrotone systems (e.g. [9,11,17,18]).

This paper explores the pros and cons of working in alternative coordinates where compactness of the carrying simplex is lost.

Definition 1.2: A map $F = (F_1, F_2) : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is retrotone (e.g. [9,18]) in a subset $D \subset \mathbb{R}^2_+$ if for $x, y \in D$ such that $F_1(x) \geq F_1(y)$ and $F_2(x) \geq F_2(y)$ but $F(x) \neq F(y)$ we have $x_1 > y_1$ provided $y_1 > 0$ and $x_2 > y_2$ provided $y_2 > 0$.

A retrotone map is sometimes also called a competitive map (see, for example, [19]). In the planar case a map satisfying Definition 1.2 has the special property that it maps the graph of a decreasing function on D to the graph of a new decreasing function on D [2,5].

The Leslie–Gower map from ecology [15] that we study in Section 3.1 is retrotone for all biologically realistic parameter values, and it is well-known that it has a unique carrying simplex [13]. On the other hand, the Ricker map is not retrotone everywhere in $\mathbb{R}^2_+$ (see for example [9,11,18]), and so existence of a carrying simplex in the standard coordinates of population densities, by means of retronicity, is only known for a limited set of parameter values.

Here we will extend the notion of the carrying simplex applied to planar systems to allow it to be non-compact, and we will call a set $\Sigma \subset \mathbb{R}^2$ a *non-compact carrying simplex* if it satisfies (CS1) without compactness and (CS3), but (CS2) is replaced by the lesser requirement that Σ globally attracts $\mathbb{R}^2$. The issue of asymptotic completeness will be addressed elsewhere.

In working with non-compact carrying simplices we may work in alternative coordinate systems for which the systems (1) that we consider here have at most one (finite) fixed point, but in so doing we lose compactness of the global attractor and asymptotic completeness. We have found that by using logarithmically transformed coordinates, we are sometimes able to obtain stronger geometrical properties for the non-compact carrying simplex, namely that it is the graph of a concave decreasing function. While the corresponding compact carrying simplices are also known to be graphs of decreasing functions, whether or not those functions are convex or concave is not generally known (for results on convexity of carrying simplices see [1,3,4,21]). Here, we will also discuss the convexity of the boundary of the basin of repulsion of infinity in the logarithmically scaled Leslie–Gower and Ricker models. When all the parameters are positive, then the maps in the logarithmically scaled versions of both models are concave (i.e. each component of the map is a concave function [14]). We take advantage of this fact to prove that the basin of repulsion of infinity is an invariant convex set. We establish a relationship between the convexity and the strict decreasingness of the members of a sequence of sets that converges to the boundary of the basin of repulsion of infinity. Then, it becomes straightforward to show that this boundary satisfies (CS3).

2. Preliminary results

In this section, we prove three lemmas that play pivotal roles. The first lemma will enable us to prove that the boundary of each of the sets we are discussing is the graph of a continuous strictly decreasing function. The second lemma shows that for given a set in a certain class of subsets of $\mathbb{R}^2$ whose members have boundary that is the graph of a continuous strictly decreasing function, that set must be convex.

Lemma 2.1: *Let $X \subset \mathbb{R}^2$ and $a, b \in \mathbb{R}$ be given. Suppose there exist two continuous functions $A : (-\infty, a) \to \mathbb{R}$ and $B : (-\infty, b) \to \mathbb{R}$ such that*

$$\{x \mid (x, c) \in X\} = \begin{cases} (-\infty, A(c)], & c \in (-\infty, a) \\ \emptyset, & \text{otherwise} \end{cases} \tag{2}$$

$$\{y \mid (d, y) \in X\} = \begin{cases} (-\infty, B(d)], & d \in (-\infty, b) \\ \emptyset, & \text{otherwise.} \end{cases} \tag{3}$$

Then $X \subset (-\infty, b) \times (-\infty, a)$, both A and B are strictly decreasing functions, and

$$\partial X = \{(A(c), c) \mid c \in (-\infty, a)\} \tag{4}$$

$$= \{(d, B(d)) \mid d \in (-\infty, b)\}. \tag{5}$$

In other words, the boundary of X is the graph of a strictly decreasing function and X is the set of all points on or under the graph of that function.

Proof: It is clear that we have

$$\{(A(c), c) \mid c \in (-\infty, a)\} \subseteq \partial X$$
$$\{(d, B(d)) \mid d \in (-\infty, b)\} \subseteq \partial X.$$

To prove (4), we observe that for each $(x, y) \in \partial X$ there exist $\{(x_n, y_n)\}_{n=1}^{\infty} \subseteq X$ and $\{(x'_n, y'_n)\}_{n=1}^{\infty} \subseteq (-\infty, b) \times (-\infty, a) - X$ such that

$$\lim_{n\to\infty} (x_n, y_n) = \lim_{n\to\infty} (x'_n, y'_n) = (x, y).$$

For each $n \in \mathbb{N}$ we have $x_n \leq A(y_n)$ and $x'_n > A(y'_n)$. It follows that since A is continuous we have

$$x = \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} A(y_n) = A(y) = \lim_{n\to\infty} A(y'_n) \leq \lim_{n\to\infty} x'_n = x.$$

Hence, $x = A(y)$ and $(x, y) \in \{(A(c), c) \mid c \in (-\infty, a)\}$. This proves (4). Proving (5) is similar. It is clear that (4) and (5) imply that A and B are inverse of each other and they are both bijective. Hence, by using the fact that they are continuous functions, we deduce that A and B are strictly decreasing functions (These functions cannot be strictly increasing since $X \subset (-\infty, b) \times (-\infty, a)$ and the boundary of X is equal to each of the graphs of A and B). ∎

Before stating Lemma 2.2, we have to define the relation '$\ll$' between some members of $\mathbb{R}^2$. Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we write $(x_1, y_1) \ll (x_2, y_2)$ if $x_1 < x_2$ and $y_1 < y_2$.

Lemma 2.2: *Let $X \subset (-\infty, b) \times (-\infty, a)$ be the set of points on or under the graph of the continuous strictly decreasing function $B : (-\infty, b) \to (-\infty, a)$. Assume that for every $\mathbf{x}, \mathbf{y} \in X$ and $0 < \lambda < 1$ there exists at least one $\mathbf{z} \in X$ such that*

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \ll \mathbf{z}. \tag{6}$$

Then X is convex.

Proof: Assume that X is not convex. Then there exist $\mathbf{x}, \mathbf{y} \in X$ and $0 < \lambda < 1$ such that

$$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \notin X.$$

Assume that $\mathbf{z}$ is as stated in the theorem. Since $z_1 > \lambda x_1 + (1 - \lambda)y_1$ and B is strictly decreasing, we have

$$B(z_1) < B(\lambda x_1 + (1 - \lambda)y_1)$$

and since $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \notin X$ and $\mathbf{z} \in X$, we have

$$z_2 \leq B(z_1)$$
$$B(\lambda x_1 + (1 - \lambda)y_1) < \lambda x_2 + (1 - \lambda)y_2.$$

Hence,

$$z_2 < \lambda x_2 + (1 - \lambda)y_2.$$

which contradicts (6). ■

Lemma 2.3: *Let $x_0 \in \mathbb{R}$ be given and suppose that $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $K : (-\infty, x_0) \to \mathbb{R}$ are continuous functions and p satisfies*

$$\lim_{x \to -\infty} \sup\{p(x, y) \mid y \in \mathbb{R}\} = -\infty. \tag{7}$$

Suppose also that there exists $G : (-\infty, y^\star) \to \mathbb{R}$ defined by

$$G(c) := \sup p\left(\left\{(x, q(x, c)) \mid x \in (-\infty, K(c)]\right\}\right).$$

Then G is continuous and

$$\Omega_c := p\left(\left\{(x, q(x, c)) \mid x \in (-\infty, K(c)]\right\}\right) = (-\infty, G(c)], \quad c \in (-\infty, y^\star).$$

Proof: Fix $c \in (-\infty, y^\star)$. Since the continuous image of a connected set is connected, we deduce that Ω_c is connected. Equation (7) implies that

$$\lim_{x \to -\infty} p(x, q(x, c)) = -\infty,$$

and hence Ω_c is unbounded below and there exists $L \in (-\infty, K(c))$ such that for every $x < L$ we have $p(x, q(x, c)) < G(c) - 1$. Therefore, by compactness of $Y_c := [L, K(c)]$ we deduce that

$$G(c) = \sup \Omega_c \in p\left(\left\{(x, q(x, c)) \mid x \in Y_c\right\}\right) \subset \Omega_c. \tag{8}$$

Connectedness of Ω_c along with the fact that it is unbounded below and $G(c) = \sup \Omega_c \in \Omega_c$ proves $\Omega_c = (-\infty, G(c)]$.

We now prove continuity of G by contradiction. Suppose that G is not continuous at some $c_0 \in (-\infty, x_0)$. Then there exist a sequence $\{a_n\}$ which converges to c_0 and $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ we have $|G(c_0) - G(a_n)| \geq \varepsilon$. By (8) we know that $G(c_0) \in \Omega_{c_0}$. Hence there exists $w_{c_0} \in Y_{c_0}$ such that $p(w_{c_0}, q(w_{c_0}, c_0)) = G(c_0)$. Similarly, for every $n \in \mathbb{N}$ there exists $w_{a_n} \in Y_{a_n}$ such that $p(w_{a_n}, q(w_{a_n}, a_n)) = G(a_n)$. Since K is continuous, we can find a sequence $\{v_n\}$ which converges to w_{c_0} and for every $n \in \mathbb{N}$ we have $v_n \in (-\infty, K(a_n)]$. By continuity of p and q we have

$$\lim_{n\to\infty} p(v_n, q(v_n, a_n)) = p(w_{c_0}, q(w_{c_0}, c_0)) = G(c_0).$$

For every $n \in \mathbb{N}$ we have

$$p(v_n, q(v_n, a_n)) \leq \sup p\left(\left\{(x, q(x, a_n)) \mid x \in (-\infty, K(a_n)]\right\}\right) = G(a_n).$$

Thus

$$G(c_0) \leq \liminf_{n\to\infty} G(a_n). \tag{9}$$

Inequality (9) along with $|G(c_0) - G(a_n)| \geq \varepsilon$ implies that there exists $M > 0$ such that for every $n > M$ we have $G(c_0) + \frac{\varepsilon}{2} < G(a_n) = p(w_{a_n}, q(w_{a_n}, a_n))$. From (7), there exists $x_1 \in \mathbb{R}$ such that for every $x < x_1$ and $y \in \mathbb{R}$ we have $p(x, y) < G(c_0) + \frac{\varepsilon}{2}$. Hence for every $n > M$ we have $x_1 \leq w_{a_n} \leq K(a_n)$. This along with the fact that K is continuous, implies that there exists $M' > 0$ such that for every $n > M'$ we have $x_1 \leq w_{a_n} \leq K(c_0) + 1$. Thus $\{w_{a_n}\}$ is bounded and has a convergent subsequence $\{w_{a_{m_n}}\}$. By the continuity of p and q we have

$$G(c_0) + \frac{\varepsilon}{2} \leq \lim_{n\to\infty} G(a_{m_n}) = \lim_{n\to\infty} p(w_{a_{m_n}}, q(w_{a_{m_n}}, a_{m_n})) = p(b, q(b, c_0)), \tag{10}$$

where $b = \lim_{n\to\infty} w_{a_{m_n}}$. But it is clear that we also have

$$p(b, q(b, c_0)) \leq \sup p\left(\left\{(x, q(x, c_0)) \mid x \in (-\infty, K(c_0)]\right\}\right) = G(c_0)$$

which contradicts (10). Therefore, G is continuous at c_0. Since $c_0 \in (-\infty, x_0)$ is arbitrary we see that G is continuous on $(-\infty, x_0)$. ■

We will now combine Lemmas 2.1, 2.2 and 2.3 to show that two well-known maps from theoretical ecology have globally attracting and invariant 1-dimensional manifolds, and also determine when they are the invariant boundary of an invariant convex set.

3. Applications to ecological models

In this section, we use the above theory to prove the convexity of a unique non-compact carrying simplex in logarithmically scaled versions of the Leslie–Gower Model and Ricker models from theoretical ecology.

3.1. The Leslie–Gower model

The planar Leslie–Gower model [7,15] is defined by the Leslie–Gower map

$$F(u,v) := \left(\frac{ru}{1+u+\alpha v}, \frac{sv}{1+v+\beta u}\right). \tag{11}$$

When $r, s < 1$ and $\alpha, \beta > 0$, then $(0,0)$ is globally asymptotically stable on $\mathbb{R}^2_+$ (see [7]). Hence, the system has no carrying simplex when $r, s < 1$ and $\alpha, \beta > 0$ since no $\Sigma \subset \mathbb{R}^2_+ - \{\mathbf{0}\}$ can satisfy (CS2).

When $r, s > 1$ the Leslie–Gower map has fixed points

$$(0,0), (r-1,0), (0,s-1) \quad \text{and, if positive,} \left(\frac{\alpha(s-1)-r+1}{\alpha\beta-1}, \frac{\beta(r-1)-s+1}{\alpha\beta-1}\right). \tag{12}$$

A number of authors [1,9,11,12] have shown that for $r, s > 1$ and $\alpha, \beta > 0$, the model (11) has a unique carrying simplex. In our approach, we use an alternative set of coordinates to those in (11): We scale (11) as follows

$$u = e^x, \quad v = e^y, \tag{13}$$

to obtain the following log-scaled version of the model:

$$f(x,y) := \left(\ln(r) + x - \ln(1+e^x+\alpha e^y), \ln(s) + y - \ln(1+e^y+\beta e^x)\right). \tag{14}$$

The only finite fixed point of the log-scale Leslie–Gower map is

$$\left(\ln\left(\frac{\alpha(s-1)-r+1}{\alpha\beta-1}\right), \ln\left(\frac{\beta(r-1)-s+1}{\alpha\beta-1}\right)\right), \tag{15}$$

when the expressions are real.

We wish to study the invariant subsets of $f : \mathbb{R}^2 \to \mathbb{R}^2$, and to this end we define

$$X_0 := \mathbb{R}^2,$$
$$X_n := \overline{f(X_{n-1})}, \quad n = 1, 2, 3, \ldots$$

and finally

$$X_\infty := \bigcap_{n=0}^{\infty} X_n. \tag{16}$$

We recall that for a non-empty set $A \subset \mathbb{R}^d$, the ω-limit set of A for a (continuous) map $T : \mathbb{R}^d \to \mathbb{R}^d$ is defined to be $\omega_T(A) := \bigcap_{n=0}^{\infty} \overline{\bigcup_{m=n}^{\infty} T^m(A)}$, (e.g. [20]). In this context X_∞ is actually $\omega_f(\mathbb{R}^2)$ for f given by (14), but we do not have compactness of any $f^n(\mathbb{R}^2)$ to apply standard results (e.g. [20, Theorem 2.11]) for ω-limit sets to conclude that X_∞ is non-empty and invariant. Instead we prove these facts directly in a series of lemmas below.

The first lemma is needed because standard theorems on non-empty intersections of decreasing sequences of compact sets cannot be applied here as our sets X_n are not compact.

Lemma 3.1: *When r, s > 1 we have* $X_\infty \neq \emptyset$.

Proof: Suppose that $\zeta_{x,y}$ and $\eta_{x,y}$ are defined as follows:

$$\zeta_{x,y} := \frac{1 - \frac{\alpha y}{y-s}}{r - x + \frac{\alpha\beta xy}{y-s}},$$

$$\eta_{x,y} := \frac{1 - \frac{\beta x}{x-r}}{s - y + \frac{\alpha\beta xy}{x-r}}.$$

Since $\lim_{(x,y)\to(0,0)}(\zeta_{x,y}, \eta_{x,y}) = (\frac{1}{r}, \frac{1}{s}) \ll (1,1)$, there exists $x', y' \in \mathbb{R}_{++}$ such that for every $(0,0) \leq (x,y) \leq (x',y')$ we have $\zeta_{x,y} < 1$, $\eta_{x,y} < 1$. Hence if $S = (0,x'] \times (0,y']$, for every $(x,y) \leq (x',y')$ we have $(x\zeta_{x,y}, y\eta_{x,y}) \in S$. It can also be easily verified that $F(x\zeta_{x,y}, y\eta_{x,y}) = (x,y)$. Thus $S \subset F(S)$ and if we define $S^* := \{(\ln(x), \ln(y)) | (x,y) \in S\}$, then $S^* \subset f(S^*)$. It means that for $n = 0, 1, 2, \ldots$ we have $S^* \subset X_n$. Therefore we have $\emptyset \neq S^* \subset X_\infty$. ∎

In the following, we rely strongly on the fact that f in (14) is invertible.

Lemma 3.2: *For the log Leslie–Gower map* (14) *the sets* $X_\infty = \omega_f(\mathbb{R}^2)$ *and* ∂X_∞ *are non-empty and invariant.*

Proof: It is well-known (e.g. [20]) that, for a given $A \subset \mathbb{R}^2$ and a continuous map $f : A \to \mathbb{R}^2$, the omega-limit set $\omega(A)$ is closed and forward-invariant under f. Therefore, $f(X_\infty) = f(\omega_f(\mathbb{R}^2)) \subset \omega_f(\mathbb{R}^2) = X_\infty$.

To prove $X_\infty \subset f(X_\infty)$, suppose for the sake of contradiction that there exists $\mathbf{x} \in X_\infty$ such that $\mathbf{x} \notin f(X_\infty)$. For every $m = 0, 1, 2, \ldots$ we have $\mathbf{x} \in X_{m+1} = \overline{f(X_m)}$ so that for every $m = 0, 1, 2, \ldots$ there exists sequence $\{y_n^{(m)}\} \subset X_m$ such that $\lim_{n\to\infty} f(\mathbf{y}_n^{(m)}) = \mathbf{x}$.

It can be easily proven that $X_1 = \overline{f(\mathbb{R}^2)} \subset (-\infty, \ln(r)) \times (-\infty, \ln(s))$. Hence, for every $m = 1, 2, \ldots$ we have $\{\mathbf{y}_n^{(m)}\} \subset X_m \subset (-\infty, \ln(r)) \times (-\infty, \ln(s))$.

There must be $a, b \in \mathbb{R}$ such that $\{\mathbf{y}_n^{(m)}\} \subset (a, \ln(r)) \times (b, \ln(s))$, because otherwise $\{f(\mathbf{y}_n^{(m)})\}$ would not be bounded below which is a contradiction to the fact that $\{f(\mathbf{y}_n^{(m)})\}$ is convergent.

Now from the fact that $\{\mathbf{y}_n^{(m)}\}$ is bounded, we deduce that it has a limit point $\mathbf{y}$. Since $\{y_n^{(m)}\} \subset X_m$ and X_m is closed, we have $\mathbf{y} \in X_m$. And from $\lim_{n\to\infty} f(\mathbf{y}_n^{(m)}) = \mathbf{x}$, we deduce $f(\mathbf{y}) = \mathbf{x}$. Now, since f is invertible, $\mathbf{y}$ is the only point whose value of f is equal to $\mathbf{x}$. It means that for every $m = 1, 2, \ldots$ we would have the same $\mathbf{y}$ for that $\mathbf{x}$. Therefore, for every $m = 1, 2, \ldots$, we have $\mathbf{y} \in X_m$, thus $\mathbf{y} \in X_\infty$ and $\mathbf{x} = f(\mathbf{y}) \in f(X_\infty)$. This proves $X_\infty \subset f(X_\infty)$. And along with the fact that X_∞ is forward invariant, we have $X_\infty = f(X_\infty)$ and X_∞ is invariant.

As f is a diffeomorphism, both f and f^{-1} map the interior of X_∞ into itself. Hence the interior of X_∞ is invariant. As X_∞ is also invariant, ∂X_∞ must be invariant. ∎

Lemma 3.3: *For any* $r, s, \alpha, \beta > 0$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $\mathbf{x} \neq \mathbf{y}$ *and* $0 < \lambda < 1$ *we have*

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \gg \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}). \tag{17}$$

Proof: Define $\xi_1 : [0,1] \to \mathbb{R}$ as follows

$$\xi_1(\lambda) := f_1(\lambda \mathbf{x} + (1-\lambda)\mathbf{y})$$

If $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, then we have

$$\begin{aligned}\xi_1''(\lambda) = &-\frac{(x_1-y_1)^2 e^{\lambda x_1+(1-\lambda)y_1} + \alpha(x_2-y_2)^2 e^{\lambda x_2+(1-\lambda)y_2}}{(1+e^{\lambda x_1+(1-\lambda)y_1}+\alpha e^{\lambda x_2+(1-\lambda)y_2})^2}\\ &-\frac{\alpha(x_1-y_1-x_2+y_2)^2 e^{\lambda x_1+(1-\lambda)y_1} e^{\lambda x_2+(1-\lambda)y_2}}{(1+e^{\lambda x_1+(1-\lambda)y_1}+\alpha e^{\lambda x_2+(1-\lambda)y_2})^2} < 0.\end{aligned}$$

Hence ξ_1 is strictly concave. Similarly $\xi_2 : [0,1] \to \mathbb{R}$ defined by $\xi_2(\lambda) := f_2(\lambda\mathbf{x} + (1-\lambda)\mathbf{y})$ is also strictly concave. The inequality (17) is now a direct result of the strict concavity of ξ_1 and ξ_2 and the following facts:

$$\xi_i(0) = f_i(\mathbf{y}), \quad \xi_i(1) = f_i(\mathbf{x}), \quad i = 1, 2.$$

■

Lemma 3.4: (a) *Let $X \subset \mathbb{R}^2$ be the set of all points on or under the graph of a continuous strictly decreasing function $B : (-\infty, b) \to \mathbb{R}$. Then for the log-scaled Leslie–Gower map $f = (f_1, f_2)$ in* (14) *we have*

$$\{x \mid (x,c) \in f(X)\} = \begin{cases} f_1\left(\{\left(x, g(x)+h(c)\right) \mid x \in (-\infty, K(c))\}\right), & c \in (-\infty, y^\star) \\ \emptyset, & \text{otherwise}\end{cases}$$

where $y^\star = \sup\{f_2(x,y) \mid (x,y) \in f(X)\}$, and

$$\begin{aligned} g(x) &= \ln(1+\beta e^x)\\ h(c) &= c - \ln(s-e^c)\\ K(c) &= H^{-1}(c - \ln(s-e^c))\end{aligned}$$

and H is the invertible continuous function defined by

$$H(x) = B(x) - \ln(1+\beta e^x).$$

(b) *We have*

$$\{x \mid (x,c) \in f(\mathbb{R}^2)\} = \begin{cases}\left(-\infty, \ln\left(\dfrac{r}{1+\frac{e^c\alpha\beta}{s-e^c}}\right)\right), & c \in (-\infty, \ln(s))\\ \emptyset, & \text{otherwise}\end{cases}$$

where h and g are as defined in part (a).

Proof: (a) For $c \in (-\infty, y^\star)$ we have

$$\begin{aligned}
\{x \mid (x,c) \in f(X)\} &= f_1\left(X \cap \left\{(x,y) \in \mathbb{R}^2 \mid f_2(x,y) = c\right\}\right) \\
&= f_1\left(X \cap \left\{(x,y) \in \mathbb{R}^2 \mid y = c - \ln(s-e^c) + \ln(1+\beta e^x)\right\}\right) \\
&= f_1\left(\left\{(x,y) \in \mathbb{R}^2 \mid y = c - \ln(s-e^c) + \ln(1+\beta e^x), y \le B(x)\right\}\right) \\
&= f_1\left(\left\{\left(x, \ln(1+\beta e^x) + c - \ln(s-e^c)\right) \mid c \right.\right. \\
&\quad \left.\left. - \ln(s-e^c) + \ln(1+\beta e^x) \le B(x)\right\}\right) \\
&= f_1\left(\left\{\left(x, g(x)+h(c)\right) \mid c - \ln(s-e^c) \le B(x) - \ln(1+\beta e^x)\right\}\right) \\
&= f_1\left(\left\{\left(x, g(x)+h(c)\right) \mid c - \ln(s-e^c) \le H(x)\right\}\right).
\end{aligned}$$

Since B is strictly decreasing, H is strictly decreasing and invertible. Hence,

$$\{x \mid (x,c) \in f(X)\} = f_1\left(\left\{\left(x, g(x)+h(c)\right) \mid x \in (-\infty, H^{-1}(c - \ln(s-e^c)))\right\}\right).$$

(b) For $c \in (-\infty, \ln(s))$ we have

$$\begin{aligned}
\{x \mid (x,c) \in f(\mathbb{R}^2)\} &= f_1\left(\left\{(x,y) \in \mathbb{R}^2 \mid f_2(x,y) = c\right\}\right) \\
&= f_1\left(\left\{(x,y) \in \mathbb{R}^2 \mid y = c - \ln(s-e^c) + \ln(1+\beta e^x)\right\}\right) \\
&= f_1\left(\left\{\left(x, \ln(1+\beta e^x) + c - \ln(s-e^c)\right) \mid x \in \mathbb{R}\right\}\right) \\
&= f_1\left(\left\{\left(x, g(x)+h(c)\right) \mid x \in \mathbb{R}\right\}\right), \\
&= \left\{f_1\left(x, g(x)+h(c)\right) \mid x \in \mathbb{R}\right\}, \\
&= \left\{\ln(r) + x - \ln(1 + e^x + \alpha e^{g(x)+h(c)}) \mid x \in \mathbb{R}\right\}, \\
&= \left\{\ln(r) + x - \ln(1 + e^x + \alpha e^{\ln(1+\beta e^x)+c-\ln(s-e^c)}) \mid x \in \mathbb{R}\right\}, \\
&= \left\{\ln(r) + x - \ln\left(1 + \frac{\alpha e^c}{s-e^c} + e^x\left(1 + \frac{e^c\alpha\beta}{s-e^c}\right)\right) \mid x \in \mathbb{R}\right\}.
\end{aligned}$$

Since $c \in (-\infty, \ln(s))$, $s - e^c$ is positive. Hence, $1 + \frac{\alpha e^c}{s-e^c} > 0$ and $1 + \frac{e^c\alpha\beta}{s-e^c} > 0$ which implies that the derivative of the function $w(x) = \ln(r) + x - \ln(1 + \frac{\alpha e^c}{s-e^c} + e^x(1 + \frac{e^c\alpha\beta}{s-e^c}))$ is always positive. Thus

$$\begin{aligned}
\left\{\ln(r) + x - \ln\left(1 + \frac{\alpha e^c}{s-e^c} + e^x\left(1 + \frac{e^c\alpha\beta}{s-e^c}\right)\right) \mid x \in \mathbb{R}\right\} &= \left(-\infty, \lim_{x\to+\infty} w(x)\right) \\
&= \left(-\infty, \ln\left(\frac{r}{1 + \frac{e^c\alpha\beta}{s-e^c}}\right)\right).
\end{aligned}$$

■

Combining the previous lemmas together we obtain.

Theorem 3.1: *For any $r, s > 1$ and $\alpha, \beta > 0$, for the log-scaled Leslie–Gower map* (14)*, the set X_∞ defined by* (16) *is convex and invariant. Moreover, ∂X_∞ is invariant and attracts $\mathbb{R}^2$.*

Proof: It is clear that X_0 is convex. We use induction to prove that for $n = 1, 2, \ldots, X_n$ is convex, from which it follows that their intersection X_∞ is convex. To prove convexity of X_1, first we observe that by Lemma 3.4(b) for every $c \in (-\infty, \ln(s))$ we have

$$\{x \mid (x,c) \in X_1\} = \overline{\{x \mid (x,c) \in f(\mathbb{R}^2)\}} = \left(-\infty, \ln\left(\frac{r}{1+\frac{e^c\alpha\beta}{s-e^c}}\right)\right].$$

Now $A(c) := \ln(\frac{r}{1+\frac{e^c\alpha\beta}{s-e^c}})$ and $X = X_1$ satisfy the conditions stated for A in Lemma 2.1.

So far, we have proven the existence of A which satisfies the conditions of Lemma 2.1 for $X = X_1$. But to apply Lemma 1 we also need to prove the existence of the second function B of that lemma. Indeed it is easy to check that B is given by $B(c) = A^{-1}(c) = \ln(\frac{s}{1+\frac{e^c\alpha\beta}{r-e^c}})$. Hence by Lemma 2.1, X_1 is the set of all points on or under the graph of a continuous strictly decreasing function. It is obvious that for every $\mathbf{x}, \mathbf{y} \in X_1$ and $0 < \lambda < 1$ we have $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \in f(\mathbb{R}^2) = X_1$. Moreover by Lemma 3.3 we have $\lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}) \ll f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y})$. Therefore, for every $\mathbf{x}, \mathbf{y} \in X_1$ and $0 < \lambda < 1$ there exists $\mathbf{z} = f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \in X_1$ such that $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \ll \mathbf{z}$. Now since X_1 satisfies the conditions of Lemma 2.2, we deduce that X_1 is convex.

Assume that for $n \geq 1$, X_n is convex and that it is the set of all points on or under the graph of a continuous strictly decreasing function $B : (-\infty, b) \to \mathbb{R}$. By Lemma 3.4(a), for every $c \in (-\infty, y^\star)$ we have

$$\{x \mid (x,c) \in X_{n+1}\} = \{x \mid (x,c) \in f(X_n)\} = f_1\left(\{\left(x, g(x) + h(c)\right) \mid x \in (-\infty, K(c))\}\right),$$

where g, h and K are as defined in Lemma 3.4. The functions $p(x,y) := f_1(x,y)$, $q(x,y) := g(x) + h(y)$ and K satisfy the conditions of Lemma 2.3, so that for every $c \in (-\infty, y^\star)$ we have $\boldsymbol{\Omega}_c = (-\infty, G(c)]$, where $G : (-\infty, y^\star) \to \mathbb{R}$ is continuous. Thus

$$X_{n+1} = \bigcup_{c \in (-\infty, y^\star)} \boldsymbol{\Omega}_c = \bigcup_{c \in (-\infty, y^\star)} (-\infty, G(c)].$$

$A{:} = G$ and $X := X_{n+1}$ satisfy the conditions stated for A in Lemma 2.1. Owing to the symmetric structure of the definition of the log-scaled Leslie–Gower map we can prove the existence of B which satisfies the conditions of Lemma 2.1 for $X = X_{n+1}$. Therefore, Lemma 2.1 shows that X_{n+1} is the set of all points on or under the graph of a continuous strictly decreasing function.

Suppose that $\mathbf{x}, \mathbf{y} \in X_{n+1}$. Since the sequence $\{X_n\}$ is decreasing, we have $\mathbf{x}, \mathbf{y} \in X_n$. Now since X_n is convex, for every $0 < \lambda < 1$ we have $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in X_n$, thus $f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \in f(X_n) = X_{n+1}$. By Lemma 3.3 we have $\lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}) \ll f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y})$. Therefore, for every $\mathbf{x}, \mathbf{y} \in X_{n+1}$ and $0 < \lambda < 1$ there exists $\mathbf{z} = f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \in X_{n+1}$ such that $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \ll \mathbf{z}$. Since X_{n+1} satisfies the conditions of Lemma 2.2, we deduce that X_{n+1} is convex. We conclude that X_∞ is convex.

∂X_∞ is invariant by Lemma 3.2. We know that ∂X_∞ is the graph of a concave and strictly decreasing function A. By invariance, $f_1(A^{-1}(y), y) = A^{-1}(f_2(A^{-1}(y), y))$, i.e. $A^{-1}(y) + \ln r - \ln(1 + \alpha e^{A^{-1}(y)} + e^y) = A^{-1}(y + \ln s - \ln(1 + e^y + \beta e^{A^{-1}(y)}))$. As A^{-1} is strictly decreasing and bounded above, $\lim_{y \to -\infty} A^{-1}(y) = x^*$ and invariance yields $x^* + \ln r -$

$\ln(1 + e^{x^*}) = x^*$, so $x^* = \ln(r - 1)$. A similar argument shows that $\lim_{x\to-\infty} A(x) = \ln(s - 1)$.

To show that ∂X_∞ attracts $\mathbb{R}^2$, first we show that any finite fixed point of (14) must belong to ∂X_∞. As there can be at most one finite fixed point, if $P = (P_1, P_2)$ is a finite fixed point not in ∂X_∞ then ∂X_∞ contains no finite fixed point and dynamics on ∂X_∞ is monotone. On ∂X_∞ we may consider the one-dimensional dynamics $x_{n+1} = f_1(x_n, A^{-1}(x_n))$ or $y_{n+1} = f_2(A(y_n), y_n)$. Suppose $x_n \to -\infty$ when $n \to \infty$:

$$0 > x_{n+1} - x_n = \ln r - \ln(1 + e^{x_n} + \alpha e^{A^{-1}(x_n)}) \to \ln r - \ln(1 + \alpha(s - 1))$$

so we need $r - 1 < \alpha(s - 1)$. On the other hand,

$$0 < y_{n+1} - y_n = \ln s - \ln(1 + e^{y_n} + \beta e^{A(y_n)}) \to \ln s - \ln(1 + \beta(r - 1))$$

so we also need $s - 1 > \beta(r - 1)$. The pair of conditions $r - 1 < \alpha(s - 1)$ and $s - 1 > \beta(r - 1)$ are incompatible with existence of a finite fixed point (15) of (14). A similar contradiction is obtained when the dynamics is monotone increasing in x_n. Hence we conclude that whenever a finite fixed point exists for (14) it must belong to ∂X_∞.

Next we recall (e.g. [7]) that all non-trivial dynamics for the unscaled Leslie–Gower map converge to a fixed point which can be $(r - 1, 0)$, $(0, s - 1)$, or $(\frac{\alpha(s-1)-r+1}{\alpha\beta-1}, \frac{\beta(r-1)-s+1}{\alpha\beta-1})$ when it is positive. Hence any orbit of (11) not convergent to the positive fixed point must converge to $(r - 1, 0)$ or $(0, s - 1)$.

Now consider an orbit (x_n, y_n) of (14) that does not converge to a finite fixed point. Then by convergence of Leslie–Gower orbits, (e^{x_n}, e^{y_n}) tends to $(r - 1, 0)$ or $(0, s - 1)$ as $n \to \infty$. Suppose (e^{x_n}, e^{y_n}) tends to $(r - 1, 0)$ as $n \to \infty$. Then $x_n \to \ln(r - 1)$, $y_n \to -\infty$ as $n \to \infty$. On the other hand, $A^{-1}(y_n) \to \ln(r - 1)$ as $n \to \infty$. Thus $\|(x_n, y_n) - (A^{-1}(y_n), y_n)\| = |x_n - A^{-1}(y_n)| \to 0$ as $n \to \infty$. Hence in this case we have $(x_n, y_n) \to \partial X_\infty$ as $n \to \infty$. The case (e^{x_n}, e^{y_n}) tends to $(0, s - 1)$ is similar.

Finally for the case that an orbit (x_n, y_n) of (14) converges a positive fixed point P, as we showed in the previous paragraph $P \in \partial X_\infty$.

Thus we conclude that ∂X_∞ is attracting. ■

3.2. *The Ricker model*

The planar Ricker model is defined by the non-invertible map

$$F(u, v) := \left(ue^{r-u-\alpha v}, ve^{s-v-\beta u}\right), \quad (u, v) \in \mathbb{R}^2_+ \tag{18}$$

where $\alpha, \beta, r, s > 0$.

With the coordinates stated in (13), we have the following log-scaled version of the model:

$$f(x, y) := \left(x + r - e^x - \alpha e^y, y + s - e^y - \beta e^x\right). \tag{19}$$

For the time being we work with the Ricker map in these standard coordinates to see when we can expect a carrying simplex to be unique when it exists. Log coordinates will be introduced later. The following points are always fixed points of F:

$$\mathbf{c}_1 := (0, 0), \quad \mathbf{c}_2 := (r, 0), \quad \mathbf{c}_3 := (0, s). \tag{20}$$

When $\alpha\beta \neq 1$ and $\mathbf{c}_4$ defined in (21) below is a member of $\mathbb{R}^2_{++}$, then F has exactly four fixed points and the fourth fixed point is:

$$\mathbf{c}_4 := \left(\frac{s\alpha - r}{\alpha\beta - 1}, \frac{r\beta - s}{\alpha\beta - 1} \right). \tag{21}$$

We will find it more convenient to now use the log-scaled version (19). We define

$$Y_0 := \mathbb{R}^2, \quad X_0 := Q_3,$$

where $Q_3 = \{(x, y) : x \leq 0 \text{ and } y \leq 0\}$ is the third quadrant. We let

$$Y_n := f(Y_{n-1}), \quad X_n := \overline{f(X_{n-1})}, \quad n = 1, 2, \ldots$$

and define the sets

$$Y_\infty := \bigcap_{n=0}^{\infty} Y_n = \bigcap_{n=0}^{\infty} f^n(\mathbb{R}^2), \quad X_\infty := \bigcap_{n=0}^{\infty} X_n = \bigcap_{n=0}^{\infty} \overline{f^n(Q_3)}. \tag{22}$$

Lemma 3.5: *If s, $r < 1$ then*

$$Y_\infty = X_\infty.$$

Proof: It is obvious that $X_\infty \subseteq Y_\infty$. To prove that we also have $Y_\infty \subseteq X_\infty$, it is sufficient to prove that for any $(x, y) \in \mathbb{R}^2$, there exists $n \in \mathbb{N}$ such that $f^n(x, y) \in Q_3$.

If $x > 0$ then we have

$$f_1(x, y) - x = r - e^x - \alpha e^y < r - 1 < 0,$$

and if $y > 0$ then

$$f_2(x, y) - y = s - e^y - \beta e^y < s - 1 < 0.$$

Therefore, if we define n as follows,

$$n = \begin{cases} 0 & x,\ y \leq 0 \\ 1 + \left\lfloor \dfrac{x}{1-r} \right\rfloor & x > 0,\ y \leq 0 \\ 1 + \left\lfloor \dfrac{y}{1-s} \right\rfloor & y > 0,\ x \leq 0 \\ 1 + \max\left\{ \left\lfloor \dfrac{x}{1-r} \right\rfloor, \left\lfloor \dfrac{y}{1-s} \right\rfloor \right\} & x,\ y > 0, \end{cases}$$

then $f^n(x, y) \in Q_3$. ■

Lemma 3.6: *When r, $s > 0$ we have $X_\infty \neq \emptyset$.*

Proof: $x', y' < 0$ can be found such that for every $(x_1, y_1) \leq (x', y')$ the following equations have at least one solution with $(x, y) \ll (x_1, y_1)$:

$$x + r - e^x - \alpha e^y = x_1,$$
$$y + s - e^y - \beta e^x = y_1.$$

Hence if $S = (-\infty, x'] \times (-\infty, y']$ then $S \subset f(S)$. Therefore for $n = 0, 1, 2, \ldots$ we have $S \subset X_n$ and $\emptyset \neq S \subset X_\infty$. ■

Now we will show that X_∞ is invariant. Since the log-scaled Ricker map f is not invertible, to show invariance of X_∞ we need a property weaker than invertibility. We use the fact that the log-scaled Ricker map f is a proper map (i.e. for every compact set $X \subset \mathbb{R}^2, f^{-1}(X)$ is compact). To see this, note that if $\{\mathbf{x}_n\}$ is a sequence such that $|\mathbf{x}_n| \to \infty$, then, according to the terms in (19), $|f(\mathbf{x}_n)| \to \infty$. Since f is continuous, we conclude that for every closed and bounded set $X, f^{-1}(X)$ is closed and bounded. Therefore, for every compact set $X, f^{-1}(X)$ is compact and we conclude that the log-scaled Ricker map f is proper.

Lemma 3.7: *When $0 < r$, $s < 1$ and f is the log-scaled Ricker map, X_∞ defined by* (22) *is invariant.*

Proof: If $\mathbf{x} \in f(X_\infty)$ then $\mathbf{x} \in f(X_n) \subset \overline{f(X_n)} = X_{n+1}$ for $n \in \mathbb{N}$. Hence $\mathbf{x} \in \bigcap_{n=1}^{\infty} X_n$ and since $X_0 = Q_3$, we have $\mathbf{x} \in \bigcap_{n=1}^{\infty} X_n = \bigcap_{n=0}^{\infty} X_n = X_\infty$. This proves $f(X_\infty) \subset X_\infty$.

We prove that $\overline{f(X_n)} = f(X_n)$ for $n \in \mathbb{N}$. It is sufficient to show that $f(X_n)$ is closed if X_n is closed. Suppose that $\mathbf{x}^*$ is a limit point of $f(X_n)$. There exists a sequence $\{\mathbf{y}_m^n\} \subset X_n$ such that $\lim_{m\to\infty} f(\mathbf{y}_m^n) = \mathbf{x}^*$. $\{\mathbf{y}_m^n\}$ is bounded since, being proper, f maps unbounded subsets of X_0 into unbounded sets whereas $\{f(\mathbf{y}_m^n)\}$ is convergent, and hence bounded. Boundedness of $\{\mathbf{y}_m^n\} \subset X_n$ and the fact that X_n is closed, imply that $\{\mathbf{y}_m^n\}$ has a limit point $\mathbf{y}^* \in X_n$. Continuity of f implies $f(\mathbf{y}^*) = \mathbf{x}^*$. Thus $\mathbf{x}^* \in f(X_n)$, which proves that $f(X_n)$ is closed and hence $\overline{f(X_n)} = f(X_n)$.

Now if $\mathbf{x} \in X_\infty$, then $\mathbf{x} \in X_{n+1} = \overline{f(X_n)} = f(X_n)$ for $n \in \mathbb{N}$. Hence for $n \in \mathbb{N}$ there exists $\mathbf{z}_n \in X_n$ such that $f(\mathbf{z}_n) = \mathbf{x}$. Since f is proper, $f^{-1}(\{\mathbf{x}\})$ is compact. Therefore, since $\{\mathbf{z}_n\} \subset f^{-1}(\{\mathbf{x}\})$, $\{\mathbf{z}_n\}$ is bounded and has at least one limit point. Let $\mathbf{z}^*$ be such a limit point. $\mathbf{z}^*$ is also a limit point of $f^{-1}(\{\mathbf{x}\})$, and since $f^{-1}(\{\mathbf{x}\})$ is closed, we have $\mathbf{z}^* \in f^{-1}(\mathbf{x})$. Thus $f(\mathbf{z}^*) = \mathbf{x}$ and since $\mathbf{z}^*$ is a limit point of $\{\mathbf{z}_n\}$, and for $n \in \mathbb{N}$ we have $\mathbf{z}_n \in X_n$ where $\{X_n\}$ is a decreasing sequence of closed sets, we conclude that $\mathbf{z}^* \in X_\infty$. This proves $X_\infty \subset f(X_\infty)$. ■

Lemma 3.8: *Lemma* 3.3 *also holds for the log-scaled Ricker map f defined in* (19).

Proof: We define $\chi_1 : [0,1] \to \mathbb{R}$ and $\chi_2 : [0,1] \to \mathbb{R}$ as follows

$$\chi_1(\lambda) := f_1(\lambda\mathbf{x} + (1-\lambda)\mathbf{y})$$
$$\chi_2(\lambda) := f_2(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}).$$

It is easy to show that both χ_1 and χ_2 are strictly concave (being the sum of linear minus exponential terms), and the rest of the proof is straightforward. ■

Lemma 3.9: (a) *Let $b \in \mathbb{R}$ and $X \subset Q_3$ be the set of all points on or under the graph of a continuous strictly decreasing function $B : (-\infty, b) \to \mathbb{R}$. Let $f = (f_1, f_2)$ denote the log-scaled Ricker map. Then we have*

$$\{x \mid (x,c) \in f(X)\} = \begin{cases} f_1\left(\{(x, P(c - s + \beta e^{x})) \mid x \in (-\infty, K(c)]\}\right), & c \in (-\infty, y^\star) \\ \emptyset, & \text{otherwise} \end{cases}$$

where $y^\star = \sup\{f_2(x,y) \mid (x,y) \in f(X)\}$,

$$P(x) := x - W_0(-e^x),$$

W_0 *is the principal branch of the Lambert W function (see, for example, [16]),*

$$K(c) := G^{-1}(c-s)$$

and G is the continuous invertible function defined by

$$G(x) := B(x) - e^{B(x)} - \beta e^x.$$

(b) *We have*

$$\{x \mid (x,c) \in f(Q_3)\} = \begin{cases} f_1\left(\{(x, P(c-s+\beta e^x)) \mid x \in (-\infty, K(c)]\}\right), & c \in (-\infty, y^\star) \\ \emptyset, & \text{otherwise} \end{cases}$$

where P is the function defined in part (a) *and*

$$y^\star = \sup\{f_2(x,y) \mid (x,y) \in f(Q_3)\},$$
$$K(c) = \min\left\{0, \ln\left(\frac{-1-c+s}{\beta}\right)\right\}.$$

Proof: (a) For $c \in (-\infty, y^\star)$ we have

$$\begin{aligned}\{x \mid (x,c) \in f(X)\} &= f_1\left(X \cap \left\{(x,y) \in Q_3 \mid f_2(x,y) = c\right\}\right) \\ &= f_1\left(X \cap \left\{(x,y) \in Q_3 \mid y + s - e^y - \beta e^x = c\right\}\right) \\ &= f_1\left(X \cap \left\{(x,y) \in Q_3 \mid y - e^y = c - s + \beta e^x\right\}\right)\end{aligned}$$

When $t < 0$, $t - e^t$ is strictly increasing. Hence, for every $l \in \mathbb{R}$, $t - e^t = l$ has at most one solution for $t \le 0$. Since $\{t - e^t \mid t \le 0\} = (-\infty, -1]$, $t - e^t = l$ has a unique solution for $t \le 0$ and $l \in (-\infty, -1]$ and no solution when $t \le 0$ and $l > -1$. We claim that $t = P(l)$ is the unique solution for $t - e^t = l$ when $t \le 0$ and $l \in (-\infty, -1]$. To prove that, we have

$$P(l) - e^{P(l)} = l - W_0(-e^l) - e^{l - W_0(-e^l)}. \tag{23}$$

By the properties of the Lambert W function W_0 we have $W_0(-e^l)e^{W_0(-e^l)} = -e^l$. Hence $-e^{l-W_0(-e^l)} = W_0(-e^l)$. This along with (23) implies $P(l) - e^{P(l)} = l$ (since $t = P(l)$ must satisfy $t < 0$, only the principal branch of the Lambert W function can be used to provide a solution). Now with $l = c - s + \beta e^x$ and $t = y$ we have

$$\begin{aligned}&f_1\left(X \cap \left\{(x,y) \in Q_3 \mid y - e^y = c - s + \beta e^x\right\}\right) \\ &= f_1\left(X \cap \left\{(x,y) \in Q_3 \mid y = P(c-s+\beta e^x)\right\}\right) \\ &= f_1\left(\left\{(x,y) \in Q_3 \mid y = P(c-s+\beta e^x), y \le B(x)\right\}\right) \\ &= f_1\left(\{\left(x, P(c-s+\beta e^x)\right) \mid P(c-s+\beta e^x) \le B(x)\}\right).\end{aligned}$$

Since $P(c-s+\beta e^x) \leq 0, B(x) \leq 0$ in the above sets, and since when $t<0$, $t-e^t$ is strictly increasing, we have

$$P(c-s+\beta e^x) \leq B(x) \Longrightarrow c-s+\beta e^x = P(c-s+\beta e^x) - e^{P(c-s+\beta e^x)} \leq B(x) - e^{B(x)}.$$

Thus

$$\begin{aligned}
&f_1\left(\left\{\left(x, P(c-s+\beta e^x)\right) \mid P(c-s+\beta e^x) \leq B(x)\right\}\right)\\
&= f_1\left(\left\{\left(x, P(c-s+\beta e^x)\right) \mid c-s+\beta e^x \leq B(x) - e^{B(x)}\right\}\right)\\
&= f_1\left(\left\{\left(x, P(c-s+\beta e^x)\right) \mid c-s \leq B(x) - e^{B(x)} - \beta e^x\right\}\right)\\
&= f_1\left(\left\{\left(x, P(c-s+\beta e^x)\right) \mid c-s \leq G(x)\right\}\right).
\end{aligned}$$

(b) For $c \in (-\infty, s-1)$ we have

$$\begin{aligned}
\{x \mid (x,c) \in f(Q_3)\} &= f_1\left(\left\{(x,y) \in Q_3 \mid f_2(x,y) = c\right\}\right)\\
&= f_1\left(\left\{(x,y) \in Q_3 \mid y+s-e^y-\beta e^x = c\right\}\right)\\
&= f_1\left(\left\{(x,y) \in Q_3 \mid y-e^y = c-s+\beta e^x\right\}\right)\\
&= f_1\left(\left\{(x,y) \in Q_3 \mid y = P(c-s+\beta e^x)\right\}\right)\\
&= f_1\left(\left\{\left(x, P(c-s+\beta e^x)\right) \mid x \in (-\infty,0], P(c-s+\beta e^x) \in (-\infty,0]\right\}\right)\\
&= f_1\left(\left\{\left(x, P(c-s+\beta e^x)\right) \mid x \in (-\infty,0], P(c-s+\beta e^x)\right.\right.\\
&\qquad \left.\left. -\ e^{P(c-s+\beta e^x)} \leq -1\right\}\right)\\
&= f_1\left(\left\{\left(x, P(c-s+\beta e^x)\right) \mid x \leq 0, \quad c-s+\beta e^x \leq -1\right\}\right)\\
&= f_1\left(\left\{\left(x, P(c-s+\beta e^x)\right) \mid x \leq 0, \quad x \leq \ln\left(\frac{-1-c+s}{\beta}\right)\right\}\right)\\
&= f_1\left(\left\{\left(x, P(c-s+\beta e^x)\right) \mid x \leq \min\left\{0, \ln\left(\frac{-1-c+s}{\beta}\right)\right\}\right\}\right).
\end{aligned}$$

■

Lemma 3.10: *For any* $0<r,\ s<1$ *and* $\alpha,\beta>0$, *and* f *is the log-scaled Ricker map,* X_∞ *defined in* (22) *is invariant and convex.*

Notice that Lemma 3.10 is not a complete analogue of Theorem 3.1 since we are not claiming that ∂X_∞ is necessarily invariant. We will address this after proving Lemma 3.10.

Lemma 3.10. Convexity of X_∞ can be proved with a similar argument to that used for the scaled Leslie–Gower model. So we explain the argument more briefly. It is obvious that X_0 is convex, and by using induction we can prove convexity of X_n for $n=1,2,\ldots$, and that implies convexity of X_∞.

To prove convexity of X_1, by Lemma 3.9(b) for $c \in (-\infty, y^\star)$ we have

$$\{x \mid (x,c) \in f(X_1)\} = \{x \mid (x,c) \in f(Q_3)\} = f_1\left(\left\{\left(x, P(c-s+\beta e^x)\right) \mid x \in (\infty, K(c)]\right\}\right)$$

where P, K and $y^\star$ are defined in that part of the lemma. Now it is easy to verify that $p = f_1$ and $q : \mathbb{R}^2 \to \mathbb{R}$ defined by $q(x, y) = P(y - s + \beta e^x)$ and K satisfy the conditions of Lemma 2.3. By Lemma 2.3 for every $c \in (-\infty, y^\star)$ we have $\Omega_c = (-\infty, G(c)]$, where $G : (-\infty, y^\star) \to \mathbb{R}$ is continuous. Thus

$$X_1 = \bigcup_{c\in(-\infty,y^\star)} \Omega_c = \bigcup_{c\in(-\infty,y^\star)} (-\infty, G(c)].$$

Now $A := G, X := X_1$ satisfy the conditions stated for A in Lemma 2.1.

Owing to the symmetric structure of the definition of the log-scaled Ricker map, we can state similar lemmas to prove that there exists B such that it satisfies the conditions of Lemma 2.1 for $X = X_1$. Now since, by Lemma 2.1, X_1 is the set of all points on or under the graph of a continuous strictly decreasing function, we can use Lemma 2.2 and Lemma 3.8 with a similar argument to that used in Theorem 1 to prove that X_1 is convex.

Assume that for $n \geq 1$, X_n is convex and it is the set of all points on or under the graph of a continuous strictly decreasing function $B : (-\infty, b) \to \mathbb{R}$. By Lemma 3.9(a) for every $c \in (-\infty, y^\star)$ we have

$$\{x \mid (x, c) \in X_{n+1}\} = \{x \mid (x, c) \in f(X_n)\} = f_1\left(\left\{\left(x, P(c - s + \beta e^x)\right) \mid x \in (-\infty, K(c))\right\}\right).$$

Then $p := f_1$ and $q : \mathbb{R}^2 \to \mathbb{R}$ defined by $q(x, y) := P(y - s + \beta e^x)$ satisfy the conditions of Lemma 2.3 and G defined in that lemma is continuous. So $A: = G$ and $X := X_{n+1}$ satisfy the conditions stated for A in Lemma 2.1. Again, owing to the symmetric structure of the definition of the log-scaled Ricker map we can prove the existence of B which satisfies the conditions of Lemma 2.1 for $X = X_{n+1}$. Therefore, by that lemma, X_{n+1} is the set of all points on or under the graph of a continuous strictly decreasing function. Now we can use Lemmas 2.2 and 3.8 and a similar argument to that used in Theorem 3.1 to prove that X_{n+1} is convex.

According to Lemma 3.7 X_∞ is invariant, and as the intersection of convex sets it is convex.

As the log-scaled Ricker map f is not invertible we cannot conclude that ∂X_∞ is also invariant. In order to prove that $\partial X_\infty \subset X_1$ is invariant, it is sufficient to show that the restriction $f|_{X_1 \to f(X_1)}$ is invertible. If the Jacobian of f is non-vanishing throughout X_1 then f is locally invertible at any point of X_1. But as is well known, locally invertibility does not always imply global invertibility. Ho [10] proved that a local homeomorphism between a pathwise connected Hausdorff space and a simply connected Hausdorff space is a global homeomorphism if and only if that map is proper. We have already established that the log-scaled Ricker map f is proper (in the paragraph preceding Lemma 3.7). Hence, if we prove that the Jacobian of f does not vanish anywhere in X_1 for a given range of parameters, then we can deduce that $\partial X_\infty \subset X_1$ is invariant for that same range of parameters.

Thus now we consider where the Jacobian vanishes.

Using [6] (which studies the unscaled Ricker map (18)) the Jacobian of the log-scaled Ricker map f only vanishes on LC_{-1} defined by

$$LC_{-1} := \{(x, y) \in Q_3 : 1 - e^x = e^y(1 - (1 - \alpha\beta)e^x)\}. \tag{24}$$

Set

$$q(t) = \frac{1 - e^t}{(\alpha\beta - 1)e^t + 1}.$$

When $\alpha\beta \geq 1$, then $q(t) > 0$ if and only if $t < 0$. If $\alpha\beta < 1$, then $q(t) > 0$ if and only if $t \in (-\infty, 0) \cup (-\ln(1-\alpha\beta), +\infty)$. In this case, LC_{-1} is the union of two connected curves. By Lemma 3.5, if $r, s < 1$, then $X_\infty = Y_\infty \subset Q_3$. So in this case we only need to consider

$$LC^1_{-1} := \{(\ln(q(t)), t) \mid t < 0\},$$

and investigate whether or not the Jacobian vanishes at some points on X_∞.

According to [6], $Y_1 := f(\mathbb{R}^2)$ is bounded by the set of points on or under LC^1_0 defined by

$$LC^1_0 := \left\{ \left(\ln\left(q(t)\right) + r - q(t) - \alpha e^t, t + s - e^t - \beta q(t)\right) \mid t < 0 \right\}. \tag{25}$$

Since $X_\infty \subset Y_1$, X_∞ is a subset of the set of points on or under LC^1_0. Hence, if $r, s < 1$ and LC^1_{-1} does not intersect that space, then the Jacobian of f does not vanish anywhere in X_∞ since $LC_{-1} \cap X_\infty = \emptyset$.

Lemma 3.11: *If $r, s < 1$ then LC^1_{-1} does not intersect the set of points on or under LC^1_0. Hence, if $r, s < 1$ then ∂X_∞ is invariant.*

Proof: It is sufficient to show that if $(x_{-1}, y) \in LC^1_{-1}$ and $(x_0, y) \in LC^1_0$, then $x_{-1} > x_0$. Since $(x_0, y) \in LC^1_0$, for some $t < 0$ we have $(x_0, y) = (\ln(q(t)) + r - q(t) - \alpha e^t, t + s - e^t - \beta q(t))$. Since $x_{-1} = \ln(q(y))$ and $y = t + s - e^t - \beta q(t)$, we have

$$x_{-1} = \ln(q(t + s - e^t - \beta q(t))).$$

We define $R(t) := x_{-1} - x_0 = \ln(q(t + s - e^t - \beta q(t))) - (\ln(q(t)) + r - q(t) - \alpha e^t)$. We have

$$q(t)\left(e^{R(t)} - 1\right) = q(t + s - e^t - \beta q(t))e^{-r+q(t)+\alpha e^t} - q(t). \tag{26}$$

It is easy to show that when $h < 0$, then $q(h) > 0$. We use this fact multiple times in this proof. From $t < 0$ we have $t - e^t < -1$. This along with $s < 1$ and $q(t) > 0$ implies $t + s - e^t - \beta q(t) < 0$. Thus $q(t + s - e^t - \beta q(t)) > 0$. Now since $e^{-r+q(t)+\alpha e^t} \geq 1 - r + q(t) + \alpha e^t$, we have

$$q(t + s - e^t - \beta q(t))e^{-r+q(t)+\alpha e^t} \geq q(t + s - e^t - \beta q(t))(1 - r + q(t) + \alpha e^t). \tag{27}$$

Now (26) and (27) imply

$$q(t)\left(e^{R(t)} - 1\right) \geq q\left(t + s - e^t - \beta q(t)\right)(1 - r + q(t) + \alpha e^t) - q(t). \tag{28}$$

For the sake of expressing equations in a simpler way, let $T := e^t$. We have $0 < T < 1$, $q(t) = q(\ln(T)) = \frac{1-T}{(\alpha\beta-1)T+1}$ and we can rewrite (28) as follows

$$q(t)\left(e^{R(t)} - 1\right) \geq q\left(\ln(T) + s - T - \beta q(\ln(T))\right)(1 - r + q(\ln(T)) + \alpha T) - q(\ln(T)). \tag{29}$$

We have

$$
\begin{aligned}
q\left(\ln(T)+s-T-\beta q(\ln(T))\right) &= \frac{1-e^{\ln(T)+s-T-\beta q(\ln(T))}}{(\alpha\beta-1)e^{\ln(T)+s-T-\beta q(\ln(T))}+1} \\
&= \frac{1-Te^{s-T-\beta\frac{1-T}{(\alpha\beta-1)T+1}}}{(\alpha\beta-1)Te^{s-T-\beta\frac{1-T}{(\alpha\beta-1)T+1}}+1} = 1 \\
&\quad - \frac{\alpha\beta Te^{s-T-\beta\frac{1-T}{(\alpha\beta-1)T+1}}}{(\alpha\beta-1)Te^{s-T-\beta\frac{1-T}{(\alpha\beta-1)T+1}}+1} \\
&= 1-\frac{\alpha\beta T}{(\alpha\beta-1)T+e^{-s+T+\beta\frac{1-T}{(\alpha\beta-1)T+1}}}. \qquad (30)
\end{aligned}
$$

As we mentioned before, $\ln(T)+s-T-\beta q(\ln(T)) = t+s-e^t-\beta q(t) < 0$. Hence, from $\alpha\beta-1 > -1$, we deduce

$$
(\alpha\beta-1)Te^{s-T-\beta\frac{1-T}{(\alpha\beta-1)T+1}}+1 = (\alpha\beta-1)e^{\ln(T)+s-T-\beta q(\ln(T))}+1 > 0,
$$

thus

$$
(\alpha\beta-1)T+e^{-s+T+\beta\frac{1-T}{(\alpha\beta-1)T+1}} = e^{-s+T+\beta\frac{1-T}{(\alpha\beta-1)T+1}}\left((\alpha\beta-1)Te^{s-T-\beta\frac{1-T}{(\alpha\beta-1)T+1}}+1\right) > 0. \qquad (31)
$$

From $T>0$, $s<1$ and $q(t) > 0$ we have

$$
(\alpha\beta-1)T+1+\left(-s+T+\beta\frac{1-T}{(\alpha\beta-1)T+1}\right) = \alpha\beta T+1-s+\beta q(t) > 0. \qquad (32)
$$

Since $e^h > 1+h$ for $h\in\mathbb{R}$, we have

$$
(\alpha\beta-1)T+e^{-s+T+\beta\frac{1-T}{(\alpha\beta-1)T+1}} \geq (\alpha\beta-1)T+1+\left(-s+T+\beta\frac{1-T}{(\alpha\beta-1)T+1}\right). \qquad (33)
$$

Now (31), (32) and (33) imply

$$
1-\frac{\alpha\beta T}{(\alpha\beta-1)T+e^{-s+T+\beta\frac{1-T}{(\alpha\beta-1)T+1}}} \geq 1-\frac{\alpha\beta T}{\alpha\beta T+1-s+\beta\frac{1-T}{(\alpha\beta-1)T+1}}.
$$

This along with (29) and (30) implies

$$
\begin{aligned}
q(t)\left(e^{R(t)}-1\right) &\geq \left(1-\frac{\alpha\beta T}{\alpha\beta T+1-s+\beta\frac{1-T}{(\alpha\beta-1)T+1}}\right) \\
&\quad\times\left(1-r+q(\ln(T))+\alpha T\right)-q(\ln(T)) \\
&= \left(1-\frac{\alpha\beta T((\alpha\beta-1)T+1)}{(\alpha\beta T+1-s)((\alpha\beta-1)T+1)+\beta(1-T)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(1 - r + \frac{1-T}{(\alpha\beta - 1)T + 1} + \alpha T\right) \\
&- \frac{1-T}{(\alpha\beta - 1)T + 1} \\
&= \frac{(1-s)((\alpha\beta - 1)T + 1) + \beta(1-T)}{(\alpha\beta T + 1 - s)((\alpha\beta - 1)T + 1) + \beta(1-T)} \\
&\times \left(1 - r + \frac{1-T}{(\alpha\beta - 1)T + 1} + \alpha T\right) - \frac{1-T}{(\alpha\beta - 1)T + 1} \\
&= \frac{E(T)}{(\alpha\beta - 1)T + 1},
\end{aligned}
$$

where

$$
\begin{aligned}
E(T) :=& \frac{(1-s)((\alpha\beta - 1)T + 1) + \beta(1-T)}{(\alpha\beta T + 1 - s)((\alpha\beta - 1)T + 1) + \beta(1-T)} \\
&\times ((1 - r + \alpha T)((\alpha\beta - 1)T + 1) + 1 - T) \\
&- (1-T) \\
=& \frac{(1-s)((\alpha\beta - 1)T + 1) + \beta(1-T)}{(\alpha\beta T + 1 - s)((\alpha\beta - 1)T + 1) + \beta(1-T)} \\
&\times ((1 - r + \alpha T)((\alpha\beta - 1)T + 1) + 1 - T) \\
&- \frac{(\alpha\beta T + 1 - s)((\alpha\beta - 1)T + 1)(1-T) + \beta(1-T)^2}{(\alpha\beta T + 1 - s)((\alpha\beta - 1)T + 1) + \beta(1-T)} \\
=& \frac{\begin{array}{c}(1-s)((\alpha\beta - 1)T + 1)(1 - r + \alpha T)((\alpha\beta - 1)T + 1) \\ +\beta(1-T)(1 - r + \alpha T)((\alpha\beta - 1)T + 1)\end{array}}{(\alpha\beta T + 1 - s)((\alpha\beta - 1)T + 1) + \beta(1-T)} \\
&+ \frac{(1-s)((\alpha\beta - 1)T + 1)(1-T) - (\alpha\beta T + 1 - s)((\alpha\beta - 1)T + 1)(1-T)}{(\alpha\beta T + 1 - s)((\alpha\beta - 1)T + 1) + \beta(1-T)}.
\end{aligned}
$$

Now by the above inequalities we have

$$
\begin{aligned}
q(t)\left(e^{R(t)} - 1\right) \geq& \frac{(1-s)(1 - r + \alpha T)((\alpha\beta - 1)T + 1) + \beta(1-T)(1 - r + \alpha T)}{(\alpha\beta T + 1 - s)((\alpha\beta - 1)T + 1) + \beta(1-T)} \\
&+ \frac{(1-s)(1-T) - (\alpha\beta T + 1 - s)(1-T)}{(\alpha\beta T + 1 - s)((\alpha\beta - 1)T + 1) + \beta(1-T)} \\
=& \frac{(1-s)(1 - r + \alpha T)((\alpha\beta - 1)T + 1) + \beta(1-T)(1-r)}{(\alpha\beta T + 1 - s)((\alpha\beta - 1)T + 1) + \beta(1-T)}.
\end{aligned}
$$

From $s < 1$, $r < 1$, $\alpha\beta - 1 > -1$ and $0 < T < 1$ we can deduce that

$$
\frac{(1-s)(1 - r + \alpha T)((\alpha\beta - 1)T + 1) + \beta(1-T)(1-r)}{(\alpha\beta T + 1 - s)((\alpha\beta - 1)T + 1) + \beta(1-T)} > 0.
$$

Therefore, $q(t)(e^{R(t)} - 1) > 0$. Now since $q(t) > 0$, we deduce that $e^{R(t)} - 1 > 0$, which implies $x_{-1} - x_0 = R(t) > 0$. This proves $x_{-1} > x_0$. Now since X_∞ is a subset of the set of points on or under LC_0^1, LC_{-1}^1 does not intersect X_∞ and by the argument we stated before, we deduce that ∂X_∞ is invariant. ■

We may now put together Lemmas 3.10 and 3.11 to obtain the analogue of Theorem 3.1 for the Ricker map:

Theorem 3.2: *For any $0 < r,\ s < 1$ and $\alpha, \beta > 0$ and f is the log-scaled Ricker map, X_∞ defined in (22) is invariant and convex, ∂X_∞ is invariant and attracts $\mathbb{R}^2$.*

Proof: All that is left to do is show that ∂X_∞ is attracting. It is proven that if $r, s < 2$, then every non-trivial orbit converges to one of the non-zero fixed points (see [5]). The possible non-zero fixed points on the x or y axis are the same as for the Leslie–Gower model with $r-1$ replaced by r and $s-1$ replaced by s. So we may use the same method as used for the log-scaled Leslie–Gower map to show attraction to ∂X_∞. ■

From this we obtain the following improvement on the known conditions

$$r + s < 1 + rs(1 - \alpha\beta) < 2,$$

(e.g. [8,9,17,18]) for the existence of a carrying simplex for the Ricker model. These inequalities fail for some $\alpha, \beta > 0$, when $r, s < 1$, namely those α, β that satisfy $r, s < 1$ and $\alpha\beta \geq (1 - 1/r)(1 - 1/s)$. However, Lemma 3.11 also shows that the unscaled Ricker map is retrotone on $[0, r] \times [0, s]$ when $r, s < 1$ so we obtain

Corollary 3.1: *When $r,\ s < 1$ and $\alpha, \beta > 0$ the Ricker map $(x, y) \in \mathbb{R}^2_+ \mapsto (xe^{r-x-\alpha y}, ye^{s-y-\beta x})$ has a (compact) carrying simplex.*

Proof: In the absence of asymptotic completeness in Theorem 3.2, we apply standard results on retrotone systems (e.g. [9,11,17,18]). ■

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