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New stability theory of model predictive control: modified stage cost approach

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ABSTRACT

This paper presents a promising new approach to establish the stability of finite receding horizon control with a terminal cost. Departing from the traditional approaches of using the property of the terminal cost or relaxed Lyapunov inequalities, this paper establishes its stability based on the property of a modified stage cost. First, we rotate the stage cost with the terminal cost. Then a one-step optimisation problem is defined based on this augmented stage cost. It is shown that a slightly modified Model Predictive Control (MPC) algorithm is stable if the value function of the augmented one-step cost (OSVF) is a Control Lyapunov Function (CLF). Stability for MPC algorithms with zero terminal cost or even negative terminal cost can be unified with this new approach. Combining it with the existing MPC stability theories, we are able to significantly relax the stability requirement on MPC and extend the stabilising MPC design space to the region that no existing MPC stability theories can cover. The proposed stage cost-based approach will help to further reduce the gap between stability theory and practical applications of MPC and other optimisation-based control methods.

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Finite horizon; optimisation; stability; model predictive control; constraints

1. Introduction

It is well known that stability of Receding Horizon Control (RHC), also known as Model Predictive Control (MPC), with a finite horizon can not be guaranteed in general, regardless of control horizon N and/or selection of a suitable stage cost, except for special cases (e.g. with an equality terminal constraint Chen & Shaw, 1982; Kwon & Pearson, 1977). The most widely used approach to guarantee the stability of a receding horizon controller is to add a terminal cost, $V_f(x_{k+N\,|\,k})$, in its cost function where $x_{k+N\,|\,k}$ is the terminal state in the prediction horizon at the time instant k (e.g. Chen et al., 2000; Fontes, 2001; Mayne et al., 2000; Primbs et al., 1999). Consequently, a well-established sufficient condition is that the system $x^+ = f(x,u)$ is stable under an MPC algorithm if, for any x different from 0, there exists some control u in the admissible set $\mathbb{U}$ such that

$$-V_f(x) + l(x,u) + V_f(f(x,u)) < 0 \tag{1}$$

where $l(x,u)$ is the stage cost. That is, a terminal cost $V_f(x_{k+N\,|\,k})$ that covers the cost-to-go (i.e. from the terminal state to the final equilibrium) shall be added into the cost function for on-line optimisation. Then together with other terminal ingredients (terminal constraints and a terminal controller), stability of a MPC scheme can be guaranteed.

Despite the elaborate and elegant theory of the terminal cost-based stability guaranteed MPC algorithms, as well explained in the survey paper (Grüne, 2013), there is a strong motivation to relax or completely get rid of these terminal ingredients so reduce the conservatism of the terminal cost-based stability guarantee framework. Significant effort and progress have been made in this aspect; for example, (Faulwasser et al., 2018; Grimm et al., 2005; Grüne & Pannek, 2017; Grüne & Rantzer, 2008; Jadbabaie & Hauser, 2005). Seminal work in Jadbabaie and Hauser (2005) presents a method to find a generalised terminal cost that is not necessary to be a Control Lyapunov Function (CLF) if certain property holds. Stability for unconstrained discrete-time systems under MPC algorithms was established in Grimm et al. (2005), where it does not require the terminal cost to be a local CLF if certain assumptions are satisfied.

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Most notably, to completely get rid of terminal ingredients including both terminal constraints and terminal cost, a variety of conditions have been developed based on relaxed Lyapunov inequalities including seminal works in Grimm et al. (2005), Grüne and Rantzer (2008), Grüne (2009) and Jadbabaie and Hauser (2005). In this approach, the optimal value function of the online optimisation involved in MPC is used to define a Lyapunov inequality, and stability can be established based on this condition, e.g. (Angeli et al., 2016; Faulwasser et al., 2018; Grüne, 2013; Grüne & Pannek, 2017). This normally requires that the horizon is *sufficiently large*, or equality terminal constraints have to be included. For example, stability can be achieved if the receding horizon is sufficiently long even in the case when the terminal cost is zero and no terminal constraints (Grüne, 2009; Grüne & Rantzer, 2008). Various methods have been investigated to give a tighter estimate of the minimum horizon required for satisfying this condition and then guaranteeing the stability of the corresponding MPC algorithms (Grimm et al., 2005; Grüne, 2009; Grüne & Rantzer, 2008). Recently, this approach was extended to the case of the MPC algorithm with a positive semi-definite (detectable) stage cost in Köhler et al. (2023). In addition, a self-tuning terminal cost is introduced to improve the average performance of MPC algorithms in Müller et al. (2013).

Due to widely spread application of MPC, recently the research in MPC stability mainly focuses on new types of MPC algorithms, most notably, economic MPC, data-driven approaches and distributed MPC arising in networked systems. The existing MPC stability analysis tools have been extended to provide stability-guaranteed MPC schemes in these new applications; see (Berberich et al., 2020, 2021; Bongard et al., 2022; Faulwasser et al., 2018; Müller & Allgöwer, 2017). For example, with the advances of machine learning and data sciences, data-driven MPC is a hot topic, attracting significant attention recently. Stability-guaranteed data-driven MPC was established first with terminal equality constraints in Berberich et al. (2020) and then extended to the use of terminal ingredients in Berberich et al. (2021). Very recently, the stability of data-driven MPC without stabilising terminal ingredients was also established in Bongard et al. (2022) under the condition that the horizon is sufficiently large. Another direction is to relax the requirement on the stage cost in the cost function, which gives rise to a group of new MPC algorithms, termed as Economic MPC (EMPC) (Faulwasser et al., 2018; Müller & Allgöwer, 2017). The key idea in establishing the stability of EMPC is to rotate the stage cost with a storage function to transform the original MPC algorithm with an indefinite stage cost to an equivalent stabilising MPC. Furthermore, it is also noted that, based on the dissipativity condition, a general storage function was recently defined in EMPC which is referred to as a control dissipativity function in Lazar (2021) and control storage function in Dong and Angeli (2017). Table 1 in Faulwasser et al. (2018) provides a nice summary of the state-of-the-art about tracking MPC and EMPC stability. Please refer to books and survey papers for a more comprehensive overview of the topic, e.g. (Faulwasser et al., 2018; Grüne, 2013; Rawlings & Mayne, 2009) and extensive references therein.

A natural question arising is, what happens if condition (1) is violated, or more precisely, if there does not exist an admissible control $u \in \mathbb{U}$ such that condition (1) holds? Formally, this can be represented as, for *all* $u \in \mathbb{U}$ and $x \neq 0$,

$$l_f(x,u) := -V_f(x) + l(x,u) + V_f(f(x,u)) > 0 \quad (2)$$

This question is much more interesting and challenging. This is because numerous *counterexamples* exist where condition (2) holds but the resulted MPC is unstable, e.g. (Chen, 2022; Kouvaritakis et al., 2002). Therefore, any stability condition to be developed must exclude these cases where condition (2) is satisfied but the resultant MPC algorithm may be unstable. So far, there is little work devoted to directly exploring the space defined by condition (2) which is *opposite* to the design space defined by (1). It shall be highlighted that condition (2) also includes the case that $V_f(x) = 0$, i.e. no terminal cost (Chen, 2022). In the case of zero terminal cost, condition (2) is satisfied as long as the stage cost, $l(x,u)$, is positive definite.

Condition (2) can be considered as a dissipativity inequality condition with a storage function $-V_f(x)$ (possibly being negative) (Willems, 1971; Zanon et al., 2014). It is equivalent to other well-known conditions such as Frequency Domain Inequalities (FDI) (Willems, 1971). Most notably, it is argued in Zanon et al. (2014) that the existence of $V_f(x)$ to satisfy condition (2) is a necessity for a stabilising LQR with $l(x,u)$ as the stage cost and infinite horizon. Instead, we are interested in finding out under what conditions an MPC algorithm with a terminal cost $V_f(x)$ satisfying (2) can guarantee stability.

The motivation of this paper is to answer the question: *under what condition is a finite receding horizon controller satisfying (2) stable?* Condition (2) is equivalent to $m(x) > 0$, where

$$m(x) := \min_{u \in \mathbb{U}} l_f(x, u) \quad (3)$$

is referred as the *One-Step Value Function (OSVF)* in this paper. It plays a key role in establishing stability of MPC in this paper. The main contribution of this paper is to develop a completely new stability theory to enable us to look into this a untouched design region in MPC defined by condition (2). To this end, we develop a new MPC stability analysis technique based on the properties of the modified stage cost, i.e. OSVF, and present a slightly modified MPC algorithm which is stable *as long as the corresponding OSVF is a CLF*.

Condition (2) is opposite to condition (1) used in establishing stability in the current terminal cost-based MPC literature for more than two decades (Grüne & Pannek, 2017; Lee, 2011; Mayne et al., 2000). The design space considered in our stage cost-based approach is *complementary* to that of the existing terminal cost-based MPC framework. That is, all cases covered by our new conditions do not satisfy the existing stability conditions developed based on (1). Our conditions allow stability to be established for MPC with zero or even a negative terminal cost (see numerical examples). Compared with the relaxed Lyapunov inequalities approach (Grimm et al., 2005; Grüne, 2009; Grüne & Rantzer, 2008; Jadbabaie & Hauser, 2005), our approach also covers the case of without terminal cost (i.e. zero terminal cost), but terminal constraints are required in our approach. However, the length of the predictive horizon in our approach could be only a few step which is important for computationally intensive MPC applications. The most distinguish feature of our approach is that, different from the terminal weight-based or the relaxed Lyapunov function-based approaches, the stability of MPC is established based on *the properties of a modified stage cost*. Our stage cost-based technique offers a complementary, promising approach, that is able to guarantee MPC stability in design space that has not been explored yet (i. e. defined by condition (2)). By combining this new analysis approach with the existing rich set of available stability conditions, we are in a much stronger position to analyse the stability of MPC and other optimisation-based control methods and design stability-guaranteed optimisation-based algorithms with more freedom and space. This is the main motivation behind this paper.

There are two key ideas behind the proposed stage-cost-based complementary approach. The first is to define a new stage cost, $l_f(x, u)$, as in (2) by rotating the stage cost $l(x, u)$ with the terminal cost $V_f(x)$. Rotating the stage cost has been widely used in economic MPC (EMPC) where the stage cost is rotated by a storage function, $\lambda(x)$, normally being non-negative, for example, see (Angeli et al., 2011; Diehl et al., 2010; Faulwasser et al., 2018). It is interesting to notice that condition (2) is equivalent to the widely used dissipativity inequality when the storage function $\lambda(x)$ is specifically chosen as a (possibly negative) $-V_f(x)$. This technique is used as an approach to establish stability and analyse the performance of EMPC. It is very important to notice that the storage function in the dissipativity inequality does not affect the behaviour and performance of the EMPC algorithm. In contrast, the stage cost is rotated by the terminal cost to be designed in our approach, and the choice of the terminal cost affects the performance and stability of the MPC algorithm since it is added to the cost function for online optimisation. We will further elaborate on the difference between these two approaches in Remark 2.1.

The second idea is to explore the observation that the change of the initial state-related cost in a cost function involved in the online optimisation of MPC does not change the optimal control sequence and the behaviour of the MPC algorithm (Chen, 2022). By combining these two ideas, we are able to modify the MPC cost function without changing its behaviour. Fundamentally, since the optimal value function of the modified cost is used as a Lyapunov function candidate to establish its stability, the combination of these two ideas enables a wide range of possible Lyapunov function candidates to be used to establish stability for the same MPC algorithm. This breaks down the rigid link between the cost function used in the optimisation of an MPC algorithm and the optimal value function used as a Lyapunov function candidate in the current approaches; for example, see (Chen, 2022; Lee, 2011; Mayne et al., 2000). The fact that changing initial state cost does not alter the optimisation involved in MPC has been exploited in Faulwasser and Bonvin (2015) and Grüne (2013). By further exploiting this property in this paper, we are able to explore new design space for stability-guaranteed MPC that was not possible previously.

Technically, to establish stability by virtue of the OSVF being a CLF, we have to slightly modify the MPC algorithm. That is, rather than using a fixed terminal constraint as in the current terminal cost-based framework, we construct a *contractive terminal set* by making use of the property that OSVF is a CLF. The idea of using a contractive terminal set can be found in de Oliveira Kothare and Morari (2000); however, the whole optimal control sequence (not only the first element) is applied to the system in de Oliveira Kothare and Morari (2000). This does not conform with the MPC setup in this paper. Furthermore, it is also noticed that an adaptive terminal cost is considered in Müller et al. (2013) to improve performance in comparison with a fixed terminal cost. It shows that a smaller terminal cost leads to a better closed-loop average performance in some cases. This also highlights the benefit of the stability condition developed in this paper where stability is established for MPC with a terminal cost smaller than the optimal cost-to-go.

This paper is organised as follows. In Section 2, for a general terminal cost-based MPC problem, we define an equivalent MPC formulation by rotating its stage cost with the terminal cost. Based on this new augmented stage cost and some mild assumptions, a new MPC algorithm with stability guarantee is proposed in Section 3. It is facilitated by a new stage cost-based MPC stability analysis technique. In Section 4, a procedure for finding a terminal cost satisfying the new stability conditions is developed. Results are provided by two illustrative examples, giving further insight into the proposed approach, in Section 5. Finally, Section 6 concludes the paper.

Notation: $\mathbb{I}$ and $\mathbb{R}$ are integers and real numbers, respectively, where subscripts may be added to give specific ranges. For any vector $x \in \mathbb{R}^n$, $|x|$ denotes the 2-norm and $|x|_P^2$ is defined by $|x|_P^2 := x^T P x$, where $P \in \mathbb{R}^{n\times n}$ is a symmetric matrix, but not necessary to be positive-definite here. The superscript * is used to indicate the optimal solution, the optimal control sequence or the state under the optimal control sequence.

2. Augmented stage cost and equivalent MPC problems

Consider a nonlinear discrete-time system described by

$$x_{k+1} = f(x_k, u_k), \quad k \in \mathbb{I}_{\geq 0} \tag{4}$$

or, more succinctly $x^+ = f(x, u)$, where x or $x_k \in \mathbb{X}$ is the current state, u or $u_k \in \mathbb{U}$ is the current control, and x^+ or x_{k+1} is the successor state. It is assumed in this paper that

(A1) $f(x, u) : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is continuous and $f(0, 0) = 0$. $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ are closed and contain the origin in the interior.

We start from a classic finite horizon optimisation problem, with the cost function

$$J(x_k, \mathbf{u}_k) := \sum_{i=0}^{N-1} l(x_{k+i\,|\,k}, u_{k+i\,|\,k}) + V_f(x_{k+N\,|\,k}), \tag{5}$$

where $x_{k\,|\,k} = x_k$ and $x_{k+i+1\,|\,k} = f(x_{k+i\,|\,k}, u_{k+i\,|\,k})$, $i \in \mathbb{I}_{0:N-1}$; the control and state sequences are[1] $\mathbf{u}_k := (u_{k\,|\,k}, \ldots, u_{k+N-1\,|\,k})$ and $\mathbf{x}_k := (x_{k\,|\,k}, \ldots, x_{k+N\,|\,k})$, subject to the constraints $\mathbb{U}$ and $\mathbb{X}$, $N \geq 1$ is the length of the control horizon, $l(x, u)$ and $V_f(x)$ are the stage cost and the terminal cost respectively. This satisfies the following assumption.

(A2) $l(x, u) : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ and $V_f(x) : \mathbb{X} \to \mathbb{R}$ are continuous. Furthermore, $l(0, 0) = 0$ and $V_f(0) = 0$.

It shall be pointed out that there is no requirement that $l(x, u)$ or $V_f(x)$ shall be positive definite in this paper.

Following the conventional MPC framework, the first control action of the optimal control sequence $\mathbf{u}_k^*$ is applied into the system, $u_k = u_{k\,|\,k}^*$, which implies that

$$x_{k+1} = f(x_k, u_k) = f\left(x_{k\,|\,k}, u_{k\,|\,k}^*\right) = x_{k+1\,|\,k}^*. \tag{6}$$

Since the initial state $x_{k\,|\,k}$ is fully independent of any control sequence, any cost that is *only* associated with the initial state, $x_{k\,|\,k}$, is not changed by a control sequence, hence completely independent from the optimisation process in MPC.

Motivated by this intuitive but important observation, we rewrite the cost function as

$$\begin{aligned}
J(x_k,\mathbf{u}_k) &= \sum_{i=0}^{N-1} l(x_{k+i\,|\,k},u_{k+i\,|\,k}) + V_f(x_{k+N\,|\,k}) \\
&= V_f(x_{k\,|\,k}) \underbrace{-V_f(x_{k\,|\,k}) + l(x_{k\,|\,k},u_{k\,|\,k}) + V_f(x_{k+1\,|\,k})}_{l_f(x_{k\,|\,k},u_{k\,|\,k})} \\
&\quad \vdots \\
&\quad \underbrace{-V_f(x_{k+N-1\,|\,k}) + l(x_{k+N-1\,|\,k},u_{k+N-1\,|\,k}) + V_f(x_{k+N\,|\,k})}_{l_f(x_{k+N-1\,|\,k},u_{k+N-1\,|\,k})} \\
&= V_f(x_{k\,|\,k}) + \sum_{i=0}^{N-1} l_f(x_{k+i\,|\,k},u_{k+i\,|\,k}), \qquad (7)
\end{aligned}$$

where $l_f(x,u)$ is the new augmented stage cost and has been defined in (2) by rotating the stage cost with the terminal cost. It shall be highlighted that the information on the system dynamics is now embedded in this new stage cost. By ignoring the initial one, $V_f(x_{k\,|\,k})$, since it does not affect the solution of the optimisation, we can define the new cost function as follows

$$\mathcal{J}(x_k,\mathbf{u}_k) := \sum_{i=0}^{N-1} l_f(x_{k+i\,|\,k},u_{k+i\,|\,k}). \qquad (8)$$

Considering the optimisation problems based on different cost functions, $J(x_k,\mathbf{u}_k)$ and $\mathcal{J}(x_k,\mathbf{u}_k)$, with the same initial state x_k and the same constraints, the optimal trajectories should be the same, regardless of the different value functions. That is

$$\begin{aligned}
\min_{\mathbf{u}_k} \mathcal{J}(x_k,\mathbf{u}_k) &= -V_f(x_k) + \min_{\mathbf{u}_k} J(x_k,\mathbf{u}_k) \\
\arg\min_{\mathbf{u}_k} \mathcal{J}(x_k,\mathbf{u}_k) &= \arg\min_{\mathbf{u}_k} J(x_k,\mathbf{u}_k).
\end{aligned} \qquad (9)$$

Considering (9), the design and analysis on the conventional MPC are equivalent to that with a new cost function $\mathcal{J}(x_k,\mathbf{u}_k)$, which is given in a form *without any terminal cost*. Therefore, the problem is converted into stability analysis of a modified MPC problem without a terminal cost.

Remark 2.1: The stage cost rotation technique has been widely used in EMPC to establish its stability; e.g. (Angeli et al., 2011; Diehl et al., 2010; Zanon & Faulwasser, 2018; Zanon et al., 2014). More specifically, a dissipativity inequality is defined with a nonnegative storage function $\lambda(x)$ (see Definition 3.1 in Faulwasser et al., 2018), that is,

$$-\lambda(x) + \lambda(f(x,u)) - l(x,u) \le -l(x_s,u_s) \qquad (10)$$

is satisfied for all x and u, where x_s and u_s are in the steady state operating condition and $l(x_s,u_s)$ can be treated as zero when the equilibrium point is at the origin. Then, to a large extent, stability of EMPC can be established by resorting to the standard MPC stability theories, i.e. extending the terminal cost-based approach or the relaxed Lyapunov inequalities approach (Faulwasser et al., 2018; Faulwasser & Zanon, 2018; Zanon & Faulwasser, 2018; Zanon et al., 2014).

Condition (2) can be regarded as a dissipativity inequality with a storage function $-V_f(x)$. However, there are 3 differences between these two approaches. First and most importantly, the main purpose of the introduction of the storage function $\lambda(x)$ in EMPC is to transform an indefinite MPC problem into an *equivalent* positive definite MPC problem or alike that is easier to analyse. The transformation must *not* change the behaviour of the MPC algorithm of concern and the choice of the storage function $\lambda(x)$ does *not* affect the behaviours of the original MPC algorithm. In contrast, condition (2) explicitly depends on the terminal cost $V_f(x)$. Since the terminal cost is added to the cost function for online optimisation, the choice of the terminal cost does affect the behaviour of the associated MPC algorithm such as stability and performance. In fact, our purpose is to determine how to choose the terminal cost such that the resultant MPC algorithm is stable by making use of condition (2). Secondly, the approach used in determining stability is different between our method and the current EMPC. The current EMPC stability analysis still resorts to the well established terminal weight-based stability theory after transforming the original EMOC problem using a storage function, while our approach is based on a modified stage cost. When the storage storage function $\lambda(x)$ is chosen as the terminal cost $-V_f(x)$, the modified terminal cost becomes $\tilde{V}_f(x) = V_f(x) + \lambda(x) = 0$ for any $V_f(x)$. By following the procedure of establishing stability in the current EMPC, the terminal weight-based MPC stability theory is not applicable any more since the terminal cost $\tilde{V}_f(x)$ of the transformed problem is zero, not a CLF as required. Thirdly, it shall be also highlighted that our MPC algorithm employs a contractive terminal constraint that is constructed based the property of OSVF, which is different from conventional EMPC (or general MPC) algorithms that have a fixed terminal set. Essentially, we propose a new MPC algorithm. with stability guarantee.

Finally, we would like to highlight that our stage cost-based approach is naturally applicable to EMPC, i.e. MPC with an indefinite stage cost, without the necessity of rotating the stage cost with a storage function $\lambda(x)$ since the original stage cost $l(x,u)$ is not necessary to be positive definite in order to satisfy condition (2).

3. MPC algorithm and stability

3.1. New contractive MPC algorithm

We first define an auxiliary optimisation problem to minimise the augmented stage cost $l_f(x,u)$ in (2) or (7)

$$\begin{aligned} m(x) &:= \min_{u\in\mathbb{U}} l_f(x,u) \\ &= \min_{u\in\mathbb{U}} -V_f(x) + l(x,u) + V_f(x^+) \\ &\text{s.t. } x \in \Omega \subseteq \mathbb{X}, \quad u \in \mathbb{U}, f(x,u) \in \Omega, \end{aligned} \quad (11)$$

where $m(x) : \Omega \subseteq \mathbb{X} \mapsto \mathbb{R}$ is the One-Step Value Function (OSVF) of the augmented stage cost and Ω is a control invariant set, which is assumed to exist and contains the equilibrium.

It shall be highlighted that $m(x) > 0$ iff condition (2) holds. In other words, $m(x) > 0$ only if there does not exist *any* control $u \in \mathbb{U}$ such that condition (1) holds. We will investigate the design space where $m(x) > 0$ is positive definite. This design space is entirely complementary to that investigated by the existing terminal cost-based MPC stability theories and has not been explored previously.

We define the sublevel set of $m(x) > 0$ as

$$\Omega(\alpha) := \left\{x \in \mathbb{R}^n : m(x) \le \alpha\right\} \subseteq \Omega \subseteq \mathbb{X}, \quad (12)$$

where $\alpha \ge 0$. As briefly mentioned in the introduction, the proposed MPC in this paper is based on the condition that OSVF is a CLF. That is, the following assumption is imposed in this paper.

(A3) The OSVF $m(x)$ is a CLF for system (4) with respect to a set $\Omega(\alpha_0)$, i.e. for any $x \in \Omega(\alpha_0)$, there exist a control $u \in \mathbb{U}$ and $\mathcal{K}_\infty$ functions $\beta_1(\cdot), \beta_2(\cdot), \beta_3(\cdot)$ such that

$$\beta_1(|x|) \le m(x) \le \beta_2(|x|) \quad (13)$$

$$m(f(x,u)) - m(x) \le -\beta_3(|x|), \quad (14)$$

where $\alpha_0 > 0$.

Condition (13) is the restatement of condition (2), which, as pointed out previously, can be considered as a dissipativity inequality condition with a storage function $-V_f(x)$ (Willems, 1971; Zanon et al., 2014). This condition is also equivalent to other well-known conditions such as Frequency Domain Inequalities (FDI) (Willems, 1971). Most notably, Zanon et al. (2014) shows that the existence of $V_f(x)$ to satisfy condition (13) is a necessity for a stabilising LQR with $l(x,u)$ as the stage cost and an infinite horizon. It is noted that a positive semi-definite inequality, rather than the positive definite inequality (13), is considered in Willems (1971) and Zanon et al. (2014). Further discussion of Assumption **(A3)** and the procedure of calculating a suitable terminal cost to satisfy Assumption **(A3)** is given in Section 4.

Remark 3.1: The approach proposed in this paper is applicable to both tracking MPC and EMPC directly. It follows from Assumption **(A3)** that the stage cost $l(x,u)$ could be positive or negative as long as condition (2) is satisfied. In that sense, we also present a complementary stability condition for EMPC where the rotated terminal cost $\tilde{V}_f(x)$ does not satisfy the requirements in the current stability conditions, e.g. to be a CLF, or the horizon is required to be sufficiently large (Faulwasser et al., 2018).

Based on Assumption **(A3)**, we are now able to construct a contractive terminal set, which is the only difference from the conventional MPC with a terminal cost. The new algorithm is described below.

Algorithm:

Offline:

Step 1 For the given system (4) subject to the constraints $\mathbb{X}$ and $\mathbb{U}$, define the stage cost $l(x,u)$ and the terminal cost $V_f(x)$, and calculate terminal constraint α_0. Initialise the time as $k=0$ and a selected small reduce rate $\delta > 0$.

Online:

Step 2 At time k, measure the current state x_k. Solve the following optimisation problem

$$\begin{aligned} V(x_k, \alpha_k) &:= \min_{\mathbf{u}_k} \mathcal{J}(x_k, \mathbf{u}_k) \\ &= \min_{\mathbf{u}_k} \sum_{i=0}^{N-1} l_f\left(x_{k+i\,|\,k}, u_{k+i\,|\,k}\right) \\ \text{s.t. } & x_{k\,|\,k} = x_k \\ & x_{k+i+1\,|\,k} = f(x_{k+i\,|\,k}, u_{k+i\,|\,k}) \\ & x_{k+i\,|\,k} \in \mathbb{X}, u_{k+i\,|\,k} \in \mathbb{U}, \quad i \in \mathbb{I}_{0:N-1} \\ & x_{k+N\,|\,k} \in \Omega(\alpha_k) \end{aligned} \tag{15}$$

and obtain the optimal solution $\mathbf{u}_k^*$ and $\mathbf{x}_k^*$. If the optimisation problem above is not feasible, update the terminal set and the associated α_k with a reduced $\delta > 0$ until the optimisation problem becomes feasible. Apply the optimal control $u_k = u^*_{k\,|\,k}$ to system (4).

Step 3 Denote the minimum between $m\left(x^*_{k+N\,|\,k}\right)$ and $m\left(x^*_{k+1\,|\,k}\right)$ as

$$\underline{m}_k := \min\left\{m\left(x^*_{k+1\,|\,k}\right), m\left(x^*_{k+N\,|\,k}\right)\right\}. \tag{16}$$

Set $\alpha_{k+1} = 0$ if $\alpha_k = 0$; otherwise, update the terminal set by the following rule

$$\alpha_{k+1} = \begin{cases} \underline{m}_k - \delta, & \text{if } \underline{m}_k \geq \delta \\ 0, & \text{if } \underline{m}_k < \delta, \end{cases} \tag{17}$$

Step 4 $k \leftarrow k+1$ and go to **Step 2**.

Remark 3.2: Compared with the conventional MPC setup, Step 3 is the *only* difference. To establish stability, there are also three key terminal elements in this algorithm: terminal cost, terminal control and terminal constraints. Instead of using a fixed terminal constraint, a contractive terminal constraint is constructed in Step 3 where one-step ahead optimisation is involved in (16) and (17). It will be shown that it is always feasible to construct a contractive terminal constraint if the corresponding OSVF is a CLF for a given terminal cost. That is, Step 3 and the new MPC algorithm are always feasible as long as it is feasible at the beginning. We will discuss how to choose a terminal cost such that the corresponding OSVF is a CLF in Section 4.

Remark 3.3: One of the drawbacks in the current terminal cost-based MPC framework is how to strike a good balance between performance and stability, particularly for nonlinear or uncertain systems. For a given stage cost, it is quite difficult to estimate the optimal cost-to-go for these systems. Since, in order to achieve stability, it requires the terminal cost to cover the optimal cost-to-go, quite often a conservative terminal cost function has to be employed to ensure the unknown optimal value function is covered. This may lead to poor performance. The total cost to be optimised in MPC consists of the summed stage cost and the terminal cost. If the chosen terminal cost is much larger than the summed stage cost, the optimisation process actually puts more effort on the terminal cost since the former carries a higher weight, despite the stage cost may represent the performance requirements better. This is confirmed by the claim and the observation in Müller et al. (2013), showing that a smaller terminal cost leads to a better closed-loop average performance in some cases. The second example in Section 5 also supports this claim. Our approach avoids the problem of using a large terminal cost in this existing terminal cost-based MPC algorithms by exploring the design space where the terminal cost is less than the optimal cost-to-go, even if it could be zero.

The main result is given by the following theorem while the detailed proof is given by the subsequent sections.

Theorem 3.1: *Suppose that*

- *Assumptions* **(A1)**–**(A3)** *are satisfied,*
- *the optimisation problem (15) is feasible at time $k = 0$,*
- *the parameter α_0 is chosen such that the terminal set $\Omega(\alpha_0) \subseteq \Omega$*

Then, the closed-loop system with the proposed MPC is recursively feasible and, eventually, asymptotically stable.

3.2. Recursive feasibility

The recursive feasibility of the proposed MPC is mainly based on the condition of OSVF being a CLF, as in Assumption **(A3)**.

Proof of the recursive feasibility in Theorem 3.1: Supposing that the optimisation (15) is feasible at time

$k \in \mathbb{I}_{\geq 0}$, the optimal control and state sequences exist, which can be denoted as

$$\mathbf{u}_k^* = \left(u_{k\,|\,k}^*, \ldots, u_{k+N-1\,|\,k}^*\right)$$
$$\mathbf{x}_k^* = \left(x_{k\,|\,k}^*, \ldots, x_{k+N\,|\,k}^*\right),$$

where the terminal state satisfies that

$$x_{k+N\,|\,k}^* \in \Omega(\alpha_k) \Leftrightarrow m\left(x_{k+N\,|\,k}^*\right) \leq \alpha_k. \tag{18}$$

In what follows, we will construct the feasible control and state sequences at time $k+1$ based on the current optimal sequences. The case when the terminal set shrinks to the origin is ignored as it degenerates into the conventional MPC with an equality terminal constraint. It is straightforward to show its recursive feasibility. To make the discussion clear, we use the superscripts, a and b, to represent two cases due to the different results of (16).

In the case of $m(x_{k+1\,|\,k}^*) \geq m(x_{k+N\,|\,k}^*)$, based on (17), if we choose $\delta \in (0, \beta_3(|x_{k+N\,|\,k}^*|)]$, then

$$\begin{aligned}\alpha_{k+1} &= m\left(x_{k+N\,|\,k}^*\right) - \delta \\ &\geq m\left(x_{k+N\,|\,k}^*\right) - \beta_3\left(\left|x_{k+N\,|\,k}^*\right|\right).\end{aligned} \tag{19}$$

By Assumption **(A3)** and since $x_{k+N\,|\,k}^* \in \Omega(\alpha_k)$, there exists a control $u_{k+N\,|\,k}^a \in \mathbb{U}$, $x_{k+N+1\,|\,k}^a = f(x_{k+N\,|\,k}^*, u_{k+N\,|\,k}^a)$ such that

$$\begin{aligned}m\left(x_{k+N+1|k}^a\right) \leq m\left(x_{k+N|k}^*\right) - \beta_3\left(\left|x_{k+N|k}^*\right|\right) \leq \alpha_{k+1} \\ \Rightarrow x_{k+N+1|k}^a \in \Omega(\alpha_{k+1}).\end{aligned} \tag{20}$$

The feasible sequences at time $k+1$ can be constructed as

$$\mathbf{u}_{k+1}^a = \left(u_{k+1\,|\,k}^*, \ldots, u_{k+N-1\,|\,k}^*, u_{k+N\,|\,k}^a\right)$$
$$\mathbf{x}_{k+1}^a = \left(x_{k+1\,|\,k}^*, \ldots, x_{k+N\,|\,k}^*, x_{k+N+1\,|\,k}^a\right).$$

In the case of $m(x_{k+1\,|\,k}^*) < m(x_{k+N\,|\,k}^*)$, and noting (18), we have that

$$\begin{aligned}m\left(x_{k+1\,|\,k}^*\right) < m\left(x_{k+N\,|\,k}^*\right) \leq \alpha_k \\ \Rightarrow x_{k+1\,|\,k}^* \in \Omega(\alpha_k).\end{aligned} \tag{21}$$

Based on (17), we have that

$$\alpha_{k+1} \leq m\left(x_{k+1\,|\,k}^*\right) < m\left(x_{k+N\,|\,k}^*\right) \leq \alpha_k. \tag{22}$$

By Assumption **(A3)**, there exists a control $u_{k+1\,|\,k}^b \in \mathbb{U}$, $x_{k+2\,|\,k}^b = f(x_{k+1\,|\,k}^*, u_{k+1\,|\,k}^b)$ such that

$$\begin{aligned}m\left(x_{k+2\,|\,k}^b\right) &\leq m\left(x_{k+1\,|\,k}^*\right) - \beta_3\left(\left|x_{k+1\,|\,k}^*\right|\right) \\ &\leq m\left(x_{k+1\,|\,k}^*\right) - \delta \\ &= \alpha_{k+1}, \quad \text{if } \delta \leq \beta_3\left(\left|x_{k+1\,|\,k}^*\right|\right) \\ &\Rightarrow x_{k+2\,|\,k}^b \in \Omega(\alpha_{k+1}) \subseteq \Omega(\alpha_k).\end{aligned} \tag{23}$$

Using Assumption **(A3)** again and noting (23), there exists a $u_{k+2\,|\,k}^b \in \mathbb{U}$, $x_{k+3\,|\,k}^b = f(x_{k+2\,|\,k}^b, u_{k+2\,|\,k}^b)$ such that

$$\begin{aligned}m\left(x_{k+3\,|\,k}^b\right) &\leq m\left(x_{k+2\,|\,k}^b\right) - \beta_3\left(\left|x_{k+2\,|\,k}^b\right|\right) \\ &\leq m\left(x_{k+2\,|\,k}^b\right) \leq \alpha_{k+1} \\ &\Rightarrow x_{k+3\,|\,k}^b \in \Omega(\alpha_{k+1}).\end{aligned} \tag{24}$$

Combing (23) and (24) together, we have that the set $\Omega(\alpha_{k+1})$ is also control invariant and hence there exists a control sequence $u_{k+i\,|\,k}^b \in \mathbb{U}$, $i \in \mathbb{I}_{0:N}$ such that

$$x_{k+1+i\,|\,k}^b = f\left(x_{k+i\,|\,k}^b, u_{k+i\,|\,k}^b\right) \in \Omega(\alpha_{k+1}). \tag{25}$$

The feasible sequences of this case can be conducted as

$$\mathbf{u}_{k+1}^b = \left(u_{k+1\,|\,k}^b, \ldots, u_{k+N-1\,|\,k}^b, u_{k+N\,|\,k}^b\right)$$
$$\mathbf{x}_{k+1}^b = \left(x_{k+1\,|\,k}^*, \ldots, x_{k+N\,|\,k}^b, x_{k+N+1\,|\,k}^b\right).$$

In conclusion, based on the above discussion on the two cases, the proposed algorithm is recursively feasible, which completes the proof. ■

Recursive feasibility guarantees that once the proposed algorithm is feasible at time $k=0$, the optimisation algorithm associated with MPC and the closed-loop system is always feasible, which implies that all the signals in the control system (e.g. α_k) are well defined, even as time goes to infinity.

3.3. Asymptotic stability

To establish stability in Theorem 3.1, several lemmas are necessary. Lemma 3.1 shows the upper bounds of the new value function $V(x, \alpha)$ in (15) while Lemma 3.2 establishes the condition for its monotonicity.

Lemma 3.1: *Suppose that Assumption* **(A3)** *holds. Then, for any feasible state x of the optimisation problem* (15) *with $\Omega(\alpha) \subseteq \Omega$, there exists a $\mathcal{K}_\infty$ function $\beta_4(\cdot)$ such that $V(x, \alpha) \leq \beta_4(|x|)$.*

Proof: We first consider a simple case, where $x \in \Omega(\alpha)$. Then, we replace the optimisation problem (15) by a suboptimal control problem, i.e. splitting it into N one-step optimisation problems (11). Recursively solving these N one-step optimisation problems as in (11) and using the optimal states and inputs as the feasible ones of the optimisation problem (15), it will give that

$$V(x, \alpha) \leq N\beta_2(|x|), \forall x \in \Omega(\alpha). \tag{26}$$

Therefore, $V(x, \alpha)$ is continuous at zero as $0 \in \Omega(\alpha)$. Following the similar analysis given in Rawlings et al. (2017, Chapter 2.4.2), we know that the set of feasible states is closed and $V(x, \alpha)$ is locally bounded by it. By applying (Rawlings et al., 2017, Proposition B.25), the upper bound conclusion can be extended to any feasible state, which completes the proof. ∎

Lemma 3.2: *Suppose that there exists a control $u_{k+N\,|\,k} \in \mathbb{U}$ such that $x_{k+N+1\,|\,k} = f(x^*_{k+N\,|\,k}, u_{k+N\,|\,k}) \in \Omega(\alpha_{k+1})$. Then*

$$V(x_{k+1}, \alpha_{k+1}) - V(x_k, \alpha_k) \leq l_f\left(x^*_{k+N\,|\,k}, u_{k+N\,|\,k}\right) - l_f\left(x^*_{k\,|\,k}, u^*_{k\,|\,k}\right).$$

Proof: We denote the optimal control and state sequences at time k as

$$\mathbf{u}^*_k = \left(u^*_{k\,|\,k}, \ldots, u^*_{k+N-1\,|\,k}\right)$$
$$\mathbf{x}^*_k = \left(x^*_{k\,|\,k}, \ldots, x^*_{k+N\,|\,k}\right).$$

The value function $V(x_k, \alpha_k)$ can be rewritten as

$$V(x_k, \alpha_k) = \sum_{i=0}^{N-1} l_f\left(x^*_{k+i\,|\,k}, u^*_{k+i\,|\,k}\right). \tag{27}$$

If there exists a control $u_{k+N\,|\,k} \in \mathbb{U}$ such that $x_{k+N+1\,|\,k} = f(x^*_{k+N\,|\,k}, u_{k+N\,|\,k}) \in \Omega(\alpha_{k+1})$, we could construct the feasible sequences at time $k+1$ as

$$\mathbf{u}_{k+1} = \left(u^*_{k+1\,|\,k}, \ldots, u^*_{k+N-1\,|\,k}, u_{k+N\,|\,k}\right)$$
$$\mathbf{x}_{k+1} = \left(x^*_{k+1\,|\,k}, \ldots, x^*_{k+N\,|\,k}, x_{k+N+1\,|\,k}\right).$$

Thus, we have that

$$\begin{aligned} & V(x_{k+1}, \alpha_{k+1}) - V(x_k, \alpha_k) \\ & \quad \leq \mathcal{J}(x_{k+1}, \mathbf{u}_{k+1}) - V(x_k, \alpha_k) \\ & \quad = \mathcal{J}\left(x^*_{k+1\,|\,k}, \mathbf{u}_{k+1}\right) - V(x_k, \alpha_k) \\ & \quad = l_f\left(x^*_{k+N\,|\,k}, u_{k+N\,|\,k}\right) - l_f\left(x^*_{k\,|\,k}, u^*_{k\,|\,k}\right), \end{aligned} \tag{28}$$

which completes the proof. ∎

We are now ready to prove Theorem 3.1 for asymptotic stability of the proposed MPC algorithm. It is established through a two stage process. In the first stage, it is shown that the terminal set continuously shrinks to the origin under the constructed update rule. In the second stage, it reduces to an MPC algorithm with an equality constraint, and its asymptotic stability can also be established using $V(x, 0)$ as a Lyapunov function.

Proof of the asymptotic stability in Theorem 3.1.: Recursive feasibility established in Section 3.2 shows that

$$\alpha_{k+1} \leq \underline{m}_k - \delta \leq m\left(x^*_{k+N\,|\,k}\right) - \delta \leq \alpha_k - \delta, \tag{29}$$

which implies that the sequence $\{\alpha_k\}$ is continuously decreasing until reaching the origin since there exists a $\delta > 0$ such that the MPC algorithm is feasible with the updated terminal set defined by α_{k+1} satisfying (29) unless $x^*_{k+N\,|\,k} = 0$. The existence is guaranteed by Assumption **(A3)**, that is, $m(x)$ satisfies the condition (14) and is positive definite. Consequently, once the terminal constraint set approaches the origin, one has

$$x^*_{k+N\,|\,k} = 0. \tag{30}$$

It follows from Assumption **(A3)** that

$$l_f\left(x^*_{k+N\,|\,k}, u_{k+N\,|\,k}\right) = 0. \tag{31}$$

Together with Step 3 in the MPC algorithm, it implies $\alpha_{k+i} = 0$ for any $i \geq 1$ if $\alpha_k = 0$. Combining them with Lemma 3.2, we have

$$V(x_{k+1},0) - V(x_k,0) \leq -l_f\left(x^*_{k\,|\,k}, u^*_{k\,|\,k}\right) \leq -m(x_{k\,|\,k}) \leq -\beta_1(|x_{k\,|\,k}|). \tag{32}$$

Combining Lemma 3.1 and noting $V(x,0) \geq m(x) \geq \beta_1(|x|)$, the closed-loop system is asymptotically stable under the proposed MPC algorithm which completes the proof. ■

Remark 3.4: To avoid calculating α_k based on $\beta_3(x^*_{k+N\,|\,k})$ at each step, the proposed MPC algorithm is initiated with a fixed small $\delta > 0$ for updating the terminal constraint. If the online optimisation problem (15) becomes infeasible with the updated terminal constraints, we reduce $\delta > 0$ until the optimisation problem (15) becomes feasible with the updated α_k. Assumption **(A3)** guarantees such a small $\delta > 0$ exists unless $x^*_{k+N\,|\,k} = 0$. In numerical examples, we found δ is not necessary to be very small to ensure the MPC algorithm under an updated terminal constraint is feasible. For example, δ is chosen from 10^{-5} to 1 in our case study, the MPC algorithm is always feasible under this range *without rescaling*. This avoids the conservativeness caused by calculating $\beta_3(x)$, speeding up the convergence of the MPC algorithm and making the MPC algorithm switch to a terminal equality constraint quicker.

Remark 3.5: There are two stages in our proposed MPC scheme. At the first stage, the terminal constraint set contracts to the origin with the help of the property of $m(x)$ being a CLF. Once the terminal set contracts to the origin, it reduces to an MPC algorithm with an equality terminal constraint in the second stage. The convergence and the stability of an MPC algorithm with an equality terminal constraint are well established in the literature as shown in Chen and Shaw (1982) and Kwon and Pearson (1977). We also can establish the stability of the proposed MPC algorithm with the help of these results.

Remark 3.6: Similar to the existing MPC theories, asymptotic stability is established also by employing an optimal value function $V(x,0)$ as a Lyapunov function candidate in our approaches. However our approach is different from the existing ones in three key aspects, e.g. (Faulwasser et al., 2018; Grüne & Pannek, 2017; Mayne et al., 2000). First, the new value function $V(x,\alpha)$ is different from the original optimal value function obtained by optimising the original cost function (5) used in the existing MPC stability theories. This is not only because of the modification of the cost function by removing the cost associated with the initial state but also the contractive constraints in the online optimisation, caused by decreasing α_k. Secondly, the stability of the MPC algorithm is established in the existing MPC theories by exploiting the monotonicity of the optimal value function during the whole MPC control process. However, we are not able to establish the monotonicity of the value function, $V(x,\alpha)$, at the beginning of the contractive MPC algorithm proposed in this paper. Actually $V(x,\alpha)$ may increase if the terminal set shrinks quite quickly based on the proposed update rule as shown by the result in the second numerical example with $\delta = 1$. We rely on the contractive terminal constraints enabled by the property of OSVF being a CLF to force the system state to approach the origin. Once the terminal set shrinks to the origin, we are able to show the monotonicity of the value function $V(x,0)$. Finally, we establish stability by *imposing a certain property on the modified stage cost* (i.e. OSVF to be a CLF) while the existing stability theories ensure stability by *imposing a condition on the terminal cost* (i.e. the terminal cost to be a CLF). In other words, we establish stability by making use of the properties of a modified stage cost.

4. Terminal cost calculation

All the assumptions used in establishing the stability of MPC are quite similar to those in the existing terminal cost-based MPC framework except Assumption **(A3)**. This section is devoted to developing a procedure for finding a terminal cost such that the corresponding OSVF $m(x)$ is a CLF. As discussed previously, the condition $m(x)$ being positive definite is opposite to the existing well-established MPC stability conditions. Furthermore, it is possible for $V_f(x)$ to be zero which includes the case of no terminal cost. Therefore, the results (Chen, 2022) for stability analysis of MPC without terminal cost can be considered as a special case in this paper. More importantly, we are able to explore the freedom of adding an appropriate terminal cost to guarantee the stability of an MPC algorithm of concern.

In what follows, we propose a design procedure to find a terminal cost to satisfy Assumption **(A3)** for linear systems, which can be extended to nonlinear systems by similar approaches in the current MPC framework, for example, linearisation of the nonlinear system at the operation condition (Rawlings et al., 2017), or using linear differential inclusion (Chen et al., 2003). It also provides more insight into the definition and the properties of OSVF $m(x)$. Here, we consider a linear system as

$$x^+ = Ax + Bu, \tag{33}$$

with the stage cost $l(x,u) = |x|_Q^2 + |u|_R^2$. We aim to find a terminal cost $V_f(x) = |x|_P^2$ satisfying Assumption **(A3)**. The modified stage cost in (2) can be specified as

$$\begin{aligned} l_f(x,u) &= -|x|_P^2 + |x|_Q^2 + |u|_R^2 + |Ax+Bu|_P^2 \\ &= |x|^2_{A^TPA+Q-P} + |u|^2_{R+B^TPB} \\ &\quad + 2x^TA^TPBu \\ &= \begin{bmatrix} x^T & u^T \end{bmatrix} M \begin{bmatrix} x \\ u \end{bmatrix}, \end{aligned} \tag{34}$$

where

$$M := \begin{bmatrix} A^TPA + Q - P & A^TPB \\ B^TPA & R + B^TPB \end{bmatrix}. \tag{35}$$

If $R + B^TPB > 0$, solving the one-step optimisation problem (11) gives

$$\begin{aligned} \arg\min_u l_f(x) &= -(R+B^TPB)^{-1}B^TPAx \\ m(x) &= \min_u l_f(x) = |x|^2_{M_P}, \end{aligned} \tag{36}$$

where

$$M_P := A^TPA + Q - P - A^TPB(R+B^TPB)^{-1}B^TPA. \tag{37}$$

Since that $m(x)$ serves as a CLF in Assumption **(A3)**, M_P should at least be positive definite. By the Schur Complement Lemma, $R + B^TPB > 0$ and $M_P > 0$ is equivalent to $M > 0$.

This result is summarised in the following lemma.

Lemma 4.1: *Consider a linear system (33) with a quadratic cost function. Then its OSVF $m(x)$ is positive definite if there exists a terminal cost $V_f(x) = |x|_P^2$ with a matrix P such that M defined in (35) is positive definite.*

Condition (35) is the same as condition (9) in Zanon et al. (2014) except that a positive semi-definite inequality is presented in (9). It has been shown in Zanon et al. (2014) that condition (9) is necessary for the existence of a stabilising LQR of the infinite horizon with the stage cost $l(x,u)$. It is also equivalent to the widely known frequency domain inequality and the dissipation inequality. The link between this condition and detectability of $(A, Q^{1/2})$ was established through the work in Willems (1971).

By now, we only need to consider condition (14). $m(x)$ is in a quadratic form in this case. Condition (14) can be relaxed as below since there exists a control u_k such that $m(x_{k+1}) < m(x_k)$ holds for any x_k.

Lemma 4.2: *$m(x_{k+1}) < m(x_k)$ holds, if and only if there exist u_k and u_{k+1} such that $l_f(x_{k+1}, u_{k+1}) < m(x_k)$ holds.*

Proof: $\Leftarrow$: If there exist u_k and u_{k+1} such that $l_f(x_{k+1}, u_{k+1}) < m(x_k)$, we then have that $m(x_{k+1}) \le l_f(x_{k+1}, u_{k+1}) < m(x_k)$.

$\Rightarrow$: If $m(x_{k+1}) < m(x_k)$, the existence of u_k is obvious. As for u_{k+1}, we can choose $u_{k+1} = \arg\min_{u_{k+1}} \times l_f(x_{k+1}, u_{k+1})$. Hence, there exist u_k and u_{k+1} such that

$$l_f(x_{k+1}, u_{k+1}) = m(x_{k+1}) < m(x_k).$$

This completes the proof. ■

Lemma 4.3: *If $R > 0$, then*

$$\begin{bmatrix} Q - SR^{-1}S^T & A \\ A^T & B \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} Q & S & A \\ S^T & R & 0 \\ A^T & 0 & B \end{bmatrix} > 0. \tag{38}$$

Proof: By using the congruent transformation, we have that

$$\begin{bmatrix} I & -SR^{-1} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} Q & S & A \\ S^T & R & 0 \\ A^T & 0 & B \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -R^{-1}S^T & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} Q - SR^{-1}S^T & 0 & A \\ 0 & R & 0 \\ A^T & 0 & B \end{bmatrix}.$$

This completes the proof. ■

Theorem 4.1: *If there exists a terminal cost P and two control gains K_1 and K_2 such that the following bilinear matrix inequality (BMI) is satisfied*

$$\begin{bmatrix} M & \begin{bmatrix} (A+BK_1)^T & K_2^T \\ 0 & 0 \end{bmatrix} M \\ M \begin{bmatrix} (A+BK_1) & 0 \\ K_2 & 0 \end{bmatrix} & M \end{bmatrix} > 0, \quad (39)$$

then $m(x)$ is a CLF for system (33).

Proof: Note that if condition (39) is satisfied, it follows that $M > 0$, which further guarantees $R + B^T PB > 0$ and $M_P > 0$ by using the Schur Complement Lemma. $R + B^T PB > 0$ implies that the one-step optimisation problem (11) is well-defined.

Letting $u_k = K_1 x_k$ and $u_{k+1} = \tilde{K}_2 x_{k+1}$, a quick calculation gives

$$\begin{aligned} & l_f(x_{k+1}, u_{k+1}) \\ & = \begin{bmatrix} x_k^T (A+BK_1)^T & x_k^T K_2^T \end{bmatrix} M \begin{bmatrix} (A+BK_1)x_k \\ K_2 x_k \end{bmatrix} \\ & = x_k^T \begin{bmatrix} (A+BK_1)^T & K_2^T \end{bmatrix} M \begin{bmatrix} A+BK_1 \\ K_2 \end{bmatrix} x_k \quad (40) \end{aligned}$$

where $K_2 = \tilde{K}_2(A + BK_1)$. Using the Schur Complement Lemma and Lemma 4.2, noting $R + B^T PB > 0$, the condition $l_f(x_{k+1}, u_{k+1}) < m(x_k)$ is satisfied if the following matrix inequality holds

$$\begin{aligned} & \begin{bmatrix} (A+BK_1)^T & K_2^T \end{bmatrix} M \begin{bmatrix} A+BK_1 \\ K_2 \end{bmatrix} < M_P \\ \Leftrightarrow & \begin{bmatrix} (A+BK_1)^T & K_2^T \end{bmatrix} M M^{-1} M \begin{bmatrix} A+BK_1 \\ K_2 \end{bmatrix} < M_P \\ \Leftrightarrow & \begin{bmatrix} M_P & \begin{bmatrix} (A+BK_1)^T & K_2^T \end{bmatrix} M \\ M \begin{bmatrix} A+BK_1 \\ K_2 \end{bmatrix} & M \end{bmatrix} > 0 \\ \Leftrightarrow & (39). \end{aligned}$$

This completes the proof. ■

Remark 4.1: It shall be highlighted that P, the associated terminal cost $V_f(x)$, is not necessary to be positive definite. The proposed BMI (39) can be solved by using the PENLAB package (Fiala et al., 2013), which is an open-source software package implemented in MATLAB for matrix inequalities.

5. Illustrative examples

5.1. A first-order example

This section is to illustrate the significance of the proposed new stability analysis approach in this paper and highlight the main differences with the existing terminal cost-based MPC stability theory by using a simple first-order system

$$x^+ = ax + bu, \quad b \neq 0 \quad (41)$$

with $l(x, u) = qx^2 + ru^2$ and $V_f(x) = px^2$, where the lowercase letters represent scalars.

More specifically, this example is used to highlight the following two points.

(1) With the new condition, the proposed MPC algorithm may be stable with a zero terminal cost or a negative terminal cost.
(2) The new stability condition is complementary to the existing terminal cost-based MPC stability conditions in the design space defined by key parameters.

It follows from the definition of the augmented stage cost that $l_f(x, u) = -px^2 + qx^2 + ru^2 + p(ax + bu)^2$ for this example. The corresponding OSVF $m(x)$ is ready to be calculated. For this first-order linear system, according to Lemma 4.1, Assumption **(A3)** is satisfied if

$$\begin{aligned} & r + b^2 p > 0 \\ & a^2 p + q - p - \frac{a^2 p^2 b^2}{r + b^2 p} > 0. \end{aligned} \quad (42)$$

Inspecting condition (42) gives

$$\begin{aligned} q & > (1 - a^2)p + \frac{a^2 p^2 b^2}{r + b^2 p} \\ & = \frac{1}{b^2}\left(z_1 + \frac{a^2 r^2}{z_1} - \left(1 + a^2\right) r\right), \end{aligned} \quad (43)$$

where $z_1 := r + b^2 p > 0$. Equation (43) gives a lower bound of q by

$$\begin{aligned} q > \underline{q} & := \frac{2|ar| - (1 + a^2)r}{b^2} \\ & = \begin{cases} \dfrac{-(|a| - 1)^2 r}{b^2}, & \text{if } r > 0 \\ 0, & \text{if } r = 0 \\ \dfrac{-(|a| + 1)^2 r}{b^2}, & \text{if } r < 0. \end{cases} \end{aligned} \quad (44)$$

It shall be highlighted that Equation (44) reveals q is possible to be negative or zero if $r \neq 0$.

Next, for the purpose of comparison, we present the conventional MPC stability conditions for this simple MPC problem. First, all the weightings are required to be at least positive semi-definite, i.e. $q > 0$, $r \geq 0$, and $p \geq 0$. Secondly, the existing condition states that the conventional MPC for this unconstrained system is stable if the famous Fake Algebraic Riccati Equation (FARE) (Bitmead et al., 1990) holds

$$b^2p^2 + (1 - b^2q - a^2)p - q \geq 0$$
$$\Leftrightarrow q \leq \frac{1}{b^2}\left(z_2 + \frac{a^2r^2}{z_2} - \left(1 + a^2\right) r\right) \tag{45}$$

If we focus on the right-side functions of (43) and (45), when p is large enough, both functions go to a linear one $q = p - \frac{ra^2}{b^2}$, which is parallel with but below the line $q = p$ if $r > 0$.

In what follows, we will make an explicit comparison of the stability conditions and regions between the proposed new approach and the existing terminal cost-based MPC framework, in terms of the control weight $r > 0$, $r < 0$, and $r = 0$. It will be shown later the system parameter a has less effect on the cases of $r = 0$ and $r < 0$.

5.1.1. Case with control weight $r > 0$

We only consider the case that $r = 1$, as the same results hold for a different r by normalising state and terminal weighting with q/r and p/r. Regarding stability condition (44), there are three cases to discuss, $|a| < 1$, $|a| = 1$, and $|a| > 1$, which correspond to open-loop stable, marginally stable, and unstable systems, respectively. The stability regions defined by the new condition (43) are given by Figure 1. These regions actually define the possible parameter combinations of the system and cost functions including the stage and terminal cost so we refer them as the feasible *design spaces*. It is interesting to see when $|a| < 1$, the proposed MPC could be stable even when both p and q are negative, which covers the situation widely studied in EMPC.

It shall be noted that the blue curve corresponds to the solution of the Riccati equations under different stage costs, i.e. q. Therefore, to ensure stability, the existing MPC stability condition requires that the terminal cost p is larger than or equal to the solution of the Riccati equation. This is shown by the region on the right side of the blue curve; that is, the system under MPC is stable when the corresponding terminal cost p covers the cost-to-go. In contrast, the stability region given by our new stability condition lies on the left side of the blue curve so covers the design space where the terminal cost p is less than the Riccati solution under the corresponding stage and control weights q and r.

It is observed that there is no overlap between the stability regions derived by the existing terminal cost-based MPC framework and the new condition proposed in this paper. Furthermore, by combining the stability regions yielded by these two methods, we are able to significantly extend the MPC design space (i.e. in terms of possible terminal costs) where stability could be guaranteed. Therefore, our approach is complementary to the existing terminal cost-based MPC framework.

Finally, it is also observed that both the lines $p = 0$ and $p = q$ are covered by the proposed MPC stability condition as shown in Figure 1. $p = 0$ and $p = q$ are both the special cases of MPC where there is no terminal cost but with slightly different Lyapunov functions. To illustrate the difference, we take an example when $N = 2$.

- If $p = 0$, the optimisation in (15) reduces to

$$\min_{u_{k|k}, u_{k+1|k}} \left(qx_{k|k}^2 + ru_{k|k}^2\right) + \left(qx_{k+1|k}^2 + ru_{k+1|k}^2\right)$$
$$\text{s.t. } x_{k|k} = x_k$$
$$x_{k+i+1|k} = ax_{k+i|k} + bu_{k+i|k}, \quad i = 0, 1$$
$$x_{k+2|k} \in \Omega(\alpha_k), \tag{46}$$

  where $M_P = a^2p + q - p - a^2p^2b^2 \times (r + b^2p)^{-1} = q$.
- If $p = q$, the optimisation in (15) becomes

$$\min_{u_{k|k}, u_{k+1|k}} \left(qx_{k|k}^2 + ru_{k|k}^2\right) + \left(qx_{k+1|k}^2 + ru_{k+1|k}^2\right) + px_{k+2|k}^2$$
$$= \min_{u_{k|k}, u_{k+1|k}, u_{k+2|k}} \sum_{i=0}^{2} \left(qx_{k+i|k}^2 + ru_{k+i|k}^2\right)$$
$$\text{s.t. } x_{k|k} = x_k$$
$$x_{k+i+1|k} = ax_{k+i|k} + bu_{k+i|k}, \quad i = 0, 1$$
$$x_{k+2|k} \in \Omega(\alpha_k), \tag{47}$$

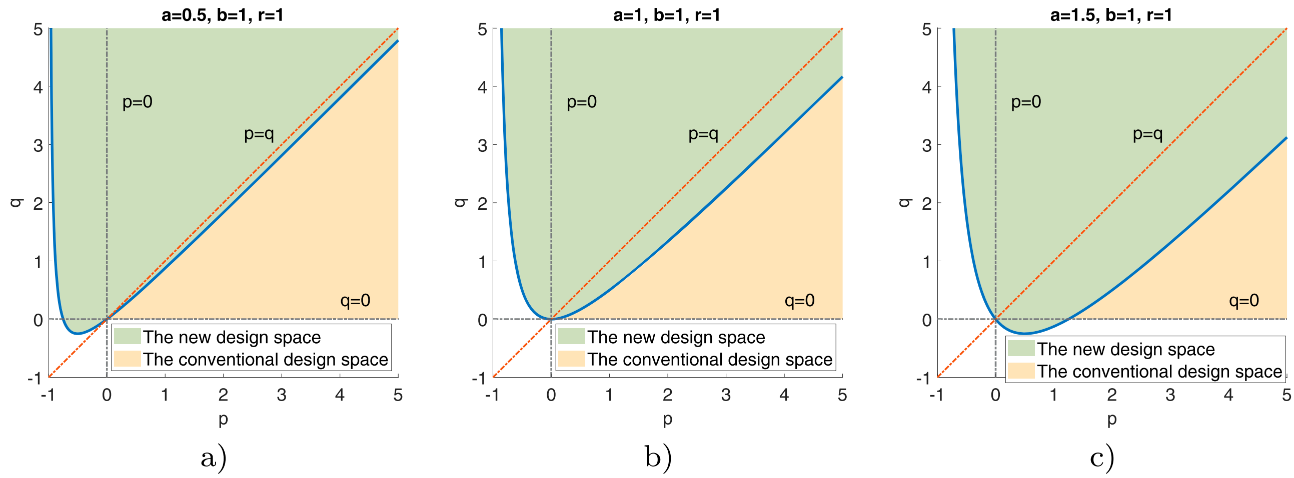

Figure 1. Stable design spaces of the proposed and conventional MPC for different open-loop systems when $r > 0$. (a) Stable, i.e. $|a| < 1$. (b) Marginally stable, i.e. $|a| = 1$ and (c) Unstable, i.e. $|a| > 1$.

where $M_P = a^2p + q - p - a^2p^2b^2(r + b^2p)^{-1} = a^2q - a^2q^2b^2(r + b^2q)^{-1} = a^2qr(r + b^2q)^{-1}$. The equivalent cost function holds as the decision variable $u_{k+2\,|\,k}$ would never affect the control and state sequences.

Therefore, they correspond to the different choices of the augmented stage cost and the associated OSVF. This also implies different terminal constraints are used in the algorithm and different Lyapunov functions are employed in establishing stability. This special case also provides more insight into the proposed new approach.

5.1.2. Case with control weight $r = 0$

The case of $r = 0$ is also interesting as both methods occupy the perfect half of the whole design space, as shown by Figure 2(a). This is because when $r = 0$, condition (43) is equivalent to $q > p > 0$ while condition (45) is $0 < q \leq p$.

5.1.3. Case with control weight $r < 0$

In this case, in a similar fashion as $r > 0$, we fix $r = -1$ and the result is given by Figure 2(b). It is worth noting that the design space of the conventional MPC framework is empty since the control weight is normally required to be positive definite.

In summary, for the cases of the control weight $r < 0$ or $r = 0$, we are able to draw the same conclusion from Figure 2 about the proposed new stability condition, and its comparison and connection with the existing terminal cost-based MPC framework, as in the case of $r > 0$. More specifically, for each pair of state and control weights q,r, the existing MPC stability theory ascertains that the corresponding MPC algorithm with the terminal cost p in the yellow region is stable while our stability condition states that the corresponding MPC algorithm with the terminal cost p in the light green area is also stable. Thus, we are able to state that an MPC algorithm is stable with any terminal cost from the combined areas of these two stability approaches. This substantially relaxes the stability requirement on MPC algorithms and increases the freedom in designing and tuning the key parameters involved in MPC. This numerical example and detailed comparison give more insight into the proposed new augmented stage cost method and its links with the existing MPC stability theory.

5.2. Simple nonlinear system study

This section is to further investigate the behaviours of the proposed contractive MPC particularly the influence of the design parameter δ, and compare it with the existing terminal cost-based MPC approach. Consider a benchmark system of a cart with a mass moving on a plane with a nonlinear spring in Magni et al. (2003). Its discretised model under the sample time of 0.4 sec is given by

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + 0.4x_{2,k} \\ x_{2,k+1} &= 0.132\,\mathrm{e}^{-x_{1,k}} + 0.56x_{2,k} + 0.4u_k, \end{aligned} \tag{48}$$

where x_1 is the displacement, x_2 the velocity and u the external force. The linearisation at the origin gives the

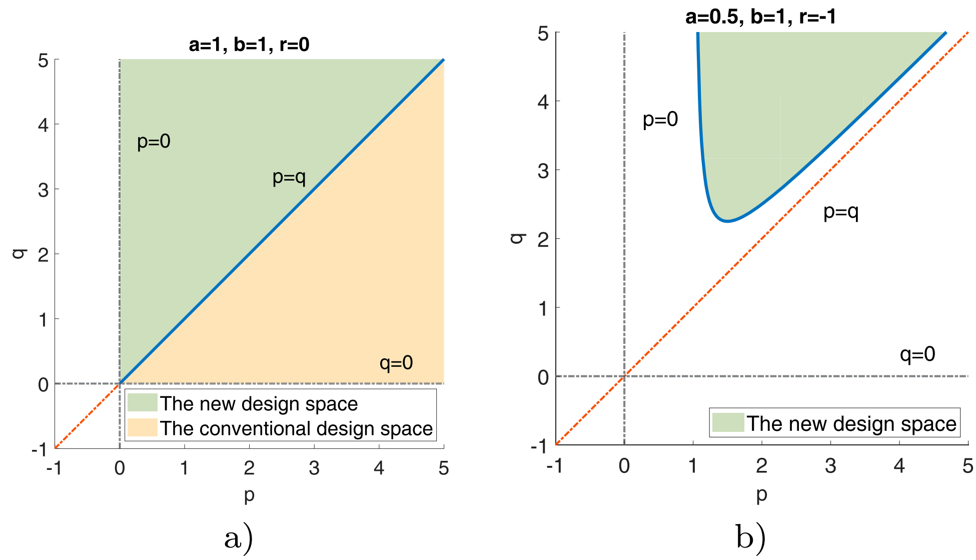

Figure 2. Stable design spaces of the proposed and conventional MPC. (a) $r = 0$ and (b) $r < 0$.

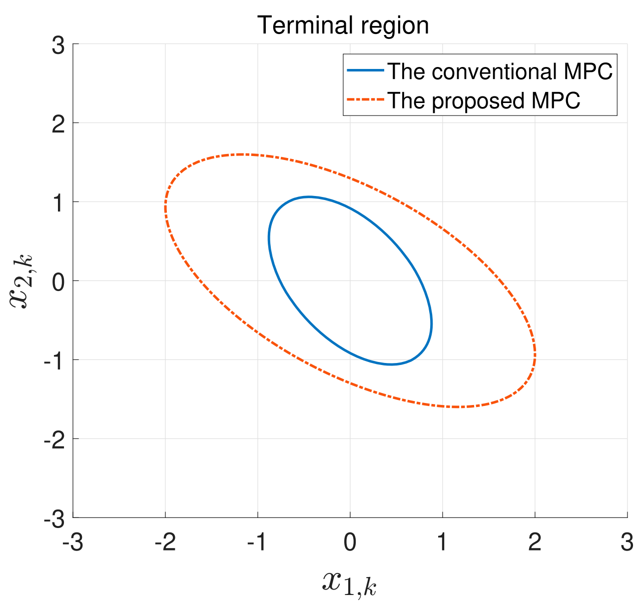

Figure 3. Terminal regions of the proposed and conventional MPC.

controllable pair

$$A = \begin{bmatrix} 1 & 0.4 \\ -0.132 & 0.56 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.4 \end{bmatrix}$$

where an offset $u_s = -0.132/0.4 = -0.33$ should be added to the MPC control input before applying to the original nonlinear system. The state and input constrains are $\mathbb{X} = [-2, 2] \times [-3, 3]$ and $\mathbb{U} = [-4, 4]$ and the initial state is $x_0 = [-2, 1]^T$. The prediction horizon is chosen to be short enough, $N = 3$, due to the fast calculation time by the motion control system. The stage and terminal costs are both given in a quadratic form, i.e. $l(x, u) = |x|_Q^2 + |u|_R^2$ and $V_f(x) = |x|_P^2$, where

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad R = 1.$$

In what follows, we will calculate the terminal costs and regions and assess their stability and performance using these two methods respectively.

- We first consider the conventional MPC design. The terminal cost is obtained by the Riccati equation for the linearisation

  $$P = \begin{bmatrix} 10.9153 & 4.5604 \\ 4.5604 & 7.5023 \end{bmatrix}$$

  and the terminal region is given by

  $$\Omega = \left\{ x \in R^2 : x^T P x \le \alpha \right\}$$

  where $\alpha = 6.3076$ is calculated by the optimisation method introduced in Chen et al. (2003).
- As for the proposed MPC design, following the procedure in Section 4, the terminal cost is obtained by solving the BMI (39)

  $$P = \begin{bmatrix} 3.5249 & -0.3522 \\ -0.3522 & 1.5731 \end{bmatrix}$$

  and the corresponding terminal region is given by

  $$\Omega = \left\{ x \in R^2 : x^T M_P x \le \alpha_0 \right\}$$

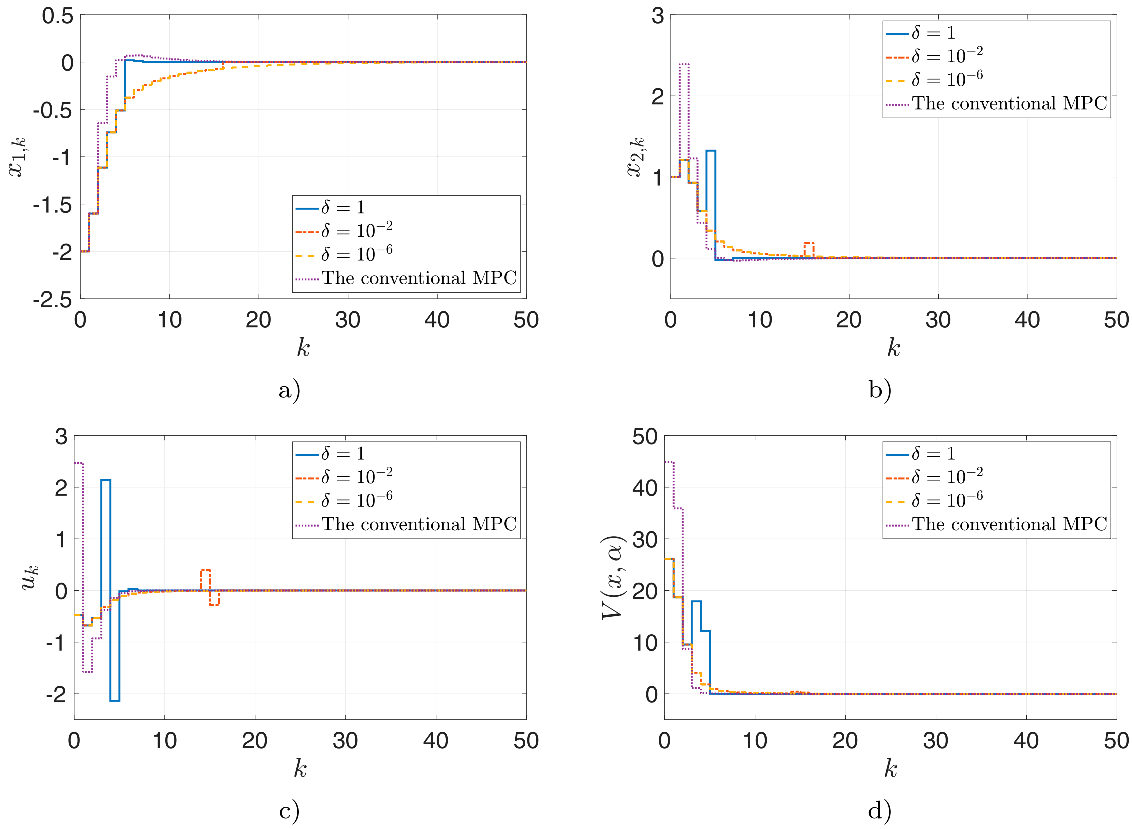

Figure 4. Performance and convergence of the proposed (with $\delta = 1, 10^{-2}, 10^{-5}$) and conventional terminal cost-based MPC. (a) state $x_{1,k}$. (b) state $x_{2,k}$. (c) control u_k and (d) optimal value function $V(x_k, \alpha_k)$.

where M_P is computed by (37) and $\alpha_0 = 5.4823$ is also calculated by the same method in Chen et al. (2003).

The corresponding terminal regions are shown in Figure 3. It shall be highlighted that different from stability regions plotted in the parameter space as in Figures 1 and 2, the terminal regions are given in state space in Figure 3.

To investigate the influence of the parameter δ, we test the proposed method under $\delta = 1, 10^{-2}, 10^{-5}$. The recursive feasibility and convergence of the proposed MPC algorithm are achieved in this nonlinear simulation case without the need to decrease the initially chosen δ under all these settings. The states, inputs, and optimal value functions are given by Figure 4. It is interesting to notice that the optimal value function $V(x_k, \alpha_k)$ may increase in the stage of contraction (e.g. when $\delta = 1$). It can be shown that a larger δ leads to faster convergence in total while a smaller δ may give strict monotony in the value function. Besides, since the terminal cost P of the conventional MPC has to cover the cost-to-go, its initial value function is much larger than the proposed one, which is clearly demonstrated by Figure 4(d). The total running cost of the different algorithms is given by Table 1. It is evident that the performance is substantially improved by our new MPC algorithm with up to near 40% improvement at $\delta = 10^{-5}$.

6. Conclusion

Stability is an important property of any control method since it helps to ensure the safety of a system. Compared with other optimisation-based methods such as supervised learning, one distinctive feature of MPC is that it is possible to establish its stability under certain conditions. However, despite its huge success, there is still room to further improve its stability analysis by developing new theory and techniques and designing new stability-guaranteed algorithms. This paper explores the design space where the current terminal cost-based MPC framework is not applicable. We propose a completely new stability theory based on the modified stage cost approach. Consequently we developed stability conditions in the unexplored

Table 1. The total running cost of different methods.

Proposed MPC			
$\delta = 1$	$\delta = 10^{-2}$	$\delta = 10^{-5}$	Conventional MPC
49.0512	35.0821	34.7469	57.3252

design space in MPC. We show that, by constructing a contractive terminal constraint, rather than a fixed terminal constraint as in the current terminal cost-based MPC framework, we are able to guarantee stability in a significantly large region that has not been covered previously. To achieve this, we construct an augmented stage cost using a terminal cost. That is, the augmented stage cost consists of a nominal stage cost rotated with a terminal cost. It is shown that the feasibility and the stability of the proposed MPC algorithm can be established under the condition that the one-step optimal value function of the augmented stage cost is a CLF. This new stage cost-based stability analysis technique provides a promising new theory and tool for stability analysis of MPC and other optimisation-based methods and is complementary to the existing stability theories.

It shall be highlighted that similar to all other stability conditions in MPC, stability conditions established by our new approach are also sufficient. That is, MPC algorithms that satisfy Condition (2) but don't meet the condition in this paper may be still stable, and more research is required to fully explore the potential of this promising new approach. The stability region of MPC can be significantly enlarged by combining the new stability condition with the existing ones; e.g. guarantee stability under a much wider range of combinations of state, control and terminal weights. The work in this paper only concerns a basic MPC regulation problem without disturbance or uncertainty.

Our future work includes a full investigation of the conservativeness of the proposed stability condition particularly Assumption **(A3)**, and further exploring and improving the (augmented) state cost-based stability analysis approach proposed in this paper. For example, one direction is to relax condition (2) to include equality, which would give rise to a positive semi-definite augmented stage cost. This is much more challenging and it is expected some extra assumption such as detectability has to be added (e.g. Köhler et al., 2023). We will investigate more efficient and less conservative approaches to design OSVF as a CLF for general nonlinear systems and to estimate the associated terminal set.

Note

1. For simplicity, when used in algebraic expressions, $\mathbf{x}_k$ denotes the column vector $(x^T_{k\,|\,k}, \ldots, x^T_{k+N\,|\,k})^T$; similarly in algebraic expressions of $\mathbf{u}_k$.

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No potential conflict of interest was reported by the author(s).

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Data availability statement

The datasets generated and(or) analysed during the current study are available from the corresponding author W-H Chen on reasonable request.

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