Foundations for an Elementary Algebraic Theory of Systems with Arbitrary Non-Relativistic Spin

Peter Thomas Joseph Bradshaw

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For Nev, Josie, and Paul

Declaration

I, Peter Thomas Joseph Bradshaw, confirm that the work presented in my thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

The description of spin in modern physics is multifaceted, and links together a broad variety of physical concepts, including angular momentum, spinors, quantum mechanics, and special relativity. However, there remain foundational aspects of the existence of spin which are not fully understood: What physical and mathematical structure is strictly necessary for arbitrary spin to exist within a general physical model? Are quantum mechanics, relativity, or notions of angular momentum essential to its existence? What are the physically distinct observables in a physical theory with spin?

In this thesis, we will address these questions by presenting a new account for the emergence of spin in non-relativistic physical theories through the mathematical language of non-commutative algebras. The structure of these algebras will fundamentally derive from the geometry of real Euclidean three-space, and reveals a geometric origin for spin which is neither classical nor quantum. We will see that spin's phenomenology as a form of angular momentum is an emergent prediction of quantum mechanics, and that spin may be a natural source of non-commutative geometry, entailing couplings between the position and spin of a system.

To achieve this, we will use limited mathematical structure to: construct a generic methodology for the elementary study of algebraic structures from their minimal polynomials; present an elementary algebraic method to derive real algebras which describe arbitrary spins in terms of the physically distinct observables of the system; and define a family of algebras of position operators whose structures encode both the geometric action of rotations, and the structure of its spin operators in terms of geometrical objects.

Impact Statement

In the mid to late twentieth century, humanity experienced perhaps the most rapid technological transformation in its history, as a result of the electronics, computing, and information revolutions. This fundamentally altered the way we live our lives, and is responsible for innumerable benefits across myriad domains. Arguably, these dramatic changes are firmly rooted in the advances of physics made in the early part of the century, in particular in the development of quantum mechanics.

Now, we are at the precipice of another such revolution heralded by the emergence of quantum computing and quantum technologies. Within these domains, the physics of spin is a critical component, underpinning the leading bleeding-edge approaches. As such, research into the foundations of quantum mechanics, in particular into the fundamental nature of spin, has become increasingly important as a route to enrich and empower advances within these fields.

This thesis advances our understanding of the phenomenon of spin through a new, more elementary method of study. In particular, this approach results in an explication of the physical characteristics of systems with arbitrary spin, as well as a relatively unattested link between the spin and position degrees of freedom of such systems. While spin is ubiquitous in modern physics, the mathematical models which result from our analysis are of direct use in a wide variety of disciplines, including: the modelling and development of quantum computers; the modelling and development of novel quantum sensors; the study of crystalline solids for materials science applications; the modelling of processes in physical chemistry; the development of novel diagnostic techniques similar to nuclear magnetic resonance spectroscopy; and the development of novel electronic components, especially in spintronics applications. Moreover, the analytical methods developed in this thesis to study spin can be utilised to perform elementary studies of other important physical properties of systems. In tandem with other avenues of foundational research, this may contribute to further advances in a wide range of technological frontiers.

Beyond these more direct usages, the insights provided by this thesis into the relationship between spin and geometry offer a glimpse of advances to our understanding of gravity. While this may not seem directly tied to improving our quality of life, note: without the robust understanding of gravity offered by Einstein's theory of General Relativity, satellite navigation would not have been possible. As with many aspects of fundamental physics research, including advances in our understanding of the relationship between quantum mechanics and gravity, it is difficult to predict exactly how such knowledge will benefit us in the long-term. But, as the developments of the twentieth century can attest, such transformations can be dramatic, far-reaching, and, once realised, impossible to imagine the world without.

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Chapter 1

Introduction

1.1 Motivation

1.1.1 Algebraic Structure in Quantum Mechanics

In quantum mechanics, one traditionally considers the physical properties of a system to be encoded within its states. Operators on the state space offer both a means to extract this information, through the use of Hermitian operators (observables), and a method of altering the physical properties of the system, for example through time evolution. From this perspective, one may surmise that knowledge of the physical properties of a system is only available by studying its possible states. However, this is not the case: essential information about the system is also encoded within the algebra of its observables. We will illustrate this point with the algebra of spin generators for a non-relativistic system with arbitrary spin. In fact, this will be the primary example of interest throughout this thesis.

1.1.1.1 Spin in Quantum Mechanics

Phenomenologically, spin presents as a form of angular momentum which is intrinsic to particles. It varies in magnitude discretely by half-integer amounts, and is most commonly observed experimentally through its associated magnetic dipole moment. The states for a spin-s system are called spinors, and are typically modelled using $2s+1$-dimensional complex vectors. The standard basis one uses to describe an arbitrary spinor is an eigenbasis for the operator associated with this intrinsic angular momentum. More precisely, one interprets each eigenstate as a configuration of the system in which this intrinsic angular momentum has a particular alignment with a given spatial direction (conventionally the z-direction). Measurement of the amount of intrinsic angular momentum along this direction for one of these eigenstates would yield a discrete value in the range $\{s, s-1, \dots, -s+1, -s\}$. This interpretation is consistent with, for example, the Stern-Gerlach experiment [1]. Further discussion of the history of spin in physics can be found in [2, 3].

Mathematically, many authors define spin to be a difference which, in principle, could exist between total and orbital angular momentum [4, 1] of a quantum mechanical system. More explicitly, the spin degrees of freedom of a fundamental non-relativistic system are traditionally modelled using irreducible representations of the Lie group $\mathrm{SU}(2, \mathbb{C})$,

$$\mathrm{SU}(2, \mathbb{C}) \coloneqq \big\{ A \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid A^\dagger A = AA^\dagger = I_{2\times 2},\, \det(A) = 1 \big\}. \tag{1.1.1}$$

The use of $\mathrm{SU}(2, \mathbb{C})$ is motivated by rotational symmetry and Wigner's theorem [5] within the Hilbert space formulation of quantum mechanics. This theorem states that all symmetry transformations, here meaning transformations which preserve transition probabilities, can be associated with either unitary or antiunitary operators. For a group of such symmetry transformations, we consider unitary "projective" representations [5] of that group, since a change of phase does not alter transition probabilities, and antiunitary operators are not closed under composition. We are physically motivated to construct quantum mechanics to be rotationally invariant, and so would utilise such representations of $\mathrm{SO}(3, \mathbb{R})$; however, since $\mathrm{SU}(2, \mathbb{C})$ is the unique "universal covering group" of $\mathrm{SO}(3, \mathbb{R})$ [6], it is mathematically equivalent and more convenient to consider "ordinary" representations of it instead [5].

Instead of considering the group $\mathrm{SU}(2, \mathbb{C})$, we may equivalently model such a system using irreducible representations of the Lie algebra which generates it, namely $\mathfrak{su}(2, \mathbb{C})$ [7],

$$\mathfrak{su}(2, \mathbb{C}) \coloneqq \big\{ A \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid A^\dagger = -A,\, \mathrm{tr}(A) = 0 \big\}. \tag{1.1.2}$$

The representations of $\mathrm{SU}(2, \mathbb{C})$ and $\mathfrak{su}(2, \mathbb{C})$ are one-to-one [7], but the latter are mathematically simpler to derive. An account of the traditional methods used to identify the irreducible representations on $\mathfrak{su}(2, \mathbb{C})$ is given in Appendix A.1.

More explicitly, representations for $\mathfrak{su}(2, \mathbb{C})$ are pairs $(\mathcal{V}, \rho)$ consisting of a finite-dimensional complex vector space $\mathcal{V}$, and a vector space homomorphism ρ satisfying,

$\forall A, B \in \mathfrak{su}(2,\mathbb{C})$,

$$\begin{gathered}\rho : \mathfrak{su}(2,\mathbb{C}) \to \mathrm{End}(\mathcal{V}) \\ \rho(A)\circ\rho(B) - \rho(B)\circ\rho(A) = \rho\big([A,B]\big),\end{gathered} \tag{1.1.3}$$

with $\circ$ composition of functions, and $\mathrm{End}(\mathcal{V})$ the space of vector space endomorphisms on $\mathcal{V}$. Such a representation is irreducible iff, for a subspace $\mathcal{U} \subseteq \mathcal{V}$, the only representations $(\mathcal{U}, \sigma)$ satisfying, $\forall A \in \mathfrak{su}(2,\mathbb{C})$,

$$\begin{gathered}\sigma : \mathfrak{su}(2,\mathbb{C}) \to \mathrm{End}(\mathcal{U}) \\ i\circ\sigma(A) = \rho(A)\circ i,\end{gathered} \tag{1.1.4}$$

are 0 and ρ, with $i : \mathcal{U} \to \mathcal{V}$ the inclusion map. In the case of $\mathfrak{su}(2,\mathbb{C})$, the irreducible representations $\{(\mathcal{V}^{(s)}, \rho^{(s)})\}$ may be parametrised by a single half integer,

$$s \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\right\}, \tag{1.1.5}$$

and have $\dim(\mathcal{V}^{(s)}) = 2s+1$. This half-integer s corresponds to the magnitude of the intrinsic angular momentum of the system modelled by the representation.

More commonly in physics, we describe $\mathfrak{su}(2,\mathbb{C})$ in terms of a basis of "spin generators" $\{\hat{S}_a\}$ for its complexification $\mathfrak{su}(2,\mathbb{C})\otimes_{\mathbb{R}}\mathbb{C}$, for which, $\forall a \in \{1,2,3\}$,

$$\left[\hat{S}_a, \hat{S}_b\right] = i\sum_{c=1}^{3} \varepsilon_{abc}\hat{S}_c. \tag{1.1.6}$$

Furthermore, we present the irreducible representations as matrices with respect to a given basis of $\mathcal{V}^{(s)}$. The vectors of $\mathcal{V}^{(s)}$ are the spinors previously mentioned, and the spinor basis we typically choose is the eigenbasis for the $\rho^{(s)}(\hat{S}_z)$ operator. General expressions for the matrices of spin generators for any spin-s representation, $s \in \{0, \frac{1}{2}, 1\ldots\}$, are known[1]. We show some examples of these in Table 1.1.

1.1.1.2 Structure in the Algebras of Spin Operators

Despite the traditional focus on spinor states, we can see that information is encoded within the matrices of spin generators themselves, both visually and in their algebraic relationships. Consider, for example, the matrix corresponding to the composition $\rho^{(s)}(\hat{S}_x)\circ\rho^{(s)}(\hat{S}_x)$. For spin-$\frac{1}{2}$ systems, we readily see that,

$$\rho^{(1/2)}(\hat{S}_x)\rho^{(1/2)}(\hat{S}_x) = \frac{1}{4}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac{1}{2}I_{2\times2}, \tag{1.1.7}$$

s	$\rho^{(s)}(\hat{S}_x)$	$\rho^{(s)}(\hat{S}_y)$	$\rho^{(s)}(\hat{S}_z)$
$\frac{1}{2}$	$\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
1	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
$\frac{3}{2}$	$\frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 0 & -i\sqrt{3} & 0 & 0 \\ i\sqrt{3} & 0 & -2i & 0 \\ 0 & 2i & 0 & -i\sqrt{3} \\ 0 & 0 & i\sqrt{3} & 0 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$

Table 1.1: Examples of Spin Representations

with $I_{2\times2}$ the two-by-two identity matrix. However, for spin-1 and spin-$\frac{3}{2}$ systems, we see that $\forall\alpha \in \mathbb{C}$,

$$\rho^{(1)}(\hat{S}_x)\rho^{(1)}(\hat{S}_x) \neq \alpha I_{3\times3}, \quad \rho^{(3/2)}(\hat{S}_x)\rho^{(3/2)}(\hat{S}_x) \neq \alpha I_{4\times4}. \tag{1.1.8}$$

This indicates that the algebra generated by $\rho^{(s)}(\hat{S}_x)$, for example, has captured some information about these spin systems; in this case, it enables us to distinguish spin-$\frac{1}{2}$ systems from others, without the need for spinor states[1].

Of course, one could argue that some information encoded in the spinor is implicit in the dimensionality and chosen basis of the matrix representation for the spin generators. However, this ability to distinguish between systems of differing spins through algebraic relations, and in so doing access information about the physical properties of the system, is independent of these choices for the representation; this means that in situations where a spin-s system has two different representations, for example a fundamental versus a composite system, these algebraic properties remain consistent. To make this more explicit, we claim, and will later prove in Chapter 4, that for each spin $s \in \{0, \frac{1}{2}, 1, \dots\}$ there exists a unital associative algebra $A^{(s)}$, containing $\mathfrak{su}(2,\mathbb{C})$ as a Lie subalgebra, and on which $\rho^{(s)}$ may be

[1]While this was simply an illustrative example, a result of this thesis is that systems of arbitrary spin can be classified and distinguished purely algebraically.

extended to a unital associative algebra representation. As we will see, the algebra $A^{(s)}$ may be defined without the need for complex numbers, spinor bases, or the notion of angular momentum, but completely determines the algebra of operators for an arbitrary spin-s system.

As such, there are a number of questions it is natural to ask:

- What kind of information about a physical system does its algebra of operators contain?
- What can we learn about a physical system by understanding the algebraic structure of its algebra of operators?
- If we consider *only* the system's algebra of operators, what models of the physical system can we still construct?

These questions will drive the developments of this thesis, which seeks to understand the fundamental nature of spin, and its relationship to the physical content of quantum mechanics, through the study of the algebras which naturally describe it.

1.1.2 Algebraic Theories in Physics

1.1.2.1 In General

Using algebras to study and construct physical theories has a long history. To begin, Maxwell famously used quaternions to unify the equations governing electromagnetism[8]. The pursuit of algebraic theories in physics was also advocated by Einstein[9] as a means to more naturally describe quanta than is achieved in a continuum theory. In fact, many aspects of algebraic approaches have been behind several seminal results in physics. For example, Dirac's standard bra/ket[10] enabled him to construct quantum theory almost entirely in terms of operators. Furthermore, his derivation of the Dirac equation from the Klein-Gordon equation necessitated the structure of a Clifford algebra to formulate. Following Dirac's method, but in the non-relativistic domain, Lévy-Leblond derived the Pauli equation from the Schrödinger equation[11, 12], again necessitating the use of an algebra. Furthermore, Von Neumann's study of the position-momentum operator algebra, and use of an algebraic idempotent, was essential in his proof of the Stone-Von Neumann theorem[13]. The algebra he described arguably anticipated the Moyal star algebra by at least a decade[14, 15].

The prevalence of algebraic models in physics has been increasing since the second half of the twentieth century. Haag and Kastler use algebras in their axiomatic framework for "Algebraic Quantum Field Theory"[16, 17]. Clifford algebras in particular are enjoying somewhat of a renaissance. For example, Trayling et al.[18] and Furey et al.[19, 20] utilise Clifford algebras to construct the particles found in the standard model of particle physics, as well as their transformations. Hestenes[21], Doran and Lasenby[22], Hiley and Callaghan[23, 24], and many others also use Clifford algebras to study the Schrödinger, Pauli, and Dirac equations, including the construction of algebraic spinors.

Beyond Clifford algebras, algebraic methods are being applied to a wide variety of fields in physics. For example, Gilmore et al. [25] use algebraic methods to study the quantum defect, and Furey et al.[19, 20] utilise the non-associative octonions to study the standard model of particle physics and symmetry breaking within it. An overview of algebraic techniques commonly used in physics can be found in Iachello[26]. The future of algebraic studies of physical theories is also promising. For example, algebras form the backbone of non-commutative geometry[27, 28, 29, 30] which underlies many modern theories of quantum gravity. They are also an important component of the "amplituhedron" formalism[31].

1.1.2.2 For Spin in Non-Relativistic Systems

Spin has been ubiquitous in quantum mechanics since its discovery; as such, it would be impossible to discuss every study into its nature. The study of spin in physics is traditionally approached analytically, and within the context of relativity: Weinberg's study of relativistic spin[32] is performed using the language of quantum field theory; as is the work of Giraud et al.[33] it inspired. Field theory also underpins the modern study of higher spins in higher-dimensional Minkowski space [34, 35].

For both non-relativistic and relativistic systems with spin, only a handful of algebraic descriptions for the structure of their spin operators are known. In the non-relativistic domain, which is our primary case of interest, there are three, which are defined by (1.1.6) together with the following identities:

For spin-0 systems, $\forall a \in \{1, 2, 3\}$,

$$\hat{S}_a = 0; \tag{1.1.9}$$

For spin-$\frac{1}{2}$ systems, $\forall a, b \in \{1, 2, 3\}$,

$$\hat{S}_a \hat{S}_b + \hat{S}_b \hat{S}_a = \frac{1}{2} \delta_{ab}; \tag{1.1.10}$$

And for spin-1 systems, $\forall a, b, c \in \{1, 2, 3\}$,

$$\hat{S}_a \hat{S}_b \hat{S}_c + \hat{S}_c \hat{S}_b \hat{S}_a = \delta_{ab} \hat{S}_c + \delta_{bc} \hat{S}_a, \quad \hat{S}^2 = \sum_{d=1}^{3} \hat{S}_d \hat{S}_d = 2. \tag{1.1.11}$$

The spin-$\frac{1}{2}$ and spin-1 algebras are called the "Pauli" and "Duffin-Kemmer-Petiau" [36, 37] algebras respectively. The Pauli algebra is also an example of a Clifford algebra[22]. The Duffin-Kemmer-Petiau algebra can also describe relativistic spin-1 (and sometimes spin-0) systems, and can be used to model mesons[37]. These algebras describe the spin operators of their respective systems completely, in the sense that any representation of these operators for spin-0, spin-$\frac{1}{2}$, or spin-1 must satisfy the corresponding algebraic identities. At time of writing and to the author's knowledge, no elementary generalisation of the Clifford and Duffin-Kemmer-Petiau algebras to describe arbitrary spin systems is known.

In an algebraic setting, higher-spin systems may be described by utilising subspaces of tensor products of the Pauli algebra, or more commonly a Clifford algebra which contains the Pauli algebra[38]. This technique also underpins the definition of many classic higher-spin models in the relativistic domain[39, 40, 41], where the Pauli algebra is replaced by the algebra of relativistic spin-$\frac{1}{2}$ operators.

Beyond these models, describing systems with spin using algebraic methods is not uncommon: Racah's spherical tensor operator formalism[42] is perhaps best known, which describes total angular momentum in quantum mechanics; Zemach[43] presents methods which use spin-tensorial objects; and Giraud et al.[33, 44] utilise similar, but distinct, objects to describe density operators.

1.1.3 The Approach of this Thesis

As we have seen, most of the major studies of spin in a physical context have been performed within theories comprised of many more physical elements than spin alone, such as quantum field theory or quantum mechanics. In a mathematical context, spin is explored using the tools of representation theory, which is summarised in Appendix A.1. These tools require the use of complex numbers to extend the scalars of $\mathfrak{su}(2, \mathbb{C})$, so this too brings in additional structure not initially present in this Lie algebra.

That isn't to say that any of these approaches are improper for doing so; indeed, much of what we know about spin originates from such theories. Nor does this imply that the additional structures added aren't important for the *phenomenology* of spin. However, I question the degree to which these methods enable us to understand the more *fundamental* nature of spin. This is because, in each of these analyses, the objects under consideration are necessarily *emergent*; they can only be studied in the context of the broader theory, which contains many mathematical and physical structures which may not be intrinsically associated with spin. Phrased differently, in such a theory we cannot be sure what is fundamental about spin, and what is a consequence of the coupling of this fundamental structure to extrinsic mathematical features present in the model. In this spirit, we will often differentiate between "spin" and "intrinsic angular momentum" in this thesis, with the former referring to the fundamental structure of the property, and the latter referring to its emergent and experimentally observed character. This distinction will be expounded further in Chapter 4.

Therefore, to attempt to study spin in its most fundamental form, we will work with a minimal number of mathematical and physical assumptions. The structures we will exclude from our analysis are motivated by the conventional understanding of spin within the physics community we have discussed. Through omission, we also seek to better understand the role of these structures in both the existence and properties of spin, and perhaps within quantum mechanics more broadly. It is worth highlighting that, while such an analysis is only possible theoretically, since experiments inherently probe physics in its complete emergent form, I argue that such work is vital for developing a richer understanding of the foundations of quantum mechanics.

1.1.4 Motivation for this Approach

The broad question which motivates this thesis is, what can we learn about non-relativistic spin systems by studying their algebra of operators? Moreover, what information about such systems is determined by this algebraic structure? To answer these questions, the following commonly considered features of non-relativistic spin systems will motivate our approach:

1.1.4.1 Complex Numbers

There are a number of links between spin and complex numbers in the standard theory: the Lie algebra $\mathfrak{su}(2, \mathbb{C})$ is formed of complex matrices; the half-integer spin representations cannot be realised as representations on a real vector space of the same dimension; and within the association between spin and quantum mechanics, with complex numbers seen by many as a hallmark of quantum theory. Despite this, the relationship between spin and complex numbers is not at all straightforward.

Firstly, though $\mathfrak{su}(2, \mathbb{C})$ may be defined in terms of complex matrices, it is a real Lie algebra, meaning it is only closed under real linear combinations of its elements. This fact may be easily verified from equation (1.1.2), and is the reason why representation theory needs to complexify it for its methods to apply (see Appendix A.1). We may equivalently define $\mathfrak{su}(2, \mathbb{C})$ in an abstract algebraic way with no complex numbers at all, $\forall a, b \in \{1, 2, 3\}$,

$$\begin{aligned} \mathfrak{su}(2, \mathbb{C}) &\coloneqq \operatorname{span}_{\mathbb{R}}(\{S_1, S_2, S_3\}) \\ S_a \times S_b &= \sum_{c=1}^{3} \varepsilon_{abc} S_c, \end{aligned} \tag{1.1.12}$$

where $\times$ is bilinear and satisfies the Jacobi identity.

Secondly, using this abstract algebraic definition, we find that the Pauli algebra identity of equation (1.1.10) becomes,

$$S_a S_b + S_b S_a + \frac{1}{2}\delta_{ab} = 0. \tag{1.1.13}$$

Therefore, even though the representation $(\mathbb{C}^2, \rho^{(1/2)})$ is canonically complex (in fact, quaternionic, as may easily be verified), the algebraic identities which distinguish it ((1.1.12) and (1.1.13)) are both real.

Finally, while there is no doubt that complex numbers are of significance to quantum mechanics, these two observations cast doubt on their essential relationship with the fundamental structures governing the spin operators of physical systems. Therefore, to better understand the role complex numbers play in the development of spin, we shall avoid introducing them into our analysis. While this will make our analysis more challenging, for myriad reasons which will be explored in Chapter 3, we will be able to see clearly what role they play in spin systems. Through this, we may also improve our understanding of the nature and physical interpretation of complex numbers within quantum mechanics more broadly.

1.1.4.2 Representation Theory

Recall that to model physical systems with spin, we use irreducible representations $(\mathcal{V}^{(s)}, \rho^{(s)})$ of $\mathfrak{su}(2,\mathbb{C})$, with $\rho^{(s)}$ satisfying (1.1.3). Utilising the definition (1.1.12) for $\mathfrak{su}(2,\mathbb{C})$, notice that in $\mathfrak{su}(2,\mathbb{C})$ the product $\times$ satisfies the Jacobi identity, whereas in $\mathrm{End}\big(\mathcal{V}^{(s)}\big)$ the product is function composition $\circ$, and is associative. The commutator of $\mathrm{End}\big(\mathcal{V}^{(s)}\big)$ satisfies the Jacobi identity, which is how we have been able to embed the structure of $\times$ into it. However, in $\mathrm{End}\big(\mathcal{V}^{(s)}\big)$ arbitrary products $\rho^{(s)}(A)\circ\rho^{(s)}(B)$ are also defined, which have no equivalent in $\mathfrak{su}(2,\mathbb{C})$.

In a sense, the representation $(\mathcal{V}^{(s)}, \rho^{(s)})$ contains additional information compared to $\mathfrak{su}(2,\mathbb{C})$ which enables arbitrary products to be well-defined. With this observation in mind, a natural question to ask is, how might we access this additional information? Since it seems to concern products of operators, might this information be available to use through studying the algebra of these operators? Moreover, can this additional information be understood in physical terms? Addressing these questions will be a focus of Chapter 4.

1.1.4.3 Spinor Eigenstates

Spin is typically understood through spinor eigenstates. Such eigenstates are usually framed by analogy using a classical angular momentum pseudovector, which has a number of possible alignments with a given spatial direction determined by the spin of the system (specifically there are $2s+1$ possible alignments for a spin-s system). There eigenstates are implicitly used whenever a matrix representation for a spin operator is stated, since one typically chooses $\rho^{(s)}(\hat{S}_z)$ to be diagonal. Thus, the $\rho^{(s)}(\hat{S}_z)$ constitute a conventional choice of basis for the state space.

That being said, it is natural to wonder if a description of spin in terms of spinor eigenstates is an essential interpretation for the phenomenon? This question is emphasised given their seemingly necessary link to spatial directions, which is not apparent in the Pauli and Duffin-Kemmer-Petiau algebraic models, nor from the matrix representations for the operators. Then, instead of understanding spin through spinors, we may ask what other physical conceptions of spin are possible which are independent of a choice of spatial direction? Furthermore, it is unclear if the pseudovector-like property that spinor eigenstates are associated with is the only physically significant property for spin systems? If not, how might we exhaustively calculate all physically distinct (not necessarily simultaneously measurable) observables for such systems?

These questions not only apply to a single spin system, but to the way in which we compare systems with differing spin. Are there other ways to contrast such systems besides the discrete set of alignments? Might such a scheme also allow us to discuss the similarities between systems with differing spin, and so isolate the properties which make them physically distinguishable?

To answer these questions, we will proceed without the use of spin eigenstates, or spinors of any kind. In particular, this means we shall not use matrices in our analysis; instead, we will utilise abstract algebraic arguments and constructions. The development of such methods will be the focus of Chapter 3, and their use will be central to Chapter 4.

1.1.4.4 Angular Momentum and Relativity

A more fundamental nature for spin is difficult to determine from the traditional formalism. As previously stated, spin presents phenomenologically as a form of angular momentum which is intrinsic to particles. Both momentum and angular momentum are physical qualities associated with the motion of a system, and the symmetries which govern this motion. In this way, they are dynamical notions, through which we associate a dynamical character to spin. Despite this association, spin may also be attributed a geometrical character through the connection between $\mathrm{SU}(2,\mathbb{C})$ and $\mathrm{SO}(3,\mathbb{R})$ discussed earlier. This is because, at their most basic, rotations relate certain configurations of objects in space in which lengths and angles remain unchanged. Indeed, they may be defined in the absence of any notion of motion using these geometric notions. To complicate matters further, there is also a belief commonly-held amongst physicists that the work of Dirac and others demonstrates spin is a consequence of the union between quantum mechanics and special relativity. Indeed, when presenting the work contained in this thesis at conferences, there is usually at least one person who asks me how my work relates to this. As such, for many, spin is also an inherently quantum and relativistic phenomenon.

Given these associations between spin and myriad physical concepts, it is not clear which of these notions are fundamental for systems with spin to exist. A motivating observation is that the Lie algebras which generate $\mathrm{SU}(2,\mathbb{C})$ and $\mathrm{SO}(3,\mathbb{R})$, namely $\mathfrak{su}(2,\mathbb{C})$ and $\mathfrak{so}(3,\mathbb{R})$ respectively, are isomorphic, and so share the same representations. This suggests the possibility of an understanding of spin based more concretely in rotations and their geometric properties. Therefore, to better understand the extent to which geometry, angular momentum, quantum mechanics, and relativity play a role in the fundamental development of spin, we shall directly explore the connections between spin and the geometry of Euclidean three-space. This means we shall not introduce the notion of angular momentum, quantum mechanical notions such as states or probabilities, nor the geometry of Minkowski space into our analysis. Furthermore, to ensure the least amount of structure is used, we shall not introduce any other dynamical concepts into our analysis including: linear momentum, energy, or time. Mathematically, this means that in developing our models of spin will avoid: derivatives, parametrised curves, covectors, and symplectic forms. We will develop such spin models in Chapters 4 and 5.

Exceptions to the above statement will seemingly occur in Chapters 2 and 3, where covectors and derivations will be utilised to develop the mathematical framework we require. The distinction between these usages and their inclusion in physical theories, such as in quantum mechanics, is that we are avoiding including such objects directly in our models for spin; including such objects would necessarily introduce additional algebraic relationships, which is something we seek to minimise.

Seeing how far we can develop our models with such limited mathematical and physical content will communicate important information about the essentiality of dynamics and relativity in defining spin. Moreover, we will see in Chapter 5 that, through our geometric models, spin may be intrinsically linked to non-commutative geometry.

1.1.4.5 Internal Degrees of Freedom

In quantum mechanics, the full state space of a physical system with spin is traditionally modelled as a tensor product between the position-momentum state space and the spinor space. As these models typically describe point-like particles, i.e. particles which can exist at a single point and possess no internal structure, it is common practice to describe the spin degrees of freedom of the system as "internal".

From this, we may observe: the physical reality of such "internal" degrees of freedom for a particle existing at a single point is not obvious; and that, in this model, a clear line is drawn between the space the system occupies and the spin degrees of freedom. Given this somewhat physically unsatisfactory notion of "internal" degrees of freedom for a point-like system, it is natural to wonder if this concept is essential for the description of particles with spin that also exist in space? Furthermore, in light of the relationship between spin and geometry we have previously motivated, is there a way to relate physical space, its geometry, and spin together within a model in which there is no "internal space" associated with the system? The development of such models is addressed in Chapter 5.

1.2 Thesis Aim and Outline

In this thesis, we will present work towards an elementary algebraic theory of non-relativistic spin. To do this, we have opted to assume as little mathematical structure as is practical, as this approach will empower a study of spin in its most fundamental form. Throughout, we will discuss the relationship between this and existing work, as well as the uses and implications of this work for wider physics. Through this approach, we also seek to understand what physical and mathematical structure is strictly necessary for arbitrary spin to exist within a general physical model. This thesis is structures as follows:

Since we are working with as little mathematical structure, and as few physical concepts, as possible, we must be especially rigorous in our definitions and working. Doing so will ensure that we do not inadvertently introduce ambiguities or implicitly assume additional mathematical features. This is especially important for the use of this work in physics, since many of the structures used have distinct meanings which are inappropriate for this work, such as: scalars, vectors, fields, and tensors. This careful overview of the required mathematical background will be given in Chapter 2. The work presented here is mostly well-known, however to the author's knowledge the result of Lemma 2.2.34 and its role in defining $\mathrm{SO}^+(\mathcal{V}, g)$ is novel.

In Chapter 3, we will explore some consequences of the choices we have made in excluding commonly utilised mathematical objects, such as complex numbers. To overcome these difficulties, we will develop a basis-independent methodology for extracting information about an element of a unital associative algebra (including operators) from its algebraic properties alone. These methods are also independent of the dimension of the algebra to which the element belongs, as well as the existence of eigenvalues. The work presented in this chapter was developed independently. During the writing of this thesis, the author became aware of the "Primary

Decomposition Theorem", which generalises the theoretical foundation on which the methods developed are built. However, to the author's knowledge, the connection developed between primary decomposition, identity resolution, and idempotents via Bézout's identity is still novel, as is the content of Sections 3.4 and 3.5.

In Chapter 4, we will utilise these methods to explore the role complex numbers, angular momentum, quantum mechanics, and special relativity play in the defining structure of systems with spin. We will do this by deriving real algebras of spin operators for non-relativistic systems of arbitrary spin using only $\mathfrak{so}(3,\mathbb{R})$, the generators of rotations of Euclidean three-space. This work also explores the relationship between representation theory and the physical properties of spin systems. In particular, the "spin algebras" we derive are constructed directly from, what emerge to be, the physically distinct observables for a system with spin. This will reveal the role of complex numbers in the phenomenology of spin, and indicate that spin is fundamentally geometric, not dynamical, in nature. The results of this chapter, as well as the method employed to derive them, are novel.

Finally, in Chapter 5, we will investigate the role of "internal space" in the description of systems with spin. We will do this by explicating the relationship between spin and geometry, by constructing algebras of position operators in which the structure of an arbitrary spin system is embedded. These algebras contain a natural action of $\mathfrak{so}(3,\mathbb{R})$ on its objects, do not require the notion of "internal space", and necessitate non-trivial couplings between the position and spin degrees of freedom of the system. Through these algebras, we will also discuss the role of spin in naturally generating non-commutative geometries for such systems. All the work presented in this chapter is novel.

Chapter 2

Mathematical Background

2.1 Chapter Aim and Outline

In this chapter, we will formally introduce the mathematical content required to understand the original work of this thesis. While this account is intended to be as complete as possible, and accessible to anyone with an undergraduate degree in physics, some basic set theory is assumed including: sets, subsets, unions, intersections, set differences, Cartesian products, and functions. The notion of a field of numbers is also assumed. While not assumed, it is advantageous to understand the notions of: vector spaces; homomorphisms between vector spaces; the usual ways in which we construct new vector spaces from existing ones; groups; and homomorphisms and antihomomorphisms between groups. Brief accounts of these concepts, as well as some well-known and useful results, are given for reference in Appendices B.1, B.2, B.3, and B.4.

The principal objects of interest in this thesis are algebras. Of particular importance are the ability to perform elementary studies of the structure of an algebra, and to construct specialised algebras from more general ones. To develop these ideas fully, a number of mathematical concepts are required. As each concept is introduced, its relevance to the work presented here will be outlined. To avoid overburdening the reader with information, I have divided this chapter into three parts:

In Section 2.2, we will outline the essential elements for a simple model of spacetime which we will use throughout this thesis. This will include a discussion of: the encoding of geometric information on a vector space through a metric; the structure that such geometries give to the functions on a vector space; the symmetries of such a geometry; and a discussion of how such symmetries are related to geometric objects in space-time. The content of this section will also be used to define general algebras which have no inherent relationship to space-time.

In Section 2.3, we will define the different species of algebraic structures which are essential to our programme. This will entail a discussion of: algebras, in their most general terms; Lie algebras, which generate geometric symmetries; the algebras of functions between vector spaces; functions between algebras; and the notions of actions and modules.

Finally, in Section 2.4, we will introduce the algebras from which we will either draw inspiration, or directly construct the main algebraic results of this thesis: the arbitrary spin algebras in Chapter 4, and the indefinite spin and spin-s position operator algebras in Chapter 5. We will discuss: the quotienting of algebras by ideals; tensor algebras; symmetric algebras; universal enveloping algebras; exterior algebras; the Clifford and Duffin-Kemmer-Petiau algebras; and univariate polynomial algebras, upon which we will base our analytical methods in Chapter 3.

Much of what we will discuss in this chapter comes from [45, 46, 47, 48, 49, 22, 50, 51, 52, 53, 54, 55, 56, 57, 36, 37], which we cite here once to avoid repetition.

2.2 Space, Geometry, and Symmetries

2.2.1 Vector Spaces with Metrics

2.2.1.1 Role in this Thesis

Metrics are perhaps the single most important mathematical object to the developments of this thesis. They encode geometric information about the elements of a vector space such as length, angle, and causal relationships; this information is otherwise absent from a vector space. We will use metrics to construct algebras whose algebraic structure naturally subsumes this geometric information, and will ultimately lead to the algebraic realisations of spin this work is building towards.

2.2.1.2 Metrics

Metrics are nothing more than a particular species of multilinear map, the properties of which determine the geometry one bestows upon our given vector space. They are essential for Special Relativity and underpin the structure of the indefinite-spin position algebra,

Definition 2.2.1 (Metric)**.** A metric is a bilinear map $g : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ on a real vector space $\mathcal{V}$ which is symmetric, $\forall a, b \in \mathcal{V}$,

$$g(b,a) = g(a,b), \tag{2.2.1}$$

and non-degenerate, $\forall a, b \in \mathcal{V}$,

$$\big[\forall b \in \mathcal{V} : g(a,b) = 0\big] \Rightarrow [a = 0]. \tag{2.2.2}$$

Remark. We restrict the definitions given in this section to real vector spaces so that we may discuss classes of metrics important to this work using the strict total ordering of the reals. In principle, one may define metrics over arbitrary fields. Note also that we are following the use of the term "metric" as in special and general relativity; in mathematics, "metric" refers to a different kind of $V \times V \to \mathbb{R}$ function which is in general non-linear.

2.2.1.3 "Geometry" in this Thesis

The word "geometry" without qualifiers can refer to myriad disciples within mathematics. In modern physics, it most commonly refers to the space-time curvature of General Relativity, and the differential geometry which underlies that theory. Many quantities encountered in physics which derive from the structure of space-time, such as lengths, areas, volumes, angles, and their generalisations, are also referred to as "geometric".

These quantities ultimately derive from the metric associated with each point on the pseudo-Riemannian manifold, on which a physical model is defined. Moreover, the "signature" of the metric used determines the precise nature of these quantities. As such, in this thesis we refer to "geometry" in the more restrictive sense of that information "encoded" by the metric of our vector space. This is reasonable in our case, as vector spaces have no natural geometric measures without defining some metric. Furthermore, we will use the term "geometric" to mean those mathematical concepts which are intrinsically linked to our choice of metric, such as objects which transform irreducibly under its symmetry group, and algebras formed from such objects. A closer connection between this notion of geometry and the more general notion encountered in differential geometry is beyond the scope of this thesis.

2.2.1.4 Orthogonality

The first notion we must define in order to understand the information encoded in the metric g is the notion of orthogonality,

Definition 2.2.2 (g-Orthogonal)**.** The vectors $a, b \in \mathcal{V}$ are g-orthogonal iff,

$$g(a,b) = 0. \tag{2.2.3}$$

Remark. Definition 2.2.1 can be understood as a statement that, for a metric g, the only vector $a \in \mathcal{V}$ which is g-orthogonal to all vectors $b \in \mathcal{V}$ (including a itself) is 0.

As we have not yet given any metric definable on $\mathcal{V}$ privilege over any other, it would be improper to not specify with respect to which metric the vectors a, b are orthogonal. However, in this thesis, we will always work with such a privileged metric, and use orthogonal to mean orthogonal with respect to this privileged metric.

2.2.1.5 Minkowski Space-Time

There is no canonical choice of metric for any given vector space. However, in a physical theory we often wish to utilise a particular metric, since it encodes important physical information about our model. To facilitate discussion of vector spaces equipped with particular choices of metrics, let us define,

Definition 2.2.3 ($(\mathcal{V}, g)$)**.** A Minkowski space-time $(\mathcal{V}, g)$ is a pair consisting of a real finite-dimensional vector space $\mathcal{V}$ and a metric $g : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$.

Remark. What we call Minkowski space-times, other authors call "Pseudo-Euclidean" spaces. We will prefer the former as it more readily evokes a connection with Special Relativity, despite generalising that setting.

2.2.1.6 Positive-Definite Metrics

An important class of metrics, which will underpin the non-relativistic spin portion of this thesis, are the positive-definite metrics,

Definition 2.2.4 (Positive-Definite Metric)**.** A metric g is positive-definite iff $\forall a \in \mathcal{V}$ with $a \neq 0$,

$$g(a,a) > 0. \tag{2.2.4}$$

It must be verified that such objects are indeed metrics,

Lemma 2.2.5. *All positive-definite metrics are non-degenerate.*

Proof. Suppose g is a positive-definite metric, and $\exists a \in \mathcal{V}$ such that, $\forall b \in \mathcal{V}$, $g(a,b) = 0$. Then, $g(a,a) = 0$, and so by the positive-definiteness of g, $a = 0$. □

Remark. Lemma 2.2.5 enables us to work with general metrics, and restrict our attention to positive-definite metrics only when necessary; we will do this when discussing non-relativistic spin.

2.2.1.7 Euclidean Three-Space

In this thesis, the most important example of a vector space equipped with a chosen positive-definite metric is Euclidean three-space. This is because its structure underpins the properties of systems with arbitrary spin,

Definition 2.2.6 ((E,δ))**.** Euclidean three-space (E,δ) is a pair consisting of a real three-dimensional vector space E and a positive-definite metric $\delta : E \times E \to \mathbb{R}$.

2.2.1.8 Null and Non-Null Vectors

While Lemma 2.2.5 is true, its converse is not in general. Typically, a metric will be positive- or negative- definite on some subspaces, and identically zero on others. This motivates some terminology,

Definition 2.2.7 (Null and Non-Null Vectors)**.** Consider a Minkowski space-time $(\mathcal{V}, g)$. A vector $a \in \mathcal{V}$ is null iff $g(a,a) = 0$. Similarly, a vector $a \in \mathcal{V}$ is non-null iff $g(a,a) \neq 0$.

Remark. Many authors further classify non-null vectors $a \in \mathcal{V}$ by the definiteness of $g(a,a)$. We shall not commit to such an extension to avoid confusion, as such schemes are usually tied to a particular choice of metric and basis.

2.2.2 Metric Adjoint Endomorphisms

2.2.2.1 Role in this Thesis

Metrics not only give structure to the vector spaces on which they act, they also distinguish many of the vector space's endomorphisms. An important way in which it does this is by identifying pairs of endomorphisms which are in a precise sense compatible with the metric. This identification is part of the definition of the Lie algebra of the symmetry group of the metric, and understanding how it works will enable us to discover the elements of the Lie algebra without the need for the machinery of differential geometry, or the theory of Lie groups.

2.2.2.2 Metric Adjoints

For every vector space endomorphism, a metric associates a unique counterpart endomorphism in a way which is naturally compatible with its structure. Understanding these counterparts will inform our study of the symmetries of the metric.

Definition 2.2.8 (g-adjoint)**.** Consider a Minkowski space-time $(\mathcal{V}, g)$. A g-adjoint of an endomorphism $A \in \mathrm{End}(\mathcal{V})$ is an endomorphism $B \in \mathrm{End}(\mathcal{V})$ such that $\forall v, w \in \mathcal{V}$,

$$g(A(v),w) = g(v,B(w)). \tag{2.2.5}$$

Lemma 2.2.9. *Consider a Minkowski space-time $(\mathcal{V}, g)$. For all $A \in \mathrm{End}(\mathcal{V})$, a g-adjoint of A exists.*

Proof. See Appendix B.5. □

Lemma 2.2.10. *For all $A \in \mathrm{End}(\mathcal{V})$, the g-adjoint of A is unique.*

Proof. Suppose $B, C \in \mathrm{End}(\mathcal{V})$ are g-adjoints of A. Then, $\forall v, w \in \mathcal{V}$,

$$g(A(v),w) = g(v,B(w)) = g(v,C(w)),$$

thus,

$$g\big(v, (B - C)(w)\big) = 0.$$

By the non-degeneracy of g, we have, $\forall w \in \mathcal{V}$,

$$(B - C)(w) = 0.$$

Therefore, $B = C$. □

Lemmas 2.2.9 and 2.2.10 enable us to define a function from $\mathrm{End}(\mathcal{V})$ to itself which yields the g-adjoint of any endomorphism,

Definition 2.2.11 (A^g)**.** Let us define $(\,\cdot\,)^g : \mathrm{End}(\mathcal{V}) \to \mathrm{End}(\mathcal{V})$ such that, $\forall A \in \mathrm{End}(\mathcal{V})$, $\forall v, w \in \mathcal{V}$,

$$g(A(v),w) = g(v,A^g(w)). \tag{2.2.6}$$

Remark. The notation employed in Definition 2.2.11 is intended to mirror the notation for matrix transposition and Hermitian conjugation, as the g-adjoint shares many similar properties to these.

The map $(\,\cdot\,)^g$ is a useful encapsulation of the notion of a g-adjoint since it respects the vector space structure of $\mathrm{End}(\mathcal{V})$, and so is an endomorphism on $\mathrm{End}(\mathcal{V})$,

Lemma 2.2.12. $(\,\cdot\,)^g \in \mathrm{End}\big(\mathrm{End}(\mathcal{V})\big)$.

Proof. For all $A, B \in \mathrm{End}(\mathcal{V})$, $\forall v, w \in \mathcal{V}$,

$$\begin{aligned} g\big(v, (A+B)^g(w)\big) &= g\big((A+B)(v), w\big) \\ &= g(A(v),w) + g(B(v),w) \\ &= g(v,A^g(w)) + g(v,B^g(w)) \\ &= g\big(v, (A^g + B^g)(w)\big), \end{aligned}$$

which by Lemma 2.2.10 establishes $(A+B)^g = A^g + B^g$. Furthermore, $\forall \alpha \in \mathbb{R}$,

$$\begin{aligned} g\big(v, (\alpha A)^g(w)\big) &= g\big((\alpha A)(v), w\big) \\ &= \alpha g(A(v),w) \\ &= \alpha g(v,A^g(w)) \\ &= g\big(v, (\alpha A^g)(w)\big), \end{aligned}$$

which again by Lemma 2.2.10 establishes that $(\alpha A)^g = \alpha A^g$. □

The map $(\,\cdot\,)^g$ is also an involution,

Lemma 2.2.13. *For all* $A \in \mathrm{End}(\mathcal{V})$,

$$(A^g)^g = A. \tag{2.2.7}$$

Proof. For all $v, w \in \mathcal{V}$,

$$g\big(v, (A^g)^g(w)\big) = g(A^g(v),w) = g(w,A^g(v)) = g(A(w),v) = g(v,A(w)),$$

which by Lemma 2.2.10 establishes the result. □

Finally, to complete our classification of the properties of $(\,\cdot\,)^g$, and for later use in our analysis of the symmetries of g, let us consider its relationship with composition,

Lemma 2.2.14. *For all* $A, B \in \mathrm{End}(\mathcal{V})$

$$(A \circ B)^g = B^g \circ A^g. \tag{2.2.8}$$

Proof. For all $v, w \in \mathcal{V}$,

$$g\big(v, (A\circ B)^g(w)\big) = g(A\circ B(v),w) = g(B(v),A^g(w)) = g\big(v, B^g\circ A^g(w)\big),$$

which by Lemma 2.2.10 establishes the result. □

2.2.2.3 Self- and Anti-Self-Metric Adjoints

The involutive nature of the map $(\,\cdot\,)^g$ enables us to study the relationship that general endomorphisms of $\mathcal{V}$ have with the metric g by decomposing them in a natural way. This will be especially useful when discussing the Lie algebra of the symmetry group of the metric g.

Lemma 2.2.15. *The map $(\,\cdot\,)^g$ only has eigenvalues ± 1.*

Proof. Consider $A \in \mathrm{End}(\mathcal{V})$ such that for some $\lambda \in \mathbb{R}$,

$$A^g = \lambda A.$$

Then, by Lemma 2.2.13,

$$A = (A^g)^g = \lambda^2 A,$$

thus, $\lambda = \pm 1$. □

Let us name the eigenstates of $(\,\cdot\,)^g$ for ease of reference,

Definition 2.2.16 (Self- and Anti-Self-g-adjoint)**.** An endomorphism $A \in \mathrm{End}(\mathcal{V})$ is self-g-adjoint when $A^g = A$, and anti-self-g-adjoint when $A^g = -A$.

We may easily convert any endomorphism into a self- or anti-self-g-adjoint one,

Definition 2.2.17. For all $A \in \mathrm{End}(\mathcal{V})$,

$$a_\pm(A) \coloneqq \frac{1}{2}\big(A \pm A^g\big). \tag{2.2.9}$$

Lemma 2.2.18. *For all $A \in \mathrm{End}(\mathcal{V})$, $a_+(A)$ is self-g-adjoint, and $a_-(A)$ is anti-self-g-adjoint.*

Proof. For all $v, w \in \mathcal{V}$,

$$\begin{aligned} g(a_\pm(A)(v),w) &= g\Big(\frac{1}{2}\big(A \pm A^g\big)(v), w\Big) \\ &= g\Big(v, \frac{1}{2}\big(A^g \pm A\big)(w)\Big) \\ &= \pm g(v,a_\pm(A)(w)). \end{aligned}$$

□

Moreover, these self- and anti-self-g-adjoint parts cannot be further decomposed,

Lemma 2.2.19. *For all* $A \in \mathrm{End}(\mathcal{V})$*,*

$$a_\pm \circ a_\pm(A) = a_\pm(A) \tag{2.2.10a}$$

$$a_\pm \circ a_\mp(A) = 0. \tag{2.2.10b}$$

Proof. Direct computation. □

Thus,

Corollary 2.2.20. *Each* $A \in \mathrm{End}(\mathcal{V})$ *may be decomposed into self-g-adjoint and anti-self-g-adjoint parts,*

$$A = a_+(A) + a_-(A). \tag{2.2.11}$$

Proof. This follows immediately from Definition 2.2.17 and Lemma 2.2.18. □

Finally,

Lemma 2.2.21. *The self- and anti-self-g-adjoint parts of an endomorphism are unique.*

Proof. Suppose for an endomorphism $A \in \mathrm{End}(\mathcal{V})$, $\exists B, C, D, E \in \mathrm{End}(\mathcal{V})$ for which $B^g = B$, $C^g = -C$, $D^g = D$, and $E^g = -E$, such that,

$$A = B + C = D + E.$$

Applying $a_\pm$ to this equation reveals $B = D$ and $C = E$. □

The ability to isolate self- and anti-self-g-adjoint parts of an endomorphism will facilitate a rapid identification of the $\mathfrak{so}(\mathcal{V}, g)$-action on vectors in Chapter 5.

2.2.3 Symmetries of the Metric

2.2.3.1 Role in this Thesis

The symmetries of a metric are automorphisms which naturally respect the geometry it encodes. Many of these automorphisms are generated by simple transformations which are intimately connected to geometric objects in our space. Understanding the structure of these generating maps is what will enable our development of new, inherently geometrical models of systems with arbitrary spin. To enable this, we will examine the symmetries of a metric in two stages. First, we will consider a general definition for these symmetries and find many insights already available to us. Following this, we will seek to understand the relationship between general endomorphisms on our space and these symmetries, and in the process reveal the role that Lie algebras play behind the scenes.

2.2.3.2 Orthogonal Endomorphisms and $\mathrm{O}(\mathcal{V}, g)$

Let us begin our analysis by giving a general definition for the symmetries of a metric,

Definition 2.2.22 (g-Symmetry Endomorphism)**.** Consider a Minkowski spacetime $(\mathcal{V}, g)$. A g-symmetry endomorphism is an endomorphism $A \in \mathrm{End}(\mathcal{V})$ such that, $\forall v, w \in \mathcal{V}$,

$$g(A(v), A(w)) = g(v, w). \tag{2.2.12}$$

Definition 2.2.23 (Orthogonal Endomorphism)**.** In this thesis, orthogonal endomorphism and g-symmetry endomorphism are synonymous. The former is more widely used.

Often we are not interested in any particular orthogonal endomorphism, so let us also define,

Definition 2.2.24 ($\mathrm{O}(\mathcal{V}, g)$)**.** $\mathrm{O}(\mathcal{V}, g)$ is the set of all orthogonal endomorphisms of g.

Remark. $\mathrm{O}(\mathcal{V}, g)$ is often denoted $\mathrm{O}(p, q, \mathbb{R})$ which explicates the "signature" of the metric g. The definition and discussion of metric signatures beyond this notation is unnecessary for this thesis, and so has been avoided.

2.2.3.3 The Properties of $\mathrm{O}(\mathcal{V}, g)$

The set of orthogonal endomorphisms $\mathrm{O}(\mathcal{V}, g)$ may naturally be given the structure of a group.

Lemma 2.2.25. $\mathrm{O}(\mathcal{V}, g)$ *forms a group under composition.*

Proof. See Appendix B.6. □

We must understand this group structure as it closely relates to the structure of its Lie algebra, which we require to develop the spin algebras.

Remark. From this point on we will abuse notation by referring to $\mathrm{O}(\mathcal{V}, g)$ as the orthogonal group for the metric g; in doing this we imply its group product is composition of homomorphisms.

2.2.3.4 A Description of $\mathrm{O}(\mathcal{V}, g)$ by Reflections

With the group properties of $\mathrm{O}(\mathcal{V}, g)$ established, let us give a prescription for constructing arbitrary orthogonal maps. Understanding this prescription will later enable us to identify the product of the Lie algebra of $\mathrm{O}(\mathcal{V}, g)$ in a simple way. We proceed by defining,

Definition 2.2.26 (Householder Reflection $R(a)$)**.** For non-null $a \in \mathcal{V}$, the Householder reflection $R(a)$ in a is,

$$R(a) := v \mapsto v - 2\frac{g(a,v)}{g(a,a)}a. \tag{2.2.13}$$

We will use the terms "Householder reflection" and "reflection" interchangeably. The reflections are a useful starting point because,

Lemma 2.2.27. *For all non-null $a \in \mathcal{V}$, $R(a) \in \mathrm{O}(\mathcal{V}, g)$.*

Proof. Direct computation. □

In fact, by the famous Cartan-Dieudonné theorem,

Theorem 2.2.28 (Cartan-Dieudonné)**.** *All transformations from $\mathrm{O}(\mathcal{V}, g)$ are compositions of at most $\dim(\mathcal{V})$ reflections.*

Proof. See [58, 59, 55]. □

Thus, we have a recipe to construct arbitrary elements of $\mathrm{O}(\mathcal{V}, g)$, and thus a means to probe its group structure. In particular, in Chapter 5, studying the properties of reflections will lead us naturally to the Lie algebra action of $\mathfrak{so}(\mathcal{V}, g)$ on $\mathcal{V}$. The ability to derive this action from spatial symmetry alone will strengthen our claim that the underlying structure of systems with spin is rooted in the geometry of Euclidean three-space.

2.2.3.5 The Relationship Between End($\mathcal{V}$) and $\mathrm{O}(\mathcal{V}, g)$

Let us now begin the second phase of our analysis by investigating the relationship between a generic endomorphism $A \in \mathrm{End}(\mathcal{V})$, and the symmetry group $\mathrm{O}(\mathcal{V}, g)$ of

$(\mathcal{V}, g)$. The argument presented here is, to his knowledge, the author's own, and offers a route to the identity-connected component of $\mathrm{O}(\mathcal{V}, g)$, and its Lie algebra, without needing to employ topology or differentiable manifolds.

Recall from Lemma 2.2.10, the metric g imparts a structure on $\mathrm{End}(\mathcal{V})$ through the g-adjoint $(\,\cdot\,)^g$, which associates to each endomorphism $A \in \mathrm{End}(\mathcal{V})$ a unique partner $A^g \in \mathrm{End}(\mathcal{V})$. To understand the significance of this association to the symmetries of g, let us first define,

Definition 2.2.29 (Endomorphism Exponential (exp))**.** The exponential $\exp(A)$ of an endomorphism $A \in \mathrm{End}(\mathcal{V})$ is,

$$\exp(A) \coloneqq \sum_{j=0}^{\infty} \frac{1}{j!} A^{\circ j}, \tag{2.2.14}$$

where $A^{\circ 0} = \mathrm{id}_{\mathcal{V}}$ and $\forall k \in \mathbb{N}$, $A^{\circ(k+1)} \coloneqq A \circ A^{\circ k}$.

Corollary 2.2.30.

$$\exp(0) = \mathrm{id}_{\mathcal{V}}\,. \tag{2.2.15}$$

Proof. Direct computation. □

Importantly,

Lemma 2.2.31. *Consider a finite-dimensional vector space* $\mathcal{V}$*. For all* $A \in \mathrm{End}(\mathcal{V})$*,* $\exp(A)$ *converges to an element of* $\mathrm{End}(\mathcal{V})$*.*

Proof. See [52]. □

Furthermore,

Lemma 2.2.32. *Consider* $A, B \in \mathrm{End}(\mathcal{V})$ *such that* $A \circ B = B \circ A$*. Then,*

$$\exp(A + B) = \exp(A) \circ \exp(B). \tag{2.2.16}$$

Proof. See [7]. □

Finally,

Lemma 2.2.33. *For all* $A \in \mathrm{End}(\mathcal{V})$*,*

$$\big(\exp(A)\big)^g = \exp(A^g). \tag{2.2.17}$$

Proof. Direct computation. □

With these Lemmas in hand, we may now show,

Lemma 2.2.34. *For all* $A \in \mathrm{End}(\mathcal{V})$, $\forall v, w \in \mathcal{V}$,

$$g\big(\exp(A)(v), \exp(-A^g)(w)\big) = g(v,w). \tag{2.2.18}$$

Proof.

$$\begin{aligned} g(v,w) &= g\big(\exp(A-A)(v), w\big) \\ &= g\big(\exp(-A)\circ\exp(A)(v), w\big) \\ &= g\big(\exp(A)(v), (\exp(-A))^g(w)\big) \\ &= g\big(\exp(A)(v), \exp(-A^g)(w)\big). \end{aligned}$$

□

2.2.3.6 The Group $\mathrm{SO}^+(\mathcal{V}, g)$

Lemma 2.2.34 reveals some structure of $\mathrm{O}(\mathcal{V}, g)$ which is of central importance to this thesis,

Corollary 2.2.35. *For all* $A \in \mathrm{End}(\mathcal{V})$ *for which* $A = -A^g$, $\exp(A) \in \mathrm{O}(\mathcal{V}, g)$.

Proof. Directly follows from Lemma 2.2.34. □

Thus, we have found some elements of $\mathrm{O}(\mathcal{V}, g)$. In general, $\exp(A)\circ\exp(B) \neq \exp(A+B)$, so we are motivated to define,

Definition 2.2.36 ($\mathrm{SO}^+(\mathcal{V}, g)$)**.** $\mathrm{SO}^+(\mathcal{V}, g)$ is the set of all finite compositions of endomorphisms $\exp(A) \in \mathrm{End}(\mathcal{V})$, $\forall A \in \{B \in \mathrm{End}(\mathcal{V}) \mid B = -B^g\}$.

As one might expect from this definition,

Lemma 2.2.37. $\mathrm{SO}^+(\mathcal{V}, g)$ *forms a group under composition.*

Proof. $\mathrm{SO}^+(\mathcal{V}, g)$ is closed under composition by definition. Since $0 = -0^g$, $\exp(0) = \mathrm{id}_{\mathcal{V}} \in \mathrm{SO}^+(\mathcal{V}, g)$. Furthermore, for all $A \in \mathrm{End}(\mathcal{V})$ such that $A = -A^g$, the inverse map of $\exp(A)$ is $\big(\exp(A)\big)^g = \exp(-A)$. Since $(-A) = -(-A)^g$, $\exp(-A) \in \mathrm{SO}^+(\mathcal{V}, g)$. □

Corollary 2.2.38. $\mathrm{SO}^+(\mathcal{V}, g)$ *is a subgroup of* $\mathrm{O}(\mathcal{V}, g)$.

Proof. By definition. □

$\mathrm{SO}^+(\mathcal{V}, g)$ is of great importance to $\mathrm{O}(\mathcal{V}, g)$, to a degree far beyond the scope of this thesis. What is of greater importance to us are the endomorphisms which "generate" $\mathrm{SO}^+(\mathcal{V}, g)$, as these are the central objects in our construction of the spin algebras. We will examine these in the next section.

2.2.3.7 The Group $\mathbf{SO}(3, \mathbb{R})$

As with $\mathrm{O}(\mathcal{V}, g)$, $\mathrm{SO}^+(\mathcal{V}, g)$ is more commonly referred to as $\mathrm{SO}^+(p, q, \mathbb{R})$. We will avoid this notation in general, except in one particular case,

Definition 2.2.39 ($\mathrm{SO}(3, \mathbb{R})$)**.** In this thesis, $\mathrm{SO}(3, \mathbb{R})$ and $\mathrm{SO}^+(E, \delta)$ are synonymous.

We will do this to ensure all discussion of the rotation group when developing the spin algebras is done using as familiar language as possible to ensure accessibility to those less familiar with metrics and the orthogonal groups in the abstract.

2.2.4 The Lie Algebra of $\mathrm{SO}^+(\mathcal{V}, g)$

2.2.4.1 Role in this Thesis

The Lie algebra which "generates" the group $\mathrm{SO}^+(\mathcal{V}, g)$ lies at the heart of two central results of this thesis: the spin algebras and the indefinite/spin-s position operator algebras. As such, understanding its structure is paramount to the success of this work.

2.2.4.2 The Lie Algebra $\mathfrak{so}(\mathcal{V}, g)$

We saw in Definition 2.2.36 that the set of all anti-self-g-adjoint endomorphisms is essential in defining $\mathrm{SO}^+(\mathcal{V}, g)$. Let us capture these endomorphisms,

Definition 2.2.40 ($\mathfrak{so}(\mathcal{V}, g)$)**.** Consider a Minkowski space $(\mathcal{V}, g)$. $\mathfrak{so}(\mathcal{V}, g)$ is the set of all $A \in \mathrm{End}(\mathcal{V})$ which are anti-self-g-adjoint, $A = -A^g$, i.e. $\forall v, w \in \mathcal{V}$,

$$g(A(v), w) + g(v, A(w)) = 0. \tag{2.2.19}$$

Just as the set $\mathrm{O}(\mathcal{V}, g)$ naturally supported a group structure, $\mathfrak{so}(\mathcal{V}, g)$ supports the structure of a Lie algebra. We will define Lie algebras formally in Section 2.3.2.

Lemma 2.2.41. *The quadruple* $(\mathfrak{so}(\mathcal{V}, g), \mathbb{R}, +, \text{“ ”})$ *is a vector space with vector addition and scalar multiplication inherited from* $\mathrm{End}(\mathcal{V})$.

Proof. This follows from $(\,\cdot\,)^g \in \mathrm{End}\big(\mathrm{End}(\mathcal{V})\big)$. □

Lemma 2.2.42. *$\mathfrak{so}(\mathcal{V},g)$ is closed under commutators.*

Proof. For all $A, B \in \mathfrak{so}(\mathcal{V},g)$,

$$[A,B]^g = (A \circ B - B \circ A)^g = B^g \circ A^g - A^g \circ B^g = -[A,B].$$

□

Lemma 2.2.43. *With respect to $(\mathfrak{so}(\mathcal{V},g), \mathbb{R}, +, \text{“ ”})$, the commutator is a bilinear map.*

Proof. This follows from the fact the commutator is a bilinear map with respect to the vector space structure of $\mathrm{End}(\mathcal{V})$. □

Corollary 2.2.44. *$(\mathfrak{so}(\mathcal{V},g), \mathbb{R}, +, \text{“ ”})$ forms a Lie algebra with Lie product the commutator.*

Proof. This follows from Lemmas 2.2.41, 2.2.42, and 2.2.43. □

Remark. From this point on we will adopt a common abuse of notation by referring to $\mathfrak{so}(\mathcal{V},g)$ as the Lie algebra formed by $(\mathfrak{so}(\mathcal{V},g), \mathbb{R}, +, \text{“ ”})$ with the commutator as its Lie product.

2.2.4.3 A Concrete Description of $\mathfrak{so}(\mathcal{V},g)$

While our definition of $\mathfrak{so}(\mathcal{V},g)$ is perfectly good, there is much to its structure which must be uncovered. One may be tempted at this point to find a matrix representation of $\mathfrak{so}(\mathcal{V},g)$ to make progress, but this is unnecessary; a complete description of the constituent endomorphisms of $\mathfrak{so}(\mathcal{V},g)$ is possible using only the metric g. To show this, let us first define,

Definition 2.2.45 ($t(a,b)$)**.** For all $a, b \in \mathcal{V}$, $t(a,b)$ is the bilinear map,

$$\begin{aligned} t : \mathcal{V} \times \mathcal{V} &\to \mathrm{End}(\mathcal{V}) \\ t(a,b) &:= v \mapsto g(a,v)b - g(b,v)a. \end{aligned} \tag{2.2.20}$$

Remark. At this point in our discussion, this definition may seem somewhat arbitrary. We shall provide a mode concrete justification for it in Chapter 5; we will demonstrate that $t(a,b)$ controls the composition of two Householder reflections. This provides a natural context, connected to the symmetries of g, in which to identify this map.

Let us first note that,

Lemma 2.2.46. *For all* $v, w \in \mathcal{V}$, $t(a,b) \in \mathfrak{so}(\mathcal{V}, g)$.

Proof. Direct computation verifies that $t(v,w)^g = -t(v,w)$. □

Lemma 2.2.47. *Any* $A \in \mathfrak{so}(\mathcal{V}, g)$ *may be written in the form,*

$$A = \sum_{j \in J} \alpha_j \, t(a_j,b_j), \tag{2.2.21}$$

where $\forall j \in J$, $\alpha_j \in \mathbb{R}$, $(a_j, b_j) \in \mathcal{V} \times \mathcal{V}$.

Proof. See Appendix B.7. □

This fact enables us to find the dimension of $\mathfrak{so}(\mathcal{V}, g)$,

Lemma 2.2.48. $\dim\big(\mathfrak{so}(\mathcal{V}, g)\big) = \frac{1}{2} \dim(\mathcal{V})\big(\dim(\mathcal{V}) - 1\big)$.

Proof. See Appendix B.7. □

It also entails a basis-independent description for the Lie product of $\mathfrak{so}(\mathcal{V}, g)$, i.e. the commutator,

Lemma 2.2.49. *For all* $a, b, c, d \in \mathcal{V}$,

$$t(a,b) \circ t(c,d) - t(c,d) \circ t(a,b) = t\big(t(a,b)(c), d\big) + t\big(c, t(a,b)(d)\big). \tag{2.2.22}$$

Proof. Direct computation. □

The concrete description of $\mathfrak{so}(\mathcal{V}, g)$ we have constructed in terms of the maps $t(a,b)$ will be important for us in establishing an algebraic realisation of $\mathfrak{so}(\mathcal{V}, g)$ in terms of bivectors, and in relating these to the generators of spin.

2.2.4.4 An Abstract Algebraic Description of $\mathfrak{so}(\mathcal{V}, g)$

So far, we have explored the Lie algebra $\mathfrak{so}(\mathcal{V}, g)$ in its form as a commutator algebra of anti-self-g-adjoint endomorphisms on $\mathcal{V}$, and established many of its important properties. We may use this information to motivate an alternative formulation of $\mathfrak{so}(\mathcal{V}, g)$ as an abstract algebra naturally defined in terms of geometrically meaningful objects. This approach is the first step towards incorporating the structure of $\mathfrak{so}(\mathcal{V}, g)$ into general algebraic settings, which we will do in Chapter 5.

To begin this process, let us define,

Definition 2.2.50 (2-Blade)**.** We define a 2-blade $a{\wedge}b \in \mathcal{V}^{\otimes 2}$, as, $\forall a, b \in \mathcal{V}$,

$$a{\wedge}b \coloneqq \frac{1}{2}(a{\otimes}b - b{\otimes}a). \tag{2.2.23}$$

Remark. We may interpret a 2-blade geometrically as an oriented area element.

Definition 2.2.51 (Bivector)**.** A bivector $B \in \mathcal{V}^{\otimes 2}$ is a finite linear combination of 2-blades.

Remark. In two- and three-dimensions, all bivectors may be written as a single 2-blade, but this ceases to be true in four-dimensions and higher. This distinction between 2-blades and more general bivectors, is consistent with, for example, [60].

Definition 2.2.52 ($\Lambda^2(\mathcal{V})$)**.** We denote the vector space of all bivectors constructed from vectors in $\mathcal{V}$ as $\Lambda^2(\mathcal{V})$.

The universal property of the tensor product of vector spaces, implies that there exists a unique extension of the antisymmetric bilinear map t to a homomorphism on bivectors,

Definition 2.2.53 ($\mu(a{\wedge}b)$)**.** We define the vector space homomorphism μ, $\forall a{\wedge}b \in \Lambda^2(\mathcal{V})$,

$$\begin{aligned} \mu : \Lambda^2(\mathcal{V}) &\to \mathfrak{so}(\mathcal{V}, g) \\ \mu(a{\wedge}b) &= v \mapsto g(a,v)b - g(b,v)a. \end{aligned} \tag{2.2.24}$$

Remark. Strictly speaking the universal property of the tensor product extends t to a linear map on $\mathcal{V}^{\otimes 2}$, but we have used the antisymmetry of t to naturally restrict μ to $\Lambda^2(\mathcal{V})$. We have also used the fact that $\forall a, b \in \mathcal{V}$, $\mu(a{\wedge}b) = t(a,b) \in \mathfrak{so}(\mathcal{V}, g)$ to naturally project $\mu(a{\wedge}b)$ into $\mathfrak{so}(\mathcal{V}, g)$.

The homomorphism μ is more than a curious repackaging of t, it enables us to precisely define the connection between bivectors and the anti-self-g-adjoint endomorphisms of $\mathfrak{so}(\mathcal{V}, g)$,

Lemma 2.2.54. *μ is a vector space isomorphism.*

Proof. See Appendix B.7. □

Lemma 2.2.54 attests a connection between bivectors and the generators of $\mathrm{SO}^+(\mathcal{V}, g)$. In fact, we may draw this connection between $\Lambda^2(\mathcal{V})$ and $\mathfrak{so}(\mathcal{V}, g)$ even closer by considering a natural question: if each element of $\mathfrak{so}(\mathcal{V}, g)$ has a corresponding element of $\Lambda^2(\mathcal{V})$, is there an analogue of the Lie product of $\mathfrak{so}(\mathcal{V}, g)$ on

$\Lambda^2(\mathcal{V})$? Pursuing this will yield an abstract realisation of the Lie algebra $\mathfrak{so}(\mathcal{V},g)$ in terms of geometrically significant objects which we will use extensively in this thesis to construct new algebras which subsume this structure.

To make progress, let us note that,

Lemma 2.2.55. *For all $a\wedge b, c\wedge d \in \Lambda^2(\mathcal{V})$,*

$$\mu(a\wedge b)\circ\mu(c\wedge d) - \mu(c\wedge d)\circ\mu(a\wedge b) = \mu\big(\mu(a\wedge b)(c)\wedge d + c\wedge\mu(a\wedge b)(d)\big). \quad (2.2.25)$$

Proof. This follows directly from Lemma 2.2.49. □

The right-hand side in Lemma 2.2.55 suggests a natural definition for a Lie product on $\Lambda^2(\mathcal{V})$ is,

Definition 2.2.56 ($\lambda(a\wedge b, c\wedge d)$)**.** For all $a\wedge b, c\wedge d \in \Lambda^2(\mathcal{V})$,

$$\begin{aligned} \lambda : \Lambda^2(\mathcal{V}) \times \Lambda^2(\mathcal{V}) &\to \Lambda^2(\mathcal{V}) \\ \lambda(a\wedge b, c\wedge d) &:= \mu(a\wedge b)(c)\wedge d + c\wedge\mu(a\wedge b)(d), \end{aligned} \quad (2.2.26)$$

extended bilinearly.

Remark. Definition 2.2.56 is consistent with the standard "orthonormal" basis-dependent forms for $\mathfrak{so}(\mathcal{V},g)$ given in the physics literature, for example in [5], where they instead use the complexified basis $\hat{J}^{\mu\nu} = i(e^\mu\wedge e^\nu)$.

This allows us to promote μ from a vector space isomorphism to a Lie algebra isomorphism, i.e. it also maps Lie products into each other,

Lemma 2.2.57. *For all $A, B \in \Lambda^2(\mathcal{V})$,*

$$\mu\big(\lambda(A,B)\big) = \big[\mu(A), \mu(B)\big]. \quad (2.2.27)$$

Proof. Directly follows from Lemma 2.2.55. □

We will fully define and generalise such homomorphisms in Section 2.3.4.

2.2.4.5 Scale Freedom in Descriptions of $\mathfrak{so}(\mathcal{V},g)$

With Lemma 2.2.57, we have completely connected the concrete description of $\mathfrak{so}(\mathcal{V},g)$ in terms of the maps $t(a,b)$ and its abstraction as the Lie algebra of bivectors $\Lambda^2(\mathcal{V})$ equipped with Lie product λ. However, many algebras which implement some version of the bivector Lie algebra do not do so exactly as in Definition 2.2.56:

there is often a change in the scale of the Lie product. We will now show that this does not affect the isomorphism as Lie algebras between $\mathfrak{so}(\mathcal{V}, g)$ and $\Lambda^2(\mathcal{V})$ that we have established. To begin,

Lemma 2.2.58. *Consider a vector space isomorphism $f : \mathcal{V} \to \mathcal{W}$ between two vector spaces over the same field $\mathbb{F}$. For all $\kappa \in \mathbb{F}$, $\kappa \neq 0$, the map $f_\kappa := v \mapsto \kappa f(v)$ is a vector space isomorphism.*

Proof. Direct computation by noting $f_\kappa^{-1} = v \mapsto \frac{1}{\kappa} f^{-1}(v)$. □

Lemma 2.2.59. *Consider a Lie algebra isomorphism $f : \mathcal{V} \to \mathcal{W}$ between Lie algebras over the same field $\mathbb{F}$ with Lie products p and q respectively. For all $\kappa \in \mathbb{F}$, $\kappa \neq 0$, the map $f_\kappa := v \mapsto \kappa f(v)$ is a Lie algebra isomorphism between $\mathcal{V}$ with Lie product $p_\kappa := \kappa p$, and $\mathcal{W}$ with Lie product q.*

Proof. For f to be a Lie algebra isomorphism, it must be a vector space isomorphism which satisfies $\forall a, b \in \mathcal{V}$,

$$f(p(a, b)) = q(f(a), f(b)).$$

Thus,

$$f_\kappa(p_\kappa(a, b)) = \kappa^2 f(p(a, b)) = \kappa^2 q(f(a), f(b)) = q(f_\kappa(a), f_\kappa(b)).$$

□

Accounting for this scale freedom between Lie products will enable us to construct general algebras which implement the bivector Lie algebra up to an arbitrary non-zero scale; this will ensure compatibility between these and other algebras which implement the bivector Lie algebra, whilst ensuring its connection to $\mathfrak{so}(\mathcal{V}, g)$ is maintained.

2.2.4.6 An Abstract Algebraic Description of $\mathfrak{so}(3, \mathbb{R})$

Of principal importance to the development of the spin algebraic results of this thesis is the Lie algebra of Euclidean three-space $\mathfrak{so}(E, \delta)$. To remove any ambiguity in our discussion, we will denote the Lie product on $\Lambda^2(E)$ by,

Definition 2.2.60 ($(a \wedge b) \times (c \wedge d)$)**.** For all $a \wedge b, c \wedge d \in \Lambda^2(E)$, $\forall \kappa \in \mathbb{R}$, $\kappa \neq 0$,

$$\begin{aligned} \times : \Lambda^2(E) \times \Lambda^2(E) &\to \Lambda^2(E) \\ (a \wedge b) \times (c \wedge d) &:= \kappa \lambda(a \wedge b, c \wedge d), \end{aligned} \tag{2.2.28}$$

where λ is defined on (E, δ) as in Definition 2.2.56.

Remark. The $\times$ notation has been chosen to reflect the fact that $\mathfrak{so}(E,\delta)$ is isomorphic to the cross-product Lie algebra in three dimensions.

We will also adopt a notation for the basis vectors of $\Lambda^2(E)$ which is consistent with the common notation used for spin generators, for example in [61],

Definition 2.2.61 (S_p)**.** Consider an orthonormal basis $\{e_a\}$ indexed over the set $\{1,2,3\}$, i.e. $\forall a,b \in \{1,2,3\}$, $\delta(e_a,e_b) = \delta_{ab}$, with δ_{ab} the Kronecker delta. Then, $\forall p \in \{1,2,3\}$, we define,

$$S_p := \frac{1}{2\kappa} \sum_{a,b=1}^{3} \varepsilon_{abp}\, e_a \wedge e_b, \tag{2.2.29}$$

which has inverse transformation,

$$e_a \wedge e_b = \kappa \sum_{p=1}^{3} \varepsilon_{abp} S_p, \tag{2.2.30}$$

where ε_{abc} is the Levi-Civita symbol.

Lemma 2.2.62. *For all* $p,q \in \{1,2,3\}$,

$$S_p \times S_q = \sum_{r=1}^{3} \varepsilon_{pqr} S_r, \tag{2.2.31}$$

Proof. Direct use of Definition 2.2.61 applied to Definition 2.2.60 evaluated on $e_a \wedge e_b$ and $e_c \wedge e_d$. □

Remark. Lemma 2.2.62 is consistent with the standard orthonormal basis-dependent forms for $\mathfrak{so}(E,\delta)$ given in the physics literature, for example in [1], where they instead use the complexified basis $\hat{S}_p = iS_p$.

Finally, to avoid as much confusion as possible, we will adopt the more familiar terminology,

Definition 2.2.63 ($\mathfrak{so}(3,\mathbb{R})$)**.** In this thesis, $\mathfrak{so}(3,\mathbb{R})$, $\mathfrak{so}(E,\delta)$ and, isomorphism implicit, the Lie algebra of generators $(\{S_a\},\times)$ are synonymous.

2.3 Algebraic Structures

Throughout Section 2.2, our developments naturally entailed algebraic structures of various species and homomorphisms between them. In this section, we will formally define these, and other algebraic structures and concepts which are essential to the work of this thesis.

2.3.1 General Algebras

2.3.1.1 Role in this Thesis

Algebras are at the heart of all the work presented in this thesis. They are both a major result from it, enabling the abstract modelling of systems with spin, and the essential component in the methods we will develop to derive and study these structures. They are utterly essential and indispensable to this programme, and most of the developments of this chapter have been building up to them.

2.3.1.2 Algebras over a Field

There are many species of algebra which are important for this thesis. Let us begin with the most general definition,

Definition 2.3.1 (Algebra over a Field)**.** An algebra over the field $\mathbb{F}$ is a pair $\mathcal{A} = (\mathcal{V}, \bullet)$ consisting of:

- a vector space $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$ over $\mathcal{V}$;
- a bilinear map $\bullet : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, called the product.

This definition is usually too general for most applications, and certainly is for the kinds of algebras we will be using in this thesis. As such, let us define some specialisations,

Definition 2.3.2 (Unital Algebra)**.** An algebra $\mathcal{A} = (\mathcal{V}, \bullet)$ is unital iff $\exists e \in \mathcal{V}$ such that $\forall a \in \mathcal{V}$,

$$e \bullet a = a \bullet e = a, \tag{2.3.1}$$

i.e. e is a two-sided identity. If an algebra contains such a two-sided identity it is unique. As such, we will denote unital algebras by the triple $\mathcal{A} = (\mathcal{V}, \bullet, e)$ both for clarity, and to enforce some restrictions on algebra homomorphisms between them later in this section.

Definition 2.3.3 (Associative Algebra)**.** An algebra $\mathcal{A} = (\mathcal{V}, \bullet)$ is associative iff $\forall a, b, c \in \mathcal{V}$,

$$(a \bullet b) \bullet c = a \bullet (b \bullet c). \tag{2.3.2}$$

We may also refer to the product $\bullet$ of an associative algebra as associative.

Definition 2.3.4 (Jacobi Algebra)**.** An algebra $\mathcal{A} = (\mathcal{V}, \bullet)$ is Jacobi iff $\forall a, b, c \in \mathcal{V}$,

$$a \bullet (b \bullet c) + b \bullet (c \bullet a) + c \bullet (a \bullet b) = 0. \tag{2.3.3}$$

We may also refer to the product $\bullet$ of a Jacobi algebra as Jacobi.

Naturally, there are many intersections between these types of algebra. Of particular importance to this thesis will be non-unital Jacobi algebras and unital associative algebras. We will also almost exclusively deal with real algebras $\mathbb{F} = \mathbb{R}$, and may write $a \in \mathcal{A}$ to mean the same as $a \in \mathcal{V}$.

There is a somewhat trivial but very important example of an algebra which must be defined,

Lemma 2.3.5. *The pair (abusing notation)* $\{0\} = (\{0\}, \bullet)$ *with,*

$$0 \bullet 0 = 0 \tag{2.3.4}$$

is an algebra.

Proof. The algebra axioms may be trivially verified. □

Definition 2.3.6 (Trivial Algebra)**.** We call the algebra $\{0\}$ from Lemma 2.3.5 the trivial algebra.

2.3.1.3 Subalgebras

The algebraic methods developed in Chapter 3, which we will use to derive the spin algebras, make use of the notion of a subalgebra. Therefore, it is important that we give subalgebras due consideration,

Lemma 2.3.7. *Consider an algebra* $\mathcal{A} = (\mathcal{V}, \bullet)$ *and a vector subspace* $\mathcal{U} \subseteq \mathcal{V}$*, for which,* $\forall a, b \in \mathcal{U}$*,*

$$a \bullet b \in \mathcal{U}. \tag{2.3.5}$$

Then $\mathcal{C} = (\mathcal{U}, \bullet_{\mathcal{C}})$ *is an algebra, where,*

$$i \circ \bullet_{\mathcal{C}} := \bullet \circ j \tag{2.3.6}$$

where $i : \mathcal{U} \to \mathcal{V}$ *and* $j : \mathcal{U} \times \mathcal{U} \to \mathcal{V} \times \mathcal{V}$ *are vector space inclusion homomorphisms.*

Proof. Immediate from the definition of an algebra. □

Accordingly,

Definition 2.3.8 (Subalgebra)**.** We call $\mathcal{C} = (\mathcal{U}, \bullet_c)$ as in Lemma 2.3.7 a subalgebra of $\mathcal{A}$.

There are some obvious, but important, subalgebras which are shared by all algebras,

Lemma 2.3.9. *Both* $\{0\}$ *and* $\mathcal{A}$ *are subalgebras of* $\mathcal{A}$*, where* $0 \in \mathcal{A}$ *is the identity element of vector addition in* $\mathcal{A}$*.*

Proof. Direct verification of the subalgebra axioms. □

In light of Lemma 2.3.9, let us distinguish,

Definition 2.3.10 (Proper Subalgebra)**.** A subalgebra $\mathcal{C}$ of $\mathcal{A}$ is proper iff $\mathcal{C} \neq \{0\}$ and $\mathcal{C} \neq \mathcal{A}$.

An important subalgebra for our analysis (which is frequently proper) is the subalgebra of a unital associative algebra "generated" by a single element,

Definition 2.3.11 ($\mathcal{C}_a$)**.** Consider a unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$ over the field $\mathbb{F}$, and an arbitrary element $a \in \mathcal{A}$. Then,

$$\mathcal{C}_a \coloneqq \operatorname{span}_{\mathbb{F}}\big(\{c_j \in \mathcal{A} \mid \forall j \in \mathbb{N}\}\big) \tag{2.3.7}$$

where, $\forall j \in \mathbb{N}$,

$$c_j \coloneqq \begin{cases} e & j = 0 \\ a \bullet c_{j-1} & j \geq 1. \end{cases} \tag{2.3.8}$$

Lemma 2.3.12. *The triple* $(\mathcal{C}_a, \bullet_{\mathcal{C}_a}, e)$ *is a commutative unital associative subalgebra of* $\mathcal{A}$*, with product defined as in Lemma 2.3.7.*

Proof. Clear from its definition. □

We will discuss these subalgebras further in Section 2.4.7 and Chapter 3.

2.3.1.4 The Centre of an Associative Algebra

The properties of subalgebras may be very different from their parent algebras. Of particular importance to the derivation of the spin algebras is the subalgebra known as the centre,

Definition 2.3.13 (Centre of an Associative Algebra)**.** Consider an associative algebra $\mathcal{A} = (\mathcal{V}, \bullet)$. The centre of $\mathcal{A}$ is the set $Z(\mathcal{A})$,

$$Z(\mathcal{A}) \coloneqq \{z \in \mathcal{A} \mid \forall a \in \mathcal{A} : z \bullet a = a \bullet z\}. \tag{2.3.9}$$

Lemma 2.3.14. *The centre $Z(\mathcal{A})$ of the associative algebra $\mathcal{A}$ is non-empty, and may be endowed with the structure of a commutative subalgebra.*

Proof. Since $\forall a \in \mathcal{A}$, $0 \in \mathcal{A}$ satisfies $0 \bullet a = a \bullet 0 = 0$, thus $0 \in Z(\mathcal{A})$, and so $Z(\mathcal{A})$ is non-empty. Since the product $\bullet$ is bilinear, it is clear that $Z(\mathcal{A})$ is a vector subspace of $\mathcal{A}$ in the usual way. That $Z(\mathcal{A})$ is a subalgebra follows from the associativity of $\mathcal{A}$, and that this subalgebra is commutative is clear from its definition. □

Remark. The centre of a Lie algebra is definable as in Definition 2.3.13, and its proof that it is a subalgebra follows using the antisymmetry and Jacobi properties of the Lie product. We will not need to consider the centres of Lie algebras in this work, so we omit a formal proof of this fact, but include this note for completeness. We will use the centre of the universal enveloping algebra of a Lie algebra, which is a unital associative algebra, extensively in this work.

2.3.1.5 Left, Right, and Two-Sided Ideals

A core part of this thesis is the ability to impose new algebraic constraints onto existing, more general algebras; in this way, we will construct the indefinite and spin-s position algebras, the spin algebras, and many of the algebras from which these are derived. The formal objects which we need to achieve this are ideals; the method by which we achieve this will be described in the next section. These come in a number of species,

Definition 2.3.15 (Left Ideal)**.** Consider an algebra $\mathcal{A} = (\mathcal{V}, \bullet)$. A vector subspace $\mathcal{I} \subseteq \mathcal{V}$ is a left ideal of $\mathcal{A}$ iff $\forall i \in I$, $\forall a \in \mathcal{V}$,

$$a \bullet i \in \mathcal{I}. \tag{2.3.10}$$

Definition 2.3.16 (Right Ideal)**.** Consider an algebra $\mathcal{A} = (\mathcal{V}, \bullet)$. A vector subspace $\mathcal{I} \subseteq \mathcal{V}$ is a right ideal of $\mathcal{A}$ iff $\forall i \in I$, $\forall a \in \mathcal{V}$,

$$i \bullet a \in \mathcal{I}. \tag{2.3.11}$$

Definition 2.3.17 (Two-Sided Ideal)**.** An ideal $\mathcal{I}$ is two-sided iff $\mathcal{I}$ is both a left ideal and a right ideal.

Remark. In this thesis, two-sided ideals are the primary species of interest. As such, we may refer to them simply as ideals when their two-sided nature is clear from context, such as when constructing a quotient algebra.

Lemma 2.3.18. *Consider a two-sided ideal $\mathcal{I}$ of an algebra $\mathcal{A} = (\mathcal{V}, \bullet)$. The pair $(\mathcal{I}, \bullet)$ is a subalgebra of $\mathcal{A}$.*

Proof. Clear from the definition. □

There are some obvious, but important, ideals which are shared by all algebras,

Lemma 2.3.19. *Both $\{0\}$ and $\mathcal{V}$ are (left, right, two-sided) ideals of $\mathcal{A}$, where $0 \in \mathcal{A}$ is the identity element of vector addition in $\mathcal{A}$.*

Proof. Direct verification of the (left, right, two-sided) ideal axioms. □

In light of Lemma 2.3.19, let us distinguish,

Definition 2.3.20 (Proper Ideal)**.** An ideal $\mathcal{I}$ of $\mathcal{A}$ is proper iff $\mathcal{I} \neq \{0\}$ and $\mathcal{I} \neq \mathcal{A}$.

2.3.1.6 Ideals Generated by a Set

The structure of an ideal determines what mathematical objects can be derived using them, and what properties they will have. Fortunately, there is a simple way to construct ideals which code for any property we desire. To show how, we must establish some facts,

Lemma 2.3.21. *Consider an algebra $\mathcal{A} = (\mathcal{V}, \bullet)$ over the vector space $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$, and a subset $U \subseteq V$. Let $\{\mathcal{I}_j\}$, indexed over a set J, denote the set of all non-empty ideals of a given species (left, right, or two-sided) such that $\forall j \in J$, $U \subseteq \mathcal{I}_j$. Then,*

$$I(U) \coloneqq \bigcap_{j \in J} \mathcal{I}_j,$$

is a non-empty ideal of the same species. Moreover, $I(U)$ is the unique smallest ideal of this species such that $U \subseteq I(U)$.

Proof. To begin, $I(U)$ is non-empty by Lemma 2.3.19. Now, $\forall p, q \in I(U)$, $\forall j \in J$, $p, q \in \mathcal{I}_j$. Thus, $(p+q) \in \mathcal{I}_j$, so $(p+q) \in I(U)$, and $\forall \alpha \in \mathbb{F}$, $\alpha p \in \mathcal{I}_j$, so $\alpha p \in I(U)$. Therefore, $I(U)$ is a vector subspace of $\mathcal{V}$. If we were considering left ideals then $\forall a \in \mathcal{A}$, $\forall j \in J$, $a \bullet p \in \mathcal{I}_j$, and thus $a \bullet p \in I(U)$, so $I(U)$ is a left ideal. Similarly, if we were considering right ideals then $\forall a \in \mathcal{A}$, $\forall j \in J$, $p \bullet a \in \mathcal{I}_j$, and thus

$p \bullet a \in I(U)$, so $I(U)$ is a right ideal. The two-sided case follows from both of these. To establish the final claim, consider an ideal $\mathcal{I}$ of the same species such that, $U \subseteq \mathcal{I} \subseteq I(U)$. By the first inclusion and the definition of $I(U)$, we must have $\mathcal{I} \cap I(U) = I(U)$, and thus $I(U) \subseteq \mathcal{I}$, so $\mathcal{I} = I(U)$. □

Definition 2.3.22 (Ideal Generated by a Subset)**.** Consider an algebra $\mathcal{A} = (\mathcal{V}, \bullet)$ over the vector space $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$, and a subset $U \subseteq V$. The (right, left, two-sided) ideal $I(U)$ as defined in Lemma 2.3.21 is the (right, left, two-sided) ideal generated by U.

When dealing with unital algebra, the ideal generated by a set is very simple,

Lemma 2.3.23. *Consider a unital algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$ over the vector space $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$, and a subset $U \subseteq V$. Then, $I(U)$ is the (left, right, two-sided) ideal containing all finite linear combinations of (left, right, two-sided) products of the elements of U by $\mathcal{A}$.*

Proof. See [49]. □

For simplicity, we will often adopt related notation for ideals generated by a subset,

Definition 2.3.24 (Ideal Generated by Indexed Elements)**.** Consider an algebra $\mathcal{A}$ and a set of elements $\{a_j \in \mathcal{A}\}$ indexed over a set J. In this thesis, $I(\{a_j\})$ and $I(a_j)$ will be synonymous.

Definition 2.3.25 (Ideal Generated by Multilinear Map)**.** Consider an algebra $\mathcal{A} = (\mathcal{V}, \bullet)$, a subspace $\mathcal{U} \subseteq \mathcal{V}$, and a multilinear map $f : \mathcal{U} \times \cdots \times \mathcal{U} \to \mathcal{A}$. In this thesis, $I(\mathrm{Im}(f))$ and $I(f(a, \ldots, z))$ will be synonymous, where $\{a, \ldots, z\}$ are indeterminate elements of $\mathcal{U}$.

2.3.1.7 Notation for Products

In this thesis, we will often encounter finite products of elements $\{a_j \in \mathcal{A}\}$ which we may index over the set $\{1, \ldots, n\}$. To write such expressions in full would be cumbersome or impossible when the number of elements is variable. Furthermore, since the algebras we will be using are mainly non-commutative, the order of the products must be specified unambiguously. With these considerations in mind, let us introduce an indexed product notation similar to the Big-Π product notation,

Definition 2.3.26 (Big-$\bullet$ Notation)**.** Consider a unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$, and a set of $\{a_j \in \mathcal{A}\}$ indexed over a set $\{1, \ldots, n \in \mathbb{Z}^+\}$. Then, we define,

$$\mathop{\huge\bullet}_{j=1}^{n} a_j := \begin{cases} e & n = 0 \\ a_1 & n = 1 \\ a_1 \bullet \left(\mathop{\huge\bullet}_{j=1}^{n-1} a_{j+1} \right) & n \geq 2, \end{cases} \tag{2.3.12}$$

where the case $n = 0$ corresponds to the empty product, and may be used to make the statement of lemmas simpler.

We will also encounter repeated products of a single element, for which we introduce,

Definition 2.3.27 ($\bullet$-Exponent Notation)**.** Consider a unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$. For all $a \in \mathcal{A}$, $n \in \mathbb{N}$,

$$a^{\bullet n} := \mathop{\huge\bullet}_{j=1}^{n} a. \tag{2.3.13}$$

Furthermore, if an element $a \in \mathcal{A}$ has a two-sided inverse $b \in \mathcal{A}$, we may denote, $\forall n \in \mathbb{N}$,

$$a^{\bullet(-n)} := \mathop{\huge\bullet}_{j=1}^{n} b. \tag{2.3.14}$$

Finally, let us use this notation to generalise the exponential of an endomorphism in Definition 2.2.29 to the case of a unital associative algebra,

Definition 2.3.28 (Algebra Exponential (exp))**.** Consider a unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$. For all $a \in \mathcal{A}$, its exponential $\exp(a)$ is the formal power series,

$$\exp(a) := \sum_{j=0}^{\infty} \frac{1}{j!} a^{\bullet j}. \tag{2.3.15}$$

Remark. In general, the convergence of exp for an arbitrary element in a unital associative algebra is not guaranteed.

The product symbol $\bullet$ will almost never be used in the main text. Instead, it will be replaced in all notation with the appropriate symbol for the product of the algebra being used. For example, $\circ$ will be used for endomorphisms, $\otimes$ will be used for the tensor algebra, and $\wedge$ will be used for the exterior algebra. For algebras which denote a product by concatenation, the usual Big-Π notation will be used, but will follow the ordering convention of Definition 2.3.26.

2.3.2 Lie Algebras

2.3.2.1 Role in this Thesis

As we saw in Section 2.2.4, Lie algebras lie at the heart of symmetry groups and control a significant amount of their structure. It is through the algebraic study of the Lie group $\mathfrak{so}(3, \mathbb{R})$ that we will derive algebras for arbitrary spin systems; as such, Lie algebras are one of the principle objects of study in this thesis.

2.3.2.2 Lie Algebras

In this thesis, we will only consider fields for which $\mathbb{Q} \subseteq \mathbb{F}$, so consider $\mathbb{F}$ to satisfy this in what follows.

Definition 2.3.29 (Lie Algebra)**.** Consider an algebra $\mathfrak{g} = (\mathcal{V}, l)$ over a field $\mathbb{F}$. $\mathfrak{g}$ is a Lie algebra iff it is Jacobi and antisymmetric, i.e. $\forall a, b \in \mathcal{V}$,

$$l(b,a) = -l(a,b). \tag{2.3.16}$$

Remark. Traditionally, the product of a Lie algebra is referred to as a "Lie bracket" and denoted $[\cdot,\cdot]$; we will instead use the term "Lie product", and avoid the bracket notation. This is to ensure $[\cdot,\cdot]$ unambiguously refers to the commutator in an associative algebra.

Unlike most other algebras we will work with in this thesis, non-trivial Lie algebras cannot contain identity elements,

Lemma 2.3.30. *Consider a unital Lie algebra $\mathfrak{g} = (\mathcal{V}, l, e)$. Then, $\mathcal{V} \cong \{0\}$.*

Proof. For all $a \in \mathfrak{g}$,

$$a = l(e,a) = l(a,e) = -l(e,a) = -a,$$

so $a = 0$. □

Some examples of a Lie algebra which are commonly encountered are the commutator subalgebras of an associative algebra,

Lemma 2.3.31. *Consider an associative algebra $\mathcal{A} = (\mathcal{V}, \bullet)$, and a vector subspace $\mathcal{U} \subseteq \mathcal{V}$ such that, $\forall a, b \in \mathcal{U}$,*

$$[a, b] \coloneqq (a \bullet b - b \bullet a) \in \mathcal{U}. \tag{2.3.17}$$

Then, $(\mathcal{U}, [\cdot,\cdot])$, is a Lie algebra.

Proof. That the commutator is bilinear is clear. Furthermore, $\forall a, b, c \in \mathcal{U}$,

$$[a, b] = a \bullet b - b \bullet a = -[b, a],$$

and we may easily calculate that,

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0,$$

so it is antisymmetric and Jacobi. □

2.3.2.3 Abelian and Non-Abelian Lie Algebras

Clearly, the properties of a Lie algebra are controlled by its product. Following the definitions for groups, we classify,

Definition 2.3.32 ((Non-)Abelian Lie Algebra)**.** A Lie algebra $\mathfrak{g} = (\mathcal{V}, l)$ is Abelian iff $l = 0$. Otherwise, $\mathfrak{g}$ is non-Abelian.

Remark. Since the Lie product is anticommutative, the Lie product of an Abelian Lie algebra is identically zero iff it is commutative. This reveals the connection to the corresponding definition for a group.

The Lie algebra $\mathfrak{so}(\mathcal{V}, g)$ is usually non-Abelian (which we will prove later), so these will be of primary interest to us. Indeed, many of the developments of this thesis directly utilise the non-Abelian nature of $\mathfrak{so}(\mathcal{V}, g)$ to form new algebras, including the arbitrary spin algebras.

2.3.2.4 Simple and Semisimple Lie Algebras

Since Abelian and non-Abelian Lie algebras are a less interesting classification than for groups, let us explore a classification scheme which considers ideals. First,

Lemma 2.3.33. *Consider an algebra $\mathcal{A}_\gamma = (\mathcal{V}, \bullet)$ whose product satisfies, $\forall a, b \in \mathcal{A}$,*

$$b \bullet a = (-1)^\gamma a \bullet b, \tag{2.3.18}$$

where $\gamma \in \{0, 1\}$. Then, all ideals of $\mathcal{A}_\gamma$ are two-sided.

Proof. Follows from the fact that all ideals are vector subspaces of their containing algebras. □

Remark. In light of Lemma 2.3.33, we shall refer to two-sided ideals of a Lie algebra simply as "ideals".

Definition 2.3.34 (Simple Lie Algebra)**.** A non-Abelian Lie algebra is simple iff it has no proper ideals.

With Lemma 2.3.18 in mind, we may consider simple Lie algebras as the "smallest" Lie algebras. Indeed, we may use them to construct larger Lie algebras,

Definition 2.3.35 (Semisimple Lie Algebra)**.** A non-Abelian Lie algebra is semisimple iff it has no proper Abelian ideals.

Not all Lie algebras are semisimple, but the Lie algebras we will use in this thesis are. This property is important when considering the centre of their universal enveloping algebras. In particular,

Lemma 2.3.36. *The Lie algebra* $\mathfrak{so}(3,\mathbb{R})$ *is simple.*

Proof. It is clear from its definition that $\mathfrak{so}(3,\mathbb{R})$ is non-Abelian. Now, consider an ideal $\mathcal{I} \subseteq \mathfrak{so}(3,\mathbb{R})$ which contains some non-zero element $v \in \mathfrak{so}(3,\mathbb{R})$, which we may write in the usual basis as $v = \sum_{a=1}^{3} \nu_a S_a$. By definition, $\forall b \in \{1,2,3\}$, $S_b \times v \in \mathcal{I}$, and we find,

$$\begin{aligned} S_1 \times (S_2 \times v) &= \nu_1 S_2 \\ S_2 \times (S_3 \times v) &= \nu_2 S_3 \\ S_3 \times (S_1 \times v) &= \nu_3 S_1. \end{aligned}$$

Thus, for at least one $b \in \{1,2,3\}$, $S_b \in \mathcal{I}$. Supposing $b = 1$, then $S_1 \times S_2 = S_3$ and $S_3 \times S_1 = S_2$, and so $\forall c \in \{1,2,3\}$ $S_c \in \mathcal{I}$. This may be further verified for the cases $b \in \{2,3\}$. Thus, $\mathcal{I} = \mathfrak{so}(3,\mathbb{R})$. □

2.3.2.5 The Adjoint Action

Since the structure of a Lie algebra is controlled by its product, we may use it to directly probe this structure. Indeed, this will be our principle mode of investigation in Chapter 4. To facilitate this, let us define,

Definition 2.3.37 (ad)**.** Consider a Lie algebra $\mathfrak{g} = (\mathcal{V}, l)$. The adjoint action on $\mathfrak{g}$ is a vector space homomorphism,

$$\begin{aligned} \mathrm{ad} &: \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) \\ \mathrm{ad} &:= v \mapsto \big(w \mapsto l(v,w)\big). \end{aligned} \tag{2.3.19}$$

The adjoint action will be extended in Section 2.4.4, and this extension will facilitate our construction of the spin algebras. Vital to this extension is the relationship the adjoint action has with the Lie product,

Lemma 2.3.38. *Consider a Lie algebra $\mathfrak{g} = (\mathcal{V}, l)$. For all $v, w \in \mathfrak{g}$,*

$$\operatorname{ad}\big(l(v,w)\big) = \operatorname{ad}(v)\circ\operatorname{ad}(w) - \operatorname{ad}(w)\circ\operatorname{ad}(v). \tag{2.3.20}$$

Proof. Recall from Definition 2.3.29 that all Lie products are antisymmetric and Jacobi, meaning $\forall a, b, c \in \mathfrak{g}$,

$$\begin{aligned} 0 &= l\big(a, l(b,c)\big) + l\big(b, l(c,a)\big) + l\big(c, l(a,b)\big) \\ &= \operatorname{ad}(a)\circ\operatorname{ad}(b)(c) - \operatorname{ad}(b)\circ\operatorname{ad}(a)(c) - \operatorname{ad}\big(l(a,b)\big)(c). \end{aligned}$$

□

We will formalise homomorphisms with similar relationships in Sections 2.3.4 and 2.3.5.

2.3.2.6 Rank of a Lie Algebra

The rank of a Lie algebra is an important concept for us, as for semisimple Lie algebras it describes the structure of the centre of their universal enveloping algebras. In Chapter 4, we will use this centre to decompose the universal enveloping algebra of $\mathfrak{so}(3, \mathbb{R})$, which leads the way to the arbitrary spin algebras.

Definition 2.3.39 (Rank of a Lie Algebra)**.** Consider a finite-dimensional Lie algebra $\mathfrak{g} = (\mathcal{V}, l)$. Its rank is a positive integer $r \in \mathbb{Z}^+$ such that,

$$r := \min\Big(\Big\{\dim\big(\operatorname{Ker}\big(\operatorname{ad}(v)\big)\big) \Big| \forall v \in \mathfrak{g}\Big\}\Big). \tag{2.3.21}$$

Note that by antisymmetry, $\forall v \in \mathfrak{g}$, $\operatorname{ad}(v)(v) = 0$, and so $r \neq 0$.

Remark. Many authors define the rank in terms of "Cartan subalgebras". As we would not discuss Cartan subalgebras anywhere else in the main body of this thesis, we have avoided this.

For our purposes, we must find the rank of $\mathfrak{so}(3, \mathbb{R})$,

Lemma 2.3.40. *The Lie algebra $\mathfrak{so}(3, \mathbb{R})$ has rank* 1.

Proof. See Appendix B.8. □

2.3.3 Algebraic Structures on Vector Space Endomorphisms

2.3.3.1 Role in this Thesis

In Chapter 3, we will develop general methods to study unital associative algebras using the algebraic properties of its elements only. In order to utilise these methods to study the universal enveloping algebra of $\mathfrak{so}(3, \mathbb{R})$, which leads the way to defining the arbitrary spin algebras, we must understand the various algebraic properties of vector space endomorphisms. These will also be useful in Section 2.3.5 to define actions and modules of algebraic structures.

2.3.3.2 Vector Space Endomorphisms as a Unital Associative Algebra

The first species of algebraic structure that vector space endomorphisms may form is a unital associative algebra,

Lemma 2.3.41. *Consider a vector space* $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$*. The triple* $(\mathrm{End}(\mathcal{V}), \circ, \mathrm{id}_{\mathcal{V}})$ *is a unital associative algebra over* $\mathbb{F}$*.*

Proof. Direct computation. □

Definition 2.3.42 $(\mathrm{End}^{\circ}(\mathcal{V}))$**.** $\mathrm{End}^{\circ}(\mathcal{V})$ is the unital associative algebra $(\mathrm{End}(\mathcal{V}), \circ, \mathrm{id}_{\mathcal{V}})$. We will often refer to $\mathrm{End}^{\circ}(\mathcal{V})$ as simply the "endomorphism algebra".

2.3.3.3 Vector Space Endomorphisms as a Lie Algebra

With $\mathrm{End}^{\circ}(\mathcal{V})$ defined, we see that a wide variety of different algebraic structures on $\mathrm{End}(\mathcal{V})$ are possible by selecting algebraic substructures of $\mathrm{End}^{\circ}(\mathcal{V})$. Of particular importance to this work is its Lie subalgebra,

Lemma 2.3.43. *Consider a vector space* $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$*, where* $\mathbb{Q} \subseteq \mathbb{F}$*. The pair* $(\mathrm{End}(\mathcal{V}), [\,\cdot\,,\cdot\,])$ *is a Lie algebra over* $\mathbb{F}$*.*

Proof. Follows from Lemma 2.3.31. □

Definition 2.3.44 $(\mathrm{End}^{[\,\cdot\,,\cdot\,]}(\mathcal{V}))$**.** $\mathrm{End}^{[\,\cdot\,,\cdot\,]}(\mathcal{V})$ is the Lie algebra $(\mathrm{End}(\mathcal{V}), [\,\cdot\,,\cdot\,])$.

2.3.3.4 Vector Space Automorphisms as a Group

Finally, let us consider $\mathrm{Aut}(\mathcal{V}) \subset \mathrm{End}(\mathcal{V})$,

Lemma 2.3.45. *Consider a vector space* $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$. *The triple* $(\mathrm{Aut}(\mathcal{V}), \circ, \mathrm{id}_{\mathcal{V}})$ *is a group.*

Proof. Direct computation. □

Definition 2.3.46 ($\mathrm{Aut}^{\circ}(\mathcal{V})$)**.** $\mathrm{Aut}^{\circ}(\mathcal{V})$ is the group $(\mathrm{Aut}(\mathcal{V}), \circ, \mathrm{id}_{\mathcal{V}})$.

2.3.4 Homomorphisms on Algebras

2.3.4.1 Role in this Thesis

Algebra homomorphisms are essential for us to understand the relationships between algebras. In particular, we will employ them extensively to study how certain elements of an algebra act on others; amongst other things, this will enable us to decompose the universal enveloping algebra of $\mathfrak{so}(3, \mathbb{R})$ in a natural way, which will lead directly to the definition of the spin algebras.

2.3.4.2 Algebra Homomorphisms and Antihomomorphisms

As with group homomorphisms, the non-commutative nature of a general algebra leads to there being two species of structure preserving functions,

Definition 2.3.47 (Algebra Homomorphism)**.** Consider two algebras $\mathcal{A} = (\mathcal{V}, \bullet)$ and $\mathcal{B} = (\mathcal{W}, \star)$ over the same field $\mathbb{F}$. An algebra homomorphism between $\mathcal{A}$ and $\mathcal{B}$ is a vector space homomorphism $p \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ such that, $\forall a, b \in \mathcal{V}$,

$$p(a \bullet b) = p(a) \star p(b). \tag{2.3.22}$$

We will denote the set of all algebra homomorphisms by $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$, and may write $p : \mathcal{A} \to \mathcal{B}$ to mean the same as an algebra homomorphism $p : \mathcal{V} \to \mathcal{W}$.

Definition 2.3.48 (Algebra Antihomomorphism)**.** Consider two algebras $\mathcal{A} = (\mathcal{V}, \bullet)$ and $\mathcal{B} = (\mathcal{W}, \star)$ over the same field $\mathbb{F}$. An algebra antihomomorphism between $\mathcal{A}$ and $\mathcal{B}$ is a vector space homomorphism $q \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ such that, $\forall a, b \in \mathcal{V}$,

$$q(a \bullet b) = q(b) \star q(a). \tag{2.3.23}$$

We will denote the set of all algebra antihomomorphisms by $\overline{\mathrm{Hom}}(\mathcal{A}, \mathcal{B})$.

Remark. Care must be taken when considering algebra homomorphisms or antihomomorphisms, since, unlike more general vector space homomorphisms, they *do not* form a vector space.

2.3.4.3 Unital Algebra (Anti)Homomorphisms

In general, we reflect the structures of the algebras related by an algebra (anti)homomorphism in its naming. For example, we call an algebra homomorphism between two associative algebras an associative algebra homomorphism. This changes nothing about the definition of such a function, except when we are mapping between two unital algebras,

Definition 2.3.49 (Unital Algebra (Anti)Homomorphism)**.** Consider two unital algebras $\mathcal{A} = (\mathcal{V}, \bullet, e)$ and $\mathcal{B} = (\mathcal{W}, \star, f)$ over the same field $\mathbb{F}$, and a vector space homomorphism $r \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$. Then, r is a unital algebra (anti)homomorphism iff r is an algebra (anti)homomorphism and,

$$r(e) = f. \tag{2.3.24}$$

2.3.4.4 Algebra Isomorphisms

An important class of algebra homomorphisms are those with two-sided inverses,

Lemma 2.3.50. *Consider two algebras $\mathcal{A} = (\mathcal{V}, \bullet)$ and $\mathcal{B} = (\mathcal{W}, \star)$ over the same field $\mathbb{F}$, and a vector space isomorphism $f \in \mathrm{Iso}(\mathcal{V}, \mathcal{W})$. Then $f \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$ iff $f^{-1} \in \mathrm{Hom}(\mathcal{B}, \mathcal{A})$.*

Proof. In the forward direction, since $\mathrm{Im}(f^{-1}) = \mathcal{V}$, $\forall a, b \in \mathcal{V}$, $\exists x, y \in \mathcal{W}$ such that $a = f^{-1}(x)$ and $b = f^{-1}(y)$, thus,

$$f^{-1}(x) \bullet f^{-1}(y) = a \bullet b = f^{-1} \circ f(a \bullet b) = f^{-1}\big(f(a) \star f(b)\big) = f^{-1}(x \star y).$$

The reverse direction follows the same logic with $\mathcal{V}$ and $\mathcal{W}$, and f^{-1} and f reversed.

□

Definition 2.3.51 (Algebra Isomorphism)**.** An algebra isomorphism between two algebras $\mathcal{A}$ and $\mathcal{B}$ over the same field $\mathbb{F}$, is a vector space isomorphism $f \in \mathrm{Iso}(\mathcal{V}, \mathcal{W})$ which is also an algebra homomorphism $f \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$. We denote the set of all algebra isomorphisms as $\mathrm{Iso}(\mathcal{A}, \mathcal{B})$

Remark. Using algebra antihomomorphisms in place of algebra homomorphisms, we may also establish the notion of an algebra anti-isomorphism. We will not formally do so, as we will make much less use of them than algebra isomorphisms.

Just as for vector spaces, the ability to discuss algebras which share compatible algebraic structure will empower us to study abstract algebras in terms of more concrete objects,

Definition 2.3.52 (Isomorphic as Algebras ($\cong$))**.** Two algebras $\mathcal{A}$ and $\mathcal{B}$ over the same field $\mathbb{F}$ are isomorphic as algebras iff there exists an algebra isomorphism f between them. We denote this fact by $\mathcal{A} \cong \mathcal{B}$. When it is clear we are referring to their algebraic structures, we may refer to this as simply isomorphic.

Remark. The notation we use to denote algebra isomorphic algebras is the same as for vector space isomorphic vector spaces. In this thesis, which we mean in any given instance will usually be clear from context and otherwise explicitly stated.

Lemma 2.3.53. *Consider the algebras* $\mathcal{A} = (\mathcal{V}, \bullet)$, $\mathcal{B} = (\mathcal{W}, \star)$, *and* $\mathcal{C} = (\mathcal{X}, *)$ *over the same field* $\mathbb{F}$. *For all* $f \in \mathrm{Iso}(\mathcal{A}, \mathcal{B})$ *and* $g \in \mathrm{Iso}(\mathcal{B}, \mathcal{C})$, *their composition,*

$$g \circ f \in \mathrm{Iso}(\mathcal{A}, \mathcal{C}). \tag{2.3.25}$$

Proof. We may immediately verify that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. □

Lemma 2.3.54. $\cong$ *is an equivalence relation between algebras.*

Proof. Identical proof to that of Lemma B.2.28. □

2.3.4.5 Derivations

Derivations are a species of vector space endomorphism defined for algebras which are neither algebra homomorphisms nor algebra antihomomorphisms; nevertheless, they will play an important role in constructing algebras in Chapters 4 and 5.

Definition 2.3.55 (Derivation on an Algebra)**.** Consider an algebra $\mathcal{A} = (\mathcal{V}, \bullet)$. Then, a vector space endomorphism $f \in \mathrm{End}(\mathcal{V})$ is a derivation on $\mathcal{A}$ iff $\forall a, b \in \mathcal{V}$,

$$f(a \bullet b) = f(a) \bullet b + a \bullet f(b). \tag{2.3.26}$$

We may denote the set of all derivations on an algebra $\mathcal{A}$ as $\mathrm{Der}(\mathcal{A})$.

Derivations enjoy many of the same properties as the derivative, some of which will now present as they will be useful in Section 3.4.

Lemma 2.3.56. *For all derivations* $D \in \mathrm{Der}(\mathcal{A})$, $\forall k \in \mathbb{N}$, $\forall a, b \in \mathcal{A}$,

$$D^{\circ k}(a \bullet b) = \sum_{j=0}^{k} \binom{k}{j} D^{\circ j}(a) \bullet D^{\circ (k-j)}(b). \tag{2.3.27}$$

Proof. This follows from induction. □

Lemma 2.3.57. *For all derivations $D \in \mathrm{Der}(\mathcal{A})$, $\forall k \in \mathbb{Z}^+$, $\forall \{a_j \in \mathcal{A}\}$ indexed over the set $\{1, \ldots, k+1\}$,*

$$D\left(\bigodot_{j=1}^{k+1} a_j\right) = D(a_1) \bullet \bigodot_{j=1}^{k} a_{j+1} + a_1 \bullet D\left(\bigodot_{j=1}^{k} a_{j+1}\right). \tag{2.3.28}$$

Proof. This follows by induction. □

Corollary 2.3.58. *Suppose for $D \in \mathrm{Der}(\mathcal{A})$, $\exists a \in \mathcal{A}$ such that,*

$$a \bullet D(a) = D(a) \bullet a. \tag{2.3.29}$$

Then, $\forall n \in \mathbb{Z}^+$,

$$D(a^{\bullet n}) = n\, a^{\bullet(n-1)} \bullet D(a). \tag{2.3.30}$$

Proof. Follows from Lemmas 2.3.56 and 2.3.57. □

Lemma 2.3.59. *If $\mathcal{A} = (\mathcal{V}, \bullet, e)$ is unital, then $\forall D \in \mathrm{Der}(\mathcal{A})$,*

$$D(e) = 0. \tag{2.3.31}$$

Proof.

$$D(e) = D(e \bullet e) = 2D(e).$$

□

Lemma 2.3.60. *Consider a unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$, and an element $a \in \mathcal{A}$ which has a two-sided inverse $a^{-1} \in \mathcal{A}$. Then, $\forall D \in \mathrm{Der}(\mathcal{A})$,*

$$D(a^{-1}) = -(a^{-1} \bullet D(a) \bullet a^{-1}) \tag{2.3.32}$$

Proof.

$$0 = D(e) = D(a \bullet a^{-1}) = D(a) \bullet a^{-1} + a \bullet D(a^{-1}).$$

□

Lemma 2.3.61. *Suppose for $D \in \mathrm{Der}(\mathcal{A})$, $\exists a \in \mathcal{A}$ such that,*

$$a \bullet D(a) = D(a) \bullet a, \tag{2.3.33}$$

with a two-sided inverse $a^{-1} \in \mathcal{A}$. Then, $\forall n \in \mathbb{Z}^+$,

$$D\big((a^{-1})^{\bullet n}\big) = -n\,(a^{-1})^{\bullet(n+1)} \bullet D(a). \tag{2.3.34}$$

Proof. Multiplying by a^{-1} on both sides of (2.3.33) reveals that $a^{-1} \bullet D(a) = D(a) \bullet a^{-1}$. Thus, the $n = 1$ case follows by Lemma 2.3.60. The remaining cases may be proved by induction. □

Lemma 2.3.62. *Consider a unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$ over $\mathbb{F}$, a non-zero polynomial $p(x)$ with coefficients in $\mathbb{F}$, and suppose there exists a derivation $D \in \mathrm{Der}(\mathcal{A})$ for which $D(p(a))$ has a left inverse. Then, the endomorphism,*

$$D_{p(a)} \coloneqq v \mapsto D(p(a))^{-1} \bullet D(v), \tag{2.3.35}$$

is also a derivation and satisfies,

$$D_{p(a)}(p(a)) = e. \tag{2.3.36}$$

Proof. Direct computation. □

It will also be useful later to note that,

Lemma 2.3.63. *In any associative algebra $\mathcal{A} = (\mathcal{V}, \bullet)$, the commutator with a fixed element $a \in \mathcal{A}$ is a derivation.*

Proof. For all $b, c \in \mathcal{A}$,

$$\begin{aligned} a \bullet (b \bullet c) - (b \bullet c) \bullet a &= a \bullet (b \bullet c) - (b \bullet c) \bullet a + (b \bullet a) \bullet c - (b \bullet a) \bullet c \\ &= (a \bullet b - b \bullet a) \bullet c + b \bullet (a \bullet c - c \bullet a). \end{aligned}$$

□

2.3.5 Actions on Vector Spaces and Modules

2.3.5.1 Role in this Thesis

Actions are a method to describe how an algebraic structure can affect a vector space. This leads naturally to the notion of a module, which generalises vector spaces. Actions are central to our study of the structure of universal enveloping algebras, and ultimately lead to a decomposition of the universal enveloping algebra of $\mathfrak{so}(3, \mathbb{R})$ into modules over its centre. This decomposition is essential in defining the arbitrary spin algebras, and thus these concepts must be understood.

2.3.5.2 Associative Algebra Actions on Vector Spaces

In our analysis, we will work primarily with associative algebras, and will use associative algebra actions to probe their structure. Due to the general non-commutativity of algebras, we must consider two species of actions,

Definition 2.3.64 (Left Associative Algebra Actions)**.** Consider an associative algebra $\mathcal{A} = (\mathcal{V}, \bullet)$ over the field $\mathbb{F}$, and a vector space $\mathcal{W}$ over $\mathbb{F}$. Then, a left associative algebra action of $\mathcal{A}$ on $\mathcal{W}$ is an algebra homomorphism $f \in \mathrm{Hom}(\mathcal{A}, \mathrm{End}^{\circ}(\mathcal{W}))$.

Similarly,

Definition 2.3.65 (Right Associative Algebra Actions)**.** Consider an associative algebra $\mathcal{A} = (\mathcal{V}, \bullet)$ over the field $\mathbb{F}$, and a vector space $\mathcal{W}$ over $\mathbb{F}$. Then, a right associative algebra action of $\mathcal{A}$ on $\mathcal{W}$ is an algebra antihomomorphism $f \in \overline{\mathrm{Hom}}(\mathcal{A}, \mathrm{End}^{\circ}(\mathcal{W}))$.

Remark. If the associative algebra $\mathcal{A}$ is commutative, then left associative algebra actions are also right associative algebra actions, and vice-versa.

We will most frequently be working with unital associative algebras, so we additionally define,

Definition 2.3.66 ((Left/Right) Unital Associative Algebra Action)**.** Consider a unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$ over the field $\mathbb{F}$, and a vector space $\mathcal{W}$ over $\mathbb{F}$. Then, a left (resp. right) unital associative algebra action is a left (resp. right) algebra action of $\mathcal{A}$ on $\mathcal{W}$ which is also a unital algebra homomorphism (resp. antihomomorphism).

2.3.5.3 Lie Algebra Actions on Vector Spaces

The adjoint action on the universal enveloping algebra of $\mathfrak{so}(3, \mathbb{R})$ will be principally important to this thesis in identifying the multipole tensors, and ultimately the arbitrary spin algebras. The structure of this action is rooted in a Lie algebra action, and so this must be clearly defined,

Definition 2.3.67 (Lie Algebra Action)**.** Consider a Lie algebra $\mathfrak{g} = (\mathcal{V}, l)$ over the field $\mathbb{F}$, and a vector space $\mathcal{W}$ over $\mathbb{F}$. Then, a Lie algebra action of $\mathfrak{g}$ on $\mathcal{W}$ is an algebra homomorphism $f \in \mathrm{Hom}(\mathfrak{g}, \mathrm{End}^{[\cdot,\cdot]}(\mathcal{W}))$.

Remark. The anticommutativity of the Lie product causes left and right actions to coincide in the case of Lie algebras, so only one definition is needed.

2.3.5.4 Group Actions on Vector Spaces

For completeness, we also define the action of a group. As with algebras, group actions come in two species,

Definition 2.3.68 (Left Group Action)**.** Consider a group G, and a vector space $\mathcal{W}$ over $\mathbb{F}$. Then, a left group action of G on $\mathcal{W}$ is a group homomorphism $f \in \mathrm{Hom}(G, \mathrm{Aut}^{\circ}(\mathcal{W}))$.

Similarly,

Definition 2.3.69 (Right Group Action)**.** Consider a group G, and a vector space $\mathcal{W}$ over $\mathbb{F}$. Then, a right group action of G on $\mathcal{W}$ is a group antihomomorphism $f \in \overline{\mathrm{Hom}}(G, \mathrm{Aut}^{\circ}(\mathcal{W}))$.

When considering continuous symmetry groups, actions of these groups and actions of their associated Lie algebras are closely related: the “derivative” of such a group action at the identity element of the group yields a Lie algebra action. Lie algebra actions may also be “exponentiated” to yield group actions; however, just as many inequivalent groups may share a given Lie algebra, not all Lie algebra actions yield group actions for a particular choice of group. To avoid introducing concepts in differential geometry which are unnecessary for the development of this thesis, we will not make these statements more precise. A fuller discussion can be found in, for example, [52, 62].

2.3.5.5 Modules

Actions allow us to consider generalisations of the notion of a vector space, where the “scalars” are elements of mathematical structures other than fields. For us, we will find that the natural decomposition of the universal enveloping algebra of $\mathfrak{so}(3, \mathbb{R})$ is in terms of modules over its centre, so for us this generalisation is natural.

Definition 2.3.70 ((Left/Right) Module over a (Unital) Associative Algebra)**.** A left (resp. right) module over a (unital) associative algebra is a triple $(\mathcal{W}, \mathcal{A}, f)$ consisting of:

- a vector space $\mathcal{W}$ over $\mathbb{F}$;
- a (unital) associative algebra $\mathcal{A}$ over a field $\mathbb{F}$;

- a left (resp. right) (unital) associative algebra action f of $\mathcal{A}$ on $\mathcal{W}$.

Definition 2.3.71 (Module over a Lie Algebra)**.** A module over a Lie algebra is a triple $(\mathcal{W}, \mathfrak{g}, f)$ consisting of:

- a vector space $\mathcal{W}$ over $\mathbb{F}$;
- a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$;
- a Lie algebra action f of $\mathfrak{g}$ on $\mathcal{W}$.

Definition 2.3.72 ((Left/Right) Module over a Group)**.** A left (resp. right) module over a group is a triple $(\mathcal{W}, G, f)$ consisting of:

- a vector space $\mathcal{W}$ over $\mathbb{F}$;
- a group G;
- a left (resp. right) group action f of G on $\mathcal{W}$.

Remark. These definitions are not quite as general as they could be; many authors replace the vector space $\mathcal{W}$ with an Abelian group, essentially "forgetting" the scalar multiplication on $\mathcal{W}$. We will always consider vector spaces, so these stricter definitions will not disadvantage us.

2.3.5.6 Direct Sum of Modules

The modules over the centre of the universal enveloping algebra of $\mathfrak{so}(3, \mathbb{R})$ which naturally decompose it may be direct summed to reconstitute the whole algebra. As such, a definition for the direct sum of modules is in order,

Definition 2.3.73 (Direct Sum of Modules)**.** Consider two left (resp. right) modules of the same species $\mathcal{M} = (\mathcal{W}, \mathcal{D}, f)$, and $\mathcal{N} = (\mathcal{X}, \mathcal{D}, g)$ whose vector spaces are over the same field $\mathbb{F}$. Then, we may form their direct sum $\mathcal{M} \oplus \mathcal{N} = (\mathcal{W} \oplus \mathcal{X}, \mathcal{D}, h)$, where h is the left (resp. right) action of $\mathcal{D}$ on $\mathcal{W} \oplus \mathcal{X}$ such that, $\forall d \in \mathcal{D}$, $\forall (w +_{\mathcal{W}\oplus\mathcal{X}} x) \in \mathcal{W} \oplus \mathcal{X}$,

$$h(d)(w +_{\mathcal{W}\oplus\mathcal{X}} x) \coloneqq f(d)(w) +_{\mathcal{W}\oplus\mathcal{X}} g(d)(x). \tag{2.3.37}$$

2.4 Foundational Algebras and Methods

2.4.1 Quotienting Algebras by Two-Sided Ideals

2.4.1.1 Role in this Thesis

In this thesis, we will be deriving algebras with the structure of arbitrary spin systems from more general ones by quotienting two-sided ideals. As such, it is imperative that we understand this process.

2.4.1.2 Quotient Algebras

Quotient algebras are constructed by the same process as quotient vector spaces. However, if a general subspace, or even a general subalgebra is used, the properties of the resulting quotient space are not strong enough to define on it the structure of an algebra. Instead, we must use two-sided ideals,

Definition 2.4.1 (Algebra Quotient)**.** Consider an algebra $\mathcal{A} = (\mathcal{V}, \bullet)$ over the field $\mathbb{F}$, and a two-sided ideal $\mathcal{I} \subseteq \mathcal{A}$. Their quotient $\mathcal{K}$ is defined,

$$\mathcal{K} \cong \frac{\mathcal{A}}{\mathcal{I}}, \tag{2.4.1}$$

and consists of all equivalence classes of elements of $\mathcal{A}$ under the equivalence relation, $\forall a, b \in \mathcal{A}$,

$$[a \sim b] \Leftrightarrow \big[(a +_{\mathcal{V}} (-b)) \in \mathcal{I}\big]. \tag{2.4.2}$$

Lemma 2.4.2. *The pair* $(\mathcal{K}, \blacksquare)$ *is an algebra over* $\mathbb{F}$*, where* $\blacksquare$ *is inherited from* $\mathcal{A}$ *up to* $\sim$*.*

Proof. From the definition, it is clear that $\mathcal{K}$ is a vector space. Let us first show that $\blacksquare$ is closed, $\forall [\,a\,], [\,b\,] \in \mathcal{K}$, $\forall i, j \in \mathcal{I}$,

$$[\,a\,] \blacksquare [\,b\,] := (a + i) \bullet (b + j) = a \bullet b + a \bullet j + i \bullet b + i \bullet j = a \bullet b + k = [\,a \bullet b\,],$$

where $k \in \mathcal{I}$. Using this, we may directly compute that $\mathcal{K}$ is well-defined and bilinear. □

Definition 2.4.3 (Quotient Algebra)**.** We call $(\mathcal{K}, \blacksquare)$ the quotient algebra of $\mathcal{A}$ by the two-sided ideal $\mathcal{I}$.

Remark. While the objects of any quotient algebra are, strictly speaking, equivalence classes of elements, we will often instead use representatives and manipulate algebraic expressions according to their equivalences.

Quotient algebras may have different properties and dimension to their parent algebra. In particular, finite-dimensional quotient algebras can be obtained from infinite-dimensional algebras.

2.4.1.3 Quotienting an Algebra by an Ideal Generated by a Set

The most important scheme for constructing quotient algebras in this thesis is using two-sided ideals $I(U)$ constructed from a subset of elements U. While there is nothing more to add mathematically, it is worth discussing the consequences of this process practically.

Lemma 2.4.4. *Consider an algebra $\mathcal{A}$, a two-sided ideal $\mathcal{I}$, and the quotient algebra,*

$$\mathcal{K} \cong \frac{\mathcal{A}}{\mathcal{I}}. \tag{2.4.3}$$

Then, $\forall a \in \mathcal{A}$, $a \in \mathcal{I}$ iff $a \in [\,0\,]$ where $[\,0\,] \in \mathcal{K}$ is the equivalence class of 0.

Proof. In the forward direction, since $\mathcal{I}$ is a vector space $0 \in \mathcal{I}$, so $a + (-0) = a \in \mathcal{I}$. Therefore, $a \sim 0$. In the reverse direction, $a \in [\,0\,]$ means $a + (-0) = a \in \mathcal{I}$. □

Corollary 2.4.5. *Suppose $\mathcal{I} = I(U)$ in the above Lemma. For all $i \in U$, $i \in [\,0\,]$.*

Proof. Directly follows from Lemma 2.4.4, noting $U \subseteq I(U)$. □

Remark. Corollary 2.4.5 presents an opportunity. If we construct U to contain elements of the form $A - B$, then in the quotient algebra by $I(U)$, we will have that $[\,A - B\,] = [\,0\,]$, and so $[\,A\,] = [\,B\,]$. This effectively imposes the identity $A = B$ in the new algebra we have constructed. This method of imposing algebraic relations will be used extensively in Chapters 4 and 5, and we will see many well-known examples of its use in this section.

2.4.2 Tensor Algebras

2.4.2.1 Role in this Thesis

The tensor algebra is the most generic unital associative algebra one can construct from a given vector space. As such, all other unital associative algebras may be derived from it by quotient, including many important to the properties of spin. Therefore, to make progress we must understand its structure.

2.4.2.2 The Tensor Algebra

Definition 2.4.6 ($T(\mathcal{V})$)**.** Consider a vector space $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$. We define its tensor algebra $T(\mathcal{V})$ to be the unital associative algebra,

$$T(\mathcal{V}) \cong \bigoplus_{j=0}^{\infty} \mathcal{V}^{\otimes j}, \tag{2.4.4}$$

with product $\otimes : T(\mathcal{V}) \times T(\mathcal{V}) \to T(\mathcal{V})$ defined bilinearly in terms of $\forall j, k \in \mathbb{N}$,

$$\begin{aligned} \otimes^{(j,k)} : \mathcal{V}^{\otimes j} \times \mathcal{V}^{\otimes k} &\to \mathcal{V}^{\otimes (j+k)} \\ (A, B) &\mapsto A \otimes B, \end{aligned} \tag{2.4.5}$$

with $1 \in \mathcal{V}^{\otimes 0} \cong \mathbb{F}$ as its identity element. We call an element of the tensor algebra a “tensor”.

In Chapter 4, we will make particular use of,

Definition 2.4.7 (k-adic Tensor)**.** For $k = 0$, an element $A \in T(\mathcal{V})$ is 0-adic iff $A = \alpha 1$, where $\alpha \in \mathbb{F}$. For $k \in \mathbb{Z}^+$, an element $A \in T(\mathfrak{so}(3, \mathbb{R}))$ is k-adic iff $\exists \{v_j \in \mathcal{V}\}$ indexed over the set $\{1, \ldots, k\}$ such that,

$$A = \bigotimes_{j=1}^{k} v_j. \tag{2.4.6}$$

It is also useful to define,

Definition 2.4.8 (Tensor Order)**.** We define the tensor order of a k-adic tensor to be k, and the tensor order of a linear combination of k-adics to be the largest tensor order amongst its terms.

Lemma 2.4.9. *For all $k \in \mathbb{N}$, consider bases $\{b_j^{(k)}\}$ for $\mathcal{V}^{\otimes k}$ indexed over the sets $J^{(k)}$. Then,*

$$\bigcup_{k=0}^{\infty} \{b_j^{(k)}\}, \tag{2.4.7}$$

is a basis for $T(\mathcal{V})$.

Proof. Follows from the definition of $T(\mathcal{V})$. □

2.4.2.3 The Universal Property of the Tensor Algebra

Like the tensor product spaces which comprise it, the tensor algebra enjoys a universal property,

Lemma 2.4.10 (Universal Property of the Tensor Algebra)**.** *Consider a vector space* $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$ *and a unital associative algebra* $\mathcal{B} = (\mathcal{W}, \star, f)$ *over* $\mathbb{F}$*. For any vector space homomorphism* $d : \mathcal{V} \to \mathcal{B}$*, there exists a unique unital associative algebra homomorphism* $\tilde{d} : T(\mathcal{V}) \to \mathcal{B}$ *such that,*

$$d = \tilde{d} \circ i, \tag{2.4.8}$$

where $i : \mathcal{V} \to T(\mathcal{V})$ *is the canonical inclusion map.*

Proof. See [49]. □

Remark. Lemma 2.4.10 means that all unital associative algebras which we can relate $\mathcal{V}$ to by some homomorphism d may be studied through the elements of the tensor algebra via $\tilde{d}$. As such, the structure of the tensor algebra forms a general base from which we may study the structure of other unital associative algebras. This will be important when discussing the multipole tensors of Chapter 4.

2.4.3 Symmetric Algebras

2.4.3.1 Role in this Thesis

The symmetric algebra is important to this thesis, since its elements form a basis for the universal enveloping algebra of a Lie algebra, which we will discuss in greater detail shortly. It also serves as a simple example of a quotient algebra, which is something we will do frequently in this work.

2.4.3.2 The Symmetric Tensors

To understand the structure of the symmetric algebra, let us first define the symmetric tensors. In what follows, we assume $\mathbb{Q} \subseteq \mathbb{F}$.

Definition 2.4.11 ($\mathrm{Sym}^k(\mathcal{V})$)**.** For all $k \in \mathbb{N}$, the kth-symmetric power of $\mathcal{V}$ is the subspace $\mathrm{Sym}^k(\mathcal{V}) \subseteq \mathcal{V}^{\otimes k}$ spanned by elements of the form, $\forall v_j \in \mathcal{V}$,

$$\bigodot_{j=1}^{k} v_j := \frac{1}{k!} \sum_{\sigma \in S_k} \bigotimes_{j=1}^{k} v_{\sigma(j)}, \tag{2.4.9}$$

where S_k is set of permutations of k symbols. We refer to the elements of $\mathrm{Sym}^k(\mathcal{V})$ as kth-order symmetric tensors.

Remark. When we discuss symmetric tensors in the context of an algebra, we are discussing elements of the form in Definition 2.4.11 with the product of the algebra replacing $\otimes$.

Lemma 2.4.12. *For all $\sigma \in S_k$, $\forall v_j \in \mathcal{V}$,*

$$\bigodot_{j=1}^{k} v_{\sigma(j)} = \bigodot_{j=1}^{k} v_j. \tag{2.4.10}$$

Proof. Follows from the fact that S_k is a finite group. □

Lemma 2.4.13. *When $\mathcal{V}$ is finite-dimensional, so is $\mathrm{Sym}^k(\mathcal{V})$ and $\dim\big(\mathrm{Sym}^k(\mathcal{V})\big) = \binom{\dim(\mathcal{V})+k-1}{k}$.*

Proof. See [49]. □

2.4.3.3 The Symmetric Algebra

Definition 2.4.14 ($\mathrm{Sym}(\mathcal{V})$)**.** Consider a vector space $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$. Its symmetric algebra $\mathrm{Sym}(\mathcal{V})$ is the quotient algebra, $\forall v, w \in \mathcal{V}$,

$$\mathrm{Sym}(\mathcal{V}) \cong \frac{T(\mathcal{V})}{I\left(v \otimes w - w \otimes v\right)}. \tag{2.4.11}$$

We denote the product of the symmetric algebra by $\odot$, and call it the "symmetric product".

Now, let us investigate the structure of $\mathrm{Sym}(\mathcal{V})$ by using the universal property of $T(\mathcal{V})$ to see what happens to a k-adic tensor after the quotient. For this algebra, there are simpler methods to achieve this, but the method we will use can be more broadly applied.

Lemma 2.4.15. *For all $k \in \mathbb{N}$, the equivalence classes of the k-adic tensors in $\mathrm{Sym}(\mathcal{V})$ are labelled by the kth-order symmetric tensors.*

Proof. Consider $\bigotimes_{j=1}^{k} v_j \in \mathcal{V}^{\otimes k}$, then,

$$\left[\bigotimes_{j=1}^{k} v_j\right] = \left[\bigotimes_{j=1}^{k} v_j - \frac{1}{k!}\sum_{\sigma \in S_k} \bigotimes_{j=1}^{k} v_{\sigma(j)}\right] + \left[\bigodot_{j=1}^{k} v_j\right].$$

By Lemma 2.3.23, in $\mathrm{Sym}(\mathcal{V})$ we have, $\forall A, B \in T(\mathcal{V})$, $\forall v, w \in \mathcal{V}$,

$$\big[A \otimes v \otimes w \otimes B\big] = \big[A \otimes w \otimes v \otimes B\big],$$

and so,

$$\left[\bigotimes_{j=1}^{k} v_j - \frac{1}{k!}\sum_{\sigma\in S_k}\bigotimes_{j=1}^{k} v_{\sigma(j)}\right] = \left[\bigotimes_{j=1}^{k} v_j - \frac{1}{k!}\sum_{\sigma\in S_k}\bigotimes_{j=1}^{k} v_j\right] = [0],$$

since $|S_k| = \frac{1}{k!}$. □

Thus, we find,

Lemma 2.4.16.

$$\mathrm{Sym}(\mathcal{V}) \cong \bigoplus_{j=0}^{\infty} \mathrm{Sym}^j(\mathcal{V}). \tag{2.4.12}$$

Proof. Follows directly from Lemma 2.4.15. □

Corollary 2.4.17. *For all vector spaces $\mathcal{V}$, $\mathrm{Sym}(\mathcal{V})$ is infinite-dimensional.*

Proof. Follows immediately from Lemmas 2.4.13 and 2.4.16. □

2.4.4 Universal Enveloping Algebras

2.4.4.1 Role in this Thesis

The universal enveloping algebra of a Lie algebra is one of the most important algebras we will consider in this thesis, as we will directly derive the arbitrary spin algebras from it by analysing its structure. The arbitrary spin algebras will later inform the construction of the arbitrary spin position operator algebras. As such, the universal enveloping algebra is essential to many of the major results of this thesis.

2.4.4.2 The Universal Enveloping Algebra

In this thesis, we will only consider fields for which $\mathbb{Q} \subseteq \mathbb{F}$, so consider $\mathbb{F}$ to satisfy this condition in what follows. The universal enveloping algebra subsumes the structure of a Lie algebra within the structure of a unital associative algebra,

Definition 2.4.18 ($U(\mathfrak{g})$)**.** Consider a Lie algebra $\mathfrak{g} = (\mathcal{V}, l)$ over the field $\mathbb{F}$. Its universal enveloping algebra is the quotient algebra, $\forall v, w \in \mathfrak{g}$,

$$U(\mathfrak{g}) \cong \frac{T(\mathcal{V})}{I\left(v{\otimes}w - w{\otimes}v - l(v,w)\right)}. \tag{2.4.13}$$

There is no consensus on which symbol should be used to denote the product of the universal enveloping algebra. In this thesis, we shall leave the product implicit.

Lemma 2.4.19. *As vector spaces,* $\mathrm{Sym}(\mathcal{V}) \cong U(\mathfrak{g})$*.*

Proof. Firstly, it is clear that no symmetric tensors are present in the ideal $I\left(v\otimes w - w\otimes v - l(v,w)\right)$. Thus, by Lemma 2.4.4 all symmetric tensors are non-zero in $U(\mathfrak{g})$. Now, consider $\bigotimes_{j=1}^{k} v_j \in \mathcal{V}^{\otimes k}$, then after the quotient,

$$\left[\bigotimes_{j=1}^{k} v_j\right] = \left[\bigotimes_{j=1}^{k} v_j - \frac{1}{k!}\sum_{\sigma\in S_k}\bigotimes_{j=1}^{k} v_{\sigma(j)}\right] + \left[\bigodot_{j=1}^{k} v_j\right].$$

By Lemma 2.3.23, in $U(\mathfrak{g})$ we have, $\forall A, B \in T(\mathcal{V})$, $\forall v, w \in \mathcal{V}$,

$$\left[A\otimes v\otimes w\otimes B\right] = \left[A\otimes w\otimes v\otimes B\right] + \left[A\otimes l(v,w)\otimes B\right],$$

noting that the final term is of a strictly lower tensor order than the others. Thus,

$$\begin{aligned}\left[\bigotimes_{j=1}^{k} v_j - \frac{1}{k!}\sum_{\sigma\in S_k}\bigotimes_{j=1}^{k} v_{\sigma(j)}\right] &= \left[\bigotimes_{j=1}^{k} v_j - \frac{1}{k!}\sum_{\sigma\in S_k}\bigotimes_{j=1}^{k} v_j + f(v_1,\ldots,v_k)\right] \\ &= \left[\, f(v_1,\ldots,v_k)\,\right],\end{aligned}$$

since $|S_k| = \frac{1}{k!}$, where $f(v_1,\ldots,v_k)$ is a tensor of order strictly less than k. Therefore, we may repeat this process for all n-adics with $n < k$ which comprise $f(v_1,\ldots,v_k)$; this process is guaranteed to terminate since tensor order is bounded from below. Thus, every element of $U(\mathfrak{g})$ may be written as an $\mathbb{F}$-linear combination of symmetric tensors, and we may establish the isomorphism as mapping a given symmetric tensor to its equivalence class in $U(\mathfrak{g})$. □

Corollary 2.4.20. *The symmetric tensors form a basis for* $U(\mathfrak{g})$*.*

Proof. This is clear from Lemma 2.4.19. □

In this thesis, we favour basis-independent arguments as they often lead to a better understanding about what we are investigating. The major exception to this rule will occur in Chapter 4, where we will use the spin generator basis of Definition 2.2.61 to probe the structure of $U(\mathfrak{so}(3,\mathbb{R}))$. We do this to ensure consistency with the published work[61]. As such, we will make use of a famous, but more basis-dependent theorem,

Theorem 2.4.21 (Poincaré-Birkhoff-Witt Theorem)**.** *Consider a basis* $\{v_j\}$ *for* $\mathfrak{g}$ *indexed over a set* J*, and a total ordering* $\leq$ *on* J*. Then,* $\forall k \in \mathbb{Z}^+$ *the mapping into* $U(\mathfrak{g})$*,*

$$(v_{j_1}, v_{j_2}, \ldots, v_{j_k}) \mapsto v_{j_1} v_{j_2} \ldots v_{j_k}, \tag{2.4.14}$$

is injective, and the set,

$$\{1, v_{j_1} v_{j_2} \dots v_{j_k} \mid \forall k \in \mathbb{Z}^+, \forall j \in \{1, 2, \dots, n\}, \forall j_n \in J : j_1 \leq j_2 \leq \dots \leq j_k\}, \tag{2.4.15}$$

forms a basis for $U(\mathfrak{g})$.

Proof. See [7, 52]. □

Remark. In Chapter 4, we will utilise Theorem 2.4.21 to improve on the bases of both Corollary 2.4.20 and Theorem 2.4.21 itself. The new basis we will construct, which we call the "multipoles", will correspond to the physically distinct observables of an arbitrary spin system, and is essential in deriving algebraic descriptions of arbitrary spin systems.

2.4.4.3 The Universal Property of the Universal Enveloping Algebra

Like the tensor algebra it derives from, the universal enveloping algebra enjoys a universal property, though its statement is somewhat subtler than for the former,

Lemma 2.4.22 (Universal Property of the Universal Enveloping Algebra)**.** *Consider a Lie algebra* $\mathfrak{g} = (\mathcal{V}, l)$ *over the field* $\mathbb{F}$*, a unital associative algebra* $\mathcal{B} = (\mathcal{W}, \star, f)$ *over* $\mathbb{F}$*, and its commutator subalgebra* $\mathcal{B}^{[\cdot,\cdot]} = (\mathcal{W}, [\cdot, \cdot])$*. For any Lie algebra homomorphism* $\lambda : \mathfrak{g} \to \mathcal{B}^{[\cdot,\cdot]}$*, there exists a unique unital associative algebra homomorphism* $\tilde{\lambda} : U(\mathfrak{g}) \to \mathcal{B}$ *such that,*

$$j \circ \lambda = \tilde{\lambda} \circ i, \tag{2.4.16}$$

where $i : \mathfrak{g} \to U(\mathfrak{g})$ *and* $j : \mathcal{B}^{[\cdot,\cdot]} = (\mathcal{W}, [\cdot, \cdot]) \to \mathcal{B}$ *are inclusion maps.*

Proof. See [54]. □

Remark. Lemma 2.4.22 means that all unital associative algebras which implement the Lie product of $\mathfrak{g}$ as a commutator may be studied through the elements of $U(\mathfrak{g})$ via $\tilde{\lambda}$. In this sense, $U(\mathfrak{g})$ is the most general unital associative algebra which subsumes the Lie product in this way. As such, $U(\mathfrak{g})$ will be our starting point for the construction of the spin algebras in Chapter 4.

2.4.4.4 Casimir Elements

In Chapter 4, we will see that the spin algebras are naturally revealed through a decomposition of $U(\mathfrak{so}(3,\mathbb{R}))$ which utilises its centre $Z(U(\mathfrak{so}(3,\mathbb{R})))$. Thus, understanding the structure of the centre $Z(U(\mathfrak{g}))$ of a universal enveloping algebra $U(\mathfrak{g})$ will be essential to make progress.

Lemma 2.4.23. *Consider a semisimple Lie algebra* $\mathfrak{g}$ *of rank* r *over the field* $\mathbb{F}$. *Then,* $Z(U(\mathfrak{g}))$ *is the* $\mathbb{F}$*-linear span of all products of the identity element* 1 *and* r *linearly independent non-scalar elements.*

Proof. See [63]. □

Following Lemma 2.4.23, we make special identification of the sets of elements of $Z(U(\mathfrak{g}))$ which can generate it,

Definition 2.4.24 (Casimir Elements)**.** Consider a semisimple Lie algebra $\mathfrak{g}$ of rank r over the field $\mathbb{F}$. A set $\{z_j\}$ indexed over the set $\{1,\ldots,r\}$ are called Casimir elements iff they generate $Z(U(\mathfrak{g}))$ as in Lemma 2.4.23.

Of particular interest to this thesis is,

Lemma 2.4.25. *Consider a semisimple Lie algebra* $\mathfrak{g} = (\mathcal{V}, l)$ *over the field* $\mathbb{F}$*, and a basis* $\{b_j\}$ *for* $\mathfrak{g}$ *indexed by the set* J*, with respect to which the Lie product takes the form,* $\forall p,q \in J$*,*

$$l(b_p,b_q) = \sum_{r\in J} f^r_{pq} b_r \tag{2.4.17}$$

for $f^r_{pq} \in \mathbb{F}$*. Then, there exists* $\Omega \in Z(U(\mathfrak{g}))$ *such that,*

$$\Omega := \sum_{p,q\in J} \beta^{pq} b_p b_q, \tag{2.4.18}$$

for $\beta^{pq} \in \mathbb{F}$*, with* $\beta^{qp} = \beta^{pq}$*, where,* $\forall a,b,c \in J$*,*

$$\sum_{d\in J} \beta^{da} f^b_{dc} = -\sum_{d\in J} \beta^{db} f^a_{dc}. \tag{2.4.19}$$

Furthermore, Ω *is independent of the chosen basis.*

Proof. See [52, 64]. □

Definition 2.4.26 (Quadratic Casimir Element)**.** The quadratic Casimir element is the element Ω in Lemma 2.4.25.

2.4.4.5 $Z(U(\mathfrak{so}(3,\mathbb{R})))$

In this thesis we are most concerned with the Lie algebra $\mathfrak{so}(3,\mathbb{R})$. As such,

Lemma 2.4.27. *For $Z(U(\mathfrak{so}(3,\mathbb{R})))$, the quadratic Casimir element is the only non-scalar element which generates it.*

Proof. Lemma 2.3.36 asserts $\mathfrak{so}(3,\mathbb{R})$ is simple and Lemma 2.3.40 asserts it has rank 1. Thus, by Lemma 2.4.23, $Z(U(\mathfrak{so}(3,\mathbb{R})))$ contains a single Casimir element. By Lemma 2.4.25, the quadratic Casimir element exists for $Z(U(\mathfrak{so}(3,\mathbb{R})))$, and thus it must be this single Casimir element up to scaling. □

Remark. While the methods we will develop in Chapter 4 may be utilised for any semisimple Lie algebra, Lemma 2.4.27 greatly simplifies the analysis for $\mathfrak{so}(3,\mathbb{R})$.

As we must fix a scaling, we define,

Definition 2.4.28 (S^2)**.** We define $S^2 \in Z(U(\mathfrak{so}(3,\mathbb{R})))$,

$$S^2 \coloneqq \sum_{a=1}^{3} S_a S_a. \tag{2.4.20}$$

2.4.4.6 Universal Enveloping Algebra Actions

In Chapter 4, we will be working mainly with the universal enveloping algebra $U(\mathfrak{so}(3,\mathbb{R}))$, using methods developed in Chapter 3. To properly apply these methods, it is necessary for us to consider actions of the universal enveloping algebra on itself. The first two we must consider are the left and right multiplication actions,

Definition 2.4.29 (Left Multiplication)**.** Consider a Lie algebra $\mathfrak{g}$ over the field $\mathbb{F}$. We define, the left multiplication action $L \in \mathrm{Hom}(U(\mathfrak{g}), \mathrm{End}^{\circ}(U(\mathfrak{g})))$,

$$L \coloneqq A \mapsto (B \mapsto AB)\,. \tag{2.4.21}$$

Definition 2.4.30 (Right Multiplication)**.** Consider a Lie algebra $\mathfrak{g}$ over the field $\mathbb{F}$. We define, the right multiplication action $L \in \overline{\mathrm{Hom}}(U(\mathfrak{g}), \mathrm{End}^{\circ}(U(\mathfrak{g})))$,

$$R \coloneqq A \mapsto (B \mapsto BA)\,. \tag{2.4.22}$$

We also wish to expand the definition of the adjoint action ad from and action of $\mathfrak{g}$ on itself to an action of $U(\mathfrak{g})$ on itself. To do this, note,

Lemma 2.4.31. *Consider the map,*

$$\begin{aligned} f &: \mathfrak{g} \to \mathrm{End}(U(\mathfrak{g})) \\ f &:= v \mapsto \big(A \mapsto (vA - Av)\big). \end{aligned} \tag{2.4.23}$$

Then, $f \in \mathrm{Hom}(\mathfrak{g}, \mathrm{End}^{[\cdot,\cdot]}(U(\mathfrak{g})))$, f is a derivation, and $\forall v, w \in \mathfrak{g}$,

$$f(a)(b) = i \circ \big(\mathrm{ad}(a)\big)(b), \tag{2.4.24}$$

where $i : \mathfrak{g} \to U(\mathfrak{g})$ is the inclusion map.

Proof. The first and third claims follow by direct computation, and the second Lemma 2.3.63. □

Lemma 2.4.31 offers a natural extension of ad as a Lie algebra action on $\mathrm{End}(U(\mathfrak{g}))$,

Definition 2.4.32 (Adjoint Action of a Lie Algebra on its Universal Enveloping Algebra)**.** We define $\mathrm{ad} \in \mathrm{Hom}(\mathfrak{g}, \mathrm{End}^{[\cdot,\cdot]}(U(\mathfrak{g})))$ to be identical to f in Lemma 2.4.31.

From this, we may utilise the universal property of $U(\mathfrak{g})$ to complete the extension,

Definition 2.4.33 (Adjoint Action of the Universal Enveloping Algebra on Itself)**.** Consider the map f defined in Lemma 2.4.31. By Lemma 2.4.22 (and abusing notation), there exists a unique unital associative algebra homomorphism $\mathrm{ad} \in \mathrm{Hom}(U(\mathfrak{g}), \mathrm{End}^{\circ}(U(\mathfrak{g})))$ for which,

$$\mathrm{ad}(A) := B \mapsto \begin{cases} A \cdot_{U(\mathfrak{g})} B & A \in \mathbb{F} \\ \mathrm{ad}(A)(B) & A \in \mathfrak{g} \\ \mathrm{ad}(C) \circ \mathrm{ad}(D)(B) & A = CD, \end{cases} \tag{2.4.25}$$

where we have left all necessary inclusion maps implicit.

2.4.5 Exterior Algebras

2.4.5.1 Role in this Thesis

The exterior algebra is important to this thesis through its relationship to both the Clifford algebra and geometry. The objects of the exterior algebra have a definite geometrical character, algebraically representing hypervolumes of all possible dimensions for a given vector space. In particular, the planar objects which we met in Section 2.2.4 will be used heavily in our construction of non-commutative position algebras in Chapter 5.

2.4.5.2 The Antisymmetric Tensors

To understand the structure of the exterior algebra, let us first define the antisymmetric tensors. In what follows, we assume $\mathbb{Q} \subseteq \mathbb{F}$.

Definition 2.4.34 ($\Lambda^k(\mathcal{V})$)**.** For all $k \in \mathbb{N}$, the kth-exterior power of $\mathcal{V}$ is the subspace $\Lambda^k(\mathcal{V}) \subseteq \mathcal{V}^{\otimes k}$ spanned by elements of the form, $\forall v_j \in \mathcal{V}$,

$$\bigwedge_{j=1}^{k} v_j := \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \bigotimes_{j=1}^{k} v_{\sigma(j)}, \tag{2.4.26}$$

where S_k is set of permutations of k symbols, and $\operatorname{sgn}(\sigma)$ the sign of the permutation σ. We refer to the elements of $\Lambda^k(\mathcal{V})$ as kth-order antisymmetric tensors, or more commonly k-vectors. We refer to elements of the form (2.4.26) as k-blades.

Remark. When we discuss antisymmetric tensors in the context of an algebra, we are discussing elements of the form in Definition 2.4.34 with the product of the algebra replacing $\otimes$.

Lemma 2.4.35. *For all $\sigma \in S_k$, $\forall v_j \in \mathcal{V}$,*

$$\bigwedge_{j=1}^{k} v_{\sigma(j)} = \operatorname{sgn}(\sigma) \bigwedge_{j=1}^{k} v_j. \tag{2.4.27}$$

Proof. Follows from the fact that S_k is a finite group, and that sgn is a group homomorphism from S_k to the group $(\{1, -1\}, \times, 1)$. □

Lemma 2.4.36. *When $\mathcal{V}$ is finite-dimensional, so is $\Lambda^k(\mathcal{V})$ and* $\dim\big(\Lambda^k(\mathcal{V})\big) = \binom{\dim(\mathcal{V})}{k}$.

Proof. See [49]. □

Remark. Lemma 2.4.36 reveals that, unlike the symmetric tensors, for $n > \dim(\mathcal{V})$, $\Lambda^n(\mathcal{V}) \cong \{0\}$.

2.4.5.3 The Exterior Algebra

Definition 2.4.37 ($\Lambda(\mathcal{V})$)**.** Consider a vector space $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$. Its exterior algebra $\Lambda(\mathcal{V})$ is the quotient algebra, $\forall v, w \in \mathcal{V}$,

$$\Lambda(\mathcal{V}) \cong \frac{T(\mathcal{V})}{I\left(v \otimes w + w \otimes v\right)}. \tag{2.4.28}$$

We denote the product of the antisymmetric algebra by $\wedge$, and call it the "wedge product".

Now, let us investigate the structure of $\Lambda(\mathcal{V})$ by using the universal property of $T(\mathcal{V})$ to see what happens to a k-adic tensor after the quotient. This follows the process we used to study the symmetric tensors.

Lemma 2.4.38. *For all $k \in \mathbb{N}$, the equivalence classes of the k-adic tensors in $\Lambda(\mathcal{V})$ are labelled by the k-blades.*

Proof. Consider $\bigotimes_{j=1}^{k} v_j \in \mathcal{V}^{\otimes k}$, then,

$$\left[\bigotimes_{j=1}^{k} v_j\right] = \left[\bigotimes_{j=1}^{k} v_j - \frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\bigotimes_{j=1}^{k} v_{\sigma(j)}\right] + \left[\bigwedge_{j=1}^{k} v_j\right].$$

By Lemma 2.3.23, in $\Lambda(\mathcal{V})$ we have, $\forall A, B \in T(\mathcal{V})$, $\forall v, w \in \mathcal{V}$,

$$\big[A\otimes v\otimes w\otimes B\big] = \big[-A\otimes w\otimes v\otimes B\big],$$

and so,

$$\left[\bigotimes_{j=1}^{k} v_j - \frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)\bigotimes_{j=1}^{k} v_{\sigma(j)}\right] = \left[\bigotimes_{j=1}^{k} v_j - \frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)^2\bigotimes_{j=1}^{k} v_j\right] = [\,0\,],$$

since $|S_k| = \frac{1}{k!}$ and $\forall\sigma \in S_k$, $\operatorname{sgn}(\sigma)^2 = 1$. □

Thus, we find,

Lemma 2.4.39.

$$\Lambda(\mathcal{V}) \cong \bigoplus_{j=0}^{\dim(\mathcal{V})} \Lambda^j(\mathcal{V}). \tag{2.4.29}$$

Proof. Follows directly from Lemma 2.4.38. □

Corollary 2.4.40. *When $\mathcal{V}$ is finite-dimensional, so is $\Lambda(\mathcal{V})$ and $\dim\big(\Lambda(\mathcal{V})\big) = 2^{\dim(\mathcal{V})}$.*

Proof. Follows immediately from Lemmas 2.4.36 and 2.4.39. □

2.4.6 Clifford and Duffin-Kemmer-Petiau Algebras

2.4.6.1 Role in this Thesis

Unlike the other algebras discussed in this section, the Clifford and Duffin-Kemmer-Petiau algebras do not play direct roles in the development of this thesis. However, they showcase many important properties that will motivate the general definition of the indefinite-spin and spin-s position algebras which we will construct in Chapter 5. As such, we shall discuss them briefly and overview their important features.

2.4.6.2 The Clifford Algebra

In this thesis, we will only consider fields for which $\mathbb{Q} \subseteq \mathbb{F}$, so consider the field $\mathbb{F}$ to satisfy this in what follows. The Clifford algebra algebraically subsumes the metric of a Minkowski space-time,

Definition 2.4.41 ($\mathrm{Cl}(\mathcal{V}, g)$)**.** Consider a Minkowski space-time $(\mathcal{V}, g)$ over the field $\mathbb{F}$. Its Clifford algebra is the quotient algebra, $\forall v, w \in \mathcal{V}$,

$$\mathrm{Cl}(\mathcal{V}, g) \cong \frac{T(\mathcal{V})}{I\left(v \otimes w + w \otimes v - 2g(v,w)\right)}. \tag{2.4.30}$$

The product of the Clifford algebra is often left implicit in the literature; we shall adopt this convention.

Remark. Clifford algebras are also definable for spaces equipped with degenerate symmetric bilinear maps, but we will not consider such maps or algebras in this thesis.

The Clifford algebra is also commonly defined using a quadratic form instead of a metric,

Lemma 2.4.42. *As $\mathbb{Q} \subseteq \mathbb{F}$, Definition 2.4.41 is equivalent to the construction of* $\mathrm{Cl}(\mathcal{V}, g)$ *using a quadratic form.*

Proof. See [57]. □

Lemma 2.4.43. *As vector spaces,* $\mathrm{Cl}(\mathcal{V}, g) \cong \Lambda(\mathcal{V})$*.*

Proof. Firstly, it is clear that no antisymmetric tensors are present in the ideal $I\left(v \otimes w + w \otimes v - 2g(v,w)\right)$. Thus, by Lemma 2.4.4 all antisymmetric tensors are non-zero in $\mathrm{Cl}(\mathcal{V}, g)$. Now, consider $\bigotimes_{j=1}^{k} v_j \in \mathcal{V}^{\otimes k}$, then after the quotient,

$$\left[\bigotimes_{j=1}^{k} v_j\right] = \left[\bigotimes_{j=1}^{k} v_j - \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \bigotimes_{j=1}^{k} v_{\sigma(j)}\right] + \left[\bigwedge_{j=1}^{k} v_j\right].$$

By Lemma 2.3.23, in $\mathrm{Cl}(\mathcal{V}, g)$ we have, $\forall A, B \in T(\mathcal{V})$, $\forall v, w \in \mathcal{V}$,

$$\left[A \otimes v \otimes w \otimes B\right] = \left[-A \otimes w \otimes v \otimes B\right] + \left[2g(v,w) A \otimes B\right],$$

noting that the final term is of a strictly lower tensor order than the others. Thus,

$$\left[\bigotimes_{j=1}^{k} v_j - \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \bigotimes_{j=1}^{k} v_{\sigma(j)}\right] = \left[\bigotimes_{j=1}^{k} v_j - \frac{1}{k!} \sum_{\sigma \in S_k} \bigotimes_{j=1}^{k} v_j + f(v_1, \ldots, v_k)\right]$$

$$= [\, f(v_1, \ldots, v_k)\,],$$

since $|S_k| = \frac{1}{k!}$ and $\mathrm{sgn}(\sigma)^2 = 1$, where $f(v_1, \ldots, v_k)$ is a tensor of order strictly less than k. Therefore, we may repeat this process for all n-adics with $n < k$ which comprise $f(v_1, \ldots, v_k)$; this process is guaranteed to terminate since tensor order is bounded from below. Thus, every element of $\mathrm{Cl}(\mathcal{V}, g)$ may be written as an $\mathbb{F}$-linear combination of antisymmetric tensors, and we may establish the isomorphism as mapping a given antisymmetric tensor to its equivalence class in $\mathrm{Cl}(\mathcal{V}, g)$. □

Corollary 2.4.44. *The antisymmetric tensors form a basis for* $\mathrm{Cl}(\mathcal{V}, g)$*.*

Proof. This is clear from Lemma 2.4.43. □

Remark. Corollary 2.4.44 shows that the $\mathrm{Cl}(\mathcal{V}, g)$ is constructed from objects which represent hypervolumes in $\mathcal{V}$. Furthermore, by its definition, the algebraic structure of the Clifford algebra is controlled entirely by the properties of the metric g. As such, it is an algebra with a natural geometric character.

2.4.6.3 The Clifford Algebra and $\mathfrak{so}(\mathcal{V}, g)$

As one might hope from its definition, the Clifford algebra has a natural relationship with the symmetries of g. To begin, $\mathrm{Cl}(\mathcal{V}, g)$ admits a natural action of bivectors on vectors,

Definition 2.4.45 (u_{cl})**.** For all $a, b, c \in \mathcal{V}$,

$$\begin{aligned} \mathrm{u}_{cl} &: \Lambda^2(\mathcal{V}) \to \mathrm{Hom}(\mathcal{V}, \mathrm{Cl}(\mathcal{V}, g)) \\ \mathrm{u}_{cl}(a\wedge b)(c) &:= (a\wedge b)c - c(a\wedge b). \end{aligned} \tag{2.4.31}$$

Lemma 2.4.46. *For all* $a, b, c \in \mathcal{V}$*,*

$$\mathrm{u}_{cl}(a\wedge b)(c) = -2\big(g(a,c)b - g(b,c)a\big) = -2\mu(a\wedge b)(c), \tag{2.4.32}$$

recalling μ *from Definition 2.2.53.*

Proof. Direct calculation. □

Lemma 2.4.47. $\mathrm{u}_{cl}(a\wedge b)$ *may be extended to an action on* $\mathrm{Cl}(\mathcal{V}, g)$ *as a derivation.*

Proof. This follows immediately from Lemma 2.3.63. □

Using this action, we find the bivectors of the Clifford algebra are closed under commutators,

Lemma 2.4.48. *For all* $a, b, c, d \in \mathcal{V} \subset \mathrm{Cl}(\mathcal{V}, g)$*,*

$$(a\wedge b)(c\wedge d) - (c\wedge d)(a\wedge b) = \mathrm{u}_{cl}(a\wedge b)(c\wedge d) = -2\mu(a\wedge b)(c\wedge d). \tag{2.4.33}$$

Proof. Direct calculation. □

Lemma 2.4.49. *The commutator Lie algebra of bivectors in* $\mathrm{Cl}(\mathcal{V}, g)$ *is Lie algebra isomorphic to* $\mathfrak{so}(\mathcal{V}, g)$*,*

Proof. This follows immediately from Lemmas 2.2.57 and 2.2.59. □

Thus, we have found that within the Clifford algebra is a natural implementation of the bivector algebra (up to a constant) from Section 2.2.4.

2.4.6.4 The Duffin-Kemmer-Petiau Algebra

Like the Clifford algebra, the Duffin-Kemmer-Petiau algebra also subsumes the metric of a Minkowski space-time, but in a slightly different way,

Definition 2.4.50 ($\mathrm{Dkp}(\mathcal{V}, g)$)**.** Consider a Minkowski space-time $(\mathcal{V}, g)$ over the field $\mathbb{F}$. Its Duffin-Kemmer-Petiau algebra is the quotient algebra, $\forall u, v, w \in \mathcal{V}$,

$$\mathrm{Dkp}(\mathcal{V}, g) \cong \frac{T(\mathcal{V})}{I\left(u\otimes v\otimes w + w\otimes v\otimes u - g(u,v)w - g(w,v)u\right)}. \tag{2.4.34}$$

The product of the Duffin-Kemmer-Petiau algebra is often left implicit in the literature; we shall adopt this convention.

For our purposes, we need not fully determine the relationship between $\Lambda(\mathcal{V})$ and $\mathrm{Dkp}(\mathcal{V}, g)$,

Lemma 2.4.51. *The vector space homomorphism,*

$$\begin{gathered}\phi : \Lambda^2(\mathcal{V}) \to \mathrm{Dkp}(\mathcal{V}, g)\\ \phi = a\wedge b \mapsto [\,a\wedge b\,],\end{gathered} \tag{2.4.35}$$

is injective.

Proof. Since no 2-blades are in the ideal,

$$I\left(u\otimes v\otimes w + w\otimes v\otimes u - g(u,v)w - g(w,v)u\right),$$

Lemma 2.4.4 implies that $\mathrm{Ker}(\phi) = \{0\}$. Thus, ϕ is injective. □

2.4.6.5 The Duffin-Kemmer-Petiau Algebra and $\mathfrak{so}(\mathcal{V}, g)$

Again, like the Clifford algebra, the Duffin-Kemmer-Petiau algebra has a natural relationship with the symmetries of g,

Definition 2.4.52 (u_{dkp})**.** For all $a, b, c \in \mathcal{V}$,

$$\begin{aligned} \mathrm{u}_{cl} &: \Lambda^2(\mathcal{V}) \to \mathrm{Hom}(\mathcal{V}, \mathrm{Dkp}(\mathcal{V}, g)) \\ \mathrm{u}_{dkp}(a \wedge b)(c) &:= (a \wedge b)c - c(a \wedge b). \end{aligned} \tag{2.4.36}$$

Lemma 2.4.53. *For all $a, b, c \in \mathcal{V}$,*

$$\mathrm{u}_{dkp}(a \wedge b)(c) = -\frac{1}{2}\big(g(a,c)b - g(b,c)a\big) = -\frac{1}{2}\mu(a \wedge b)(c), \tag{2.4.37}$$

recalling μ from Definition 2.2.53.

Proof.

$$\begin{aligned} (a \wedge b)c - c(a \wedge b) &= \frac{1}{2}\big((abc + cba) - (bac + cab)\big) \\ &= \frac{1}{2}\big(g(a,b)c + g(c,b)a - g(b,a)c - g(c,a)b\big) \\ &= -\frac{1}{2}\big(g(a,c)b - g(b,c)a\big). \end{aligned}$$

□

From this point, our analysis follows that of the Clifford algebra,

Lemma 2.4.54. $\mathrm{u}_{dkp}(a \wedge b)$ *may be extended to an action on* $\mathrm{Dkp}(\mathcal{V}, g)$ *as a derivation.*

Proof. This follows immediately from Lemma 2.3.63. □

Lemma 2.4.55. *For all $a, b, c, d \in \mathcal{V} \subset \mathrm{Dkp}(\mathcal{V}, g)$,*

$$(a \wedge b)(c \wedge d) - (c \wedge d)(a \wedge b) = \mathrm{u}_{dkp}(a \wedge b)(c \wedge d) = -\frac{1}{2}\mu(a \wedge b)(c \wedge d). \tag{2.4.38}$$

Proof. Direct calculation. □

Lemma 2.4.56. *The commutator Lie algebra of bivectors in* $\mathrm{Dkp}(\mathcal{V}, g)$ *is Lie algebra isomorphic to* $\mathfrak{so}(\mathcal{V}, g)$*,*

Proof. This follows immediately from Lemmas 2.2.57 and 2.2.59. □

Once again, we have found that, like the Clifford algebra, there is also a natural implementation of the bivector algebra (up to a constant) from Section 2.2.4 within Duffin-Kemmer-Petiau algebra. These observations will be instructive for us in Chapter 5.

2.4.7 Univariate Polynomial Algebras

2.4.7.1 Role in this Thesis

The methods developed in Chapter 3 directly utilise a natural connection between the unital associative algebra generated by a single element and quotient rings of univariate polynomial algebras. As such, we must understand these structures, and this connection, to define and utilise these methods in the rest of the thesis.

2.4.7.2 Univariate Polynomial Algebras

Definition 2.4.57 ($\mathbb{F}[x]$)**.** The univariate polynomial algebra over the field $\mathbb{F}$ is the unital associative algebra of all polynomials in a single indeterminate x,

$$p(x) = \sum_{j=0}^{k} \alpha_j x^j, \tag{2.4.39}$$

$\forall k \in \mathbb{N}$, with coefficients $\forall j \in \mathbb{N}$, $\alpha_j \in \mathbb{F}$, and identifying $x^0 = 1$ consistently with Definition 2.3.13. The product for this algebra is polynomial multiplication, and the constant polynomial 1 the identity element. We denote the univariate polynomial algebra over $\mathbb{F}$ as $\mathbb{F}[x]$.

Remark. Unlike polynomial functions, the indeterminate x is not an as yet undetermined scalar in $\mathbb{F}$, but simply a non-zero, non-constant (i.e. of the form $\alpha 1$ for some $\alpha \in \mathbb{F}$) element in the algebra.

Lemma 2.4.58. *The countably infinite set* $\{x^j \mid j \in \mathbb{N}\}$ *is a basis for* $\mathbb{F}[x]$ *as a vector space over* $\mathbb{F}$.

Proof. This follows from the definition of polynomial equality. □

To ease discussion of polynomials, it is often useful to classify them by their "largest" non-zero terms,

Lemma 2.4.59. *Consider a non-zero polynomial* $p(x) \in \mathbb{F}[x]$ *written as,*

$$p(x) = \sum_{j=0}^{k} \alpha_j x^j. \tag{2.4.40}$$

Then, there exists a unique $n \in \mathbb{N}$ *such that* $\alpha_n \neq 0$ *but* $\forall m \in \mathbb{Z}^+$, $\alpha_{n+m} = 0$.

Proof. Existence follows from the definition of a polynomial. Uniqueness follows from the definition of polynomial equality. □

Definition 2.4.60 (Polynomial Order)**.** Consider a polynomial $p(x) \in \mathbb{F}[x]$. Then: if $p(x) \neq 0$, the polynomial order of $p(x)$ is the unique natural n defined in Lemma 2.4.59; if $p(x) = 0$, the polynomial order is 0. We denote the polynomial order of $p(x)$ by $|p(x)|$.

Remark. We use the term "order" here, instead of the more usual "degree", to align our terminology for polynomials with what we will later define for tensors.

Lemma 2.4.61. *Consider two polynomials $p(x), q(x) \in \mathbb{F}[x]$. Then, $|p(x)q(x)| = |p(x)| + |q(x)|$.*

Proof. Writing $p(x) = \alpha_{|p(x)|}x^{|p(x)|} + r(x)$ and $q(x) = \beta_{|q(x)|}x^{|q(x)|} + s(x)$, we find,

$$p(x)q(x) = \alpha_{|p(x)|}\beta_{|q(x)|}x^{(|p(x)|+|q(x)|)} + t(x).$$

□

Since $\mathbb{F}$ is a field and $\alpha_{|p(x)|} \neq 0$ for non-zero $p(x) \in \mathbb{F}[x]$, it is simpler to consider polynomials which have been normalised by dividing by $\alpha_{|p(x)|}$,

Definition 2.4.62 (Monic Polynomial)**.** A non-zero polynomial $p(x) \in \mathbb{F}[x]$ is monic iff its component $\alpha_{|p(x)|}$ in $x^{|p(x)|}$ satisfies $\alpha_{|p(x)|} = 1$.

2.4.7.3 Polynomial Factorisation and Irreducible Polynomials

Just as prime factorisation enables a consistent description of positive integers, we may factorise polynomials in much the same way to understand their structure. This will be of central importance to the methods developed in Chapter 3. First, let us define some terms,

Definition 2.4.63 (Polynomial Factor)**.** Consider two polynomials $p(x), q(x) \in \mathbb{F}[x]$. Then, $q(x)$ is a polynomial factor of $p(x)$ iff $\exists r(x) \in \mathbb{F}[x]$ such that $p(x) = q(x)r(x)$. We denote this relationship as $q(x) \mid p(x)$. We may also refer to $q(x)$ as a divisor of $p(x)$, and $p(x)$ as a multiple of $q(x)$.

Lemma 2.4.64. *Consider three polynomials $f(x), g(x), h(x) \in \mathbb{F}[x]$ such that $f(x) \mid g(x)$ and $g(x) \mid h(x)$. Then, $f(x) \mid h(x)$.*

Proof. There exist $a(x), b(x) \in \mathbb{F}[x]$ such that $g(x) = a(x)f(x)$ and $h(x) = b(x)g(x)$. Thus, $h(x) = b(x)a(x)f(x)$. □

Lemma 2.4.65. *For all $p(x) \in \mathbb{F}$, $p(x) \mid 0$, and $0 \mid p(x)$ iff $p(x) = 0$.*

Proof. $0\,p(x) = 0$, thus $p(x) \mid 0$. If $0 \mid p(x)$, $\exists r(x) \in \mathbb{F}[x]$ such that $p(x) = 0\,r(x) = 0$. The reverse is trivial. □

Lemma 2.4.66. *Consider non-zero polynomials $p(x), q(x) \in \mathbb{F}[x]$ such that $q(x) \mid p(x)$. Then, $|q(x)| \le |p(x)|$.*

Proof. Direct consequence of Lemma 2.4.61. □

Lemma 2.4.67. *Consider two polynomials $p(x), q(x) \in \mathbb{F}[x]$ such that $q(x) \mid p(x)$ and $p(x) \mid q(x)$. Then, $p(x) = q(x) = 0$, or $\exists \alpha \in \mathbb{F}$, $\alpha \neq 0$ such that $q(x) = \alpha p(x)$.*

Proof. For the first claim, without loss of generality, suppose $p(x) = 0$. Then, by Lemma 2.4.65, $q(x) = 0$. Thus, let us assume $p(x) \neq 0$ and $q(x) \neq 0$. By Lemma 2.4.66, we have $|q(x)| \le |p(x)|$ and $|p(x)| \le |q(x)|$, so $|q(x)| = |p(x)|$. But since $\exists a(x), b(x) \in \mathbb{F}$ such that $q(x) = a(x)p(x)$ and $p(x) = b(x)q(x)$, we must have $|a(x)| = |b(x)| = 0$, so $a(x), b(x) \in \mathbb{F}$. Since $p(x) \neq 0$ and $q(x) \neq 0$, then $a(x) \neq 0$ and $b(x) = a(x)^{-1}$. □

Of greatest significance to the present work are polynomials which have no interesting polynomial factors,

Definition 2.4.68 (Irreducible Polynomial over $\mathbb{F}$)**.** A non-zero, non-constant polynomial $p(x) \in \mathbb{F}[x]$ is irreducible over $\mathbb{F}$ iff,

$$[p(x) = q(x)r(x)] \Rightarrow \big[[q(x) = \beta 1] \vee [r(x) = \gamma 1],\big] \tag{2.4.41}$$

for some $\beta, \gamma \in \mathbb{F}$.

Remark. Note that the irreducibility of a polynomial is dependent on the field of scalars $\mathbb{F}$ one is working with.

For convenient discussion, let us also define,

Definition 2.4.69 (Irreducible Power)**.** A non-zero polynomial $p(x) \in \mathbb{F}[x]$ is an irreducible power iff $\exists q(x) \in \mathbb{F}[x]$ such that $q(x)$ is irreducible over $\mathbb{F}$ and $\exists k \in \mathbb{Z}^+$ such that $p(x) = q(x)^k$.

Much like the prime numbers, there is a deep relationship between arbitrary and irreducible polynomials,

Definition 2.4.70 (Irreducible Power Form)**.** Consider a non-zero polynomial $p(x) \in \mathbb{F}[x]$. An irreducible power form for $p(x)$ is a set $\{f_j(x)^{d_j} \in \mathbb{F}[x]\}$ indexed over a set J such that,

$$p(x) = \alpha_{|p(x)|} \prod_{j \in J} f_j(x)^{d_j}, \tag{2.4.42}$$

and:

- For all $j \in J$, $f_j(x)^{d_j}$ is monic and an irreducible power;
- $\forall j, k \in J$, $j \neq k$, $\gcd(f_j(x), f_k(x)) = 1$.

Lemma 2.4.71. *For all $p(x) \in \mathbb{F}[x]$, the irreducible factor form of $p(x)$ is unique.*

Proof. This follows from the fact that $\mathbb{F}[x]$ is a unique factorisation domain [53]. □

We will use irreducible power forms of polynomials extensively in Chapter 3 to develop our methods.

2.4.7.4 Greatest Common Divisors and Least Common Multiples

In Chapter 3, we will make heavy use of shared factors and multiples of a family of polynomials. Thus, we must understand the structure of these shared polynomials. To progress, we first define,

Definition 2.4.72 (Greatest Common Divisor (gcd))**.** Consider two polynomials $f(x), g(x) \in \mathbb{F}[x]$ not both zero. Their greatest common divisor $\gcd(f(x), g(x)) \in \mathbb{F}[x]$ is a polynomial such that:

1. $\gcd(f(x), g(x)) \mid f(x)$ and $\gcd(f(x), g(x)) \mid g(x)$;
2. $\forall s(x) \in \mathbb{F}[x]$ such that $s(x) \mid f(x)$ and $s(x) \mid g(x)$, then $s(x) \mid \gcd(f(x), g(x))$.

When $\gcd(f(x), g(x)) \in \mathbb{F}[x] \neq 0$, we scale it to be *monic*.

Definition 2.4.73 (Least Common Multiple (lcm))**.** Consider two polynomials $f(x), g(x) \in \mathbb{F}[x]$ not both zero. Their least common multiple $\operatorname{lcm}(f(x), g(x)) \in \mathbb{F}[x]$ is a *monic* polynomial such that:

1. $f(x) \mid \operatorname{lcm}(f(x), g(x))$ and $g(x) \mid \operatorname{lcm}(f(x), g(x))$;
2. $\forall s(x) \in \mathbb{F}[x]$ such that $f(x) \mid s(x)$ and $g(x) \mid s(x)$, then $\operatorname{lcm}(f(x), g(x)) \mid s(x)$.

When $\operatorname{lcm}(f(x), g(x)) \in \mathbb{F}[x] \neq 0$, we scale it to be *monic*.

Remark. From Definition 2.4.63, any non-zero constant multiple of a polynomial factor is also a polynomial factor. This is also the case for multiples of a polynomial. We have chosen gcd and lcm to yield strictly monic polynomials to avoid this ambiguity, and to reduce the number of unnecessary qualifications in our later arguments.

2.4.7.5 Properties of gcd and lcm

The gcd and lcm have a number of important properties which we will rely on in Chapter 3. First, we must understand when they exist, their uniqueness, and when they are zero,

Lemma 2.4.74. *Consider two polynomials $f(x), g(x) \in \mathbb{F}[x]$ not both zero. Then,* $\gcd(f(x), g(x))$ *and* $\operatorname{lcm}(f(x), g(x))$ *exists and is unique.*

Proof. For existence, see [49]. For uniqueness, consider two monic polynomials $p(x), q(x) \in \mathbb{F}[x]$ which satisfy Definition 2.4.72. Thus, we have $p(x) \mid q(x)$ and $q(x) \mid p(x)$. Since $p(x)$ and $q(x)$ are monic, by Lemma 2.4.67 we must have $p(x) = q(x)$. The argument is identical for $\operatorname{lcm}(f(x), g(x))$. □

Lemma 2.4.75. *Consider two polynomials $f(x), g(x) \in \mathbb{F}[x]$ not both zero. Then,* $\gcd(f(x), g(x)) \neq 0$.

Proof. Let us prove the contraposition of the claim. If $\gcd(f(x), g(x)) = 0$, then applying Lemma 2.4.65 to Definition 2.4.72, we must have $f(x) = 0$ and $g(x) = 0$. □

Lemma 2.4.76. *Consider two polynomials $f(x), g(x) \in \mathbb{F}[x]$ not both zero. Then* $\operatorname{lcm}(f(x), g(x)) = 0$ *implies $f(x) = 0$ or $g(x) = 0$.*

Proof. Since, $f(x) \mid f(x)g(x)$ and $g(x) \mid f(x)g(x)$, then $\operatorname{lcm}(f(x), g(x)) = 0$ implies $0 \mid f(x)g(x)$. Thus, by Lemma 2.4.65, we have $f(x)g(x) = 0$, and so $f(x) = 0$ or $g(x) = 0$. □

Now, we must consider the functional properties of gcd and lcm,

Corollary 2.4.77. *Consider two polynomials $f(x), g(x) \in \mathbb{F}[x]$ not both zero. Then,* $\gcd(f(x), g(x))$ *has the greatest polynomial order amongst all divisors of $f(x)$ and $g(x)$. Similarly,* $\operatorname{lcm}(f(x), g(x))$ *has the least polynomial order amongst all multiples of $f(x)$ and $g(x)$.*

Proof. Consider a divisor $a(x) \in \mathbb{F}[x]$ of $f(x)$. By definition, $a(x) \mid \gcd(f(x), g(x))$, which means, by Lemma 2.4.66, $|a(x)| \leq |\gcd(f(x), g(x))|$. The argument is identical for $\mathrm{lcm}(f(x), g(x))$. □

Corollary 2.4.78. *Consider two polynomials $f(x), g(x) \in \mathbb{F}[x]$ not both zero. Then, $f(x) \mid g(x)$ iff $\gcd(f(x), g(x)) = \alpha f(x)$, where $\alpha \in \mathbb{F}$, $\alpha \neq 0$, and $\alpha f(x)$ is monic. Similarly, $g(x) \mid f(x)$ iff $\gcd(f(x), g(x)) = \beta g(x)$, where $\beta \in \mathbb{F}$, $\beta \neq 0$, and $\beta g(x)$ is monic.*

Proof. If $f(x) \mid g(x)$, then from Definition 2.4.72, we find $\gcd(f(x), g(x)) \mid f(x)$ and $f(x) \mid \gcd(f(x), g(x))$. Then, by Lemmas 2.4.75 and 2.4.67 we must have $\gcd(f(x), g(x)) = \alpha f(x)$ for $\alpha \in \mathbb{F}$, $\alpha \neq 0$, such that $\alpha f(x)$ is monic. The same argument applies to the case when $g(x) \mid f(x)$. The reverse directions for both cases follow from Definition 2.4.72. □

Corollary 2.4.79. *Consider two non-zero polynomials $f(x), g(x) \in \mathbb{F}[x]$. Then, $f(x) \mid g(x)$ iff $\mathrm{lcm}(f(x), g(x)) = \alpha g(x)$, where $\alpha \in \mathbb{F}$, $\alpha \neq 0$, and $\alpha g(x)$ is monic. Similarly, $g(x) \mid f(x)$ iff $\mathrm{lcm}(f(x), g(x)) = \beta f(x)$, where $\beta \in \mathbb{F}$, $\beta \neq 0$, and $\beta f(x)$ is monic.*

Proof. If $f(x) \mid g(x)$, then from Definition 2.4.73, we find $f(x) \mid \mathrm{lcm}(f(x), g(x))$ and $\mathrm{lcm}(f(x), g(x)) \mid f(x)$. Then, by 2.4.67 we must have $\mathrm{lcm}(f(x), g(x)) = \alpha g(x)$ for $\alpha \in \mathbb{F}$, $\alpha \neq 0$, such that $\alpha g(x)$ is monic. The same argument applies to the case when $g(x) \mid f(x)$. The reverse directions for both cases follow from Definition 2.4.73. □

Finally,

Lemma 2.4.80. *For all $f(x), g(x), h(x) \in \mathbb{F}[x]$ not all zero,*

1. $\gcd(g(x), f(x)) = \gcd(f(x), g(x))$;
2. $\mathrm{lcm}(g(x), f(x)) = \mathrm{lcm}(f(x), g(x))$;
3. $\gcd\big(f(x), \gcd(g(x), h(x))\big) = \gcd\big(\gcd(f(x), g(x)), h(x)\big)$;
4. $\mathrm{lcm}\big(f(x), \mathrm{lcm}(g(x), h(x))\big) = \mathrm{lcm}\big(\mathrm{lcm}(f(x), g(x)), h(x)\big)$;

Proof. The first two properties follow from the logically symmetric definition of gcd and lcm. To prove the third property, let us denote $a(x) :=$

$\gcd\big(f(x), \gcd(g(x), h(x))\big)$ and $b(x) := \gcd\big(\gcd(f(x), g(x)), h(x)\big)$. Since $a(x) \mid \gcd(g(x), h(x))$, by Lemma 2.4.64 we have $a(x) \mid g(x)$ and $a(x) \mid h(x)$. Therefore, by $a(x) \mid f(x)$ and the definition of $\gcd(f(x), g(x))$, we have $a(x) \mid \gcd(f(x), g(x))$. Thus, $a(x) \mid b(x)$. By a similar argument we conclude $b(x) \mid a(x)$. Since $a(x)$ and $b(x)$ are monic, by Lemma 2.4.67 we must have $a(x) = b(x)$. The proof of the fourth property follows the same logic as for the third. □

2.4.7.6 gcd and lcm for Finite Sets of Polynomials

We may use Lemma 2.4.80 to extend gcd and lcm to finitely many non-zero polynomials,

Definition 2.4.81 (Greatest Common Divisor (gcd) (Finite Set))**.** Consider a set of polynomials $\{f_j(x) \in \mathbb{F}[x]\}$ not all zero, indexed over the set $\{1, \ldots, n\}$, where $n \in \mathbb{Z}^+$, $n \geq 2$. Then, their greatest common divisor $\gcd(f_1, \ldots, f_n) \in \mathbb{F}[x]$ is the polynomial,

$$\gcd(f_1, f_2, \ldots, f_n) := \gcd\big(f_1, \gcd(f_2, \ldots, f_n)\big). \tag{2.4.43}$$

We may also write gcd as a function taking the set of functions $\gcd\big(\{f_j(x)\}\big) \in \mathbb{F}[x]$.

Definition 2.4.82 (Least Common Multiple (lcm) (Finite Set))**.** Consider a set of polynomials $\{f_j(x) \in \mathbb{F}[x]\}$ not all zero, indexed over the set $\{1, \ldots, n\}$, where $n \in \mathbb{Z}^+$, $n \geq 2$. Then, their least common multiple $\operatorname{lcm}(f_1, \ldots, f_n) \in \mathbb{F}[x]$ is the polynomial,

$$\operatorname{lcm}(f_1, f_2, \ldots, f_n) := \operatorname{lcm}\big(f_1, \operatorname{lcm}(f_2, \ldots, f_n)\big). \tag{2.4.44}$$

We may also write lcm as a function taking the set of functions $\operatorname{lcm}\big(\{f_j(x)\}\big) \in \mathbb{F}[x]$.

Using these definitions,

Lemma 2.4.83. *Consider a set of non-zero polynomials $\{f_j(x) \in \mathbb{F}[x]\}$, indexed over the set $\{1, \ldots, n\}$, where $n \in \mathbb{Z}^+$, $n \geq 2$. Then,*

$$|\gcd\big(\{f_j(x)\}\big)| \leq \min\big(\{|f_j(x)|\}\big) \leq \max\big(\{|f_j(x)|\}\big) \leq |\operatorname{lcm}\big(\{f_j(x)\}\big)|, \tag{2.4.45}$$

with equality in the first and last inequalities iff $\beta \gcd\big(\{f_j(x)\}\big) \in \{f_j(x) \in \mathbb{F}[x]\}$ and $\beta \operatorname{lcm}\big(\{f_j(x)\}\big) \in \{f_j(x) \in \mathbb{F}[x]\}$ respectively, for some $\beta \in \mathbb{F}$, $\beta \neq 0$.

Proof. By definition, $\forall k \in \{1, \ldots, n\}$, $\gcd\big(\{f_j(x)\}\big) \mid f_k$, so by Lemma 2.4.66 we establish the first inequality. Also by definition, $\forall k \in \{1, \ldots, n\}$, $f_k \mid \operatorname{lcm}\big(\{f_j(x)\}\big)$, so by Lemma 2.4.66 we establish the last inequality. The middle inequality is trivial.

If $|\gcd(\{f_j(x)\})| = \min(\{|f_j(x)|\})$, then for some $m \in \{1, \ldots, n\}$ we must have $\gcd(\{f_j(x)\}) = \alpha f_m(x)$ for $\alpha \in \mathbb{F}$, $\alpha \neq 0$, such that $\alpha f_m(x)$ is monic. Similarly, if $\max(\{|f_j(x)|\}) = |\operatorname{lcm}(\{f_j(x)\})|$, then for some $m \in \{1, \ldots, n\}$ we must have $\operatorname{lcm}(\{f_j(x)\}) = \alpha f_m(x)$ for $\alpha \in \mathbb{F}$, $\alpha \neq 0$, such that $\alpha f_m(x)$ is monic. □

2.4.7.7 Quotients of Univariate Polynomial Algebras

The quotient algebras formed from $\mathbb{F}[x]$ are of considerable importance to this thesis: in Chapter 3, they will allow us to directly probe the properties of an endomorphism in a basis-independent and dimensionally agnostic way. We are particularly interested in the properties of quotient algebras resulting from a two-sided ideal generated by a given polynomial,

Definition 2.4.84. Consider a polynomial $p(x) \in \mathbb{F}[x]$. We define the commutative unital associative algebra,

$$\mathcal{Q}_{p(x)}[x] \cong \frac{\mathbb{F}[x]}{I\,(p(x))}, \tag{2.4.46}$$

whose equivalence classes $[r(x)]$ are labelled by the remainder polynomials $r(x)$ under polynomial division by $p(x)$, i.e. $\forall f(x) \in \mathbb{F}$, we may uniquely write,

$$f(x) = s(x)p(x) + r(x), \tag{2.4.47}$$

where $|s(x)| = |f(x)| - |p(x)|$ and $|r(x)| < |p(x)|$.

Definition 2.4.85 (Polynomial Order in $\mathcal{Q}_{p(x)}[x]$)**.** We define the polynomial order of an equivalence class $|[r(x)]|$ to be the polynomial order of the unique (up to scaling) remainder polynomial $|r(x)|$ which labels it.

Remark. We will often abuse notation and discuss the elements of $\mathcal{Q}_{p(x)}[x]$ in terms of the remainder polynomials which label their equivalence classes. This renders the $\mathbb{F}[x]$ and $\mathcal{Q}_{p(x)}[x]$ definitions of polynomial order identical.

Lemma 2.4.86. *Consider $p(x) \in \mathbb{F}[x]$. The quotient algebra $\mathcal{Q}_{p(x)}[x]$ has dimension $\dim(\mathcal{Q}_{p(x)}[x]) = |p(x)|$.*

Proof. Since $\forall r(x) \in \mathcal{Q}_{p(x)}[x]$ satisfy $|r(x)| < |p(x)|$, $\{1, x, x^2, \ldots, x^{|p(x)|-1}\}$ forms a basis for $\mathcal{Q}_{p(x)}[x]$. □

2.4.7.8 Univariate Polynomial Algebras and Unital Associative Algebras

Far from being an abstract consideration, univariate polynomial algebras, and its quotient algebras, frequently arise from other unital associative algebras. To explore these connections, let us first define the canonical evaluation map,

Definition 2.4.87 (Polynomial Evaluation)**.** Consider the polynomial algebra $\mathbb{F}[x]$ and a unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$ over $\mathbb{F}$. Polynomial evaluation by an element $a \in \mathcal{A}$ is the unique unital associative algebra homomorphism $\phi(a) : \mathbb{F}[x] \to \mathcal{A}$ for which $\phi(a)(1) = e$ and $\phi(a)(x) = a$.

Remark. In this thesis, $\forall p(x) \in \mathbb{F}[x]$ and $a \in \mathcal{A}$, we will often write $p(a)$ to mean $\phi(a)\big(p(x)\big)$ for notational clarity.

Now, considering a general unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$ and an arbitrary element $a \in \mathcal{A}$, there are two possibilities for the subalgebra $\mathcal{C}_a$ generated by a. The first is that $\mathcal{C}_a$ is infinite-dimensional,

Lemma 2.4.88. *Suppose $\mathcal{C}_a$ is infinite-dimensional. Then, $\mathcal{C}_a \cong \mathbb{F}[x]$ as unital associative algebras.*

Proof. We may easily verify that the unital associative algebra homomorphism $f : \mathcal{C}_a \to \mathbb{F}[x]$ for which $f(e) = 1$ and $f(a) = x$ is a two-sided inverse of the polynomial evaluation map $\phi(a)$. □

2.4.7.9 Annihilating and Minimal Polynomials

Now, let us consider the case where $\mathcal{C}_a$ is finite-dimensional,

Lemma 2.4.89. *Suppose $\mathcal{C}_a$ is finite-dimensional. Then, $\exists n(x) \in \mathbb{F}[x]$ such that $\phi(a)\big(n(x)\big) = 0$.*

Proof. Since $\mathcal{C}_a$ is finite-dimensional, $\{e, a, a^{\bullet 2}, \ldots, a^{\bullet(\dim(\mathcal{C}_a)-1)}\}$ must be linearly independent. Since this is a set of $\dim(\mathcal{C}_a)$ vectors, it must form a basis for $\mathcal{C}_a$. This means that the set $\{e, a, a^{\bullet 2}, \ldots, a^{\bullet \dim(\mathcal{C}_a)}\}$ is linearly dependent, i.e. $\exists \alpha_j \in \mathbb{F}$ not all zero such that,

$$0 = a^{\bullet(\dim \mathcal{C}_a)} - \sum_{j=0}^{\dim(\mathcal{C}_a)-1} \alpha_j a^{\bullet j} = \phi(a)\left(x^{\dim(\mathcal{C}_a)} - \sum_{j=0}^{\dim(\mathcal{C}_a)-1} \alpha_j x^j \right).$$

□

The kind of polynomials revealed in Lemma 2.4.89 are of central importance to this thesis. To further relate $\mathcal{C}_a$ to $\mathbb{F}[x]$, we must understand their properties further,

Definition 2.4.90 (Annihilating Polynomial)**.** Consider a unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$, and an element $a \in \mathcal{A}$. A polynomial $n(x) \in \mathbb{F}[x]$ is an annihilating polynomial for a iff $\phi(a)\big(n(x)\big) = 0$.

There is no guarantee that an annihilating polynomial will exist for an arbitrary element of a general unital associative algebra. However, we are often interested in finite-dimensional algebras for which,

Lemma 2.4.91. *Suppose the unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$ is finite-dimensional. Then, an annihilating polynomial exists for each $a \in \mathcal{A}$.*

Proof. Since $\mathcal{C}_a$ is a subalgebra of $\mathcal{A}$, it must be finite-dimensional. The result then follows from Lemma 2.4.89. □

Clearly, annihilating polynomials are not unique, as we could always multiply one by an arbitrary polynomial to yield another. However, there is privileged annihilating polynomial which is unique up to scaling, and represents the most compact description of the properties of the element a available,

Definition 2.4.92 (Minimal Polynomial)**.** Consider the set A of all annihilating polynomials for an element $a \in \mathcal{A}$. If A is non-empty, a *monic* annihilating polynomial $m(x)$ is the minimal polynomial for a iff,

$$|m(x)| = \min\big(\big\{|n(x)| \,\big|\, \forall n(x) \in A\big\}\big). \tag{2.4.48}$$

Remark. We define the minimal polynomial to be monic to simplify later discussion. Note, this convention is not always followed by other authors.

Lemma 2.4.93. *Consider an element $a \in \mathcal{A}$ which has an annihilating polynomial. Then, the minimal polynomial for a is unique.*

Proof. Suppose we have two monic annihilating polynomials for a, $f(x), g(x) \in \mathbb{F}[x]$ of minimal polynomial order. Therefore, we must have $|f(g)| = |g(x)|$, and we consider their difference $h(x) := f(x) - g(x)$. Suppose $h(x) \neq 0$, then $|h(x)| < |f(x)|$ since both $f(x)$ and $g(x)$ have identical leading-order terms. This contradicts the minimal polynomial order of $f(x)$, and so $h(x) = 0$. □

In fact, the minimal polynomial is the reason for an annihilating polynomial's properties,

Lemma 2.4.94. *Consider an element $a \in \mathcal{A}$ which has an annihilating polynomial, and let $m(x)$ denote its minimal polynomial. Then, for all annihilating polynomials $n(x) \in \mathbb{F}[x]$, $m(x) \mid n(x)$.*

Proof. The case where $|n(x)| = |m(x)|$ follows from the uniqueness of the minimal polynomial. When $|n(x)| > |m(x)|$, we may use polynomial division to write,

$$n(x) = q(x)m(x) + r(x),$$

for $q(x), r(x) \in \mathbb{F}[x]$, where $|r(x)| < |m(x)|$. Since $n(x)$ and $m(x)$ are both annihilating this implies that $r(x)$ is also annihilating. If $r(x) \neq 0$, by $|r(x)| < |m(x)|$ this contradicts the minimality of $m(x)$. Therefore, $r(x) = 0$. $\square$

Remark. Lemma 2.4.94 is very useful: given any annihilating polynomial $n(x)$ for a, we may find the minimal polynomial for a by writing $n(x)$ in irreducible power form and finding the lowest polynomial order product of factors which is annihilating for a. This method will be employed at times in this thesis.

With these facts in hand, we may finally show,

Lemma 2.4.95. *Suppose $\mathcal{C}_a$ is finite-dimensional. Then, $\mathcal{C}_a \cong \mathcal{Q}_{m(x)}[x]$ as commutative unital associative algebras, where $m(x)$ is the minimal polynomial for a.*

Proof. Since $m(x)$ is minimal, we must have $\dim(\mathcal{C}_a) = |m(x)|$, so $\mathcal{C}_a \cong \mathcal{Q}_{m(x)}[x]$ as vector spaces over $\mathbb{F}$. Let us define, $\forall j \in \big\{0, 1, \ldots, \big(|m(x)| - 1\big)\big\}$,

$$f : \mathcal{C}_a \to \mathcal{Q}_{m(x)}[x]$$
$$f(a^{\bullet j}) = x^j,$$

and extended linearly. We may easily verify that f is a commutative unital associative algebra isomorphism with the obvious inverse. $\square$

The relationship proven in Lemma 2.4.95 underpins the methods developed in Chapter 3, which are used extensively in this thesis to probe the properties of algebras important to the structure of systems with arbitrary spin.

Chapter 3

Minimal Polynomial Methods

3.1 Chapter Aim and Outline

In this chapter, we will develop elementary methods which utilise the minimal polynomials of elements in an algebra to probe that algebra's structure. The content of this chapter first appeared in a publication by the author [65], but is given a more general, formal, and complete treatment here. This work is structured as follows:

First, in Section 3.2, we will motivate the need for such methods in the absence of algebraic closure or explicit bases. Then, in Section 3.3, we will formulate our methods in full via Bézout's identity, and describe the advantages and limitations of our approach. We will also highlight the use of these methods to naturally decompose a vector space through algebraic resolution of the identity. Following this, in Section 3.4, we will present a generalisation of Taylor and formal power series for arbitrary irreducible polynomials, and highlight the use of such series to calculate the required coefficient polynomials in the aforementioned identity resolutions. Finally, in Section 3.5, we will give explicit forms for series form from arbitrary real irreducible polynomials. This will provide us with all we require to develop the results of Chapter 4.

Before proceeding further, the author would like to highlight that during the writing of this thesis he became aware of the "Primary Decomposition Theorem" [49] for torsion modules over principle ideal domains. This theorem provides a more general theoretical backbone for the minimal polynomial methods developed in this chapter than he presents in Section 3.3; however, what is presented represents his own work, and the connection developed between primary decomposition, identity resolution, and idempotents via Bézout's identity is, to his knowledge, still novel.

Furthermore, the content of Sections 3.4 and 3.5 remain novel, as well as the application of these methods to the (algebraic) representation theory of $\mathfrak{so}(3,\mathbb{R})$ as a means to further foundational physics in Chapter 4.

3.2 Motivation

3.2.1 Challenges to the Use of Traditional Methods

The aim of this thesis is to understand what physical and mathematical structure is strictly necessary for arbitrary spin to exist within a general physical model. To facilitate this, we will include as little structure as possible in our initial model, so that clear and direct connections between assumed physical structures and spin can be drawn. In particular, we shall assume only the structure of a real Euclidean three-space (E,δ) (which we will generalise to a Minkowski space-time $(\mathcal{V},g)$ where possible). From this initial model, we shall follow quantum mechanics and explore spin through the unital associative algebras of physical properties associated with our system; in our case, this means higher-order spatial tensors. Thus, we must understand how to analyse general algebras of these tensors so that specialised algebras with the structure of arbitrary spin systems can be constructed from them. Performing this analysis with limited assumed mathematical structure presents a number of challenges.

3.2.1.1 Complex Numbers and Representation Theory

The first and most significant challenge to this programme is our choice to work exclusively with a real physical model. Unlike the complex numbers, the reals lack algebraic closure,

Definition 3.2.1 (Algebraic Closure [49])**.** A field $\mathbb{F}$ is algebraically closed iff $\forall p(x)\in\mathbb{F}[x]$, $\exists\lambda\in\mathbb{F}$ such that $p(\lambda)=0$.

Lemma 3.2.2. *Consider a finite-dimensional vector space $\mathcal{V}$ over an algebraically closed field $\mathbb{F}$. Then, $\forall A\in\mathrm{End}(\mathcal{V})$, $\exists\lambda\in\mathbb{F}$, $\exists v\in\mathcal{V}$, such that $A(v)=\lambda v$.*

Proof. See [46]. □

If the field $\mathbb{F}$ in Lemma 3.2.2 was not algebraically closed, this would no longer be true in general, for example a planar rotation in two dimensions. Thus, our choice to work exclusively with real models means we cannot generally apply methods that require eigenvalues to exist. In particular, this means the traditional weight space approach to representation theory is unavailable to us.

Besides the mathematical considerations, our aim is not only to model systems of arbitrary spin, but to describe them physically, and reveal the similarities and differences between systems of different spin in elementary terms. In this regard, the traditional methods do not offer the kinds of physical insights we seek; often, their physical interpretations are tied to the additional complex structure that is included, or reliant on their phenomenological status within a theory with considerably more structure that we will assume, such as quantum mechanics. As such, we need to take a more elementary approach to the study of spin systems.

3.2.1.2 Analytic and Matrix Representations

The considerations of the previous section lead us to consider systems with spin in their most general terms, starting with a tensor algebra. In this thesis, we will consider both the tensor algebras $T(\mathfrak{so}(3,\mathbb{R}))$ in Chapter 4 and $T(E)$ in Chapter 5 respectively. As both of these algebras are infinite-dimensional, we cannot use matrix representations to describe them. We could utilise analytical representations for them, but doing so would necessitate introducing the structure of a differentiable manifold

Alternatively, we could employ some representation to study subspaces of these tensor algebras. This would not be practical for us as we will frequently impose more algebraic structure on our algebras to eventually specialise them to describe systems with spin: whatever representation we choose would require alteration at each of these stages, and contribute an unnecessary source of work. Instead, we will work directly with the algebraic structure itself. Doing so has the added benefit of empowering our use of basis-independent arguments, which will yield important insights into the structure we are working with. One major exception to this rule is the introduction of a basis for $\mathfrak{so}(3,\mathbb{R})$ in Chapter 4. This is because that work, as originally presented in [61] makes use of a basis; however, the results of Chapter 4 can easily be rendered basis independent.

3.2.2 Motivating Example

To achieve the aims we have set out in this thesis, we must use an elementary algebraic approach. Such an approach is well-known in the case of real symmetric matrices, which will serve as an example to develop our methods. A real symmetric matrix A enjoys an eigenspectrum $\{\lambda_j\}$ indexed over a set J, and a set of orthogonal eigenvectors [60]. Their minimal polynomials are of the form,

$$m(x) = \prod_{j \in J} (x - \lambda_j), \tag{3.2.1}$$

such that $\forall j, k \in J$, $j \neq k$, $\lambda_j \neq \lambda_k$, and we may construct projectors into each of its eigenspaces algebraically as,

$$\Pi_j \coloneqq \prod_{k \in J,\, k \neq j} \frac{A - \lambda_k}{\lambda_j - \lambda_k}, \tag{3.2.2}$$

which satisfy, $\forall j \in J$,

$$\left[0 = m(A) \prod_{k \in J,\, k \neq j} \frac{1}{\lambda_j - \lambda_k} = (A - \lambda_j)\Pi_j\right] \Rightarrow \left[A\,\Pi_j = \lambda_j \Pi_j\right]. \tag{3.2.3}$$

These projectors resolve the identity matrix,

$$I = \sum_{j \in J} \Pi_j,$$

and also satisfy, $\forall j, k \in J$, $j \neq k$,

$$\Pi_j \Pi_k = \Pi_k \Pi_j = 0. \tag{3.2.4}$$

This algebraic description of the properties of A is highly compelling. The projectors Π_j are naturally basis-independent, and allow us to work with arbitrary eigenvectors directly without needing to calculate them. The resolution of the identity also allows us to investigate the actions of another matrix B relative to the eigenspaces of A in a basis-independent way,

$$B = IBI = \sum_{j,k \in J} \Pi_j B\, \Pi_k. \tag{3.2.5}$$

This naturally supports a description of B in terms of intrinsically meaningful objects to A, offering significant flexibility and interpretive power. Therefore, we will make progress towards the aims of this thesis, by generalising this method for real symmetric matrices to apply to any element with a minimal polynomial from a unital associative algebra over a field $\mathbb{F}$ for which $\mathbb{Q} \subseteq \mathbb{F}$.

3.3 Identity Resolution using Minimal Polynomials

3.3.1 Bézout's Identity

3.3.1.1 Bézout's Identity for Two Polynomials

To resolve the identity of a unital associative algebra using the minimal polynomial of one of its elements, we must utilise Bézout's identity.

Theorem 3.3.1 (Bézout's Identity [66]). *Consider two polynomials* $f(x), g(x) \in \mathbb{F}[x]$ *not both zero. Then,* $\exists a(x), b(x) \in \mathbb{F}[x]$ *such that,*

$$a(x)f(x) + b(x)g(x) = \gcd(f(x), g(x)). \tag{3.3.1}$$

Proof. See [49]. □

Remark. The $a(x)$ and $b(x)$ are not unique in general.

Bézout's Identity for two polynomials will underpin all of the methods developed in this chapter, and its generalisation will ultimately give us the tools we need to analyse $U(\mathfrak{so}(3, \mathbb{R}))$ and derive real algebraic descriptions for the structure of systems with arbitrary spin.

Definition 3.3.2 (Bézout Polynomials/Coefficients). We may refer to the polynomials $a(x)$ and $b(x)$ which appear in Theorem 3.3.1 as either "Bézout Polynomials" or "Bézout Coefficients".

To use this identity to resolve the identity element of a unital associative algebra, we must understand when the polynomials $a(x)$, $b(x)$ are zero for distinct $f(x)$, $g(x)$,

Lemma 3.3.3. *Consider two polynomials* $f(x), g(x) \in \mathbb{F}[x]$ *not both zero, and* $(a(x), b(x)) \in \mathbb{F}[x] \times \mathbb{F}[x]$ *satisfying Theorem 3.3.1. Then,* $[a(x) \neq 0] \vee [b(x) \neq 0]$.

Proof. Suppose $a(x) = 0$ and $b(x) = 0$. Then, $\gcd(f(x), g(x)) = 0$, contradicting Lemma 2.4.75. □

Lemma 3.3.4. *Consider two polynomials* $f(x), g(x) \in \mathbb{F}[x]$ *not both zero. Then:*

1. $\exists (a(x), 0) \in \mathbb{F}[x] \times \mathbb{F}[x]$ *satisfying Theorem 3.3.1 iff* $f(x) \mid g(x)$*;*

2. $\exists (0, b(x)) \in \mathbb{F}[x] \times \mathbb{F}[x]$ *satisfying Theorem 3.3.1 iff* $g(x) \mid f(x)$*.*

Proof. By the symmetry of gcd, we need only prove the first claim without loss of generality. In the forward direction, we have $a(x)f(x) = \gcd(f(x), g(x))$, so $f(x) \mid \gcd(f(x), g(x))$. Therefore, by Lemma 2.4.64, $f(x) \mid g(x)$. In the reverse direction, $f(x) \mid g(x)$ implies $\exists s(x) \in \mathbb{F}[x]$ such that $g(x) = s(x)f(x)$. Considering $(a(x), b(x)) \in \mathbb{F}[x]\times\mathbb{F}[x]$ satisfying Theorem 3.3.1, we have $\big(a(x) + b(x)s(x)\big)f(x) = \gcd(f(x), g(x))$. Defining $a'(x) \coloneqq a(x)+b(x)s(x)$, we see that $(a'(x), 0)$ also satisfies Theorem 3.3.1. □

Corollary 3.3.5. *Consider two polynomials $f(x), g(x) \in \mathbb{F}[x]$ not both zero:*

1. *If $\exists(a(x), 0) \in \mathbb{F}[x]\times\mathbb{F}[x]$ satisfying Theorem 3.3.1, then $a(x) \in \mathbb{F}$, $a(x) \neq 0$;*
2. *If $\exists(0, b(x)) \in \mathbb{F}[x]\times\mathbb{F}[x]$ satisfying Theorem 3.3.1, then $b(x) \in \mathbb{F}$, $b(x) \neq 0$.*

Proof. By the symmetry of gcd, we need only prove the first claim without loss of generality. In this case, $a(x)f(x) = \gcd(f(x), g(x))$, and so $f(x) \mid \gcd(f(x), g(x))$. Therefore, by Lemmas 2.4.67, 2.4.75, and Definition 2.4.72, we have $\gcd(f(x), g(x)) = \alpha f(x)$, $\alpha \neq 0$, and so $\alpha = a(x)$. □

3.3.1.2 Bézout's Identity for a Finite Set of Polynomials

Using the properties of gcd, we may extend Theorem 3.3.1 to a finite set of polynomials,

Theorem 3.3.6 (Bézout's Identity (Finite Set))**.** *Consider a finite set of polynomials $\{f_j(x) \in \mathbb{F}[x]\}$ not all zero, indexed over the set $\{1, \dots, n\}$, where $n \in \mathbb{Z}^+$, $n \geq 2$. Then, there exists a set of polynomials $\{a_j(x) \in \mathbb{F}[x]\}$ indexed over $\{1, \dots, n\}$ such that,*

$$\sum_{j=1}^{n} a_j(x)f_j(x) = \gcd\big(\{f_j(x)\}\big). \tag{3.3.2}$$

Proof. Let us proceed by induction. For $n = 2$, Theorem 3.3.1 holds. Assuming the case $n = k$, $k \in \mathbb{Z}^+$, $k \geq 2$, we apply Theorem 3.3.1 to f_{k+1} and $\gcd(f_1, \dots, f_k)$, thus,

$$\begin{aligned} \gcd(f_1, \dots, f_{k+1}) &= a'(x)f_{k+1}(x) + b'(x)\gcd(f_1, \dots, f_k) \\ &= a'(x)f_{k+1}(x) + b'(x)\sum_{j=1}^{k} a_j(x)f_j(x). \end{aligned}$$

□

Theorem 3.3.6 is the version of Bézout's Identity which is of most use to us. The behaviour of many important elements in unital associative algebras, such as the spin generators, can be characterised by their minimal polynomials. We will soon see that this information is most readily accessed via this theorem.

Since Theorem 3.3.6 is built from recursion using Theorem 3.3.1, we may assert,

Corollary 3.3.7. *Consider a finite set of polynomials* $\{f_j(x) \in \mathbb{F}[x]\}$ *not all zero, indexed over the set* $\{1, \ldots, n\}$*, where* $n \in \mathbb{Z}^+$*,* $n \geq 3$*. If* $\forall k \in \{2, \ldots, n-1\}$*,*

$$\big[\gcd(f_1, \ldots, f_k) \nmid f_{k+1}\big] \wedge \big[f_{k+1} \nmid \gcd(f_1, \ldots, f_k)\big], \tag{3.3.3}$$

then $\nexists\{a_j(x) \in \mathbb{F}[x]\}$ *indexed over* $\{1, \ldots, n\}$ *satisfying Theorem 3.3.6, such that* $a_k(x) = 0$ *for some* $k \in \{1, \ldots, n\}$*.*

Proof. Since Theorem 3.3.6 is build recursively using Theorem 3.3.1, we apply Lemma 3.3.4 at each step to ensure each new $a'(x), b'(x) \in \mathbb{F}[x]$ are non-zero. □

3.3.1.3 Polynomial Order of Bézout Polynomials

The method of construction in Theorem 3.3.6 does not typically yield low polynomial order Bézout polynomials. Fortunately, we may always reduce this order,

Lemma 3.3.8. *Consider a finite set of non-zero polynomials* $\{f_j(x) \in \mathbb{F}[x]\}$*, indexed over the set* $\{1, \ldots, n\}$*, where* $n \in \mathbb{Z}^+$*,* $n \geq 2$*, such that* $\nexists k \in \{1, \ldots, n\}$ *such that* $f_k(x) = \beta \gcd\big(\{f_j(x)\}\big)$ *or* $f_k(x) = \beta \operatorname{lcm}\big(\{f_j(x)\}\big)$*, for* $\beta \in \mathbb{F}$*,* $\beta \neq 0$*. Then, we may always find a set* $\{a_j(x)\}$ *indexed over* $\{1, \ldots, n\}$ *satisfying Theorem 3.3.6 such that* $\forall j \in \{1, \ldots, n\}$*,*

$$|a_j(x)| + |f_j(x)| < |\operatorname{lcm}(\{f_j(x)\})|. \tag{3.3.4}$$

Proof. Consider a set $\{a_j^{(0)}(x) \in \mathbb{F}[x]\}$ indexed over $\{1, \ldots, n\}$ which satisfy Theorem 3.3.6 but not the condition (3.3.4). Taking this together with Lemma 2.4.83, $\exists m \in \{1, \ldots, n\}$ such that,

$$|\gcd\big(\{f_j(x)\}\big)| < |\operatorname{lcm}\big(\{f_j(x)\}\big)| \leq |a_m^{(0)}(x) f_m(x)|. \tag{3.3.5}$$

Thus, we construct the set,

$$K \coloneqq \Big\{0, \ldots, \max\big(\{|a_j^{(0)}(x) f_j(x)|\}\big) - \big|\operatorname{lcm}\big(\{f_j(x)\}\big)\big|\Big\},$$

and define $\forall p \in K$, $\kappa_p := \max(K) - p$. Let us define the polynomials, $\forall p \in K$, $\forall j \in \{0, \ldots, n\}$,

$$a_j^{(p+1)}(x) := a_j^{(p)}(x) - \alpha_j^{(\kappa_p)} x^{\kappa_p} g_j(x),$$

where $g_j(x) \in \mathbb{F}[x]$ satisfies $\operatorname{lcm}\big(\{f_j(x)\}\big) = g_j(x) f_j(x)$, and $\alpha_j^{(\kappa_p)} \in \mathbb{F}$ is the coefficient of the $\big(\kappa_p + \big|\operatorname{lcm}\big(\{f_j(x)\}\big)\big|\big)$-order term in $a_j^{(p)}(x) f_j(x)$. We claim that the set $\{a_j^{(\max(K)+1)}\}$ satisfies both Theorem 3.3.6 and the inequality (3.3.4).

To see this, first note that the inequality (3.3.5) implies $\forall p \in K$,

$$\sum_{j=1}^{n} \alpha_j^{(\kappa_p)} = 0. \tag{3.3.6}$$

Thus, $\forall p \in K$,

$$\begin{aligned}\sum_{j=1}^{n} a_j^{(p+1)}(x) f_j(x) &= \sum_{j=1}^{n} a_j^{(p)}(x) f_j(x) - \Big(\sum_{j=1}^{n} \alpha_j^{(\kappa_p)}\Big) x^{\kappa_p} \operatorname{lcm}\big(\{f_j(x)\}\big) \\ &= \sum_{j=1}^{n} a_j^{(p)}(x) f_j(x),\end{aligned}$$

so the set $\{a_j^{(p+1)}(x)\}$ satisfies Theorem 3.3.6 iff $\{a_j^{(p)}(x)\}$ does. Therefore, since $\{a_j^{(0)}\}$ satisfies Theorem 3.3.6, so does $\{a_j^{(\max(K)+1)}\}$ by induction. Furthermore, since $\forall p \in K$, $\forall j \in \{1, \ldots, n\}$,

$$a_j^{(p+1)}(x) f_j(x) = a_j^{(p)}(x) f_j(x) - \alpha_j^{(\kappa_p)} x^{\kappa_p} \operatorname{lcm}\big(\{f_j(x)\}\big),$$

$a_j^{(p+1)}(x) f_j(x)$ has zero $\big(\kappa_p + \big|\operatorname{lcm}\big(\{f_j(x)\}\big)\big|\big)$-order term. Thus, by induction, $\forall p \in K$, $\forall j \in \{1, \ldots, n\}$,

$$|a_j^{(p+1)}(x) f_j(x)| < \kappa_p + \big|\operatorname{lcm}\big(\{f_j(x)\}\big)\big|.$$

Hence, $\{a_j^{(\max(K)+1)}\}$ satisfy the inequality (3.3.4). □

3.3.2 Identity Resolution from a Monic Polynomial

We will now describe the general procedure to produce a resolution of the identity from an arbitrary monic polynomial. This, and its application to elements of unital associative algebras with minimal polynomials, will be used heavily in Chapter 4.

3.3.2.1 Resolution via Irreducible Power Form

Definition 3.3.9 ($p_k(x)$, $q_k(x)$)**.** Consider a monic polynomial $m(x) \in \mathbb{F}[x]$ and its irreducible power form,

$$m(x) = \prod_{j \in J} f_j(x)^{d_j}. \tag{3.3.7}$$

For all $k \in J$, we define,

$$p_k(x) := f_k(x)^{d_k} \tag{3.3.8a}$$

$$q_k(x) := \prod_{j \in J \setminus \{k\}} f_j(x)^{d_j}. \tag{3.3.8b}$$

Lemma 3.3.10. *For all $k \in J$, $p_k(x)q_k(x) = m(x)$.*

Proof. By definition. □

To understand how we may use the irreducible power form of $m(x)$ to resolve the identity, we must first describe some important properties of the $q_k(x)$. We begin with their gcds,

Lemma 3.3.11. *For all $S \subseteq J$,*

$$\gcd\big(\{q_k(x)\}_{k \in S}\big) = \prod_{j \in J \setminus S} f_j(x)^{d_j}. \tag{3.3.9}$$

Proof. For all $j, k \in J$, $j \neq k$, we have,

$$\gcd(q_j(x), q_k(x)) = \prod_{l \in J \setminus \{j,k\}} f_l(x)^{d_l}.$$

The claim follows by induction. □

Corollary 3.3.12. $\gcd\big(\{q_k(x)\}\big) = 1.$

Proof. For $S = J$,

$$\gcd\big(\{q_k(x)\}_{k \in J}\big) = \prod_{j \in J \setminus J} f_j(x)^{d_j} = 1.$$

□

Lemma 3.3.13. *For all $k \in J$, $\forall S \subseteq J \setminus \{k\}$,*

$$\big[\gcd(\{q_j(x)\}_{j \in S}) \nmid q_k(x)\big] \wedge \big[q_k(x) \nmid \gcd(\{q_j(x)\}_{j \in S})\big]. \tag{3.3.10}$$

Proof. By Definition 3.3.9, $\forall k \in J$, $\forall S \in J \setminus \{k\}$,

$$\big[p_k(x) \nmid q_k(x)\big] \wedge \Big[\prod_{j \in S} q_j(x) \mid q_k(x)\Big].$$

By Lemma 3.3.11, $\forall k \in J$, $\forall S \subseteq J \setminus \{k\}$,

$$\big[p_k(x) \mid \gcd(\{q_j(x)\}_{j \in S})\big] \wedge \Big[\prod_{j \in S} q_j(x) \nmid \gcd(\{q_j(x)\}_{j \in S})\Big].$$

Thus, if $\gcd(\{q_j(x)\}_{j \in S}) \mid q_k(x)$, then by Lemma 2.4.64, $p_k(x) \mid q_k(x)$, which is a contradiction. Similarly, if $q_k(x) \mid \gcd(\{q_j(x)\}_{j \in S})$ then by Lemma 2.4.64, $\prod_{j \in S} q_j(x) \mid \gcd(\{q_j(x)\}_{j \in S})$, which is a contradiction. □

Now, we consider their lcm,

Lemma 3.3.14. $\mathrm{lcm}\big(\{q_j(x)\}\big) = m(x)$.

Proof. For all $j, k \in J$, $j \neq k$, we have,

$$\mathrm{lcm}(q_j(x), q_k(x)) = \prod_{l \in J} f_l(x)^{d_l} = m(x).$$

Thus, by Lemma 2.4.79, we are done. □

With these properties in hand, we may present our identity resolution,

Theorem 3.3.15. *Consider an arbitrary monic polynomial* $m(x) \in \mathbb{F}[x]$. *Then,* $\exists\{a_j(x) \in \mathbb{F}[x]\}$, *such that,*

$$\sum_{j \in J} a_j(x) q_j(x) = 1, \tag{3.3.11}$$

and $\forall j \in J$, $a_j(x) \neq 0$ *with,*

$$|a_j(x)| + |q_j(x)| < |m(x)|. \tag{3.3.12}$$

Proof. By Theorem 3.3.6 and Corollary 3.3.12, we establish the existence of $\{a_j(x)\}$ such that (3.3.11) holds. That $\forall j \in J$, $a_j(x) \neq 0$ is a consequence of Corollary 3.3.7 and Lemma 3.3.13. That $\forall j \in J$, $a_j(x)$ satisfies the inequality (3.3.12) follows from Lemmas 3.3.8 and 3.3.14. □

Theorem 3.3.15 demonstrates how to utilise the information contained in a polynomial to resolve the identity. In Chapter 4, we will apply this method to the minimal polynomials of $\mathrm{ad}(S^2)$ when acting on arbitrary elements in $U(\mathfrak{so}(3, \mathbb{R}))$, and in so doing uncover important information about the physically distinct observables for a system of arbitrary spin.

Example 3.3.16. Consider the monic polynomial $m(x) = (x-1)^3(x^2+x+3)^2(x+5)$ and define,

$$q_1(x) = (x-1)^3(x^2+x+3)^2$$
$$q_2(x) = (x-1)^3(x+5)$$
$$q_3(x) = (x^2+x+3)^2(x+5).$$

One may easily verify that,

$$a_1(x) = -\frac{1}{529000}$$

$$a_2(x) = -\frac{1998}{330625}x^3 - \frac{2999}{330625}x^2 - \frac{404}{13225}x - \frac{1163}{330625}$$
$$a_3(x) = \frac{15989}{2645000}x^2 - \frac{609}{28750}x + \frac{57619}{2645000},$$

satisfy Theorem 3.3.15.

Remark. We have so far only proved the *existence* of $\{a_j(x)\}$ satisfying Theorem 3.3.15. In Section 3.4, we shall give a general method for determining them for an arbitrary identity resolution.

3.3.3 Identity Resolution in Quotient Algebras of $\mathbb{F}[x]$

In this thesis, we wish to exploit the algebraic structure of the minimal polynomials of elements in unital associative algebras to study important algebraic structures. To do this, noting Lemma 2.4.89, we must understand how the resolution of the identity of Theorem 3.3.15 behaves in a quotient algebra of $\mathbb{F}[x]$. This will be of central importance to the developments of Chapter 4.

3.3.3.1 Identity Resolution by $m(x)$

Lemma 3.3.17. *Consider an arbitrary monic polynomial $m(x) \in \mathbb{F}[x]$, and the quotient algebra,*

$$\mathcal{Q}_{m(x)}[x] \cong \frac{\mathbb{F}[x]}{I\,(m(x))}. \tag{3.3.13}$$

Then, the resolution of the identity (3.3.11) *of $\mathbb{F}[x]$ from Theorem 3.3.15 remains unaltered in $\mathcal{Q}_{m(x)}[x]$.*

Proof. Since the inequality (3.3.12) holds, by Lemmas 2.3.23 and 2.4.4, no term in (3.3.11) becomes zero or is altered in $\mathcal{Q}_{m(x)}[x]$. □

While the expression (3.3.11) is not altered by the quotient, its terms acquire useful algebraic properties in $\mathcal{Q}_{m(x)}[x]$. First, we note,

Lemma 3.3.18. *In $\mathcal{Q}_{m(x)}[x]$,*

$$m(x) = 0. \tag{3.3.14}$$

Proof. Direct consequence of Lemma 2.4.4. □

Lemma 3.3.19. *In $\mathcal{Q}_{m(x)}[x]$, $\forall k \in J$,*

$$p_k(x)a_k(x)q_k(x) = 0. \tag{3.3.15}$$

Proof. By Lemma 3.3.18,

$$p_k(x)a_k(x)q_k(x) = a_k(x)p_k(x)q_k(x) = a_k(x)m(x) = 0.$$

□

Lemma 3.3.20. *In* $\mathcal{Q}_{m(x)}[x]$, $\forall j, k \in J$, $j \neq k$,

$$a_j(x)q_j(x)a_k(x)q_k(x) = 0. \tag{3.3.16}$$

Proof. From their definitions, $\forall l \in J\backslash\{k\}$, $p_k(x) \mid q_l(x)$. Thus, $\exists r_l(x) \in \mathbb{F}[x]$ such that $q_l(x) = p_k(x)r_l(x)$. So, by Lemma 3.3.19,

$$a_j(x)q_j(x)a_k(x)q_k(x) = a_j(x)p_k(x)r_j(x)a_k(x)q_k(x) = a_j(x)r_j(x)p_k(x)a_k(x)q_k(x) = 0.$$

□

Lemma 3.3.21. *In* $\mathcal{Q}_{m(x)}[x]$, $\forall k \in J$,

$$(a_k(x)q_k(x))^2 = a_k(x)q_k(x). \tag{3.3.17}$$

Proof. By Theorem 3.3.15 and Lemma 3.3.20, we have, $\forall k \in J$,

$$\begin{aligned} a_k(x)q_k(x) &= a_k(x)q_k(x)\Big(\sum_{j\in J} a_j(x)q_j(x)\Big) \\ &= \sum_{j\in J} a_k(x)q_k(x)a_j(x)q_j(x) \\ &= (a_k(x)q_k(x))^2. \end{aligned}$$

□

Thus, we have found that the $\{a_k(x)q_k(x)\}$ are idempotent and mutually "orthogonal". Accordingly, let us introduce some useful notation,

Definition 3.3.22 ($\Pi_{p_k(x)}(x)$)**.** For all $k \in J$,

$$\Pi_{p_k(x)}(x) \coloneqq a_k(x)q_k(x). \tag{3.3.18}$$

Theorem 3.3.23. *Consider an arbitrary monic polynomial* $m(x)$*, and the quotient algebra* $\mathcal{Q}_{m(x)}[x]$*. Then,*

$$\sum_{j\in J} \Pi_{p_j(x)}(x) = 1, \tag{3.3.19}$$

is a resolution of the identity of $\mathcal{Q}_{m(x)}[x]$ *in terms of mutually "orthogonal" idempotent elements* $\Pi_{p_k(x)}(x) \in \mathcal{Q}_{m(x)}[x]$*, defined by Definition 3.3.22, such that,* $\forall j, k \in J$*,* $j \neq k$*,*

$$|\Pi_{p_k(x)}(x)| < |m(x)| \tag{3.3.20a}$$

$$p_k(x)\Pi_{p_k(x)}(x) = 0 \tag{3.3.20b}$$

$$\Pi_{p_k(x)}(x)\Pi_{p_k(x)}(x) = \Pi_{p_k(x)}(x) \tag{3.3.20c}$$

$$\Pi_{p_j(x)}(x)\Pi_{p_k(x)}(x) = 0. \tag{3.3.20d}$$

Proof. This follows taking Theorem 3.3.15, and Lemmas 3.3.19, 3.3.20, and 3.3.21 together. □

Theorem 3.3.23 shows that the terms in identity resolutions behave exactly as a set of mutually "orthogonal" idempotents which are each annihilated by a different irreducible power from the minimal polynomial use to generate the resolution. We shall later see that these idempotents are exactly the generalisation of projectors into eigenspaces that we require for a complete analysis of a real operator. This also confirms that much of the essential information about, for example, an operator, is accessible algebraically and basis-independently, without the need to add additional mathematical structure. This supports our efforts to work entirely without complex numbers, and will empower our analysis of the structure of arbitrary spin systems in Chapter 4.

Example 3.3.24. Consider the matrix,

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

We may easily verify that the minimal polynomial for A is $m(x) = (x-2)(x-3)(x^2+1)$, and so the algebra generated by A is isomorphic to $\mathcal{Q}_{m(x)}[x]$. Furthermore, we may see that,

$$\Pi_{x-2}(A) = -\frac{1}{5}(A - 3I_{5\times5})(A^2 + I_{5\times5})$$
$$\Pi_{x-3}(A) = \frac{1}{10}(A - 2I_{5\times5})(A^2 + I_{5\times5})$$

$$\Pi_{x^2+1}(A) = \frac{1}{10}(A + I_{5\times5})(A - 2I_{5\times5})(A - 3I_{5\times5}),$$

satisfy Theorem 3.3.23, where $I_{5\times5}$ is the 5×5 identity matrix.

Theorem 3.3.23 will be used extensively in Chapter 4.

3.3.3.2 Identity Resolution by a Multiple of $m(x)$

When utilising Theorem 3.3.23, as we later will, to decompose a vector space into a direct sum of components, we may find we make use of an annihilating polynomial for our chosen endomorphism which is not minimal. Let us consider this possibility, and find that nothing important will change in doing so.

By Lemma 2.4.94, the minimal polynomial is always a divisor of any annihilating polynomial. Therefore, let us consider a further quotient to $\mathcal{Q}_{n(x)}[x]$ by one of its divisors,

Lemma 3.3.25. *Consider the quotient algebra $\mathcal{Q}_{n(x)}[x]$ and a polynomial $m(x)$ such that $m(x) \mid n(x)$. Then, the resolution of the identity of Theorem 3.3.23 by $m(x)$ in $\mathcal{Q}_{m(x)}[x]$ is identical to the resolution of the identity by $n(x)$ in,*

$$\mathcal{K}[x] \cong \frac{\mathcal{Q}_{n(x)}[x]}{I\left(m(x)\right)}, \tag{3.3.21}$$

up to equivalence of Bézout polynomials.

Proof. First, note that in $\mathcal{K}[x]$ we have $\Pi_{p_k(x)} = 0$ when $m(x) \mid \Pi_{p_k(x)}$, where the $\Pi_{p_k(x)}$ are defined using the factors of $n(x)$. If $m(x) \nmid \Pi_{p_k(x)}$, we may use Lemma 3.3.8 to reduce their polynomial order until $|\Pi_{p_k(x)}(x)| < |m(x)|$. That these idempotents are identical, up to equivalence, to those derived in $\mathcal{Q}_{m(x)}[x]$ is assured by the "Third Isomorphism Theorem" for algebras (see [49]). □

Example 3.3.26. Consider again the matrix A from Example 3.3.24. We may easily verify that $n(x) = (x-1)(x-2)(x-3)^2(x^2+1)$ is an annihilating polynomial for A. Resolving the identity by this polynomial, we find,

$$\begin{aligned}
\Pi'_{x-1}(x) &= -\frac{1}{8}(x-2)(x-3)^2(x^2+1) \\
\Pi'_{x-2}(x) &= \frac{1}{5}(x-1)(x-3)^2(x^2+1) \\
\Pi'_{(x-3)^2}(x) &= -\frac{1}{200}(21x-73)(x-1)(x-2)(x^2+1) \\
\Pi'_{x^2+1}(x) &= \frac{1}{100}(3x-1)(x-1)(x-2)(x-3)^2.
\end{aligned}$$

It can easily be verified that,

$$\Pi'_{x-1}(A) = 0$$
$$\Pi'_{x-2}(A) = \Pi_{x-2}(A)$$
$$\Pi'_{(x-3)^2}(A) = \Pi_{x-3}(A)$$
$$\Pi'_{x^2+1}(A) = \Pi_{x^2+1}(A).$$

3.3.4 Identity Resolution in General Unital Associative Algebras

Before discussing the case of endomorphism algebras, which will be our primary case of interest in this thesis, it is important to state how Theorem 3.3.23 relates to general unital associative algebras.

3.3.4.1 Single Element Identity Resolutions

Combining Lemma 2.4.95 and Theorem 3.3.23, we immediately gain resolutions of the identity element $e \in \mathcal{A}$ by any element $a \in \mathcal{A}$ of a unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$ which has a minimal polynomial. This can be also be used to probe the structure of another element $b \in \mathcal{A}$ in terms of information we have about a,

Corollary 3.3.27. *Consider two elements $a, b \in \mathcal{A}$ of a unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$, for which a has minimal polynomial $m(x)$, and the resolution of the identity for a given by Theorem 3.3.23. Then,*

$$b = \sum_{j,k \in J} \Pi_{p_j(a)}(a) \bullet b \bullet \Pi_{p_k(a)}(a). \tag{3.3.22}$$

Proof. Since $e = \sum_{k \in J} \Pi_{p_k(a)}(a)$,

$$b = e \bullet b \bullet e = \Big(\sum_{j \in J} \Pi_{p_j(a)}(a)\Big) \bullet b \bullet \Big(\sum_{k \in J} \Pi_{p_k(a)}(a)\Big) = \sum_{j,k \in J} \Pi_{p_j(a)}(a) \bullet b \bullet \Pi_{p_k(a)}(a).$$

□

3.3.4.2 Simultaneous Identity Resolutions

When attempting to resolve the identity element $e \in \mathcal{A}$ of a unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$ by a collection of elements, the situation is much richer. However, we may access one important case immediately,

Corollary 3.3.28. *Consider a unital associative algebra $\mathcal{A} = (\mathcal{V}, \bullet, e)$ and a set of elements $\{a_n \in \mathcal{A}\}$ indexed over a set N, for which each element has a minimal polynomial $m_n(x)$ and, $\forall m, n \in N$,*

$$a_m \bullet a_n - a_n \bullet a_m = 0. \tag{3.3.23}$$

Let us also denote the indexing set for the irreducible powers of the minimal polynomial $m_n(x)$ by J_n, and define the set,

$$L \coloneqq \bigtimes_{n \in N} J_n, \tag{3.3.24}$$

to be the Cartesian product of the indexed sets $\{J_n\}$. Then, there is a unique simultaneous resolution of the identity element $e \in \mathcal{A}$ by $\{a_n\}$,

$$e = \sum_{l \in L} P_l\big(\{a_n\}\big), \tag{3.3.25}$$

where, $\forall l \in L$,

$$P_l\big(\{a_n\}\big) \coloneqq \mathop{\Large\bullet}_{n \in N} \Pi_{p_{l(n)}(a_{l(n)})}(a_{l(n)}), \tag{3.3.26}$$

with $l(n)$ the n component of l.

Proof. For all $n \in N$, we have resolutions of the identity,

$$e = \sum_{j \in J_n} \Pi_{p_j(a_n)}(a_n).$$

Utilising the idempotence of the identity element $e \bullet e = e$, we may combine these,

$$\begin{aligned} e &= \mathop{\Large\bullet}_{n \in N} \sum_{j \in J_n} \Pi_{p_j(a_n)}(a_n) \\ &= \sum_{l \in L} \mathop{\Large\bullet}_{n \in N} \Pi_{p_{l(n)}(a_{l(n)})}(a_{l(n)}). \end{aligned}$$

The uniqueness of this identity resolution follows from the mutual commutativity of the $\{\Pi_{p_{l(n)}(a_{l(n)})}(a_{l(n)})\}$, $\forall l \in L$. □

Remark. Unlike the single element case, there is no guarantee that a given $P_l\big(\{a_n\}\big)$ is non-zero. Which of the $\{P_l\big(\{a_n\}\big)\}$ are non-zero depends on the precise algebraic relationships between the $\{a_n\}$.

Lemma 3.3.29. *For all $l, l' \in L$,*

$$P_l\big(\{a_n\}\big) \bullet P_{l'}\big(\{a_n\}\big) = \begin{cases} P_l\big(\{a_n\}\big) & l = l' \\ 0 & l \neq l'. \end{cases} \tag{3.3.27}$$

Proof. If $l = l'$, this can be directly computed. If $l \neq l'$, $\exists m \in N$ such that $l(m) \neq l'(m)$. Thus,

$$P_l(\{a_n\}) \bullet P_{l'}(\{a_n\}) = f(\{a_n\}_{n\neq m}) \bullet \Pi_{p_{l(m)}(a_m)}(a_m) \bullet \Pi_{p_{l'(m)}(a_m)}(a_m) = 0,$$

by Theorem 3.3.23. □

If the $\{a_n\}$ was not mutually commuting, then we may still construct simultaneous resolutions of the identity as in Corollary 3.3.28. However, such a resolution is no longer unique, nor will the idempotents which comprise it necessarily satisfy Lemma 3.3.29.

In this thesis, we will not need to utilise simultaneous identity resolutions of either variety. This content has been included here since understanding how to perform simultaneous identity resolutions is essential to extend the results of Chapter 4 to more structurally rich Lie algebras.

3.3.5 Identity Resolution in Endomorphism Algebras

The methods we have developed in this section can be applied to any unital associative algebra. In this thesis, we will utilise them exclusively as they apply to endomorphism algebras. In this context, we may utilise a given identity resolution to decompose the domain of our endomorphisms of interest in a basis-independent way.

3.3.5.1 A-Orthogonality and A-Orthogonal Decompositions

To understand how an identity resolution from a given endomorphism $A \in \mathrm{End}(\mathcal{V})$ imparts structure on its domain $\mathcal{V}$, let us first define some terminology,

Definition 3.3.30 (A-Orthogonal)**.** Consider a vector space $\mathcal{V}$ over $\mathbb{F}$, two subspaces $\mathcal{T},\mathcal{U} \subset \mathcal{V}$, and $A \in \mathrm{End}(\mathcal{V})$. Then $\mathcal{T}$ and $\mathcal{U}$ are A-orthogonal iff $\mathcal{T} \cap \mathcal{U} = \{0\}$ and both $\mathcal{T}$ and $\mathcal{U}$ are closed under the action of A.

This usage of the word orthogonal may seem in tension with the notion of g-orthogonality established in Definition 2.2.2; in fact, the two notions are related.

Lemma 3.3.31. *Consider a Minkowski space-time $(\mathcal{V}, g)$ and let us define,*

$$\begin{aligned} P &: \mathcal{V} \to \mathrm{End}(\mathcal{V}) \\ P &:= a \mapsto \big(b \mapsto g(a,b)a\big). \end{aligned} \tag{3.3.28}$$

Then, $\forall a, b \in \mathcal{V}$ such that $\{a, b\}$ is linearly independent, a and b are g-orthogonal, i.e. $g(a,b) = 0$, iff $\mathrm{span}_{\mathbb{R}}(\{a\})$ and $\mathrm{span}_{\mathbb{R}}(\{b\})$ are $P(a)$-orthogonal. Furthermore, $\mathrm{span}_{\mathbb{R}}(\{a\})$ and $\mathrm{span}_{\mathbb{R}}(\{b\})$ are $P(a)$-orthogonal iff they are $P(b)$-orthogonal.

Proof. First, note that linear independence of $\{a, b\}$ implies,

$$\mathrm{span}_{\mathbb{R}}(\{a\}) \cap \mathrm{span}_{\mathbb{R}}(\{b\}) = \{0\}.$$

To see this, consider $c \in \mathrm{span}_{\mathbb{R}}(\{a\}) \cap \mathrm{span}_{\mathbb{R}}(\{b\})$, then $\exists \alpha, \beta \in \mathbb{R}$ such that $c = \alpha a = \beta b$. Thus, $\alpha a - \beta b = 0$, and so by the linear independence of $\{a, b\}$, we have $\alpha = \beta = 0$.

Now, in the forward direction, we have $P(a)(a) \in \mathrm{span}_{\mathbb{R}}(\{a\})$ and $P(a)(b) = 0$. Thus, $\mathrm{span}_{\mathbb{R}}(\{a\})$ and $\mathrm{span}_{\mathbb{R}}(\{b\})$ are closed under $P(a)$, so, given they intersect trivially, we conclude that they are $P(a)$-orthogonal. In the reverse direction, closure of our spans under $P(a)$ entails $P(a)(b) \in \mathrm{span}_{\mathbb{R}}(\{b\})$. Therefore, $\exists \alpha \in \mathbb{R}$ such that $g(a,b)a = \alpha b$. Since $\mathrm{span}_{\mathbb{R}}(\{a\}) \cap \mathrm{span}_{\mathbb{R}}(\{b\}) = \{0\}$, we conclude $g(a,b) = 0$. The final claim follows from the symmetry of g. □

The notion of A-orthogonality may be extended to a set of endomorphisms,

Definition 3.3.32 ($\{A_j\}$-Orthogonal)**.** Consider a vector space $\mathcal{V}$ over $\mathbb{F}$, two subspaces $\mathcal{T}, \mathcal{U} \subset \mathcal{V}$, and a set of endomorphisms $\{A_j \in \mathrm{End}(\mathcal{V})\}$ indexed over a set J. Then $\mathcal{T}$ and $\mathcal{U}$ are $\{A_j\}$-orthogonal iff $\forall j \in J$ they are A_j-orthogonal.

Remark. When the endomorphism or set of endomorphisms is clear from context, we may refer to the subspaces simply as "orthogonal".

With the notion on A-orthogonality in hand, we may consider decompositions of a vector space compatible with this notion,

Definition 3.3.33 (A-Orthogonal Decomposition)**.** Consider a vector space $\mathcal{V}$ over $\mathbb{F}$, an endomorphism $A \in \mathrm{End}(\mathcal{V})$, and a set of subspaces $\{\mathcal{U}_j \subset \mathcal{V}\}$ indexed over a set J which are each closed under the action of A. Then, $\{\mathcal{U}_j\}$ is an A-orthogonal decomposition for $\mathcal{V}$ iff,

$$\mathcal{V} \cong \bigoplus_{j \in J} \mathcal{U}_j. \tag{3.3.29}$$

As before, this notion extends to a set of endomorphisms,

Definition 3.3.34 ($\{A_j\}$-Orthogonal Decomposition)**.** Consider a vector space $\mathcal{V}$ over $\mathbb{F}$, a set of endomorphisms $\{A_j \in \mathrm{End}(\mathcal{V})\}$ indexed over a set J, and a set of subspaces $\{\mathcal{U}_k \subset \mathcal{V}\}$ indexed over a set K which, $\forall j \in J$, are each closed under the action of A_j. Then, $\{\mathcal{U}_j\}$ is a $\{A_k\}$-orthogonal decomposition for $\mathcal{V}$ iff,

$$\mathcal{V} \cong \bigoplus_{j \in J} \mathcal{U}_j. \tag{3.3.30}$$

Remark. $\{A_j\}$-orthogonal decompositions are a basis-independent generalisation of simultaneous diagonalisation of diagonalisable matrices to all endomorphisms with minimal polynomials.

When the endomorphism or set of endomorphisms is clear from context, we may refer these decompositions simply as "orthogonal". A-orthogonal decompositions are vital to the decomposition of $U(\mathfrak{so}(3, \mathbb{R}))$ in Chapter 4.

3.3.5.2 A-Orthogonal Decompositions and Identity Resolution

When considering an endomorphism $A \in \mathrm{End}(\mathcal{V})$ which has a minimal polynomial $m(x)$, a resolution of the identity by A naturally yields an A-orthogonal decomposition of $\mathcal{V}$. To prove this, first note,

Lemma 3.3.35. *Consider two endomorphisms $A, B \in \mathrm{End}(\mathcal{V})$ satisfying $[A, B] = 0$. Then, $\mathrm{Im}(A \circ B) \subseteq \mathrm{Im}(B)$ and $\mathrm{Im}(B \circ A) \subseteq \mathrm{Im}(A)$.*

Proof. Follows from $\mathrm{Im}(B \circ A) \subseteq \mathrm{Im}(B)$ and $\mathrm{Im}(A \circ B) \subseteq \mathrm{Im}(A)$. □

Thus,

Theorem 3.3.36. *Consider a vector space $\mathcal{V}$ over $\mathbb{F}$, an endomorphism $A \in \mathrm{End}(\mathcal{V})$ with minimal polynomial $m(x)$, and its associated identity resolution,*

$$\mathrm{id} = \sum_{j \in J} \Pi_{p_j(A)}(A), \tag{3.3.31}$$

given by Theorem 3.3.23 applied to the subalgebra $\mathcal{C}_A \subset \mathrm{End}^\circ(\mathcal{V})$. Then, $\{\mathrm{Im}(\Pi_{p_j(A)}(A))\}$ is an A-orthogonal decomposition for $\mathcal{V}$, i.e.,

$$\mathcal{V} \cong \bigoplus_{j \in J} \mathrm{Im}(\Pi_{p_j(A)}(A)), \tag{3.3.32}$$

with each summand closed under the action of A.

Proof. For compactness, let us denote $\mathcal{U}_j \coloneqq \mathrm{Im}(\Pi_{p_j(A)}(A))$ and $\mathcal{W} \coloneqq \bigoplus_{j \in J} \mathcal{U}_j$, and define,

$$\iota : \mathcal{W} \to \mathcal{V}$$

$$\iota := \sum_{j\in J}{}_{\mathcal{W}} v_j \mapsto \sum_{j\in J}{}_{\mathcal{V}} v_j,$$

and the vector space homomorphism,

$$\eta : \mathcal{V} \to \mathcal{W}$$
$$\eta := v \mapsto \sum_{j\in J}{}_{\mathcal{W}} \Pi_{p_j(A)}(A)(v).$$

We will now show that these maps are two-sided inverses for each other. First, $\forall v_j \in \mathcal{U}_j$, we have $\Pi_{p_j(A)}(A)(v_j) = v_j$, and so,

$$\begin{aligned}
\eta\circ\iota\Big(\sum_{j\in J}{}_{\mathcal{W}} v_j\Big) &= \eta\Big(\sum_{j\in J}{}_{\mathcal{V}} v_j\Big) \\
&= \eta\Big(\sum_{j\in J}{}_{\mathcal{V}} \Pi_{p_j(A)}(A)(v_j)\Big) \\
&= \sum_{k\in J}{}_{\mathcal{W}} \Pi_{p_k(A)}(A)\Big(\sum_{j\in J}{}_{\mathcal{V}} \Pi_{p_j(A)}(A)(v_j)\Big) \\
&= \sum_{k\in J}{}_{\mathcal{W}} \sum_{j\in J}{}_{\mathcal{U}_k} \Pi_{p_k(A)}(A)\circ\Pi_{p_j(A)}(A)(v_j) \\
&= \sum_{k\in J}{}_{\mathcal{W}} \Pi_{p_k(A)}(A)(v_k) \\
&= \sum_{k\in J}{}_{\mathcal{W}} v_k.
\end{aligned}$$

Furthermore, $\forall v \in \mathcal{V}$,

$$\iota\circ\eta(v) = \iota\Big(\sum_{j\in J}{}_{\mathcal{W}} \Pi_{p_j(A)}(A)(v)\Big) = \sum_{j\in J}{}_{\mathcal{V}} \Pi_{p_j(A)}(A)(v) = \mathrm{id}(v) = v.$$

Therefore, $\eta\circ\iota = \mathrm{id}_{\mathcal{W}}$ and $\iota\circ\eta = \mathrm{id}_{\mathcal{V}}$. Thus, $\mathcal{V} \cong \mathcal{W}$, i.e. $\mathcal{V}$ is an internal direct sum of the $\{\mathrm{Im}(\Pi_{p_j(A)}(A))\}$. To see this is an A-orthgonal decomposition, note that by Lemma 3.3.35, $\forall j \in J$, $\mathrm{Im}(A\circ\Pi_{p_j(A)}(A)) \subseteq \mathrm{Im}(\Pi_{p_j(A)}(A))$, so each summand is closed under the action of A. □

The A-orthogonal decomposition of $\mathcal{V}$ which results from resolutions of the identity by A will be of central importance to our decomposition of $U(\mathfrak{so}(3,\mathbb{R}))$ in Chapter 4. In fact, this procedure will enable us to identify the physically distinct observables of a system of arbitrary spin, from which we shall construct our real algebraic descriptions for such systems. Describing our systems in this manner will also give us considerable descriptive power and insight into these systems.

3.3.5.3 Minimal Polynomials on A-Orthogonal Decompositions

In light of Theorem 3.3.36, it is natural to consider the properties of the endomorphism A restricted to $\mathrm{Im}(\Pi_{p_j(A)}(A))$,

Lemma 3.3.37. *Consider an endomorphism* $A \in \mathrm{End}(\mathcal{V})$ *with minimal polynomial* $m(x)$ *on* $\mathcal{V}$*, and its* A*-orthogonal decomposition of* $\mathcal{V}$ *by Theorem 3.3.36. Then,* $\forall k \in J$*,* A *has minimal polynomial* $p_k(x)$ *when restricted to* $\mathrm{Im}(\Pi_{p_k(A)}(A))$*.*

Proof. By Theorem 3.3.23, $p_k(A)$ is annihilating on $\mathrm{Im}(\Pi_{p_k(A)}(A))$. Suppose $p_k(x)$ is non-minimal. Let us denote the minimal polynomial of A on $\mathrm{Im}(\Pi_{p_k(A)}(A))$ by $r(x)$. We can see that $r(x)q_k(x)$ is annihilating on $\mathcal{V}$, so we must have $r(x)q_k(x) \mid m(x)$. This contradicts the minimality of $m(x)$. □

3.3.6 Advantages of Identity Resolution Using Minimal Polynomials

It would be useful at this stage to discuss some advantages of the methods presented in this section for the elementary study of algebras. In particular, let us discuss their benefits when applied to the algebra of endomorphisms.

Firstly, because the method relies only on the algebraic properties of the element under study, the resulting idempotents are inherently basis-independent and independent of the dimension of the algebra which contains them. This enables the consistent construction of the idempotents independently of the context or size of the model. It also means that (simultaneous) diagonalisation of our endomorphism(s) of interest can be avoided whilst still enabling theoretical predictions to be made.

Secondly, it allows us to construct projectors into eigenspaces without first having to find eigenvectors, and in the absence of a metric on the vector space. Doing so is independent of the degeneracy of the eigenspace, allowing us considerable modelling power in the presence of highly degenerate systems. This advantage in particular will be exploited in Chapter 4.

Finally, it allows us to construct projectors into both generalised eigenspaces, and other subspaces closed under the action of A in which non-zero eigenvectors do not exist. This is achieved without the need to find the Jordan Normal Form of A, which is especially useful when this isn't possible. Furthermore, we have added no additional structure to our model of study to achieve this; this avoids questions about the physicality of such additional structure, and empowers the study of endomorphisms on vector spaces over non-algebraically closed fields. This approach will be foundational to the methods developed in Chapter 4.

3.3.7 Limitations of Identity Resolution Using Minimal Polynomials

No single method is panacea for all mathematical woes, and so let us consider some limitations of these methods as presented here, again with particular focus on endomorphisms.

Firstly, because the minimal polynomial of an endomorphism $A \in \mathrm{End}(\mathcal{V})$ is largely insensitive to the dimension of $\mathcal{V}$, our methods cannot determine the dimension of each space in the A-orthogonal decomposition for $\mathcal{V}$. For similar reasons, we cannot find projectors into different subspaces of a degenerate eigenspace, as the endomorphism A sees all of these as equivalent. These limitations are technical but not unexpected: if the algebraic structure of A is not sensitive to such details, then methods which utilise this structure will not be either.

Finally, our methods cannot algebraically construct projectors into different subspaces of a generalised eigenspace. To see this, consider breaking the minimal polynomial $(x-2)(x-1)^2$ into three as $(x-2)$, $(x-2)(x-1)$, $(x-1)^2$ and applying Bézout's Identity. The greatest common divisor of the three polynomials together is 1, but in $\mathcal{Q}_{(x-2)(x-1)^2}$ the object $a(x)(x-2)(x-1)$ is nilpotent, not idempotent.

These issues can be circumvented by performing a simultaneous resolution of the identity with an endomorphism B that, in the first case, distinguishes subspaces that A does not, or, in the second case, can resolve subspaces which A cannot. This solution itself is limited by our knowledge of the algebraic relationship between A and B, as this determines which idempotents in the simultaneous identity resolution are non-zero. This information is unavailable in the minimal polynomials of A and B, and must be understood separately. Therefore, to fully resolve these issues, the methods of this section must be extended to utilise this additional algebraic information, which must be understood in some algebraic form.

For the developments of this thesis, neither of these limitations will cause any problems: the degeneracy of eigenstates is unimportant for the arguments of Chapter 4, and all minimal polynomials used in that chapter have only first-polynomial-order irreducible powers.

3.4 Bézout Coefficients and Formal Irreducible Power Series

In Section 3.3, we saw that in $\mathcal{Q}_{m(x)}[x]$ the terms in an identity resolution $\{a_k(x)q_k(x)\}$ become idempotents. While the $q_k(x)$ are known upfront in any identity resolution derived from a polynomial $m(x)$, the Bézout coefficients $\{a_k(x)\}$ are not. Since these idempotent elements are essential to the developments of Chapter 4, we must give an accounting for how to compute the Bézout coefficients in an arbitrary identity resolution. In what follows, we will assume $\mathbb{Q} \subseteq \mathbb{F}$.

3.4.1 Properties of Bézout Coefficients

Let us begin by noting some general properties of Bézout coefficients,

Lemma 3.4.1. *For all $j, k \in J$, $j \neq k$, $p_k(x) \mid q_j(x)$.*

Proof. By definition. □

Lemma 3.4.2. *For all $j \in J$,*

$$|a_j(x)| < |p_j(x)|. \tag{3.4.1}$$

Proof. Since by definition $|p_j(x)| + |q_j(x)| = |m(x)|$, this follows from Theorem 3.3.15. □

To access the algebraic properties of the Bézout coefficients, we must introduce some homomorphisms to and from the quotient algebra $\mathcal{Q}_{p_k(x)}[x]$,

Definition 3.4.3 ($\pi'_{p_k(x)}$)**.** We denote the canonical surjection,

$$\begin{aligned} \pi'_{p_k(x)} &: \mathbb{F}[x] \to \mathcal{Q}_{p_k(x)}[x] \\ \pi'_{p_k(x)} &:= g(x) \mapsto [\, g(x) \,]. \end{aligned} \tag{3.4.2}$$

Definition 3.4.4 ($\chi'_{p_k(x)}$)**.** Let us define,

$$\begin{aligned} \chi'_{p_k(x)} &: \mathcal{Q}_{p_k(x)}[x] \to \mathbb{F}[x] \\ \chi'_{p_k(x)} &:= [\, g(x) \,] \mapsto r(x), \end{aligned} \tag{3.4.3}$$

where $r(x)$ is the unique representative of $[\, g(x) \,]$ with minimal polynomial order.

Lemma 3.4.5. *For all $k \in J$, in $\mathcal{Q}_{p_k(x)}[x]$,*

$$\chi'_{p_k(x)} \circ \pi'_{p_k(x)}(a_k(x)) = a_k(x). \tag{3.4.4}$$

Proof. By Lemma 3.4.2, all elements $f(x) \in [\, a_k(x)\,]$ have form $f(x) = a_k(x) + g(x)p_k(x)$, where $g(x) \in \mathbb{F}[x]$. □

Lemma 3.4.5 indicates that we may understand the $a_k(x)$ by considering $\mathcal{Q}_{p_k(x)}[x]$. This reveals a simple way to compute the $\{a_k(x)\}$,

Lemma 3.4.6. *For all $k \in J$,*

$$\pi'_{p_k(x)}(a_k(x))\pi'_{p_k(x)}(q_k(x)) = 1. \tag{3.4.5}$$

Proof. By Theorem 3.3.15 we have,

$$\sum_{j \in J} a_j(x) q_j(x) = 1.$$

Thus, Lemma 3.4.1 entails that $\forall j \in J$, $j \neq k$, $\pi'_{p_k(x)}(q_j(x)) = 0$. □

Lemma 3.4.6 reveals a simple relationship between $a_k(x)$ and $q_k(x)$: in $\mathcal{Q}_{p_k(x)}[x]$, they are multiplicative inverses. Furthermore, once a two-sided inverse element to $q_k(x)$ is found, Lemma 3.4.5 ensures it can be recovered whole and utilised in, for example, $\mathcal{Q}_{m(x)}[x]$.

We will use this relationship to calculate the $\{a_k(x)\}$ for any identity resolution. Computing multiplicative inverses in $\mathcal{Q}_{p_k(x)}[x]$ may be achieved by, for example, the Extended Euclidean Algorithm. However, this method does not offer us deeper insights into the structure of the $\{a_k(x)\}$ and how they relate to $\{q_k(x)\}$. In the remainder of this section, we will present a method for computing $\{a_k(x)\}$ for a general identity resolution which will explicate this connection.

3.4.2 Formal Power Series

In our endeavours, we are ultimately seeking to understand the inverse relationship between $\{a_k(x)\}$ and $\{q_k(x)\}$ in a given context. Therefore, before we develop our methods further, it is instructive to study the existence of inverse elements for a broader algebra of objects than the polynomials. Specifically, let us consider generalisations of polynomials which include expressions containing a countably infinite number of terms,

Definition 3.4.7 (Formal Power Series [67])**.** Consider a countable set $\{\alpha_n \in \mathbb{F}\}$ indexed over a subset $J \subseteq \mathbb{N}$. A formal power series $f(x)$ in x is an expression,

$$f(x) := \sum_{n \in J} \alpha_n x^n. \tag{3.4.6}$$

Definition 3.4.8 ($\mathbb{F}[[x]]$)**.** We denote the set of all formal power series in x with coefficients in $\mathbb{F}$ by $\mathbb{F}[[x]]$.

Remark. Clearly $\mathbb{F}[x] \subset \mathbb{F}[[x]]$, but in $\mathbb{F}[[x]]$ there also exist "infinite polynomials". It is important to note that there is no requirement for an arbitrary formal power series to converge to an element of $\mathbb{F}$ when evaluated.

Lemma 3.4.9. *The set of formal power series may be given the structure of a unital, commutative, associative algebra over* $\mathbb{F}$*, with addition, scalar multiplication, and product defined as,* $\forall f(x), g(x) \in \mathbb{F}[[x]]$*,* $\forall \gamma \in \mathbb{F}$*,*

$$f(x) = \sum_{j=0}^{\infty} \alpha_j x^j, \quad g(x) = \sum_{j=0}^{\infty} \beta_j x^j,$$

by,

$$\begin{aligned} f(x) + g(x) &:= \sum_{j=0}^{\infty} (\alpha_j + \beta_j) x^j \\ \gamma f(x) &:= \sum_{j=0}^{\infty} (\gamma \alpha_j) x^j \\ f(x) g(x) &:= \sum_{j=0}^{\infty} \left(\sum_{m=0}^{j} \alpha_m \beta_{j-m} \right) x^j. \end{aligned}$$

Proof. The axioms for both vector spaces and unital associative algebras may be easily verified. □

Remark. We will abuse notation by also denoting the algebra of Lemma 3.4.9 by $\mathbb{F}[[x]]$.

Unlike $\mathbb{F}[x]$, many non-constant elements of $\mathbb{F}[[x]]$ have multiplicative inverses,

Lemma 3.4.10. *Consider a formal power series* $f(x) \in \mathbb{F}[[x]]$*, with constant coefficient* α_0*. Then,* $\exists g(x) \in \mathbb{F}[[x]]$ *such that* $f(x)g(x) = g(x)f(x) = 1$ *iff* $\alpha_0 \neq 0$*.*

Proof. See [67]. □

Remark. Despite the existence of multiplicative inverses in $\mathbb{F}[[x]]$ for some formal power series, there are important identity resolutions involving $q_k(x)$ for which a multiplicative inverse does not exist in $\mathbb{F}[[x]]$, such as when $q_k(x) = x$.

3.4.3 Formal Irreducible Power Series

To overcome the non-existence of inverses elements in $\mathbb{F}[[x]]$ for $\{q_k(x)\}$ is some situations, we must generalise the notion of a formal power series to expressions involving any arbitrary irreducible polynomial.

3.4.3.1 Formal Irreducible Power Series

Definition 3.4.11 ($M_{f(x)}$)**.** Consider an irreducible polynomial $f(x) \in \mathbb{F}[x]$. Then, we denote by $M_{f(x)}$ the vector subspace of $\mathbb{F}[x]$ for which $\forall a(x) \in M_{f(x)}$, $|a(x)| < |f(x)|$.

Definition 3.4.12 (Formal Irreducible Power Series)**.** Consider an irreducible polynomial $f(x) \in \mathbb{F}[x]$, and a countable set $\{\mu_j(x) \in M_{f(x)}\}$ indexed over a subset $J \subseteq \mathbb{N}$. A formal irreducible power series in $f(x)$ is an expression,

$$\sum_{j \in J} \mu_j(x) f(x)^j. \tag{3.4.7}$$

Definition 3.4.13. We denote the set of all formal irreducible power series in $f(x)$ with coefficients in $M_{f(x)}$ by $S(\mathbb{F}[x], f(x), M_{f(x)})$.

Remark. This definition mirrors that of formal power series, except we allow the coefficients of the series to be polynomials of strictly lower degree than, but in the same element x as, the irreducible polynomial $f(x)$. This will substantially alter the algebraic properties of $S(\mathbb{F}[x], f(x), M_{f(x)})$ when we define it.

It is useful to identify,

Definition 3.4.14 ((In)Finite-Order Series)**.** Consider $g(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$, which we write as,

$$g(x) = \sum_{j=0}^{\infty} \mu_j(x) f(x). \tag{3.4.8}$$

Then, $g(x)$ is a finite-order series iff $\exists k \in \mathbb{N}$ such that $\mu_k(x) \neq 0$ and $\forall n \in \mathbb{Z}^+$, $\mu_{k+n}(x) = 0$. In this case we say that $|g(x)| = k$. Otherwise, $g(x)$ is an infinite-order series.

As with formal power series, $S(\mathbb{F}[x], f(x), M_{f(x)})$ admits a natural vector space structure,

Lemma 3.4.15. *The set of formal irreducible power series may be given the structure of a vector space over* $\mathbb{F}$*, with addition and scalar multiplication defined* $\forall g(x), h(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$, $\forall \gamma \in \mathbb{F}$,

$$g(x) = \sum_{j=0}^{\infty} \mu_j(x) f(x)^j, \quad h(x) = \sum_{j=0}^{\infty} \nu_j(x) f(x)^j, \tag{3.4.9}$$

by,

$$g(x) + h(x) \coloneqq \sum_{j=0}^{\infty} (\mu_j(x) + \nu_j(x)) f(x)^j \tag{3.4.10a}$$

$$\gamma g(x) \coloneqq \sum_{j=0}^{\infty} (\gamma \mu_j(x)) f(x)^j. \tag{3.4.10b}$$

Proof. The axioms for a vector space are easily verified. □

Remark. As before, we shall abuse notation by denoting by $S(\mathbb{F}[x], f(x), M_{f(x)})$ both the set of formal irreducible power series, and the vector space on this set defined in Lemma 3.4.15.

3.4.3.2 Algebra of Formal Irreducible Power Series

$S(\mathbb{F}[x], f(x), M_{f(x)})$ also admits an algebraic structure; its definition is as intuitive as for $\mathbb{F}[[x]]$, but is mathematically more subtle due to the polynomial nature of the coefficients. To facilitate this definition, let us first note,

Lemma 3.4.16. *Consider a polynomial* $f(x) \in \mathbb{F}[x]$ *and the quotient algebra* $\mathcal{Q}_{f(x)}[x]$*. Then, all* $g(x) \in \mathcal{Q}_{f(x)}[x]$, $g(x) \neq 0$ *have two-sided inverses in* $\mathcal{Q}_{f(x)}[x]$ *iff* $f(x)$ *is irreducible over* $\mathbb{F}$.

Proof. See [49]. □

Using Lemma 3.4.16, we may define,

Definition 3.4.17 ($\pi_{f(x)}^{(j)}$)**.** Consider an irreducible polynomial $f(x) \in \mathbb{F}[x]$, and a polynomial $g(x) \in \mathbb{F}[x]$ written in the form,

$$g(x) = a_0(x) + a_1(x) f(x) + a_2(x) f(x)^2 + B(x) f(x)^3, \tag{3.4.11}$$

with $\forall j \in \{0, 1, 2\}$, $a_j(x) \in M_{f(x)}$ and $B(x) \in \mathbb{F}[x]$. Then, $\forall j \in \{0, 1, 2\}$, we define,

$$\begin{aligned} \pi_{f(x)}^{(j)} &: \mathbb{F}[x] \to M_{f(x)} \\ \pi_{f(x)}^{(j)}(g(x)) &\coloneqq a_j(x). \end{aligned} \tag{3.4.12}$$

Remark. Writing an arbitrary polynomial in the form in Definition 3.4.17 is always possible using, for example, iterated polynomial division. Some more computational definitions for $\{\pi^{(0)}_{f(x)}, \pi^{(1)}_{f(x)}, \pi^{(2)}_{f(x)}\}$, and an account of their properties and relationships, can be found in Appendix C.1.

We may use these functions to define an associative product on $S(\mathbb{F}[x], f(x), M_{f(x)})$,

Definition 3.4.18 (Product on $S(\mathbb{F}[x], f(x), M_{f(x)})$)**.** For all $g(x), h(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,

$$g(x) = \sum_{j=0}^{\infty} \mu_j(x) f(x)^j, \quad h(x) = \sum_{j=0}^{\infty} \nu_j(x) f(x)^j,$$

we define their product,

$$g(x)h(x) := \sum_{p=0}^{\infty} \eta_p(x) f(x)^p, \tag{3.4.13}$$

with, $\forall p \in \mathbb{N}$,

$$\eta_p(x) := \begin{cases} \pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big) & p = 0 \\ \sum_{q=0}^{p} \pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big) + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{p-q-1}(x)\big) & p \in \mathbb{Z}^+, \end{cases} \tag{3.4.14}$$

where we have suppressed the natural injections from $M_{f(x)}$ to $\mathbb{F}[x]$.

Clearly this product is well-defined on all pairs of elements in $S(\mathbb{F}[x], f(x), M_{f(x)})$. Furthermore,

Lemma 3.4.19. *The product in Definition 3.4.18 is bilinear, commutative, and associative, with two-sided identity element* 1*.*

Proof. See Appendix C.1. □

Remark. We shall further abuse notation by denoting by $S(\mathbb{F}[x], f(x), M_{f(x)})$ the algebra formed with the product defined in Lemma 3.4.18.

Much like the polynomial and formal power series algebras, $S(\mathbb{F}[x], f(x), M_{f(x)})$ contains no non-trivial zero divisors,

Lemma 3.4.20. *For all $g(x), h(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$, $g(x)h(x) = 0$ implies $g(x) = 0$ or $h(x) = 0$.*

Proof. We may proceed by contraposition. Suppose $g(x) \neq 0$ and $h(x) \neq 0$, so $\exists m, n \in \mathbb{N}$ such that,

$$g(x) = \sum_{j=m}^{\infty} \mu_j(x) f(x)^j, \quad h(x) = \sum_{j=n}^{\infty} \nu_j(x) f(x)^j,$$

with $\mu_m(x) \neq 0$ and $\nu_n(x) \neq 0$. Then, we have,

$$g(x)h(x) = \sum_{p=m+n}^{\infty} \eta_p(x) f(x)^p,$$

particularly,

$$\eta_{m+n}(x) = \begin{cases} \pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big) & m+n=0 \\ \displaystyle\sum_{q=0}^{m+n} \pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{m+n-q}(x)\big) & \\ \qquad + \displaystyle\sum_{q=0}^{m+n-1} \pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{m+n-q-1}(x)\big) & m+n \in \mathbb{Z}^+. \end{cases}$$

When $m+n=0$, we must have $\eta_0(x) \neq 0$ by Lemma 3.4.16. Since $\mu_j(x) = 0$, $\nu_k(x) = 0$ for all $j, k \in \mathbb{N}$ with $j < m$, $k < n$, when $m+n \in \mathbb{Z}^+$ we have,

$$\eta_{m+n}(x) = \pi^{(0)}_{f(x)}\big(\mu_m(x)\nu_n(x)\big).$$

Thus, we must have $\eta_{m+n}(x) \neq 0$ by Lemma 3.4.16. Therefore, in any case, $g(x)h(x) \neq 0$. □

Furthermore, it ensures all formal irreducible power series behave in the way we expect of a series expansion in $f(x)$,

Lemma 3.4.21. *For all* $h(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,

$$h(x) = \sum_{j=0}^{\infty} \nu_j(x) f(x)^j, \tag{3.4.15}$$

we have,

$$f(x)h(x) = \sum_{j=0}^{\infty} \nu_j(x) f(x)^{j+1}. \tag{3.4.16}$$

Proof. We may write $f(x)$ as,

$$f(x) = \sum_{j=0}^{\infty} \mu_j(x) f(x)^j,$$

with $\mu_1(x) = 1$, and $\forall j \in \mathbb{N}\backslash\{1\}$, $\mu_j(x) = 0$. Writing,

$$f(x)h(x) = \sum_{p=0}^{\infty} \eta_p(x) f(x)^p,$$

we have, $\forall p \in \mathbb{N}$,

$$\eta_p = \begin{cases} 0 & p = 0 \\ \nu_{p-1}(x) & p \in \mathbb{Z}^+. \end{cases}$$

Reindexing,

$$f(x)h(x) = \sum_{p=1}^{\infty} \nu_{p-1}(x) f(x)^p = \sum_{p=0}^{\infty} \nu_p(x) f(x)^{p+1}.$$

□

In fact, the relationship between $S(\mathbb{F}[x], f(x), M_{f(x)})$ and $\mathbb{F}[x]$ is more extensive that it may first appear. To begin with,

Lemma 3.4.22. *Consider two finite-order series $g(x), h(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$. Then, $g(x)h(x)$ is equivalent to their multiplication as polynomials in $\mathbb{F}[x]$.*

Proof. See Appendix C.1. □

From this, we see that,

Lemma 3.4.23. *The set of all finite-order series forms a subalgebra of $S(\mathbb{F}[x], f(x), M_{f(x)})$ which is unital associative algebra isomorphic to $\mathbb{F}[x]$.*

Proof. Closure can be seen by direct computation, and the isomorphism may be established through polynomial division. □

3.4.3.3 Inverses of Formal Irreducible Power Series

Like the algebra of formal power series, many elements in the algebra of formal irreducible power series admit two-sided inverse elements. In fact, the condition for such an inverse to exist is similar to the same condition for $\mathbb{F}[[x]]$ in Lemma 3.4.10,

Lemma 3.4.24. *Consider a formal irreducible power series $g(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,*

$$g(x) = \sum_{j=0}^{\infty} \mu_j(x) f(x)^j. \tag{3.4.17}$$

Then, $g(x)$ has a two-sided inverse with respect to the product of $S(\mathbb{F}[x], f(x), M_{f(x)})$ iff $\mu_0(x) \neq 0$.

Proof. See Appendix C.1. □

For our purposes, this is an essential result,

Theorem 3.4.25. *Consider a polynomial $m(x) \in \mathbb{F}[x]$, and the sets $\{p_k(x) = f_k(x)^{d_k} \in \mathbb{F}[x]\}$ and $\{q_k(x) \in \mathbb{F}[x]\}$ as defined in Definition 3.3.9 indexed over a set J. For all $k \in J$, $\exists q_k(x)^{-1} \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)})$ such that $q_k(x)^{-1}q_k(x) = q_k(x)q_k(x)^{-1} = 1$.*

Proof. Consider some $k \in J$. We may write any $g(x) \in \mathbb{F}[x]$ as an irreducible power series in $f_k(x)$ by iterated polynomial division. Thus, let us write,

$$q_k(x) = \sum_{j=0}^{n} \kappa_j(x) f_k(x)^j \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)}),$$

where $n = \left\lfloor \frac{|q_k(x)|}{|f_k(x)|} \right\rfloor$. By definition, $f_k(x) \nmid q_k(x)$, therefore $\kappa_0(x) \neq 0$. By Lemma 3.4.24, we are done. □

We will soon see that the inverse formal irreducible power series $q_k(x)^{-1}$ is exactly what we need to calculate the Bézout coefficients needed to construct resolutions of the identity. This will be essential to probe the structure of $U(\mathfrak{so}(3, \mathbb{R}))$ in Chapter 4.

3.4.4 Inverse Formal Irreducible Power Series and Bézout Coefficients

Theorem 3.4.25 establishes the existence of an inverse formal irreducible power series for each $q_k(x)$ in an identity resolution, regardless of their form. We will now show that truncations of this inverse series to a given order in $f(x)$ are exactly the Bézout coefficients we seek.

3.4.4.1 Quotient Algebras of $S(\mathbb{F}[x], f(x), M_{f(x)})$

To do this, we must first discuss some important quotient algebras of the formal irreducible power series. To begin, let us define,

Definition 3.4.26. For all $k \in \mathbb{Z}^+$,

$$S\mathcal{Q}_{f(x)^k}[x] \cong \frac{S(\mathbb{F}[x], f(x), M_{f(x)})}{I\left(f(x)^k\right)}. \tag{3.4.18}$$

Definition 3.4.27 (π_k)**.** For all $k \in \mathbb{Z}^+$, we denote the canonical surjection,

$$\begin{aligned} \pi_k &: S(\mathbb{F}[x], f(x), M_{f(x)}) \to S\mathcal{Q}_{f(x)^k}[x] \\ \pi_k &:= g(x) \mapsto [\, g(x) \,]. \end{aligned} \tag{3.4.19}$$

Lemma 3.4.28. *For all $k \in \mathbb{Z}^+$, $\forall [\, g(x)\,] \in S\mathcal{Q}_{f(x)^k}[x]$, $\exists s(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$ of finite-order such that $[\, s(x)\,] = [\, g(x)\,]$. Furthermore, there exists a unique finite-order series $r(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$ of least order such that $[\, r(x)\,] = [\, g(x)\,]$*

Proof. By Lemma 3.4.21, we may write,

$$g(x) = \sum_{j=0}^{k-1} \mu_j(x) f(x)^j + f(x)^k \sum_{j=0}^{\infty} \mu_{j+k}(x) f(x)^j,$$

and so,

$$s(x) \coloneqq \sum_{j=0}^{k-1} \mu_j(x) f(x)^j,$$

is in $[\, g(x)\,]$.

Now, suppose $\exists r(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$ of finite-order such that, suppressing inclusions, $[\, r(x)\,] = [\, g(x)\,]$ with $|r(x)| \leq |s(x)|$. Then, $[\, r(x)\,] = [\, s(x)\,]$ implies $s(x) - r(x) \in I\left(f(x)^j\right)$. Since by Lemma 2.3.23, $|s(x) - r(x)| \leq |f(x)^k| - 1$ and $I\left(f(x)^j\right)$ is proper, we must have $s(x) = r(x)$. Thus, $s(x)$ is of minimal polynomial order and unique. □

Lemmas 3.4.23 and 3.4.28 motivate us to define,

Definition 3.4.29 (χ_k)**.** For all $k \in \mathbb{Z}^+$, let us define,

$$\begin{aligned} \chi_k &: S\mathcal{Q}_{f(x)^k}[x] \to \mathbb{F}[x] \\ \chi_k &\coloneqq [\, g(x)\,] \mapsto r(x), \end{aligned} \tag{3.4.20}$$

where $r(x)$ is the unique finite-order series representative of $[\, g(x)\,]$ with minimal polynomial order.

Remark. When working with χ_k, we will often leave implicit inclusions from $\mathbb{F}[x]$ to $S(\mathbb{F}[x], f(x), M_{f(x)})$.

Lemma 3.4.30. *For all $k \in \mathbb{Z}^+$, χ_k is a vector space homomorphism.*

Proof. Clear from the definition. □

Lemma 3.4.31. *Denoting by i the inclusion,*

$$i : \mathbb{F}[x] \to S(\mathbb{F}[x], f(x), M_{f(x)}), \tag{3.4.21}$$

$\forall k \in \mathbb{Z}^+$,

$$\pi_k \circ i \circ \chi_k = \operatorname{id}. \tag{3.4.22}$$

Furthermore, χ_k is injective.

Proof. The first claim is clear from the definition. Injectivity follows from Lemma 3.4.31. □

3.4.4.2 The Relationship between Quotients of Formal Irreducible Power Series and Quotients of Polynomials

In particular, we must understand how $S\mathcal{Q}_{f(x)^k}[x]$ relates to $\mathcal{Q}_{f(x)^k}[x]$. To do this, let us define,

Definition 3.4.32 (π'_k)**.** For all $k \in \mathbb{Z}^+$, we denote the canonical surjection,

$$\begin{aligned} \pi'_k &: \mathbb{F}[x] \to \mathcal{Q}_{f(x)^k}[x] \\ \pi'_k &:= g(x) \mapsto [\, g(x) \,]. \end{aligned} \tag{3.4.23}$$

Definition 3.4.33 (χ'_k)**.** For all $k \in \mathbb{Z}^+$, let us define,

$$\begin{aligned} \chi'_k &: \mathcal{Q}_{f(x)^k}[x] \to \mathbb{F}[x] \\ \chi'_k &:= [\, g(x) \,] \mapsto r(x), \end{aligned} \tag{3.4.24}$$

where $r(x)$ is the unique representative of $[\, g(x) \,]$ with minimal polynomial order.

Lemma 3.4.34. *For all $k \in \mathbb{Z}^+$,*

$$\pi'_k \circ \chi'_k = id. \tag{3.4.25}$$

Proof. Clear from the definition. □

Then,

Lemma 3.4.35. *Denoting by i the inclusion,*

$$i : \mathbb{F}[x] \to S(\mathbb{F}[x], f(x), M_{f(x)}), \tag{3.4.26}$$

$\forall k \in \mathbb{Z}^+$, we have,

$$\chi'_k \circ \pi'_k = \chi_k \circ \pi_k \circ i \tag{3.4.27}$$

Proof. This follows from Lemma 3.4.23. □

Definition 3.4.36 (ϕ_k)**.** For all $k \in \mathbb{Z}^+$,

$$\begin{aligned} \phi_k &: S\mathcal{Q}_{f(x)^k}[x] \to \mathcal{Q}_{f(x)^k}[x] \\ \phi_k &:= \pi'_k \circ \chi_k. \end{aligned} \tag{3.4.28}$$

Lemma 3.4.37. *For all $k \in \mathbb{Z}^+$, ϕ_k is a unital associative algebra isomorphism.*

Proof. That ϕ_k is unital and a vector space homomorphism is clear. Let us first show that ϕ_k is a vector space isomorphism. Consider $[\,g(x)\,],[\,h(x)\,] \in S\mathcal{Q}_{f(x)^k}[x]$ such that, $\phi_k\big([\,g(x)\,]\big) = \phi_k\big([\,h(x)\,]\big)$. By Lemmas 3.4.28, 3.4.31 and 3.4.35, we have,

$$\begin{aligned}
&\big[\pi'_k\circ\chi_k\circ\pi_k(g(x)) = \pi'_k\circ\chi_k\circ\pi_k(h(x))\big] \\
&\qquad\Rightarrow \big[\chi'_k\circ\pi'_k\circ\chi_k\circ\pi_k(g(x)) = \chi'_k\circ\pi'_k\circ\chi_k\circ\pi_k(h(x))\big] \\
&\qquad\Rightarrow \big[\chi_k\circ\pi_k(g(x)) = \chi_k\circ\pi_k(h(x))\big] \\
&\qquad\Rightarrow \big[\pi_k(g(x)) = \pi_k(h(x))\big],
\end{aligned}$$

so ϕ_k is injective. Furthermore, Lemma 3.4.28 implies that $\dim(S\mathcal{Q}_{f(x)^k}[x]) = \dim(\mathcal{Q}_{f(x)^k}[x]) = |f(x)^k| - 1$, i.e. finite-dimensional. Thus, ϕ_k is a vector space isomorphism.

Now let us show that ϕ_k is an algebra homomorphism. For all $g(x), h(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,

$$\begin{aligned}
\phi_k\circ\pi_k(g(x))\phi_k\circ\pi_k(h(x)) &= \pi'_k\circ\chi_k\circ\pi_k(g(x))\pi'_k\circ\chi_k\circ\pi_k(h(x)) \\
&= \pi'_k\big(\chi_k\circ\pi_k(g(x))\chi_k\circ\pi_k(h(x))\big),
\end{aligned}$$

which, by polynomial division, Lemmas 3.4.23, 3.4.31, and 3.4.35, and suppressing inclusions, we may write as,

$$\begin{aligned}
&= \pi'_k\circ\chi'_k\circ\pi'_k\big(\chi_k\circ\pi_k(g(x))\chi_k\circ\pi_k(h(x))\big) \\
&= \pi'_k\circ\chi_k\circ\pi_k\big(\chi_k\circ\pi_k(g(x))\chi_k\circ\pi_k(h(x))\big) \\
&= \pi'_k\circ\chi_k\big(\pi_k\circ\chi_k\circ\pi_k(g(x))\pi_k\circ\chi_k\circ\pi_k(h(x))\big) \\
&= \pi'_k\circ\chi_k\big(\pi_k(g(x))\pi_k(h(x))\big) \\
&= \phi_k\circ\pi_k\big(g(x)h(x)\big).
\end{aligned}$$

□

3.4.4.3 Bézout Coefficients via $S\mathcal{Q}_{f(x)^k}[x]$

With the relationship between $S\mathcal{Q}_{f(x)^k}[x]$ and $\mathcal{Q}_{f(x)^k}[x]$ established, we may now conclude,

Theorem 3.4.38. *Denoting by i the inclusion,*

$$i : \mathbb{F}[x] \to S(\mathbb{F}[x], f(x), M_{f(x)}), \tag{3.4.29}$$

$\forall k \in J$, $\forall d_k \in \mathbb{Z}^+$, with $p_k(x) = f_k(x)^{d_k}$ and $i(q_k(x)) \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)})$,

$$\phi_{d_k}\circ\pi_{d_k}\big(q_k(x)^{-1}\big) = \pi'_{p_k(x)}(a_k(x)). \tag{3.4.30}$$

Proof.

$$\begin{aligned}\left[q_k(x)^{-1}q_k(x) = 1\right] &\Rightarrow \left[\phi_{d_k}\circ\pi_{d_k}\left(q_k(x)^{-1}q_k(x)\right) = \phi_{d_k}\circ\pi_{d_k}(1)\right]\\ &\Rightarrow \left[\phi_{d_k}\circ\pi_{d_k}\left(q_k(x)^{-1}\right)\phi_{d_k}\circ\pi_{d_k}(q_k(x)) = 1\right]\\ &\Rightarrow \left[\phi_{d_k}\circ\pi_{d_k}\left(q_k(x)^{-1}\right)\pi'_{d_k}(q_k(x)) = 1\right].\end{aligned}$$

Once we recognise that $\pi'_{p_k(x)} = \pi'_{d_k}$, the claim follows by uniqueness of two-sided inverses. □

As promised, Theorem 3.4.38 reveals the connection between $q_k(x)^{-1}$ and the Bézout coefficient $a_k(x)$: $a_k(x)$ is simply the truncation of $q_k(x)^{-1}$ to the appropriate finite order. Thus, if we can calculate $q_k(x)^{-1}$ in full for arbitrary $q_k(x)$, we can find its Bézout coefficient, and we will have everything we need to analyse $U(\mathfrak{so}(3,\mathbb{R}))$ by completely algebraic means in Chapter 4.

3.4.5 Computing Components of Formal Irreducible Power Series

Since Theorem 3.4.25 only demonstrates the *existence* of $q_k(x)^{-1} \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)})$, we must now develop a method for its computation. To do this, we require a number of homomorphisms on $S(\mathbb{F}[x], f(x), M_{f(x)})$ which manipulate its series in various ways.

3.4.5.1 Leading Coefficients of Formal Irreducible Power Series

An operation essential to our goal is the extraction of the leading coefficient of a formal irreducible power series,

Definition 3.4.39 ($\tau_{f(x)}$)**.** Denoting by i the injection,

$$i : M_{f(x)} \to \mathbb{F}[x], \tag{3.4.31}$$

and defining implicitly,

$$\begin{gathered}\chi' : \mathcal{Q}_{f(x)}[x] \to M_{f(x)}\\ i\circ\chi' \coloneqq \chi'_1,\end{gathered} \tag{3.4.32}$$

we define,

$$\begin{gathered}\tau_{f(x)} : S(\mathbb{F}[x], f(x), M_{f(x)}) \to M_{f(x)}\\ \tau_{f(x)} \coloneqq \chi'\circ\phi_1\circ\pi_1.\end{gathered} \tag{3.4.33}$$

Lemma 3.4.40. *$\tau_{f(x)}$ is a vector space homomorphism.*

Proof. Clear by the definition. □

Finally,

Lemma 3.4.41. *For all $g(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,*

$$g(x) = \sum_{j=0}^{\infty} \mu_j(x) f(x)^j, \tag{3.4.34}$$

we have,

$$\tau_{f(x)}(g(x)) = \mu_0(x). \tag{3.4.35}$$

Proof. By Lemmas 2.3.23 and 3.4.21,

$$f(x) \sum_{j=0}^{\infty} \mu_{j+1}(x) f(x)^j = \sum_{j=1}^{\infty} \mu_j(x) f(x)^j \in I\left(f(x)\right).$$

Therefore, $(g(x) - \mu_0(x)) \in I\left(f(x)\right)$, and so in $\mathcal{Q}_{f(x)}[x]$, $[\,g(x)\,] = [\,\mu_0(x)\,]$. It is clear that $\mu_0(x)$ is the unique finite-order representative with minimal polynomial order, so we are done. □

Consequently,

Lemma 3.4.42. *For all $g(x) \in \mathbb{F}[x]$, and suppressing inclusions,*

$$\tau_{f(x)}(g(x)) = \pi^{(0)}_{f(x)}(g(x)). \tag{3.4.36}$$

Proof. This follows from Lemma 3.4.41. □

Thus,

Lemma 3.4.43. *For all $g(x), h(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,*

$$\tau_{f(x)}\big(g(x)h(x)\big) = \tau_{f(x)}\big(\tau_{f(x)}(g(x))\tau_{f(x)}(h(x))\big). \tag{3.4.37}$$

Proof. Writing,

$$g(x) = \sum_{j=0}^{\infty} \mu_j(x) f(x)^j, \quad h(x) = \sum_{j=0}^{\infty} \nu_j(x) f(x)^j,$$

by Definition 3.4.18 and Lemmas 3.4.41 and 3.4.42 we have, suppressing inclusions,

$$\begin{aligned}
\tau_{f(x)}\big(g(x)h(x)\big) &= \pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big) \\
&= \pi^{(0)}_{f(x)}\big(\tau_{f(x)}(g(x))\tau_{f(x)}(h(x))\big) \\
&= \tau_{f(x)}\big(\tau_{f(x)}(g(x))\tau_{f(x)}(h(x))\big),
\end{aligned}$$

where we have suppressed the inclusions from $M_{f(x)}$ into $\mathbb{F}[x]$. □

3.4.5.2 Derivations on Formal Irreducible Power Series

As with polynomials and formal power series, there are non-trivial derivations on $S(\mathbb{F}[x], f(x), M_{f(x)})$. We will use these later to reduce higher-order coefficients in the series to leading-order. First,

Definition 3.4.44 (∂_x)**.** We define $\partial_x \in \mathrm{Der}(\mathbb{F}[x])$ by,

$$\partial_x(x) \coloneqq 1. \tag{3.4.38}$$

Now, note that,

Lemma 3.4.45. *For all* $\mu(x) \in M_{f(x)}$*,*

$$\partial_x(\mu(x)) \in M_{f(x)}. \tag{3.4.39}$$

Furthermore,

$$\partial_x(f(x)) \in M_{f(x)}. \tag{3.4.40}$$

Proof. See Appendix C.1. □

Thus, we may define,

Definition 3.4.46 (D_x)**.** For all $g(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,

$$g(x) = \sum_{j=0}^{\infty} \mu_j(x) f(x)^j, \tag{3.4.41}$$

we define the function,

$$\begin{aligned} D_x : S(\mathbb{F}[x], f(x), M_{f(x)}) &\to S(\mathbb{F}[x], f(x), M_{f(x)}) \\ D_x(g(x)) &\coloneqq \sum_{p=0}^{\infty} \delta_p(x) f(x)^p, \end{aligned} \tag{3.4.42}$$

where, $\forall p \in \mathbb{N}$,

$$\delta_p \coloneqq \partial_x(\mu_p(x)) + (p+1)\pi^{(0)}_{f(x)}\big(\mu_{p+1}(x)\partial_x(f(x))\big) + p\,\pi^{(1)}_{f(x)}\big(\mu_p(x)\partial_x(f(x))\big), \tag{3.4.43}$$

and we have suppressed the natural inclusions from $M_{f(x)}$ to $\mathbb{F}[x]$.

Lemma 3.4.47. $D_x \in \mathrm{Der}(S(\mathbb{F}[x], f(x), M_{f(x)}))$.

Proof. See Appendix C.1. □

As with the product on $\mathrm{Der}(S(\mathbb{F}[x], f(x), M_{f(x)}))$, D_x reduces to familiar operations on finite-order series,

Lemma 3.4.48. *For all finite-order series $g(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,*

$$D_x(g(x)) = \partial_x(g(x)). \tag{3.4.44}$$

Proof. See Appendix C.1. □

We may utilise D_x to define a convenient derivation for series manipulation. To do this, let us note,

Lemma 3.4.49. *$D_x(f(x)) \in S(\mathbb{F}[x], f(x), M_{f(x)})$ has a two-sided inverse.*

Proof. Since $f(x)$ is non-constant, $D_x f(x) \neq 0$. Thus, by Lemmas 3.4.24 and 3.4.45 we are done. □

Accordingly, we may define,

Definition 3.4.50 ($D_{f(x)}$)**.**

$$\begin{aligned} D_{f(x)} &: S(\mathbb{F}[x], f(x), M_{f(x)}) \to S(\mathbb{F}[x], f(x), M_{f(x)}) \\ D_{f(x)} &:= g(x) \mapsto D_x(f(x))^{-1} D_x. \end{aligned} \tag{3.4.45}$$

Lemma 3.4.51. *$D_{f(x)} \in \mathrm{Der}(S(\mathbb{F}[x], f(x), M_{f(x)}))$ and satisfies, $\forall j \in \mathbb{Z}^+$,*

$$\begin{aligned} D_{f(x)}(1) &= 0 \\ D_{f(x)}\big(f(x)^j\big) &= j f(x)^{j-1}. \end{aligned}$$

Proof. Follows from Corollary 2.3.58 and Lemmas 2.3.59, 2.3.61, and 2.3.62. □

Remark. This derivation is the formal equivalent of the usual chain rule on real functions, but does not require continuity to define.

3.4.5.3 Computation of Series Coefficients

We may use $\tau_{f(x)}$ and $D_{f(x)}$ in tandem to find the coefficients of any formal irreducible power series,

Lemma 3.4.52. *Consider $g(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,*

$$g(x) = \sum_{j=0}^{\infty} \mu_j(x) f(x)^j. \tag{3.4.47}$$

Then, $\forall k \in \mathbb{N}$,

$$\mu_j(x) = \begin{cases} \tau_{f(x)}\big(g(x)\big) & k = 0 \\ \dfrac{1}{k!}\tau_{f(x)} \circ D_{f(x)}^{\circ k}\big(g(x)\big) - \displaystyle\sum_{j=0}^{k-1} \frac{1}{(k-j)!}\tau_{f(x)} \circ D_{f(x)}^{\circ(k-j)}\big(\mu_j(x)\big) & k \in \mathbb{Z}^+. \end{cases} \tag{3.4.48}$$

Proof. See Appendix C.1. □

Remark. It may appear as if we require the complete series for $D_x f(x)^{-1}$ to proceed, but we may iteratively use Lemma 3.4.43 to reduce to calculation to one involving products of $\tau_{f(x)}\big(D_x f(x)^{-1}\big)$ and $\tau_{f(x)}\big(D_x^{\circ j}(f(x))\big)$.

Therefore, we may easily compute the coefficient of an inverse formal irreducible power series from,

Theorem 3.4.53. *For all $\{q_k(x) \in \mathbb{F}[x]\}$ in an identity resolution, there exists a multiplicative inverse,*

$$q_k(x)^{-1} = \sum_{n=0}^{\infty} \alpha_n(x) f_k(x)^n \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)}), \tag{3.4.49}$$

where, $\forall n \in \mathbb{N}$,

$$\alpha_n(x) = \begin{cases} \tau_{f_k(x)}\big(q_k(x)^{-1}\big) & n = 0 \\ \dfrac{1}{n!}\tau_{f_k(x)} \circ D_{f_k(x)}^{\circ n}\big(q_k(x)^{-1}\big) - \displaystyle\sum_{j=0}^{n-1} \frac{1}{(n-j)!}\tau_{f_k(x)} \circ D_{f_k(x)}^{\circ(n-j)}\big(\alpha_j(x)\big) & n \in \mathbb{Z}^+. \end{cases} \tag{3.4.50}$$

Proof. This follows from Theorem 3.4.25, and 3.4.52. □

Remark. As before, we may use Lemma 3.4.43 to reduce these expressions to contain only $\tau_{f(x)}\big(q_k(x)^{-1}\big)$ and $\tau_{f(x)}\big(D_x^{\circ j}(q_k(x))\big)$, in addition to derivations of the previously calculated coefficients.

Thus, we may calculate the inverse formal irreducible power series, and therefore the Bézout coefficients, for any $q_k(x)$ in any identity resolution from the minimal polynomial. Thus, we have a complete algebraic formalism for analysing $U(\mathfrak{so}(3,\mathbb{R}))$, which will ultimately empower our derivation of real algebraic descriptions for the structure of systems with arbitrary spin. The formal irreducible power series for $q_k(x)$ which we require to do this may also be computed by this method from the polynomial form of $q_k(x)$, or directly though iterated polynomial division by $f(x)$.

3.5 Minimal Polynomial Methods in Real Unital Associative Algebras

The methods developed in Sections 3.3 and 3.4 are applicable for a wide range of fields $\mathbb{F}$. In this thesis, we are exclusively concerned with the case $\mathbb{F} = \mathbb{R}$. As such, let us specialise our discussion and derive some results for this case.

3.5.1 Irreducible Powers in $\mathbb{R}[x]$

Before we can proceed with resolving the identity, we must understand which polynomials of $\mathbb{R}[x]$ are irreducible,

Lemma 3.5.1. *Consider a monic polynomial $p(x) \in \mathbb{R}[x]$ which is irreducible over $\mathbb{R}$. Then, exactly one of the following is true:*

1. $p(x) = x - \lambda$, *where* $\lambda \in \mathbb{R}$;
2. $p(x) = (x - b)^2 + \lambda^2$, *where* $b, \lambda \in \mathbb{R}$, $\lambda \neq 0$.

Proof. To see only first and second order polynomials can be irreducible, see [68]. That the only quadratic polynomials which are irreducible are the ones above follows from the quadratic formula. □

Corollary 3.5.2. *Consider a monic, irreducible power $p(x) \in \mathbb{R}[x]$. Then, exactly one of the following is true:*

1. $p(x) = (x - \lambda)^n$, *where* $\lambda \in \mathbb{R}$, *and* $n \in \mathbb{Z}^+$;
2. $p(x) = \left((x - b)^2 + \lambda^2\right)^n$, $\lambda \neq 0$, *where* $b, \lambda \in \mathbb{R}$, *and* $n \in \mathbb{Z}^+$.

Proof. Follows from Lemma 3.5.1. □

3.5.2 Bézout Coefficients for Powers of a Real Linear Polynomial

Let us calculate an explicit expression for the inverse formal irreducible power series of $q_k(x)$ when $p_k(x)$ is an irreducible power of a linear polynomial. This is the most common case we will encounter in this thesis. We will find, in this case, the formal irreducible power series is equivalent to the Taylor series, as expected,

Lemma 3.5.3. *Consider an identity resolution in which,*

$$\begin{aligned} p_k(x) &= f_k(x)^{d_k} \\ f_k(x) &= x - \lambda, \end{aligned} \tag{3.5.1}$$

where $d_k \in \mathbb{Z}^+$, $\lambda \in \mathbb{R}$. Then, with $q_k(x)$ defined in the usual way, $q_k(x)^{-1} \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)})$ has form,

$$q_k(x)^{-1} = \sum_{n=0}^{\infty} \alpha_n(x)(x - \lambda)^n, \tag{3.5.2}$$

with, $\forall n \in \mathbb{N}$,

$$\alpha_n(x) = \begin{cases} \dfrac{1}{q_k(\lambda)} & n = 0 \\ \dfrac{1}{n!}\tau_{x-\lambda} \circ \partial_x^{\circ n}\left(q_k(x)^{-1}\right) & n \in \mathbb{Z}^+, \end{cases} \tag{3.5.3}$$

with $\tau_{x-\lambda}$ equivalent to evaluation by λ.

Proof. First, note $\partial_x(x-\lambda) = 1$. Thus, $\left(\partial_x(f_k(x))\right)^{-1} = 1$, and so $\partial_{f_k(x)} = \partial_x$. Next, note by definition $\forall p \in \mathbb{N}$, $|\alpha_p(x)| < |f_k(x)|$, and so $\partial_x(\alpha_p(x)) = 0$. Finally, since $\tau_{x-\lambda}(q_k(x)) = q_k(\lambda)$ and $\tau_{x-\lambda}(q_k(x)^{-1}q_k(x)) = 1$, we have $\tau_{x-\lambda}(q_k(x)^{-1}) = \frac{1}{q_k(\lambda)}$. Using these observations, the result follows from Theorem 3.4.53. □

Example 3.5.4. Consider the polynomial $m(x) = (x-2)^3(x^2+1)^2$. Defining,

$$p(x) = (x-2)^3$$
$$q(x) = (x^2+1)^2,$$

working within $S(\mathbb{F}[x], x-2, M_{x-2})$,and utilising Lemmas 2.3.61 and 3.4.43, we have,

$$\alpha_0(x) = \frac{1}{q(2)} = \frac{1}{25}$$
$$\alpha_1(x) = \frac{1}{1!}\tau_{x-\lambda}\big((-1)(q^{-1}(x))^2\partial_x(q(x))\big) = -\frac{8}{125}$$
$$\alpha_2(x) = \frac{1}{2!}\tau_{x-\lambda}\big(2(q^{-1}(x))^3(\partial_x(q(x)))^2 - (q^{-1}(x))^2\partial_x^{\circ 2}(q(x))\big) = \frac{38}{625},$$

and so,

$$q^{-1}(x) = \frac{1}{25} - \frac{8}{125}(x-2) + \frac{38}{625}(x-2)^2 + (x-2)^3R(x),$$

with $R(x) \in S(\mathbb{F}[x], x-2, M_{x-2})$. As such, the Bézout coefficient for $q(x)$ in the identity resolution by $m(x)$ is,

$$a(x) = \frac{1}{25} - \frac{8}{125}(x-2) + \frac{38}{625}(x-2)^2.$$

3.5.3 Bézout Coefficients for Powers of a Real Irreducible Quadratic Polynomial

Now, for completeness, let us calculate an explicit expression for the inverse formal irreducible power series of $q_k(x)$ when $p_k(x)$ is an irreducible power of a quadratic polynomial,

Lemma 3.5.5. *Consider an identity resolution in which,*

$$\begin{aligned} p_k(x) &= f_k(x)^{d_k} \\ f_k(x) &= (x-b)^2 + \lambda^2, \end{aligned} \tag{3.5.4}$$

where $d_k \in \mathbb{Z}^+$, $b, \lambda \in \mathbb{R}$, $\lambda \neq 0$. Then, with $q_k(x)$ defined in the usual way, and denoting $q_k(x)^{-1} \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)})$ by,

$$q_k(x)^{-1} = \sum_{n=0}^{\infty} \alpha_n(x)\big((x-b)^2 + \lambda^2\big)^n, \tag{3.5.5}$$

and, $\forall n \in \mathbb{N}$,

$$\tau_{f_k(x)}\big(q_k(x)\big) = \kappa_0 + \kappa_1(x-b) \tag{3.5.6a}$$

$$\alpha_n(x) = \gamma_n + \delta_n(x-b), \tag{3.5.6b}$$

we have, $\forall n \in \mathbb{N}$,

$$\alpha_n(x) = \begin{cases} \dfrac{\kappa_0 - \kappa_1(x-b)}{\kappa_0^2 + \lambda^2\kappa_1^2} & n = 0 \\ \dfrac{1}{n!}\tau_{f_k(x)} \circ D^{\circ n}_{f_k(x)}\big(q_k(x)^{-1}\big) & \\ \qquad + (x-b)\displaystyle\sum_{m=1}^{n} \frac{1}{m}\binom{2m-2}{m-1}\frac{\delta_{n-m}}{2^{2m-1}\lambda^{2m}} & n \in \mathbb{Z}^+. \end{cases} \tag{3.5.7}$$

Proof. See Appendix C.2. □

Example 3.5.6. Consider the polynomial $m(x) = (x-2)^3(x^2+1)^2$. Defining,

$$\begin{aligned} p(x) &= (x^2+1)^2 \\ q(x) &= (x-2)^3, \end{aligned}$$

and working within $S(\mathbb{F}[x], x^2+1, M_{x^2+1})$, we find $\tau_{x^2+1}(q(x)) = -2 + 11x$, and $\tau_{x^2+1}((2x)^{-1}) = -\frac{1}{2}x$. Thus, utilising Lemmas 2.3.61 and 3.4.43, we have,

$$\begin{aligned} \alpha_0(x) &= -\frac{1}{125}(2+11x) \\ \alpha_1(x) &= \frac{1}{1!}\tau_{x^2+1}\big((-1)(q^{-1}(x))^2(2x)^{-1}\partial_x(q(x))\big) + x\frac{\delta_0}{2} = -\frac{2}{625}(18+19x), \end{aligned}$$

and so,

$$q^{-1}(x) = -\frac{1}{125}(2+11x) - \frac{2}{625}(18+19x)\big(x^2+1\big) + (x^2+1)^2R(x),$$

with $R(x) \in S(\mathbb{F}[x], x^2+1, M_{x^2+1})$. As such, the Bézout coefficient for $q(x)$ in the identity resolution by $m(x)$ is,

$$a(x) = -\frac{1}{125}(2+11x) - \frac{2}{625}(18+19x)\big(x^2+1\big).$$

This result can be combined with that of Example 3.5.4 to construct the full resolution of the identity by $m(x)$.

3.6 Chapter Summary

In this chapter, we have developed a methodology for extracting information about an element of a unital associative algebra from its minimal polynomial. In so doing, we have revealed that minimal polynomials encode much of the important information about the way an operator behaves without the need for complex numbers or explicit bases. Let us now summarise the important findings of this chapter, and highlight where these results will be used in the remainder of this thesis.

Firstly, in Definition 3.3.9, we divided up a polynomial $m(x)$ into its irreducible powers,

Definition. Consider a monic polynomial $m(x) \in \mathbb{F}[x]$ and its irreducible power form,

$$m(x) = \prod_{j \in J} f_j(x)^{d_j}.$$

For all $k \in J$, we define,

$$p_k(x) \coloneqq f_k(x)^{d_k}$$
$$q_k(x) \coloneqq \prod_{j \in J \setminus \{k\}} f_j(x)^{d_j}.$$

Using these, we can construct a resolution of the identity from Definition 3.3.22, Theorem 3.3.23, and Lemma 3.4.2,

Theorem. *Consider an arbitrary monic polynomial $m(x)$, and the quotient algebra $\mathcal{Q}_{m(x)}[x]$. Then,*

$$\sum_{j \in J} \Pi_{p_j(x)}(x) = 1,$$

is a resolution of the identity of $\mathcal{Q}_{m(x)}[x]$ in terms of mutually "orthogonal" idempotent elements $\Pi_{p_k(x)}(x) \in \mathcal{Q}_{m(x)}[x]$, such that, $\forall j, k \in J$, $j \neq k$,

$$\Pi_{p_k(x)}(x) \coloneqq a_k(x) q_k(x)$$
$$|\Pi_{p_k(x)}(x)| < |m(x)|$$
$$p_k(x)\Pi_{p_k(x)}(x) = 0$$
$$\Pi_{p_k(x)}(x)\Pi_{p_k(x)}(x) = \Pi_{p_k(x)}(x)$$
$$\Pi_{p_j(x)}(x)\Pi_{p_k(x)}(x) = 0,$$

and $\forall j \in J$, $a_j(x) \neq 0$ with,

$$|a_j(x)| < |p_j(x)|.$$

We will utilise such identity resolutions extensively in Chapter 4, where we will resolve the identity using the minimal polynomials of important endomorphisms on the universal enveloping algebra $U(\mathfrak{so}(3,\mathbb{R}))$ and algebras derived from it.

To evaluate the coefficients $\{a_k(x)\}$ in such identity resolutions, in Definition 3.4.12 we defined a generalisation of the Taylor series which gives a series in any irreducible polynomial $f(x)$,

Definition. Consider an irreducible polynomial $f(x) \in \mathbb{F}[x]$, and a countable set $\{\mu_j(x) \in M_{f(x)}\}$ indexed over a subset $J \subseteq \mathbb{N}$. A formal irreducible power series in $f(x)$ is an expression $g(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,

$$g(x) = \sum_{j \in J} \mu_j(x) f(x)^j.$$

Using this, we found an explicit construction for $q_k(x)^{-1} \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)})$ from Theorems 3.4.25 and 3.4.53,

Theorem. *For all $\{q_k(x) \in \mathbb{F}[x]\}$ in an identity resolution, there exists a multiplicative inverse,*

$$q_k(x)^{-1} = \sum_{n=0}^{\infty} \alpha_n(x) f_k(x)^n \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)}),$$

where, $\forall n \in \mathbb{N}$,

$$\alpha_n(x) = \begin{cases} \tau_{f_k(x)}\big(q_k(x)^{-1}\big) & n = 0 \\ \dfrac{1}{n!}\tau_{f_k(x)} \circ D^{\circ n}_{f_k(x)}\big(q_k(x)^{-1}\big) - \displaystyle\sum_{j=0}^{n-1} \frac{1}{(n-j)!}\tau_{f_k(x)} \circ D^{\circ(n-j)}_{f_k(x)}\big(\alpha_j(x)\big) & n \in \mathbb{Z}^+. \end{cases}$$

From Theorem 3.4.38, we may calculate the $a_k(x)$ by truncation to order d_k.

Using this framework, we gave explicit expressions for the $q_k(x)^{-1}$ for both classes of real irreducible polynomials; this is the case we will require for the developments of Chapter 4. Explicitly, the case of a power of a linear polynomial is described in Lemma 3.5.3,

Lemma. *Consider an identity resolution in which,*

$$f_k(x) = x - \lambda,$$

where $\lambda \in \mathbb{R}$. Then, $q_k(x)^{-1} \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)})$ has form,

$$q_k(x)^{-1} = \sum_{n=0}^{\infty} \alpha_n(x)(x - \lambda)^n,$$

with, $\forall n \in \mathbb{N}$,

$$\alpha_n(x) = \begin{cases} \dfrac{1}{q_k(\lambda)} & n = 0 \\ \dfrac{1}{n!}\tau_{x-\lambda} \circ \partial_x^{\circ n}\big(q_k(x)^{-1}\big) & n \in \mathbb{Z}^+, \end{cases}$$

with $\tau_{x-\lambda}$ *equivalent to evaluation by* λ,

and the case of a power of an irreducible quadratic polynomial is described in Lemma 3.5.5,

Lemma. *Consider an identity resolution in which,*

$$f_k(x) = (x-b)^2 + \lambda^2,$$

where $b, \lambda \in \mathbb{R}$, $\lambda \neq 0$. *Then, denoting* $q_k(x)^{-1} \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)})$ *by,*

$$q_k(x)^{-1} = \sum_{n=0}^{\infty} \alpha_n(x)\big((x-b)^2 + \lambda^2\big)^n,$$

and, $\forall n \in \mathbb{N}$,

$$\begin{aligned} \tau_{f_k(x)}(q_k(x)) &= \kappa_0 + \kappa_1(x-b) \\ \alpha_n(x) &= \gamma_n + \delta_n(x-b), \end{aligned}$$

we have, $\forall n \in \mathbb{N}$,

$$\alpha_n(x) = \begin{cases} \dfrac{\kappa_0 - \kappa_1(x-b)}{\kappa_0^2 + \lambda^2\kappa_1^2} & n = 0 \\ \dfrac{1}{n!}\tau_{f_k(x)} \circ D_{f_k(x)}^{\circ n}\big(q_k(x)^{-1}\big) & \\ \qquad + (x-b)\displaystyle\sum_{m=1}^{n} \frac{1}{m}\binom{2m-2}{m-1}\frac{\delta_{n-m}}{2^{2m-1}\lambda^{2m}} & n \in \mathbb{Z}^+. \end{cases}$$

Chapter 4

Arbitrary Spin Algebras

4.1 Chapter Aim and Outline

In this chapter, we will show that the structure of an arbitrary spin system is fundamentally geometric, not dynamical, in character, and due entirely to the symmetries of Euclidean three-space. Furthermore, our approach will yield a complete accounting of the physically distinct observables for an arbitrary spin system, as well as explicate the physical similarities and differences between systems of differing spins. To do this, we will use the elementary formalism of Chapter 3 to derive a family of real finite-dimensional algebras which describe the structure of a system with arbitrary non-relativistic spin; we will do this directly from the Lie algebra of Euclidean three-space without the need to introduce: complex numbers; eigenstates; analytic representations; magnetic moments; energy; momentum; angular momentum; or time. The methods developed to construct these algebras are completely general, and may be adapted to derive unital associative algebras which describe systems conforming to irreducible representations of arbitrary real semisimple Lie algebras. The content of this chapter first appeared in a publication by the author [61] and, besides some notational changes, remains unaltered here. This work is structured as follows:

First, in Section 4.2, we will highlight important differences between the real Lie algebra uses in this work and the complexified Lie algebra used more commonly in physics. We will also describe how to convert the results of this chapter for use in such physical theories. Then, in Section 4.3, we will motivate the need for this algebraic analysis on physical grounds, and indicate the relationship between this work and the traditional representations used to model systems with spin. We will also motivate the use of the universal enveloping algebra in this work. Following this, in Section 4.4, we will present a method for decomposing the universal enveloping algebra using the techniques developed in Chapter 3 using the algebra's Casimir elements. This will result in a decomposition of the universal enveloping algebra into the irreducible "multipole" tensors, which form a natural basis for it and, later, will form a complete set of physically distinct observables for spin systems.

Subsequently, in Section 4.5, we will use this decomposition to derive our sought after "spin algebras" and justify our claim that they indeed describe all spins. In so doing, we will find that each spin is completely determined by its largest non-zero multipole tensor. Finally, in Section 4.6, we will contrast these new algebras with many of the existing formalisms which seek to describe arbitrary spin systems, and discuss the implications of this work on the interpretation and origin of spin in quantum mechanics.

4.2 Notation

To avoid confusion, the author wishes to restate our definition for the *real* three-dimensional Lie algebra $\mathfrak{so}(3,\mathbb{R})$,

$$\mathfrak{so}(3,\mathbb{R}) \cong (\mathrm{span}_{\mathbb{R}}(\{S_1, S_2, S_3\}), \times), \tag{4.2.1}$$

with Lie product,

$$S_a \times S_b = \sum_{c=1}^{3} \varepsilon_{abc} S_c. \tag{4.2.2}$$

This is done to avoid potential confusion by the reader, as this differs from the more familiar definition for the *complexified* three-dimensional Lie algebra $\mathfrak{so}(3,\mathbb{R})_{\mathbb{C}}$,

$$\mathfrak{so}(3,\mathbb{R})_{\mathbb{C}} \cong (\mathrm{span}_{\mathbb{C}}(\{\hat{S}_1, \hat{S}_2, \hat{S}_3\}), \times'), \tag{4.2.3}$$

with Lie product,

$$\hat{S}_a \times' \hat{S}_b = i \sum_{c=1}^{3} \varepsilon_{abc} \hat{S}_c. \tag{4.2.4}$$

The results of this and subsequent chapters are given in terms of the generators of $\mathfrak{so}(3,\mathbb{R})$ as we have defined it. In order to use these results, in physical theories where the generators of $\mathfrak{so}(3,\mathbb{R})_{\mathbb{C}}$ have instead been used, one may rigorously convert between them using the injective unital associative algebra homomorphism,

$$\begin{aligned} \phi : U(\mathfrak{so}(3,\mathbb{R})) &\to U(\mathfrak{so}(3,\mathbb{R})_{\mathbb{C}}) \\ \phi(S_a) &= -i\hat{S}_a. \end{aligned} \tag{4.2.5}$$

This map is well-defined, since it restricts to a Lie algebra homomorphism between $\mathfrak{so}(3,\mathbb{R})$ and $\mathfrak{so}(3,\mathbb{R})_{\mathbb{C}}$, $\forall a, b \in \{1,2,3\}$,

$$\phi(S_a \times S_b) = \phi(S_a) \times' \phi(S_b). \tag{4.2.6}$$

It is worth highlighting that, since we are using the real $\mathfrak{so}(3,\mathbb{R})$ we could use its isomorphism with the cross-product algebra $(\mathbb{R}^3, \times)$ to recast all the objects in this chapter in terms of tensors of three-dimensional Euclidean vectors. However, this approach does not generalise well to higher-dimensional orthogonal Lie algebras. Instead, expressing the generators of $\mathfrak{so}(3,\mathbb{R})$ in terms of bivectors as in Section 2.2.4 offers a realisation in terms of geometrical objects which generalises to arbitrary spatial dimensions. We will expand on this more in Chapter 5.5.

4.3 Motivation

4.3.1 Spin Through Representation Theory

The usual description of a spin-s system is as a finite-dimensional, irreducible representation $(\mathcal{V}^{(s)}, \rho^{(s)})$ of the real Lie algebra $\mathfrak{su}(2,\mathbb{C})$,

$$\mathfrak{su}(2,\mathbb{C}) := \operatorname{span}_{\mathbb{R}}(\{S_1, S_2, S_3\}), \tag{4.3.1}$$

of the Lie group $\mathrm{SU}(2,\mathbb{C})$, with Lie product,

$$S_a \times S_b = \sum_{c=1}^{3} \varepsilon_{abc} S_c, \tag{4.3.2}$$

and,

$$\rho^{(s)} : \mathfrak{su}(2,\mathbb{C}) \to \operatorname{End}^{[\cdot,\cdot]}(\mathcal{V}^{(s)}), \tag{4.3.3}$$

is a Lie algebra homomorphism. From here, the process for finding the irreducible representations progresses as outlined in Appendix A.1, which we will briefly summarise here.

The traditional approach[69, 1] considers instead the complexification $\mathfrak{su}(2,\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$, and uses the ladder operator basis,

$$\rho^{(s)}(S_\pm) := -i(\rho^{(s)}(S_x) \pm i\rho^{(s)}(S_y)) \tag{4.3.4a}$$

$$\rho^{(s)}(H) := -i(\rho^{(s)}(S_z)), \tag{4.3.4b}$$

to explore the representation's root system. All finite-dimensional, irreducible representations are found this way, and are labelled in relation to the value the Casimir operator,

$$S^2_{\rho^{(s)}} := \sum_{a=1}^{3} \rho^{(s)}(S_a)\circ\rho^{(s)}(S_a) = -s(s+1)\ \mathrm{id}, \tag{4.3.5}$$

takes in that representation.

The operators (4.3.4a) and (4.3.4b) have a physical interpretation through their action on the spinor space $\mathcal{V}^{(s)}$: considering spin as a form of angular momentum, the (4.3.4a) increase/decrease the amount of alignment or anti-alignment the spin angular momentum has with the z-spatial-direction, which (4.3.4b) reveals. $S^2_{\rho^{(s)}}$ is then said to represent (the negative of) the total spin angular momentum of the representation.

4.3.2 The Case for an Elementary Study of Spin

Despite the success of the traditional methods of representation theory, they yield minimal insight into the fundamental physical properties of the systems the representations describe.

Firstly, it tells us very little about the meaningful observables of the representation. The ladder operators are not observable, so the only observables the formalism highlights are $\rho^{(s)}(S_z)$ and $S^2_{\rho^{(s)}}$. Since in this picture these two observables are sufficient to characterise each representation, one might consider these to be the only relevant ones. However, it is known that there are higher-order observables within the representation, such as the quadrupole and higher-order moments[70]. As these play no role in this formalism, it is not clear if their existence is significant.

Secondly, there are issues surrounding the standard interpretation of the operators (4.3.4a), (4.3.4b), and (4.3.5). It is certainly a valid one in the complexified theory, but cannot be considered so for the original real Lie algebra $\mathfrak{su}(2,\mathbb{C})$, since (4.3.4a) nor (4.3.4b) can be defined there. This raises questions regarding the role of representations in the interpretation of physics, and the meaning of the complex numbers brought into the formalism.

Finally, the physical significance of the group $\mathrm{SU}(2,\mathbb{C})$ is also unclear. Some insight can be gained by recognising that $\mathrm{SU}(2,\mathbb{C})$ is the double cover (and in this case also universal cover) of the homogeneous symmetry group of Euclidean three-space $\mathrm{SO}(3,\mathbb{R})$. Unlike $\mathrm{SU}(2,\mathbb{C})$, $\mathrm{SO}(3,\mathbb{R})$ has a direct physical interpretation as the group of rotations. Furthermore, $\mathfrak{su}(2,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{R})$ as Lie algebras. This connection between spin and geometric symmetry is not just mathematical, it is demonstrably non-trivial: fermionic systems require a 4π rotation to return to their original state.

4.3.3 Spin from Real Euclidean Geometry

These observations lead us to motivate a more physically grounded and mathematically elementary study of spin, working exclusively with the real structures associated with the rotation group $\mathrm{SO}(3,\mathbb{R})$. As such, notions of dynamics shall be avoided. It is worth highlighting that the rotations of $\mathrm{SO}(3,\mathbb{R})$ are not fundamentally dynamical, i.e. gradual transformations of a physical space over time; they are atemporal maps encoding a relationship between two configurations of the physical space.

Since rotations preserve the Euclidean metric between any two elements of the space, they are fundamentally geometric in character, in the sense of Section 2.2.1.3. Furthermore, both the $\mathfrak{so}(3,\mathbb{R})$-action which underpins the $\mathrm{SO}(3,\mathbb{R})$-action on the space and the structure of $\mathfrak{so}(3,\mathbb{R})$ itself can be defined using only the structure of the Euclidean metric, as in Definition 2.2.45, and Definitions 2.2.60 and 2.2.61 respectively. Therefore, the mathematical structure responsible for giving rise to the irreducible spin representations is a direct result of the Euclidean metric. Thus, studying spin through the rotation group alone will directly connect it to Euclidean geometry. It is in this sense that we claim that spin is geometric in nature; we will expound this relationship in Chapter 5.

4.3.4 Spin Through Algebras

Our decision to avoid introducing complex numbers presents an immediate challenge: without their algebraic closure, we have no guarantee of eigenvalues for our operators. This makes the usual root system analysis inaccessible to us. The solution to this problem can be seen by noting that the map $\rho^{(s)}$ from the usual representation theory is mapping elements from an algebra in which only the Jacobi identity is satisfied, to elements from an associative algebra. Since, associativity is a stronger condition than satisfying the Jacobi identity, the map $\rho^{(s)}$ is effectively introducing new structure to turn the Lie algebra into an associative algebra.

Combining these observations with the presence of the identity (4.3.5), we are compelled to seek unital associative algebras $A^{(s)}$ which factor the map $\rho^{(s)}$,

$$\begin{aligned} \rho^{(s)} &= \mu^{(s)} \circ \alpha^{(s)} \\ \alpha^{(s)} &: V \to A^{(s)} \\ \mu^{(s)} &: A^{(s)} \to \mathrm{End}(\mathcal{V}^{(s)}), \end{aligned} \tag{4.3.6}$$

for which,

$$\alpha^{(s)}(a \times b) = \alpha^{(s)}(a) \bullet \alpha^{(s)}(b) - \alpha^{(s)}(b) \bullet \alpha^{(s)}(a), \tag{4.3.7}$$

holds, where $\bullet$ is the product of $A^{(s)}$. This splits the problem of representation theory in half: first one finds all unital associative algebras which embed the structure of the Lie algebra, then, if desired, one finds associative algebra representations of these. The initial step in this splitting can be considered an algebraic approach to the representation theory of Lie algebras, and shall be the focus of this chapter. We will show that such an approach yields more meaningful descriptions of all spins entirely in terms of their physical observables. From this point, the maps $\alpha^{(s)}$ we will be developing will not be written explicitly for the sake of readability.

4.3.5 The Role of $U(\mathfrak{so}(3, \mathbb{R}))$

As with representations, there are infinitely many unital associative algebras on which (4.3.7) holds, and every algebra we seek to construct satisfies this identity. This immediately compels us to consider the universal enveloping algebra $U(\mathfrak{so}(3, \mathbb{R}))$ of $\mathfrak{so}(3, \mathbb{R})$, which by Definition 2.4.18 is,

Definition 4.3.1 ($U(\mathfrak{so}(3, \mathbb{R}))$)**.** For all $v, w \in \mathfrak{so}(3, \mathbb{R})$,

$$U(\mathfrak{so}(3, \mathbb{R})) \cong \frac{T(\mathfrak{so}(3, \mathbb{R}))}{I\,(v \otimes w - w \otimes v - v \times w)}. \tag{4.3.8}$$

The universal property of $U(\mathfrak{so}(3, \mathbb{R}))$, as stated in Lemma 2.4.22 entails that it is the "most general" unital associative algebra satisfying (4.3.7); thus, all other unital associative algebras sharing this property must derive from it. Therefore, it is the natural initial object to consider for our constructions.

To progress, note by Lemma 2.4.19 that $U(\mathfrak{so}(3, \mathbb{R}))$ is infinite-dimensional. Since all representations for particle spin are finite-dimensional, it is clear that $U(\mathfrak{so}(3, \mathbb{R}))$ is too general to correspond to any of the algebras we seek. Considering the known spin representations as faithful associative algebra representations, we imply the existence of finite-dimensional associative algebras with the same structure. But since these algebras must derive from $U(\mathfrak{so}(3,\mathbb{R}))$, the question is how to construct them?

4.4 Decomposition of $U(\mathfrak{so}(3,\mathbb{R}))$

4.4.1 Considerations for the Decomposition of $U(\mathfrak{so}(3,\mathbb{R}))$

To derive finite-dimensional algebras from $U(\mathfrak{so}(3,\mathbb{R}))$, we must decompose it in a manner compatible with its algebraic structure. A priori, it is unclear how this may be achieved. To understand some of the challenges inherent in this undertaking, let us first consider how we may construct finite-dimensional algebras from the tensor algebra $T(\mathfrak{so}(3,\mathbb{R}))$.

In the case of $T(\mathfrak{so}(3,\mathbb{R}))$, we may find finite-dimensional algebras by quotienting all tensors above a certain tensor order k[45],

$$\frac{T(\mathfrak{so}(3,\mathbb{R}))}{I\left(\mathfrak{so}(3,\mathbb{R})^{\otimes(k+1)}\right)}, \tag{4.4.1}$$

where $I\left(\mathfrak{so}(3,\mathbb{R})^{\otimes(k+1)}\right)$ is the ideal generated[45] by the order-$(k+1)$ tensors. However, we cannot expect this method to produce meaningful algebras from $U(\mathfrak{so}(3,\mathbb{R}))$, since in $U(\mathfrak{so}(3,\mathbb{R}))$ tensor order is not well-defined. This can be seen from the defining identity of the algebra, $\forall a,b \in \mathfrak{so}(3,\mathbb{R})$,

$$\mathfrak{so}(3,\mathbb{R})^{\otimes 2} \ni ab - ba = a \times b \in \mathfrak{so}(3,\mathbb{R}). \tag{4.4.2}$$

To progress, we must find a scheme for constructing ideals which depends on some well-defined property of the objects of $U(\mathfrak{so}(3,\mathbb{R}))$.

To facilitate the development of such a scheme, recall Definition 2.4.33 of the adjoint action ad on the universal enveloping algebra $U(\mathfrak{so}(3,\mathbb{R}))$,

Definition.

$$\mathrm{ad}(S_a)(S_b) = S_a S_b - S_b S_a = S_a \times S_b$$

$$\mathrm{ad}(A) := B \mapsto \begin{cases} AB & A \in \mathbb{R} \\ AB - BA & A \in \mathfrak{so}(3,\mathbb{R}) \\ \mathrm{ad}(C)\circ\mathrm{ad}(D)(B) & A = CD. \end{cases}$$

Since the action of ad is entirely determined by the algebraic properties of $U(\mathfrak{so}(3,\mathbb{R}))$, it allows us to probe structures within $U(\mathfrak{so}(3,\mathbb{R}))$ which are inherently compatible with this structure. More precisely: we will use the adjoint action to find an appropriate $\mathrm{ad}(U(\mathfrak{so}(3,\mathbb{R})))$-orthogonal decomposition of $U(\mathfrak{so}(3,\mathbb{R}))$,

$$U(\mathfrak{so}(3,\mathbb{R})) \cong \bigoplus_{j=0}^{\infty} \mathcal{M}^{(j)}. \tag{4.4.3}$$

In fact, we may simplify this requirement by noting that to be closed under $\mathrm{ad}(U(\mathfrak{so}(3,\mathbb{R})))$, it is sufficient for a subspace to be closed under $\mathrm{ad}(\mathfrak{so}(3,\mathbb{R}))$. Furthermore, from its definition, it is clear that $\forall k \in \mathbb{N}$, $\mathfrak{so}(3,\mathbb{R})^{\otimes k} \subset U(\mathfrak{so}(3,\mathbb{R}))$ is closed under $\mathrm{ad}(\mathfrak{so}(3,\mathbb{R}))$. Thus, we may focus our efforts on decomposing each $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$.

If we were to use a matrix representation for $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$ as an $\mathrm{ad}(\mathfrak{so}(3,\mathbb{R}))$-module[47], we could do this by changing to a basis where $\mathrm{ad}(S_a)$ were in block diagonal form $\forall S_a$. This would become exponentially difficult however as $\dim(\mathfrak{so}(3,\mathbb{R})^{\otimes k}) = 3^k$. What this does indicate however is that the action of $\mathrm{ad}(\mathfrak{so}(3,\mathbb{R}))$ contains all the information required to isolate these closed subspaces. If we had extended scalars to $\mathbb{C}$, we could make some progress simply finding each eigenspace of a given $\mathrm{ad}(S_a)$, as these must be orthogonal. The core of these ideas is using the properties of $\mathrm{ad}(\mathfrak{so}(3,\mathbb{R}))$ to understand the structure of the space on which we are acting, i.e. $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$; we may access this information using the methods developed in Chapter 3.

4.4.2 Method of Decomposition

A priori, it is not clear how we must apply the formalism of Chapter 3 to a set of non-commuting operators such as $\mathrm{ad}(S_a), \forall S_a$. Fortunately, we may avoid this complication entirely by instead working with the commuting set of operators $\mathrm{ad}(Z(U(\mathfrak{so}(3,\mathbb{R}))))$, where,

Definition 4.4.1 ($Z(U(\mathfrak{so}(3,\mathbb{R})))$)**.** The centre $Z(U(\mathfrak{so}(3,\mathbb{R})))$ of $U(\mathfrak{so}(3,\mathbb{R}))$ is the subalgebra,

$$Z(U(\mathfrak{so}(3,\mathbb{R}))) = \{z \in U(\mathfrak{so}(3,\mathbb{R})) \,|\, zA = Az,\ \forall A \in U(\mathfrak{so}(3,\mathbb{R}))\}\,. \tag{4.4.4}$$

Lemma 4.4.2. *The $\mathbb{R}$-linear span of k-adic tensors $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$ is closed under $\mathrm{ad}(U(\mathfrak{so}(3,\mathbb{R})))$, $\forall k \in \mathbb{N}$.*

Proof. It is sufficient to show $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$ is closed under $\mathrm{ad}(\mathfrak{so}(3,\mathbb{R}))$. On 0-adics, $\mathrm{ad}(\mathfrak{so}(3,\mathbb{R}))$ acts as 0. From Lemma 2.3.63, $\forall a \in \mathfrak{so}(3,\mathbb{R})$, $\mathrm{ad}(a)$ is a derivation on $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$. Furthermore, by its definition, $\mathrm{ad}(a)$ on a 1-adic tensor gives a 1-adic tensor. Thus, $\mathrm{ad}(a)$ preserves tensor order for $k \geq 1$. □

Remark. We will use the derivation property of ad together with Lemma 4.4.2 to allow the adjoint action to act on $\mathfrak{so}(3,\mathbb{R})^{\otimes k} \subset T(\mathfrak{so}(3,\mathbb{R}))$. When we are using the adjoint action in this way will be clear from context.

Lemma 4.4.3. *$\forall z \in Z(U(\mathfrak{so}(3,\mathbb{R})))$, $\forall k \in \mathbb{N}$, an $\mathrm{ad}(z)$-orthogonal decomposition of $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$ as defined in Theorem 3.3.36 is also an $\mathrm{ad}(U(\mathfrak{so}(3,\mathbb{R})))$-orthogonal decomposition.*

Proof. $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$ is finite-dimensional, and so all operators on it have minimal polynomials by Lemma 2.4.91. Since $\forall z \in Z(U(\mathfrak{so}(3,\mathbb{R})))$, $A \in U(\mathfrak{so}(3,\mathbb{R}))$, $[z, A] = 0$ we have that $[\mathrm{ad}(z), \mathrm{ad}(A)] = \mathrm{ad}([z, A]) = 0$. Now, fix an element $w \in Z(U(\mathfrak{so}(3,\mathbb{R})))$ and use Theorem 3.3.36 to $\mathrm{ad}(w)$-orthogonally decompose $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$, resulting in,

$$\mathrm{id} = \sum_{j=1}^{n} \Pi_j(\mathrm{ad}(w))$$

Therefore, $[\Pi_j(\mathrm{ad}(w)), \mathrm{ad}(A)] = 0$, $\forall j$. Since by Lemma 4.4.2 $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$ is closed under $\mathrm{ad}(U(\mathfrak{so}(3,\mathbb{R})))$, this implies that $\mathrm{Im}(\Pi_j(\mathrm{ad}(w)))$ is closed under the action of $\mathrm{ad}(U(\mathfrak{so}(3,\mathbb{R})))$. As our choice of $w \in Z(U(\mathfrak{so}(3,\mathbb{R})))$ was arbitrary, this completed the proof. □

Thus, using $\mathrm{ad}(Z(U(\mathfrak{so}(3,\mathbb{R}))))$, we may find $\mathrm{ad}(\mathfrak{so}(3,\mathbb{R}))$-orthogonal subspaces of $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$ while avoiding the complication of non-commutativity amongst the $\mathrm{ad}(\mathfrak{so}(3,\mathbb{R}))$. It is also unnecessary to consider the whole of $Z(U(\mathfrak{so}(3,\mathbb{R})))$: it can be generated by algebraic combinations of $\mathbb{R}$ and the Casimir element,

Definition 4.4.4 (S^2)**.** The quadratic Casimir element $S^2 \in Z(U(\mathfrak{so}(3,\mathbb{R})))$ of $U(\mathfrak{so}(3,\mathbb{R}))$ is,

$$S^2 := \sum_{a=1}^{3} S_a S_a. \tag{4.4.5}$$

Therefore, only the action of $\mathrm{ad}(S^2)$ need be considered (since all $\mathrm{ad}(\mathbb{R})$ just act as scalings).

Definition 4.4.5 (E and $\mathcal{E}(k)$)**.** To simplify the following mathematics, let us introduce the notation,

$$E \coloneqq \mathrm{ad}(S^2) \tag{4.4.6a}$$

$$\mathcal{E}(k) \coloneqq \mathrm{ad}(S^2) + k(k+1)\ \mathrm{id}_{U(\mathfrak{so}(3,\mathbb{R}))}\,. \tag{4.4.6b}$$

We will defer explaining the $k(k+1)$ term in (4.4.6b) until slightly later.

Since there are infinitely many $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$ we cannot simply find the minimal polynomial of E on each of them. Instead, let us work iteratively and build up our decomposition order by order.

Definition 4.4.6 (Left multiplication)**.** We define the left multiplication action $L(A)$ on $U(\mathfrak{so}(3,\mathbb{R}))$, $\forall A \in U(\mathfrak{so}(3,\mathbb{R}))$,

$$\begin{gathered} L : U(\mathfrak{so}(3,\mathbb{R})) \to \mathrm{End}(U(\mathfrak{so}(3,\mathbb{R}))) \\ L(A) \coloneqq B \mapsto AB. \end{gathered} \tag{4.4.7}$$

We will also abuse notation slightly and use $L(a)$, $\forall a \in \mathfrak{so}(3,\mathbb{R})$, as a map $\mathfrak{so}(3,\mathbb{R})^{\otimes k} \to \mathfrak{so}(3,\mathbb{R})^{\otimes k+1}$, $\forall k \in \mathbb{N}$, or as a map $T(\mathfrak{so}(3,\mathbb{R})) \to T(\mathfrak{so}(3,\mathbb{R}))$. When we are using the left multiplication in these ways will be clear from context.

Theorem 4.4.7. *Given an E-orthogonal decomposition of $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$, we may not only promote this to an E-orthogonal decomposition of $\mathfrak{so}(3,\mathbb{R})^{\otimes k+1}$, but may refine it to isolate more* $\mathrm{ad}(U(\mathfrak{so}(3,\mathbb{R})))$*-closed subspaces.*

Proof. See Appendix D.1. □

Thus, Theorem 4.4.7 reveals that we may decompose all $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$ by starting with a scalar, then repeatedly applying $L(\mathfrak{so}(3,\mathbb{R}))$ and decomposing the result according to the action of E at each step. Along the way, we will discover relationships between families of these orthogonal subspaces which will indicate a natural set of ideals, and therefore a natural set of finite-dimensional algebras, to construct from $U(\mathfrak{so}(3,\mathbb{R}))$. This procedure will ultimately empower our derivation of real algebraic descriptions for systems with arbitrary spin.

4.4.3 Relationship Between $L(v)$ and E

To successfully execute the scheme outlined in Section 4.4.2, we must understand how $L(\mathfrak{so}(3,\mathbb{R}))$ and E interact.

Lemma 4.4.8. *$\forall v \in \mathfrak{so}(3,\mathbb{R})$,*

$$[E,[E,[E,L(v)]]] + 2\big[E^2, L(v)\big] = 0_{U(\mathfrak{so}(3,\mathbb{R}))}, \tag{4.4.8}$$

where $[\cdot,\cdot]$ is the commutator, and every implied product is composition. This identity holds on the whole of $U(\mathfrak{so}(3,\mathbb{R}))$.

Proof. See Appendix D.2. □

Definition 4.4.9. Let us denote by $\Pi_{f(A)} \in \mathrm{End}(U(\mathfrak{so}(3,\mathbb{R})))$ an idempotent for which $f(x)$ is the minimal polynomial for A on its image. Equivalently, $f(x)$ is the polynomial of least order such that,

$$f(A)\circ\Pi_{f(A)} = 0.$$

To understand the consequences of Lemma (4.4.8) it is instructive to consider its action on a subspace $\mathrm{Im}(\Pi_{E+t})$.

Lemma 4.4.10. *On $\mathrm{Im}(L(v)\circ\Pi_{E+t})$, with $t \geq -\frac{1}{4}$, E has annihilating polynomial,*

$$p(x) = (x+t)\big(x+(t+1+\sqrt{4t+1})\big)\big(x+(t+1-\sqrt{4t+1})\big). \tag{4.4.9}$$

Proof. Directly apply Lemma 4.4.8 to Π_{E+t} and factorise. □

Lemma 4.4.11. *$\sqrt{4t+1} \in \mathbb{N}$ iff $t = m(m+1)$, $m \in \mathbb{N}$.*

Proof. The backwards direction is trivial. For the forwards direction, our assumption entails $n^2 = 4t+1$ for some $n \in \mathbb{N}$. Therefore, n^2 is odd, so n is odd, and we may write $n = 2m+1$, $m \in \mathbb{N}$. Thus, $t = m(m+1)$. □

Lemma 4.4.12. *When $t = m(m+1)$, $m \in \mathbb{N}$,*

$$p(x) = \big(x+m(m+1)\big)\big(x+(m+1)(m+2)\big)\big(x+(m-1)m\big). \tag{4.4.10}$$

Proof. Direct substitution. □

Remark. For $m \neq 0$ all three roots are consecutive naturals of the form $m(m+1)$, and when $m = 0$ the roots are 0, 0, and 2.

To see the significance of this observation, let us begin our order-by-order decomposition.

Lemma 4.4.13. *On $\mathbb{R}$ and $\mathfrak{so}(3, \mathbb{R})$ the action of E has minimal polynomials,*

$$m(x) = x \tag{4.4.11a}$$

$$m'(x) = x + 2, \tag{4.4.11b}$$

respectively.

Proof. Direct application of $E = \mathrm{ad}(S^2)$ on $\alpha \in \mathbb{R}$ and $v \in \mathfrak{so}(3, \mathbb{R})$. □

Remark. Since these contain a power of a single irreducible polynomial, no further decomposition can be made here.

Lemma 4.4.14. *On all subspaces $S \subset U(\mathfrak{so}(3, \mathbb{R}))$ on which E has a minimal polynomial, its minimal polynomial must be of the form,*

$$m(x) = \prod_{n \in N} \big(x + n(n+1)\big), \tag{4.4.12}$$

for some finite $N \subset \mathbb{N}$.

Proof. Noting that $0 = 0(0 + 1)$ and $2 = 1(1 + 1)$, we see that the minimal polynomials (4.4.11a) and (4.4.11b) are of this form. Therefore, our iterative process of E-orthogonal decomposition outlined in Section 4.4.2 is initialised by subspaces annihilated by linear polynomials with natural number constant terms. Therefore, by Lemma 4.4.12, on every subspace we reach, E has minimal polynomial of the form (4.4.12). By Theorem, 4.4.7, E-orthogonal decompositions of every order of tensor in $U(\mathfrak{so}(3, \mathbb{R}))$ can be achieved by our process when starting from $\mathbb{R}$. Thus, since all elements of $U(\mathfrak{so}(3, \mathbb{R}))$ are linear combinations of k-adic tensors, on all subspaces generated by a single element E has minimal polynomials of the form (4.4.12). Therefore, if a subspace has a minimal polynomial it must be of the form (4.4.12). □

Remark. This accounts for the constant in (4.4.6b).

Remark. While (4.4.10) is always an annihilating polynomial on $\mathrm{Im}(L(v) \circ \Pi_{E+t})$, we can see from (4.4.11a) and (4.4.11b) that it is not always minimal, and that its minimality depends on the particular subspace we are acting on. Regardless, we may use it to decompose $\mathrm{Im}(L(v) \circ \Pi_{E+t})$ according to the methods of Section 3.3. If indeed the polynomial is not minimal on a given subspace, we will produce idempotents that are $0_{U(\mathfrak{so}(3,\mathbb{R}))}$ along with the complete set of non-zero ones, as in Lemma 3.3.25; as such, this will not cause any issues.

4.4.4 Recursive Decomposition of k-adic Tensors

Since we have already found $\mathbb{R}$ and $\mathfrak{so}(3,\mathbb{R})$ to be undecomposable, we may focus on decomposing k-adic tensors for $k \geq 2$. It is convenient to capture each subspace we derive as the image of a map $\mathfrak{so}(3,\mathbb{R})^{\otimes k} \to U(\mathfrak{so}(3,\mathbb{R}))$, which will usually be expressed as,

$$v_1 \otimes v_2 \otimes \ldots \otimes v_k \mapsto f(E) \circ L(v_1) \circ \Pi_{\varepsilon(m)}(v_2 \otimes \ldots \otimes v_k), \tag{4.4.13}$$

for some $m \in \mathbb{N}$ and some function f of E. The order k will correspond to the total number of $L(v_j)$ that we have applied in our recursive scheme. For these maps to make sense, their domains are $\mathfrak{so}(3,\mathbb{R})^{\otimes k} \subset T(\mathfrak{so}(3,\mathbb{R}))$, not $\mathfrak{so}(3,\mathbb{R})^{\otimes k} \subset U(\mathfrak{so}(3,\mathbb{R}))$.

We may describe $\mathrm{Im}(L(v) \circ \Pi_{\varepsilon(m)})$ conveniently by combining the left multiplication and decomposition by E steps into three new actions.

Lemma 4.4.15. *Let $\Pi_{\varepsilon(m)} \in \mathrm{End}(U(\mathfrak{so}(3,\mathbb{R})))$ be a map on whose image E has minimal polynomial $\varepsilon(m)$, $m \in \mathbb{N}$, then $\forall v \in \mathfrak{so}(3,\mathbb{R}), B \in \mathfrak{so}(3,\mathbb{R})^{\otimes k-1}$,*

$$L(v) \circ \Pi_{\varepsilon(m)}(B) = \big(L^{\downarrow}(v) + L^{-}(v) + L^{\uparrow}(v)\big) \circ \Pi_{\varepsilon(m)}(B) \tag{4.4.14}$$

where,

$$L^{\downarrow}(v) \circ \Pi_{\varepsilon(m)}(B) \coloneqq \begin{cases} 0 & m = 0 \\ \dfrac{\varepsilon(m) \circ \varepsilon(m+1)}{4m(2m+1)} \circ L(v) \circ \Pi_{\varepsilon(m)}(B) & m \in \mathbb{Z}^+ \end{cases} \tag{4.4.15a}$$

$$L^{-}(v) \circ \Pi_{\varepsilon(m)}(B) \coloneqq \begin{cases} \dfrac{(E - 2\,\mathrm{id}) \circ \varepsilon(1)}{-4} \circ L(v) \circ \Pi_{\varepsilon(0)}(B) & m = 0 \\ \dfrac{\varepsilon(m-1) \circ \varepsilon(m+1)}{-4m(m+1)} \circ L(v) \circ \Pi_{\varepsilon(m)}(B) & m \in \mathbb{Z}^+ \end{cases} \tag{4.4.15b}$$

$$L^{\uparrow}(v) \circ \Pi_{\varepsilon(m)}(B) \coloneqq \begin{cases} \dfrac{\varepsilon(0) \circ \varepsilon(0)}{4} \circ L(v) \circ \Pi_{\varepsilon(0)}(B) & m = 0 \\ \dfrac{\varepsilon(m-1) \circ \varepsilon(m)}{4(m+1)(2m+1)} \circ L(v) \circ \Pi_{\varepsilon(m)}(B) & m \in \mathbb{Z}^+. \end{cases} \tag{4.4.15c}$$

Proof. This follows directly from E-orthogonally decomposing $\mathrm{Im}(L(v) \circ \Pi_{\varepsilon(m)})$ using Lemma 4.4.8. □

Definition 4.4.16. We will call the $L^{\downarrow}(v)/L^{-}(v)/L^{\uparrow}(v)$ the "step-down/step-level/step-up by $v \in \mathfrak{so}(3,\mathbb{R})$" operators respectively.

Corollary 4.4.17. $\forall m \in \mathbb{N}$,

$$\mathcal{E}(m-1)\circ L^{\downarrow}(v)\circ\Pi_{\mathcal{E}(m)}(B) = 0 \tag{4.4.16a}$$

$$\left.\begin{array}{ll} \varepsilon(0)^2\circ L^{-}(v)\circ\Pi_{\mathcal{E}(0)}(B) & m=0 \\ \varepsilon(m)\circ L^{-}(v)\circ\Pi_{\mathcal{E}(m)}(B) & m\in\mathbb{Z}^+ \end{array}\right\} = 0 \tag{4.4.16b}$$

$$\mathcal{E}(m+1)\circ L^{\uparrow}(v)\circ\Pi_{\mathcal{E}(m)}(B) = 0. \tag{4.4.16c}$$

Proof. This follows by construction. □

Remark. We shall soon see that, in fact, $L^{-}(v)\circ\Pi_{\mathcal{E}(0)} = 0_{U(\mathfrak{so}(3,\mathbb{R}))}$ due to (4.4.10) not being minimal on $\mathrm{Im}(L(v)\circ\Pi_{\mathcal{E}(m)})$.

Using the step operators, our scheme of decomposition for $U(\mathfrak{so}(3,\mathbb{R}))$ is equivalent to applying all sequences of step operations by $v_j \in \mathfrak{so}(3,\mathbb{R})$ to 1 and keeping those that yield non-trivial subspaces. We are guaranteed to decompose $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$ fully in terms of the non-trivial subspaces reached after k steps due to (4.4.14) holding at every step. This process is summarised graphically in Figure 4.1.

4.4.5 Decomposition into Multipoles

Figure 4.1 shows a part of the E-orthogonal decomposition of $U(\mathfrak{so}(3,\mathbb{R}))$, and combines results from the whole of this Chapter. This shown part, which may be verified through explicit computation, reveals that $\forall k \in \{0,\dots,4\}$ there is exactly one subspace annihilated by $\varepsilon(k)$ amongst all subspaces reached in k steps from 1. This subspace was reached by stepping-up from the unique subspace reached in $k-1$ steps that is annihilated by $\varepsilon(k-1)$. Furthermore, there are no subspaces annihilated by $\varepsilon(k)$ reachable in fewer than k steps. Let us extend these observations to $\forall k \in \mathbb{N}$.

Definition 4.4.18 (Multipoles)**.** Let us recursively define a family of maps,

$$\{M^{(k)} : \mathfrak{so}(3,\mathbb{R})^{\otimes k} \to U(\mathfrak{so}(3,\mathbb{R}))|\, k \in \mathbb{N}\},$$

such that,

$$\begin{aligned} M^{(0)}(\alpha) &= \alpha \\ M^{(k+1)}(v\otimes B) &= L^{\uparrow}(v)\circ M^{(k)}(B). \end{aligned} \tag{4.4.17}$$

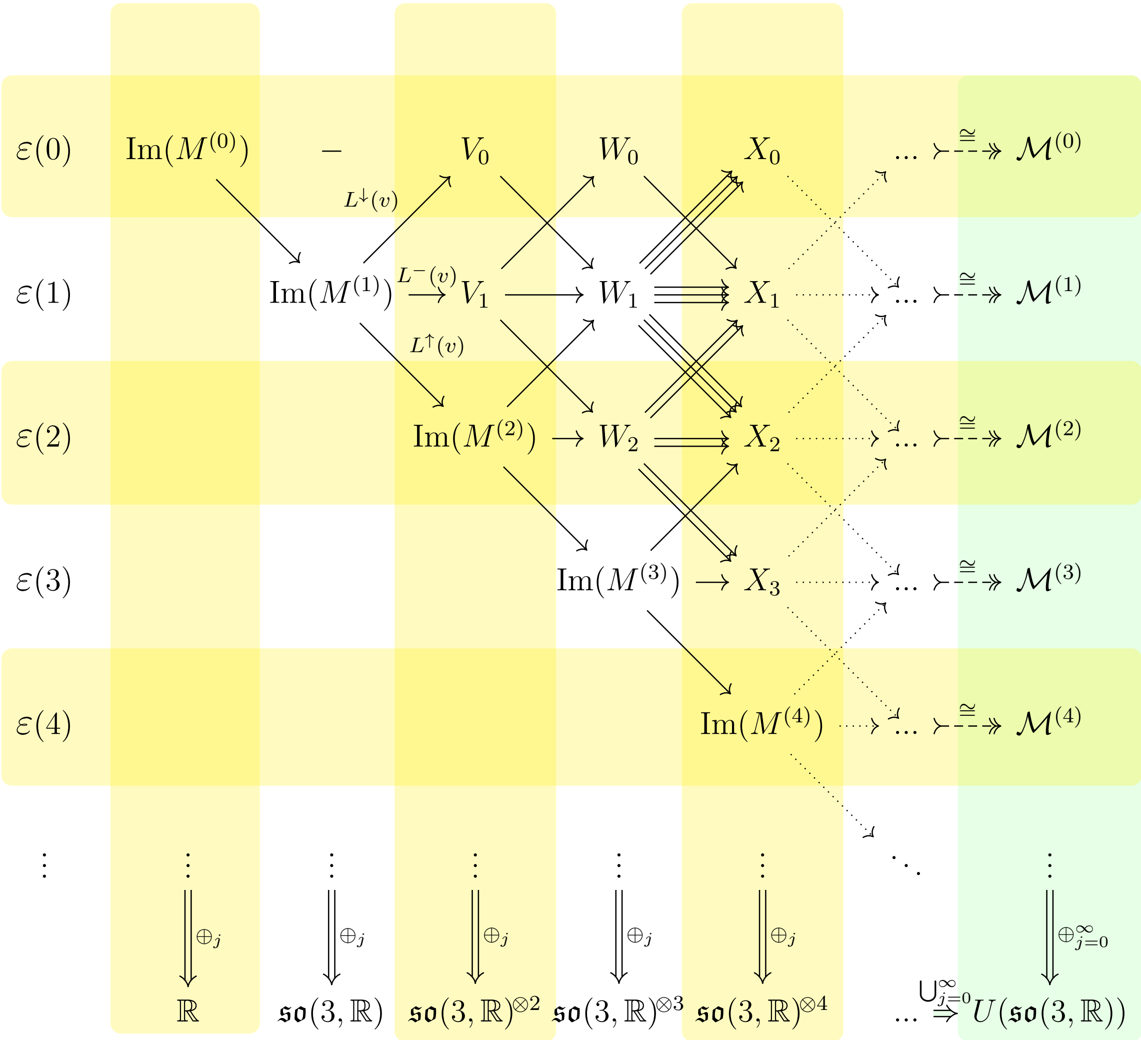

Figure 4.1: Diagramatic representation of the decomposition of $U(\mathfrak{so}(3,\mathbb{R}))$, where $\varepsilon(j)(V_j) = 0$. The vertical yellow bands contain subspaces of a given tensor order. The horizontal yellow bands contain subspaces annihilated by a given polynomial of E. The green vertical band contains the closures of the unions of all subspaces in each yellow band, expressed as modules $\mathcal{M}^{(k)}$ of multipoles $\mathrm{Im}(M^{(k)})$ over the centre $Z(U(\mathfrak{so}(3,\mathbb{R})))$. These objects will be defined in Section 4.4.5.

We refer both to $M^{(k)}$ (and its image) as the "multipole(s) of order k", and the family $\{M^{(k)}\}$ as the "multipoles". That the multipoles exist is given by Lemma 4.4.15. We will also use a component notation,

$$M_{a_1 a_2 \dots a_k} := M^{(k)}(S_{a_1} \otimes S_{a_2} \otimes \dots \otimes S_{a_k}). \qquad (4.4.18)$$

Remark. We see by definition, $\forall k \in \mathbb{N}$,

$$\mathcal{E}(k) \circ M^{(k)} = 0. \qquad (4.4.19)$$

Lemma 4.4.19. $\forall p, q \in \mathbb{N}$, $p \neq q$,

$$\mathrm{Im}(M^{(p)}) \cap \mathrm{Im}(M^{(q)}) = \{0\}.$$

Proof. Suppose $d \in \mathrm{Im}(M^{(p)}) \cap \mathrm{Im}(M^{(q)})$. Then,

$$\mathcal{E}(p)(d) = \mathcal{E}(q)(d) = 0,$$

and so,

$$\big(p(p+1) - q(q+1)\big)d = 0.$$

Since $p \neq q$, $d = 0$. □

Lemma 4.4.20 (Properties of Multipoles)**.** *$\forall k \in \mathbb{N}$, each multipole $M^{(k)}$, is closed under the adjoint action of $U(\mathfrak{so}(3,\mathbb{R}))$,*

$$\forall A \in U(\mathfrak{so}(3,\mathbb{R})),\ \mathrm{ad}(A)\circ M^{(k)} = M^{(k)}\circ \mathrm{ad}(A), \tag{4.4.20a}$$

is totally symmetric,

$$\forall \tau \in S_k,\ \ M^{(k)}\circ\tau = M^{(k)}, \tag{4.4.20b}$$

and for $k \geq 2$, is contractionless,

$$\forall m \neq n \in \{1,...,k\},\ \sum_{a_m,a_n=1}^{3} \delta_{a_m a_n} M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{a_j}\Big) = 0. \tag{4.4.20c}$$

Proof. See Appendix D.3. □

Lemma 4.4.21. *Step-levels and step-downs from multipoles $M^{(k)}$, $\forall k \in \mathbb{N}$, can be written in terms of lower order multipoles,*

$$L^{\downarrow}(S_a)\circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) = \begin{cases} \dfrac{L\big(4S^2 + (k-1)(k+1)\big)}{4(4k^2-1)}\circ\displaystyle\sum_{p=1}^{k}\Bigg(& \\ \quad (2k-1)\delta_{ab_p} M^{(k-1)}\Big(\displaystyle\bigotimes_{j\neq p} S_{b_j}\Big) & k \geq 2 \\ \quad -\displaystyle\sum_{q=1,q\neq p}^{k} \delta_{b_p b_q} M^{(k-1)}\Big(S_a \otimes \bigotimes_{j\neq p,q} S_{b_j}\Big)\Bigg) & \\ \dfrac{1}{3}S^2\delta_{ab_1} & k = 1 \\ 0 & k = 0 \end{cases} \tag{4.4.21a}$$

$$L^{-}(S_a)\circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) = \begin{cases} \frac{1}{2}\sum_{p=1}^{k}\sum_{c=1}^{3}\varepsilon_{ab_pc}M^{(k)}\Big(S_c\otimes\bigotimes_{j\neq p}S_{b_j}\Big) & k\geq 2 \\ \frac{1}{2}\sum_{c=1}^{3}\varepsilon_{ab_1c}S_c & k=1 \\ 0 & k=0. \end{cases} \tag{4.4.21b}$$

Proof. See Appendix D.4. □

Lemma 4.4.22 (Minimal Polynomial for E on Multipoles)**.** (4.4.10) *is minimal on* $\mathrm{Im}(L(v)\circ M^{(k)})$ *for* $k\in\mathbb{Z}^+$*, and* (4.4.11b) *is minimal on* $\mathrm{Im}(L(v)\circ M^{(0)})$.

Proof. See Appendix D.5. □

Corollary 4.4.23. *Lemma 4.4.22 implies,*

$$L^{-}(v)\circ M^{(0)} = 0 \tag{4.4.22a}$$

$$L^{\uparrow}(v)\circ M^{(0)} = L(v)\circ M^{(0)}. \tag{4.4.22b}$$

Proof. (4.4.22a) follows from $\varepsilon(2)\circ M^{(1)}=0$, and (4.4.22b) from applying (4.4.22a) to (4.4.14). □

Table 4.1 shows the image of $M^{(k)}$ on a k-adic tensor for the first five values of k. These objects look like non-commutative generalisations of the Cartesian multipole tensors, which supports our naming of the maps $M^{(k)}$ "multipoles". The multipoles agree with the forms implied by [71], though their algebraic properties and interrelationships are much clearer from this method.

The significance of the multipoles to our decomposition can be seen by considering the following.

Theorem 4.4.24 (Multipole Decomposition of $U(\mathfrak{so}(3,\mathbb{R}))$)**.**

$$U(\mathfrak{so}(3,\mathbb{R})) \cong \bigoplus_{j=0}^{\infty}\mathcal{M}^{(j)}, \tag{4.4.23}$$

where,

$$\mathcal{M}^{(j)} = \mathrm{span}_{\mathbb{R}[S^2]}(\mathrm{Im}(M^{(j)})), \tag{4.4.24}$$

and $\mathbb{R}[S^2]$ *is the real polynomial ring over* S^2.

Proof. Lemma (4.4.15) and Lemma 4.4.21 applied to a multipole $M^{(k)}$, $\forall k\in\mathbb{N}$, reveals that left multiplication of a multipole by $\mathfrak{so}(3,\mathbb{R})$ can be written as $\mathbb{R}[S^2]$-linear combinations of multipoles. Since we have defined $M^{(0)}=\mathbb{R}$, we may therefore

Multipole	Explicit Form
M	1
M_a	S_a
M_{ab}	$\sum_{\sigma\in\mathrm{Perm}(\{a,b\})} \frac{1}{2}S_{\sigma(a)}S_{\sigma(b)} - \frac{1}{6}\delta_{\sigma(a)\sigma(b)}S^2$
M_{abc}	$\sum_{\sigma\in\mathrm{Perm}(\{a,b,c\})} \frac{1}{6}S_{\sigma(a)}S_{\sigma(b)}S_{\sigma(c)} - \frac{1}{30}\delta_{\sigma(a)\sigma(b)}(3S^2+1)S_{\sigma(c)}$
M_{abcd}	$\sum_{\sigma\in\mathrm{Perm}(\{a,b,c,d\})} \frac{1}{24}S_{\sigma(a)}S_{\sigma(b)}S_{\sigma(c)}S_{\sigma(d)} - \frac{1}{168}\delta_{\sigma(a)\sigma(b)}(6S^2+5)S_{\sigma(c)}S_{\sigma(d)} + \frac{1}{280}\delta_{\sigma(a)\sigma(b)}\delta_{\sigma(c)\sigma(d)}(S^2+2)S^2$

Table 4.1: Images of the multipoles $k = 0, \ldots, 4$ on k-adic tensors.

write any k-adic tensors, and thus any finite $\mathbb{R}$-linear combination of k-adic tensors, using multipoles. Therefore, $U(\mathfrak{so}(3,\mathbb{R}))$ is a sum space of the $\{\mathcal{M}^{(j)}\}$. That this is a direct sum follows from Lemma 4.4.19. □

Remark. Theorem 4.4.24 enables us to find the minimal polynomial of E on any subspace of $U(\mathfrak{so}(3,\mathbb{R}))$ via Lemma 4.4.12. It also gives us a description of $U(\mathfrak{so}(3,\mathbb{R}))$ of exactly the right form to derive real algebraic descriptions for systems with arbitrary spin; we will do so in the next section.

4.5 The Spin Algebras

Unlike a tensor order decomposition of $U(\mathfrak{so}(3,\mathbb{R}))$, each summand in the multipole decomposition (4.4.24) is trivially intersecting. Thus, we may derive a family of algebras from it.

Definition 4.5.1 (Spin-$\frac{k}{2}$ Algebra)**.**

$$A^{(\frac{k}{2})} := \frac{U(\mathfrak{so}(3,\mathbb{R}))}{I\left(\mathrm{Im}(M^{(k+1)})\right)}. \tag{4.5.1}$$

Lemma 4.5.2. $\forall k \in \mathbb{N}$, $A^{(\frac{k}{2})}$ *is finitely generated.*

Proof. The quotient by $I\left(M^{(k+1)}\right)$ imposes the identity $M^{(k+1)} = 0$ on $A^{(\frac{k}{2})}$. By (4.4.17), this implies $M^{(n)} = 0$, $\forall n \geq k+1$. $\square$

What is not obvious is that this process will yield a finite-dimensional real algebra, since each summand in (4.4.24) is a module over $Z(U(\mathfrak{so}(3,\mathbb{R})))$.

Lemma 4.5.3. *In* $A^{(\frac{h}{2})}$, $\forall k \in \mathbb{N}$,

$$L(S^2) = L\left(\frac{-k(k+2)}{4}\right). \tag{4.5.2}$$

Proof. See Appendix D.6. $\square$

Remark. Since, (4.5.2) applies to the whole of $A^{(\frac{k}{2})}$, this means that in $A^{(\frac{k}{2})}$ the Casimir element S^2 becomes a real scalar. Reindexing $k = 2s$, we see that $S^2 = -s(s+1)$ in $A^{(s)}$, exactly what is expected for the spin-s representation.

In fact, this connection is more extensive than this.

Lemma 4.5.4.

$$\dim\left(\mathrm{Im}(M^{(k)})\right) = 2k+1 \tag{4.5.3}$$

Proof. See Appendix D.7. $\square$

Remark. Accordingly, $A^{(\frac{k}{2})}$ has dimension $\sum_{j=0}^{k} 2j+1 = (k+1)^2 = (2s+1)^2$. This is exactly the complex dimension of the operators in the usual complexified spin-s representation.

Conjecture 4.5.5. $\mathrm{Im}(M^{(k+1)}) = \{0\}$ *implies that if k is odd,*

$$\prod_{j=0}^{\frac{1}{2}(k-1)} \left(S_a S_a + (j+\frac{1}{2})^2\right) = 0, \tag{4.5.4}$$

and if k is even,

$$S_a \prod_{j=1}^{\frac{1}{2}k} \left(S_a S_a + j^2\right) = 0. \tag{4.5.5}$$

Evidence. Using the Table 4.1, we may calculate $M^{(k+1)}\left(\bigotimes_{j=1}^{k+1} S_a\right)$ in $A^{(\frac{k}{2})}$. This is in full agreement with the predicted spectra above. Further results may be easily found using Corollary D.3.9, from which we find the recurrence relation,

$$\begin{aligned} M^{(k+1)}\Big(\bigotimes_{j=1}^{k+1} S_a\Big) &= L(S_a)\circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_a\Big) \\ &\quad - \frac{k^2}{4(4k^2-1)} L\big(4S^2 + (k-1)(k+1)\big) M^{(k-1)}\Big(\bigotimes_{j=1}^{k-1} S_a\Big), \end{aligned}$$

where S^2 will depend on the algebra we are working in.

Remark. This successfully predicts the complete eigenspectrum expected for our basis.

Due to this correspondence we are justified in naming the algebra $A^{(s)}$ the "spin-s algebra". More concretely,

Conjecture 4.5.6. *The spin-s representation is simply an associative algebra representation of $A^{(s)}$, and derives its bulk structure from it.*

In deriving the algebras $A^{(s)}$, we have established that any spin may be specified entirely by its largest non-zero multipole, and completely described by: a finite collection of multipoles $\{M^{(n)}|\, n \in \{0, ..., 2s\}\}$; the identity $L(S^2) = L(-s(s+1))$; and a multiplication table, the start of which is given in Table 4.2. Such a table may be extended using the explicit forms of multipoles in Table 4.1, and the step identities of Section (4.4.5).

This characterisation of the spin of a system in terms of its physical properties offers a more fundamental picture of the phenomenon than has been possible up to now. In traditional quantum mechanical formulations, spin is usually considered instead in terms of the eigenstates of the dipole operator. Assuming the truth of Conjecture 4.5.5, it is clear that the traditional view mathematically requires complex structure for these eigenstates to be realisable; furthermore, this description requires some direction in space to be used as a reference to describe the alignment of these eigenstates. The multipole formulation requires neither of these additions, and empowers a powerful yet descriptive approach to modelling systems with spin.

4.6 Discussion

4.6.1 Mathematical Comparison with Other Non-Relativistic Models

The spin algebras $A^{(s)}$ with $s = \frac{1}{2}$ and $s = 1$ are completely consistent with the Clifford and Kemmer algebras on $\{S_a\}$ defined by,

$$S_a S_b + S_b S_a + \frac{1}{2}\delta_{ab} = 0 \tag{4.6.1a}$$

$$S_a S_b S_c + S_c S_b S_a + \delta_{ab} S_c + \delta_{bc} S_a = 0, \tag{4.6.1b}$$

RC	M	M_e	M_{ef}
M	M	M_e	M_{ef}
M_a	M_a	$\frac{1}{3}\delta_{ae}S^2M + \frac{1}{2}\varepsilon_{sae}M_s + M_{ae}$	$\frac{1}{60}(3+4S^2)(-2\delta_{ef}M_a + 3\delta_{ae}M_f + 3\delta_{af}M_e) + \frac{1}{2}(\varepsilon_{sae}M_{sf} + \varepsilon_{saf}M_{se}) + M_{aef}$
M_{ab}	M_{ab}	$\frac{1}{60}(3+4S^2)(-2\delta_{ab}M_e + 3\delta_{ae}M_b + 3\delta_{be}M_a) + \frac{1}{2}(\varepsilon_{sae}M_{sb} + \varepsilon_{sbe}M_{sa}) + M_{abe}$	$\frac{1}{180}S^2(3+4S^2)(-2\delta_{ab}\delta_{ef} + 3\delta_{ae}\delta_{bf} + 3\delta_{af}\delta_{be})M + \frac{1}{40}(3+4S^2)(\delta_{ae}\varepsilon_{sbf} + \delta_{af}\varepsilon_{sbe} + \delta_{be}\varepsilon_{saf} + \delta_{bf}\varepsilon_{sae})M_s + \frac{1}{84}(15+4S^2)(-4\delta_{ab}M_{ef} - 4\delta_{ef}M_{ab} + 3\delta_{ae}M_{bf} + 3\delta_{af}M_{be} + 3\delta_{be}M_{af} + 3\delta_{bf}M_{ae}) + \frac{1}{2}(\varepsilon_{sae}M_{sbf} + \varepsilon_{saf}M_{sbe} + \varepsilon_{sbe}M_{saf} + \varepsilon_{sbf}M_{sae}) + M_{abef}$

Table 4.2: A partial table of multiplication for multipoles. Repeated indices in the same term are summed over.

respectively. The identity (4.6.1a) for $\{S_a\}$ is directly implied by $M^{(2)} = 0$ in $A^{(\frac{1}{2})}$, and the identity (4.6.1b) for $\{S_a\}$ is a consequence of $M^{(3)} = 0$ in $A^{(1)}$, but requires some manipulation of the decomposed identity map for 3-adic tensors to prove. Larger Clifford and Kemmer algebras which contain their $\{S_a\}$ counterparts are definable on the physical three-space[22, 36, 37] using Definitions 2.4.41 and 2.4.50, however it is unclear if any instructive comparisons can be made between them and the spin algebras.

The counterparts to the Clifford and Kemmer algebras for spins greater than 1 are mostly unknown. The traditional method for constructing real, algebraic theories for arbitrary spins is to form $\mathrm{Cl}_3(\mathbb{R})^{\otimes k}$ from the Clifford algebra $\mathrm{Cl}_3(\mathbb{R})$[38], wherein we may find a subalgebra which describes spin-$\frac{k}{2}$; this is similar to forming a single system of arbitrary spin from multiple spin-$\frac{1}{2}$ systems. However, this approach is wasteful in the sense that the majority of $\text{Cl}_3(\mathbb{R})^{\otimes k}$ does not describe our desired spin, and consistently isolating the correct subalgebra can be challenging a priori. This process also introduces an interpretational problem: if this is necessary to describe a higher spin system algebraically, are all such systems necessarily composite? This challenges the notion that, for example, spin-1 particles like the photon can be fundamental. If they can be, then does this mean the algebraic approach is inappropriate for such systems? The $A^{(s)}$ provide a solution to these problems, as they: can be constructed systematically without needing an algebra for a lower spin; avoid the algebraic substructure the traditional method necessarily imparts.

Racah's spherical tensor operator formalism[42] is also related to the spin algebras: if we complexify $U(\mathfrak{so}(3,\mathbb{R}))$ and choose some preferred primary element of $\mathfrak{so}(3,\mathbb{R})$, we can associate each of the $2k+1$ independent components of the multipole $M^{(k)}$ with a unique component of the rank-k spherical tensor operator. The commutator between two spherical tensor operators is given by the adjoint action of one upon the other. As a particular choice of basis, the complexified spin algebras could be written in terms of the spherical tensor operators. However, since defining the spherical tensor operators requires complex numbers and a preferred element of $\mathfrak{so}(3,\mathbb{R})$, the multipole spin algebras are more elementary and general.

The most important comparison yet to be made is with the standard matrix representations[1] of a spin-s system. As discussed in Section 4.5, the matrix representations are associative algebra representations of the $A^{(s)}$, and thus the underlying structures of both theories are identical. However, there are certain aspects of the standard formalism that are implied but not immediately accessible in the real algebraic theory. For example, the non-zero eigenvalues of S_a predicted by (4.5.4) and (4.5.5) are pure imaginary, so projectors into the corresponding eigenspaces cannot be constructed within the real $A^{(s)}$. Since this is the case $\forall M^{(2m+1)}$, $m \in \mathbb{N}$, it suggests these multipoles are not observable, unlike the $M^{(2m)}$, $\forall m \in \mathbb{N}$, which have real eigenvalues. In the usual matrix representations, the customary extension of scalars to $\mathbb{C}$ enables the definition of a complete set of observable multipoles with constructible eigenstate projectors.

Though it may appear so, this is not a defect within the spin algebra formalism, nor is it an indication that the transition to matrix representations is essential to do physics. Instead, it indicates that observable multipoles and spin eigenstates are not fundamental in the description of spin; rather, they are an emergent phenomenon, predicted by a larger algebraic theory, specifically one with some real, central, algebraic element which squares to a negative real. Though this might sound contrived, quantum mechanics expressed as a real algebra contains such a structure; this implies that, in physics, observable spin multipole moments and eigenstates are an emergent, non-trivial prediction of quantum mechanics.

4.6.2 A Comment on some Relativistic Models

Since we have developed the spin algebras in a very restricted, non-relativistic setting, any comparison of them with objects from relativistic or more physically complete domains will be qualitative at best. However, in the case of Weinberg[32] and later Giraud et al.[33, 44] the comparison is more precise. They define tensorial objects which are totally symmetric, and contractionless out of spin operators extended to Minkowski space. These spin operators have the identity as their time-like element, and the resulting tensors are similar to our multipoles, but lack any particular relationship with E. This makes the significance of these objects to the symmetries of Minkowski space much less clear than the relationship the multipoles have to rotational symmetry.

4.6.3 Applications of Spin Algebras

The spin algebras $A^{(s)}$ derived in this chapter are both of theoretical and practical use. Firstly, as has already been discussed, once complexified in some way, each $A^{(s)}$ offers a comprehensive account of all the meaningful, orthogonal observables associated with the spin of the system we are modelling. As such, we may use the multipoles to exhaustively construct arbitrary Hamiltonians for the system by linear combination. These Hamiltonians include all possible interactions between, for example, the system's spin and its environment.

For example, in spin-1 systems, spin quadrupole interactions are often included by squares of a single spin generator[72, 73]; examining $A^{(1)}$ reveals that terms like M_{xy} are missing from both single- and two-site interactions, and that using S_z^2 instead of M_{zz} in the Hamiltonian inadvertently introduces a constant energy shift due to the presence of a monopole M term. This empowers the model builder to more systematically and precisely explore the resulting physics.

Similarly, considering a model of a composite system formed from a tensor product of two or more spin algebras, we can easily account for all possible interactions between the multipoles of the systems in the Hamiltonian for the combined system. In both of these cases, the orthogonal nature of the multipoles ensures that the resulting parametrisation of the Hamiltonian is physically relevant.

Secondly, as the product between two multipoles is knowable in closed-form in terms of other multipoles, as in Table 4.2, a system of arbitrary spin can be modelled on a computer without the use of matrices. This description may be particularly advantageous when describing large collections of systems with spin, as the dimensionality of matrices grows exponentially with particle number, whereas the number of multipoles from different particles in a term of the combined algebra grows only linearly.

4.6.4 Applications of the Decomposition Procedure

It is also worth highlighting that the methods developed in this chapter to derive the spin algebras may also be applied in other areas of physics to study more extensive or exotic symmetries. In particular, relativistic space-time symmetries such as $\mathrm{SO}(3,1,\mathbb{R})$, gauge symmetries such as the $\mathrm{SU}(3,\mathbb{C})$ which governs the strong force, and dynamical symmetries such as the $\mathrm{Sp}(6,\mathbb{R})$ which governs the linear symmetries of both Hamilton's equations[74] and the non-relativistic canonical commutation relations[75], are obvious candidates which immediate applications within physics. Conducting studies of this kind may yield new insights into systems governed by these symmetries, as have been gained through the present study.

4.6.5 Interpretational Comparison of Spin with Standard Descriptions

In this work, we have established that $A^{(s)}$ captures the essential structure of a spin-s system. By using real algebraic methods, we have shown that this structure can be derived without the use of dynamics, matrix representations, or complex numbers, amongst other things; they require only those structures naturally associated with the geometric symmetry group $\mathrm{SO}(3,\mathbb{R})$ of Euclidean three-space. As such, we must conclude that spin is fundamentally a geometric quality, not a dynamical one.

In most formulations of quantum mechanics, it is assumed that spin is an intrinsic form of angular momentum. Other more conservative descriptions, regard it as some fundamental property that *transforms like* and phenomenologically *behaves like* angular momentum. In Section 4.6.1, we highlighted the differences between our description and the standard matrix theory these formulations refer to. Accordingly, the present work supports the latter view, insofar as it has revealed that there are actually two distinct, but equally relevant, notions of spin in physics.

The first notion of spin is the dynamical "phenomenological spin", which is the angular momentum-like property familiar to all physicists. This spin is of primary interest to those wishing to make phenomenological predictions of a physical system from a model. The second notion of spin is the geometric "fundamental spin", which is derived solely from rotational symmetry and determines the basic properties of the phenomenological spin. This spin is of primary interest to those interested in quantum foundations due to the insight it offers into our physical theories. The study of this spin, and the fundamental structures that underpin other properties of physical objects, reveal which aspects of a phenomenon are fundamental, and which are actually emergent, and in so doing enrich our understanding of physics.

4.7 Chapter Summary

In this chapter, we discovered that complex numbers, angular momentum, quantum mechanics, and special relativity are not required to develop the structure underlying a system of arbitrary spin. In fact, only the structure of the generators of rotations $\mathfrak{so}(3,\mathbb{R})$ of Euclidean three-space is required. This suggests that spin is fundamentally geometric, not dynamical, in character. Let us now summarise the important findings of this chapter, and highlight where these results will be used in the remainder of this thesis.

Through Definitions 2.4.28 and 4.4.18, and Theorem 4.4.24, and the methods derived in Chapter 3, we developed a method of decomposition for the universal enveloping algebra $U(\mathfrak{so}(3,\mathbb{R}))$ of $\mathfrak{so}(3,\mathbb{R})$ using its centre $Z(U(\mathfrak{so}(3,\mathbb{R})))$ under the adjoint action ad in terms of the physically distinct multipole tensors,

Theorem.

$$U(\mathfrak{so}(3,\mathbb{R})) \cong \bigoplus_{j=0}^{\infty} \mathcal{M}^{(j)},$$

where,

$$\mathcal{M}^{(j)} = \mathrm{span}_{\mathbb{R}[S^2]}(\mathrm{Im}(M^{(j)})),$$

with $\forall k \in \mathbb{N}$, $\forall \alpha \in \mathbb{R}$, $\forall v \in \mathfrak{so}(3, \mathbb{R})$, $\forall B_k \in \mathfrak{so}(3, \mathbb{R})^{\otimes k}$,

$$M^{(k)} : \mathfrak{so}(3, \mathbb{R})^{\otimes k} \to U(\mathfrak{so}(3, \mathbb{R}))$$
$$M^{(0)}(\alpha) = \alpha$$
$$M^{(k+1)}(v \otimes B_k) = \frac{\mathrm{ad}(S^2 + k(k-1)) \circ \mathrm{ad}(S^2 + k(k+1))}{4(k+1)(2k+1)} \big(v \otimes M^{(k)}(B_k)\big),$$

and $\mathbb{R}[S^2]$ *is the real polynomial ring over* S^2.

From this decomposition, in Definition 4.5.1 we derived the spin algebras, which by Lemma 4.5.3 are real unital associative algebras that capture the properties of the algebra of spin operators for an arbitrary spin system,

Definition. For all $k \in \mathbb{N}$,

$$A^{(\frac{k}{2})} := \frac{U(\mathfrak{so}(3, \mathbb{R}))}{I\left(\mathrm{Im}(M^{(k+1)})\right)},$$

Lemma. *In* $A^{(\frac{k}{2})}$, $\forall k \in \mathbb{N}$,

$$L(S^2) = L\left(\frac{-k(k+2)}{4}\right).$$

By their construction, these algebras reveal that the spin of a system is determined by their highest-order non-zero multipole. With the addition of complex numbers, or complex structure from an enveloping theory, the multipoles constitute the complete set of physical observables for a system with arbitrary spin. This complex structure is also required to access the non-zero spin eigenstates of the system. We will utilise the real spin algebras in Chapter 5 to construct position operator algebras which naturally subsume the structure of a system of arbitrary spin.

Chapter 5

Indefinite-Spin and Arbitrary-Spin Position Operator Algebras

5.1 Chapter Aim and Outline

In this chapter, we will explore the fundamental relationship between spin and the geometry of space. To do this, we will use the spin algebras derived in Chapter 4 to construct a family of real infinite-dimensional algebras of position operators, within the algebraic structure which is encoded the algebraic structure for the spin degrees of freedom of the system. This will demonstrate that the geometry of Euclidean three-space (E, δ) is all that is required to construct algebras which naturally model and incorporate the structure of arbitrary spin systems. Furthermore, we will show that spin is a necessary consequence of geometry and that its presence naturally generates non-commutative geometries.

For the majority of its development, the approach presented here is applicable to space-times with non-degenerate metric signatures, laying the groundwork for future relativistic studies. The algebras we construct by these methods generalise the Clifford and Duffin-Kemmer-Petiau algebras, and offer a path towards finite-dimensional counterparts of the same with arbitrary spin. The content of this chapter first appeared in a publication by the author [65]; besides notational changes and restructuring of the content to fit better within this thesis, this work remains faithful to that originally published. This chapter is structured as follows:

First, in Section 5.2, we will motivate our approach by relating spin to the symmetries of Euclidean three-space, and how non-commutative geometry may play a central role in realising the structure of arbitrary spin systems through geometry alone. We will also state a list of properties which we will demand of position operator algebras of the kind we seek. Then, in Section 5.3, we shall consider the Clifford algebra of Euclidean three-space, and show that it satisfies two of the three properties we desire. We will also show that its algebraic structure necessarily prevents it from supporting the structure of an arbitrary system.

Following this, in Section 5.4, we will adopt a more synthetic approach to the construction of our desired algebras. We will first derive the Lie algebra action for the symmetries of a general Minkowski space-time $(\mathcal{V}, g)$ through elementary arguments using Householder reflections. Then, we will prove that this $\mathfrak{so}(\mathcal{V}, g)$-action enjoys a unique algebraic form within any unital associative algebra. This will ensure that the algebraic form of this action is completely general and that our knowledge of the Clifford algebra form does not bias our construction. In Section 5.5, we will use this result to construct the "Indefinite-Spin Position Algebra" $\mathrm{P}_\kappa(\mathcal{V}, g)$: a general position operator algebra which supports an $\mathfrak{so}(\mathcal{V}, g)$-action but contains no inherent spin structure. This fact will be proven by considering the structure of the arbitrary spin algebras.

Consequently, in Section 5.6, we shall restrict our attention to Euclidean three-space (E, δ) and combine the spin algebras $A^{(s)}$ with the Indefinite-Spin Position Algebra $\mathrm{P}_\kappa(E, \delta)$ to construct the "Spin-s Position Algebras" $\mathrm{P}_\kappa^{(s)}(E, \delta)$ for all values of spin. We will show that our construction is consistent with the commutative position operator algebra of quantum mechanics when $s = 0$, and otherwise yields novel non-commutative algebras whose spin generators are spatial bivectors. We will then discuss the degree to which these algebras constitute non-commutative geometries whose structures are generated by the spin degrees of freedom of our system. Finally, in Section 5.7, we will contrast the $\mathrm{P}_\kappa^{(s)}(E, \delta)$ with other higher spin models, and consider the implications of these algebras for quantum mechanics.

Note: especially in Section 5.5.2, the Kronecker delta δ_{ab} will not, in general, be an alternative notation for the metric δ applied to unit length basis vectors. When this is the case will be clear from context. We will also develop our ideas using non-degenerate space-times as much as possible, and specialise to Euclidean three-space only when necessary to connect to these algebras of generators.

5.2 Motivation

5.2.1 Spin and Geometry

In non-relativistic physics, spin presents phenomenologically as a form of angular momentum which is intrinsic to some quantum mechanical systems. Typically, we model such systems using a tensor product between the position-momentum state space and the "internal" spin state space. The latter is taken to be an irreducible representation of the real Lie algebra $\mathfrak{su}(2,\mathbb{C})$. In this model, a clear line is drawn between the geometry of the space the system occupies, which is captured in the position-momentum degrees of freedom, and the spin degrees of freedom. However, the two are more connected than they might first appear.

To see how, let us recall from Definition 2.2.63,

Definition ($\mathfrak{so}(3,\mathbb{R})$)**.** The real Lie algebra $\mathfrak{so}(3,\mathbb{R})$,

$$\mathfrak{so}(3,\mathbb{R}) \coloneqq \operatorname{span}_{\mathbb{R}}(\{S_1, S_2, S_3\}),$$

with Lie product,

$$S_a \times S_b = \sum_{c=1}^{3} \varepsilon_{abc} S_c.$$

It is well-known that $\mathfrak{so}(3,\mathbb{R})$ generates the Lie group $\mathrm{SO}(3,\mathbb{R})$, which is the group of rotations in Euclidean three-space [7]. These rotations preserve the Euclidean geometry of the space on which they act, directly connecting the structure of $\mathfrak{so}(3,\mathbb{R})$ to this geometry. On the other hand, $\mathfrak{su}(2,\mathbb{C})$ is isomorphic as a Lie algebra to $\mathfrak{so}(3,\mathbb{R})$ [7]. Thus, in principle, it should be possible to relate the spin degrees of freedom of a system, modelled by representations of $\mathfrak{su}(2,\mathbb{C})$ (equiv. $\mathfrak{so}(3,\mathbb{R})$), to the geometry of Euclidean three-space.

A connection between spin and geometry has been investigated in myriad ways by many authors. In the relativistic domain: Savasta et al.[76, 77] associate spin with the $\mathrm{SO}(4,\mathbb{R})$ symmetry present between three-velocities and rates of change of proper time; whereas Kaparulin et al.[78] relate spin to geometric qualities of a particle worldline. In non-relativistic physics, Bühler[79] instead connects spin to polarisations of a wave in Euclidean three-space via the Lie group $\mathrm{SL}(2,\mathbb{R})$. Many of these models also seek to describe spin without the need for internal degrees of freedom.

Of principle interest to this chapter is the approach taken by Colatto et al.[80], who introduce a connection between spin generators and non-commutative position operator algebras in Euclidean three-space,

Definition 5.2.1 (Position Operator Algebras)**.** Consider the set $\{\hat{x}_a\}$ of position operators in a physical model indexed over a set J. The $\{\hat{x}_a\}$ form a non-commutative position operator algebra iff $\exists a, b \in J$ such that,

$$\left[\hat{x}_a, \hat{x}_b\right] \neq 0. \tag{5.2.1}$$

Otherwise, the $\{\hat{x}_a\}$ form a commutative position operator algebra.

Non-commutative position operator algebras constitute non-commutative geometries in the sense of [28, 29, 30], which are employed by many as a way to incorporate gravity into quantum mechanics. In the case of Colatto et al., they consider a non-commutative position operator algebra satisfying, $\forall p, q \in \{1, 2, 3\}$,

$$\left[\hat{x}_p, \hat{x}_q\right] = -\frac{i\hbar}{m^2c^2}\sum_{r=1}^{3}\varepsilon_{pqr}\hat{S}_r, \tag{5.2.2}$$

with $\hat{S}_r$ the usual spin generators for a spin-$\frac{1}{2}$ system. Thus, encoding spin into quantum mechanical systems in this way invites the possibility of novel interactions between spin and gravity.

We will generalise this model of Colatto et al. in a number of ways. Firstly, we will construct real non-commutative position operator algebras. Secondly, our algebras will be independent of any notion of "internal" space relied on in most descriptions of spin. Finally, our non-commutative position operator algebras will model the position and spin degrees of freedom of a system with arbitrary spin. With these generalisations, we will establish elementary connections between arbitrary spin and the geometry of Euclidean three-space through non-commutative geometry. Exploring this connection will contribute to ongoing efforts to unify quantum mechanics and gravity, and is paramount to the efficacy of such models in the future.

5.2.2 Desired Properties for a Non-Commutative Position Operator Algebra

Within this chapter's approach to realising spin within a non-commutative geometry, there are a number of properties that the author wishes a non-commutative position operator algebra to have:

1. An action of the connected symmetry group for the space(-time) should be encoded within the algebra, enabling us to algebraically transform its elements;

2. The generators of the connected symmetry group for the space(-time) should exist as elements of the algebra and;

3. The structure of a system with arbitrary spin should be subsumed within the algebra of the generators.

While this may appear to be a challenging list of requirements, the Clifford and Duffin-Kemmer-Petiau[36, 37, 81] algebras famously possess all of these properties. Therefore, to gain a general understanding of the kind of algebras we wish to construct, let us first study the Clifford algebra.

5.3 The Clifford Algebra as a Position Operator Algebra

5.3.1 The Clifford Algebra

Let us recall Definition 2.4.41 for the Clifford algebra,

Definition (Clifford Algebra $\mathrm{Cl}(\mathcal{V}, g)$)**.** Given a Minkowski space-time $(\mathcal{V}, g)$, its Clifford algebra[57] $\mathrm{Cl}(\mathcal{V}, g)$ is the quotient algebra, $\forall v, w \in \mathcal{V}$,

$$\mathrm{Cl}(\mathcal{V}, g) \cong \frac{T(\mathcal{V})}{I\,(v \otimes w + w \otimes v - 2g(v,w))},$$

by the two-sided ideal $I\,(v \otimes w + w \otimes v - 2g(v,w))$ generated by all tensors of the given form, $\forall v, w \in \mathcal{V}$. We will follow the community by leaving the product of $\mathrm{Cl}(\mathcal{V}, g)$ implicit.

Defining the Clifford algebra in the above way explicates its defining algebraic identity, $\forall v, w \in \mathcal{V}$,

$$vw + wv = 2g(v,w). \tag{5.3.1}$$

Also recall Lemma 2.4.43, which shows that,

Lemma. *As a vector space,* $\mathrm{Cl}(\mathcal{V}, g) \cong \Lambda(\mathcal{V})$*, and is spanned by the* k*-blades.*

Remark. From Lemma 2.4.43, we see the Clifford algebra is constructed from objects which each have a definite geometric character. Furthermore, the defining algebraic structure of the Clifford algebra (5.3.1) is controlled entirely by the properties of the metric g. As such, it is an algebra with a strong and natural geometric character.

5.3.2 The Desired Properties in the Clifford Algebra

With the structure of the Clifford algebra understood, let us see how each of the properties defined in Section 5.2.2 emerge within it.

5.3.2.1 Generators and the Action of $\mathrm{SO}^+(\mathcal{V},g)$ in $\mathrm{Cl}(\mathcal{V},g)$

In Lemma 2.4.49, we already saw that the algebra of bivectors under commutator in $\mathrm{Cl}(\mathcal{V},g)$ is Lie algebra isomorphic to $\mathfrak{so}(\mathcal{V},g)$. Thus, in $\mathrm{Cl}(\mathcal{V},g)$ the generators of $\mathrm{SO}^+(\mathcal{V},g)$ are the bivectors $\Lambda^2(\mathcal{V})$. Furthermore, we recall from Definition 2.4.45 the action of bivectors on the whole Clifford algebra has the form,

Definition (u_{cl}[1])**.** For all $a, b, c \in \mathcal{V}$,

$$\mathrm{u}_{cl}(a{\wedge}b)(c) := (a{\wedge}b)c - c(a{\wedge}b) = -2\big(g(a,c)b - g(b,c)a\big).$$

As such, we see our first two required properties are satisfied in $\mathrm{Cl}(\mathcal{V},g)$.

5.3.2.2 The Role of Algebraic Structure in the Action of $\mathrm{SO}^+(\mathcal{V},g)$

Before moving on, it is important to note the algebraic form of $\mathrm{u}_{cl}(a{\wedge}b)$ is a result of the algebraic structure of $\mathrm{Cl}(\mathcal{V},g)$. To explore this connection further, recall that, from Lemma 2.4.53, the Duffin-Kemmer-Petiau algebra on $\mathcal{V}$ admits a similar $\mathfrak{so}(\mathcal{V},g)$-action, $\forall a, b, c \in \mathcal{V}$,

$$\mathrm{u}_{dkp}(a{\wedge}b)(c) := (a{\wedge}b)c - c(a{\wedge}b) = -\frac{1}{2}\big(g(a,c)b - g(b,c)a\big),$$

which again is derived from the structure of its algebra. The properties of these actions, such as being a derivation, are determined by their algebraic forms, which necessarily affects the character of their induced $\mathrm{SO}^+(\mathcal{V},g)$-actions. As such, to make general statements about the relationship between spin, symmetries, and geometry, we must determine if $(a{\wedge}b)c - c(a{\wedge}b)$ is the general form for the $\mathfrak{so}(\mathcal{V},g)$-action in an arbitrary unital associative algebra, independent of any additional structure in the algebra. This shall be confirmed in Section 5.4.2.

[1]We have chosen the symbol u for this and related actions for no other reason than its relationship to the maps t and μ defined in Definitions 2.2.45 and 2.2.53 respectively.

5.3.2.3 Spin-$\frac{1}{2}$ Structure in Cl(E, δ)

Restricting our attention now to the Clifford algebra $\mathrm{Cl}(E, \delta)$ of Euclidean three-space, we may discover the structure of a spin-$\frac{1}{2}$ system embedded within the algebraic structure of its bivectors. To explicate this, let us introduce some notation.

Definition 5.3.1 (S'_p)**.** Consider a Euclidean three-space (E, δ) and a basis $\{e_a\}$ which is orthonormal with respect to δ. Then, in its Clifford algebra $\mathrm{Cl}(E, \delta)$, $\forall p \in \{1, 2, 3\}$,

$$S'_p := -\frac{1}{4} \sum_{a,b=1}^{3} \varepsilon_{abp}\, e_a \wedge e_b. \tag{5.3.2}$$

Lemma 5.3.2. *In* $\mathrm{Cl}(E, \delta)$*,* $\forall p, q \in \{1, 2, 3\}$*,*

$$S'_p S'_q - S'_q S'_p = \sum_{r=1}^{3} \varepsilon_{pqr} S'_r. \tag{5.3.3}$$

Proof. Consider equation (2.4.33) on basis vectors. Then transform both bivectors on the left-hand side using (5.3.2) and simplify the right-hand side. □

Having recovered the canonical form of $\mathfrak{so}(3, \mathbb{R})$ from the bivectors of $\mathrm{Cl}(E, \delta)$, we quickly discover that $\mathbb{R} \oplus \Lambda^2(E) \subset \mathrm{Cl}(E, \delta)$ is an associative subalgebra.

Lemma 5.3.3. *In* $\mathrm{Cl}(E, \delta)$*,* $\forall p, q \in \{1, 2, 3\}$*,*

$$\frac{1}{2}(S'_p S'_q + S'_q S'_p) + \frac{1}{4}\delta_{pq} = 0. \tag{5.3.4}$$

Proof. See Appendix E.1. □

Corollary 5.3.4. *In* $\mathrm{Cl}(E, \delta)$*,* $\forall p, q \in \{1, 2, 3\}$*,*

$$S'_p S'_q = -\frac{1}{4}\delta_{pq} + \frac{1}{2} \sum_{r=1}^{3} \varepsilon_{pqr} S'_r. \tag{5.3.5}$$

Proof. Follows directly from Lemmas 5.3.2 and 5.3.3. □

And so we find,

Theorem 5.3.5. *The subalgebra* $\mathbb{R} \oplus \Lambda^2(E) \subset \mathrm{Cl}(E, \delta)$ *under the Clifford product is isomorphic as an algebra to the real Pauli algebra,*

$$1 \mapsto I_2, \quad S'_x \mapsto \frac{1}{2} i\sigma_x, \quad S'_y \mapsto -\frac{1}{2} i\sigma_y, \quad S'_z \mapsto \frac{1}{2} i\sigma_z,$$

where I_2 *is the* 2×2 *identity matrix and* $\{\sigma_j\}$ *are the* 2×2 *Pauli matrices.*

Proof. This can be verified directly using Corollary 5.3.4. □

Remark. Theorem 5.3.5 shows that $\mathrm{Cl}(E, \delta)$ contains the structure of a spin-$\frac{1}{2}$ system algebraically within $\mathbb{R} \oplus \Lambda^2(E)$. This is widely known, for example in [22, 82, 83].

5.3.2.4 The Origin of these Properties in $\mathbf{Cl}(E,\delta)$

It is clear from the outset that, in the case of the Clifford algebra, these properties are connected to its defining identity (5.3.1). However, if we are to generalise these properties to arbitrary spin systems, we must understand the extent of this connection.

Lemma 5.3.6. *There is a hierarchy of identities within* $\mathrm{Cl}(\mathcal{V},g)$*,* $\forall a,b,c,d\in\mathcal{V}$*,*

$$\begin{gathered} ab+ba = 2g(a,b) \\ \Downarrow \\ (a\wedge b)c - c(a\wedge b) = -2\big(g(a,c)b - g(b,c)a\big) \\ \Downarrow \\ (a\wedge b)(c\wedge d) - (c\wedge d)(a\wedge b) = \mathrm{u}_{cl}(a\wedge b)(c\wedge d), \end{gathered} \tag{5.3.6}$$

where the implication $A\Rightarrow B$ *indicates that identity* B *may be derived within* $\mathrm{Cl}(\mathcal{V},g)$ *assuming identity* A *alone.*

Proof. The second implication follows from the proof of Lemma 2.4.48, and the first from the proof of Lemma 2.4.46. □

Theorem 5.3.7. *The spin-*$\frac{1}{2}$ *structure of* $\mathrm{Cl}(E,\delta)$ *is a direct result of its defining relation,* $\forall a,b\in E$*,*

$$ab+ba = 2\delta(a,b). \tag{5.3.7}$$

Proof. The proof of Lemma 5.3.3 in Appendix E.1 derives Equation (5.3.4) directly from (5.3.7) on basis vectors, utilising elements of $\mathrm{Cl}(E,\delta)$ whose algebraic properties are also the result of (5.3.7). Since (5.3.3) is also a consequence of (5.3.7) by Lemma 5.3.6 we see Corollary 5.3.4 is as well. By Theorem 5.3.5, we are done. □

Theorem 5.3.7 shows that one cannot realise the structure of an arbitrary spin system within a Clifford algebra.

5.3.3 Limitations of Clifford Algebra-Based Approach

As an algebra of position operators satisfying the requirements of Section 5.2.2, the Clifford algebra is fundamentally limited to describing a spin-$\frac{1}{2}$ system by Theorem 5.3.7. As such, we cannot hope to retain its algebraic structure and describe a system with arbitrary spin. A standard approach to overcome this problem is to construct a tensor power algebra $\mathrm{Cl}(E, \delta)^{\otimes k}$; this algebra contains a Clifford substructure yet contains subalgebras with the structure of arbitrary spin systems[38] for large enough k. Indeed, subspaces of tensor products of this algebra also underpin the definition of many classic higher spin models[39, 40, 41].

However, considered as an algebra of position operators, this method also does not meet our requirements. To see why, first note that we are ultimately looking for algebras which may be compatible with quantum mechanical theories of systems with arbitrary spin. In principle, such theories should be able to accommodate Hamiltonians with arbitrary position-dependent potentials, including, for example, all polynomials of position operators. We can see from this consideration that the space of all position-dependent potentials is infinite-dimensional. However, regardless of the size of the tensor power, $\mathrm{Cl}(E, \delta)^{\otimes k}$ is always finite-dimensional; this severely limits the position-dependent potentials which may be algebraically realised within the Hamiltonians of the model.

As such, we cannot use the Clifford algebra as a base for the algebras of position operators that we seek. This immediately raises a number of challenges. Firstly, without the structure of the Clifford algebra, we cannot guarantee that our algebras will naturally entail an action of $\mathfrak{so}(\mathcal{V}, g)$ within their structure. Secondly, we have no guarantee that the algebraic form of this action in our algebras will follow the commutator form of Definition 2.4.45; this forces us to consider whether the generators of spin are bivectors for arbitrary spin systems. Finally, it is unclear at this stage if algebras containing the structure of arbitrary spin systems can be constructed in a consistent and compatible way with the $\mathfrak{so}(\mathcal{V}, g)$-action we are seeking to implement. To address these challenges in a way independent of the Clifford algebra, we must adopt a more synthetic approach.

5.4 The General Algebraic $\mathfrak{so}(\mathcal{V}, g)$-action

In order to have a natural $\mathrm{SO}^+(\mathcal{V}, g)$ action in our position operator algebras, we must first establish the general form of an $\mathfrak{so}(\mathcal{V}, g)$-action within it. So far, we have only seen the $\mathfrak{so}(\mathcal{V}, g)$-action in the context of the Clifford algebra, wherein it has a particular form and algebraic implementation. A priori, it is unclear which aspects of this are particular to the Clifford algebra. Therefore, to ensure the generality of the non-commutative position operator algebra we are constructing, it is prudent to derive both the form and algebraic implementation of the $\mathfrak{so}(\mathcal{V},g)$-action by elementary means and using only the properties of g to guide us.

5.4.1 The $\mathfrak{so}(\mathcal{V},g)$-action on Vectors from Reflections

By the Cartan-Dieudonné Theorem 2.2.28, every element of $\mathrm{O}(\mathcal{V},g)$ is a composition of reflections. Thus, we should be able to access the entirety of $\mathrm{SO}^+(\mathcal{V},g)$ through them. Thus, we should be able to identify $\mathfrak{so}(\mathcal{V},g)$ by studying the algebraic structure of reflections. To begin, let us recall Definition 2.2.8 of metric adjoints,

Definition (g-adjoint)**.** Consider a Minkowski space-time $(\mathcal{V},g)$. A g-adjoint of an endomorphism $A\in\mathrm{End}(\mathcal{V})$ is an endomorphism $A^g\in\mathrm{End}(\mathcal{V})$ such that $\forall v,w\in\mathcal{V}$,

$$g(A(v),w)=g(v,A^g(w)).$$

We must also recall Definition 2.2.16 which gives the notions of self- and anti-self-metric-adjoints,

Definition (Self- and Anti-Self-g-adjoint)**.** An endomorphism $A\in\mathrm{End}(\mathcal{V})$ is self-g-adjoint when $A^g=A$, and anti-self-g-adjoint when $A^g=-A$.

Using these concepts we find,

Lemma 5.4.1. *All reflections $R(a)$ are self-g-adjoint.*

Proof. For all $v,w\in\mathcal{V}$,

$$\begin{aligned} g\big(R(a)(v),w\big) &= g\Big(v-2\frac{g(a,v)}{g(a,a)}a,w\Big)\\ &= g\Big(v,w-2\frac{g(a,v)}{g(a,a)}a\Big)\\ &= g\big(v,R(a)(w)\big). \end{aligned}$$

□

With this in mind, let us consider the properties of a composition of two reflections $R(a)\circ R(b)$.

Lemma 5.4.2. *For all $v\in\mathcal{V}$,*

$$a_-\big(R(a)\circ R(b)\big)(v)=-\frac{2g(a,b)}{g(a,a)g(b,b)}\big(g(a,v)b-g(b,v)a\big) \tag{5.4.1a}$$

$$\begin{aligned} a_+\big(R(a)\circ R(b)\big)(v)=v&+\frac{2g(a,v)}{g(a,a)g(b,b)}\big(g(a,b)b-g(b,b)a\big)\\ &-\frac{2g(b,v)}{g(a,a)g(b,b)}\big(g(a,a)b-g(b,a)a\big). \end{aligned} \tag{5.4.1b}$$

Proof. Direct computation. □

This immediately reveals that,

Corollary 5.4.3. *The set of all reflections is not closed under composition.*

Proof. We note that for two non-null vectors a and b,

$$a_-\big(R(a)\circ R(b)\big) \neq 0,$$

when $g(a,b) \neq 0$, and thus $R(a)\circ R(b)$ cannot be a reflection. □

Furthermore,

Corollary 5.4.4. *For* $\dim(\mathcal{V}) \geq 3$*,* $\mathrm{O}(\mathcal{V}, g)$ *is non-Abelian.*

Proof. Follows directly from (5.4.1a). □

Corollary 5.4.3 is well-known, but our method of proving it has revealed the internal structure of a product of reflections. In particular, Lemma 5.4.2 reveals the $\mathfrak{so}(\mathcal{V}, g)$-action is present in both expressions, recalling Definition 2.2.45,

Definition ($t(a,b)$)**.** For all $a, b \in \mathcal{V}$, $t(a,b)$ is the bilinear map,

$$t : \mathcal{V}\times\mathcal{V} \to \mathrm{End}(\mathcal{V})$$
$$t(a,b) \coloneqq v \mapsto g(a,v)b - g(b,v)a.$$

Remark. We are not ready to write $t(a,b)$ as the action of a bivector $\mathrm{u}(a\wedge b)$; first, we would like to be certain that the bivector $a\wedge b$ itself enters the algebraic form of $t(a,b)$.

Corollary 5.4.5. *The product of any two reflections is,*

$$R(a)\circ R(b) = \mathrm{id} - \frac{2g(a,b)}{g(a,a)g(b,b)}t(a,b) + \frac{2}{g(a,a)g(b,b)}t(a,b)\circ t(a,b). \tag{5.4.2}$$

Proof. This is the sum of the results in Lemma 5.4.2 with a notational change to $t(a,b)$. □

Remark. Corollary 5.4.5 shows that the product in the algebra of reflections is controlled by the $\mathfrak{so}(\mathcal{V}, g)$-action, hinting at the relationship between compositions of reflections and the elements of $\mathrm{SO}^+(\mathcal{V}, g)$.

5.4.2 The Algebraic Form of the $\mathfrak{so}(\mathcal{V}, g)$-action on Vectors

Having motivated the $\mathfrak{so}(\mathcal{V}, g)$-action on $\mathcal{V}$ in the form of the map t from reflections, we must consider how to implement it as an algebraic identity within an associative algebra. To achieve this aim we seek a third-order tensor $f(a, b, c)$

Definition 5.4.6. For all $a, b, c \in \mathcal{V}$, and some $\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in \mathbb{R}$,

$$\begin{aligned} f : \mathcal{V}\times\mathcal{V}\times\mathcal{V} &\to T(\mathcal{V}) \\ f(a, b, c) =& \alpha(a\otimes b\otimes c) + \beta(a\otimes c\otimes b) + \gamma(b\otimes c\otimes a) \\ &+ \delta(b\otimes a\otimes c) + \epsilon(c\otimes a\otimes b) + \zeta(c\otimes b\otimes a), \end{aligned} \tag{5.4.3}$$

whose properties match those of $t(a,b)(c)$; this will enable us to construct a quotient of $T(\mathcal{V})$ by the two-sided ideal $\forall a, b, c \in \mathcal{V}$,

$$I\left(f(a, b, c) - t(a,b)(c)\right) \tag{5.4.4}$$

yielding the most general non-trivial algebra possible which contains the action of $\mathfrak{so}(\mathcal{V}, g)$ within its structure.

Theorem 5.4.7. *For arbitrary Minkowski space-times $(\mathcal{V}, g)$, the only $f(a, b, c)$ which shares all properties of $t(a,b)(c)$ is,*

$$f(a, b, c) = k\big((a\otimes b - b\otimes a)\otimes c - c\otimes(a\otimes b - b\otimes a)\big), \tag{5.4.5}$$

where $k \in \mathbb{R}$.

Proof. See Appendix E.2. □

Remark. Theorem 5.4.7 shows that, algebraically and independently of any additional algebraic structure, the action of $\mathfrak{so}(\mathcal{V}, g)$ on $\mathcal{V}$ is always implemented as a commutator between its bivector generators and the vector being transformed. This foreshadows the fact that when implemented within an algebra of position operators, all spin generators: may be realised as bivectors; and transform position operators in the same way, regardless of the size of the spin. This suggests a universality in any couplings between position and spin degrees of freedom predicted by these models, and may offer an explanation as to why no such couplings have been measured so far: they are universally small.

Theorem 5.4.7 also shows that $f(a, b, c)$ is defined only up to an arbitrary scaling. This is expected and unproblematic. Understanding the equivalence of quotient algebras defined using different values of k is beyond the scope of this thesis.

5.5 The Indefinite-Spin Position Algebra

5.5.1 The Indefinite-Spin Position Algebra $\mathrm{P}_\kappa(\mathcal{V}, g)$

Using Theorem 5.4.7 we may now define the most general algebra of position operators which contains a natural $\mathfrak{so}(\mathcal{V}, g)$-action on vectors, or more precisely a family of such algebras,

Definition 5.5.1 ($\mathrm{P}_\kappa(\mathcal{V}, g)$)**.** Given a Minkowski space-time $(\mathcal{V}, g)$, and choosing a value of $k = \frac{1}{2}$ in (5.4.5), we define the "Indefinite-Spin Position Algebra" $\mathrm{P}_\kappa(\mathcal{V}, g)$ as the quotient, $\forall a, b, c \in \mathcal{V}$,

$$\mathrm{P}_\kappa(\mathcal{V}, g) \cong \frac{T(\mathcal{V})}{I\left((a \wedge b) \otimes c - c \otimes (a \wedge b) - \kappa(g(a,c)b - g(b,c)a)\right)}, \tag{5.5.1}$$

by the two-sided ideal (5.4.4), for some $\kappa \in \mathbb{R}$. We shall leave the product of $\mathrm{P}_\kappa(\mathcal{V}, g)$ implicit by concatenation.

Remark. We include the κ in this definition to ensure these algebras are eventually consistent with the Clifford and Duffin-Kemmer-Petiau algebras defined on $(\mathcal{V}, g)$, for which $\kappa = -2$ and $\kappa = -\frac{1}{2}$ respectively. More precisely, for a family of Clifford algebras defined by, $\forall a, b, c \in \mathcal{V}$,

$$ab + ba = 2\kappa_{cl} g(a,b),$$

and a family of Duffin-Kemmer-Petiau algebras defined by, $\forall a, b, c \in \mathcal{V}$,

$$abc + cba = \kappa_{dkp}(g(a,b)c + g(c,b)a),$$

for some $\kappa_{cl}, \kappa_{dkp} \in \mathbb{R}$, then their $\mathfrak{so}(\mathcal{V}, g)$-actions are consistent with the $\mathrm{P}_\kappa(\mathcal{V}, g)$ when $\kappa = -2\kappa_{cl}$ and $\kappa = -\frac{1}{2}\kappa_{dkp}$ respectively.

Remark. The term "indefinite-spin" here foreshadows Theorem 5.5.17, where we will show that $\mathrm{P}_\kappa(E, \delta)$ does not already contain the structure of any particular spin. As such, in Definition 5.6.1 we will quotient it further to define a new family of algebras each subsuming a particular spin structure.

We must now ensure that our algebraic implementation of the $\mathfrak{so}(\mathcal{V}, g)$-action in $\mathrm{P}_\kappa(\mathcal{V}, g)$ extends to an $\mathrm{SO}^+(\mathcal{V}, g)$-action. To facilitate this discussion, let us capture the $\mathfrak{so}(\mathcal{V}, g)$-action on vectors in $\mathrm{P}_\kappa(\mathcal{V}, g)$ through the multilinear map,

Definition 5.5.2 (u)**.**

$$\mathrm{u}(a \wedge b) := c \mapsto (a \wedge b)c - c(a \wedge b) = \kappa t(a,b)(c). \tag{5.5.2}$$

Now, let us consider its properties,

Lemma 5.5.3. *For all* $a, b \in \mathcal{V}$, $\mathrm{u}(a\wedge b)$ *may be naturally extended to the whole of* $\mathrm{P}_\kappa(\mathcal{V}, g)$ *as a derivation. Furthermore,*

$$(a\wedge b)(c\wedge d) - (c\wedge d)(a\wedge b) = \mathrm{u}(a\wedge b)(c\wedge d). \tag{5.5.3}$$

Proof. Noting that $\mathrm{u}(a\wedge b)$ and $\mathrm{u}_{cl}(a\wedge b)$ (recall Definition 2.4.45) have the same implementation within their respective algebras (up to scaling), the derivation property and (5.5.3) follow the same proofs as for Lemmas 2.4.47 and 2.4.48 respectively. □

Remark. Since $\mathrm{u}(a\wedge b)$ is the unique tensorial implementation of $t(a,b)$ in $\mathrm{P}_\kappa(\mathcal{V}, g)$ (up to scaling), we may consider its properties to be the natural extension of $t(a,b)$ to $\mathrm{P}_\kappa(\mathcal{V}, g)$.

Lemma 5.5.4. *With* $\mathrm{u}(a\wedge b)$ *extended as a derivation to all of* $\mathrm{P}_\kappa(\mathcal{V}, g)$*, there exists a natural action of* $\mathrm{SO}^+(\mathcal{V}, g)$ *on* $\mathrm{P}_\kappa(\mathcal{V}, g)$ *which distributes over tensor products.*

Proof. See [52]. □

With Lemma 5.5.4, we have succeeded in constructing a general non-commutative algebra of position operators which encodes an action of $\mathrm{SO}^+(\mathcal{V}, g)$ and contains its generators as algebraic elements, in this case the bivectors $\Lambda^2(\mathcal{V})$. This construction holds for an arbitrary Minkowski space-time.

5.5.2 The Algebraic Structure of Arbitrary Spin Systems

To achieve our final aim from Section 5.2.2, to construct non-commutative position operators algebras which subsume the structure of an arbitrary spin system, we will specialise our arguments to Euclidean three-space (E, δ). We hope to complete this task by utilising the real arbitrary spin algebras developed in Chapter 4 along with the indefinite spin position algebra $\mathrm{P}_\kappa(\mathcal{V}, g)$ defined in this section. Before we proceed further with this, let us recall some important elements of the spin algebras' construction.

We derive the spin algebras directly from the universal enveloping algebra of $\mathfrak{so}(3, \mathbb{R})$, given by Definition 4.3.1 as,

Definition $(U(\mathfrak{so}(3, \mathbb{R})))$**.** Given the Lie algebra $\mathfrak{so}(3, \mathbb{R})$, its universal enveloping algebra[64] $U(\mathfrak{so}(3, \mathbb{R}))$ is the quotient algebra,

$$U(\mathfrak{so}(3, \mathbb{R})) \cong \frac{T(\mathfrak{so}(3, \mathbb{R}))}{I\left(S_a \otimes S_b - S_b \otimes S_a - S_a \times S_b\right)},$$

by the two-sided ideal $I\left(S_a \otimes S_b - S_b \otimes S_a - S_a \times S_b\right)$ generated by all tensors of the given form, $\forall S_a, S_b \in \mathfrak{so}(3,\mathbb{R})$.

More precisely, we decompose $U(\mathfrak{so}(3,\mathbb{R}))$ using the adjoint action, given by Definition 2.4.33 as,

Definition (Adjoint action)**.**

$$\mathrm{ad} : U(\mathfrak{so}(3,\mathbb{R})) \to \mathrm{End}(U(\mathfrak{so}(3,\mathbb{R})))$$

$$\mathrm{ad}(u) := v \mapsto \begin{cases} uv & u \in \mathbb{R} \\ uv - vu & u \in \mathfrak{so}(3,\mathbb{R}) \\ \mathrm{ad}(a) \circ \mathrm{ad}(b)(v) & u = ab, \end{cases}$$

of the Casimir element $S^2 \in U(\mathfrak{so}(3,\mathbb{R}))$, given by Definition 2.4.28 as,

Definition (S^2)**.**

$$S^2 := \sum_{a=1}^{3} S_a \otimes S_a. \tag{5.5.4}$$

In so doing, we identify the "multipoles" of Definition 4.4.18,

Definition (Multipoles)**.** For all $k \in \mathbb{N}$, the multipole tensors[61] are defined recursively, $\alpha \in \mathbb{R}$, $v \in \mathfrak{so}(3,\mathbb{R})$, $B_k \in \mathfrak{so}(3,\mathbb{R})^{\otimes k}$,

$$M^{(k)} : \mathfrak{so}(3,\mathbb{R})^{\otimes k} \to U(\mathfrak{so}(3,\mathbb{R}))$$

$$M^{(0)}(\alpha) = \alpha$$

$$M^{(k+1)}(v \otimes B_k) = \frac{\mathrm{ad}(S^2 + k(k-1)) \circ \mathrm{ad}(S^2 + k(k+1))}{4(k+1)(2k+1)} \big(v \otimes M^{(k)}(B_k)\big).$$

Crucially,

Lemma 5.5.5. *All elements of $U(\mathfrak{so}(3,\mathbb{R}))$ can be written as an $\mathbb{R}[S^2]$-linear combination of objects from $\mathrm{Im}(\{M^{(k)}\})$, where $\mathbb{R}[S^2]$ is the ring of real polynomials of S^2.*

Proof. See Theorem 4.4.24. □

We utilised this fact in Definition 4.5.1 of,

Definition ($A^{(s)}$)**.** The spin algebra for spin s is the quotient algebra,

$$A^{(s)} := \frac{U(\mathfrak{so}(3,\mathbb{R}))}{I\left(\mathrm{Im}(M^{(2s+1)})\right)}.$$

This quotient by a given multipole entails that,

Lemma 5.5.6. *For all $k \in \mathbb{Z}^+$ such that $k \geq 2s+1$, $M^{(k)} = 0$ in $A^{(s)}$.*

Proof. The case $k = 2s+1$ follows from the definition of $A^{(s)}$, and the case $k > 2s+1$ follows from the recursive relationship between multipoles of Definition 4.4.18. □

Furthermore,

Lemma 5.5.7. *In $A^{(s)}$, $\forall s \in \{0, \frac{1}{2}, 1, \frac{3}{2} \dots\}$,*

$$S^2 = -s(s+1). \tag{5.5.5}$$

Proof. See Lemma 4.5.3. □

Corollary 5.5.8. *For all $s \in \{0, \frac{1}{2}, 1, \frac{3}{2} \dots\}$, $A^{(s)}$ is a finite-dimensional real algebra spanned by $\{\mathrm{Im}(M^{(k)}) \mid 0 \leq k \leq 2s\}$ with dimension $\dim(A^{(s)}) = (2s+1)^2$.*

Proof. Follows from Lemmas 4.5.3 and 4.5.4. □

Remark. The algebraic structure of the spin algebras is exactly that of the unital associative algebra of generators for a spin-s system, and as such all irreducible representations of $\mathfrak{so}(3, \mathbb{R})$ derive their structure from them. In a full quantum mechanical theory, the multipoles $M^{(k)}$ are the physically distinct observables for an arbitrary spin system. See Chapter 4 for further discussion.

5.5.3 Suitability of $\mathrm{P}_\kappa(E, \delta)$ to Subsume Arbitrary Spin Algebras

In light of the previous section, if we hope to use $\mathrm{P}_\kappa(E, \delta)$ to define new algebras containing arbitrary spin structure, then we must ensure that it contains no existing spin structure. More precisely, we must ensure that it contains a subalgebra isomorphic as unital associative algebras to the whole of $U(\mathfrak{so}(3, \mathbb{R}))$.

5.5.3.1 Bivectors and Spin Generators

To begin, let us recall the notation of Definition 2.2.61,

Definition (S_p)**.** Given an orthonormal basis $\{e_a\}$ of (E, δ), $a \in \{1, 2, 3\}$, we define, $\forall p \in \{1, 2, 3\}$,

$$S_p := \frac{1}{2\kappa} \sum_{a,b=1}^{3} \varepsilon_{abp}\, e_a \wedge e_b,$$

which has inverse transformation,

$$e_a \wedge e_b = \kappa \sum_{p=1}^{3} \varepsilon_{abp} S_p.$$

Lemma 5.5.9. *In* $\mathrm{P}_\kappa(E,\delta)$, $\forall p,q \in \{1,2,3\}$

$$S_p S_q - S_q S_p = \sum_{r=1}^{3} \varepsilon_{pqr} S_r. \tag{5.5.6}$$

Proof. Apply (2.2.29) to (5.5.3) evaluated on basis vectors $\{e_a\}$. □

Lemma 5.5.10. *In* $\mathrm{P}_\kappa(E,\delta)$,

$$S^2 = \frac{1}{2\kappa^2} \sum_{a,b=1}^{3} (e_a \wedge e_b)(e_a \wedge e_b), \tag{5.5.7}$$

and commutes with all bivectors.

Proof. The form of S^2 can be seen by applying (2.2.29) to Definition 2.4.28. That it commutes with all bivectors can be directly computed. □

5.5.3.2 $\mathbf{P}_\kappa(E,\delta)$ and $U(\mathfrak{so}(3,\mathbb{R}))$

Now that we have established an explicit Lie algebra isomorphism between the bivectors and spin generators, we must determine if the algebra of bivectors contained in $\mathrm{P}_\kappa(E,\delta)$ is isomorphic to $U(\mathfrak{so}(3,\mathbb{R}))$. This is important, as our algebraic implementation of the $\mathfrak{so}(\mathcal{V},g)$-action may have unintentionally restricted this algebra à la Theorem 5.3.7.

First, let us establish that the Casimir element S^2 is not a scalar, as this would certainly indicate that the structure of $U(\mathfrak{so}(3,\mathbb{R}))$ has been disturbed in some way.

Lemma 5.5.11. *In* $\mathrm{P}_\kappa(E,\delta)$, S^2 *is not equal to a scalar.*

Proof. Considering the commutator of S^2 with an arbitrary basis vector e_d we find, $\forall d \in \{1,2,3\}$,

$$S^2 e_d - e_d S^2 = -\frac{1}{2\kappa} \sum_{a=1}^{3} \big((e_a \odot e_a) e_d - e_d (e_a \odot e_a)\big), \tag{5.5.8}$$

where we have introduced the symmetric product,

$$a \odot b := \frac{1}{2}(ab + ba).$$

The right-hand side is clearly non-zero in $T(E)$ and not an element of the ideal, $\forall a, b, c \in E$,

$$I\left((a \wedge b) \otimes c - c \otimes (a \wedge b) - \kappa(g(a,c)b - g(b,c)a)\right),$$

used to construct $\mathrm{P}_\kappa(E, \delta)$. Thus, it is non-zero in $\mathrm{P}_\kappa(E, \delta)$, and so S^2 does not generally commute with vectors $v \in E \subset \mathrm{P}_\kappa(E, \delta)$. Therefore, it cannot be a scalar. □

Remark. Notice that in $\mathrm{Cl}(E, \delta)$ the right-hand side of (5.5.8) is zero, so S^2 commutes with the whole algebra. This is a clear example of how the hierarchy of Lemma 5.3.6 induces differences in the algebraic structures of $\mathrm{Cl}(E, \delta)$ and $\mathrm{P}_\kappa(E, \delta)$.

With that in hand, let us abuse notation to define multipoles of bivectors,

Definition 5.5.12. The multipoles $M^{(k)}$ written in terms of bivectors are, $\forall n \in \mathbb{Z}^+$, $\forall j \in \{1, \ldots, n\}$, $\forall a_j, b_j \in \{1, 2, 3\}$,

$$M^{(n)}\Big(\bigotimes_{j=1}^{n} e_{a_j} \wedge e_{b_j}\Big) = \kappa^n \sum_{p_1,\ldots,p_n=1}^{3} \prod_{j=1}^{n} \varepsilon_{a_j b_j p_j} M^{(n)}\Big(\bigotimes_{m=1}^{n} S_{p_m}\Big), \tag{5.5.9}$$

and $M^{(0)}(\alpha) = \alpha$, $\forall \alpha \in \mathbb{R}$.

For example, $\forall a, b, c, d \in E$,

$$\begin{aligned} M^{(2)}\big((a \wedge b) \otimes (c \wedge d)\big) = &\frac{1}{2}\big((a \wedge b)(c \wedge d) + (c \wedge d)(a \wedge b)\big) \\ &- \frac{\kappa^2}{3} S^2 \big(\delta(a,c)\delta(b,d) - \delta(a,d)\delta(b,c)\big). \end{aligned} \tag{5.5.10}$$

Now, let us explore the possibility that some multipoles of bivectors may have become zero during the quotient, which would limit the kinds of spin structures that we could implement within $\mathrm{P}_\kappa(E, \delta)$.

Lemma 5.5.13. $\nexists n \in \mathbb{Z}^+ : M^{(n)} = 0$.

Proof. [61] shows that in $U(\mathfrak{so}(3, \mathbb{R}))$, $M^{(n)} = 0$ entails $S^2 = -\frac{(n-1)(n+1)}{4}$. By Lemma 5.5.11, S^2 is not a scalar in $\mathrm{P}_\kappa(E, \delta)$, so by contraposition no multipoles are zero in $\mathrm{P}_\kappa(E, \delta)$. □

With Lemma 5.5.13 established, we may now show that,

Lemma 5.5.14. *The unital associative algebra homomorphism,* $\forall p \in \{1, 2, 3\}$,

$$\begin{aligned} &\phi : U(\mathfrak{so}(3, \mathbb{R})) \to \mathrm{P}_\kappa(E, \delta) \\ &\phi := S_p \mapsto \frac{1}{2\kappa} \sum_{a,b=1}^{3} \varepsilon_{abp}\, e_a \wedge e_b \end{aligned} \tag{5.5.11}$$

is injective.

Proof. That ϕ as defined on generators can be extended to a unital associative algebra homomorphism follows from Lemma 5.5.9. Furthermore, Lemma 5.5.13 demonstrates that no multipoles in $\mathrm{P}_\kappa(E,\delta)$ are zero, and so by Lemma 5.5.5 we are done. □

We may draw this connection between $U(\mathfrak{so}(3,\mathbb{R}))$ and $\mathrm{P}_\kappa(E,\delta)$ closer by relating the actions of their generators. First,

Definition 5.5.15. Let us extend u to be a unital associative algebra action, $\forall\alpha\in\mathbb{R}$, $A,B\in T(\Lambda^2(\mathcal{V}))$,

$$\mathrm{u}:T(\Lambda^2(\mathcal{V}))\to\mathrm{End}(\mathrm{P}_\kappa(\mathcal{V},g))$$
$$\mathrm{u}(A):=\begin{cases}D\mapsto AD & A\in\mathbb{R}\\ \mathrm{u}(B)\circ\mathrm{u}(C) & A=B\otimes C.\end{cases}\tag{5.5.12}$$

Then,

Corollary 5.5.16. *For all, $A\in T(\Lambda^2(E))$,*

$$\mathrm{u}(A)\circ i=\mathrm{ad}(A),\tag{5.5.13}$$

where $i:T(\Lambda^2(\mathcal{V}))\to\mathrm{End}(\mathrm{P}_\kappa(\mathcal{V},g))$, and it is understood that we translate bivectors into spin generators and back where appropriate using Definition 2.2.61.

Proof. We need only compare the definitions of u and ad in light of Lemma 5.5.14. □

Finally, we are in a position where we may conclude,

Theorem 5.5.17. $\mathrm{P}_\kappa(E,\delta)$ *can support any spin structure.*

Proof. Since by Lemma 5.5.14 $U(\mathfrak{so}(3,\mathbb{R}))\subset\mathrm{P}_\kappa(E,\delta)$, $\mathrm{P}_\kappa(E,\delta)$ lacks the additional structure of any spin algebra $A^{(s)}$. Therefore, we are not limited in the kinds of spin structures we can impose on $\mathrm{P}_\kappa(E,\delta)$. □

Theorem 5.5.17 shows us that, unlike the Clifford algebra $\mathrm{Cl}(E,\delta)$, the Indefinite-Spin Position algebra $\mathrm{P}_\kappa(E,\delta)$ does not inherently contain the structure of a spin algebra within its algebra of bivectors. This means that it is possible to construct physical models for systems with arbitrary spin from this algebra; we will soon see that such models predict non-trivial relationships between the position and spin degrees of freedom of the system.

5.6 Arbitrary Spin Position Operator Algebras

5.6.1 The Indefinite-Spin Position Algebra

The aim of this chapter has been to explore the relationship between spin and geometry by constructing algebras of position operators from a Minkowski space-time $(\mathcal{V}, g)$ which meet three requirements as outlined in Section 5.2.2. The first two of these requirements are: algebraically containing an action of the connected symmetry group for the metric g; and containing the generators of this symmetry group as elements of the algebra. These goals are achieved for general Minkowski space-times $(\mathcal{V}, g)$ within the Indefinite-Spin Position Algebra, for $\kappa \in \mathbb{R}$, $\forall a, b, c \in \mathcal{V}$,

$$\mathrm{P}_\kappa(\mathcal{V}, g) \cong \frac{T(\mathcal{V})}{I\left((a{\wedge}b){\otimes}c - c{\otimes}(a{\wedge}b) - \kappa(g(a,c)b - g(b,c)a)\right)}. \tag{5.6.1}$$

Our third requirement is to subsume within such an algebra the structure of a system with arbitrary spin. To do this, we restrict our attention to Euclidean three-space (E, δ), where we may utilise the spin algebras $A^{(s)}$ described in Section 5.5.2. In Theorem 5.5.17, we showed that $\mathrm{P}_\kappa(E, \delta)$ has no pre-existing spin structure, and so is the ideal foundational algebra within which to achieve this aim for arbitrary spin (except for $s = 0$, which will be discussed shortly).

5.6.2 The Spin-s Position Algebras

5.6.2.1 General Definition

We may now implement the final property from Section 5.2.2 and complete our construction of non-commutative position operator algebras which subsume the structure of an arbitrary spin system,

Definition 5.6.1 ($\mathrm{P}_\kappa^{(s)}(E, \delta)$)**.** The Spin-$s$ Position Algebra $\mathrm{P}_\kappa^{(s)}(E, \delta)$ for $s \in \{0, \frac{1}{2}, 1, \dots\}$ is the quotient algebra,

$$\mathrm{P}_\kappa^{(s)}(E, \delta) \cong \begin{cases} \dfrac{\mathrm{P}_\kappa(E, \delta)}{I\left(\mathrm{Im}(M^{(2s+1)})\right)} & s \neq 0 \\[2ex] \dfrac{T(E)}{\mathrm{Im}(M^{(1)})} & s = 0, \end{cases} \tag{5.6.2}$$

with the multipole $M^{(2s+1)}$ understood to be tensors of bivectors as in Lemma 5.5.13.

Remark. When $s \neq 0$, by Lemma 5.5.14 we are effectively embedding the structure of $A^{(s)}$ within the algebra of bivectors in $\mathrm{P}_\kappa(E, \delta)$. When $s = 0$, we embed the structure of $A^{(0)}$ into $T(E)$ instead (this difference shall be explained shortly). Therefore, we find the geometric realisation of spin generators as bivectors is robust in the presence of arbitrary spin.

Let us explore some immediate consequences of this definition,

Lemma 5.6.2. *For all* $s \in \{0, \frac{1}{2}, 1, \dots\}$, *in* $\mathrm{P}_\kappa^{(s)}(E, \delta)$, $S^2 = -s(s+1)$.

Proof. When $s = 0$, this is clear from the definition, and the case $s \neq 0$ follows from Lemmas 5.5.7 and 5.5.14. □

Lemma 5.6.3.

$$\mathrm{P}_\kappa^{(0)}(E, \delta) \cong \mathrm{Sym}(E). \tag{5.6.3}$$

Proof. The quotient by the ideal $I\left(\mathrm{Im}(M^{(1)})\right)$ in Definition 5.6.1 entails that in $\mathrm{P}_\kappa^{(0)}(E, \delta)$,

$$[a \wedge b = 0] \Rightarrow [ab = ba].$$

Thus, the definition of $\mathrm{P}_\kappa^{(0)}(E, \delta)$ is identical to that of the symmetric algebra $\mathrm{Sym}(E)$. □

Remark. For $\mathrm{P}_\kappa^{(0)}(E, \delta)$, the value of κ changes nothing since it is only defined in the $\mathfrak{so}(3, \mathbb{R})$-action on vectors.

Lemma 5.6.4. *For all* $s \in \{0, \frac{1}{2}, 1, \dots\}$, $\mathrm{P}_\kappa^{(s)}(E, \delta)$ *is infinite-dimensional.*

Proof. It is easy to see that, $\forall s \in \{0, \frac{1}{2}, 1, \dots\}$, $\forall n \in \mathbb{Z}^+$, $\forall j \in \{1, \dots, n\}$, $\forall v_j \in E$, the vector space homomorphism,

$$\begin{gathered} f : \mathrm{Sym}(E) \to \mathrm{P}_\kappa^{(s)}(E, \delta) \\ f \coloneqq 1 \mapsto 1 \\ f \coloneqq v_1 \mapsto v_1 \\ f \coloneqq v_1 \odot \dots \odot v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \dots v_{\sigma(n)}, \end{gathered}$$

is injective. □

5.6.2.2 $\mathrm{P}_\kappa^{(0)}(E, \delta)$ and Commutative Geometry

To begin our analysis of the $\mathrm{P}_\kappa^{(s)}(E, \delta)$, let us consider the case when $s = 0$. In Definition 5.6.1, the $s = 0$ case derived from $T(E)$ rather than $\mathrm{P}_\kappa(E, \delta)$; this is to ensure that $\mathrm{P}_\kappa^{(0)}(E, \delta)$ is identical to the standard commutative position operator algebra of quantum mechanics. In terms of geometry,

Corollary 5.6.5. *In the sense of commuting position operators, the structure of a spin $s = 0$ system subsumed within $\mathrm{P}_\kappa^{(0)}(E, \delta)$ generates a commutative geometry.*

Proof. The commutative structure in Lemma 5.6.3 is a direct consequence of the defining property of $A^{(0)}$, i.e. that all spin generators are zero. □

The zeroing of spin generators in an $s = 0$ system requires us to treat this case separately, to ensure consistency with the standard position operator algebra of quantum mechanics. Otherwise,

Lemma 5.6.6. *Suppose we define,*

$$\widetilde{\mathrm{P}}_\kappa^{(0)}(E, \delta) \cong \frac{\mathrm{P}_\kappa(E, \delta)}{I\left(\mathrm{Im}(M^{(1)})\right)},$$

consistently with the $\mathrm{P}_\kappa^{(s\neq 0)}(E, \delta)$*. Then, for* $\kappa \neq 0$*,* $\forall a, b \in E$*,*

$$\widetilde{\mathrm{P}}_\kappa^{(0)}(E, \delta) \cong \mathbb{R}[x]. \tag{5.6.4}$$

Proof. In $\widetilde{\mathrm{P}}_\kappa^{(0)}(E, \delta)$, the quotient entails, $\forall a, b \in E$,

$$\kappa(a{\wedge}b) = 0,$$

and thus, $\forall a, b, c \in E$,

$$(a{\wedge}b)c - c(a{\wedge}b) = 0 = \kappa\big(\delta(a,c)b - \delta(b,c)a\big).$$

Let us now choose $\{a, b\}$ linearly independent in the above. Then, since $\kappa \neq 0$, $\forall c \in E$,

$$[\delta(a,c) = 0] \wedge [\delta(b,c) = 0] \Rightarrow [a = 0] \wedge [b = 0],$$

by the non-degeneracy of δ, contradicting our assumption. Therefore, no such linearly independent sets of $a, b \in E$ exist in $\widetilde{\mathrm{P}}_\kappa^{(0)}(E, \delta)$, so all $a \in E \subset \widetilde{\mathrm{P}}_\kappa^{(0)}(E, \delta)$ are linearly dependent. The isomorphism with $\mathbb{R}[x]$ then follows. □

Remark. Basing $\mathrm{P}_\kappa^{(0)}(E, \delta)$ on $T(E)$ instead of $\mathrm{P}_\kappa(E, \delta)$ means that it does not contain an $\mathfrak{so}(3, \mathbb{R})$-action on its elements. Therefore, no rotations of its elements are possible within $\mathrm{P}_\kappa^{(0)}(E, \delta)$. While this might seem alarming at first, recall that in quantum mechanics we may only rotate position operators using generators of *orbital* angular momentum, i.e. tensors of both position and momentum operators, with the action determined by the Heisenberg algebra. This structure is not present in $\mathrm{P}_\kappa^{(0)}(E, \delta)$, and so there is no contradiction.

5.6.2.3 $\mathbf{P}_\kappa^{(s\neq 0)}(E,\delta)$ and Non-Commutative Geometry

We established in Corollary 5.6.5 that the structure of a spin-0 system entails a commutative geometry within our algebra of position operators. We may generalise this statement to all spins,

Theorem 5.6.7. *In the sense of non-commuting position operators[28, 29, 30], the structure of a spin-s system subsumed within* $\mathrm{P}_\kappa^{(s)}(E,\delta)$ *generates a non-commutative geometry within* $\mathrm{P}_\kappa^{(s)}(E,\delta)$ *iff* $s \neq 0$*.*

Proof. We may prove the above statement by proving its contraposition. One direction of this is proven in Corollary 5.6.5, so let us prove its converse. Assume a commutative geometry within $T(E)$, so, $\forall a, b \in E$

$$\Lambda^2(E) = \{0\}.$$

Since $\mathrm{Im}(M^{(1)}) \cong \Lambda^2(E)$, our algebra must subsume the structure of a spin-0 system. □

This is the central result of this chapter: the presence of spin in a system entails a natural, spin-dependent non-commutative geometry for that system. Such non-commutative geometries are much weaker than those common to the literature[28, 29, 30], which typically place the position operators into a Heisenberg-like[84, 75] algebra.

The physics of such models are beyond the scope of this thesis, but we may appreciate the significance of their structure by noting that both the spin and position degrees of freedom do not commute, both within themselves and with each other. This implies that systems cannot be localised to a single point in space in this model, nor can arbitrary spin and position degrees of freedom be simultaneously measured to arbitrary precision. Neither of these constraints are present in traditional quantum mechanical models for systems with spin.

5.7 Discussion

5.7.1 Scope and Extensions of Model

As a non-commutative algebra of position operators, the $\mathrm{P}_\kappa^{(s)}(E,\delta)$ form the foundation for a fundamentally new approach to incorporating spin into quantum mechanical theories. Of course, each $\mathrm{P}_\kappa^{(s)}(E,\delta)$ is not a complete model for a quantum mechanical point particle; it contains only the position and spin degrees of freedom, and so lacks the momentum degrees of freedom.

However, the structure of $\mathrm{P}_\kappa^{(s)}(E,\delta)$ is amenable to extensions; in fact, this was a consideration from the beginning. For example, it may be possible to extend the model to not only include momentum degrees of freedom, but to do so in a way that enables dynamical symmetries, i.e. symplectic transformations[69, 57], to also be realised within the algebra, as we have achieved for the special orthogonal transformations.

5.7.2 Comparison with Existing Models of Spin Systems

Despite the limited scope, it is important that we contrast the $\mathrm{P}_\kappa^{(s)}(E,\delta)$ with standard quantum mechanical models of arbitrary spin systems. In such a model, position and spin degrees of freedom always commute[85], as do the position degrees of freedom within themselves. This shows that these observables are not independent in any $\mathrm{P}_\kappa^{(s\neq 0)}(E,\delta)$, and necessarily entails higher-order correlations between position and spin, and between position observables themselves. Such couplings are naturally motivated in our approach; the only exception to this statement is the spin-0 model, which is completely equivalent to the standard quantum mechanical model.

Clearly such corrections to the commuting theory must be small to have not yet been observed. Indeed, many models with non-commutative position operators, for example [86, 80], include a small area scale in the position operator commutator, which is accommodated in our model through the parameter κ. In an extended model which includes momentum, we would also expect weak couplings involving of these and the other degrees of freedom in the system not present in traditional descriptions.

5.7.3 Comparison with other Algebraic Models

It is also instructive to contrast the $\mathrm{P}_\kappa^{(s\neq 0)}(E,\delta)$ with other known descriptions of spin-s systems with $s\neq 0$, principally between $\mathrm{P}_\kappa^{(1/2)}(E,\delta)$ and the Clifford, and $\mathrm{P}_\kappa^{(1)}(E,\delta)$ and the Duffin-Kemmer-Petiau algebras for Euclidean three-space respectively. The Clifford and Duffin-Kemmer-Petiau algebras are both finite-dimensional whereas both $\mathrm{P}_\kappa^{(1/2)}(E,\delta)$ and $\mathrm{P}_\kappa^{(1)}(E,\delta)$ are infinite-dimensional by Lemma 5.6.4.

This makes the $\mathrm{P}_{\kappa}^{(s)}(E,\delta)$ more appropriate for constructing models with position-dependent potentials. Furthermore, we may always find a value of κ such that $\mathfrak{so}(3,\mathbb{R})$-actions are identical between $\mathrm{P}_{\kappa}^{(1/2)}(E,\delta)$ and the Clifford algebra, and between $\mathrm{P}_{\kappa}^{(1)}(E,\delta)$ and the Duffin-Kemmer-Petiau algebra. This suggests the Clifford and Duffin-Kemmer-Petiau algebras for Euclidean three-space may be recovered from $\mathrm{P}_{\kappa}^{(1/2)}(E,\delta)$ and $\mathrm{P}_{\kappa}^{(1)}(E,\delta)$ respectively (for particular values of κ). This may be achieved by further quotienting $\mathrm{P}_{\kappa}^{(1/2)}(E,\delta)$ and $\mathrm{P}_{\kappa}^{(1)}(E,\delta)$ by ideals generated by their defining algebraic relations. Furthermore, this raises the possibility that $\mathrm{P}_{\kappa}^{(s)}(E,\delta)$ may be used to derive finite-dimensional, higher-spin generalisations of both the Clifford and Duffin-Kemmer-Petiau algebras.

5.7.4 Beyond Non-Relativistic Models

Beyond non-relativistic quantum mechanics, the non-commutative geometric aspects of relativistic generalisations of the $\mathrm{P}_{\kappa}^{(s\neq 0)}(E,\delta)$ may prove useful in the construction of theories of quantum gravity which incorporate both non-commutative geometry and spin. Generalisations of the present model provide both motivation for a non-commutative geometric approach, and constraints for the theory to conform to under suitable limits.

5.8 Chapter Summary

In this chapter, we demonstrated that the notion of "internal space" is not required to model the structure of an arbitrary spin system in space. To do this, we sought to explicate the connection between spin and geometry hinted at in Chapter 4 by constructing algebras of position operators in which the spin structure of the system is encoded within its algebra of bivectors. Let us now summarise the important findings of this chapter.

We first showed, in Theorem 5.3.7, that the Clifford algebra cannot be altered to support an arbitrary spin structure,

Theorem. *The spin-$\frac{1}{2}$ structure of* $\mathrm{Cl}(E,\delta)$ *is a direct result of its defining relation,* $\forall a,b\in E$*,*

$$ab+ba=2\delta(a,b),$$

and so cannot form the basis for the algebras we wished to construct.

Next, in Theorem 5.4.7, we proved that any algebraic structure which implements an action of $\mathfrak{so}(\mathcal{V}, g)$ must do so via a commutator with a bivector,

Theorem. *For arbitrary Minkowski space-times* $(\mathcal{V}, g)$*, the only* $f(a, b, c)$ *which shares all properties of* $t(a,b)(c)$ *is,*

$$f(a, b, c) = k\big((a\otimes b - b\otimes a)\otimes c - c\otimes(a\otimes b - b\otimes a)\big),$$

where $k \in \mathbb{R}$.

From this result, in Definition 5.5.1, we defined a general algebra of position operators which does so,

Definition. Given a Minkowski space-time $(\mathcal{V}, g)$, we define the "Indefinite-Spin Position Algebra" $\mathrm{P}_\kappa(\mathcal{V}, g)$ as the quotient, $\forall a, b, c \in \mathcal{V}$,

$$\mathrm{P}_\kappa(\mathcal{V}, g) \cong \frac{T(\mathcal{V})}{I\left((a\wedge b)\otimes c - c\otimes(a\wedge b) - \kappa(g(a,c)b - g(b,c)a)\right)},$$

for some $\kappa \in \mathbb{R}$.

We also proved that, unlike the Clifford algebra, this algebra does not inherently contain any spin structure.

Using $\mathrm{P}_\kappa(\mathcal{V}, g)$ as a base, in Definition 5.6.1, we defined a family of position operator algebras in which the spin degrees of freedom are encoded as desired,

Definition. The Spin-s Position Algebra $\mathrm{P}^{(s)}_\kappa(E, \delta)$ for $s \in \{0, \frac{1}{2}, 1, \dots\}$ is the quotient algebra,

$$\mathrm{P}^{(s)}_\kappa(E, \delta) \cong \begin{cases} \dfrac{\mathrm{P}_\kappa(E, \delta)}{I\left(\mathrm{Im}(M^{(2s+1)})\right)} & s \neq 0 \\ \dfrac{T(E)}{\mathrm{Im}(M^{(1)})} & s = 0, \end{cases}$$

with the multipole $M^{(2s+1)}$ understood to be tensors of bivectors.

When $s = 0$, this structure is identical to the position operator algebra of a spin-0 particle. These algebras suggest that there should exist non-trivial couplings between the spin and position degrees of freedom in a system, and that spin may be a natural source of non-commutative geometry in quantum mechanics.

Conclusion

In this thesis, we have sought to develop elements of an algebraic theory for the spin of non-relativistic systems. Through these developments, we have aimed to better understand canonical aspects of the theory of spin by attempting to exclude their associated mathematical structures, and in so doing discover what may still be formulated about spin using elementary algebraic arguments alone. In particular, we have undertaken this work excluding, amongst other things: complex numbers; angular momentum; quantum mechanics; relativity; energy; and time. Indeed, we have developed this theory using only the geometry and rotational symmetry of Euclidean three-space as encoded by the Euclidean metric; this demonstrates that these concepts, and not the dynamical notions or symmetries of the model, determine the fundamental structure of a system of arbitrary spin. In this sense, we have shown that the existence and properties of the spin of a system is fundamentally a consequence of the geometry of the Euclidean space the system inhabits.

We developed a methodology which allowed us to extract meaningful information about an element of a unital associative algebra in the absence of basis choice or the existence of eigenvalues. In particular, this yielded a resolution of the identity of the algebra, and enabled us to decompose arbitrary elements of the algebra, or in the case of operators arbitrary vector subspaces of their domains, using the information contained in the minimal polynomial alone. To determine the necessary coefficients of this identity resolution, we generalised Taylor series to series of arbitrary irreducible polynomials, and proved the form for these coefficients in general. This supported our claim that such an elementary algebraic analysis of spin is possible.

Using this algebraic framework, we demonstrated that complex numbers, angular momentum, quantum mechanics, and special relativity, are not necessary to formulate spin in its most fundamental form. We achieved this by developing a formalism to decompose the universal enveloping algebra of a semisimple Lie algebra in a manner inherently compatible with its algebraic structure. Using this decomposition for the generators of rotational symmetry in Euclidean three-space, we derived real algebraic descriptions for the structure of the generators of a system of arbitrary spin. In so doing, we discovered that a spin-s system is a finite collection of non-commutative generalisations of Cartesian multipole tensors, and is completely determined by specifying only the largest non-zero multipole. These multipoles constitute a complete set of physically distinct observables for the system. This also revealed that complex numbers, or complex structure, combined with the structure of the spin algebras, realises the angular momentum-like phenomenology of spin as an emergent prediction of quantum mechanics.

Finally, we demonstrated that the notion of "internal degrees of freedom" are not necessary to model a system with both spin and position. We did this by seeking to explicate the relationship between spin and geometry hinted by our previous construction, by defining a family of position operator algebras in which the spin structures for systems of arbitrary spin are encoded in their algebras of bivectors. To achieve this, we first constructed the most general algebra of position operators which contains a natural action for the generators of orthogonal transformations, but contains no inherent spin structure. Using this, we were able to construct a family of position operator algebras each embedding the algebraic structure for a system of arbitrary spin. These algebras are consistent with spin-0 quantum mechanical systems when the spin structure is trivial. For non-zero spin however, we demonstrated that these algebras entail couplings between the position and spin degrees of freedom of the system, and indicate that spin may be a natural source of non-commutative geometry in quantum mechanics.

Outlook

In this thesis, many results and methods have been derived which offer not only a novel and informative perspective on spins systems, but also a valuable toolset for exploring the fundamental structures entailed by physical assumptions in our models. The promise of this work is that such insights may be gained for a much wider range of physical phenomena than presented here. As such, let us now consider what the future of this work might look like.

Algebraic Methods

The algebraic methods developed in Chapter 3 are general and can be used for elementary analyses in myriad settings within mathematics. For example, the resolutions of the identity we derived yield an algebraic description of eigenspace projectors; given an interpretation for the elements such a projector contains, this formalism offers expressive power in interpreting the meaning of the eigenvectors of the space in such terms. We may also use the formal irreducible power series to give closed-form expressions for functions of algebraic elements commonly used in physics, such as exponentials; this would allow for a completely algebraic treatment of many physical problems such as the dynamics of spin systems in quantum mechanics, and enable a richer accounting of the processes involved. These methods may also be generalised further to accommodate, for example, finite fields.

Spin Algebras

Chapter 4 presented an algebraic framework for studying real Lie algebras. The output of this method was a family of real associative algebras which captured the structure of the Lie algebra's real irreducible representations. These algebras offered a rich description of such systems in terms of their physical properties which does not require eigenstates, unlike traditional formalisms. It also revealed that systems of arbitrary spin may be characterised exactly in terms of what set of multipoles it possesses, which does not require a choice of spatial direction, unlike traditional formalisms. The structure of these algebras revealed both the direct consequences of our physical assumptions, and the limits of what phenomena can be described from them alone. They also offered a route towards exhaustively accounting for all possible interactions between the modelled system and its environment.

Beyond the results of that chapter, the framework we defined to derive these algebras may be applied to other important symmetries in physics, and in so doing enhance our understanding of a broad range of phenomena. For example, we may study: the Lorentz group $\mathrm{SO}(3,1,\mathbb{R})$ to investigate relativistic systems with spin, which may allow us better understand the difficulties in modelling higher spin systems in quantum field theories; the gauge group of the strong interaction $\mathrm{SU}(3,\mathbb{C})$, which may lead to both a deeper knowledge of hadrons through their observables, and the improvements in hadronisation simulations that such insights bring; dynamical symmetry groups such as $\mathrm{Sp}(6,\mathbb{R})$, which may reveal previously unknown relationships between dynamical systems through their masses.

The algebras which would result from such investigations contain both the physical properties of the system as elements, and the relationships between them in closed-form through its product. This information may be utilised to model collections of such systems on a computer without the use of matrices. This description may offer scaling advantages over traditional methods, empowering a considerable improvement in our phenomenological capabilities.

Non-Commutative Position Operator Algebras

Chapter 5 offered a new model of systems of arbitrary spin which did not require the notion of "internal" degrees of freedom, and predicted novel coupling phenomena between the position and spin degrees of freedom of a system not present in traditional quantum mechanics. As such, these models present a rich phenomenology to explore for systems of arbitrary spin, which may lead to the design of new experiments to detect such effects.

The methods for constructing these non-commutative position algebra models are also broadly applicable in other settings, such as relativistic systems with spin in Minkowski space, or non-relativistic systems which also have momentum. The non-trivial couplings entailed by such models will also recur in these more general settings, yielding a wide variety of new physical phenomena to explore. These additional couplings between physical observables may also motivate and constrain models of quantum gravity which seek to incorporate both non-commutative geometry and spin.

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Appendix A

Introduction

A.1 Representation Theory in Brief

This section provides an overview of the weight space approach to the representation theory of a Lie algebra. All concepts utilised here are defined rigorously in Chapter 2.

A.1.1 Representations, Subrepresentations, and Irreducible Representations

In general, a "representation" of a mathematical structure $\mathcal{D}$ is a left-$\mathcal{D}$ module $\mathcal{M} = (\mathcal{W}, \mathcal{D}, f)$. Typically, only representations for which $\mathcal{W}$ is a complex vector space are considered, as this may be assumed without loss of generality [87, 7]. When we discuss the dimension of a representation $\mathcal{M} = (\mathcal{W}, \mathcal{D}, f)$, we are implicitly referring to the dimension of the vector space $\mathcal{W}$.

It is important to note that representations for a given mathematical structure are never unique, nor are there ever finitely many of them. Despite this, "representation theory" as a discipline seeks to tame this infinity by systematically cataloguing these representations. For many mathematical structures, such as the Heisenberg group[75, 84], this problem is made all the more challenging by the lack of any finite-dimensional representations, which prevents matrix methods from being used. We will discuss methods for finding finite-dimensional representations in the case of Lie algebras in Section A.1.2.

Given a representation $\mathcal{M} = (\mathcal{W}, \mathcal{D}, f)$ for some mathematical structure $\mathcal{D}$, suppose there exists a subspace $\mathcal{U} \subseteq \mathcal{W}$ such that $\forall a \in \mathcal{D}$,

$$\mathrm{Im}(\rho(a)\circ i) \subseteq \mathcal{U}, \tag{A.1.1}$$

with $i : \mathcal{U} \to \mathcal{W}$ the inclusion map. Then, we may define a "subrepresentation"[50] $\mathcal{P} = (\mathcal{U}, \mathcal{D}, \tau)$ of $\mathcal{M}$, where $\forall a \in \mathcal{D}$, $i\circ\tau(a) := \rho(a)\circ i$. Since, by this definition, every representation is a subrepresentation of itself, we call a subrepresentation which is not equal to the original representation "proper".

We note that the trivial representation $(\{0\}, \mathcal{D}, \rho)$ of $\mathcal{D}$ is a proper subrepresentation for all other representations $\mathcal{M} = (\mathcal{W}, \mathcal{D}, f)$ of $\mathcal{D}$. If the trivial representation is the **only** proper subrepresentation of $\mathcal{M} = (\mathcal{W}, \mathcal{D}, f)$, we call $\mathcal{M}$ "irreducible"[7, 64]. Irreducible representations are particularly important to the study of Lie algebra representations due to "Weyl's Theorem on Complete Reducibility"[7, 64], which states that any representation of a semisimple Lie algebra can be written as a direct sum of irreducible representations, provided that repeated addition of the identity element in the field of scalars of the Lie algebra never equals zero.

A.1.2 Representation Theory of $\mathfrak{so}(3,\mathbb{R})$ Through Weight Spaces

As the main results of this thesis are most closely related to Lie algebra representations, it is worth considering the mathematical machinery most commonly used to do representation theory: "Root Systems" and "Weight Spaces". While the precise details of this method are largely unimportant to this thesis, we will present the method applied to $\mathfrak{su}(2,\mathbb{C}) \cong \mathfrak{so}(3,\mathbb{R})$ to enable comparison with our results. A more general account of the method may be found in [55, 7, 64, 50, 54].

The Lie algebra $\mathfrak{so}(3,\mathbb{R})$ is Lie algebra isomorphic to,

$$\mathfrak{so}(3,\mathbb{R}) \cong (\mathrm{span}_{\mathbb{R}}(\{S_1, S_2, S_3\}), \times) \tag{A.1.2}$$

with Lie product,

$$S_a \times S_b = \sum_{c=1}^{3} \varepsilon_{abc} S_c. \tag{A.1.3}$$

Consider an arbitrary irreducible Lie algebra representation $\mathcal{M} = (\mathcal{W}, \mathfrak{so}(3,\mathbb{R}), \rho)$ over a complex finite-dimensional vector space $\mathcal{W}$. For simplicity of discussion, let us also assume that ρ is injective. At this point, most authors complexify their real Lie algebra,

$$\mathfrak{so}(3,\mathbb{R})_{\mathbb{C}} \cong \mathfrak{so}(3,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C} \cong (\mathrm{span}_{\mathbb{C}}(\{S_1, S_2, S_3\}), \times'), \tag{A.1.4}$$

where the new Lie product $\times'$ is the unique complex bilinear extension to the original Lie product $\times$[7]. This is harmless, since all complex irreducible representations of the real Lie algebra $\mathcal{M} = (\mathcal{W}, \mathfrak{so}(3,\mathbb{R}), \rho)$ extend uniquely to complex irreducible representations $\mathcal{M}' = (\mathcal{W}, \mathfrak{so}(3,\mathbb{R})_{\mathbb{C}}, \rho')$ of the complexified Lie algebra[7]. We see that ρ' is also injective.

One may have noticed that the Lie product $\times$ in (A.1.3) and, its extension $\times'$, are a factor i from the usual expression,

$$\hat{S}_a \times' \hat{S}_b = i \sum_{c=1}^{3} \varepsilon_{abc} \hat{S}_c, \tag{A.1.5}$$

found in the physics literature. It is worth highlighting at the Lie product (A.1.5) is only valid when discussing $\mathfrak{so}(3,\mathbb{R})_{\mathbb{C}}$, as most authors implicitly do. All work presented in this thesis will utilise the real $\mathfrak{so}(3,\mathbb{R})$ and thus the product (A.1.3) unless explicitly stated.

From here we adopt a special basis,

$$S_{\pm} := i(S_x \pm i S_y) \tag{A.1.6a}$$

$$H := i S_z, \tag{A.1.6b}$$

referred to in the physics literature as a basis of "ladder operators"[1]. In more mathematical terms this basis reconstructs a "Cartan decomposition"[69] of $\mathfrak{so}(3,\mathbb{R})_{\mathbb{C}}$, where H and its generalisations are referred to as "roots". While the precise definition of a Cartan decomposition is irrelevant for our purposes, its existence relies on the guarantee of eigenvalues for $\mathrm{ad}(H)$ that the complexification (A.1.4) brings[1]. Using this basis, we find,

$$\begin{aligned} H \times' S_{\pm} &= \pm S_{\pm} \\ S_+ \times' S_- &= 2H. \end{aligned} \tag{A.1.7}$$

As we are working with a complex vector space $\mathcal{W}$, we are guaranteed that $\exists v_\lambda \in \mathcal{W}$ such that,

$$\rho'(H)(v_\lambda) = \lambda\, v_\lambda, \tag{A.1.8}$$

for some "weight" $\lambda \in \mathbb{C}$. From (A.1.7), and the fact that ρ' is a Lie algebra homomorphism, we find a family of eigenstates with distinct eigenvalues related by all possible combinations of applications of $\rho'(S_{\pm})$,

$$\rho'(H) \circ \rho'(S_{\pm})(v_\lambda) = (\lambda \pm 1)\rho'(S_{\pm})(v_\lambda). \tag{A.1.9}$$

[1]More specifically, the algebraic closure of $\mathbb{C}$ is what guarantees the existence of at least one eigenvalue for endomorphisms on a complex vector space[7, 46].

By construction, this family forms a subrepresentation $\mathcal{P}'$ of $\mathcal{M}'$. However, since $\mathcal{M}'$ is irreducible, and the fact that $\mathcal{P}'$ is non-trivial by the injectivity of ρ', we must have $\mathcal{P}' = \mathcal{M}'$. Furthermore, since $\mathcal{M}'$ is finite-dimensional, there must be some non-zero $v_{\lambda'} \in \mathcal{W}$ for which,

$$\rho'(S_+)(v_{\lambda'}) = 0. \tag{A.1.10}$$

This $v_{\lambda'}$ is called the "highest weight vector" and its eigenvalue λ' the "highest weight" for the representation $\mathcal{M}'$. By the same argument, we must have,

$$\rho'(S_-)^{\circ \dim(\mathcal{W})}(v_{\lambda'}) = 0, \tag{A.1.11}$$

but,

$$\rho'(S_-)^{\circ(\dim(\mathcal{W})-1)}(v_{\lambda'}) \neq 0, \tag{A.1.12}$$

Following [7], we may define,

$$v_{\lambda'-k} \coloneqq \rho'(S_-)^{\circ k}(v_{\lambda'}), \tag{A.1.13}$$

and can easily show both,

$$\rho'(H)(v_{\lambda'-k}) = (\lambda' - k)v_{\lambda'-k}, \tag{A.1.14}$$

and,

$$\rho'(S_+)(v_{\lambda'-k}) = k(2\lambda' - (k-1))v_{\lambda'-(k-1)} \tag{A.1.15}$$

Combining (A.1.13) with (A.1.15) when $k = \dim(\mathcal{W})$ we find,

$$0 = (\dim(\mathcal{W}))(2\lambda' + 1 - \dim(\mathcal{W}))v_{\lambda'-(\dim(\mathcal{W})-1)}. \tag{A.1.16}$$

By (A.1.12) and $\dim(\mathcal{W}) \geq 1$, we must have $2\lambda' + 1 = \dim(\mathcal{W}) \in \mathbb{Z}^+$, and so $2\lambda' \in \mathbb{N}$.

Thus, we find we may label these irreducible representations by a half-natural $s = \lambda'$. For more complicated Lie algebras, we require multiple numbers associated with the action of all Casimir elements on the representation[63]. For $\mathfrak{so}(3, \mathbb{R})$, we identify the highest weight s with the spin s of the system described by the representation $\mathcal{M}'$, and the vector space $\mathcal{W}$ as the space of spinors. In more general settings, the interpretation of these labelling numbers and the vectors of the representation is much less straightforward.

Appendix B

Mathematical Background

B.1 Vector Spaces in Brief

B.1.1 Role in this Thesis

In this thesis, we will use objects of definite geometrical character to construct algebras which encode the structure of arbitrary spin systems. In order to do this we must first understand the concept and structure of a vector space, since all algebras are build from them. This section is also included to help disambiguate the terms "vector" and "scalar", since these have many different meanings across physics especially in, for example, high-energy physics.

B.1.2 Vector Spaces

Definition B.1.1 (Vector Space). A vector space $\mathcal{V}$ over $\mathbb{F}$ is a quadruple $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$ consisting of:

- a set V, whose elements we call vectors;
- a field of numbers $\mathbb{F}$, whose elements we call scalars;
- an associative, commutative map $+_{\mathcal{V}} : V \times V \to V$, which we call vector addition, for which, $\forall a \in V$,

$$\exists 0 \in V : 0 +_{\mathcal{V}} a = a \tag{B.1.1a}$$

$$\exists (-a) \in V : (-a) +_{\mathcal{V}} a = 0; \tag{B.1.1b}$$

- and a map $\cdot_{\mathcal{V}} : \mathbb{F} \times V \to V$, which we call scalar multiplication, such that, $\forall a, b \in V$, $\forall \alpha, \beta \in \mathbb{F}$,

$$\alpha \cdot_{\mathcal{V}} (\beta \cdot_{\mathcal{V}} a) = (\alpha \times_{\mathbb{F}} \beta) \cdot_{\mathcal{V}} a \tag{B.1.2a}$$

$$1 \cdot_{\mathcal{V}} a = a \tag{B.1.2b}$$

$$\alpha \cdot_{\mathcal{V}} (a +_{\mathcal{V}} b) = (\alpha \cdot_{\mathcal{V}} a) +_{\mathcal{V}} (\alpha \cdot_{\mathcal{V}} b) \tag{B.1.2c}$$

$$(\alpha +_{\mathbb{F}} \beta) \cdot_{\mathcal{V}} a = (\alpha \cdot_{\mathcal{V}} a) +_{\mathcal{V}} (\beta \cdot_{\mathcal{V}} a), \tag{B.1.2d}$$

where $1 \in \mathbb{F}$ is the multiplicative identity, and $+_{\mathbb{F}} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and $\times_{\mathbb{F}} : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ are the addition and product of the field $\mathbb{F}$ respectively.

There is a somewhat trivial but very important example of a vector space which must be defined,

Lemma B.1.2. *The quadruple (abusing notation)* $\{0\} = (\{0\}, \mathbb{F}, +, \cdot)$ *with* $+$ *and* $\cdot$ *defined as,* $\forall \alpha \in \mathbb{F}$,

$$0 + 0 = 0 \tag{B.1.3a}$$

$$\alpha \cdot 0 = 0, \tag{B.1.3b}$$

is a vector space.

Proof. The vector space axioms may be trivially verified. □

Definition B.1.3 (Trivial Vector Space)**.** We call the vector space $\{0\}$ from Lemma B.1.2 the trivial vector space.

In this thesis: the field of scalars $\mathbb{F}$ will almost always be the real numbers $\mathbb{R}$; we will often omit subscripts on vector additions and imply scalar multiplications by juxtaposition to aid readability; and we will write $v \in \mathcal{V}$ to mean the same as $v \in V$, where $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$.

B.1.3 Linear Combinations and Linear Independence

To solve problems in the context of a vector space $\mathcal{V}$, we must understand the structure of expressions in $\mathcal{V}$. Let us first, define,

Definition B.1.4 (Linear Combination)**.** An $\mathbb{F}$-linear combination of a finite set of vectors $\{a_j \in \mathcal{V}\}$ indexed over a set J is an expression,

$$\sum_{j \in J} \alpha_j \cdot_{\mathcal{V}} a_j, \tag{B.1.4}$$

where $\forall j \in J : \alpha_j \in \mathbb{F}$. We also refer to this expression as simply a linear combination when the field of scalars is clear from context.

Of particular importance are linear combinations for which the vectors $\{a_j \in \mathcal{V}\}$ are "independent" of each other. Let us formalise this notion,

Definition B.1.5 (Linear Independence (Finite)). A finite set of vectors $\{a_j \in \mathcal{V}\}$ indexed over a set J is linearly independent iff,

$$\left[\sum_{j \in J} \alpha_j \cdot_{\mathcal{V}} a_j = 0\right] \Rightarrow \left[\forall j \in J : \alpha_j = 0\right]. \tag{B.1.5}$$

Linear independence for a finite set of vectors is a strong condition, in particular,

Lemma B.1.6. *If a finite set of vectors $\{a_j \in \mathcal{V}\}$ indexed over a set J is linearly independent, every non-empty finite proper subset of $\{a_j \in \mathcal{V}\}$ is linearly independent.*

Proof. See [46]. □

We may use Lemma B.1.6 to extend the notion of linear independence to infinite sets,

Definition B.1.7 (Linear Independence (Infinite)). An infinite set of vectors $\{a_j \in \mathcal{V}\}$ indexed over a set J is linearly independent iff every non-empty finite subset of $\{a_j \in \mathcal{V}\}$ is linearly independent.

In many cases, a set of vectors will not be linearly independent as one or more of them may be written as a linear combination of the others. When this happens we say,

Definition B.1.8 (Linear Dependence). A set of vectors $\{a_j \in \mathcal{V}\}$ indexed over a set J is linearly dependent iff it is not linearly independent.

B.1.4 Subspaces

It is usually the case that we may learn a lot about the vector space $\mathcal{V}$ representing some physical system by considering subsets of its vectors which respect vector addition and scalar multiplication.

Lemma B.1.9. *Consider a subset of vectors* $U \subseteq V$ *from a vector space* $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$ *for which,* $\forall a, b \in U$ *and* $\forall \alpha \in \mathbb{F}$*,*

$$a +_{\mathcal{V}} b \in U \tag{B.1.6a}$$

$$\alpha \cdot_{\mathcal{V}} a \in U. \tag{B.1.6b}$$

Then $\mathcal{U} = (U, \mathbb{F}, +_{\mathcal{U}}, \cdot_{\mathcal{U}})$ *is a vector space, where,*

$$i \circ +_{\mathcal{U}} := +_{\mathcal{V}} \circ j \tag{B.1.7a}$$

$$i \circ \cdot_{\mathcal{U}} := \cdot_{\mathcal{V}} \circ j, \tag{B.1.7b}$$

where $i : U \to V$ *and* $j : U \times U \to V \times V$ *are inclusion maps, and* $\circ$ *is function composition.*

Proof. For all $a \in U$, we have $0 \cdot_{\mathcal{V}} a = 0 \in U$ and $-1 \cdot_{\mathcal{V}} a = (-a) \in U$. The other vector space axioms may be trivially checked. □

Definition B.1.10 (Subspace)**.** We call $\mathcal{U} = (U, \mathbb{F}, +_{\mathcal{U}}, \cdot_{\mathcal{U}})$ as in Lemma B.1.9 a subspace of $\mathcal{V}$, and write $\mathcal{U} \subseteq \mathcal{V}$.

Remark. Often in this thesis we will prove that a subset of vectors is a subspace of a vector space $\mathcal{V}$ without specifying the vector addition and scalar multiplication explicitly. In these cases, the implied vector addition and scalar multiplication are inherited from $\mathcal{V}$ in the sense of Lemma B.1.9.

There are some obvious, but important, subspaces which are shared by all vector spaces,

Lemma B.1.11. *Both* $\{0\} \subseteq \mathcal{V}$ *and* $\mathcal{V} \subseteq \mathcal{V}$*, where* $0 \in \mathcal{V}$ *is the identity element of vector addition in* $\mathcal{V}$*.*

Proof. Direct verification of the subspace axioms. □

In light of Lemma B.1.11, let us distinguish,

Definition B.1.12 (Proper Subspace)**.** A subspace $\mathcal{U} \subseteq \mathcal{V}$ is proper iff $\mathcal{U} \neq \{0\}$ and $\mathcal{U} \neq \mathcal{V}$.

B.1.5 Linear Span

Slightly less obvious but important subspaces of a vector space arise from arbitrary linear combinations of a set of vectors. Such subspaces enable us to probe the interrelationship between vectors in a vector space. To this end, let us define,

Definition B.1.13 (Linear Span)**.** The $\mathbb{F}$-linear span of a set of vectors $A = \{a_j \in \mathcal{V}\}$ indexed over a set J is the set of all finite linear combinations,

$$\operatorname{span}_{\mathbb{F}}(A) \coloneqq \left\{ \sum_{j \in K} \alpha_j \cdot_{\mathcal{V}} a_j \,\middle|\, \forall \alpha_j \in \mathbb{F}, K \subseteq J : |K| \in \mathbb{Z}^+ \right\}. \tag{B.1.8}$$

When the field of scalars is clear from context we may omit the subscript $\mathbb{F}$ from the notation, and the prefix $\mathbb{F}$- from the name.

Remark. This definition for linear span excludes infinite linear combinations like one might find in a Fourier series. Such a restriction will not limit the developments of this thesis.

Lemma B.1.14. *Given a set of vectors $A = \{a_j \in \mathcal{V}\}$ indexed over a set J, $\mathcal{S} = (\operatorname{span}_{\mathbb{F}}(A), \mathbb{F}, +_{\mathcal{S}}, \cdot_{\mathcal{S}}) \subseteq \mathcal{V}$, with vector addition and scalar multiplication defined as in Lemma B.1.9.*

Proof. The vector space axioms may be easily verified for $\mathcal{S}$. □

B.1.6 Basis and Dimension

The definition of $\operatorname{span}_{\mathbb{F}}(A)$ invokes a naturally prompts us to consider spans when A is linearly independent. Such A are of particular importance to the study of vector spaces, as they empower the description of the whole vector space in a simple way. To see how, we define,

Definition B.1.15 (Basis)**.** A basis B of $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$ is a set of linearly independent vectors $\{a_j \in \mathcal{V}\}$ indexed over a set J such that $\operatorname{span}_{\mathbb{F}}(B) = \mathcal{V}$.

Now, note,

Lemma B.1.16. *All vectors $v \in \mathcal{V}$ may be written as a unique linear combination of finitely-many basis vectors $\{b_j \in B\}$, which are indexed over a set J.*

Proof. Since $\text{span}_{\mathbb{F}}(B) = \mathcal{V}$ and by the definition of the span, every vector $v \in \mathcal{V}$ may be written as a linear combination of finitely-many basis vectors $\{b_j \in B\}$. To establish uniqueness, consider two linear combinations,

$$v = \sum_{k \in K} \alpha_k \cdot_{\mathcal{V}} b_k = \sum_{l \in L} \beta_l \cdot_{\mathcal{V}} b_l,$$

where $K \subseteq J$ and $L \subseteq J$ are finite indexing sets. Computing their difference we find,

$$0 = \sum_{p \in K \setminus L} \alpha_p \cdot_{\mathcal{V}} b_p - \sum_{q \in L \setminus K} \beta_q \cdot_{\mathcal{V}} b_q + \sum_{i \in K \cap L} (\alpha_i - \beta_i) \cdot_{\mathcal{V}} b_i,$$

with $A \setminus B$ the set difference. By the linear independence of B, we see that both linear combinations are the same. □

Amongst other things, bases allow us to categorise vector spaces according to the number of vectors they contain,

Definition B.1.17 (Dimensionality)**.** A vector space $\mathcal{V}$ is finite-dimensional iff it contains a basis B such that $|B| \in \mathbb{N}$. Otherwise, $\mathcal{V}$ is infinite-dimensional.

This is a useful classification, since,

Lemma B.1.18. *All bases in a finite-dimensional vector space contain the same number of vectors.*

Proof. See [46]. □

Thus, we may associate to each finite-dimensional vector space a well-defined number which characterises it,

Definition B.1.19 (Dimension)**.** The dimension $\dim(\mathcal{V})$ of a finite-dimensional vector space $\mathcal{V}$ is the cardinality of any one of its bases.

Many important physical objects, including physical space, may be modelled using finite-dimensional vector spaces. Furthermore, many infinite-dimensional vector spaces important to this thesis may be understood in terms of finite-dimensional vector spaces.

Bases are invaluable tools in a variety of applications across physics and mathematics. However, in this thesis we will be working as much as possible without utilising a particular basis, as doing so will yield clear and valuable insights into the relationship between algebraic objects and the geometry of space.

B.2 Vector Space Homomorphisms in Brief

B.2.1 Role in this Thesis

In this thesis, vector space homomorphisms underpin all the methods used to probe the structure of symmetry transformations and decompose important algebras. They also make precise the notion of equivalence between two vector spaces, which is essential for us to study some infinite-dimensional vector spaces. As such, it is vital that they are discussed.

B.2.2 Vector Space Homomorphisms

The basic idea of a vector space homomorphism is that it is a map between vector spaces which respect their structure.

Definition B.2.1 (Vector Space Homomorphism)**.** Consider two vector spaces $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$ and $\mathcal{W} = (W, \mathbb{F}, +_{\mathcal{W}}, \cdot_{\mathcal{W}})$ over the same field $\mathbb{F}$. A vector space homomorphism between $\mathcal{V}$ and $\mathcal{W}$ is a function $f : V \to W$ such that, $\forall a, b \in V$ and $\alpha \in \mathbb{F}$,

$$f(a +_{\mathcal{V}} b) = f(a) +_{\mathcal{W}} f(b) \tag{B.2.1a}$$

$$f(\alpha \cdot_{\mathcal{V}} a) = \alpha \cdot_{\mathcal{W}} f(a). \tag{B.2.1b}$$

We may also refer to a vector space homomorphism as a linear map, or simply as a homomorphism when the context is clear. We may also write $f : \mathcal{V} \to \mathcal{W}$ to mean the same as a homomorphism $f : V \to W$.

Often, we are not concerned with a particular linear map and wish to consider an arbitrary one. To facilitate discussing such arbitrary maps, let us define,

Definition B.2.2 ($\mathrm{Hom}(\mathcal{V}, \mathcal{W})$)**.** $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$ is the set of all vector space homomorphisms between $\mathcal{V}$ and $\mathcal{W}$.

$\mathrm{Hom}(\mathcal{V}, \mathcal{W})$ naturally inherits a vector space structure from the codomain of its elements,

Lemma B.2.3. *The quadruple (abusing notation),*

$$\mathrm{Hom}(\mathcal{V}, \mathcal{W}) = (\mathrm{Hom}(\mathcal{V}, \mathcal{W}), \mathbb{F}, +, \text{“ \quad ”}), \tag{B.2.2}$$

with $+$ *and* "$\cdot$" *defined as,* $\forall f, g \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$*,* $\forall \alpha \in \mathbb{F}$*,*

$$f + g \coloneqq a \mapsto f(a) +_{\mathcal{W}} g(a) \tag{B.2.3a}$$

$$\alpha f \coloneqq a \mapsto \alpha \cdot_{\mathcal{W}} f(a), \tag{B.2.3b}$$

is a vector space.

Proof. First, we may verify that the definitions of vector addition and scalar multiplication both produce elements of $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$. Second, note that the zero map $0 \coloneqq (a \mapsto 0) \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ and is an additive identity for $+$. Third, note that for every map $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$, the map $g \coloneqq (a \mapsto -f(a)) \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ and is an additive inverse for f. With these facts identified, verifying the vector space axioms may be done directly. □

When $\mathcal{V}$ and $\mathcal{W}$ are finite-dimensional, the dimension of $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$ can be determined,

Lemma B.2.4. *If both* $\mathcal{V}$ *and* $\mathcal{W}$ *are finite-dimensional, then so is* $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$*, and has dimension* $\dim(\mathrm{Hom}(\mathcal{V}, \mathcal{W})) = \dim(\mathcal{V}) \dim(\mathcal{W})$*.*

Proof. Consider a basis $\{b_j\}$ for $\mathcal{V}$ indexed over the set J, and a basis $\{c_k\}$ for $\mathcal{W}$ indexed over the set K. Let us define a set of homomorphisms $\{f_{jk}\}$ indexed over the set $J \times K$, by,

$$f_{ab} \coloneqq b_j \mapsto \begin{cases} c_b & j = a \\ 0 & j \neq a, \end{cases}$$

and extended linearly. By considering a general linear combination of the $\{f_{jk}\}$ on each basis vector $\{b_j\}$ and setting to zero, we see that $\{f_{jk}\}$ is linearly independent. We see that its span is $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$ by inspecting the output of an arbitrary homomorphism acting on each basis vector $\{b_j\}$, written in $\{c_k\}$. Thus, $\{f_{jk}\}$ is a basis for $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$ containing $\dim(\mathcal{V}) \dim(\mathcal{W})$ vectors. □

B.2.3 Kernel and Image of a Homomorphism

Frequently in this work, we are interested in subspaces associated with a particular linear map $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$, as these tell us important information about the way the map behaves. There are two canonical such subspaces associated with every linear map f. The first of these is the kernel, which captures the vectors of $\mathcal{V}$ that f "discards",

Definition B.2.5 (Kernel)**.** The kernel $\mathrm{Ker}(f)$ of a homomorphism $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ is the set,

$$\mathrm{Ker}(f) \coloneqq \{v \in \mathcal{V} \mid f(v) = 0\}. \tag{B.2.4}$$

The second is the image, which captures the vectors of $\mathcal{W}$ that f maps to,

Definition B.2.6. The image $\mathrm{Im}(f)$ of a homomorphism $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ is the set,

$$\mathrm{Im}(f) \coloneqq \{w \in \mathcal{W} \mid \exists v \in \mathcal{V} : w = f(v)\}. \tag{B.2.5}$$

Lemma B.2.7. *The kernel* $\mathrm{Ker}(f)$ *and image* $\mathrm{Im}(f)$ *of a homomorphism* $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ *are subspaces* $\mathrm{Ker}(A) \subseteq \mathcal{V}$ *and* $\mathrm{Im}(A) \subseteq \mathcal{W}$ *respectively.*

Proof. See [46]. □

The kernel and image will be of particular importance in developing the methods of Chapter 3.

B.2.4 Composition of Homomorphisms

Understanding the properties of a composition of homomorphisms will be essential to the algebraic methods we will use throughout this thesis. Importantly,

Lemma B.2.8. *Consider the vector spaces* $\mathcal{V} = (V, \mathbb{F}, +_\mathcal{V}, \cdot_\mathcal{V})$, $\mathcal{W} = (W, \mathbb{F}, +_\mathcal{W}, \cdot_\mathcal{W})$, *and* $\mathcal{X} = (X, \mathbb{F}, +_\mathcal{X}, \cdot_\mathcal{X})$, *and the spaces of homomorphisms* $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$ *and* $\mathrm{Hom}(\mathcal{W}, \mathcal{X})$*. For all* $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ *and* $g \in \mathrm{Hom}(\mathcal{W}, \mathcal{X})$*, their composition,*

$$g \circ f \coloneqq (a \mapsto g(f(a))) \in \mathrm{Hom}(\mathcal{V}, \mathcal{X}). \tag{B.2.6}$$

Proof. Direct computation. □

Composition also has a simple relationship with kernels and images enabling us to extend the usefulness of both concepts,

Lemma B.2.9. *For all* $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ *and* $g \in \mathrm{Hom}(\mathcal{W}, \mathcal{X})$*,*

$$\mathrm{Ker}(f) \subseteq \mathrm{Ker}(g \circ f) \tag{B.2.7a}$$

$$\mathrm{Im}(g \circ f) \subseteq \mathrm{Im}(g). \tag{B.2.7b}$$

Proof. Suppose $v \in \mathrm{Ker}(f)$, then $g \circ f(v) = g(0) = 0$, which proves the first statement. Now suppose $x \in \mathrm{Im}(g \circ f)$, then $\exists v \in \mathcal{V}$ such that $x = g \circ f(v) = g(f(v))$, which proves the second. □

B.2.5 Injective and Surjective Homomorphisms

Understanding the properties of homomorphisms enables us to keep track of how particular vectors are transformed. We are particularly interested in two kinds of homomorphisms: those which maintain the distinctness of input vectors from one another,

Definition B.2.10 (Injective)**.** A homomorphism $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ is injective iff, $\forall a, b \in \mathcal{V}$,

$$[f(a) = f(b)] \Rightarrow [a = b]; \tag{B.2.8}$$

and those that yield all possible output vectors,

Definition B.2.11 (Surjective)**.** A homomorphism $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ is surjective iff $\forall w \in \mathcal{W}$, $\exists v \in \mathcal{V}$ such that,

$$w = f(v). \tag{B.2.9}$$

For a homomorphism, injectivity and surjectivity naturally relate to their kernel and image.

Lemma B.2.12. *A homomorphism* $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ *is injective iff,*

$$\mathrm{Ker}(f) = \{0\}. \tag{B.2.10}$$

Proof. In the forward direction, since $0 \in \mathrm{Ker}(f)$, we have $\forall v \in \mathrm{Ker}(f)$,

$$[0 = f(v) = f(0)] \Rightarrow [v = 0].$$

In the reverse direction, suppose for $a, b \in \mathcal{V}$ that $f(a) = f(b)$, then,

$$f(a - b) = 0,$$

and so $a - b = 0$, thus $a = b$. □

Lemma B.2.13. *A homomorphism* $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ *is surjective iff,*

$$\mathrm{Im}(f) = \mathcal{W}. \tag{B.2.11}$$

Proof. This follows from the definition of the image. □

B.2.6 Identity, Left-, and Right-Inverse Homomorphisms

There is a useful relationship between injective and surjective homomorphisms and composition which we will occasionally exploit: there exist homomorphisms capable of undoing their actions. To explore this, we must first define a mundane but special map,

Definition B.2.14 (Identity Map)**.** The identity map $\mathrm{id}_{\mathcal{V}} \in \mathrm{Hom}(\mathcal{V}, \mathcal{V})$ is defined as,

$$\mathrm{id}_{\mathcal{V}} \coloneqq a \mapsto a. \tag{B.2.12}$$

Importantly,

Lemma B.2.15. *For all* $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$*,*

$$\mathrm{id}_{\mathcal{W}} \circ f = f \tag{B.2.13a}$$

$$f \circ \mathrm{id}_{\mathcal{V}} = f \tag{B.2.13b}$$

Proof. Direct computation. □

When the vector space in question is clear, we will often drop the subscript on the identity map id. Using the identity map, we may define,

Definition B.2.16 (Left- and Right-Inverse)**.** A homomorphism $g \in \mathrm{Hom}(\mathcal{W}, \mathcal{V})$ is called a left-inverse for a homomorphism $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ iff,

$$g \circ f = \mathrm{id}_{\mathcal{V}}, \tag{B.2.14}$$

and a right-inverse for f iff,

$$f \circ g = \mathrm{id}_{\mathcal{W}}. \tag{B.2.15}$$

Such maps, if they exist, can be useful in manipulating compositions but also for determining the properties of a homomorphism, since,

Lemma B.2.17. *A homomorphism* $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ *has a left-inverse iff* f *is injective.*

Proof. In the forward direction, suppose $g \in \mathrm{Hom}(\mathcal{W}, \mathcal{V})$ is a left-inverse for f, then,

$$[f(a) = f(b)] \Rightarrow [g \circ f(a) = g \circ f(b)] \Rightarrow [a = b].$$

In the reverse direction, f injective implies for each $w \in \mathrm{Im}(f)$ there is exactly one $v_w \in \mathcal{V}$ for which $w = f(v_w)$. Therefore, it may be easily verified that any homomorphism $g \in \mathrm{Hom}(\mathcal{W}, \mathcal{V})$ such that $\forall w \in \mathcal{W}$, $g(w) = v_w$ is a left-inverse for f. □

Furthermore,

Lemma B.2.18. *A homomorphism f has a right-inverse iff f is surjective.*

Proof. In the forward direction, suppose $g \in \mathrm{Hom}(\mathcal{W}, \mathcal{V})$ is a right-inverse for f, then, $\forall w \in \mathcal{W}$,

$$[v = g(w)] \Rightarrow [f(v) = f \circ g(w) = w],$$

which shows that $w \in \mathrm{Im}(f)$, thus $\mathcal{W} \subseteq \mathrm{Im}(f)$. Since $\mathrm{Im}(f) \subseteq \mathcal{W}$ by definition, we have surjectivity by Lemma B.2.13. In the reverse direction, for some $w \in \mathcal{W}$ consider,

$$A_w := \{v \in \mathcal{V} \mid w = f(v)\}.$$

By the surjectivity of f, A_w is non-empty $\forall w \in \mathcal{W}$. Thus, it may easily be verified that any homomorphism $g \in \mathrm{Hom}(\mathcal{W}, \mathcal{V})$ such that $\forall w \in \mathcal{W}$, $g(w) \in A_w$ is a right-inverse for g. □

B.2.7 Isomorphisms

In general, left- and right-inverse homomorphisms are not unique. However, there are an essential class of homomorphisms for which they are: the isomorphisms. Isomorphisms enable us to identify wildly different looking vector spaces as essentially the same.

Definition B.2.19 (Vector Space Isomorphism)**.** A homomorphism $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ is an isomorphism iff it has both a left-inverse and a right-inverse.

To understand which homomorphisms are isomorphisms, let us define,

Definition B.2.20 (Bijective)**.** A homomorphism $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ is bijective iff it is both injective and surjective.

Lemma B.2.21. *A homomorphism $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ is an isomorphism iff it is bijective.*

Proof. This follows from Lemmas B.2.17 and B.2.18. □

Isomorphisms tame the non-uniqueness of left- and right-inverses. To see how, let us define,

Definition B.2.22 (Two-Sided Inverse)**.** A homomorphism $g \in \mathrm{Hom}(\mathcal{W}, \mathcal{V})$ is called a two-sided inverse for a homomorphism $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ if it is both a left-inverse and a right-inverse for f.

Lemma B.2.23. *If a homomorphism $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ has both a left- and a right-inverse, its left- and right- inverses are unique and equal to a unique two-sided inverse.*

Proof. Suppose we have two left-inverses $g, h \in \mathrm{Hom}(\mathcal{W}, \mathcal{V})$, and two right-inverses $p, q \in \mathrm{Hom}(\mathcal{W}, \mathcal{V})$. Then,

$$[\mathrm{id}_{\mathcal{V}} = g \circ f = h \circ f] \Rightarrow [g \circ f \circ p = h \circ f \circ p] \Rightarrow [g = h],$$

and similarly,

$$[\mathrm{id}_{\mathcal{W}} = f \circ p = f \circ q] \Rightarrow [g \circ f \circ p = g \circ f \circ q] \Rightarrow [p = q].$$

Finally,

$$f \circ g = f \circ g \circ f \circ p = \mathrm{id}_{\mathcal{W}} .$$

□

To reduce notational complexity in light of Lemma B.2.23, let us define,

Definition B.2.24 (f^{-1})**.** We denote by $f^{-1} \in \mathrm{Iso}(\mathcal{W}, \mathcal{V})$ the unique two-sided inverse of the isomorphism $f \in \mathrm{Iso}(\mathcal{V}, \mathcal{W})$.

Often, we are not interested in a particular isomorphism but wish to discuss arbitrary ones. To this end, let us define,

Definition B.2.25 ($\mathrm{Iso}(\mathcal{V}, \mathcal{W})$)**.** $\mathrm{Iso}(\mathcal{V}, \mathcal{W})$ is the set of all vector space isomorphisms between $\mathcal{V}$ and $\mathcal{W}$.

Remark. $\mathrm{Iso}(\mathcal{V}, \mathcal{W})$ is not a subspace of $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$, since the zero map has no inverse, so $0 \notin \mathrm{Iso}(\mathcal{V}, \mathcal{W})$.

B.2.8 Composition of Isomorphisms

Despite sets of isomorphisms not forming subspaces of their containing homomorphisms, they enjoy a simple interrelationship under composition,

Lemma B.2.26. *Consider the vector spaces* $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$, $\mathcal{W} = (W, \mathbb{F}, +_{\mathcal{W}}, \cdot_{\mathcal{W}})$, *and* $\mathcal{X} = (X, \mathbb{F}, +_{\mathcal{X}}, \cdot_{\mathcal{X}})$, *and the sets of isomorphisms* $\mathrm{Iso}(\mathcal{V}, \mathcal{W})$ *and* $\mathrm{Iso}(\mathcal{W}, \mathcal{X})$. *For all* $f \in \mathrm{Iso}(\mathcal{V}, \mathcal{W})$ *and* $g \in \mathrm{Iso}(\mathcal{W}, \mathcal{X})$, *their composition,*

$$g \circ f \in \mathrm{Iso}(\mathcal{V}, \mathcal{X}). \tag{B.2.16}$$

Proof. We may immediately verify that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. □

This compositional property will be essential in our later discussion of symmetry groups.

B.2.9 Isomorphic Vector Spaces

Isomorphisms grant us the ability to formally state when two different vector spaces have equivalent structure.

Definition B.2.27 (Isomorphic as Vector Spaces ($\cong$))**.** Two vector spaces $\mathcal{V}$ and $\mathcal{W}$ are isomorphic as vector spaces iff there exists a vector space isomorphism $f \in \mathrm{Iso}(\mathcal{V}, \mathcal{W})$ between them. We denote this fact by $\mathcal{V} \cong \mathcal{W}$. When it is clear we are referring to their vector space structures, we may refer to this as simply isomorphic.

Lemma B.2.28. $\cong$ *is an equivalence relation between vector spaces.*

Proof. It is clear that since $\mathrm{id}_{\mathcal{V}}$ is its own inverse we have $\mathcal{V} \cong \mathcal{V}$. Thus, $\cong$ is reflexive. Next, if $\mathcal{V} \cong \mathcal{W}$ with isomorphism f, this implies $\mathcal{W} \cong \mathcal{V}$ with isomorphism f^{-1}. Thus, $\cong$ is symmetric. Finally, by Lemma B.2.26 we establish if $\mathcal{V} \cong \mathcal{W}$ with isomorphism f and $\mathcal{W} \cong \mathcal{X}$ with isomorphism g, then $\mathcal{V} \cong \mathcal{X}$ with isomorphism $g \circ f$. Thus, $\cong$ is transitive. □

That being isomorphic as vector spaces is an equivalence relation massively simplifies the work ahead: we need only establish something for an example of a class of vector spaces and that statement will hold for all equivalent vector spaces.

Given the power of isomorphisms to extend any statement we make beyond any concrete setting, it is important to understand when two vector spaces are isomorphic. For finite-dimensional vector spaces, this is simple,

Lemma B.2.29. *Two finite-dimensional vector spaces* $\mathcal{V}$ *and* $\mathcal{W}$ *are isomorphic iff* $\dim(\mathcal{V}) = \dim(\mathcal{W})$.

Proof. In the forward direction, consider $f \in \mathrm{Iso}(\mathcal{V}, \mathcal{W})$ and a basis $\{b_j\}$ of $\mathcal{V}$. We find directly that $\{f(b_j)\}$ is linearly independent in $\mathcal{W}$, and so $\dim(\mathcal{V}) \leq \dim(\mathcal{W})$. Repeating this argument for a basis of $\mathcal{W}$ under f^{-1} shows $\dim(\mathcal{V}) \geq \dim(\mathcal{W})$. Thus, $\dim(\mathcal{V}) = \dim(\mathcal{W})$. In the reverse direction, we may index any two bases $\{b_j\}$ of $\mathcal{V}$, and $\{c_j\}$ of $\mathcal{W}$ respectively, over the same set J. For all $a \in J$, construct the homomorphisms $b_a \mapsto c_a$ and $c_a \mapsto b_a$ extended linearly. It is clear that these maps are inverses. □

A particularly useful extension of Lemma B.2.29 allows us to see that injective homomorphisms between vector spaces of the same dimension are isomorphisms,

Lemma B.2.30. *Consider an injective homomorphism $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ between two finite-dimensional vector spaces $\mathcal{V}$ and $\mathcal{W}$ with $\dim(\mathcal{V}) = \dim(\mathcal{W})$. Then, $f \in \mathrm{Iso}(\mathcal{V}, \mathcal{W})$.*

Proof. Consider a basis $\{b_j\}$ of $\mathcal{V}$ indexed over a set J. By injectivity, $\{f(b_j)\}$ is linearly independent and $|\{f(b_j)\}| = |\{b_j\}|$. Thus, $\mathrm{span}(\{f(b_j)\}) = \mathcal{W}$, and so $\{f(b_j)\}$ is a basis for $\mathcal{W}$. Therefore, $\mathrm{Im}(f) = \mathcal{W}$, and so by Lemma B.2.13, f is surjective. Thus, f is bijective, so by Lemma B.2.21 it is an isomorphism. □

Lemma B.2.30 will be useful in our analysis of the structure of the symmetries of a metric.

B.2.10 Endomorphisms and Automorphisms

All we have learned about vector space homomorphisms so far also applies when we are considering homomorphisms from a vector space to itself, which we will frequently do to study a particular vector space. In this case, we apply some special terminology,

Definition B.2.31 (Vector Space Endomorphism)**.** An endomorphism is a homomorphism $f \in \mathrm{Hom}(\mathcal{V}, \mathcal{V})$.

Definition B.2.32 ($\mathrm{End}(\mathcal{V})$)**.** $\mathrm{End}(\mathcal{V}) = \mathrm{Hom}(\mathcal{V}, \mathcal{V})$.

Definition B.2.33 (Vector Space Automorphism)**.** An automorphism is an isomorphism $f \in \mathrm{Iso}(\mathcal{V}, \mathcal{V})$.

Definition B.2.34 ($\mathrm{Aut}(\mathcal{V})$)**.** $\mathrm{Aut}(\mathcal{V}) = \mathrm{Iso}(\mathcal{V}, \mathcal{V})$.

B.2.11 Multilinear Maps

One of the central objects of this thesis is the metric g. It is a two-argument function which has the structure of a vector space homomorphism in each. A natural name for such a function might be a "multihomomorphism", but I will try not to introduce new terminology unnecessarily. Instead, let us name such a function in the conventional way,

Definition B.2.35 (Multilinear Map)**.** Consider a finite family of vector spaces $\{\mathcal{V}_j = (V_j, \mathbb{F}, +_{\mathcal{V}_j}, \cdot_{\mathcal{V}_j})\}$ indexed over the set $\{1, \ldots, n\}$ and a vector space $\mathcal{W} = (W, \mathbb{F}, +_{\mathcal{W}}, \cdot_{\mathcal{W}})$. A multilinear map is a function $h : V_1 \times \cdots \times V_n \to W$ which, $\forall k \in \{1, \ldots, n\}$, reduces to a homomorphism $\mathrm{Hom}(\mathcal{V}_k, \mathcal{W})$ when all but the $\mathcal{V}_k$ argument is partially applied.

We will encounter many multilinear maps in this thesis, as well as useful ways to redefine them as more conventional homomorphisms over tensors.

B.3 Constructing Vector Spaces in Brief

B.3.1 Role in this Thesis

In this thesis, we must understand how to form new vector spaces from existing ones, particularly using direct sums and tensor products, as many vector spaces important to this work may be decomposed in this manner, such as the tensor algebra. We will exploit this fact to decompose the universal enveloping algebra in Chapter 4. Furthermore, extensions of the notion of quotient spaces will underpin our ability to create bespoke algebras from more general ones, which is essential to define algebras for arbitrary spin systems.

B.3.2 External Direct Sum Spaces

In Chapter 3, we will define a general formalism for decomposing a vector space into a direct sum of vector spaces according to the properties of a vector space endomorphism. Such decompositions will also inform the structure of the tensor algebra, and so the direct sum is of great significance to this thesis.

Definition B.3.1 (External Direct Sum ($\oplus$))**.** Consider two vector spaces $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$ and $\mathcal{W} = (W, \mathbb{F}, +_{\mathcal{W}}, \cdot_{\mathcal{W}})$. Then, we may form their direct sum

$\mathcal{V} \oplus \mathcal{W} = (V \times W, \mathbb{F}, +_{\mathcal{V}\oplus\mathcal{W}}, \cdot_{\mathcal{V}\oplus\mathcal{W}})$, with its vector addition and scalar multiplication defined as, $\forall (a, b), (c, d) \in V \times W$, $\forall \alpha \in \mathbb{F}$,

$$(a, b) +_{\mathcal{V}\oplus\mathcal{W}} (c, d) \coloneqq (a +_{\mathcal{V}} c, b +_{\mathcal{W}} d) \tag{B.3.1a}$$

$$\alpha \cdot_{\mathcal{V}\oplus\mathcal{W}} (a, b) \coloneqq (\alpha \cdot_{\mathcal{V}} a, \alpha \cdot_{\mathcal{W}} b), \tag{B.3.1b}$$

respectively.

Remark. Direct sums are often defined in two species: "internal" and "external". Where there is overlap between these constructions, their results are isomorphic. For simplicity, we shall consider the "external" direct sum and later indicate its connection with the "internal" direct sum. In this thesis, which species is meant will be clear from context.

Lemma B.3.2. *The direct sum* $\mathcal{V} \oplus \mathcal{W} = (V \times W, \mathbb{F}, +_{\mathcal{V}\oplus\mathcal{W}}, \cdot_{\mathcal{V}\oplus\mathcal{W}})$ *is a vector space over* $\mathbb{F}$*.*

Proof. Direct computation, noting that $(0, 0)$ is the additive identity. □

The direct sum $\oplus$ has a convenient relationship with the trivial space $\{0\}$, which can help to simplify larger direct sums,

Lemma B.3.3. *For all vector spaces* $\mathcal{V}$*,*

$$\mathcal{V} \oplus \{0\} \cong \{0\} \oplus \mathcal{V} \cong \mathcal{V}. \tag{B.3.2}$$

Proof. The required isomorphisms are clear. □

Often in this thesis, we will consider direct sums of more than two vector spaces. Conveniently, direct sums are both commutative and associative,

Lemma B.3.4. *For all vector spaces* $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$*,* $\mathcal{W} = (W, \mathbb{F}, +_{\mathcal{W}}, \cdot_{\mathcal{W}})$*, and* $\mathcal{X} = (X, \mathbb{F}, +_{\mathcal{X}}, \cdot_{\mathcal{X}})$*,*

$$\mathcal{V} \oplus \mathcal{W} \cong \mathcal{W} \oplus \mathcal{V} \tag{B.3.3a}$$

$$(\mathcal{V} \oplus \mathcal{W}) \oplus \mathcal{X} \cong \mathcal{V} \oplus (\mathcal{W} \oplus \mathcal{X}), \tag{B.3.3b}$$

Proof. The required isomorphisms are clear. □

Lemma B.3.4 allows us to disregard the order in which our spaces have been direct summed. It also motivates an additive (abuse of) notation for such spaces adopted by many authors,

Definition B.3.5 (Additive notation for $\mathcal{V} \oplus \mathcal{W}$)**.** For all $(a, b) \in V \times W$,

$$(a, b) = (a, 0) +_{\mathcal{V}\oplus\mathcal{W}} (0, b) \equiv a +_{\mathcal{V}\oplus\mathcal{W}} b. \tag{B.3.4}$$

We shall also adopt this notation for readability.

In light of Lemma B.3.4, let us define,

Definition B.3.6 (Finite Direct Sum)**.** Consider a finite set of vector spaces $\{\mathcal{V}_j = (V_j, \mathbb{F}, +_{\mathcal{V}_j}, \cdot_{\mathcal{V}_j})\}$ indexed over the set $\{1, \dots, N\}$ with $N \subseteq \mathbb{Z}^+$. We may define the direct sum of $\{\mathcal{V}_j\}$ recursively,

$$\bigoplus_{j=1}^{N} \mathcal{V}_j := \begin{cases} \mathcal{V}_1 & N = 1 \\ \mathcal{V}_1 \oplus \bigoplus_{j=1}^{N-1} \mathcal{V}_{j+1} & N \geq 2. \end{cases} \tag{B.3.5}$$

In fact, we will encounter both finite- and countably infinite-dimensional direct sum spaces in this thesis,

Definition B.3.7 (Countably Infinite Direct Sum)**.** Consider a countable set of vector spaces $\{\mathcal{V}_j = (V_j, \mathbb{F}, +_{\mathcal{V}_j}, \cdot_{\mathcal{V}_j})\}$ indexed over a set J. Their direct sum,

$$\bigoplus_{j\in J} \mathcal{V}_j, \tag{B.3.6}$$

consists of all tuples of vectors $a \in \bigtimes_{j\in J} V_j$ which have only finitely many non-zero components. Vector addition and scalar multiplication are defined component-wise as in Definition B.3.1.

Remark. The notation for countably infinite direct sums may also be used for finite direct sums, as the former generalises the latter.

When a direct sum space is finite-dimensional, its dimension is easily understood in terms of its summands,

Lemma B.3.8. *If $\mathcal{V}$ and $\mathcal{W}$ are both finite-dimensional, then so is $\mathcal{V} \oplus \mathcal{W}$, and has dimension* $\dim(\mathcal{V} \oplus \mathcal{W}) = \dim(\mathcal{V}) + \dim(\mathcal{W})$.

Proof. Consider a basis $\{b_j\}$ for $\mathcal{V}$ indexed over a set J, and a basis $\{c_k\}$ for $\mathcal{W}$ indexed over a set K. It is easy to verify that $\{(b_j, 0), (0, c_k) \mid \forall j \in J, \forall k \in K\}$ is a basis for $\mathcal{V} \oplus \mathcal{W}$. □

B.3.3 Internal Direct Sum Spaces

When considering a set of subspaces of a single vector space, we may relate their external direct sum to this common superspace,

Definition B.3.9. Consider a vector space $\mathcal{V}$, and a countable set of subspaces $\{\mathcal{U}_j \subseteq \mathcal{V}\}$ indexed over a set J. We define,

$$\begin{aligned} \iota : \bigoplus_{j\in J} \mathcal{U}_j &\to \mathcal{V} \\ \iota := \sum_{j\in J}{}_{\oplus\mathcal{U}_\mathrm{k}} v_j &\mapsto \sum_{j\in J}{}_{\mathcal{V}} v_j, \end{aligned} \tag{B.3.7}$$

i.e. all elements are mapped to the sum of their components. This map is well-defined as all elements in $\bigoplus_{j\in J} \mathcal{U}_j$ have only finitely many non-zero components.

Lemma B.3.10. *Consider a vector space $\mathcal{V}$, and a countable set of subspaces $\{\mathcal{U}_j \subseteq \mathcal{V}\}$ indexed over a set J. The map ι of Definition B.3.9 is a vector space homomorphism.*

Proof. This follows directly from the component-wise definition of vector addition and scalar multiplication. □

Definition B.3.11 (Internal Direct Sum)**.** Consider a vector space $\mathcal{V}$, a countable set of subspaces $\{\mathcal{U}_j \subseteq \mathcal{V}\}$ indexed over a set J, and their direct sum $\bigoplus_{j\in J} \mathcal{U}_j$. When ι is an isomorphism, we may say $\mathcal{V}$ is the "internal" direct sum of $\{\mathcal{U}_j\}$.

B.3.4 Tensor Product Spaces

Tensor product spaces form the backbone of the tensor algebra, which we will use extensively to construct a wide variety of unital associative algebras, such as the universal enveloping algebras of Section 2.4.4, and the spin algebras, the indefinite position algebras, and the spin-s position algebras of Chapters 4 and 5, which are the main focus of this work. To avoid confusion, it is important to note that the tensor product between vector spaces defined here is distinct from the product of the tensor algebra between its elements, though they share notation and are closely related.

To begin, we must define an equivalence relation on $V \times W$,

Lemma B.3.12. *Consider two vector spaces* $\mathcal{V} = (V, \mathbb{F}, +_\mathcal{V}, \cdot_\mathcal{V})$ *and* $\mathcal{W} = (W, \mathbb{F}, +_\mathcal{W}, \cdot_\mathcal{W})$*, and a subfield* $\mathbb{G}$ *of* $\mathbb{F}$*. For all* $(a, c), (b, d) \in V \times W$*, the relation,*

$$\big[(a,c) \sim (b,d)\big] \Leftrightarrow \Big[\exists \alpha \in \mathbb{G} : \big[[a = \alpha \cdot_\mathcal{V} b] \wedge [d = \alpha \cdot_\mathcal{W} c]\big] \vee \big[[b = \alpha \cdot_\mathcal{V} a] \wedge [c = \alpha \cdot_\mathcal{W} d]\big]\Big] \tag{B.3.8}$$

is an equivalence relation.

Proof. First, $(a, c) \sim (a, c)$ since $1 \in \mathbb{G}$, so $\sim$ is reflexive. Next, by the commutativity of logical disjunction $\vee$, we see immediately that $\big[(b, d) \sim (a, c)\big] \Leftrightarrow \big[(a, c) \sim (b, d)\big]$. Thus, $\sim$ is symmetric. Finally, suppose $\big[(a, d) \sim (b, e)\big] \wedge \big[(b, e) \sim (c, f)\big]$. Showing this statement entails $(a, d) \sim (c, f)$ requires a number of cases to be considered:

1. Consider $\exists \alpha, \beta \in \mathbb{G}$ such that $[a = \alpha \cdot_\mathcal{V} b] \wedge [e = \alpha \cdot_\mathcal{W} d] \wedge [b = \beta \cdot_\mathcal{V} c] \wedge [f = \beta \cdot_\mathcal{W} e]$. Thus, $[a = (\alpha\beta) \cdot_\mathcal{V} c] \wedge [f = (\beta\alpha) \cdot_\mathcal{W} d]$. Since $\mathbb{G}$ is commutative, $\beta\alpha = \alpha\beta$.

2. Consider $\exists \alpha, \beta \in \mathbb{G}$ such that $[b = \alpha \cdot_\mathcal{V} a] \wedge [d = \alpha \cdot_\mathcal{W} e] \wedge [c = \beta \cdot_\mathcal{V} b] \wedge [e = \beta \cdot_\mathcal{W} f]$. Thus, $[c = (\beta\alpha) \cdot_\mathcal{V} a] \wedge [d = (\alpha\beta) \cdot_\mathcal{W} f]$. Since $\mathbb{G}$ is commutative, $\beta\alpha = \alpha\beta$.

3. Consider $\exists \alpha, \beta \in \mathbb{G}$ such that $[a = \alpha \cdot_\mathcal{V} b] \wedge [e = \alpha \cdot_\mathcal{W} d] \wedge [c = \beta \cdot_\mathcal{V} b] \wedge [e = \beta \cdot_\mathcal{W} f]$. Using the commutativity of $\mathbb{G}$, we have $[\beta \cdot_\mathcal{V} a = \alpha \cdot_\mathcal{V} c] \wedge [\alpha \cdot_\mathcal{W} d = \beta \cdot_\mathcal{W} f]$. If $\alpha \neq 0, \beta \neq 0$, since $\alpha^{-1}, \beta^{-1} \in \mathbb{G}$, we are done. If $\alpha = 0$, $\beta \neq 0$, then $a = f = 0$, and we may directly verify that $(0, d) \sim (c, 0)$. If $\alpha \neq 0$, $\beta = 0$, then $c = d = 0$, and we may directly verify that $(a, 0) \sim (0, f)$. If $\alpha = \beta = 0$, the conditions are trivially satisfied, and we must have $a = b = c = 0$, $d = e = f = 0$, and thus $(0, 0) \sim (0, 0)$ by the idempotence of the logical conjunction $\wedge$.

4. Consider $\exists \alpha, \beta \in \mathbb{G}$ such that $[b = \alpha \cdot_\mathcal{V} a] \wedge [d = \alpha \cdot_\mathcal{W} e] \wedge [b = \beta \cdot_\mathcal{V} c] \wedge [f = \beta \cdot_\mathcal{W} e]$. Using the commutativity of $\mathbb{G}$, we have $[\alpha \cdot_\mathcal{V} a = \beta \cdot_\mathcal{V} c] \wedge [\beta \cdot_\mathcal{W} d = \alpha \cdot_\mathcal{W} f]$. If $\alpha \neq 0, \beta \neq 0$, since $\alpha^{-1}, \beta^{-1} \in \mathbb{G}$, we are done. If $\alpha = 0$, $\beta \neq 0$, then $c = d = 0$, and we may directly verify that $(a, 0) \sim (0, f)$. If $\alpha \neq 0$, $\beta = 0$, then $a = f = 0$, and we may directly verify that $(0, d) \sim (c, 0)$. If $\alpha = \beta = 0$, the conditions are trivially satisfied, and we must have $a = b = c = 0$, $d = e = f = 0$, and thus $(0, 0) \sim (0, 0)$ by the idempotence of the logical conjunction $\wedge$.

Thus, $\sim$ is transitive. □

We may use the equivalence classes of this equivalence relation to define the tensor product of two vector spaces,

Definition B.3.13 (Tensor Product ($\otimes_{\mathbb{G}}$))**.** Consider two vector spaces $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$ and $\mathcal{W} = (W, \mathbb{F}, +_{\mathcal{W}}, \cdot_{\mathcal{W}})$, a subfield $\mathbb{G}$ of $\mathbb{F}$, and the set of all equivalence classes of $V \times W$ under the equivalence relation $\sim$ in Lemma B.3.12, which we denote by $S_{\mathbb{G}}(\mathcal{V}, \mathcal{W})$. Then, we may form the tensor product of $\mathcal{V}$ and $\mathcal{W}$ over $\mathbb{G}$, $\mathcal{V} \otimes_{\mathbb{G}} \mathcal{W} = (\mathrm{Lin}(S_{\mathbb{G}}(\mathcal{V}, \mathcal{W})), \mathbb{G}, +_{\mathcal{V}\otimes_{\mathbb{G}}\mathcal{W}}, \cdot_{\mathcal{V}\otimes_{\mathbb{G}}\mathcal{W}})$, where: the vector addition and scalar multiplication are defined by, $\forall \big[(a,c)\big], \big[(b,d)\big] \in S_{\mathbb{G}}(\mathcal{V}, \mathcal{W})$, $\forall \alpha \in \mathbb{G}$,

$$\big[(a,c)\big] +_{\mathcal{V}\otimes_{\mathbb{G}}\mathcal{W}} \big[(a,d)\big] \coloneqq \big[(a, c +_{\mathcal{W}} d)\big] \tag{B.3.9a}$$

$$\big[(a,c)\big] +_{\mathcal{V}\otimes_{\mathbb{G}}\mathcal{W}} \big[(b,c)\big] \coloneqq \big[(a +_{\mathcal{V}} b, c)\big] \tag{B.3.9b}$$

$$\alpha \cdot_{\mathcal{V}\otimes_{\mathbb{G}}\mathcal{W}} \big[(a,b)\big] \coloneqq \big[(\alpha \cdot_{\mathcal{V}} a, b)\big] = \big[(a, \alpha \cdot_{\mathcal{W}} b)\big], \tag{B.3.9c}$$

respectively, with the final equality coming from the definition of $S_{\mathbb{G}}(\mathcal{V}, \mathcal{W})$; and $\mathrm{Lin}(S_{\mathbb{G}}(\mathcal{V}, \mathcal{W}))$ is the set of all finite $\mathbb{G}$-linear combinations of $S_{\mathbb{G}}(\mathcal{V}, \mathcal{W})$ under $+_{\mathcal{V}\otimes_{\mathbb{G}}\mathcal{W}}$ and $\cdot_{\mathcal{V}\otimes_{\mathbb{G}}\mathcal{W}}$.

The attentive reader will have noticed how horrible the notation we have introduced for the elements of $\mathcal{V} \otimes_{\mathbb{G}} \mathcal{W}$ is. This motivates us to adopt an alternative, but widely used, notation,

Definition B.3.14 (Multiplicative notation for $\mathcal{V} \otimes_{\mathbb{G}} \mathcal{W}$)**.** For all $\big[(a,c)\big] \in S_{\mathbb{G}}(\mathcal{V}, \mathcal{W})$,

$$\big[(a,c)\big] \equiv a \otimes c. \tag{B.3.10}$$

Remark. While this notation may suggest we have introduced a product between the elements of $\mathcal{V}$ and $\mathcal{W}$, or between the elements of $\mathcal{V} \otimes_{\mathbb{G}} \mathcal{W}$; this is not the case. The notation we have used here is identical to the notation for the product of the tensor algebra, which is a genuine product between its elements. The relationship between these two uses of $\otimes$ will become apparent in Section 2.4.2.

With this notation in hand, it is easy to show that,

Lemma B.3.15. *The tensor product* $\mathcal{V} \otimes_{\mathbb{G}} \mathcal{W} = (\mathrm{Lin}(S_{\mathbb{G}}(\mathcal{V}, \mathcal{W})), \mathbb{G}, +_{\mathcal{V}\otimes_{\mathbb{G}}\mathcal{W}}, \cdot_{\mathcal{V}\otimes_{\mathbb{G}}\mathcal{W}})$ *is a vector space.*

Proof. Direct computation, noting that $0 \otimes 0$ is the additive identity. □

The structure of a tensor product space is markedly richer than that of a direct sum space. However, the tensor product of vector spaces shares many similar properties as a binary operator as the direct sum.

Lemma B.3.16. *For all vector spaces $\mathcal{V}$ over $\mathbb{F}$,*

$$\mathcal{V} \otimes_{\mathbb{F}} \mathbb{F} \cong \mathbb{F} \otimes_{\mathbb{F}} \mathcal{V} \cong \mathcal{V}. \tag{B.3.11}$$

Proof. The required isomorphisms are clear once one notices, $v \otimes \alpha \in \mathcal{V} \otimes_{\mathbb{F}} \mathbb{F}$ is equal to $(\alpha \cdot_{\mathcal{V}} v) \otimes 1$. □

Often in this thesis, we will consider tensor products of more than two vector spaces. Conveniently, tensor products of vector spaces, like direct sums, are commutative and associative,

Lemma B.3.17. *For all vector spaces $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$, $\mathcal{W} = (W, \mathbb{F}, +_{\mathcal{W}}, \cdot_{\mathcal{W}})$, and $\mathcal{X} = (X, \mathbb{F}, +_{\mathcal{X}}, \cdot_{\mathcal{X}})$,*

$$\mathcal{V} \otimes_{\mathbb{G}} \mathcal{W} \cong \mathcal{W} \otimes_{\mathbb{G}} \mathcal{V} \tag{B.3.12a}$$

$$(\mathcal{V} \otimes_{\mathbb{G}} \mathcal{W}) \otimes_{\mathbb{G}} \mathcal{X} \cong \mathcal{V} \otimes_{\mathbb{G}} (\mathcal{W} \otimes_{\mathbb{G}} \mathcal{X}). \tag{B.3.12b}$$

Proof. The required isomorphisms are clear. □

Lemma B.3.17 allows us to disregard the order in which our spaces have been tensor producted.

Remark. Though the tensor product of vector spaces is commutative, when considering a tensor product such as $\mathcal{V} \otimes_{\mathbb{G}} \mathcal{V}$ in general,

$$a \otimes b \neq b \otimes a, \tag{B.3.13}$$

for elements $a \otimes b, b \otimes a \in \mathcal{V} \otimes_{\mathbb{G}} \mathcal{V}$.

Unlike the case of direct sum spaces, we will encounter only finite-dimensional tensor product spaces in this thesis. Nevertheless, some compact notation is in order in light of Lemma B.3.17,

Definition B.3.18 (Finite Tensor Product)**.** Consider a finite family of vector spaces $\{\mathcal{V}_j = (V_j, \mathbb{F}, +_{\mathcal{V}_j}, \cdot_{\mathcal{V}_j})\}$ indexed over a set J, and a subfield $\mathbb{G}$ of $\mathbb{F}$. We denote their tensor product over $\mathbb{G}$ by,

$$\bigotimes_{j \in J}{}_{\mathbb{G}} \mathcal{V}_j. \tag{B.3.14}$$

We will also frequently construct tensor products of a single vector space,

Definition B.3.19 (Tensor Power)**.** Consider a vector space $\mathcal{V} = (V, \mathbb{F}, +_\mathcal{V}, \cdot_\mathcal{V})$, a subfield $\mathbb{G}$ of $\mathbb{F}$, and $k \in \mathbb{N}$. The kth tensor power over $\mathbb{G}$ of $\mathcal{V}$ is,

$$\mathcal{V}^{\otimes_\mathbb{G} k} := \begin{cases} \bigotimes_{j=1}^{k}{}_\mathbb{G} \mathcal{V} & k \neq 0 \\ \mathbb{G} & k = 0. \end{cases} \tag{B.3.15}$$

In this thesis, we will mostly take $\mathbb{G} = \mathbb{F}$ i.e. the whole field, and so will often omit the subscripts on tensor products, tensor powers, and so forth when the field over which we are taking the tensor product is clear from context.

B.3.5 The Universal Property of Tensor Product Spaces

The tensor product plays a special role in understanding a set of vector spaces $\{\mathcal{V}_j\}$: it is the bridge between multilinear maps and more typical vector space homomorphisms, and enables us to reduce the complexity of objects we will encounter in this thesis. This is formalised in its "universal property",

Lemma B.3.20 (Universal Property of Tensor Product)**.** *Consider the vector spaces* $\mathcal{V} = (V, \mathbb{F}, +_\mathcal{V}, \cdot_\mathcal{V})$, $\mathcal{W} = (W, \mathbb{F}, +_\mathcal{W}, \cdot_\mathcal{W})$, *and* $\mathcal{X} = (X, \mathbb{F}, +_\mathcal{X}, \cdot_\mathcal{X})$, *and the subfield* $\mathbb{G} \subseteq \mathbb{F}$. *For any bilinear map* $h : \mathcal{V} \times \mathcal{W} \to \mathcal{X}$ *there exists a unique vector space homomorphism* $\tilde{h} : \mathcal{V} \otimes_\mathbb{G} \mathcal{W} \to \mathcal{X}$ *such that,*

$$h = \tilde{h} \circ \phi, \tag{B.3.16}$$

where,

$$\phi := (v, w) \mapsto v \otimes w, \tag{B.3.17}$$

is the canonical endomorphism from $V \times W$ *into* $\mathcal{V} \otimes_\mathbb{G} \mathcal{W}$.

Proof. See [49]. □

The universal property is useful in a number of ways. It allows us to understand the properties of the tensor product, for example,

Lemma B.3.21. *Suppose* $\{b_j\}$, *indexed over a set* J, *is a basis for* $\mathcal{V} = (V, \mathbb{F}, +_\mathcal{V}, \cdot_\mathcal{V})$ *when considered as a vector space over the subfield* $\mathbb{G}$ *of* $\mathbb{F}$, *and* $\{c_k\}$, *indexed over a set* K, *is a basis for* $\mathcal{W} = (W, \mathbb{F}, +_\mathcal{W}, \cdot_\mathcal{W})$ *when also considered as a vector space over* $\mathbb{G}$. *Then,* $\{b_j \otimes c_k\}$ *indexed over the set* $J \times K$ *is a basis for* $\mathcal{V} \otimes_\mathbb{G} \mathcal{W}$.

Proof. This proof is adapted from [48], and presented to underscore the universal property as the property of the tensor product responsible for its structure. By the definition of $\mathcal{V} \otimes_{\mathbb{G}} \mathcal{W}$, $\{b_j \otimes c_k\}$ clearly spans $\mathcal{V} \otimes_{\mathbb{G}} \mathcal{W}$. To establish linear independence, let us first define, $\forall p \in J$, $\forall q \in K$, $f_p \in \operatorname{Hom}(\mathcal{V}, \mathbb{G})$, $g_q \in \operatorname{Hom}(\mathcal{W}, \mathbb{G})$ such that,

$$f_p(b_j) := \begin{cases} 1 & j = p \\ 0 & j \neq p, \end{cases} \qquad g_q(c_k) := \begin{cases} 1 & k = q \\ 0 & k \neq q, \end{cases}$$

where $1 \in \mathbb{G}$ is the multiplicative identity of $\mathbb{G}$. Let us also define the bilinear maps, $\forall v \in \mathcal{V}$, $\forall w \in \mathcal{W}$,

$$h_{pq} : \mathcal{V} \times \mathcal{W} \to \mathbb{G}$$
$$h_{pq}(v, w) := f_p(v) g_q(w).$$

By Lemma B.3.20, there exist unique maps $\tilde{h}_{pq} \in \operatorname{Hom}(\mathcal{V} \otimes_{\mathbb{G}} \mathcal{W}, \mathbb{G})$ such that $\forall v, w \in \mathcal{V}$,

$$\tilde{h}_{pq}(v \otimes w) = h_{pq}(v, w).$$

Thus, for all non-empty finite subsets $L \subseteq J \times K$, and $\forall (p, q) \in L$,

$$\Big[0 = \sum_{(j,k) \in L} \alpha_{jk}\, b_j \otimes c_k\Big] \Rightarrow \Big[0 = \sum_{(j,k) \in L} \alpha_{jk}\, \tilde{h}_{pq}(b_j \otimes c_k) = \alpha_{pq}\Big].$$

Therefore, all non-empty finite subsets of $\{b_j \otimes c_k\}$ are linearly independent, and so $\{b_j \otimes c_k\}$ is linearly independent. □

Corollary B.3.22. *If $\mathcal{V}$ and $\mathcal{W}$ are both finite-dimensional as vector spaces over $\mathbb{G}$, then so is $\mathcal{V} \otimes_{\mathbb{G}} \mathcal{W}$ and $\dim_{\mathbb{G}}(\mathcal{V} \otimes_{\mathbb{G}} \mathcal{W}) = \dim_{\mathbb{G}}(\mathcal{V}) \dim_{\mathbb{G}}(\mathcal{W})$, where $\dim_{\mathbb{G}}$ is the dimension of each vector space considered as a vector space over $\mathbb{G}$.*

Proof. Follows immediately from Lemma B.3.21. □

More specifically for this work, the universal property enables us to match the properties of a multilinear map to a tensorial object with the same properties. This will be invaluable when searching for natural algebraic objects to generate a transformation, such as in Section 2.2.4.4. For example,

Example B.3.23. Consider a bilinear map $b : \mathcal{V} \times \mathcal{V} \to \mathcal{W}$ which is antisymmetric, i.e. $\forall x, y \in \mathcal{V}$,

$$b(y, x) = -b(x, y). \tag{B.3.18}$$

By the universal property B.3.20, there is a corresponding homomorphism $\tilde{b} : \mathcal{V}\otimes\mathcal{V} \to \mathcal{W}$ which satisfies,

$$[\tilde{b}(x\otimes y) = -\tilde{b}(y\otimes x)] \Rightarrow [\tilde{b}(x\otimes y + y\otimes x) = 0]. \tag{B.3.19}$$

This reveals that $\tilde{b}$ is naturally defined on antisymmetric tensors $x\otimes y - y\otimes x$, allowing us to move the properties of the map b onto the objects that $\tilde{b}$ acts on. We may now restrict $\tilde{b}$ to the space of these objects if desired.

B.3.6 Direct Sums and Tensor Products

Many spaces of interest to us, such as the tensor algebra, include both tensor products and direct sums in their structure. To fully understand such a space, we must therefore understand the relationship between them. This turns out to be distributive,

Lemma B.3.24. *For all vector spaces* $\mathcal{V} = (V, \mathbb{F}, +_{\mathcal{V}}, \cdot_{\mathcal{V}})$, $\mathcal{W} = (W, \mathbb{F}, +_{\mathcal{W}}, \cdot_{\mathcal{W}})$, *and* $\mathcal{X} = (X, \mathbb{F}, +_{\mathcal{X}}, \cdot_{\mathcal{X}})$,

$$\mathcal{V}\otimes_{\mathbb{G}}(\mathcal{W} \oplus \mathcal{X}) \cong (\mathcal{V}\otimes_{\mathbb{G}}\mathcal{W}) \oplus (\mathcal{V}\otimes_{\mathbb{G}}\mathcal{X}). \tag{B.3.20}$$

Proof. See [49]. □

B.3.7 Quotient Spaces

In contrast to the direct sum and tensor product, which construct larger vector spaces from smaller ones, a quotient produces a smaller vector space from a larger one. Quotients are particularly useful if one wishes to remove uninteresting details from a vector space. Furthermore, generalisations of this idea form the backbone of the methods we will use to derive specialised algebras from more general ones.

Definition B.3.25 (Vector Space Quotient)**.** Consider a vector space $\mathcal{V}$ and a subspace $\mathcal{U} \subseteq \mathcal{V}$. The quotient $\mathcal{Q}$ is defined,

$$\mathcal{Q} \cong \frac{\mathcal{V}}{\mathcal{U}}, \tag{B.3.21}$$

and consists of all equivalence classes of elements of $\mathcal{V}$ under the equivalence relation, $\forall a, b \in \mathcal{V}$,

$$[a \sim b] \Leftrightarrow \big[(a +_{\mathcal{V}} (-b)) \in \mathcal{U}\big]. \tag{B.3.22}$$

Lemma B.3.26. *The quadruple $\mathcal{Q} = (Q, \mathbb{F}, +_{\mathcal{Q}}, \cdot_{\mathcal{Q}})$ is a vector space, where the vector addition and scalar multiplication are inherited from $\mathcal{V}$ up to $\sim$.*

Proof. Direct computation reveals not only are the vector space axioms satisfied, but vector addition and scalar multiplication are well-defined, i.e. independent of the chosen representatives of the equivalence classes we are manipulating. □

Definition B.3.27 (Quotient Space)**.** We call $\mathcal{Q} = (Q, \mathbb{F}, +_{\mathcal{Q}}, \cdot_{\mathcal{Q}})$ the quotient space of $\mathcal{V}$ by $\mathcal{U}$.

Remark. While the objects of any quotient space are, strictly speaking, equivalence classes of elements, we will often instead use representatives and manipulate expressions according to their equivalences.

B.3.8 Direct Sums and Quotient Spaces

Despite their markedly different appearance, direct sums and quotients are closely related,

Lemma B.3.28. *Consider a vector space $\mathcal{V}$, a subspace $\mathcal{U} \subseteq \mathcal{V}$, and their quotient space $\mathcal{Q}$. Then,*

$$\mathcal{V} \cong \mathcal{U} \oplus \mathcal{Q}. \tag{B.3.23}$$

Proof. This follows from the fact that the category of vector spaces is an Abelian category[88, 89]. □

Lemma B.3.28 provides us another way to decompose a given space, which will be useful in understanding the structure of vector spaces with metrics, which are objects of central importance to this work.

B.4 Groups and Group Homomorphisms in Brief

B.4.1 Role in this Thesis

In Section 2.2.3, the symmetry endomorphisms of a metric g naturally forms a group. This section is intended to formally define the notion of a group so that this may be made precise.

B.4.2 General Groups

Definition B.4.1 (Group)**.** A group is a triple $G = (X, \diamond, e)$ consisting of:

- a set X;
- a binary function $\diamond : X \times X \to X$, which we call the group product, for which, $\forall a, b, c \in X$,
$$(a \diamond b) \diamond c = a \diamond (b \diamond c); \tag{B.4.1}$$
- a two-sided identity element $e \in X$, i.e. $\forall a \in X$,
$$e \diamond a = a \diamond e = a; \tag{B.4.2}$$
- a two-sided inverse element $a^{-1} \in X$ for each element $a \in X$,
$$a^{-1} \diamond a = a \diamond a^{-1} = e. \tag{B.4.3}$$

Remark. Most authors axiomatise only the existence of left-inverse elements and the left-identity, as that axiomatic system is strong enough to entail these elements also being right-inverses and right-identities respectively. As such, our definition is equivalent but avoids some additional lemmas for the sake of clarity.

Lemma B.4.2. *In a group $G = (X, \diamond, e)$, the identity element and all inverse elements are unique.*

Proof. See [53]. □

As with vector spaces, we shall often write $a \in G$ to mean the same as $a \in X$.

B.4.3 Abelian and Non-Abelian Groups

The attentive reader will have noticed that the Definition B.4.1 makes no assumption about the commutativity of the group product. Indeed, this compels us to consider two classes of groups,

Definition B.4.3 ((Non-)Abelian Groups)**.** A group $G = (X, \diamond, e)$ is Abelian iff $\forall a, b \in G$,
$$a \diamond b = b \diamond a. \tag{B.4.4}$$

Otherwise, the group G is non-Abelian.

The most important groups for this thesis, $\mathrm{SO}^+(\mathcal{V}, g)$ and $\mathrm{O}(\mathcal{V}, g)$, are usually non-Abelian (this will be proven in Corollary 5.4.4), but Abelian groups form the backbone of many structures we use to study them, for example any vector space under addition only is an Abelian group.

B.4.4 Subgroups

As we saw with $\mathrm{SO}^+(\mathcal{V}, g)$ and $\mathrm{O}(\mathcal{V}, g)$, we are often interested in only a part of a larger group which is nevertheless closed under the group product,

Lemma B.4.4. *Consider a subset of elements $W \subseteq X$ from a group $G = (X, \diamond, e)$, for which, $\forall a, b \in W$,*

$$e \in W \tag{B.4.5a}$$

$$[a \in W] \Leftrightarrow [a^{-1} \in W] \tag{B.4.5b}$$

$$a \diamond b \in W. \tag{B.4.5c}$$

Then $F = (W, \square, e)$ is a group, where,

$$i \circ \square := \diamond \circ j \tag{B.4.6}$$

where $i : W \to X$ and $j : W \times W \to X \times X$ are inclusion maps.

Proof. Immediate from the definition of a group. □

Accordingly,

Definition B.4.5 (Subgroup)**.** We call $F = (W, \square, e)$ as in Lemma B.4.4 a subgroup of G.

B.4.5 Group Homomorphisms and Antihomomorphisms

Just like with vector spaces, we are often interested in understanding the relationships that two groups can have with each other. This leads us to consider functions between groups which are compatible with their group structures. However, unlike with vector spaces, the non-Abelian nature of general groups causes such functions to come in two species. The first is perhaps what you might expect,

Definition B.4.6 (Group Homomorphism)**.** Consider two groups $G = (X, \diamond, e)$ and $H = (Y, *, f)$. A group homomorphism between G and H is a function $p : X \to Y$ such that, $\forall a, b \in G$,

$$p(a \diamond b) = p(a) * p(b). \tag{B.4.7}$$

We call the set of all group homomorphisms $\mathrm{Hom}(G, H)$. We may also write $p : G \to H$ to mean the same as $p : X \to Y$.

The second species instead reverses the order of all products,

Definition B.4.7 (Group Antihomomorphism)**.** Consider two groups $G = (X, \diamond, e)$ and $H = (Y, *, f)$. A group antihomomorphism between G and H is a function $p : X \to Y$ such that, $\forall a, b \in G$,

$$p(a \diamond b) = p(b) * p(a). \tag{B.4.8}$$

We call the set of all group antihomomorphisms $\overline{\mathrm{Hom}}(G, H)$. We may also write $p : G \to H$ to mean the same as $p : X \to Y$.

Remark. As with vector space homomorphisms, there exist the notions of group (anti-)isomorphisms, (anti-)endomorphisms, and (anti-)automorphisms. These have the definitions you might expect, but as they are not needed for the developments in this thesis, we shall not consider them further.

In our definitions, we have not specified how either of these species of functions should act on the identity or inverses. Regardless of which species of group structure-preserving functions we use, we find,

Lemma B.4.8. *Consider two groups $G = (X, \diamond, e)$ and $H = (Y, *, f)$ and the functions $p \in \mathrm{Hom}(G, H)$, $q \in \overline{\mathrm{Hom}}(G, H)$. Then, $\forall a \in G$,*

$$p(e) = q(e) = f \tag{B.4.9a}$$

$$p(a^{-1}) = p(a)^{-1} \tag{B.4.9b}$$

$$q(a^{-1}) = q(a)^{-1}. \tag{B.4.9c}$$

Proof. By definition, $\forall a \in \mathcal{V}$,

$$p(a) = p(e \diamond a) = p(e) * p(a)$$

$$q(a) = q(e \diamond a) = q(a) * q(e)$$

Thus, by Lemma B.4.2, we have (B.4.9a). Similarly, $\forall a \in G$,

$$f = p(e) = p(a^{-1} \diamond a) = p(a^{-1}) * p(a)$$
$$f = q(e) = q(a^{-1} \diamond a) = q(a) * q(a^{-1}).$$

Thus, by Lemma B.4.2, we have (B.4.9b) and (B.4.9c). □

B.5 Proof that the Metric Adjoint in a Minkowski Space Exists

Consider a Minkowski space-time $(\mathcal{V}, g)$. To prove that a g-adjoint exists $\forall A \in \mathrm{End}(\mathcal{V})$, we must introduce some additional concepts. The argument I present here is a variation on the proof for the Riesz Representation Theorem [90].

Definition B.5.1 (Dual Vector Space)**.** The dual vector space of $\mathcal{V}$ is $\mathrm{Hom}(\mathcal{V}, \mathbb{R})$.

The metric g induces a natural relationship with the dual space $\mathrm{Hom}(\mathcal{V}, \mathbb{R})$,

Lemma B.5.2. *The map* $\Phi_g : \mathcal{V} \to \mathrm{Hom}(\mathcal{V}, \mathbb{R}) : v \mapsto \big(w \mapsto g(v,w)\big)$ *is an isomorphism.*

Proof. It is clear from its definition that Φ_g is a vector space homomorphism. Suppose $\exists a, b \in \mathcal{V}$ such that $\Phi_g(a) = \Phi_g(b)$, then,

$$0 = \Phi_g(a) - \Phi_g(b) = w \mapsto g\big((a-b), w\big),$$

which by the non-degeneracy of g implies $a = b$. Thus, Φ_g is injective. By Lemma B.2.4, $\dim(\mathcal{V}) = \dim(\mathrm{Hom}(\mathcal{V}, \mathbb{R}))$, and so by Lemma B.2.30, Φ_g is an isomorphism. □

Lemma B.5.3. *For all* $f \in \mathrm{Hom}(\mathcal{V}, \mathbb{R})$*,* $\exists v_f \in \mathcal{V}$ *such that,* $\forall w \in \mathcal{V}$*,*

$$f(w) = g(v_f, w). \tag{B.5.1}$$

Proof. For any $a \in \mathcal{V}$, consider $f = \Phi_g(a)$. By definition, $\forall w \in \mathcal{V}$,

$$\big(\Phi_g(a)\big)(w) = g(a,w),$$

and therefore,

$$f(w) = g(\Phi_g^{-1}(f), w).$$

We are done since $\mathrm{Im}(\Phi_g) = \mathrm{Hom}(\mathcal{V}, \mathbb{R})$. □

Lemma B.5.4. *For all $f \in \mathrm{Hom}(\mathcal{V}, \mathbb{R})$, the vector $v_f \in \mathcal{V}$ such that, $\forall w \in \mathcal{V}$,*

$$f(w) = g(v_f,w), \tag{B.5.2}$$

is unique.

Proof. Consider $v_f, w_f \in \mathcal{V}$ such that, $\forall w \in \mathcal{V}$,

$$\big[g(v_f,w) = f(w) = g(w_f,w)\big] \Rightarrow \big[g\big((v_f - w_f, w)\big) = 0\big].$$

Thus, the non-degeneracy of g implies that $v_f = w_f$. □

Now we have all we need to prove the existence of g-adjoints,

Lemma B.5.5. *For all $A \in \mathrm{End}(\mathcal{V})$, $\exists B \in \mathrm{End}(\mathcal{V})$, such that, $\forall v, w \in \mathcal{V}$,*

$$g(A(v),w) = g(v,B(w)). \tag{B.5.3}$$

Proof. Consider the map $f \in \mathrm{Hom}(\mathcal{V}, \mathbb{R})$ defined for some $v \in \mathcal{V}$ by $f = \Phi_g(v) \circ A$. By Lemmas B.5.3 and B.5.4, there exists a unique $v_f \in \mathcal{V}$, such that, $\forall w \in \mathcal{V}$,

$$g(v_f,w) = \big(\Phi_g(v) \circ A\big)(w) = g(v,A(w)).$$

Thus, we may define a function,

$$\varphi : V \to V$$
$$v \mapsto v_f,$$

so that,

$$g(\varphi(v),w) = g(v,A(w)).$$

All that remains it to verify that $\varphi \in \mathrm{End}(\mathcal{V})$. This follows directly from the fact that Φ_g is a vector space homomorphism, and the definitions of homomorphism addition and scalar multiplication. □

B.6 Proof that $\mathrm{O}(\mathcal{V}, g)$ forms a Group under Composition

Lemma B.6.1. $\mathrm{id}_{\mathcal{V}} \in \mathrm{O}(\mathcal{V}, g)$.

Proof. For all $v, w \in \mathcal{V}$,

$$g(\mathrm{id}_{\mathcal{V}}(v), \mathrm{id}_{\mathcal{V}}(w)) = g(v,w)$$

□

Lemma B.6.2. *For all* $A, B \in \mathrm{O}(\mathcal{V}, g)$, $A \circ B \in \mathrm{O}(\mathcal{V}, g)$.

Proof. For all $v, w \in \mathcal{V}$,

$$g\big((A \circ B)(v), (A \circ B)(w)\big) = g\big(A(B(v)), A(B(w))\big) = g(B(v),B(w)) = g(v,w)$$

□

This next property is not itself a requirement for $\mathrm{O}(\mathcal{V}, g)$ to be a group, but reveals some of the required structure,

Lemma B.6.3. *For all* $A \in \mathrm{O}(\mathcal{V}, g)$,

$$A^g \circ A = A \circ A^g = \mathrm{id}_{\mathcal{V}}\,. \tag{B.6.1}$$

Proof. First, note, $\forall v, w \in \mathcal{V}$,

$$g(v,w) = g(A(v),A(w)) = g\big(v, (A^g \circ A)(w)\big).$$

Thus, by the non-degeneracy of g, $A^g \circ A = \mathrm{id}_{\mathcal{V}}$, so A^g is a left-inverse for A. By Lemma B.2.17, this means A is injective. Therefore, by Lemma B.2.30, A is an isomorphism. So, by Lemma B.2.23, A^g is the unique two-sided inverse for A. □

Thus,

Lemma B.6.4. $A^g \in \mathrm{O}(\mathcal{V}, g)$.

Proof. By Lemma B.6.3, $\forall v, w \in \mathcal{V}$,

$$g(v,w) = g(\mathrm{id}_{\mathcal{V}}(v), \mathrm{id}_{\mathcal{V}}(w)) = g\big((A \circ A^g)(v), (A \circ A^g)(w)\big) = g(A^g(v),A^g(w)).$$

□

Lemma B.6.5. $\mathrm{O}(\mathcal{V}, g)$ *forms a group under composition.*

Proof. Lemma B.6.2 shows that $\mathrm{O}(\mathcal{V}, g)$ is closed under composition, and Lemma B.6.1 shows it contains its identity element. Lemma B.6.3 shows that the g-adjoint of an orthogonal map is its two-sided inverse, and Lemma B.6.4 shows this inverse is also an orthogonal map. Noting that composition of homomorphisms is associative, we are done. □

B.7 The Isomorphism Between $\mathfrak{so}(\mathcal{V},g)$ and $\Lambda^2(\mathcal{V})$

B.7.1 Proof that $\mathfrak{so}(\mathcal{V},g)$ is Spanned by $t(a,b)$ Endomorphisms

First, let us define some bilinear maps which will be useful in our proof,

Definition B.7.1 ($s(a,b)$). For all $a, b \in \mathcal{V}$, $s(a,b)$ is the bilinear map,

$$\begin{aligned} s : \mathcal{V}\times\mathcal{V} &\to \mathrm{End}(\mathcal{V}) \\ s(a,b) &:= v \mapsto g(a,v)b. \end{aligned} \tag{B.7.1}$$

Lemma B.7.2. *Consider a basis $\{b_j\}$ for $\mathcal{V}$ indexed over a set J, and the inclusion map,*

$$i : \{b_j\}\times\{b_k\} \to \mathcal{V}\times\mathcal{V}. \tag{B.7.2}$$

Then, the map $s\circ i$ is injective.

Proof. Suppose $\exists(b_p, b_q), (b_r, b_s) \in \{b_j\}\times\{b_k\}$ such that $s\circ i(b_p, b_q) = s\circ i(b_r, b_s)$. Then, $\forall v \in \mathcal{V}$,

$$0 = g(b_p,v)b_q - g(b_r,v)b_s.$$

If $b_q \neq b_s$, then $\{b_q, b_s\}$ is linearly independent, and so $g(b_p,v) = -g(b_r,v) = 0$. By the non-degeneracy of g this implies $b_p = b_r = 0$ which contradicts the linear independence of the basis $\{b_j\}$. Therefore, $b_q = b_s$. Thus,

$$0 = \big(g(b_p,v) - g(b_r,v)\big)b_q = g\big(b_p - b_r, v\big)b_q,$$

which by the non-degeneracy of g implies $b_p = b_r$. So, $(b_p, b_q) = (b_r, b_s)$. □

Lemma B.7.3. *The set $\{s\circ i(b_j, b_k)\}$ indexed over the set $J\times J$ is linearly independent in* $\mathrm{End}(\mathcal{V})$.

Proof. Consider $\forall(j, k) \in J \times J$, $\{\alpha_{jk} \in \mathbb{R}\}$ such that,

$$0 = \sum_{(j,k)\in J\times J} \alpha_{jk}\, s\circ i(b_j, b_k).$$

Thus, $\forall v \in \mathcal{V}$,

$$0 = \sum_{(j,k)\in J\times J} \alpha_{jk}\, g(b_j,v)b_k = \sum_{k\in J}\Big(\sum_{j\in J} \alpha_{jk}\, g(b_j,v)\Big)b_k = \sum_{k\in J} \beta_k b_k,$$

where we have defined $\beta_k \coloneqq \sum_{j\in J} \alpha_{jk}\, g(b_j,v)$. Since $\{b_k\}$ is linearly independent, we must have, $\forall k \in J$,

$$0 = \beta_k = \sum_{j\in J} \alpha_{jk}\, g(b_j,v) = g\Big(\sum_{j\in J} \alpha_{jk}\, b_j, v\Big).$$

Thus, by the non-degeneracy of g, we must have, $\forall k \in J$,

$$\sum_{j\in J} \alpha_{jk}\, b_j = 0.$$

Again, since $\{b_j\}$ is linearly independent, we must have, $\forall (j,k) \in J\times J$, $\alpha_{jk} = 0$. □

Lemma B.7.4. *The set* $\{s\circ i(b_j, b_k)\}$ *indexed over the set* $J\times J$ *is a basis for* $\mathrm{End}(\mathcal{V})$.

Proof. Since $s\circ i$ is injective by Lemma B.7.2, $|\{s\circ i(b_j, b_k)\}| = |\{b_j\}\times\{b_k\}| = \dim(\mathcal{V})^2$. By Lemma B.2.4, $\dim(\mathrm{End}(\mathcal{V})) = \dim(\mathcal{V})^2$. Since $s\circ i$ is linearly independent by Lemma B.7.3, it is therefore a basis for $\mathrm{End}(\mathcal{V})$[46]. □

This basis is particularly useful for us since it behaves nicely under $(\,\cdot\,)^g$,

Lemma B.7.5. *For all* $a, b \in \mathcal{V}$, $\big(s(a,b)\big)^g = s(b,a)$.

Proof. For all $v, w \in \mathcal{V}$,

$$g\big(s(a,b)(v), w\big) = g\big(g(a,v)b, w\big) = g\big(v, g(b,w)a\big) = g\big(v, s(b,a)(w)\big).$$

□

With a natural basis for $\mathrm{End}(\mathcal{V})$ identified in the presence of the metric g, let us now probe $\mathfrak{so}(\mathcal{V}, g)$,

Lemma B.7.6. *The set* $\{t\circ i(b_j, b_k)\}$ *indexed over the set* $J \times J$ *spans* $\mathfrak{so}(\mathcal{V}, g)$.

Proof. By definition $\forall A \in \mathfrak{so}(\mathcal{V}, g)$, $A = -A^g$, or equivalently $a_-(A) = A$. Writing,

$$A = \sum_{(j,k)\in J\times J} \alpha_{jk}\, s(b_j,b_k),$$

we find,

$$a_-(A) = A = \sum_{(j,k)\in J\times J} \alpha_{jk}\, a_-\big(s(b_j,b_k)\big) = \sum_{(j,k)\in J\times J} \frac{1}{2}\alpha_{jk}\, t(b_j,b_k).$$

□

Corollary B.7.7. *Any* $A \in \mathfrak{so}(\mathcal{V}, g)$ *may be written in the form,*

$$A = \sum_{k\in K} \gamma_k\, t(c_k,d_k), \tag{B.7.3}$$

where $\forall k \in K$, $\gamma_k \in \mathbb{R}$, $(c_k, d_k) \in \mathcal{V}\times\mathcal{V}$.

Proof. This follows directly from Lemma B.7.6. □

Lemma B.7.8. *Suppose the indexing set J can be given a strict total order $<$. Then, the set,*

$$\{t \circ i(b_j, b_k) \mid \forall j, k \in J : j < k\} \tag{B.7.4}$$

is a basis for $\mathfrak{so}(\mathcal{V}, g)$.

Proof. By the definition of $t(a,b)$, $\forall j, k \in J$, $t(b_k,b_j) = -t(b_j,b_k)$, which implies $\forall j = k$, $t(b_j,b_j) = 0$. Thus, the set (B.7.4) contains no obvious linear dependence. Let us now prove that this set is linearly independent, $\forall v \in \mathcal{V}$,

$$0 = \sum_{j,k \in J, j<k} \alpha_{jk}\, t(b_j,b_k)(v) = \sum_{j,k \in J, j<k} \alpha_{jk} \big(s(b_j,b_k)(v) - s(b_k,b_j)(v)\big),$$

and so by Lemma B.7.4, $\forall j, k \in J$, $j < k$, $\alpha_{jk} = 0$. □

Corollary B.7.9. $\dim\big(\mathfrak{so}(\mathcal{V}, g)\big) = \frac{1}{2}\dim(\mathcal{V})\big(\dim(\mathcal{V}) - 1\big)$.

Proof. For all $j \in J$, $t(b_j,b_j) = 0$, and $j \neq k \in J$, $t(b_k,b_j) = -t(b_j,b_k)$. Thus, of the $\dim(\mathcal{V})^2$ endomorphisms $\{t \circ i(b_j, b_k)\}$ indexed over the set $J \times J$, $\dim(\mathcal{V})$ are zero and $\frac{1}{2}\dim(\mathcal{V})\big(\dim(\mathcal{V}) - 1\big)$ are linearly dependent on the others. Thus, there are,

$$\dim(\mathcal{V})^2 - \dim(\mathcal{V}) - \frac{1}{2}\dim(\mathcal{V})\big(\dim(\mathcal{V}) - 1\big) = \frac{1}{2}\dim(\mathcal{V})\big(\dim(\mathcal{V}) - 1\big),$$

linearly independent $\{t \circ i(b_j, b_k)\}$. □

B.7.2 Proof that μ is a Vector Space Isomorphism

To establish this claim, we need to assert some properties about the bivectors $\Lambda^2(\mathcal{V})$,

Lemma B.7.10. *Consider a basis $\{b_j\}$ for $\mathcal{V}$ indexed over a set J which is equipped with a strict total order $<$. Then,*

$$\{b_j \wedge b_k \mid \forall j, k \in J : j < k\}, \tag{B.7.5}$$

is a basis for $\Lambda^2(\mathcal{V})$.

Proof. By Lemma B.3.21, the set $\{b_j \otimes b_k\}$ indexed over the set $J \times J$ is a basis for $\mathcal{V}^{\otimes 2}$. Thus, for every bivector $B \in \Lambda^2(\mathcal{V})$,

$$B = \sum_{j,k \in J} \alpha_{jk}\, b_j \wedge b_k,$$

so the set $\{b_j \wedge b_k\}$ spans $\Lambda^2(\mathcal{V})$. From their definition as tensors, $\forall j,k \in J$ we have $b_k \wedge b_j = -b_j \wedge b_k$; we may use the strict partial order $<$ to remove these linearly dependent elements from $\{b_j \wedge b_k\}$, resulting in (B.7.5). All that remains is to prove the linear independence of (B.7.5),

$$0 = \sum_{j,k \in J, j<k} \alpha_{jk}\, b_j \wedge b_k = \sum_{j,k \in J, j<k} \frac{1}{2} \alpha_{jk} \left(b_j \otimes b_k - b_k \otimes b_j \right),$$

from which $\alpha_{jk} = 0$ follows from the linear independence of $\{b_j \otimes b_k\}$. □

Lemma B.7.11. *The vector space homomorphism μ is injective.*

Proof. Consider $A, B \in \Lambda^2(\mathcal{V})$ such that $\mu(A) = \mu(B)$. Then, using the basis (B.7.5) to write,

$$A = \sum_{j,k \in J, j<k} \alpha_{jk}\, b_j \wedge b_k, \quad B = \sum_{j,k \in J, j<k} \beta_{jk}\, b_j \wedge b_k,$$

we have,

$$\big[\mu(A) = \mu(B)\big] \Rightarrow \Big[0 = \sum_{j,k \in J, j<k} (\alpha_{jk} - \beta_{jk})\, \mu(b_j \wedge b_k) = \sum_{j,k \in J, j<k} (\alpha_{jk} - \beta_{jk})\, t(b_j, b_k)\Big].$$

Thus, by the linear independence of $\{t(b_j, b_k)\}$ we have $\alpha_{jk} = \beta_{jk}$. □

Lemma B.7.12. *The vector space homomorphism μ is surjective.*

Proof. Consider $f \in \mathfrak{so}(\mathcal{V}, g)$. By Lemma 2.2.47, there exists some expansion,

$$f = \sum_{j \in J} \alpha_j\, t(a_j, b_j) = \sum_{j \in J} \alpha_j\, \mu(a_j \wedge b_j) = \mu\Big(\sum_{j \in J} \alpha_j a_j \wedge b_j \Big).$$

Thus, $f \in \mathrm{Im}(\mu)$. □

B.8 Proof that $\mathfrak{so}(3, \mathbb{R})$ has Rank One

First note that,

Lemma B.8.1. *For all $v, w \in \mathfrak{so}(3, \mathbb{R})$, $\{v, w\}$ is linearly independent iff $v \times w \neq 0$.*

Proof. We will prove the contrapositive of this claim. If $\{v, w\}$ is linearly dependent then by antisymmetry $v \times w = 0$. If $v \times w = 0$ and $[v = 0] \vee [w = 0]$, then $\{v, w\}$ is clearly linearly dependent, so let us consider the case $[v \neq 0] \wedge [w \neq 0]$. Then, $v \times w = 0$ implies,

$$\nu_1 \omega_2 - \nu_2 \omega_1 = 0$$

$$\nu_2\omega_3 - \nu_3\omega_2 = 0$$
$$\nu_3\omega_1 - \nu_1\omega_3 = 0.$$

Since, $w \neq 0$, consider the case $\omega_1 \neq 0$. Then, $\nu_1 = 0$ implies $v = 0$, thus $\nu_1 \neq 0$, and we must have,

$$v = \nu_1 S_1 + \frac{\nu_1\omega_2}{\omega_1} S_2 + \frac{\nu_1\omega_3}{\omega_1} S_3 = \frac{\nu_1}{\omega_1} w.$$

The cases $\omega_2 \neq 0$ and $\omega_3 \neq 0$ similarly yield $v = \frac{\nu_2}{\omega_2} w$ and $v = \frac{\nu_3}{\omega_3} w$ respectively. □

Lemma B.8.2. *For all $v, w \in \mathfrak{so}(3,\mathbb{R})$, written in the usual basis as $v = \sum_{a=1}^{3} \nu_a S_a$ and $w = \sum_{b=1}^{3} \omega_b S_b$,*

$$v\times(v\times w) = -\left(\sum_{a=1}^{3} \nu_a^2\right) w + \left(\sum_{a=1}^{3} \nu_a\omega_a\right) v. \tag{B.8.1}$$

Proof. Direct computation. □

Lemma B.8.3. *For all $v, w \in \mathfrak{so}(3,\mathbb{R})$, $\{v, w\}$ is linearly independent iff $\{v, w, v\times w\}$ is linearly independent.*

Proof. The reverse direction follows from Lemma B.1.6. In the forwards direction consider $\alpha, \beta, \gamma \in \mathbb{F}$ such that,

$$\alpha v + \beta w + \gamma v\times w = 0.$$

Thus, writing $v = \sum_{a=1}^{3} \nu_a S_a$ and $w = \sum_{b=1}^{3} \omega_b S_b$,

$$\begin{aligned}
0 &= \alpha v\times(v\times w) + \beta w\times(v\times w)\\
&= -\alpha\left(\sum_{a=1}^{3} \nu_a^2\right) w + \alpha\left(\sum_{a=1}^{3} \nu_a\omega_a\right) v + \beta\left(\sum_{a=1}^{3} \omega_a^2\right) v - \beta\left(\sum_{a=1}^{3} \omega_a\nu_a\right) w\\
&= \left(\alpha\sum_{a=1}^{3} \nu_a\omega_a + \beta\sum_{a=1}^{3} \omega_a^2\right) v - \left(\alpha\sum_{a=1}^{3} \nu_a^2 + \beta\sum_{a=1}^{3} \omega_a\nu_a\right) w.
\end{aligned}$$

By the linear independence of $\{v, w\}$ we must have,

$$\begin{gathered}
\left[\alpha\sum_{a=1}^{3} \nu_a\omega_a + \beta\sum_{a=1}^{3} \omega_a^2 = 0\right] \wedge \left[\alpha\sum_{a=1}^{3} \nu_a^2 + \beta\sum_{a=1}^{3} \omega_a\nu_a = 0\right]\\
\Rightarrow \alpha\left(\left(\sum_{a=1}^{3} \nu_a^2\right)\left(\sum_{b=1}^{3} \omega_b^2\right) - \left(\sum_{a=1}^{3} \omega_a\nu_a\right)^2\right) = 0.
\end{gathered}$$

Recognising the form of the standard Euclidean metric on an orthonormal basis, the second factor is non-zero by Lemma B.8.4, and so $\alpha = \beta = 0$. Thus, $\gamma = 0$ by Lemma B.8.1. □

To conclude the proof, we need,

Lemma B.8.4 (Cauchy-Schwarz Inequality)**.** *For any positive-definite metric g, $\forall a, b \in \mathcal{V}$,*

$$g(a,a)g(b,b) \geq \big(g(a,b)\big)^2, \tag{B.8.2}$$

with equality iff $\{a, b\}$ is linearly dependent.

Proof. See [22]. □

Now, we may find,

Lemma B.8.5. *The Lie algebra $\mathfrak{so}(3, \mathbb{R})$ has rank 1.*

Proof. Consider two non-zero elements $v, w \in \mathfrak{so}(3, \mathbb{R})$ such that $\{v, w\}$ is linearly independent. By Lemmas B.8.3 and B.1.18, $\{v, w, v \times w\}$ is a basis for $\mathfrak{so}(3, \mathbb{R})$. Suppose $\exists \alpha, \beta \in \mathbb{F}$ such that,

$$\mathrm{ad}(v)\big(\alpha w + \beta v \times w\big) = 0.$$

By Lemma B.8.2 we have,

$$\alpha v \times w - \beta \left(\sum_{a=1}^{3} \nu_a^2\right) w + \beta \left(\sum_{a=1}^{3} \nu_a \omega_a\right) v = 0,$$

which by Lemma B.8.3 implies that $\alpha = \beta = 0$. Therefore, $\forall v \in \mathfrak{so}(3, \mathbb{R})$, $\mathrm{Ker}(\mathrm{ad}(v)) = \mathrm{span}_{\mathbb{F}}(\{v\})$, thus $\dim\big(\mathrm{Ker}(\mathrm{ad}(v))\big) = 1$. □

Appendix C

Minimal Polynomial Methods

C.1 Proofs for Formal Irreducible Power Series

In this section, we will often leave implicit inclusions from $M_{f(x)}$ to $\mathbb{F}[x]$.

C.1.1 The Algebra of Formal Irreducible Power Series

C.1.1.1 The Homomorphism $\pi^{(0)}_{f(x)}$

Definition C.1.1. Let π' denote the canonical projection,

$$\begin{aligned} \pi' &: \mathbb{F}[x] \to \mathcal{Q}_{f(x)}[x] \\ \pi' &:= g(x) \mapsto [\, g(x) \,]. \end{aligned} \tag{C.1.1}$$

Definition C.1.2.

$$\begin{aligned} \chi' &: \mathcal{Q}_{f(x)}[x] \to M_{f(x)} \\ \chi' &:= [\, g(x) \,] \mapsto r(x), \end{aligned} \tag{C.1.2}$$

where $r(x)$ is the unique least-order representative of $[\, g(x) \,]$.

Lemma C.1.3. *For all $h(x) \in [\, g(x) \,]$, suppressing inclusions,*

$$h(x) = \chi'(h(x)) + k(x)f(x), \tag{C.1.3}$$

for some $k(x) \in \mathbb{F}[x]$.

Proof. Clear from polynomial division. □

Definition C.1.4 ($\pi^{(0)}_{f(x)}$).

$$\begin{aligned} \pi^{(0)}_{f(x)} &: \mathbb{F}[x] \to M_{f(x)} \\ \pi^{(0)}_{f(x)} &:= \chi' \circ \pi'. \end{aligned} \tag{C.1.4}$$

Lemma C.1.5. *For all* $g(x) \in \mathbb{F}[x]$,

$$g(x) = a(x) + B(x)f(x), \tag{C.1.5}$$

with $a(x) \in M_{f(x)}$, $B(x) \in \mathbb{F}[x]$, *we have,*

$$\pi^{(0)}_{f(x)}(g(x)) = a(x). \tag{C.1.6}$$

Proof. Clear from the definition and Lemma C.1.3. □

Lemma C.1.6.

$$\pi^{(0)}_{f(x)} \circ \pi^{(0)}_{f(x)} = \pi^{(0)}_{f(x)} \tag{C.1.7}$$

Proof. Follows from Lemma C.1.5. □

Lemma C.1.7. *For all* $g(x), h(x) \in \mathbb{F}[x]$,

$$\pi^{(0)}_{f(x)}\big(g(x)h(x)\big) = \pi^{(0)}_{f(x)}\big(\pi^{(0)}_{f(x)}(g(x))\pi^{(0)}_{f(x)}(h(x))\big). \tag{C.1.8}$$

Proof. Writing,

$$g(x) = a(x) + B(x)f(x), \quad h(x) = c(x) + D(x)f(x),$$

with $a(x), c(x) \in M_{f(x)}$, $B(x), D(x) \in \mathbb{F}[x]$, we see,

$$g(x)h(x) = a(x)c(x) + E(x)f(x),$$

thus,

$$\pi^{(0)}_{f(x)}\big(g(x)h(x)\big) = \pi^{(0)}_{f(x)}\big(a(x)c(x)\big) = \pi^{(0)}_{f(x)}\big(\pi^{(0)}_{f(x)}(g(x))\pi^{(0)}_{f(x)}(h(x))\big).$$

□

C.1.1.2 The Homomorphism $\pi^{(1)}_{f(x)}$

Lemma C.1.8. *For all polynomials* $\mu(x) \in M_{f(x)}$, $\mu(x) \neq 0$,

$$\pi'(\mu(x)) \neq \pi'(0). \tag{C.1.9}$$

Proof. For all $\mu(x) \in M_{f(x)}$, $|\mu(x)| < |f(x)|$, and thus $\mu(x) \notin I\,(f(x))$. Thus, the claim follows from Lemma 2.4.4. □

Lemma C.1.9. *For all* $\mu(x) \in M_{f(x)}$,

$$\partial_x(\mu(x)) \in M_{f(x)}. \tag{C.1.10}$$

Furthermore,

$$\partial_x(f(x)) \in M_{f(x)}. \tag{C.1.11}$$

Proof. For all polynomials $g(x) \in \mathbb{F}[x]$ of the form,

$$g(x) = \sum_{j=0}^{|f(x)|} \alpha_j x^j,$$

by Corollary 2.3.58 and Lemma 2.3.59, we have,

$$\partial_x(g(x)) = \sum_{j=0}^{|f(x)|-1} (j+1)\alpha_{j+1} x^j \in M_{f(x)}.$$

□

Definition C.1.10 (∂_x)**.** We define $\partial_x \in \mathrm{Der}(\mathbb{F}[x])$ by,

$$\partial_x(x) \coloneqq 1. \tag{C.1.12}$$

Lemma C.1.11. *There exists $t(x) \in M_{f(x)}$ such that,*

$$\pi'(t(x)) = \pi'(\partial_x(f(x)))^{-1}. \tag{C.1.13}$$

Proof. Direct consequence of Lemmas C.1.9, C.1.8, and 3.4.16. □

Definition C.1.12 ($\partial_{f(x)}$)**.**

$$\begin{aligned} \partial_{f(x)} &: \mathbb{F}[x] \to \mathbb{F}[x] \\ \partial_{f(x)} &\coloneqq g(x) \mapsto t(x)\partial_x(g(x)). \end{aligned} \tag{C.1.14}$$

Lemma C.1.13.

$$\pi^{(0)}_{f(x)}\big(\partial_{f(x)}(f(x))\big) = 1. \tag{C.1.15}$$

Proof.

$$\pi^{(0)}_{f(x)}\big(\partial_{f(x)}(f(x))\big) = \chi' \circ \pi'\big(t(x)\partial_x(f(x))\big) = \chi'\big(\pi'(t(x))\pi'(\partial_x(f(x)))\big) = \chi' \circ \pi'(1) = 1.$$

□

Definition C.1.14.

$$\begin{aligned} \pi^{(1)}_{f(x)} &: \mathbb{F}[x] \to M_{f(x)} \\ \pi^{(1)}_{f(x)} &\coloneqq \pi^{(0)}_{f(x)} \circ \partial_{f(x)} \circ \big(\mathrm{id} - \pi^{(0)}_{f(x)}\big). \end{aligned} \tag{C.1.16}$$

Lemma C.1.15. *For all $g(x) \in \mathbb{F}[x]$,*

$$g(x) = a(x) + b(x)f(x) + C(x)f(x)^2, \tag{C.1.17}$$

with $a(x), b(x) \in M_{f(x)}$, $C(x) \in \mathbb{F}[x]$, we have,

$$\pi^{(1)}_{f(x)}(g(x)) = b(x). \tag{C.1.18}$$

Proof.

$$\begin{aligned}\pi^{(1)}_{f(x)}(g(x)) &= \pi^{(0)}_{f(x)}\circ\partial_{f(x)}\big(b(x)f(x)+C(x)f(x)^2\big)\\ &= \pi^{(0)}_{f(x)}\big(b(x)\partial_{f(x)}(f(x))+D(x)f(x)\big)\\ &= b(x),\end{aligned}$$

with the final line following from Lemmas C.1.7 and C.1.13. □

C.1.1.3 The Homomorphism $\pi^{(2)}_{f(x)}$

Definition C.1.16 (P)**.** For all $g(x)\in\mathbb{F}[x]$, let us define,

$$\begin{aligned}P &: \mathbb{F}[x]\to\mathrm{End}(\mathbb{F}[x])\\ P(g(x)) &\coloneqq h(x)\mapsto g(x)h(x).\end{aligned} \tag{C.1.19}$$

Definition C.1.17 ($\pi^{(2)}_{f(x)}$)**.**

$$\begin{aligned}\pi^{(2)}_{f(x)} &: \mathbb{F}[x]\to M_{f(x)}\\ \pi^{(2)}_{f(x)} &\coloneqq \frac{1}{2}\pi^{(0)}_{f(x)}\circ\partial^{\circ 2}_{f(x)}\circ\big(\mathrm{id}-\pi^{(0)}_{f(x)}-P(f(x))\circ\pi^{(1)}_{f(x)}\big).\end{aligned} \tag{C.1.20}$$

Lemma C.1.18. *For all* $g(x)\in\mathbb{F}[x]$,

$$g(x)=a(x)+b(x)f(x)+c(x)f(x)^2+D(x)f(x)^3, \tag{C.1.21}$$

with $a(x),b(x),c(x)\in M_{f(x)}$, $D(x)\in\mathbb{F}[x]$, *we have,*

$$\pi^{(1)}_{f(x)}(g(x))=c(x). \tag{C.1.22}$$

Proof.

$$\begin{aligned}\pi^{(2)}_{f(x)}(g(x)) &= \frac{1}{2}\pi^{(0)}_{f(x)}\circ\partial^{\circ 2}_{f(x)}\circ\big(\mathrm{id}-\pi^{(0)}_{f(x)}-P(f(x))\circ\pi^{(1)}_{f(x)}\big)(g(x))\\ &= \frac{1}{2}\pi^{(0)}_{f(x)}\circ\partial^{\circ 2}_{f(x)}\big(c(x)f(x)^2+D(x)f(x)^3\big)\\ &= \frac{1}{2}\pi^{(0)}_{f(x)}\circ\partial_{f(x)}\big(2c(x)\partial_{f(x)}(f(x))f(x)+E(x)f(x)^2\big)\\ &= \pi^{(0)}_{f(x)}\Big(c(x)\big(\partial_{f(x)}(f(x))\big)^2+F(x)f(x)\Big)\\ &= c(x),\end{aligned}$$

with the final line following from Lemmas C.1.7 and C.1.13. □

C.1.1.4 Relationships Between the $\pi^{(j)}_{f(x)}$

Lemma C.1.19. *For all $\mu(x), \nu(x) \in M_{f(x)}$, suppressing inclusions into $\mathbb{F}[x]$,*

$$\mu(x)\nu(x) = \pi^{(0)}_{f(x)}\big(\mu(x)\nu(x)\big) + \pi^{(1)}_{f(x)}\big(\mu(x)\nu(x)\big)f(x). \tag{C.1.23}$$

Proof. Since $|\mu(x)| \leq |f(x)| - 1$ and $|\nu(x)| \leq |f(x)| - 1$, we must have $|\mu(x)\nu(x)| \leq |f(x)^2| - 2$. Thus, by polynomial division $\exists a(x), b(x) \in M_{f(x)}$ such that,

$$\mu(x)\nu(x) = a(x) + b(x)f(x).$$

The result follows from Lemmas C.1.5 and C.1.15. □

Lemma C.1.20. *For all $j, k \in \{0, 1, 2\}$,*

$$\pi^{(j)}_{f(x)} \circ \pi^{(k)}_{f(x)} = \begin{cases} \pi^{(k)}_{f(x)} & j = 0 \\ 0 & j \neq 0. \end{cases} \tag{C.1.24}$$

Proof. Follows from Lemmas C.1.5, C.1.15, and C.1.18. □

Lemma C.1.21. *For all $p(x), q(x), r(x) \in \mathbb{F}[x]$,*

$$\begin{aligned} &\pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}(p(x))\pi^{(0)}_{f(x)}(q(x))\pi^{(0)}_{f(x)}(r(x))\Big) \\ &\qquad = \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\pi^{(0)}_{f(x)}(p(x))\pi^{(0)}_{f(x)}(q(x))\big)\pi^{(0)}_{f(x)}(r(x))\Big) \\ &\qquad = \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}(p(x))\pi^{(0)}_{f(x)}\big(\pi^{(0)}_{f(x)}(q(x))\pi^{(0)}_{f(x)}(r(x))\big)\Big). \end{aligned} \tag{C.1.25}$$

Proof. By associativity of the product in $\mathbb{F}[x]$, we may use Lemma C.1.7 to expand each side of,

$$\begin{aligned} &\pi^{(0)}_{f(x)}\Big(\big(\pi^{(0)}_{f(x)}(p(x))\pi^{(0)}_{f(x)}(q(x))\big)\pi^{(0)}_{f(x)}(r(x))\Big) \\ &\qquad = \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}(p(x))\big(\pi^{(0)}_{f(x)}(q(x))\pi^{(0)}_{f(x)}(r(x))\big)\Big), \end{aligned}$$

and simplify the resulting expressions using Lemma C.1.6. □

Lemma C.1.22. *For all $g(x), h(x) \in \mathbb{F}[x]$,*

$$\begin{aligned} \pi^{(1)}_{f(x)}\big(g(x)h(x)\big) = {} & \pi^{(1)}_{f(x)}\big(\pi^{(0)}_{f(x)}(g(x))\pi^{(0)}_{f(x)}(h(x))\big) \\ & + \pi^{(0)}_{f(x)}\big(\pi^{(1)}_{f(x)}(g(x))\pi^{(0)}_{f(x)}(h(x))\big) \\ & + \pi^{(0)}_{f(x)}\big(\pi^{(0)}_{f(x)}(g(x))\pi^{(1)}_{f(x)}(h(x))\big). \end{aligned} \tag{C.1.26}$$

Proof. Writing,

$$g(x) = a(x) + b(x)f(x) + P(x)f(x)^2$$
$$h(x) = c(x) + d(x)f(x) + Q(x)f(x)^2,$$

with $a(x), b(x), c(x), d(x) \in M_{f(x)}$, $P(x), Q(x) \in \mathbb{F}[x]$, we have,

$$\begin{aligned}\pi^{(1)}_{f(x)}\big(g(x)h(x)\big) &= \pi^{(1)}_{f(x)}\Big(\big(a(x) + b(x)f(x)\big)\big(c(x) + d(x)f(x)\big)\Big) \\ &= \pi^{(1)}_{f(x)}\big(a(x)c(x) + b(x)c(x)f(x) + a(x)d(x)f(x)\big)\end{aligned}$$

and utilising Lemma C.1.19,

$$= \pi^{(1)}_{f(x)}\big(a(x)c(x)\big) + \pi^{(0)}_{f(x)}\big(b(x)c(x)\big) + \pi^{(0)}_{f(x)}\big(a(x)d(x)\big).$$

□

Lemma C.1.23. *For all $p(x), q(x), r(x) \in \mathbb{F}[x]$,*

$$\begin{aligned}&\pi^{(1)}_{f(x)}\Big(\pi^{(0)}_{f(x)}(p(x))\pi^{(0)}_{f(x)}(q(x))\pi^{(0)}_{f(x)}(r(x))\Big) \\ &\quad = \pi^{(1)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\pi^{(0)}_{f(x)}(p(x))\pi^{(0)}_{f(x)}(q(x))\big)\pi^{(0)}_{f(x)}(r(x))\Big) \\ &\qquad + \pi^{(0)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\pi^{(0)}_{f(x)}(p(x))\pi^{(0)}_{f(x)}(q(x))\big)\pi^{(0)}_{f(x)}(r(x))\Big) \\ &\quad = \pi^{(1)}_{f(x)}\Big(\pi^{(0)}_{f(x)}(p(x))\pi^{(0)}_{f(x)}\big(\pi^{(0)}_{f(x)}(q(x))\pi^{(0)}_{f(x)}(r(x))\big)\Big) \\ &\qquad + \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}(p(x))\pi^{(1)}_{f(x)}\big(\pi^{(0)}_{f(x)}(q(x))\pi^{(0)}_{f(x)}(r(x))\big)\Big).\end{aligned} \tag{C.1.27}$$

Proof. By associativity of the product in $\mathbb{F}[x]$, we may use Lemma C.1.22 to expand each side of,

$$\begin{aligned}&\pi^{(1)}_{f(x)}\Big(\big(\pi^{(0)}_{f(x)}(p(x))\pi^{(0)}_{f(x)}(q(x))\big)\pi^{(0)}_{f(x)}(r(x))\Big) \\ &\quad = \pi^{(1)}_{f(x)}\Big(\pi^{(0)}_{f(x)}(p(x))\big(\pi^{(0)}_{f(x)}(q(x))\pi^{(0)}_{f(x)}(r(x))\big)\Big),\end{aligned}$$

and simplify the resulting expressions using Lemma C.1.20. □

Lemma C.1.24. *For all $\mu(x), \nu(x) \in M_{f(x)}$,*

$$\pi^{(2)}_{f(x)}\big(\mu(x)\nu(x)\big) = 0.$$

Proof. Direct consequence of Lemma C.1.19. □

Lemma C.1.25. *For all* $g(x), h(x) \in \mathbb{F}[x]$,

$$\begin{aligned}\pi^{(2)}_{f(x)}\big(g(x)h(x)\big) =&\pi^{(1)}_{f(x)}\big(\pi^{(1)}_{f(x)}(g(x))\pi^{(0)}_{f(x)}(h(x))\big) + \pi^{(1)}_{f(x)}\big(\pi^{(0)}_{f(x)}(g(x))\pi^{(1)}_{f(x)}(h(x))\big)\\ &+ \pi^{(0)}_{f(x)}\big(\pi^{(2)}_{f(x)}(g(x))\pi^{(0)}_{f(x)}(h(x))\big) + \pi^{(0)}_{f(x)}\big(\pi^{(0)}_{f(x)}(g(x))\pi^{(2)}_{f(x)}(h(x))\big)\\ &+ \pi^{(0)}_{f(x)}\big(\pi^{(1)}_{f(x)}(g(x))\pi^{(1)}_{f(x)}(h(x))\big).\end{aligned} \tag{C.1.28}$$

Proof. Writing,

$$\begin{aligned}g(x) &= a(x) + b(x)f(x) + c(x)f(x)^2 + D(x)f(x)^3\\ h(x) &= p(x) + q(x)f(x) + r(x)f(x)^2 + S(x)f(x)^3,\end{aligned}$$

with $a(x), b(x), c(x), p(x), q(x), r(x) \in M_{f(x)}$, $D(x), S(x) \in \mathbb{F}[x]$, we have,

$$\begin{aligned}\pi^{(2)}_{f(x)}\big(g(x)h(x)\big) &= \pi^{(2)}_{f(x)}\Big(\big(a(x) + b(x)f(x) + c(x)f(x)^2\big)\\ &\qquad\qquad \big(p(x) + q(x)f(x) + r(x)f(x)^2\big)\Big)\\ &= \pi^{(2)}_{f(x)}\big(a(x)p(x) + a(x)q(x)f(x) + a(x)r(x)f(x)^2\\ &\qquad + b(x)p(x)f(x) + b(x)q(x)f(x)^2 + c(x)p(x)f(x)^2\big)\end{aligned}$$

and utilising Lemmas C.1.19 and C.1.24,

$$\begin{aligned}&= \pi^{(1)}_{f(x)}\big(a(x)q(x)\big) + \pi^{(0)}_{f(x)}\big(a(x)r(x)\big) + \pi^{(1)}_{f(x)}\big(b(x)p(x)\big)\\ &\quad + \pi^{(0)}_{f(x)}\big(b(x)q(x)\big) + \pi^{(0)}_{f(x)}\big(c(x)p(x)\big).\end{aligned}$$

□

Lemma C.1.26. *For all* $p(x), q(x), r(x) \in \mathbb{F}[x]$,

$$\begin{aligned}&\pi^{(2)}_{f(x)}\Big(\pi^{(0)}_{f(x)}(p(x))\pi^{(0)}_{f(x)}(q(x))\pi^{(0)}_{f(x)}(r(x))\Big)\\ &\qquad = \pi^{(1)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\pi^{(0)}_{f(x)}(p(x))\pi^{(0)}_{f(x)}(q(x))\big)\pi^{(0)}_{f(x)}(r(x))\Big)\\ &\qquad = \pi^{(1)}_{f(x)}\Big(\pi^{(0)}_{f(x)}(p(x))\pi^{(1)}_{f(x)}\big(\pi^{(0)}_{f(x)}(q(x))\pi^{(0)}_{f(x)}(r(x))\big)\Big).\end{aligned} \tag{C.1.29}$$

Proof. By associativity of the product in $\mathbb{F}[x]$, we may use Lemma C.1.25 to expand each side of,

$$\begin{aligned}&\pi^{(2)}_{f(x)}\Big(\big(\pi^{(0)}_{f(x)}(p(x))\pi^{(0)}_{f(x)}(q(x))\big)\pi^{(0)}_{f(x)}(r(x))\Big)\\ &\qquad = \pi^{(2)}_{f(x)}\Big(\pi^{(0)}_{f(x)}(p(x))\big(\pi^{(0)}_{f(x)}(q(x))\pi^{(0)}_{f(x)}(r(x))\big)\Big),\end{aligned}$$

and simplify the resulting expressions using Lemmas C.1.20 and C.1.24. □

C.1.1.5 The Product of $S(\mathbb{F}[x], f(x), M_{f(x)})$

In this section, unless otherwise stated, we will consider $g(x), h(x), \in S(\mathbb{F}[x], f(x), M_{f(x)})$,

$$g(x) = \sum_{j=0}^{\infty} \mu_j(x) f(x)^j, \quad h(x) = \sum_{j=0}^{\infty} \nu_j(x) f(x)^j,$$

with product,

$$g(x)h(x) := \sum_{p=0}^{\infty} \eta_p(x) f(x)^p, \tag{C.1.30}$$

with, $\forall p \in \mathbb{N}$,

$$\eta_p(x) := \begin{cases} \pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big) & p = 0 \\ \sum_{q=0}^{p} \pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big) + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{p-q-1}(x)\big) & p \in \mathbb{Z}^+. \end{cases} \tag{C.1.31}$$

Lemma C.1.27. *The product in Definition 3.4.18 is commutative.*

Proof. For all $p \in \mathbb{N}$, we may reindex,

$$\begin{aligned} \eta_p(x) &= \begin{cases} \pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big) & p = 0 \\ \sum_{q=0}^{p} \pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big) + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{p-q-1}(x)\big) & p \in \mathbb{Z}^+. \end{cases} \\ &= \begin{cases} \pi^{(0)}_{f(x)}\big(\nu_0(x)\mu_0(x)\big) & p = 0 \\ \sum_{r=0}^{p} \pi^{(0)}_{f(x)}\big(\nu_r(x)\mu_{p-r}(x)\big) + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\big(\nu_r(x)\mu_{p-r-1}(x)\big) & p \in \mathbb{Z}^+, \end{cases} \end{aligned}$$

thus,

$$g(x)h(x) = h(x)g(x).$$

□

Lemma C.1.28. *The product in Definition 3.4.18 is bilinear.*

Proof. By Lemma C.1.27, we need only show linearity in the first argument. For all $\gamma \in \mathbb{F}$, by definition,

$$\gamma(g(x)h(x)) = \sum_{p=0}^{\infty} (\gamma \eta_p(x)) f(x)^p.$$

Thus, $\forall p \in \mathbb{N}$,

$$\gamma\eta_p(x) = \begin{cases} \gamma\pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big) & p = 0 \\ \displaystyle\sum_{q=0}^{p} \gamma\pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big) + \sum_{q=0}^{p-1} \gamma\pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{p-q-1}(x)\big) & p \in \mathbb{Z}^+ \end{cases}$$
$$= \begin{cases} \pi^{(0)}_{f(x)}\big((\gamma\mu_0(x))\nu_0(x)\big) & p = 0 \\ \displaystyle\sum_{q=0}^{p} \pi^{(0)}_{f(x)}\big((\gamma\mu_q(x))\nu_{p-q}(x)\big) + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\big((\gamma\mu_q(x))\nu_{p-q-1}(x)\big) & p \in \mathbb{Z}^+. \end{cases}$$

Therefore,

$$\gamma(g(x)h(x)) = (\gamma g(x))h(x).$$

Now, consider $r(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,

$$r(x) = \sum_{j=0}^{\infty} \rho_j(x) f(x)^j.$$

Then, by definition,

$$(g(x) + r(x))h(x) = \sum_{p=0}^{\infty} \epsilon_p(x) f(x)^p,$$

with, $\forall p \in \mathbb{N}$,

$$\epsilon_p(x) = \begin{cases} \pi^{(0)}_{f(x)}\big((\mu_0(x) + \rho_0(x))\nu_0(x)\big) & p = 0 \\ \displaystyle\sum_{q=0}^{p} \pi^{(0)}_{f(x)}\big((\mu_q(x) + \rho_q(x))\nu_{p-q}(x)\big) \\ \qquad + \displaystyle\sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\big((\mu_q(x) + \rho_q(x))\nu_{p-q-1}(x)\big) & p \in \mathbb{Z}^+ \end{cases}$$
$$= \begin{cases} \pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big) + \pi^{(0)}_{f(x)}\big(\rho_0(x)\nu_0(x)\big) & p = 0 \\ \displaystyle\sum_{q=0}^{p}\pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big) + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{p-q-1}(x)\big) \\ \qquad + \displaystyle\sum_{q=0}^{p} \pi^{(0)}_{f(x)}\big(\rho_q(x)\nu_{p-q}(x)\big) + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\big(\rho_q(x)\nu_{p-q-1}(x)\big) & p \in \mathbb{Z}^+. \end{cases}$$

Thus,

$$(g(x) + r(x))h(x) = g(x)h(x) + r(x)h(x).$$

□

Lemma C.1.29. *The product in Definition 3.4.18 has two-sided identity element* 1.

Proof. By Lemma C.1.27, we need only show that 1 is a left identity for the product. We may write,

$$1 = \sum_{j=0}^{\infty} \iota_j(x) f(x)^j,$$

with,

$$\iota_j(x) := \begin{cases} 1 & j = 0 \\ 0 & j \in \mathbb{Z}^+. \end{cases}$$

Therefore, $1\,h(x)$ has coefficients,

$$\begin{aligned}
\eta_p(x) &= \begin{cases} \pi^{(0)}_{f(x)}\big(\iota_0(x)\nu_0(x)\big) & p = 0 \\ \sum_{q=0}^{p} \pi^{(0)}_{f(x)}\big(\iota_q(x)\nu_{p-q}(x)\big) + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\big(\iota_q(x)\nu_{p-q-1}(x)\big) & p \in \mathbb{Z}^+ \end{cases} \\
&= \begin{cases} \pi^{(0)}_{f(x)}\big(\nu_0(x)\big) & p = 0 \\ \pi^{(0)}_{f(x)}\big(\nu_p(x)\big) + \pi^{(1)}_{f(x)}\big(\nu_{p-1}(x)\big) & p \in \mathbb{Z}^+ \end{cases} \\
&= \begin{cases} \pi^{(0)}_{f(x)}\big(\nu_0(x)\big) & p = 0 \\ \pi^{(0)}_{f(x)}\big(\nu_p(x)\big) & p \in \mathbb{Z}^+. \end{cases}
\end{aligned}$$

Therefore,

$$1\,h(x) = h(x).$$

□

Lemma C.1.30. *The product in Definition 3.4.18 is associative.*

Proof. Let $r(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,

$$r(x) = \sum_{j=0}^{\infty} \rho_j(x) f(x)^j, \quad h(x)r(x) = \sum_{j=0}^{\infty} \sigma_j(x) f(x)^j$$

$$(g(x)h(x))r(x) = \sum_{p=0}^{\infty} \epsilon_p(x) f(x)^p, \quad g(x)(h(x)r(x)) = \sum_{p=0}^{\infty} \epsilon'_p(x) f(x)^p$$

When $p = 0$,

$$\begin{aligned}
\epsilon_0(x) &= \pi^{(0)}_{f(x)}\big(\eta_0(x)\rho_0(x)\big) \\
&= \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big)\rho_0(x)\Big),
\end{aligned}$$

applying Lemma C.1.21,

$$= \pi^{(0)}_{f(x)}\Big(\mu_0(x)\nu_0(x)\rho_0(x)\Big)$$

$$
\begin{aligned}
&= \pi^{(0)}_{f(x)}\Big(\mu_0(x)\pi^{(0)}_{f(x)}\big(\nu_0(x)\rho_0(x)\big)\Big) \\
&= \pi^{(0)}_{f(x)}\Big(\mu_0(x)\sigma_0(x)\Big) \\
&= \epsilon'_0(x).
\end{aligned}
$$

When $p = 1$,

$$
\begin{aligned}
\epsilon_1(x) &= \pi^{(0)}_{f(x)}\big(\eta_0(x)\rho_1(x) + \eta_1(x)\rho_0(x)\big) + \pi^{(1)}_{f(x)}\big(\eta_0(x)\rho_0(x)\big) \\
&= \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big)\rho_1(x)\Big) + \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_1(x)\big)\rho_0(x)\Big) \\
&\quad + \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_1(x)\nu_0(x)\big)\rho_0(x)\Big) + \pi^{(0)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big)\rho_0(x)\Big) \\
&\quad + \pi^{(1)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big)\rho_0(x)\Big),
\end{aligned}
$$

applying Lemmas C.1.21 and C.1.23,

$$
\begin{aligned}
&= \pi^{(0)}_{f(x)}\Big(\mu_0(x)\pi^{(0)}_{f(x)}\big(\nu_0(x)\rho_1(x)\big)\Big) + \pi^{(0)}_{f(x)}\Big(\mu_0(x)\pi^{(0)}_{f(x)}\big(\nu_1(x)\rho_0(x)\big)\Big) \\
&\quad + \pi^{(0)}_{f(x)}\Big(\mu_1(x)\pi^{(0)}_{f(x)}\big(\nu_0(x)\rho_0(x)\big)\Big) + \pi^{(0)}_{f(x)}\Big(\mu_0(x)\pi^{(1)}_{f(x)}\big(\nu_0(x)\rho_0(x)\big)\Big) \\
&\quad + \pi^{(1)}_{f(x)}\Big(\mu_0(x)\pi^{(0)}_{f(x)}\big(\nu_0(x)\rho_0(x)\big)\Big), \\
&= \pi^{(0)}_{f(x)}\Big(\mu_0(x)\sigma_1(x)\Big) + \pi^{(0)}_{f(x)}\Big(\mu_1(x)\sigma_0(x)\Big) + \pi^{(1)}_{f(x)}\Big(\mu_0(x)\sigma_0(x)\Big) \\
&= \epsilon'_1(x).
\end{aligned}
$$

When $p \in \mathbb{Z}^+$, $p \geq 2$,

$$
\begin{aligned}
\epsilon_p(x) &= \pi^{(0)}_{f(x)}\big(\eta_0(x)\rho_p(x)\big) + \pi^{(1)}_{f(x)}\big(\eta_0(x)\rho_{p-1}(x)\big) \\
&\qquad + \sum_{q=1}^{p} \pi^{(0)}_{f(x)}\big(\eta_q(x)\rho_{p-q}(x)\big) + \sum_{q=1}^{p-1} \pi^{(1)}_{f(x)}\big(\eta_q(x)\rho_{p-q-1}(x)\big) \\
&= \sum_{q=0}^{p}\sum_{r=0}^{q} \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_r(x)\nu_{q-r}(x)\big)\rho_{p-q}(x)\Big) \\
&\quad + \sum_{q=0}^{p-1}\sum_{r=0}^{q} \pi^{(1)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_r(x)\nu_{q-r}(x)\big)\rho_{p-q-1}(x)\Big) \\
&\quad + \sum_{q=1}^{p}\sum_{r=0}^{q-1} \pi^{(0)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_r(x)\nu_{q-r-1}(x)\big)\rho_{p-q}(x)\Big) \\
&\quad + \sum_{q=1}^{p-1}\sum_{r=0}^{q-1} \pi^{(1)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_r(x)\nu_{q-r-1}(x)\big)\rho_{p-q-1}(x)\Big).
\end{aligned}
$$

Reversing the order of summation and reindexing,

$$\begin{aligned}
&= \sum_{r=0}^{p}\sum_{q=r}^{p} \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_r(x)\nu_{q-r}(x)\big)\rho_{p-q}(x)\Big) \\
&\qquad + \sum_{r=0}^{p-1}\sum_{q=r}^{p-1} \pi^{(1)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_r(x)\nu_{q-r}(x)\big)\rho_{p-q-1}(x)\Big) \\
&\qquad + \sum_{r=0}^{p-1}\sum_{q=r+1}^{p} \pi^{(0)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_r(x)\nu_{q-r-1}(x)\big)\rho_{p-q}(x)\Big) \\
&\qquad + \sum_{r=0}^{p-2}\sum_{q=r+1}^{p-1} \pi^{(1)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_r(x)\nu_{q-r-1}(x)\big)\rho_{p-q-1}(x)\Big) \\
&= \sum_{r=0}^{p}\sum_{m=0}^{p-r} \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_r(x)\nu_{m}(x)\big)\rho_{p-r-m}(x)\Big) \\
&\qquad + \sum_{r=0}^{p-1}\sum_{m=0}^{p-r-1} \pi^{(1)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_r(x)\nu_{m}(x)\big)\rho_{p-r-m-1}(x)\Big) \\
&\qquad + \sum_{r=0}^{p-1}\sum_{m=0}^{p-r-1} \pi^{(0)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_r(x)\nu_{m}(x)\big)\rho_{p-r-m-1}(x)\Big) \\
&\qquad + \sum_{r=0}^{p-2}\sum_{m=0}^{p-r-2} \pi^{(1)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_r(x)\nu_{m}(x)\big)\rho_{p-r-m-2}(x)\Big),
\end{aligned}$$

and applying Lemmas C.1.21, C.1.23, and C.1.26,

$$\begin{aligned}
&= \sum_{r=0}^{p}\sum_{m=0}^{p-r} \pi^{(0)}_{f(x)}\Big(\mu_r(x)\pi^{(0)}_{f(x)}\big(\nu_{m}(x)\rho_{p-r-m}(x)\big)\Big) \\
&\qquad + \sum_{r=0}^{p-1}\sum_{m=0}^{p-r-1} \pi^{(1)}_{f(x)}\Big(\mu_r(x)\pi^{(0)}_{f(x)}\big(\nu_{m}(x)\rho_{p-r-m-1}(x)\big)\Big) \\
&\qquad + \sum_{r=0}^{p-1}\sum_{m=0}^{p-r-1} \pi^{(0)}_{f(x)}\Big(\mu_r(x)\pi^{(1)}_{f(x)}\big(\nu_{m}(x)\rho_{p-r-m-1}(x)\big)\Big) \\
&\qquad + \sum_{r=0}^{p-2}\sum_{m=0}^{p-r-2} \pi^{(1)}_{f(x)}\Big(\mu_r(x)\pi^{(1)}_{f(x)}\big(\nu_{m}(x)\rho_{p-r-m-2}(x)\big)\Big) \\
&= \pi^{(0)}_{f(x)}\Big(\mu_p(x)\pi^{(0)}_{f(x)}\big(\nu_0(x)\rho_0(x)\big)\Big) + \pi^{(1)}_{f(x)}\Big(\mu_{p-1}(x)\pi^{(0)}_{f(x)}\big(\nu_0(x)\rho_0(x)\big)\Big) \\
&\quad + \sum_{r=0}^{p-1}\pi^{(0)}_{f(x)}\Bigg(\mu_r(x)\Bigg(\sum_{m=0}^{p-r}\pi^{(0)}_{f(x)}\big(\nu_m(x)\rho_{p-r-m}(x)\big) \\
&\qquad\qquad\qquad + \sum_{m=0}^{p-r-1}\pi^{(1)}_{f(x)}\big(\nu_m(x)\rho_{p-r-m-1}(x)\big)\Bigg)\Bigg) \\
&\quad + \sum_{r=0}^{p-2}\pi^{(1)}_{f(x)}\Bigg(\mu_r(x)\Bigg(\sum_{m=0}^{p-r-1}\pi^{(0)}_{f(x)}\big(\nu_m(x)\rho_{p-r-m-1}(x)\big) \\
&\qquad\qquad\qquad + \sum_{m=0}^{p-r-2}\pi^{(1)}_{f(x)}\big(\nu_m(x)\rho_{p-r-m-2}(x)\big)\Bigg)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \pi^{(0)}_{f(x)}\big(\mu_p(x)\sigma_0(x)\big) + \pi^{(1)}_{f(x)}\big(\mu_{p-1}(x)\sigma_0(x)\big) \\
&\qquad + \sum_{r=0}^{p-1} \pi^{(0)}_{f(x)}\big(\mu_r(x)\sigma_{p-r}(x)\big) + \sum_{r=0}^{p-2} \pi^{(1)}_{f(x)}\big(\mu_r(x)\sigma_{p-r-1}(x)\big) \\
&= \sum_{r=0}^{p} \pi^{(0)}_{f(x)}\big(\mu_r(x)\sigma_{p-r}(x)\big) + \sum_{r=0}^{p-1} \pi^{(1)}_{f(x)}\big(\mu_r(x)\sigma_{p-r-1}(x)\big) \\
&= \epsilon'_p(x).
\end{aligned}
$$

□

Lemma C.1.31. *Consider $g(x), h(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$, both with only finitely many non-zero terms. Then, $g(x)h(x)$ is equivalent to their multiplication as polynomials in $\mathbb{F}[x]$.*

Proof. Writing,

$$
g(x) = \sum_{j=0}^{m} \mu_j(x) f(x)^j, \quad h(x) = \sum_{j=0}^{n} \nu_j(x) f(x)^j,
$$

with $m, n \in \mathbb{N}$, and defining $k := m + n + 1$, consider their product in $S(\mathbb{F}[x], f(x), M_{f(x)})$,

$$
\begin{aligned}
g(x)h(x) &= \sum_{p=0}^{k} \eta_p(x) f(x)^p \\
&= \pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big) + \sum_{p=1}^{k}\left(\sum_{q=0}^{p} \pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{p-q-1}(x)\big)\right) f(x)^p \\
&= \sum_{p=0}^{k}\sum_{q=0}^{p} \pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big) f(x)^p + \sum_{p=1}^{k}\sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{p-q-1}(x)\big) f(x)^p \\
&= \sum_{p=0}^{k}\sum_{q=0}^{p} \pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big) f(x)^p + \sum_{r=0}^{k-1}\sum_{q=0}^{r} \pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{r-q}(x)\big) f(x)^{r+1} \\
&= \sum_{q=0}^{k} \pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{k-q}(x)\big) f(x)^k \\
&\qquad + \sum_{p=0}^{k-1}\sum_{q=0}^{p}\Big(\pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big) + \pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big) f(x)\Big) f(x)^p \\
&= \sum_{q=0}^{k} \pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{k-q}(x)\big) f(x)^k + \sum_{p=0}^{k-1}\sum_{q=0}^{p}\big(\mu_q(x)\nu_{p-q}(x)\big) f(x)^p \\
&= \sum_{p=0}^{k-1}\sum_{q=0}^{p}\big(\mu_q(x)\nu_{p-q}(x)\big) f(x)^p,
\end{aligned}
$$

with the final equality following from $\mu_q(x) = 0$ when $q > m$ and $\nu_{k-q}(x) = 0$ when $m + n + 1 - q > n$ implying the first term is zero. Noting, $\nu_s(x) = 0$ when $s > n$ and reindexing,

$$= \sum_{s=0}^{m+n} \sum_{r=0}^{m+n-s} \big(\mu_r(x)\nu_s(x)\big) f(x)^{r+s}$$
$$= \sum_{s=0}^{n} \sum_{r=0}^{m+n-s} \big(\mu_r(x)\nu_s(x)\big) f(x)^{r+s}.$$

Now, $0 \leq s \leq n$ implies $m \leq m + n - s \leq m + n$, and so,

$$= \sum_{s=0}^{n} \sum_{r=0}^{m} \big(\mu_r(x)\nu_s(x)\big) f(x)^{r+s} + 0$$
$$= \left(\sum_{r=0}^{m} \mu_r(x) f(x)^r\right) \left(\sum_{s=0}^{n} \nu_s(x) f(x)^s\right).$$

□

C.1.2 Existence of Inverse Formal Power Series

Lemma C.1.32. *Consider a formal irreducible power series* $g(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,

$$g(x) = \sum_{j=0}^{\infty} \mu_j(x) f(x)^j. \tag{C.1.32}$$

Then, $g(x)$ *has a two-sided inverse with respect to the product of* $S(\mathbb{F}[x], f(x), M_{f(x)})$ *iff* $\mu_0(x) \neq 0$.

Proof. This proof is similar to that for $\mathbb{F}[[x]]$ in [67]. In the forward direction, consider $\exists h(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$ such that $g(x)h(x) = h(x)g(x) = 1$. Therefore, $\eta_0(x) = \pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big) = 1$ and so we must have $\mu_0(x) \neq 0$. In the reverse direction, suppose $\mu_0(x) \neq 0$. Then, by Definition 3.4.17 and Lemmas C.1.8 and 3.4.16, $\forall \mu_0 \in M_{f(x)}$, $\exists \mu_0^{-1}(x) \in M_{f(x)}$ such that,

$$\pi^{(0)}_{f(x)}\big(\mu_0(x)\mu_0^{-1}(x)\big) = 1.$$

Therefore, let us define $h(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,

$$h(x) := \sum_{p=0}^{\infty} \omega_p(x) f(x)^p,$$

with, $\forall p \in \mathbb{N}$,

$$\omega_p(x) \coloneqq \begin{cases} \mu_0^{-1}(x) & p = 0 \\ -\sum_{q=1}^{p} \pi_{f(x)}^{(0)}\Big(\mu_0^{-1}(x)\pi_{f(x)}^{(0)}\big(\mu_q(x)\omega_{p-q}(x)\big)\Big) & \\ \qquad -\sum_{q=0}^{p-1} \pi_{f(x)}^{(0)}\Big(\mu_0^{-1}(x)\pi_{f(x)}^{(1)}\big(\mu_q(x)\omega_{p-q-1}(x)\big)\Big) & p \in \mathbb{Z}^+. \end{cases}$$

Note, that, $\forall p \in \mathbb{Z}^+$,

$$\begin{aligned} \pi_{f(x)}^{(0)}\big(\mu_0(x)\omega_p(x)\big) &= -\sum_{q=1}^{p} \pi_{f(x)}^{(0)}\left(\mu_0(x)\pi_{f(x)}^{(0)}\Big(\mu_0^{-1}(x)\pi_{f(x)}^{(0)}\big(\mu_q(x)\omega_{p-q}(x)\big)\Big)\right) \\ &\qquad -\sum_{q=0}^{p-1} \pi_{f(x)}^{(0)}\left(\mu_0(x)\pi_{f(x)}^{(0)}\Big(\mu_0^{-1}(x)\pi_{f(x)}^{(1)}\big(\mu_q(x)\omega_{p-q-1}(x)\big)\Big)\right) \\ &= -\sum_{q=1}^{p} \pi_{f(x)}^{(0)}\left(\mu_0(x)\mu_0^{-1}(x)\pi_{f(x)}^{(0)}\big(\mu_q(x)\omega_{p-q}(x)\big)\right) \\ &\qquad -\sum_{q=0}^{p-1} \pi_{f(x)}^{(0)}\left(\mu_0(x)\mu_0^{-1}(x)\pi_{f(x)}^{(1)}\big(\mu_q(x)\omega_{p-q-1}(x)\big)\right) \\ &= -\sum_{q=1}^{p} \pi_{f(x)}^{(0)}\left(\pi_{f(x)}^{(0)}\big(\mu_q(x)\omega_{p-q}(x)\big)\right) \\ &\qquad\qquad -\sum_{q=0}^{p-1} \pi_{f(x)}^{(0)}\left(\pi_{f(x)}^{(1)}\big(\mu_q(x)\omega_{p-q-1}(x)\big)\right) \\ &= -\sum_{q=1}^{p} \pi_{f(x)}^{(0)}\big(\mu_q(x)\omega_{p-q}(x)\big) - \sum_{q=0}^{p-1} \pi_{f(x)}^{(1)}\big(\mu_q(x)\omega_{p-q-1}(x)\big). \end{aligned}$$

Then, writing,

$$g(x)h(x) = \sum_{p=0}^{\infty} \eta_p f(x)^p,$$

we find,

$$\eta_0(x) = \pi_{f(x)}^{(0)}\big(\mu_0(x)\omega_0(x)\big) = \pi_{f(x)}^{(0)}\big(\mu_0(x)\mu_0^{-1}(x)\big) = 1,$$

and, $\forall p \in \mathbb{Z}^+$,

$$\begin{aligned} \eta_p(x) &= \sum_{q=0}^{p} \pi_{f(x)}^{(0)}\big(\mu_q(x)\omega_{p-q}(x)\big) + \sum_{q=0}^{p-1} \pi_{f(x)}^{(1)}\big(\mu_q(x)\omega_{p-q-1}(x)\big) \\ &= \pi_{f(x)}^{(0)}\big(\mu_0(x)\omega_p(x)\big) + \sum_{q=1}^{p} \pi_{f(x)}^{(0)}\big(\mu_q(x)\omega_{p-q}(x)\big) + \sum_{q=0}^{p-1} \pi_{f(x)}^{(1)}\big(\mu_q(x)\omega_{p-q-1}(x)\big) \\ &= 0. \end{aligned}$$

Therefore, $h(x)$ is a two-sided inverse for $g(x)$. □

C.1.3 Derivations on Formal Power Series

Lemma C.1.33. *For all $g(x) \in \mathbb{F}[x]$,*

$$\left[\pi^{(0)}_{f(x)}, \partial_x\right](g(x)) = \pi^{(0)}_{f(x)}\left(\pi^{(1)}_{f(x)}(g(x))\partial_x(f(x))\right). \tag{C.1.33}$$

Proof. Writing,

$$g(x) = a(x) + b(x)f(x) + C(x)f(x)^2,$$

with $a(x), b(x) \in M_{f(x)}$, $C(x) \in \mathbb{F}[x]$, and using Lemma C.1.9,

$$\begin{aligned}\left[\pi^{(0)}_{f(x)}, \partial_x\right](g(x)) &= \partial_x(a(x)) + \pi^{(0)}_{f(x)}\big(b(x)\partial_x(f(x))\big) - \partial_x(a(x)) \\ &= \pi^{(0)}_{f(x)}\big(b(x)\partial_x(f(x))\big).\end{aligned}$$

□

Lemma C.1.34. *For all $g(x) \in \mathbb{F}[x]$,*

$$\left[\pi^{(1)}_{f(x)}, \partial_x\right](g(x)) = \pi^{(1)}_{f(x)}\left(\pi^{(1)}_{f(x)}(g(x))\partial_x(f(x))\right) + 2\pi^{(0)}_{f(x)}\left(\pi^{(2)}_{f(x)}(g(x))\partial_x(f(x))\right). \tag{C.1.34}$$

Proof. Writing,

$$g(x) = a(x) + b(x)f(x) + c(x)f(x)^2 + D(x)f(x)^3,$$

with $a(x), b(x), c(x) \in M_{f(x)}$, $D(x) \in \mathbb{F}[x]$, and using Lemma C.1.9 and C.1.19,

$$\begin{aligned}\left[\pi^{(1)}_{f(x)}, \partial_x\right](g(x)) &= \partial_x(b(x)) + \pi^{(1)}_{f(x)}\big(b(x)\partial_x(f(x))\big) \\ &\qquad + \pi^{(0)}_{f(x)}\big(2c(x)\partial_x(f(x))\big) - \partial_x(b(x)) \\ &= \pi^{(1)}_{f(x)}\big(b(x)\partial_x(f(x))\big) + \pi^{(0)}_{f(x)}\big(2c(x)\partial_x(f(x))\big).\end{aligned}$$

□

Lemma C.1.35. *For all $g(x), h(x) \in \mathbb{F}[x]$, $\forall j \in \{0, 1\}$,*

$$\begin{aligned}&\left[\pi^{(j)}_{f(x)}, \partial_x\right]\left(\pi^{(0)}_{f(x)}(g(x))\pi^{(0)}_{f(x)}(h(x))\right) \\ &\quad = \pi^{(j)}_{f(x)}\left(\pi^{(1)}_{f(x)}\left(\pi^{(0)}_{f(x)}(g(x))\pi^{(0)}_{f(x)}(h(x))\right)\partial_x(f(x))\right).\end{aligned} \tag{C.1.35}$$

Proof. Follows immediately from Lemmas C.1.24, C.1.33, and C.1.34. □

For the remaining proofs, unless otherwise stated, we will consider $g(x), h(x), \in S(\mathbb{F}[x], f(x), M_{f(x)})$,

$$g(x) = \sum_{j=0}^{\infty} \mu_j(x) f(x)^j, \quad h(x) = \sum_{j=0}^{\infty} \nu_j(x) f(x)^j,$$

with derivations,

$$D_x(g(x)) = \sum_{p=0}^{\infty} \theta_p(x) f(x)^p, \quad D_x(h(x)) = \sum_{p=0}^{\infty} \phi_p(x) f(x)^p, \tag{C.1.36}$$

with, for example, $\forall p \in \mathbb{N}$,

$$\theta_p(x) \coloneqq \partial_x(\mu_p(x)) + (p+1)\pi^{(0)}_{f(x)}\big(\mu_{p+1}(x)\partial_x(f(x))\big) + p\,\pi^{(1)}_{f(x)}\big(\mu_p(x)\partial_x(f(x))\big). \tag{C.1.37}$$

Lemma C.1.36. $D_x \in \mathrm{End}(S(\mathbb{F}[x], f(x), M_{f(x)}))$.

Proof. For all $\gamma \in \mathbb{F}$, by definition,

$$\gamma D_x(g(x)) = \sum_{p=0}^{\infty} (\gamma\theta_p(x)) f(x)^p.$$

Thus, $\forall p \in \mathbb{N}$,

$$\begin{aligned}\gamma\theta_p(x) &= \gamma\partial_x(\mu_p(x)) + \gamma(p+1)\pi^{(0)}_{f(x)}\big(\mu_{p+1}(x)\partial_x(f(x))\big) + \gamma p\,\pi^{(1)}_{f(x)}\big(\mu_p(x)\partial_x(f(x))\big)\\ &= \partial_x((\gamma\mu_p(x))) + (p+1)\pi^{(0)}_{f(x)}\big((\gamma\mu_{p+1}(x))\partial_x(f(x))\big)\\ &\quad + p\,\pi^{(1)}_{f(x)}\big((\gamma\mu_p(x))\partial_x(f(x))\big),\end{aligned}$$

and thus,

$$\gamma D_x(g(x)) = D_x(\gamma g(x)).$$

Now, by definition,

$$D_x(g(x)) + D_x(h(x)) = \sum_{p=0}^{\infty} (\theta_p(x) + \phi_p(x)) f(x)^p,$$

so, $\forall p \in \mathbb{N}$,

$$\begin{aligned}\theta_p(x) + \phi_p(x) &= \partial_x(\mu_p(x)) + (p+1)\pi^{(0)}_{f(x)}\big(\mu_{p+1}(x)\partial_x(f(x))\big)\\ &\quad + p\,\pi^{(1)}_{f(x)}\big(\mu_p(x)\partial_x(f(x))\big) + \partial_x(\nu_p(x))\\ &\quad + (p+1)\pi^{(0)}_{f(x)}\big(\nu_{p+1}(x)\partial_x(f(x))\big) + p\,\pi^{(1)}_{f(x)}\big(\nu_p(x)\partial_x(f(x))\big)\\ &= \partial_x\big((\mu_p(x) + \nu_p(x))\big) + (p+1)\pi^{(0)}_{f(x)}\Big(\big(\mu_{p+1}(x) + \nu_{p+1}(x)\big)\partial_x(f(x))\Big)\\ &\quad + p\,\pi^{(1)}_{f(x)}\Big(\big(\mu_p(x) + \nu_p(x)\big)\partial_x(f(x))\Big).\end{aligned}$$

Thus,

$$D_x(g(x)) + D_x(h(x)) = D_x(g(x) + h(x)).$$

□

Lemma C.1.37. $D_x \in \mathrm{Der}(S(\mathbb{F}[x], f(x), M_{f(x)}))$.

Proof. Writing,

$$g(x)h(x) = \sum_{j=0}^{\infty} \eta_j(x) f(x)^j, \quad D_x\big(g(x)h(x)\big) = \sum_{p=0}^{\infty} \epsilon_p(x) f(x)^p$$

$$\big(D_x(g(x))\big)h(x) = \sum_{j=0}^{\infty} \rho_j f(x)^j, \quad g(x)\big(D_x(h(x))\big) = \sum_{j=0}^{\infty} \sigma_j f(x)^j,$$

we have, $\forall p \in \mathbb{N}$,

$$\epsilon_p(x) = \partial_x(\eta_p(x)) + (p+1)\pi^{(0)}_{f(x)}\big(\eta_{p+1}(x)\partial_x(f(x))\big) + p\,\pi^{(1)}_{f(x)}\big(\eta_p(x)\partial_x(f(x))\big).$$

When $p = 0$,

$$\begin{aligned}
\epsilon_0(x) &= \partial_x(\eta_0(x)) + \pi^{(0)}_{f(x)}\big(\eta_1(x)\partial_x(f(x))\big) \\
&= \partial_x \circ \pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big) + \pi^{(0)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_0(x)\nu_0(x)\big)\partial_x(f(x))\Big) \\
&\quad + \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_0(x)\nu_1(x)\big)\partial_x(f(x))\Big) \\
&\quad + \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_1(x)\nu_0(x)\big)\partial_x(f(x))\Big).
\end{aligned}$$

Thus, utilising Lemma C.1.35 and C.1.21,

$$\begin{aligned}
&= \pi^{(0)}_{f(x)} \circ \partial_x\big(\mu_0(x)\nu_0(x)\big) \\
&\qquad + \pi^{(0)}_{f(x)}\Big(\mu_0(x)\pi^{(0)}_{f(x)}\big(\nu_1(x)\partial_x(f(x))\big)\Big) \\
&\qquad + \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_1(x)\partial_x(f(x))\big)\nu_0(x)\Big) \\
&= \pi^{(0)}_{f(x)}\left(\Big(\partial_x(\mu_0(x)) + \pi^{(0)}_{f(x)}\big(\mu_1(x)\partial_x(f(x))\big)\Big)\nu_0(x)\right) \\
&\qquad + \pi^{(0)}_{f(x)}\left(\mu_0(x)\Big(\partial_x(\nu_0(x)) + \pi^{(0)}_{f(x)}\big(\nu_1(x)\partial_x(f(x))\big)\Big)\right) \\
&= \pi^{(0)}_{f(x)}\big(\theta_0(x)\nu_0(x)\big) + \pi^{(0)}_{f(x)}\big(\mu_0(x)\phi_0(x)\big) \\
&= \rho_0(x) + \sigma_0(x).
\end{aligned}$$

When $p \in \mathbb{Z}^+$,

$$\begin{aligned}
\epsilon_p(x) &= \partial_x(\eta_p(x)) + (p+1)\pi^{(0)}_{f(x)}\big(\eta_{p+1}(x)\partial_x(f(x))\big) + p\,\pi^{(1)}_{f(x)}\big(\eta_p(x)\partial_x(f(x))\big) \\
&= \sum_{q=0}^{p} \partial_x \circ \pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big) + \sum_{q=0}^{p-1} \partial_x \circ \pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{p-q-1}(x)\big) \\
&\qquad + (p+1)\sum_{q=0}^{p+1} \pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q+1}(x)\big)\partial_x(f(x))\Big)
\end{aligned}$$

$$+ (p+1)\sum_{q=0}^{p} \pi^{(0)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big)\partial_x(f(x))\Big)$$
$$+ p\sum_{q=0}^{p} \pi^{(1)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big)\partial_x(f(x))\Big)$$
$$+ p\sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{p-q-1}(x)\big)\partial_x(f(x))\Big).$$

Applying Lemma C.1.35, and writing the prefactors to match the indices on the $\{\nu_j(x)\}$,

$$\cdots = \sum_{q=0}^{p}\pi^{(0)}_{f(x)}\circ\partial_x\big(\mu_q(x)\nu_{p-q}(x)\big) + \sum_{q=0}^{p-1}\pi^{(1)}_{f(x)}\circ\partial_x\big(\mu_q(x)\nu_{p-q-1}(x)\big)$$
$$+ \sum_{q=0}^{p+1}(p-q+1+q)\pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q+1}(x)\big)\partial_x(f(x))\Big)$$
$$+ \sum_{q=0}^{p}(p-q+q)\pi^{(0)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big)\partial_x(f(x))\Big)$$
$$+ \sum_{q=0}^{p}(p-q+q)\pi^{(1)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_q(x)\nu_{p-q}(x)\big)\partial_x(f(x))\Big)$$
$$+ \sum_{q=0}^{p-1}(p-q-1+q)\pi^{(1)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_q(x)\nu_{p-q-1}(x)\big)\partial_x(f(x))\Big).$$

Now, expanding all but the first and second terms, and applying Lemmas C.1.21, C.1.23, and C.1.26 to them,

$$\cdots = \sum_{q=0}^{p}\pi^{(0)}_{f(x)}\circ\partial_x\big(\mu_q(x)\nu_{p-q}(x)\big) + \sum_{q=0}^{p-1}\pi^{(1)}_{f(x)}\circ\partial_x\big(\mu_q(x)\nu_{p-q-1}(x)\big)$$
$$+ \sum_{q=0}^{p}(p-q+1)\pi^{(0)}_{f(x)}\Big(\mu_q(x)\pi^{(0)}_{f(x)}\big(\nu_{p-q+1}(x)\partial_x(f(x))\big)\Big)$$
$$+ \sum_{q=1}^{p+1} q\,\pi^{(0)}_{f(x)}\Big(\pi^{(0)}_{f(x)}\big(\mu_q(x)\partial_x(f(x))\big)\nu_{p-q+1}(x)\Big)$$
$$+ \sum_{q=0}^{p-1}(p-q-1)\pi^{(1)}_{f(x)}\Big(\mu_q(x)\pi^{(1)}_{f(x)}\big(\nu_{p-q-1}(x)\partial_x(f(x))\big)\Big)$$
$$+ \sum_{q=0}^{p-1} q\,\pi^{(1)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_q(x)\partial_x(f(x))\big)\nu_{p-q-1}(x)\Big)$$
$$+ \sum_{q=0}^{p}(p-q)\pi^{(0)}_{f(x)}\Big(\mu_q(x)\pi^{(1)}_{f(x)}\big(\nu_{p-q}(x)\partial_x(f(x))\big)\Big)$$
$$+ \sum_{q=0}^{p}(p-q)\pi^{(1)}_{f(x)}\Big(\mu_q(x)\pi^{(0)}_{f(x)}\big(\nu_{p-q}(x)\partial_x(f(x))\big)\Big)$$
$$+ \sum_{q=0}^{p} q\,\pi^{(0)}_{f(x)}\Big(\pi^{(1)}_{f(x)}\big(\mu_q(x)\partial_x(f(x))\big)\nu_{p-q}(x)\Big)$$

$$+ \sum_{q=1}^{p} q\, \pi^{(1)}_{f(x)} \Big(\pi^{(0)}_{f(x)} \big(\mu_q(x) \partial_x(f(x)) \big) \nu_{p-q}(x) \Big).$$

Then reindexing and gathering,

$$\begin{aligned}
\cdots = {} & \sum_{q=0}^{p} \pi^{(0)}_{f(x)} \circ \partial_x \big(\mu_q(x) \nu_{p-q}(x) \big) + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)} \circ \partial_x \big(\mu_q(x) \nu_{p-q-1}(x) \big) \\
& + \sum_{q=0}^{p} (p-q+1) \pi^{(0)}_{f(x)} \Big(\mu_q(x) \pi^{(0)}_{f(x)} \big(\nu_{p-q+1}(x) \partial_x(f(x)) \big) \Big) \\
& + \sum_{r=0}^{p} (r+1)\, \pi^{(0)}_{f(x)} \Big(\pi^{(0)}_{f(x)} \big(\mu_{r+1}(x) \partial_x(f(x)) \big) \nu_{p-r}(x) \Big) \\
& + \sum_{q=0}^{p-1} (p-q-1) \pi^{(1)}_{f(x)} \Big(\mu_q(x) \pi^{(1)}_{f(x)} \big(\nu_{p-q-1}(x) \partial_x(f(x)) \big) \Big) \\
& + \sum_{q=0}^{p-1} q\, \pi^{(1)}_{f(x)} \Big(\pi^{(1)}_{f(x)} \big(\mu_q(x) \partial_x(f(x)) \big) \nu_{p-q-1}(x) \Big) \\
& + \sum_{q=0}^{p} (p-q) \pi^{(0)}_{f(x)} \Big(\mu_q(x) \pi^{(1)}_{f(x)} \big(\nu_{p-q}(x) \partial_x(f(x)) \big) \Big) \\
& + \sum_{q=0}^{p} (p-q) \pi^{(1)}_{f(x)} \Big(\mu_q(x) \pi^{(0)}_{f(x)} \big(\nu_{p-q}(x) \partial_x(f(x)) \big) \Big) \\
& + \sum_{q=0}^{p} q\, \pi^{(0)}_{f(x)} \Big(\pi^{(1)}_{f(x)} \big(\mu_q(x) \partial_x(f(x)) \big) \nu_{p-q}(x) \Big) \\
& + \sum_{r=0}^{p-1} (r+1)\, \pi^{(1)}_{f(x)} \Big(\pi^{(0)}_{f(x)} \big(\mu_{r+1}(x) \partial_x(f(x)) \big) \nu_{p-r-1}(x) \Big).
\end{aligned}$$

Thus,

$$\begin{aligned}
\cdots = {} & \sum_{q=0}^{p} \pi^{(0)}_{f(x)} \Bigg(\Big(\partial_x(\mu_q(x)) + q \pi^{(1)}_{f(x)} \big(\mu_q(x) \partial_x(f(x)) \big) \\
& \qquad + (q+1) \pi^{(0)}_{f(x)} \big(\mu_{q+1}(x) \partial_x(f(x)) \big) \Big) \nu_{p-q}(x) \Bigg) \\
& + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)} \Bigg(\Big(\partial_x(\mu_q(x)) + q \pi^{(1)}_{f(x)} \big(\mu_q(x) \partial_x(f(x)) \big) \\
& \qquad + (q+1) \pi^{(0)}_{f(x)} \big(\mu_{q+1}(x) \partial_x(f(x)) \big) \Big) \nu_{p-q-1}(x) \Bigg) \\
& + \sum_{q=0}^{p} \pi^{(0)}_{f(x)} \Bigg(\mu_q(x) \Big(\partial_x(\nu_{p-q}(x)) + (p-q) \pi^{(1)}_{f(x)} \big(\nu_{p-q}(x) \partial_x(f(x)) \big) \\
& \qquad + (p-q+1) \pi^{(0)}_{f(x)} \big(\nu_{p-q+1}(x) \partial_x(f(x)) \big) \Big) \Bigg) \\
& + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)} \Bigg(\mu_q(x) \Big(\partial_x(\nu_{p-q-1}(x)) + (p-q-1) \pi^{(1)}_{f(x)} \big(\nu_{p-q-1}(x) \partial_x(f(x)) \big) \\
& \qquad + (p-q) \pi^{(0)}_{f(x)} \big(\nu_{p-q}(x) \partial_x(f(x)) \big) \Big) \Bigg),
\end{aligned}$$

and so,

$$\begin{aligned}\dots &= \sum_{q=0}^{p} \pi^{(0)}_{f(x)}\big(\theta_q(x)\nu_{p-q}(x)\big) + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\big(\theta_q(x)\nu_{p-q-1}(x)\big) \\ &\quad + \sum_{q=0}^{p} \pi^{(0)}_{f(x)}\big(\mu_q(x)\phi_{p-q}(x)\big) + \sum_{q=0}^{p-1} \pi^{(1)}_{f(x)}\big(\mu_q(x)\phi_{p-q-1}(x)\big) \\ &= \rho_p(x) + \sigma_p(x).\end{aligned}$$

□

Lemma C.1.38. *For all finite-order series $g(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,*

$$D_x(g(x)) = \partial_x(g(x)). \tag{C.1.38}$$

Proof. Writing,

$$g(x) = \sum_{j=0}^{m} \mu_j(x) f(x)^j,$$

we see that, $\forall n \in \mathbb{Z}^+$,

$$\begin{aligned}\theta_{m+n}(x) &= \partial_x(\mu_{m+n}(x)) + (m+n+1)\pi^{(0)}_{f(x)}\big(\mu_{m+n+1}(x)\partial_x(f(x))\big) \\ &\quad + (m+n)\,\pi^{(1)}_{f(x)}\big(\mu_{m+n}(x)\partial_x(f(x))\big) \\ &= 0,\end{aligned}$$

and,

$$\begin{aligned}\theta_m(x) &= \partial_x(\mu_m(x)) + (m+1)\pi^{(0)}_{f(x)}\big(\mu_{m+1}(x)\partial_x(f(x))\big) + m\,\pi^{(1)}_{f(x)}\big(\mu_m(x)\partial_x(f(x))\big) \\ &= \partial_x(\mu_m(x)) + m\,\pi^{(1)}_{f(x)}\big(\mu_m(x)\partial_x(f(x))\big).\end{aligned}$$

Thus,

$$\begin{aligned}D_x(g(x)) &= \sum_{p=0}^{m} \theta_p(x) f(x)^p \\ &= \sum_{p=0}^{m-1}\Bigg(\Big(\partial_x(\mu_p(x)) + (p+1)\pi^{(0)}_{f(x)}\big(\mu_{p+1}(x)\partial_x(f(x))\big) \\ &\qquad + p\,\pi^{(1)}_{f(x)}\big(\mu_p(x)\partial_x(f(x))\big)\Big) f(x)^p\Bigg) \\ &\qquad + \Big(\partial_x(\mu_m(x)) + m\,\pi^{(1)}_{f(x)}\big(\mu_m(x)\partial_x(f(x))\big)\Big) f(x)^m. \\ &= \sum_{p=0}^{m} \partial_x(\mu_p(x)) f(x)^p + \sum_{p=1}^{m} p\,\pi^{(1)}_{f(x)}\big(\mu_p(x)\partial_x(f(x))\big) f(x)^p \\ &\qquad + \sum_{p=0}^{m-1}\Big((p+1)\pi^{(0)}_{f(x)}\big(\mu_{p+1}(x)\partial_x(f(x))\big)\Big) f(x)^p\end{aligned}$$

$$= \sum_{p=0}^{m}\partial_x(\mu_p(x))f(x)^p + \sum_{r=0}^{m-1}(r+1)\,\pi^{(1)}_{f(x)}\big(\mu_{r+1}(x)\partial_x(f(x))\big)f(x)^{r+1}$$
$$+ \sum_{p=0}^{m-1}(p+1)\pi^{(0)}_{f(x)}\big(\mu_{p+1}(x)\partial_x(f(x))\big)f(x)^p,$$

so utilising Lemma C.1.19 and reindexing,

$$\begin{aligned}
\ldots &= \sum_{p=0}^{m}\partial_x(\mu_p(x))f(x)^p + \sum_{p=0}^{m-1}(p+1)\mu_{p+1}(x)\partial_x(f(x))f(x)^p \\
&= \sum_{p=0}^{m}\partial_x(\mu_p(x))f(x)^p + \sum_{r=0}^{m}r\mu_r(x)\partial_x(f(x))f(x)^{r-1} \\
&= \sum_{p=0}^{m}\partial_x\Big(\mu_p(x)f(x)^p\Big) \\
&= \partial_x(g(x)).
\end{aligned}$$

□

C.1.4 Calculating the Coefficients of a Formal Irreducible Power Series

Lemma C.1.39. *Consider* $g(x) \in S(\mathbb{F}[x], f(x), M_{f(x)})$,

$$g(x) = \sum_{j=0}^{\infty}\mu_j(x)f(x)^j. \tag{C.1.39}$$

Then, $\forall k \in \mathbb{N}$,

$$\mu_k(x) = \begin{cases} \tau_{f(x)}\big(g(x)\big) & k=0 \\ \dfrac{1}{k!}\tau_{f(x)}\circ D^{\circ k}_{f(x)}\big(g(x)\big) - \displaystyle\sum_{j=0}^{k-1}\frac{1}{(k-j)!}\tau_{f(x)}\circ D^{\circ(k-j)}_{f(x)}\big(\mu_j(x)\big) & k \in \mathbb{Z}^+. \end{cases} \tag{C.1.40}$$

Proof. By Lemma 3.4.21, we may write,

$$g(x) = \sum_{j=0}^{k}\mu_j(x)f(x)^j + f(x)^{k+1}\sum_{j=0}^{\infty}\mu_{j+k+1}(x)f(x)^j,$$

and so, by Lemmas 2.3.56 and 3.4.51,

$$\frac{1}{k!}D^{\circ k}_{f(x)}\big(g(x)\big) = \sum_{j=0}^{k}\sum_{p=0}^{k}\frac{1}{k!}\binom{k}{p}D^{\circ p}_{f(x)}\big(\mu_j(x)\big)D^{\circ(k-p)}_{f(x)}\big(f(x)^j\big)$$
$$+ \sum_{p=0}^{k}D^{\circ p}_{f(x)}\big(f(x)^{k+1}\big)D^{\circ(k-p)}_{f(x)}\big(\sum_{j=0}^{\infty}\mu_{j+k+1}(x)f(x)^j\big)$$

$$= \sum_{j=0}^{k} \sum_{p=k-j}^{k} \frac{1}{p!(k-p)!} D_{f(x)}^{\circ p}\big(\mu_j(x)\big) \frac{j!}{(j-(k-p))!} f(x)^{j-(k-p)}$$

$$+ f(x) \sum_{p=0}^{k} \frac{(k+1)!}{(k+1-p)!} f(x)^{k-p} D_{f(x)}^{\circ(k-p)}\big(\sum_{j=0}^{\infty} \mu_{j+k+1}(x) f(x)^j\big)$$

$$= \frac{1}{k!} D_{f(x)}^{\circ k}\big(\mu_0(x)\big) + \sum_{j=1}^{k} \frac{1}{(k-j)!} D_{f(x)}^{\circ(k-j)}\big(\mu_j(x)\big)$$

$$+ \sum_{j=1}^{k} \sum_{p=k-j+1}^{k} \frac{1}{p!(k-p)!} D_{f(x)}^{\circ p}\big(\mu_j(x)\big) \frac{j!}{(j-(k-p))!} f(x)^{j-(k-p)}$$

$$+ f(x) \sum_{p=0}^{k} \frac{(k+1)!}{(k+1-p)!} f(x)^{k-p} D_{f(x)}^{\circ(k-p)}\big(\sum_{j=0}^{\infty} \mu_{j+k+1}(x) f(x)^j\big)$$

$$= \sum_{j=0}^{k} \frac{1}{(k-j)!} D_{f(x)}^{\circ(k-j)}\big(\mu_j(x)\big)$$

$$+ f(x) \sum_{m=0}^{k-1} \sum_{q=0}^{m} \Bigg(\frac{(m+1)!}{(q+k-m)!} \frac{1}{(m-q)!(q+1)!} f(x)^q D_{f(x)}^{\circ(q+k-m)}\big(\mu_{m+1}(x)\big) \Bigg)$$

$$+ f(x) \sum_{p=0}^{k} \frac{(k+1)!}{(k+1-p)!} f(x)^{k-p} D_{f(x)}^{\circ(k-p)}\big(\sum_{j=0}^{\infty} \mu_{j+k+1}(x) f(x)^j\big).$$

Therefore, by Lemma 3.4.41,

$$\frac{1}{k!} \tau_{f(x)} \circ D_{f(x)}^{\circ k}\big(g(x)\big) = \sum_{j=0}^{k} \frac{1}{(k-j)!} \tau_{f(x)} \circ D_{f(x)}^{\circ(k-j)}\big(\mu_j(x)\big),$$

and so,

$$\mu_k(x) = \frac{1}{k!} \tau_{f(x)} \circ D_{f(x)}^{\circ k}\big(g(x)\big) - \sum_{j=0}^{k-1} \frac{1}{(k-j)!} \tau_{f(x)} \circ D_{f(x)}^{\circ(k-j)}\big(\mu_j(x)\big).$$

□

C.2 Proof for Quadratic Irreducible Power Series

In this section: we will denote,

$$p_k(x) = f_k(x)^{d_k}$$
$$f_k(x) = (x-b)^2 + \lambda^2, \tag{C.2.1}$$

with $d_k \in \mathbb{Z}^+$, $b, \lambda \in \mathbb{R}$, $\lambda \neq 0$; $q_k(x)$ is defined in the usual way; $q_k(x)^{-1} \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)})$ with,

$$q_k(x)^{-1} = \sum_{n=0}^{\infty} \alpha_n(x)\big((x-b)^2 + \lambda^2\big)^n; \tag{C.2.2}$$

and, $\forall n \in \mathbb{N}$,

$$\tau_{f_k(x)}(q_k(x)) = \kappa_0 + \kappa_1(x-b) \tag{C.2.3}$$

$$\alpha_n(x) = \gamma_n + \delta_n(x-b). \tag{C.2.4}$$

Lemma C.2.1. *For all $g(x) \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)})$ with a two-sided inverse, and writing,*

$$\tau_{f_k(x)}(g(x)) = \alpha + \beta(x-b), \tag{C.2.5}$$

then,

$$\tau_{f_k(x)}(g(x)^{-1}) = \frac{\alpha - \beta(x-b)}{\alpha^2 + \lambda^2\beta^2}. \tag{C.2.6}$$

Proof. Noting $\tau_{f_k(x)}\big((x-b)^2 + \lambda^2\big) = 0$, and writing,

$$\tau_{f_k(x)}\big(g(x)^{-1}\big) = \gamma + \delta(x-b),$$

$$\begin{aligned}
1 &= \tau_{f_k(x)}\big(g(x)^{-1}g(x)\big) \\
&= \tau_{f_k(x)}\Big(\tau_{f_k(x)}\big(g(x)^{-1}\big)\tau_{f_k(x)}(g(x))\Big) \\
&= \tau_{f_k(x)}\Big((\gamma + \delta(x-b))(\alpha + \beta(x-b))\Big) \\
&= \alpha\gamma + \big(\alpha\delta + \beta\gamma\big)(x-b) + \beta\delta\tau_{f_k(x)}\big((x-b)^2\big) \\
&= \big(\alpha\gamma - \lambda^2\beta\delta\big) + \big(\alpha\delta + \beta\gamma\big)(x-b),
\end{aligned}$$

which may be easily solved. □

Lemma C.2.2.

$$\tau_{f_k(x)}\big(q_k(x)^{-1}\big) = \frac{\kappa_0 - \kappa_1(x-b)}{\kappa_0^2 + \lambda^2\kappa_1^2}. \tag{C.2.7}$$

Proof. Follows from Lemma C.2.1. □

Lemma C.2.3.

$$\tau_{f_k(x)}\Big(\big(D_x(f_k(x))\big)^{-1}\Big) = -\frac{1}{2\lambda^2}(x-b). \tag{C.2.8}$$

Proof. Follows from $\partial_x\big((x-b)^2 + \lambda^2\big) = 2(x-b)$ and Lemma C.2.1. □

Lemma C.2.4. *For all $n \in \mathbb{N}$, $\forall m \in \mathbb{Z}^+$,*

$$D^{\circ m}_{f_k(x)}\big(\alpha_n(x)\big) = (-1)^{m-1}\delta_n\frac{(2m-2)!}{(m-1)!}\big(D_x(f_k(x))\big)^{-(2m-1)}. \tag{C.2.9}$$

Proof. We proceed by induction. When $m = 1$, the expression coincides with the result from direct computation,

$$D_{f_k(x)}(\alpha_n(x)) = \big(D_x(f_k(x))\big)^{-1} D_x\big(\gamma_n + \delta_n(x-b)\big) = \delta_n \big(D_x(f_k(x))\big)^{-1}.$$

Assuming the expression holds for $m = l$, and noting that,

$$D_x^{\circ 2}(f_k(x)) = 2,$$

we find,

$$\begin{aligned}
D_{f_k(x)}^{\circ l+1}\big(\alpha_n(x)\big) &= D_{f_k(x)}\Big((-1)^{l-1}\delta_n \frac{(2l-2)!}{(l-1)!}\big(D_x(f_k(x))\big)^{-(2l-1)}\Big) \\
&= (-1)^{l-1}\delta_n \frac{(2l-2)!}{(l-1)!}\big(D_x(f_k(x))\big)^{-1} D_x\Big(\big(D_x(f_k(x))\big)^{-(2l-1)}\Big) \\
&= (-1)^{l-1}\delta_n \frac{(2l-2)!}{(l-1)!}(-1)(2l-1) D_x^{\circ 2}(f_k(x))\big(D_x(f_k(x))\big)^{-(2l+1)} \\
&= (-1)^{l}\delta_n \frac{(2l)!}{l!}\big(D_x(f_k(x))\big)^{-(2l+1)} \\
&= (-1)^{(l+1)-1}\delta_n \frac{(2(l+1)-2)!}{((l+1)-1)!}\big(D_x(f_k(x))\big)^{-(2(l+1)-1)}.
\end{aligned}$$

□

Lemma C.2.5. *For all $n \in \mathbb{N}$,*

$$\tau_{f_k(x)}\Big(\big(D_x(f_k(x))\big)^{-n}\Big) = \begin{cases} \dfrac{(-1)^m}{2^{2m}\lambda^{2m}} & n = 2m,\ m \in \mathbb{N} \\[2ex] \dfrac{(-1)^{m+1}}{2^{2m+1}\lambda^{2m+2}}(x-b) & n = 2m+1,\ m \in \mathbb{N}. \end{cases} \tag{C.2.10}$$

Proof. By Lemma 3.4.43,

$$\begin{aligned}
&\tau_{f_k(x)}\Big(\big(D_x(f_k(x))\big)^{-n}\Big) \\
&\quad = \tau_{f_k(x)}\Big(\Big(-\frac{1}{2\lambda^2}(x-b)\Big)^n\Big) \\
&\quad = \begin{cases} \dfrac{(-1)^{2m}}{2^{2m}\lambda^{4m}}\tau_{f_k(x)}\Big(\big((x-b)^2\big)^m\Big) & n = 2m,\ m \in \mathbb{N} \\[2ex] \dfrac{(-1)^{2m+1}}{2^{2m+1}\lambda^{4m+2}}\tau_{f_k(x)}\Big(\big((x-b)^2\big)^m(x-b)\Big) & n = 2m+1,\ m \in \mathbb{N}. \end{cases} \\
&\quad = \begin{cases} \dfrac{1}{2^{2m}\lambda^{4m}}(-1)^m\lambda^{2m} & n = 2m,\ m \in \mathbb{N} \\[2ex] \dfrac{(-1)}{2^{2m+1}\lambda^{4m+2}}(-1)^m\lambda^{2m}(x-b) & n = 2m+1,\ m \in \mathbb{N}. \end{cases}
\end{aligned}$$

□

Lemma C.2.6. *For all $n \in \mathbb{N}$, $m \in \mathbb{Z}^+$,*

$$\tau_{f_k(x)} \circ D^{\circ m}_{f_k(x)}\big(\alpha_n(x)\big) = -\frac{\delta_n}{2^{2m-1}\lambda^{2m}} \frac{(2m-2)!}{(m-1)!}(x-b). \tag{C.2.11}$$

Proof. Follows from Lemmas C.2.4 and C.2.5 noting $2m-1 = 2(m-1)+1$. □

Lemma C.2.7. *Consider an identity resolution in which,*

$$\begin{aligned} p_k(x) &= f_k(x)^{d_k} \\ f_k(x) &= (x-b)^2 + \lambda^2, \end{aligned} \tag{C.2.12}$$

where $d_k \in \mathbb{Z}^+$, $b, \lambda \in \mathbb{R}$, $\lambda \neq 0$. Then, with $q_k(x)$ defined in the usual way, and denoting $q_k(x)^{-1} \in S(\mathbb{F}[x], f_k(x), M_{f_k(x)})$ by,

$$q_k(x)^{-1} = \sum_{n=0}^{\infty} \alpha_n(x)\big((x-b)^2 + \lambda^2\big)^n, \tag{C.2.13}$$

and, $\forall n \in \mathbb{N}$,

$$\tau_{f_k(x)}(q_k(x)) = \kappa_0 + \kappa_1(x-b) \tag{C.2.14a}$$

$$\alpha_n(x) = \gamma_n + \delta_n(x-b), \tag{C.2.14b}$$

we have, $\forall n \in \mathbb{N}$,

$$\alpha_n(x) = \begin{cases} \dfrac{\kappa_0 - \kappa_1(x-b)}{\kappa_0^2 + \lambda^2\kappa_1^2} & n = 0 \\ \dfrac{1}{n!}\tau_{f_k(x)} \circ D^{\circ n}_{f_k(x)}\big(q_k(x)^{-1}\big) & n \in \mathbb{Z}^+. \\ \qquad + (x-b)\displaystyle\sum_{m=1}^{n} \frac{1}{m}\binom{2m-2}{m-1}\frac{\delta_{n-m}}{2^{2m-1}\lambda^{2m}} & \end{cases} \tag{C.2.15}$$

Proof. The $n = 0$ case follows from Lemma C.2.2. The remaining case follows from Lemma C.2.6 after reindexing $m = n - j$ in (3.4.50). □

Appendix D

Arbitrary Spin Algebras

D.1 Proof of the Promotion and Refinement of an E-Orthogonal Decomposition

Definition D.1.1 ($\mathcal{L}$)**.** Let us abuse notation to define the linear maps, $\forall k \in \mathbb{N}$, using the isomorphism $\mathfrak{so}(3,\mathbb{R})^{\otimes k+1} \cong \mathfrak{so}(3,\mathbb{R}) \otimes \mathfrak{so}(3,\mathbb{R})^{\otimes k}$,

$$\begin{aligned} \mathcal{L} &: \mathrm{End}(\mathfrak{so}(3,\mathbb{R})^{\otimes k}) \to \mathrm{End}(\mathfrak{so}(3,\mathbb{R})^{\otimes k+1}) \\ \mathcal{L}(A) &:= \mathrm{id}_{\mathfrak{so}(3,\mathbb{R})} \otimes A = \big(v \otimes X \mapsto v \otimes A(X)\big). \end{aligned} \tag{D.1.1}$$

Lemma D.1.2. *For all $k \in \mathbb{N}$,*

$$\mathrm{id}_{\mathfrak{so}(3,\mathbb{R})^{\otimes k+1}} = \mathcal{L}(\mathrm{id}_{\mathfrak{so}(3,\mathbb{R})^{\otimes k}}). \tag{D.1.2}$$

Proof. Clear from the definition of $\mathcal{L}$. □

As it is slightly less straightforward than for $U(\mathfrak{so}(3,\mathbb{R}))$, let us be precise in our definition of the adjoint action $\mathrm{ad}(U(\mathfrak{so}(3,\mathbb{R})))$ on $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$,

Definition D.1.3 (Adjoint action on $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$)**.** For $k = 0$,

$$\begin{aligned} \mathrm{ad}^{(0)} &: U(\mathfrak{so}(3,\mathbb{R})) \to \mathrm{End}(\mathbb{R}) \\ \mathrm{ad}^{(0)}(u) &:= \alpha \mapsto \begin{cases} u\alpha & u \in \mathbb{R} \\ 0 & u \notin \mathbb{R}. \end{cases} \end{aligned} \tag{D.1.3}$$

For all $k \geq 1$, we use the isomorphism $\mathfrak{so}(3,\mathbb{R})^{\otimes k} \cong \mathfrak{so}(3,\mathbb{R}) \otimes \mathfrak{so}(3,\mathbb{R})^{\otimes k-1}$ to define,

$$\mathrm{ad}^{(k)} : U(\mathfrak{so}(3,\mathbb{R})) \to \mathrm{End}(\mathfrak{so}(3,\mathbb{R})^{\otimes k})$$
$$\mathrm{ad}^{(k)}(u) \coloneqq v \otimes X \mapsto \begin{cases} u(v \otimes X) & u \in \mathbb{R} \\ (u \times v) \otimes X + v \otimes \mathrm{ad}^{(k-1)}(u)(X) & u \in \mathfrak{so}(3,\mathbb{R}) \\ \mathrm{ad}(a) \circ \mathrm{ad}(b)(v \otimes X) & u = a \otimes b, \end{cases} \tag{D.1.4}$$

All definitions are extended by linearity in the first argument.

Remark. The properties of $\mathrm{ad}^{(k)}$ are identical to ad defined on $U(\mathfrak{so}(3,\mathbb{R}))$. We will abuse notation by dropping the superscript when the tensor order is clear.

Lemma D.1.4. *Consider $A \in \mathrm{End}(\mathfrak{so}(3,\mathbb{R})^{\otimes k})$ such that $\mathrm{Im}(A)$ is closed under the action of $\mathrm{ad}(U(\mathfrak{so}(3,\mathbb{R})))$. Then, $\mathrm{Im}(\mathcal{L}(A))$ is closed under the action of $\mathrm{ad}(U(\mathfrak{so}(3,\mathbb{R})))$.*

Proof. As in the main text, we need only show that $\mathrm{Im}(\mathcal{L}(A))$ is closed under the action of $\mathrm{ad}(\mathfrak{so}(3,\mathbb{R}))$. Using the isomorphism $\mathfrak{so}(3,\mathbb{R})^{\otimes k+1} \cong \mathfrak{so}(3,\mathbb{R}) \otimes \mathfrak{so}(3,\mathbb{R})^{\otimes k}$, $\forall u \in \mathfrak{so}(3,\mathbb{R})$,

$$\begin{aligned} \mathrm{ad}(u)(v \otimes A(X)) &= (u \times v) \otimes A(X) + v \otimes \mathrm{ad}(u) \circ A(X) \\ &= (u \times v) \otimes A(X) + v \otimes A(Y) \\ &\in \mathrm{Im}(\mathcal{L}(A)). \end{aligned}$$

□

Lemma D.1.5. *Suppose we have an E-orthogonal decomposition for $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$ derived from Theorem 3.3.36, with resolution of the identity,*

$$\mathrm{id}_{\mathfrak{so}(3,\mathbb{R})^{\otimes k}} = \sum_{j=1}^{n} \Pi_j(E), \tag{D.1.5}$$

Then $\forall k \in \mathbb{N}$, we may extend this to an E-orthogonal decomposition on $\mathfrak{so}(3,\mathbb{R})^{\otimes k+1}$ with resolution of the identity,

$$\mathrm{id}_{\mathfrak{so}(3,\mathbb{R})^{\otimes k+1}} = \sum_{j=1}^{n} \mathcal{L}(\Pi_j(E)). \tag{D.1.6}$$

Proof. By Lemma D.1.2, (D.1.6) follows by applying $\mathcal{L}$ to (D.1.5). We must now verify that it constitutes an E-orthogonal decomposition. First, note that $\forall p \neq q$,

$$\mathcal{L}(\Pi_p(E)) \circ \mathcal{L}(\Pi_p(E)) = \mathrm{id}_{\mathfrak{so}(3,\mathbb{R})} \otimes (\Pi_p(E) \circ \Pi_p(E)) = \mathcal{L}(\Pi_p(E))$$

$$\mathcal{L}(\Pi_p(E))\circ\mathcal{L}(\Pi_q(E)) = \mathrm{id}_{\mathfrak{so}(3,\mathbb{R})}\otimes(\Pi_p(E)\circ\Pi_q(E)) = 0,$$

and so by the arguments made in the proof of Theorem 3.3.36,

$$\mathfrak{so}(3,\mathbb{R})^{\otimes k+1} \cong \bigoplus_{j=1}^{n} \mathrm{Im}(\mathcal{L}(\Pi_j(E))).$$

By Lemma D.1.4, each $\mathrm{Im}(\mathcal{L}(\Pi_p(E)))$ is closed under E, so we are done. □

So we have shown that we may promote any E-orthogonal decomposition of $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$ to one of $\mathfrak{so}(3,\mathbb{R})^{\otimes k+1}$. Let us now show this decomposition may be refined.

Lemma D.1.6. *Consider* $A \in \mathrm{End}(\mathfrak{so}(3,\mathbb{R})^{\otimes k})$. *If* $\left[\mathrm{ad}^{(k)}(B), A\right] = 0$ *for some* $B \in U(\mathfrak{so}(3,\mathbb{R}))$, *then* $\left[\mathrm{ad}^{(k+1)}(B), \mathcal{L}(A)\right] = 0$.

Proof. We need only show this is the case for $\mathrm{ad}^{(k+1)}(v)$, $\forall v \in \mathfrak{so}(3,\mathbb{R})$. Thus,

$$\begin{aligned}
\mathrm{ad}^{(k+1)}(v)\circ\mathcal{L}(A) &= \mathrm{ad}^{(k+1)}(v)\circ(\mathrm{id}_{\mathfrak{so}(3,\mathbb{R})}\otimes A)\\
&= \mathrm{ad}^{(1)}(v)\otimes A + \mathrm{id}_{\mathfrak{so}(3,\mathbb{R})}\otimes(\mathrm{ad}^{(k)}(v)\circ A)\\
&= \mathrm{ad}^{(1)}(v)\otimes A + \mathrm{id}_{\mathfrak{so}(3,\mathbb{R})}\otimes(A\circ\mathrm{ad}^{(k)}(v))\\
&= (\mathrm{id}_{\mathfrak{so}(3,\mathbb{R})}\otimes A)\circ(\mathrm{ad}^{(1)}(v)\otimes\mathrm{id}_{\mathfrak{so}(3,\mathbb{R})^{\otimes k}} + \mathrm{id}_{\mathfrak{so}(3,\mathbb{R})}\otimes\mathrm{ad}^{(k)}(v))\\
&= \mathcal{L}(A)\circ\mathrm{ad}^{(k+1)}(v)\circ(\mathrm{id}_{\mathfrak{so}(3,\mathbb{R})}\otimes\mathrm{id}_{\mathfrak{so}(3,\mathbb{R})^{\otimes k}})\\
&= \mathcal{L}(A)\circ\mathrm{ad}^{(k+1)}(v).
\end{aligned}$$

□

Theorem D.1.7. *The E-orthogonal decomposition of* $\mathfrak{so}(3,\mathbb{R})^{\otimes k+1}$ *from Lemma D.1.5,*

$$\mathfrak{so}(3,\mathbb{R})^{\otimes k+1} \cong \bigoplus_{j=1}^{n} \mathrm{Im}(\mathcal{L}(\Pi_j(E))),$$

may be refined to an E-orthogonal decomposition,

$$\mathfrak{so}(3,\mathbb{R})^{\otimes k+1} \cong \bigoplus_{j=1}^{n}\left(\bigoplus_{l_j=1}^{m_j} \mathrm{Im}(P_{(j,l_j)}(E))\right),$$

where $m_j \in \mathbb{Z}^+$,

$$\mathrm{Im}(\mathcal{L}(\Pi_j(E))) \cong \bigoplus_{l_j=1}^{m_j} \mathrm{Im}(P_{(j,l_j)}(E)),$$

is an E-orthogonal decomposition for $\mathrm{Im}(\mathcal{L}(\Pi_j(E)))$, *and* $\forall (p, r_p) \neq (q, s_q)$ *the* $\{P_{(j,l_j)}\}$ *satisfy,*

$$\mathrm{id}_{\mathfrak{so}(3,\mathbb{R})^{\otimes k+1}} = \sum_{j=1}^{n}\sum_{l_j=1}^{m_j} P_{(j,l_j)}(E) \tag{D.1.7a}$$

$$P_{(p,r_p)}(E)\circ P_{(p,r_p)}(E) = P_{(p,r_p)}(E) \tag{D.1.7b}$$

$$P_{(p,r_p)}(E)\circ P_{(q,s_q)}(E) = P_{(q,s_q)}(E)\circ P_{(p,r_p)}(E) = 0. \tag{D.1.7c}$$

Proof. As $\mathrm{Im}(\mathcal{L}(\Pi_j(E)))$ is finite-dimensional there must exist an E-orthogonal decomposition on it yielding a resolution of $\mathcal{L}(\Pi_j(E))$,

$$\mathcal{L}(\Pi_j(E)) = \sum_{l_j=1}^{m_j} P_{(j,l_j)}(E).$$

The positive integers m_j and the forms of the $\{P_{(j,l_j)}\}$ depend on the form of the minimal polynomial for E restricted to $\mathrm{Im}(\mathcal{L}(\Pi_j(E)))$. Given such E-orthogonal decompositions $\forall j$, it suffices for us to verify the properties (D.1.7a)-(D.1.7c). The identity resolution (D.1.7a) is obvious, and so by Lemma 3.3.21, (D.1.7b) follows from (D.1.7c).

To prove (D.1.7c) note that from Theorem 3.3.23,

$$P_{(j,l_j)}(E) = f_{(j,l_j)}(E)\circ\mathcal{L}(\Pi_j(E)),$$

for determined polynomial functions $f_{(j,l_j)}(E)$. When $p = q$, $\forall r_p \neq s_p$,

$$P_{(p,r_p)}(E)\circ P_{(p,s_p)}(E) = 0,$$

by definition. When $p \neq q$, we may use Lemma D.1.6, to show $\forall r_p, s_q$

$$\begin{aligned} P_{(p,r_p)}(E)\circ P_{(q,s_q)}(E) &= f_{(p,r_p)}(E)\circ\mathcal{L}(\Pi_p(E))\circ f_{(q,s_q)}(E)\circ\mathcal{L}(\Pi_q(E)) \\ &= f_{(p,r_p)}(E)\circ f_{(q,s_q)}(E)\circ\mathcal{L}(\Pi_p(E)\circ\Pi_q(E)) \\ &= 0. \end{aligned}$$

□

D.2 Proof of Left Action Identity

To facilitate this and other proofs we must first discuss some identities. Let us recall the definition of the right multiplication,

$$R(v) = A \mapsto Av,$$

and note this is not a Lie algebra action on $U(\mathfrak{so}(3,\mathbb{R}))$, since,

$$R(a)\circ R(b) - R(b)\circ R(a) = R(b\times a) \neq R(a\times b). \tag{D.2.1}$$

We may use it to describe the adjoint action of $v \in \mathfrak{so}(3,\mathbb{R})$,

$$\mathrm{ad}(v) = L(v) - R(v). \tag{D.2.2}$$

Noting $\forall A, B \in U(\mathfrak{so}(3,\mathbb{R}))$,

$$[L(A), R(B)] = 0, \tag{D.2.3}$$

we easily see the commutators,

$$[\mathrm{ad}(S_a), L(S_b)] = [L(S_a), L(S_b)] = L(S_a \times S_b) \tag{D.2.4}$$

$$[\mathrm{ad}(S_a), R(S_b)] = [-R(S_a), R(S_b)] = R(S_a \times S_b). \tag{D.2.5}$$

Next, let us examine E more closely. For central elements $z \in Z(U(\mathfrak{so}(3,\mathbb{R})))$,

$$[L(z), A] = 0 \tag{D.2.6}$$

$$L(z) = R(z). \tag{D.2.7}$$

Then,

$$E = \sum_{a=1}^{3} \mathrm{ad}(S_a) \circ \mathrm{ad}(S_a) = 2L(S^2) - 2\sum_{a=1}^{3} L(S_a) \circ R(S_a). \tag{D.2.8}$$

We may now proceed with the proof.

$$\begin{aligned}
[E, L(S_b)] &= \sum_{a=1}^{3} [\mathrm{ad}(S_a) \circ \mathrm{ad}(S_a), L(S_b)] \\
&= \sum_{a=1}^{3} \mathrm{ad}(S_a) \circ [\mathrm{ad}(S_a), L(S_b)] + [\mathrm{ad}(S_a), L(S_b)] \circ \mathrm{ad}(S_a) \\
&= \sum_{a=1}^{3} \mathrm{ad}(S_a) \circ L(S_a \times S_b) + L(S_a \times S_b) \circ \mathrm{ad}(S_a) \\
&= \sum_{a,c=1}^{3} \varepsilon_{abc} \big(\mathrm{ad}(S_a) \circ L(S_c) + L(S_c) \circ \mathrm{ad}(S_a)\big) \\
&= \sum_{a,c=1}^{3} \varepsilon_{abc} \Big(\big(L(S_a) - R(S_a)\big) \circ L(S_c) + L(S_c) \circ \big(L(S_a) - R(S_a)\big)\Big) \\
&= \sum_{a,c=1}^{3} \varepsilon_{abc} \big(L(S_a) \circ L(S_c) + L(S_c) \circ L(S_a)\big) - 2\sum_{a,c=1}^{3} \varepsilon_{abc}\, L(S_c) \circ R(S_a).
\end{aligned}$$

Since total contraction of a symmetric and antisymmetric object yields zero we find,

$$[E, L(S_b)] = -2\sum_{c,a=1}^{3} \varepsilon_{bca}\, L(S_c) \circ R(S_a) := -2F(S_b). \tag{D.2.9}$$

If we instead had calculated $[E, R(S_a)]$ we would discover that,

$$[E, L(S_a)] = [E, R(S_a)]. \tag{D.2.10}$$

Next let us consider,

$$[E, F(S_b)] = \sum_{c,a=1}^{3} \varepsilon_{bca}[E, L(S_c)\circ R(S_a)].$$

From (D.2.6) and (D.2.8) we see,

$$\begin{aligned}
[E, F(S_b)] &= -2 \sum_{d,c,a=1}^{3} \varepsilon_{bca}[L(S_d)\circ R(S_d), L(S_c)\circ R(S_a)] \\
&= -2 \sum_{d,c,a=1}^{3} \varepsilon_{bca}\big(L(S_d)\circ L(S_c)\circ [R(S_d), R(S_a)] \\
&\qquad\qquad + [L(S_d), L(S_c)]\circ R(S_a)\circ R(S_d)\big) \\
&= -2 \sum_{d,c,a=1}^{3} \varepsilon_{bca}\big(L(S_d)\circ L(S_c)\circ R(S_a\times S_d) + L(S_d\times S_c)\circ R(S_a)\circ R(S_d)\big) \\
&= -2 \sum_{e,d,c,a=1}^{3} \varepsilon_{bca}\varepsilon_{ade} L(S_d)\circ L(S_c)\circ R(S_e) \\
&\qquad\qquad - 2 \sum_{e,d,c,a=1}^{3} \varepsilon_{bca}\varepsilon_{dce} L(S_e)\circ R(S_a)\circ R(S_d).
\end{aligned}$$

Utilising,

$$\sum_{x=1}^{3} \varepsilon_{xpq}\varepsilon_{xrs} = \delta_{pr}\delta_{qs} - \delta_{ps}\delta_{qr}, \tag{D.2.11}$$

we find,

$$\begin{aligned}
[E, F(S_b)] &= -2 \sum_{e,d,c=1}^{3} (\delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd}) L(S_d)\circ L(S_c)\circ R(S_e) \\
&\qquad - 2 \sum_{e,d,a=1}^{3} (\delta_{bd}\delta_{ae} - \delta_{be}\delta_{ad}) L(S_e)\circ R(S_a)\circ R(S_d) \\
&= -2\sum_{c=1}^{3} L(S_b)\circ L(S_c)\circ R(S_c) + 2\sum_{c=1}^{3} L(S_c)\circ L(S_c)\circ R(S_b) \\
&\qquad - 2\sum_{a=1}^{3} L(S_a)\circ R(S_a)\circ R(S_b) + 2\sum_{a=1}^{3} L(S_b)\circ R(S_a)\circ R(S_a) \\
&= L(S_b)\circ(E - 2L(S^2)) + 2L(S^2)\circ R(S_b) \\
&\qquad + (E - 2L(S^2))\circ R(S_b) + 2L(S_b)\circ L(S^2),
\end{aligned}$$

and thus,

$$[E, F(S_b)] = L(S_b)\circ E + E\circ R(S_b) = R(S_b)\circ E + E\circ L(S_b), \tag{D.2.12}$$

with the final equality following from (D.2.10).

Hence, combining (D.2.9) and (D.2.12),

$$\begin{aligned}[E,[E,[E,L(S_b)]]] &= -2[E,[E,F(S_b)]] \\ &= -2\big[E,\big(L(S_b)\circ E + E\circ R(S_b)\big)\big] \\ &= -2[E,L(S_b)]\circ E - 2E\circ[E,R(S_b)] \\ &= -2\big([E,L(S_b)]\circ E + E\circ[E,L(S_b)]\big),\end{aligned}$$

thus, finally,

$$[E,[E,[E,L(S_b)]]] = -2\big[E^2, L(S_b)\big] \quad \blacksquare. \tag{D.2.13}$$

D.3 Proof of the Properties of the Multipoles

D.3.1 Closed under ad$(U(\mathfrak{so}(3,\mathbb{R})))$

Lemma D.3.1. *$\forall v \in \mathfrak{so}(3,\mathbb{R})$, $\forall A \in U(\mathfrak{so}(3,\mathbb{R}))$,*

$$[\mathrm{ad}(v), L(A)] = L(\mathrm{ad}(v)(A)).$$

Proof. $\forall W \in U(\mathfrak{so}(3,\mathbb{R}))$,

$$\begin{aligned}[\mathrm{ad}(v), L(A)](W) &= \mathrm{ad}(v)(AW) - A\big(\mathrm{ad}(v)(W)\big) \\ &= \Big(\big(\mathrm{ad}(v)(A)\big)W + A\big(\mathrm{ad}(v)(W)\big)\Big) - A\big(\mathrm{ad}(v)(W)\big) \\ &= L(\mathrm{ad}(v)(A))(W)\end{aligned}$$

where we have used the derivation property of $\mathrm{ad}(v)$ in the second line. □

Theorem D.3.2. *$\forall k \in \mathbb{N}$, $A \in U(\mathfrak{so}(3,\mathbb{R}))$,*

$$\forall A \in U(\mathfrak{so}(3,\mathbb{R})),\ \mathrm{ad}(A)\circ M^{(k)} = M^{(k)}\circ\mathrm{ad}(A). \tag{D.3.1}$$

Proof. As ever we need only prove that (D.3.1) is true $\forall v \in \mathfrak{so}(3,\mathbb{R})$. When $k = 0$, $\forall \alpha \in \mathbb{R}$,

$$M^{(0)}(\alpha) = \alpha,$$

and so (D.3.1) follows from,

$$\mathrm{ad}(v)(\alpha) = 0.$$

When $k \geq 1$, we may use induction. For $k = 1$, $\forall w \in \mathfrak{so}(3,\mathbb{R})$,

$$M^{(1)}(w) = w,$$

and so (D.3.1) is clear. The, assuming the claim is true for the case $k = m$, by definition we have, $\forall w_j \in \mathfrak{so}(3,\mathbb{R})$,

$$M^{(m+1)}\Big(w_1 \otimes \bigotimes_{j=2}^{m+1} w_j\Big) = \frac{\varepsilon(m-1)\circ\varepsilon(m)}{4(m+1)(2m+1)} \circ L(w_1)\circ M^{(m)}\Big(\bigotimes_{j=2}^{m+1} w_j\Big)$$

and so, using Lemma D.3.1,

$$\begin{aligned}
\mathrm{ad}(v)\circ M^{(m+1)}\Big(w_1 \otimes \bigotimes_{j=2}^{m+1} w_j\Big) &= \mathrm{ad}(v)\circ \frac{\varepsilon(m-1)\circ\varepsilon(m)}{4(m+1)(2m+1)} \circ L(w_1)\circ M^{(m)}\Big(\bigotimes_{j=2}^{m+1} w_j\Big) \\
&= \frac{\varepsilon(m-1)\circ\varepsilon(m)}{4(m+1)(2m+1)} \circ \mathrm{ad}(v)\circ L(w_1)\circ M^{(m)}\Big(\bigotimes_{j=2}^{m+1} w_j\Big) \\
&= \frac{\varepsilon(m-1)\circ\varepsilon(m)}{4(m+1)(2m+1)} \circ \Big([\mathrm{ad}(v), L(w_1)] + L(w_1)\circ \mathrm{ad}(v)\Big)\circ M^{(m)}\Big(\bigotimes_{j=2}^{m+1} w_j\Big) \\
&= \frac{\varepsilon(m-1)\circ\varepsilon(m)}{4(m+1)(2m+1)} \circ \Big(L(\mathrm{ad}(v)(w_1))\circ M^{(m)}\Big(\bigotimes_{j=2}^{m+1} w_j\Big) \\
&\qquad + L(w_1)\circ M^{(m)}\circ \mathrm{ad}(v)\Big(\bigotimes_{j=2}^{m+1} w_j\Big)\Big) \\
&= M^{(m+1)}\Big(\mathrm{ad}(v)(w_1)\otimes \bigotimes_{j=2}^{m+1} w_j + w_1 \otimes \mathrm{ad}(v)\Big(\bigotimes_{j=2}^{m+1} w_j\Big)\Big) \\
&= M^{(m+1)}\circ \mathrm{ad}(v)\Big(w_1 \otimes \bigotimes_{j=2}^{m+1} w_j\Big),
\end{aligned}$$

where the final line follows from the derivation property of $\mathrm{ad}(v)$. □

D.3.2 Totally Symmetric and Contractionless

Lemma D.3.3. *The quadrupole* $M^{(2)}$ *satisfies properties* (4.4.20b) *and* (4.4.20c).

Proof. By explicit computation we find,

$$M^{(2)}(S_a \otimes S_b) = \frac{1}{2}(S_a S_b + S_b S_a) - \frac{1}{3}\delta_{ab} S^2.$$

Verifying the properties on this tensor is trivial. □

Definition D.3.4. For $M^{(k)}$ satisfying (4.4.20b) and (4.4.20c),

$$A := L(S_a)\circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) \tag{D.3.2}$$

$$B\circ L(S_a)\circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) := \sum_{p=1}^{k}\sum_{d=1}^{3} \delta_{ab_p} L(S_d)\circ M^{(k)}\Big(S_d \otimes \bigotimes_{j=1, j\neq p}^{k} S_{b_j}\Big) \tag{D.3.3}$$

$$C \circ L(S_a) \circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) := \sum_{p=1}^{k} L(S_{b_p}) \circ M^{(k)}\Big(S_a \otimes \bigotimes_{j=1, j\neq p}^{k} S_{b_j}\Big) \tag{D.3.4}$$

$$D \circ L(S_a) \circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) := \begin{cases} \displaystyle\sum_{p=1}^{k} \sum_{q=1, q\neq p}^{k} \sum_{d=1}^{3} \Big(\\ \qquad \delta_{b_p b_q} L(S_d) \circ M^{(k)}\Big(S_a \otimes S_d \otimes \bigotimes_{j=1, j\neq p,q}^{k} S_{b_j}\Big)\Big) & k \geq 2 \\ 0 & k = 1. \end{cases} \tag{D.3.5}$$

Lemma D.3.5. *For $M^{(k)}$ satisfying* (4.4.20b) *and* (4.4.20c),

$$E \circ A = \Big(\big(-2 - k(k+1)\big) \operatorname{id} + 2B - 2C\Big) \circ A \tag{D.3.6}$$

$$E \circ E \circ A = \Big((k+1)^2(k^2+4) \operatorname{id} - 4(k^2+2)B + 4(k+1)^2 C - 4D\Big) \circ A. \tag{D.3.7}$$

Proof. First, we see from (D.2.4),

$$\operatorname{ad}(S_c) \circ L(S_a) \circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) = L(\operatorname{ad}(S_c)(S_a)) \circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) + L(S_a) \circ \operatorname{ad}(S_c) \circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big),$$

and so,

$$\begin{aligned} &E \circ L(S_a) \circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) \\ &= \sum_{c=1}^{3} \operatorname{ad}(S_c) \circ \operatorname{ad}(S_c) \circ L(S_a) \circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) \\ &= (-2 - k(k+1)) L(S_a) \circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) \\ &\qquad + 2 \sum_{c=1}^{3} L(\operatorname{ad}(S_c)(S_a)) \circ \operatorname{ad}(S_c) \circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big). \end{aligned}$$

We may evaluate this second term with the help of (D.2.11),

$$\begin{aligned} &2 \sum_{c=1}^{3} L(\operatorname{ad}(S_c)(S_a)) \circ \operatorname{ad}(S_c) \circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) \\ &= 2 \sum_{p=1}^{k} \sum_{c,d,e=1}^{3} \varepsilon_{cad} \varepsilon_{c b_p e} L(S_d) \circ M^{(k)}\Big(S_e \otimes \bigotimes_{j=1, j\neq p}^{k} S_{b_j}\Big) \\ &= 2 \sum_{p=1}^{k} \sum_{d=1}^{3} \delta_{a b_p} L(S_d) \circ M^{(k)}\Big(S_d \otimes \bigotimes_{j=1, j\neq p}^{k} S_{b_j}\Big) - 2 \sum_{p=1}^{k} L(S_{b_p}) \circ M^{(k)}\Big(S_a \otimes \bigotimes_{j=1, j\neq p}^{k} S_{b_j}\Big). \end{aligned}$$

The expression for $E\circ A$ then follows. Applying a second E, we may reuse the results so far to find,

$$E\circ B\circ A = -k(k-1)B\circ A$$
$$E\circ C\circ A = \big(-2k\,\mathrm{id} + 2B - k(k+3)C + 2D\big)\circ A.$$

The expression for $E\circ E\circ A$ then follows. □

Lemma D.3.6. *For $M^{(k)}$ satisfying* (4.4.20b) *and* (4.4.20c),

$$M^{(k+1)}\Big(\bigotimes_{j=1}^{k+1} S_{a_j}\Big) = \frac{1}{k+1}\sum_{p=1}^{k+1} L(S_{a_p})\circ M^{(k)}\Big(\bigotimes_{j=1, j\neq p}^{k+1} S_{a_j}\Big) - \frac{1}{(k+1)(2k+1)}\sum_{p=1}^{k+1}\sum_{q=1,q\neq p}^{k+1}\sum_{d=1}^{3}\Big(\delta_{a_p a_q} L(S_d)\circ M^{(k)}\Big(S_d\otimes\bigotimes_{j=1,j\neq p,q}^{k} S_{b_j}\Big)\Big).$$

Proof. By Lemma D.3.5, we see that for $k\geq 1$,

$$\begin{aligned} M^{(k+1)}\Big(\bigotimes_{j=1}^{k+1} S_{a_j}\Big) &= \frac{\varepsilon(k-1)\circ\varepsilon(k)}{4(k+1)(2k+1)}\circ L(S_{a_1})\circ M^{(k)}\Big(\bigotimes_{j=2}^{k+1} S_{a_j}\Big)\\ &= \frac{1}{4(k+1)(2k+1)}\big(4(2k+1)\,\mathrm{id} - 8B + 4(2k+1)C - 4D\big)\\ &\qquad\circ L(S_{a_1})\circ M^{(k)}\Big(\bigotimes_{j=2}^{k+1} S_{a_j}\Big). \end{aligned}$$

Noting that,

$$\begin{aligned} (2B+D)\circ L(S_{a_1})\circ M^{(k)}\Big(\bigotimes_{j=2}^{k+1} S_{a_j}\Big) &= \Bigg(\Big(\sum_{p=1}^{1}\sum_{q=1,q\neq p}^{k+1} + \sum_{p=2}^{k+1}\sum_{q=1,q\neq p}^{1}\Big) + \sum_{p=2}^{k+1}\sum_{q=2,q\neq p}^{k+1}\Bigg)\sum_{d=1}^{3}\\ &\qquad\Big(\delta_{a_p a_q} L(S_d)\circ M^{(k)}\Big(S_d\otimes\bigotimes_{j=1,j\neq p,q}^{k} S_{b_j}\Big)\Big)\\ &= \sum_{p=1}^{k+1}\sum_{q=1,q\neq p}^{k+1}\sum_{d=1}^{3}\delta_{a_p a_q} L(S_d)\circ M^{(k)}\Big(S_d\otimes\bigotimes_{j=1,j\neq p,q}^{k} S_{b_j}\Big), \end{aligned}$$

we are done. □

Lemma D.3.7. $\forall k\in\mathbb{Z}^+$,

$$\sum_{e=1}^{3} L(S_e)\circ M^{(k)}\Big(S_e\otimes\bigotimes_{j=1}^{k-1} S_{c_j}\Big) = \frac{k}{4(2k-1)} L\big(4S^2 + (k-1)(k+1)\big)\circ M^{(k-1)}\Big(\bigotimes_{j=1}^{k-1} S_{c_j}\Big)$$

Proof. For $k = 1$, we see,

$$\sum_{e=1}^{3} L(S_e)\circ M^{(1)}(S_e) = \sum_{e=1}^{3} L(S_e)\circ L(S_e)\circ M^{(0)}(1) = L(S^2)\circ M^{(0)}(1).$$

For $k > 1$, consider,

$$\begin{aligned}
&\sum_{e=1}^{3} L(S_e)\circ M^{(k)}\Big(S_e\otimes\bigotimes_{j=1}^{k-1} S_{c_j}\Big)\\
&\qquad = \frac{1}{4k(2k-1)}\sum_{e=1}^{3} L(S_e)\circ\mathcal{E}(k-2)\circ\mathcal{E}(k-1)\circ L(S_e)\circ M^{(k-1)}\Big(\bigotimes_{j=1}^{k-1} S_{c_j}\Big)\\
&\qquad = \frac{1}{4k(2k-1)}\sum_{e=1}^{3} L(S_e)\circ\mathcal{E}(k-2)\circ[E, L(S_e)]\circ M^{(k-1)}\Big(\bigotimes_{j=1}^{k-1} S_{c_j}\Big) + 0\\
&\qquad = \frac{1}{4k(2k-1)}\sum_{e=1}^{3} L(S_e)\circ\Big([E,[E,L(S_e)]]\\
&\qquad\qquad\qquad - 2(k-1)[E, L(S_e)]\Big)\circ M^{(k-1)}\Big(\bigotimes_{j=1}^{k-1} S_{c_j}\Big),
\end{aligned}$$

and by (D.4.1), (D.2.9) and (D.2.8) we find,

$$\begin{aligned}
... &= \frac{1}{4k(2k-1)}\sum_{e=1}^{3} L(S_e)\circ\Big(-2\big(L(S_e)+R(S_e)\big)\circ E\\
&\qquad\qquad - 2k[E, L(S_e)]\Big)\circ M^{(k-1)}\Big(\bigotimes_{j=1}^{k-1} S_{c_j}\Big)\\
... &= \frac{1}{4k(2k-1)}\Big(4k(k-1)L(S^2) - k(k-1)E\\
&\qquad + 4k\sum_{e,g,a=1}^{3}\varepsilon_{ega} L(S_e)\circ L(S_g)\circ R(S_a)\Big)\circ M^{(k-1)}\Big(\bigotimes_{j=1}^{k-1} S_{c_j}\Big)\\
... &= \frac{1}{4k(2k-1)}\Big(4k(k-1)L(S^2) + k^2(k-1)^2\,\mathrm{id}\\
&\qquad + 4k\sum_{e=1}^{3} L(S_e)\circ R(S_e)\Big)\circ M^{(k-1)}\Big(\bigotimes_{j=1}^{k-1} S_{c_j}\Big)\\
... &= \frac{1}{4k(2k-1)}\Big(4k(k-1)L(S^2) + k^2(k-1)^2\,\mathrm{id}\\
&\qquad - 2kE + 4kL(S^2)\Big)\circ M^{(k-1)}\Big(\bigotimes_{j=1}^{k-1} S_{c_j}\Big)\\
... &= \frac{k}{4(2k-1)} L\big(4S^2 + (k-1)(k+1)\big)\circ M^{(k-1)}\Big(\bigotimes_{j=1}^{k-1} S_{c_j}\Big).
\end{aligned}$$

This result is consistent with the $k = 1$ case. □

Corollary D.3.8. $\forall k \in \mathbb{Z}^+$,

$$B \circ L(S_a) \circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) := \frac{k}{4(2k-1)} L\big(4S^2 + (k-1)(k+1)\big) \circ \sum_{p=1}^{k} \delta_{ab_p} M^{(k-1)}\Big(\bigotimes_{j=1, j\neq p}^{k} S_{b_j}\Big) \tag{D.3.8}$$

$$D \circ L(S_a) \circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) := \begin{cases} \frac{k}{4(2k-1)} L\big(4S^2 + (k-1)(k+1)\big) \circ \sum_{p=1}^{k} \sum_{q=1, q\neq p}^{k} \Big(\delta_{b_p b_q} M^{(k-1)}\Big(S_a \otimes \bigotimes_{j=1, j\neq p,q}^{k} S_{b_j}\Big)\Big) & k \geq 2 \\ 0 & k = 1. \end{cases} \tag{D.3.9}$$

Proof. Direct substitution. □

Corollary D.3.9. *For* $M^{(k)}$ *satisfying* (4.4.20b) *and* (4.4.20c),

$$M^{(k+1)}\Big(\bigotimes_{j=1}^{k+1} S_{a_j}\Big) = \frac{1}{k+1} \sum_{p=1}^{k+1} L(S_{a_p}) \circ M^{(k)}\Big(\bigotimes_{j=1, j\neq p}^{k+1} S_{a_j}\Big) - \frac{k}{4(k+1)(4k^2-1)} L\big(4S^2 + (k-1)(k+1)\big) \circ \sum_{p=1}^{k+1} \sum_{q=1, q\neq p}^{k+1} \delta_{a_p a_q} M^{(k-1)}\Big(\bigotimes_{j=1, j\neq p,q}^{k+1} S_{a_j}\Big).$$

Proof. Direct substitution. □

Theorem D.3.10. $\forall k \in \mathbb{N}$, $k \geq 2$, $M^{(k)}$ *satisfies* (4.4.20b) *and* (4.4.20c).

Proof. Our proof is by induction. The base case $k = 2$ is proven in Lemma D.3.3. Assuming the theorem is true $\forall M^{(m)}$, $m \in \{2, \ldots, n\}$, for some $n \in \mathbb{N}$, then $M^{(m+1)}$ has form given in Corollary D.3.9. That $M^{(m+1)}\Big(\bigotimes_{j=1}^{k+1} S_{a_j}\Big)$ is totally symmetric is clear. To establish contractionlessness, fix two indices $r \neq s \in \{1, \ldots, k+1\}$. If a contraction occurs between two indices within $M^{(k)}$ or $M^{(k-1)}$, then by assumption these give zero. Therefore, we need only check contractions occurring partially inside or completely outside $M^{(k)}$ and $M^{(k-1)}$. Let us first consider the first term in Corollary D.3.9,

$$\frac{1}{k+1} \sum_{a_r, a_s=1}^{3} \delta_{a_r a_s} \sum_{p=1}^{k+1} L(S_{a_p}) \circ M^{(k)}\Big(\bigotimes_{j=1, j\neq p}^{k+1} S_{a_j}\Big) = \frac{2}{k+1} \sum_{d=1}^{3} L(S_d) \circ M^{(k)}\Big(S_d \otimes \bigotimes_{j=1, j\neq p,s}^{k+1} S_{a_j}\Big),$$

which by Lemma D.3.7 becomes,

$$\cdots = \frac{k}{2(k+1)(2k-1)} L\big(4S^2 + (k-1)(k+1)\big) \circ M^{(k-1)} \Big(\bigotimes_{j=1, j\neq p,s}^{k+1} S_{a_j} \Big). \tag{D.3.10}$$

Now consider the second term in Corollary D.3.9,

$$-\frac{k}{4(k+1)(4k^2-1)} \sum_{a_r, a_s = 1}^{3} \delta_{a_r a_s} L\big(4S^2 + (k-1)(k+1)\big) \circ \sum_{p=1}^{k+1} \sum_{q=1, q\neq p}^{k+1} \delta_{a_p a_q} M^{(k-1)} \Big(\bigotimes_{j=1, j\neq p,q}^{k+1} S_{a_j} \Big) =$$

$$= -\frac{k}{4(k+1)(4k^2-1)} L\big(4S^2 + (k-1)(k+1)\big) \circ \sum_{d=1}^{3} \Bigg(2\delta_{dd} M^{(k-1)} \Big(\bigotimes_{j=1, j\neq r,s}^{k+1} S_{a_j} \Big) +$$
$$+ \sum_{q=1, q\neq r,s}^{k+1} \delta_{d a_q} M^{(k-1)} \Big(S_d \otimes \bigotimes_{j=1, j\neq r,q,s}^{k+1} S_{a_j} \Big)$$
$$+ \sum_{q=1, q\neq s,r}^{k+1} \delta_{d a_q} M^{(k-1)} \Big(S_d \otimes \bigotimes_{j=1, j\neq s,q,r}^{k+1} S_{a_j} \Big)$$
$$+ \sum_{p=1, p\neq r,s}^{k+1} \delta_{a_p d} M^{(k-1)} \Big(S_d \otimes \bigotimes_{j=1, j\neq r,p,s}^{k+1} S_{a_j} \Big)$$
$$+ \sum_{p=1, p\neq s,r}^{k+1} \delta_{a_p d} M^{(k-1)} \Big(S_d \otimes \bigotimes_{j=1, j\neq s,p,r}^{k+1} S_{a_j} \Big) \Bigg)$$
$$= -\frac{k}{2(k+1)(2k-1)} L\big(4S^2 + (k-1)(k+1)\big) \circ M^{(k-1)} \Big(\bigotimes_{j=1, j\neq r,s}^{k+1} S_{a_j} \Big).$$

Thus, we get exact cancellation between the two terms, and so $\forall r, s \in \{1, \ldots, k+1\}$, $r \neq s$,

$$\sum_{a_r, a_s = 1}^{3} \delta_{a_r a_s} M^{(k+1)} \Big(\bigotimes_{j=1}^{k+1} S_{a_j} \Big) = 0.$$

□

D.4 Derivation of the Images of Multipoles under Step-Level and Step-Down

Here we will prove the results given in (4.4.21a) and (4.4.21b).

D.4.1 Step-Level Image

The step-level by $v \in \mathfrak{so}(3, \mathbb{R})$ of a multipole $M^{(k)}$ is given by,

$$L^{-}(v) \circ M^{(k)} = \frac{\varepsilon(k-1) \circ \varepsilon(k+1)}{-4k(k+1)} \circ L(v) \circ M^{(k)}.$$

Commuting through the $\varepsilon(\cdot)$ we find,

$$... = \frac{1}{-4k(k+1)}\Big([E,[E,L(v)]] + 2[E,L(v)] - 4k(k+1)L(v)\Big)\circ M^{(k)}.$$

From (D.2.12) we see that,

$$\begin{aligned}[E,[E,L(v)]] &= -2\big(R(v)\circ E + E\circ L(v)\big)\\ &= -2[E,L(v)] - 2\big(R(v)\circ E + L(v)\circ E\big),\end{aligned} \tag{D.4.1}$$

which we combine with the previous equation and (D.2.2) to find,

$$\begin{aligned}L^{-}(v)\circ M^{(k)} &= \frac{1}{-4k(k+1)}\Big(-2\big(R(v)+L(v)\big)\circ E - 4k(k+1)L(v)\Big)\circ M^{(k)}\\ &= \frac{-2k(k+1)}{-4k(k+1)}\big(L(v)-R(v)\big)\circ M^{(k)}\\ &= \frac{1}{2}\mathrm{ad}(v)\circ M^{(k)}\\ &= \frac{1}{2}M^{(k)}\circ \mathrm{ad}(v),\end{aligned} \tag{D.4.2}$$

from which (4.4.21b) follows.

D.4.2 Step-Down Image

The step-down by $S_a \in \mathfrak{so}(3,\mathbb{R})$ of a multipole $M^{(k)}$ is given by,

$$\begin{aligned}L^{\downarrow}(S_a)\circ M^{(k)} &= \frac{\varepsilon(k)\circ\varepsilon(k+1)}{4k(2k+1)}\circ L(S_a)\circ M^{(k)}\\ &= \frac{\big(E^2 + 2(k+1)^2E + k(k+1)^2(k+2)\,\mathrm{id}\big)}{4k(2k+1)}\circ L(S_a)\circ M^{(k)}.\end{aligned}$$

Using Lemma D.3.5 we find,

$$L^{\downarrow}(S_a)\circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_{b_j}\Big) = \frac{\big((2k-1)B - D\big)}{k(2k+1)}\circ A. \tag{D.4.3}$$

The identity (4.4.21a) follows by direct substitution using Lemma D.3.8.

D.4.3 Right Multiplication Images

The results of the previous subsections may be utilised to derive the form of a right multiplication of a multipole. This is essential to expand the Table 4.2.

We observe that from the definition of ad(v) where $v \in \mathfrak{so}(3,\mathbb{R})$,

$$0 = \mathrm{ad}(v)\circ\mathcal{E}(k)\circ M^{(k)} = \mathcal{E}(k)\circ\mathrm{ad}(v)\circ M^{(k)} = \mathcal{E}(k)\circ\big(L(v) - R(v)\big)\circ M^{(k)},$$

and so,

$$\mathcal{E}(k)\circ L(v)\circ M^{(k)} = \mathcal{E}(k)\circ R(v)\circ M^{(k)},$$

which implies that,

$$R^{\downarrow}(v)\circ M^{(k)} = L^{\downarrow}(v)\circ M^{(k)} \tag{D.4.4}$$

$$R^{\uparrow}(v)\circ M^{(k)} = L^{\uparrow}(v)\circ M^{(k)}. \tag{D.4.5}$$

While,

$$\begin{aligned} R^{-}(v)\circ M^{(k)} &= \frac{\varepsilon(k-1)\circ\varepsilon(k+1)}{-4k(k+1)}\circ R(v)\circ M^{(k)} \\ &= \frac{\varepsilon(k-1)\circ\varepsilon(k+1)}{-4k(k+1)}\circ\big(L(v)-\mathrm{ad}(v)\big)\circ M^{(k)} \\ &= L^{-}(v)\circ M^{(k)} - \mathrm{ad}(v)\circ M^{(k)} \\ &= -\frac{1}{2}\mathrm{ad}(v)\circ M^{(k)}, \end{aligned}$$

which from (D.4.2) gives,

$$R^{-}(v)\circ M^{(k)} = -L^{-}(v)\circ M^{(k)}. \tag{D.4.6}$$

D.5 Proof of the Minimal Polynomial of E on Left Multiplied Multipoles

Lemma D.5.1. *$M^{(0)}$ and $M^{(1)}$ are non-zero.*

Proof. By definition. □

Lemma D.5.2. *If $M^{(k)} \neq 0$ is non-zero from some $k \in \mathbb{N}$, $\forall n \leq k$ $M^{(n)} \neq 0$ is also non-zero.*

Proof. If $M^{(n)} = 0$ is zero, then $M^{(k)} = 0$ by their relationship via step-ups. The claim is the contraposition of this fact. □

Lemma D.5.3. *If for some $k \in \mathbb{N}$, $\exists a \in \{1,2,3\}$, such that,*

$$M^{(k)}\Big(\bigotimes_{j=1}^{k} S_a\Big) \neq 0.$$

Then,

$$M^{(k+1)}\Big(\bigotimes_{j=1}^{k+1} S_a\Big) \neq 0.$$

Proof. The case $k = 0$ is trivial, so consider $M^{(k+1)}$ with $k \geq 1$ written as in Lemma D.3.6. Then we may write,

$$\begin{aligned} M^{(k+1)}\Big(\bigotimes_{j=1}^{k+1} S_a\Big) &= L(S_a)\circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_a\Big) - \frac{k}{(2k+1)}\sum_{d=1}^{3} L(S_d)\circ M^{(k)}\Big(S_d\otimes\bigotimes_{j=1}^{k-1} S_a\Big) \\ &= \frac{k+1}{2k+1} S_a M^{(k)}\Big(\bigotimes_{j=1}^{k} S_a\Big) - \frac{k}{(2k+1)}\sum_{d\neq a} S_d M^{(k)}\Big(S_d\otimes\bigotimes_{j=1}^{k-1} S_a\Big). \end{aligned}$$

That each of these terms is linearly independent follows from the Poincaré-Birkhoff-Witt theorem[7]. Thus, since the first term is non-zero by assumption, we have established the claim. □

Lemma D.5.4. (4.4.11b) *is minimal on* $\mathrm{Im}(L(v)\circ M^{(0)})$.

Proof. This follows from Lemma 4.4.13. □

Theorem D.5.5. (4.4.10) *is minimal on* $\mathrm{Im}(L(v)\circ M^{(k)})$, $\forall k \in \mathbb{Z}^+$.

Proof. Consider that the step operators contain pairs of factors from the annihilating polynomial $\varepsilon(k-1)\circ\varepsilon(k)\circ\varepsilon(k+1)$ of (4.4.12). Lemma D.5.3 established that $M^{(k)} \neq 0$, $\forall k \in \mathbb{Z}^+$, and so,

$$M^{(k+1)}\Big(w_1\otimes\bigotimes_{j=2}^{k+1} w_j\Big) = \frac{\varepsilon(k-1)\circ\varepsilon(k)}{4(k+1)(2k+1)}\circ L(w_1)\circ M^{(k)}\Big(\bigotimes_{j=2}^{k+1} w_j\Big) \neq 0$$

Thus, $\varepsilon(k-1)\circ\varepsilon(k)$ is not annihilating on $\mathrm{Im}(L(v)\circ M^{(k)})$. Now consider the step-down from the multipole $M^{(k)}$ of equation (4.4.21a) on the tensor $\bigotimes_{j=1}^{k+1} S_a$,

$$L^{\downarrow}(S_a)\circ M^{(k)}\Big(\bigotimes_{j=1}^{k} S_a\Big) = \begin{cases} \dfrac{k^2}{4(4k^2-1)}\circ L\big(4S^2 \\ \qquad\qquad + (k-1)(k+1)\big)\circ M^{(k-1)}\Big(\bigotimes_{j=1}^{k-1} S_a\Big) & k \geq 2 \\ \dfrac{1}{3}S^2 & k = 1. \end{cases}$$

This is also non-zero, so,

$$L^{\downarrow}(S_{w_1})\circ M^{(k)}\Big(\bigotimes_{j=2}^{k+1} S_{w_j}\Big) = \frac{\varepsilon(k)\circ\varepsilon(k+1)}{4k(2k+1)}\circ L(w_1)\circ M^{(k)}\Big(\bigotimes_{j=2}^{k+1} S_{w_j}\Big) \neq 0.$$

Thus, $\varepsilon(k)\circ\varepsilon(k+1)$ is not annihilating on $\mathrm{Im}(L(v)\circ M^{(k)})$. Finally, consider the form of the step-level from the multipole $M^{(k)}$ of equation (D.4.2),

$$L^{-}(S_a)\circ M^{(k)} = \frac{1}{2}\mathrm{ad}(S_a)\circ M^{(k)}.$$

If $L^{-}(S_a)\circ M^{(k)} = 0$, then $\mathrm{ad}(\mathfrak{so}(3,\mathbb{R}))$ must annihilate $M^{(k)}$. This means that E must also annihilate $M^{(k)}$. But by definition $\varepsilon(k)$ annihilates $M^{(k)}$, so,

$$\mathcal{E}(k)\circ M^{(k)} = 0 = E\circ M^{(k)},$$

i.e.

$$k(k+1)M^{(k)} = 0.$$

This implies $M^{(k)} = 0$, which contradicts Lemma D.5.3. So from,

$$L^{-}(S_{w_1})\circ M^{(k)}\Big(\bigotimes_{j=2}^{k+1} S_{w_j}\Big) = \frac{\varepsilon(k-1)\circ\varepsilon(k+1)}{-4k(k+1)}\circ L(w_1)\circ M^{(k)}\Big(\bigotimes_{j=2}^{k+1} S_{w_j}\Big) \neq 0,$$

$\varepsilon(k-1)\circ\varepsilon(k+1)$ is not annihilating on $\mathrm{Im}(L(v)\circ M^{(k)})$. Therefore, no pair of factors from (4.4.12) is annihilating on $\mathrm{Im}(L(v)\circ M^{(k)})$. Therefore, there is no annihilating polynomial of lower order within (4.4.12). Thus, from Lemma 3.4.51, (4.4.12) is the minimal polynomial. □

D.6 Proof of Scalar Multiple Casimir Action

Here we will prove that on the quotient algebra $A^{(\frac{k}{2})}$,

$$A^{(\frac{k}{2})} := \frac{U(\mathfrak{so}(3,\mathbb{R}))}{I\left(\mathrm{Im}(M^{(k+1)})\right)},$$

of $U(\mathfrak{so}(3,\mathbb{R}))$ by the ideal generated by $\mathrm{Im}(M^{(k+1)})$, $k \in \mathbb{N}$, that the Casimir element S^2 acts as a scalar,

$$L(S^2) = L\left(\frac{-k(k+2)}{4}\right).$$

To do this, consider $\mathrm{Im}(f)$ where,

$$f := S_a\otimes\bigotimes_{j=1}^{k+1} S_{b_j} \mapsto L^{\downarrow}(S_a)\circ L^{\uparrow}(S_{b_1})\circ M^{(k)}\Big(\bigotimes_{j=2}^{k+1} S_{b_j}\Big). \tag{D.6.1}$$

From (4.4.17) we know,

$$L^{\uparrow}(S_{b_1})\circ M^{(k)}(A) = M^{(k+1)}(S_{b_1}\otimes A) = 0, \tag{D.6.2}$$

with the final equality following from $M^{(k+1)} = 0$ in $A^{(\frac{k}{2})}$, and thus $f = 0$. However, we know from the main analysis of $U(\mathfrak{so}(3,\mathbb{R}))$ that $\mathrm{Im}(f)$ can be written as a linear combination of central multiples of $M^{(k)}$. Since $M^{(k)}$ is non-zero in $A^{(\frac{k}{2})}$, and $\mathrm{Im}(f)$

is non-trivial in $U(\mathfrak{so}(3,\mathbb{R}))$, there must be some new identity amongst the central multiples causing $\mathrm{Im}(f)$ to be trivial.

Equations (D.6.1) and (D.6.2) show that we are studying a step-down from the multipole $M^{(k+1)}$. When $k=0$,

$$0 = f(S_a \otimes S_b) = \frac{1}{3} S^2 \delta_{ab} = \frac{1}{3} L(S^2) \circ M^{(0)}(1), \tag{D.6.3}$$

from (4.4.21a). Since $A^{(0)}$ is spanned by $M^{(0)}$ we conclude that on the whole of $A^{(0)}$,

$$L(S^2) = 0. \tag{D.6.4}$$

If $k > 0$ we use (4.4.21a) to find,

$$\begin{aligned} 0 = \frac{L\big(4S^2 + k(k+2)\big)}{4(2k+3)(2k+1)} \circ \sum_{p=1}^{k+1} \Bigg(&(2k+1)\delta_{ab_p} M^{(k)}\Big(\bigotimes_{j\neq p} S_{b_j}\Big) \\ &- \sum_{q=1, q\neq p}^{k+1} \delta_{b_p b_q} M^{(k)}\Big(S_a \otimes \bigotimes_{j\neq p,q} S_{b_j}\Big)\Bigg) \\ = \frac{L\big(4S^2 + k(k+2)\big)}{4(2k+3)(2k+1)} \circ M^{(k)} \Bigg(&\sum_{p=1}^{k+1} \sum_{q=1,q\neq p}^{k+1} \Big(\frac{2k+1}{k}\delta_{ab_p} S_{b_q} - \delta_{b_p b_q} S_a\Big) \otimes \bigotimes_{j\neq p,q} S_{b_j}\Bigg). \end{aligned}$$

In $U(\mathfrak{so}(3,\mathbb{R}))$ the prefactor in the above has trivial kernel, since $U(\mathfrak{so}(3,\mathbb{R}))$ contains no zero divisors[54]. Furthermore, it is clear that there are enough arguments $\{C_j\}$ of the form in the argument of $M^{(k)}$ above such that $\{M^{(k)}(C_j)\}$ spans $\mathrm{Im}(f)$, and therefore $\mathrm{Im}(M^{(k)})$. Since $\mathrm{Im}(M^{(k)})$ is non-trivial in $A^{(\frac{k}{2})}$, and $k > 0$ we must have that,

$$L\big(4S^2 + k(k+2)\big) \circ M^{(k)} = 0. \tag{D.6.5}$$

As $4S^2 + k(k+2)$ is central we may repeat the process by stepping-down (D.6.5), producing a family of identities $\forall n \in \{0, ..., k\}$,

$$\left(\bigcirc_{j=n}^{k} L\big(4S^2 + j(j+2)\big) \right) \circ M^{(n)} = 0, \tag{D.6.6}$$

where $\bigcirc_{j=n}^{k}$ denotes composition over the indexed maps.

A priori, any combination of these $L\big(4S^2 + j(j+2)\big)$ could be responsible for annihilating $\mathrm{Im}(M^{(n)})$. To make progress, let us first consider a non-empty subset $I \subset \{0, ..., k-1\}$ and suppose,

$$\left(\bigcirc_{j\in I} L\big(4S^2 + j(j+2)\big) \right) \circ M^{(n)} = 0, \tag{D.6.7}$$

for some $n \in \{0, ..., k\}$. Since $M^{(k)}$ may be written as a series of step-ups from $M^{(n)}$ by (4.4.17), we may use the fact that the composition in (D.6.7) commutes with step-ups to find,

$$\left(\bigcirc_{j\in I} L\big(4S^2 + j(j+2)\big) \right) \circ M^{(k)} = 0. \tag{D.6.8}$$

Now, observe that for any p we may write,

$$L\big(4S^2 + k(k+2)\big) = L\big(4S^2 + p(p+2)\big) + L\big((k-p)(k+p+2)\big), \tag{D.6.9}$$

which we may use to rewrite (D.6.8) as,

$$\left(\bigcirc_{j\in I} \Big(L\big(4S^2 + k(k+2)\big) - L\big((k-j)(k+j+2)\big)\Big) \right) \circ M^{(k)} = 0. \tag{D.6.10}$$

We note that since $k \notin I$ that there is a left multiplication of a non-zero scalar in (D.6.10). Thus, from (D.6.5) we find,

$$\Big(\prod_{j\in I} -(k-j)(k+j+2) \Big) M^{(k)} = 0, \tag{D.6.11}$$

which implies $\mathrm{Im}(M^{(k)})$ is trivial. This is in contradiction with our construction of $A^{(\frac{k}{2})}$ and thus (D.6.7) must be impossible $\forall n \in \{0, ..., k\}$. This means any annihilating action of a composition of factors $L\big(4S^2 + p(p+2)\big)$ must include the factor with $p = k$.

With this in hand, let us consider the identity (D.6.6) on $M^{(0)}$,

$$\left(\bigcirc_{j=0}^{k} L\big(4S^2 + j(j+2)\big) \right) \circ M^{(0)} = 0, \tag{D.6.12}$$

and notice that it is an annihilating polynomial for $L(S^2)$ on $\mathrm{Im}(M^{(0)})$. Thus, using the results of Section 3.3 we may resolve the identity on $\mathrm{Im}(M^{(0)})$,

$$M^{(0)} = \sum_{j=0}^{k} \left(\bigcirc_{p=0, p\neq j}^{k} \frac{L\big(4S^2 + p(p+2)\big)}{-j(j+2) + p(p+2)} \right) \circ M^{(0)} = \sum_{j=0}^{k} \Pi_j. \tag{D.6.13}$$

From our earlier argument, no annihilating composition like (D.6.7) can exist, and thus we must conclude that $\mathrm{Im}(M^{(0)}) \cap \mathrm{Im}(\Pi_k) \neq \{0\}$, since Π_k contains no factor $L\big(4S^2+k(k+2)\big)$ by definition. However, $\dim(\mathrm{Im}(M^{(0)})) = 1$, and since Π_k is linear, we must conclude that $\mathrm{Im}(M^{(0)}) \subset \mathrm{Im}(\Pi_k)$ and $\mathrm{Im}(M^{(0)}) \cap \mathrm{Im}(\Pi_j) = \{0\}$ for $j \neq k$ by orthogonality.

Thus, for $j \neq k$, $\Pi_j = 0$, and so we have a family of annihilating polynomials for $L(S^2)$ on $\mathrm{Im}(M^{(0)})$, $\forall j \in \{0, ..., k-1\}$,

$$\left(\bigcirc_{p=0, p\neq j}^{k} L\big(4S^2 + p(p+2)\big) \right) \circ M^{(0)} = 0. \tag{D.6.14}$$

But every annihilating polynomial must be a polynomial multiple of the minimal polynomial[46]. Since the family (D.6.14) have only one factor in common, we must conclude that,

$$L\big(4S^2 + k(k+2)\big)\circ M^{(0)} = 0. \tag{D.6.15}$$

By the recursive relationship between the multipoles (4.4.17), all $M^{(k)}$ begin from repeated stepping-up from $M^{(0)}$. $L\big(4S^2 + k(k+2)\big)$ is commutative with step-ups, thus we find $\forall n \in \{0, ..., k\}$,

$$L\big(4S^2 + k(k+2)\big)\circ M^{(n)} = 0. \tag{D.6.16}$$

Since the multipoles $\{M^{(n)} |\, n \in \{0, ..., k\}\}$ form a basis for $A^{(\frac{k}{2})}$, from (D.6.16) we must finally conclude that on the whole of $A^{(\frac{k}{2})}$,

$$L\big(4S^2 + k(k+2)\big) = 0 \quad \blacksquare. \tag{D.6.17}$$

D.7 Proof of the Dimension of the Multipoles

Lemma D.7.1. *Let $f : \mathfrak{so}(3,\mathbb{R})^{\otimes k} \to \mathfrak{so}(3,\mathbb{R})^{\otimes k}$ be a projector into a subspace of kth-order tensors which satisfy the properties 4.4.20. Then,* $\mathrm{Im}(f)$ *is annihilated by* $\varepsilon(k)$.

Proof.

$$\begin{aligned}
E\circ f\Big(\bigotimes_{j=1}^{k} S_{a_j}\Big) &= \sum_{c=1}^{3} \mathrm{ad}(S_c)\circ\mathrm{ad}(S_c)\circ f\Big(\bigotimes_{j=1}^{k} S_{a_j}\Big)\\
&= \sum_{p=1}^{k}\sum_{c,d=1}^{3} \varepsilon_{ca_pd}\, \mathrm{ad}(S_c)\circ f\Big(S_d\otimes\bigotimes_{j=1,j\neq p}^{k} S_{a_j}\Big)\\
&= \sum_{p=1}^{k}\sum_{c,d,e=1}^{3} \varepsilon_{ca_pd}\varepsilon_{cde} f\Big(S_e\otimes\bigotimes_{j=1,j\neq p}^{k} S_{a_j}\Big)\\
&\qquad\qquad + \sum_{p=1}^{k}\sum_{q=1,q\neq p}^{k}\sum_{c,d,e=1}^{3} \varepsilon_{ca_pd}\varepsilon_{ca_qe} f\Big(S_d\otimes S_e\otimes\bigotimes_{j=1,j\neq p,q}^{k} S_{a_j}\Big)\\
&= -2\sum_{p=1}^{k} f\Big(\bigotimes_{j=1}^{k} S_{a_j}\Big)\\
&\quad + \sum_{p=1}^{k}\sum_{q=1,q\neq p}^{k}\sum_{c,d,e=1}^{3} (\delta_{a_pa_q}\delta_{de} - \delta_{a_pe}\delta_{da_q}) f\Big(S_d\otimes S_e\otimes\bigotimes_{j=1,j\neq p,q}^{k} S_{a_j}\Big)\\
&= (-2k - k(k-1)) f\Big(\bigotimes_{j=1}^{k} S_{a_j}\Big)
\end{aligned}$$

$$= -k(k+1)f\Big(\bigotimes_{j=1}^{k} S_{a_j}\Big).$$

□

Lemma D.7.2. *In $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$, the maximal subspace of totally symmetric contractionless tensors has dimension $2k+1$.*

Proof. There are $\binom{k+2}{k} = \frac{(k+2)(k+1)}{2}$ symmetric tensors within $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$. Every possible contraction between these tensors reduces the number of linearly independent tensors by 1. There are $\frac{k(k-1)}{2}$ ways to contract a kth-order tensor, since contraction is a symmetric process. Therefore, there are,

$$\frac{(k+2)(k+1)}{2} - \frac{k(k-1)}{2} = 2k+1,$$

linearly independent, totally symmetric, contractionless tensors. □

Lemma D.7.3. *$\forall k \in \mathbb{N}$, $\mathrm{Im}(M^{(k)})$ is unique subspace annihilated by $\varepsilon(k)$ within the E-orthogonal decomposition of $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$. Furthermore, there are no non-trivial subspaces within $\mathfrak{so}(3,\mathbb{R})^{\otimes k}$ annihilated by $\varepsilon(n)$ for $n \in \mathbb{N}$, $n > k$.*

Proof. The case $k = 0$ is trivial. We may proceed by induction. Suppose the statement is true for $k = m$, then by Lemma 4.4.22 and Theorem 4.4.24 (applied to $T(\mathfrak{so}(3,\mathbb{R}))$), stepping-up from any subspace of $\mathfrak{so}(3,\mathbb{R})^{\otimes m}$ other than $\mathrm{Im}(M^{(m)})$ results in a subspace annihilated by $\varepsilon(n)$ with $n < k$. On the other hand, by assumption, there are no subspaces annihilated by $\varepsilon(m+1)$ or $\varepsilon(m+2)$ that we may step-level or step-down from respectively. Thus, the only subspace which steps-up to one annihilated by $\varepsilon(m+1)$ is $\mathrm{Im}(M^{(m)})$. This subspace is by definition $\mathrm{Im}(M^{(m+1)})$, and from this argument is clearly unique and has the largest n amongst subspaces annihilated by $\varepsilon(n)$. □

Lemma D.7.4. *No totally symmetric contractionless tensors are forced to zero during the quotient from $T(\mathfrak{so}(3,\mathbb{R}))$ to $U(\mathfrak{so}(3,\mathbb{R}))$.*

Proof. This is a consequence of the Poincaré-Birkhoff-Witt theorem[7]. □

Corollary D.7.5. *$\mathrm{Im}(M^{(k)})$ contains all the totally symmetric, contractionless tensors of order k, and therefore has dimension $2k+1$.*

Proof. This follows easily from the lemmas established in this appendix. □

Appendix E

Indefinite-Spin and Arbitrary-Spin Position Operator Algebras

E.1 Proof that $M^{(2)}(S'_p \otimes S'_q) = 0$ in $\mathbf{Cl}(E, \delta)$

Definition E.1.1 (I)**.** Consider a Euclidean three-space (E, δ) and a basis $\{e_a\}$ which is orthonormal with respect to δ. Then, we define $I \in \mathrm{Cl}(E, \delta)$,

$$I \coloneqq e_1 \wedge e_2 \wedge e_3. \tag{E.1.1}$$

This element I is called the "pseudoscalar" element[22] of $\mathrm{Cl}(E, \delta)$.

Lemma E.1.2. *In* $\mathrm{Cl}(E, \delta)$*, the orthonormal set of vectors* $\{e_a\}$ *satisfy,* $\forall a, b \in \{1, 2, 3\}$*,*

$$\frac{1}{2}\big(e_a e_b + e_b e_a\big) = \delta_{ab}. \tag{E.1.2}$$

Proof. By the definition of $\mathrm{Cl}(E, \delta)$. □

Corollary E.1.3.

$$I = e_1 e_2 e_3. \tag{E.1.3}$$

Proof. By Lemma E.1.2 $\forall \sigma \in S_k$,

$$\mathrm{sgn}(\sigma) e_{\sigma(1)} e_{\sigma(2)} e_{\sigma(3)} = e_1 e_2 e_3.$$

□

Corollary E.1.4.

$$II = -1. \tag{E.1.4}$$

Proof. Direct computation. □

Lemma E.1.5. *For all $a \in \{1,2,3\}$,*

$$e_a I = I e_a. \tag{E.1.5}$$

Proof. Fix $k \in \{1,2,3\}$, then,

$$e_k I = e_k e_1 e_2 e_3 = (-1)^2 e_1 e_2 e_3 e_k = I e_k.$$

□

Corollary E.1.6. *For all $A \in \mathrm{Cl}(E,\delta)$,*

$$AI = IA. \tag{E.1.6}$$

Proof. This follows directly from E.1.5 as all elements of $\mathrm{Cl}(E,\delta)$ are algebraic combinations of $\mathbb{R}$ and the $\{e_a\}$. □

Lemma E.1.7. *For all $a \in \{1,2,3\}$,*

$$I e_a = \frac{1}{2} \sum_{b,c=1}^{3} \varepsilon_{abc}\, e_b \wedge e_c. \tag{E.1.7}$$

Proof. Direct computation. □

Corollary E.1.8. *For all $b, c \in \{1,2,3\}$,*

$$e_b \wedge e_c = \sum_{a=1}^{3} \varepsilon_{abc}\, I e_a. \tag{E.1.8}$$

Proof. Immediate consequence of Lemma E.1.7. □

Lemma E.1.9. *For all $p, q \in \{1,2,3\}$,*

$$\frac{1}{2}(S'_p S'_q + S'_q S'_p) + \frac{1}{4}\delta_{pq} = 0. \tag{E.1.9}$$

Proof. From Lemma E.1.2 we apply,

$$\frac{1}{2} \sum_{a=1}^{3} \sum_{b=1}^{3} \varepsilon_{acd} \varepsilon_{bfg}\, II \big(e_a e_b + e_b e_a\big) = \sum_{a=1}^{3} \sum_{b=1}^{3} \varepsilon_{acd} \varepsilon_{bfg}\, II \delta_{ab}.$$

Applying Corollaries E.1.6 and E.1.8 to the left-hand side, and Corollary E.1.4 to the right, we find,

$$\frac{1}{2}\big((e_c \wedge e_d)(e_f \wedge e_g) + (e_f \wedge e_g)(e_c \wedge e_d)\big) = -(\delta_{cf}\delta_{dg} - \delta_{cg}\delta_{df}).$$

Applying the transformation (5.3.2) to both sides yields,

$$\frac{1}{2}(S'_p S'_q + S'_q S'_p) = -\frac{1}{4}\delta_{pq}.$$

□

E.2 Proof of the Algebraic Form of $t(a,b)(c)$

Let us begin by showing some important properties of $t(a,b)(c)$.

Lemma E.2.1. *For all $a, b, e \in \mathcal{V}$,*

$$t(b,a)(e) = -t(a,b)(e). \tag{E.2.1}$$

Proof. Direct computation. □

Lemma E.2.2. *For all $a, b, e \in \mathcal{V}$,*

$$t(a,b)(e) + t(b,e)(a) + t(e,a)(b) = 0. \tag{E.2.2}$$

Proof. Direct computation. □

Lemma E.2.3. *The properties of Lemmas E.2.1, E.2.2, and 2.2.49 entail the constraints, $\forall a, b, c, d, e \in \mathcal{V}$,*

$$f(b, a, e) + f(a, b, e) = 0 \tag{E.2.3a}$$

$$f(a, b, e) + f(b, e, a) + f(e, a, b) = 0 \tag{E.2.3b}$$

$$\begin{aligned} &f(a, b, f(c, d, e)) - f(c, d, f(a, b, e)) \\ &- f(f(a, b, c), d, e) - f(c, f(a, b, d), e) = 0 \end{aligned} \tag{E.2.3c}$$

respectively.

Proof. In any algebra where $f(a, b, c) = t(a,b)(c)$ (replacing the tensor products in f with the product of the algebra), we may translate the properties of Lemmas E.2.1, E.2.2, and 2.2.49 into constraints on f by direct substitution. □

Theorem E.2.4. *For arbitrary Minkowski space-times, the system of constraints in Lemma E.2.3 is satisfied only when,*

$$f(a, b, c) = k\big((a\otimes b - b\otimes a)\otimes c - c\otimes(a\otimes b - b\otimes a)\big),$$

where $k \in \mathbb{R}$.

Proof. In general, the form of f which satisfies the constraints (E.2.3a), (E.2.3b), and (E.2.3c) will depend on the dimension of $\mathcal{V}$; this is because in low dimension we cannot assume that the tensors in our expressions are all linearly independent.

When $\dim(\mathcal{V}) = 1$, all non-zero vectors are linearly dependent, so writing $a = k_a x$ etc. for some non-zero $x \in \mathcal{V}$ we have,

$$f(a, b, c) = (\alpha + \beta + \gamma + \delta + \epsilon + \zeta)k_a k_b k_c\, x\otimes x\otimes x.$$

Applying the constraint (E.2.3a), we find,

$$\alpha + \beta + \gamma + \delta + \epsilon + \zeta = 0,$$

and so $f(a,b,c) = 0$. This trivially satisfies (E.2.3b) and (E.2.3c), and is of the form (E.2.4).

When imposing the constraints (E.2.3a)-(E.2.3c) for $\dim(\mathcal{V}) \geq 2$, we will assume where possible that our set of input vectors are linearly independent. However, linear independence of this set may fail when the dimension of the space considered is too low. To overcome this difficulty, we will employ the following combinatorial argument. Consider a basis $I = \{b_j\}$ for $\mathcal{V}$ indexed over a set J, and a set of parameterised linear combinations $D = \{\sum_{j\in J} \lambda_j^{(k)} b_j\}$ indexed over a set K. Then, to impose a constraint defined with n vectors for $n > \dim(\mathcal{V})$, let us fix $|D| = n - \dim(\mathcal{V})$ and evaluate it using all n-tuples of vectors $(\sigma(1), \ldots, \sigma(n))$ for all bijective $\sigma : \{1, \ldots, n\} \to I \cup D$. We consider valid only those solutions which are independent of the parameterisation of the elements of D, and independent of the choice of σ.

Now let us continue with the proof, and implicitly utilise either linear independence or this combinatorial argument when necessary. When $\dim(\mathcal{V}) \geq 3$, imposing the constraint (E.2.3a) entails,

$$[\alpha = -\delta] \wedge [\beta = -\gamma] \wedge [\epsilon = -\zeta], \tag{E.2.4}$$

and so,

$$f(a,b,c) = \alpha(a\otimes b - b\otimes a)\otimes c + \epsilon\, c\otimes(a\otimes b - b\otimes a) + \beta\big(a\otimes c\otimes b - b\otimes c\otimes a\big). \tag{E.2.5}$$

Next, imposing (E.2.3b) on (E.2.5) entails,

$$[\alpha + \epsilon = \beta], \tag{E.2.6}$$

thus,

$$\begin{aligned} f(a,b,c) &= \alpha(a\otimes b - b\otimes a)\otimes c + \epsilon\, c\otimes(a\otimes b - b\otimes a) \\ &\quad + (\alpha+\epsilon)\big(a\otimes c\otimes b - b\otimes c\otimes a\big). \end{aligned} \tag{E.2.7}$$

Finally, imposing (E.2.3c) on (E.2.7) entails,

$$[\alpha^2 + \alpha\epsilon = 0] \wedge [\epsilon^2 + \alpha\epsilon = 0] \wedge [(\alpha+\epsilon)^2 = 0],$$

which reduces to,

$$[\alpha = -\epsilon], \tag{E.2.8}$$

yielding (E.2.4).

There is an edge case when $\dim(\mathcal{V}) = 2$ which we shall now address. Imposing (E.2.3a) yields (E.2.4) as before, however, in two-dimensions, all tensors of the form (E.2.5) satisfy the constraint (E.2.3b). Thus, we must impose (E.2.3c) directly on (E.2.5), yielding,

$$\big([\alpha = -\epsilon] \wedge [\beta = 0]\big) \vee \big([\alpha = \epsilon] \wedge [\beta = -\epsilon]\big).$$

The first solution in the above gives the general case, and the second solution yields only a trivial solution,

$$f(a, b, c) = \alpha\big(a \wedge b \wedge c\big) = 0,$$

with the final equality following from the necessary linear dependence of $\{a, b, c\}$. $\square$